`SpinPN` notations and definitions

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The purpose of this note is to give all the relevant definitions for the post-Newtonian results gathered in the repository `SpinPN`.

I. GUIDE TO PUBLISHED REFERENCES

We refer the user to the review [1] for a comprehensive overview of the approach used for our post-Newtonian (PN) calculations.

The purpose of the SpinPN repository is to gather in one place the results of a number of post-Newtonian computations of higher-order spin effects. The articles these results were presented in are, in chronological order:

- [2]: SO dynamics at next-to-next-to leading order (3.5PN)
- [3]: SO dynamics at next-to-next-to leading order (3.5PN): center-of-mass frame and conserved quantities
- [4]: SO emitted energy flux at next-to-next-to leading order (3.5PN)
- [5]: SO tail terms in the waveform at next-to-leading order (4PN)
- [6]: SSS leading-order terms in the dynamics and energy flux (3.5PN)
- [7]: SS next-to-leading order in the dynamics and energy flux (3PN)

The nomenclature used here is SO for spin-orbit terms (linear in spin), SS for quadratic-in-spin terms (including both self-spin terms S_1^2 and cross terms $S_1 S_2$) and similarly SSS for cubic terms. This repository does not (yet) include files for the non-spinning expressions. We will not give a literature review here, and we will simply note that, to this date, these results extend to 4PN for the dynamics and 3.5PN for the energy flux and phase evolution. The review [1] gives many useful references.

II. DEFINITIONS

The results presented here were obtained in a harmonic gauge, $\partial_\nu h^{\mu\nu} = 0$ for the gravitational perturbation $h^{\mu\nu} = \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}$. The formalism used to represent the spinning compact objects is the one of point particules carrying spin-induced multipolar structure. We refer to Section II in [7] and Sections II-III in [6] for a summary of the multipolar formalism. Along with the worldlines tracing the position of the compact objects $y_A^i(t)$, the formalism equips them with an antisymmetric spin tensor $S_A^{\mu\nu}$ for $A = 1, 2$. The spin supplementary condition used in this work is the covariant one (or Tulczyjew) $S^{\mu\nu} p_\nu = 0$, with p_μ the particle's linear momentum.

We scale out a factor c of all spin variables to be able to treat the spin as a Newtonian quantity for rapidly-spinning compact objects, according to

$$S = cS_{physical} = Gm^2\chi\,, \tag{1}$$

where S_{physical} has the dimension of an angular momentum and $\chi \in [-1, 1]$ is the dimensionless spin.

We present our results under three forms:

- in a general frame, before any transformation has been performed (no file suffix);
- in the center-of-mass frame (file suffix `_CM_`); for the energy flux, only instantaneous terms are given for general orbits (additional file suffix `_inst_`)
- after reduction of the results to circular orbits in the spin-aligned configuration (file suffix `_circ_`) – note that these results do not cover quasi-circular precessing orbits with misaligned spins.

In the general frame, we use the notations summarized in Table I. Results are given either in terms of the spatial components of the spin tensor, S_A^{ij} with $A = 1, 2$ (the mixed components of the tensor, S_A^{0i}, have been eliminated with the help of the SSC), or in terms of conserved-norm spin vectors S_A^i. The construction for these spin vectors is explained e.g. in (II C) of [7], and their norm is conserved in the Euclidean sense,

$$S_{Ai}S_A^i = \delta_{ij}S_A^iS_A^j = S^2 = \text{const}\,. \tag{2}$$

Here, the conserved spin norm matches the conserved norm for the covariant formalism, i.e. $S^2 = S_{\mu\nu}S^{\mu\nu}/2$. The spin vector and the spin tensor are essentially duals (see (**??**) below), and the relation between the two involves the Levi-Civita tensor ε_{ijk}. The choice of using either representation is made to simplify the expressions, e.g. the energy is given with the spin tensor and the angular momentum is given with the spin vector. The magnitude of the spin-induced quadrupole and octupole are parameterized by the constant coefficients κ_A and λ_A, which take the value 1 for Kerr black holes but can vary for neutron stars according to the equation of state.

In the center-of-mass frame (COM) frame, we use the notations summarized in Table II. The transformation to the COM frame is done by cancelling the center-of-mass integral of motion G_i which is such that $\ddot{G}_i = 0$. In practice, this amounts to transforming the position of the two bodies according to

$$\boldsymbol{y}_1 = \frac{m_2}{m}r\boldsymbol{n} + \boldsymbol{z}\,, \tag{3a}$$

$$\boldsymbol{y}_2 = -\frac{m_1}{m}r\boldsymbol{n} + \boldsymbol{z}\,, \tag{3b}$$

with $\boldsymbol{z}$ starting at 1PN. Instead of the individual spin vectors, we will use the combinations

$$\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2\,, \tag{4a}$$

$$\boldsymbol{\Sigma} = \frac{m}{m_2}\boldsymbol{S}_2 - \frac{m}{m_1}\boldsymbol{S}_1\,, \tag{4b}$$

which are respectively symmetric and antisymmetric under the exchange of the two bodies. These combinations are built from the conserved-norm spin vectors $\boldsymbol{S}_A$, but they are not of conserved norm themselves for generic precessing orbits. For the coefficients of the spin-induced quadrupole and octupole, we use the symmatric and antisymmetric combinations

$$\kappa_\pm = \kappa_1 \pm \kappa_2 \,, \tag{5a}$$
$$\lambda_\pm = \lambda_1 \pm \lambda_2 \,. \tag{5b}$$

Note that this definition (maybe not optimal, but kept for historical reasons) does *not* include the customary factor $1/2$. In particular, for binary black hole systems with $\kappa_1 = \kappa_2 = 1$, we have $\kappa_- = 0$ and $\kappa_+ = 2$.

Finally, we will also give results after reduction to the circular orbit limit in the spin-aligned case, with notations given in Table III. This limit is appropriate to represent systems having radiated away their excentricity, with a departure from exactly circular orbits solely caused by radiation reaction. Radiation reaction enters at the 2.5PN relative order, which for spin effects would be at least 4PN (1.5PN SO + 2.5PN), a higher order than what we consider in the dynamics. With both spins are aligned with the normal to the orbital plane, orbital plane precession does not occur. Note that these results do not cover precessing, quasi-circular orbits (as investigated in [5, 8]). We will represent the spins by their components on $\boldsymbol{\ell}$ as $\boldsymbol{S}_A = (\boldsymbol{S}_A \cdot \boldsymbol{\ell})\boldsymbol{\ell}$, with

$$\boldsymbol{\ell} = \frac{\boldsymbol{n} \times \boldsymbol{v}}{|\boldsymbol{n} \times \boldsymbol{v}|} \,. \tag{6}$$

Denoting by φ the orbital phase and $\omega = \dot{\varphi}$ the orbital frequency, we use for circular-orbits results the PN parameter (formally of order $1/c^2$)

$$x = \left(\frac{Gm\omega}{c^3}\right)^{\frac{2}{3}} \,. \tag{7}$$

To express the modified Kepler relation, we will also use the PN parameter $\gamma = Gm/rc^2$. Given the emitted energy flux $\mathcal{F}(x)$ and the conserved energy $E(x)$, we can write the balance equation

$$\mathcal{F} = -\frac{dE}{dt} \tag{8}$$

to deduce the evolution of the orbital frequency and phase over time. Rewriting this evolution by using ω as a tracking parameter, one can obtain the orbital phasing of the system as a function of x, $\varphi(x)$.

Note that our harmonic gauge differs from the one used for ADM-type computations, and the construction of our conserved-norm spin vectors differs from the construction of canonical spin variables in the ADM approach. The map from one set of results to the other includes a contact transformation for the position of the particles as well as a transformation of the spin vectors. This map is not given in the present repository, we instead refer the user to Section VII D in [5], App. D in [7] and App. B in [6].

III. STRUCTURE OF THE POST-NEWTONIAN EXPANSION

In this section, we give the post-Newtonian structure of the results we present, merely pointing out the orders at which corrections start and what is included in the repository. Here we use the convention $\mathcal{O}(n) = \mathcal{O}(c^{-n})$, formally treating $1/c$ itself as the perturbative parameter, and so that $1\text{PN} = \mathcal{O}(2)$.

Having a description of the structure of the PN expansion for spin effects is useful in the following sense. We can think of the PN results as a double formal series both in spin and in the PN expansion parameter. In general, performing operations on PN quantities (e.g. taking a time derivative where the accelerations and derivatives of the spins have to be evaluated) means replacing some quantities with their PN expression. This leads to mixing between the spin orders: for instance, taking the derivative of a SO $\mathcal{O}(3)$ quantity generates, through the SS equations of motion, some SO$\times$SS terms that are SSS $\mathcal{O}(7)$. In the following, we give the structure as a formal decomposition, with the spin order meaning the number of explicit occurences of the spins.

Let us begin with the relation between the spin tensor and the spin vector. At leading order, the relation between the spin tensor spatial components S_A^{ij} and the conserved-norm spin vectors S_A^i reads as the duality

$$S_{Ai} = \frac{1}{2}\varepsilon_{ijk}S_A^{jk} + \mathcal{O}(2)\,, \tag{9a}$$

$$S_A^{ij} = \varepsilon^{ijk}S_{Ak} + \mathcal{O}(2)\,. \tag{9b}$$

The PN structure of the higher-order corrections in the relation between spin variables is

$$\begin{aligned}\boldsymbol{S} = \boldsymbol{S}^0_{SO} + \frac{1}{c^2}\boldsymbol{S}^2_{SO} + \frac{1}{c^4}\boldsymbol{S}^4_{SO} + \frac{1}{c^5}\boldsymbol{S}^5_{SO} + \frac{1}{c^6}\boldsymbol{S}^6_{SO} \\ + \frac{1}{c^3}\boldsymbol{S}^3_{SS} + \frac{1}{c^5}\boldsymbol{S}^5_{SS} + \mathcal{O}(7)\,,\end{aligned} \tag{10}$$

with and identical structure for either (9a) or (9b). Note here the occurence of an odd term at $\mathcal{O}(5)$. This term is responsible for differences in structure according to which spin variables is used, and using the conserved-norm spins yields more natural structure. For instance, the conserved angular momentum would show unexpected odd (in relative sense) SO terms at $\mathcal{O}(6)$ if expressed with S_A^{ij}, while it has the more usual even-relative PN structure when expressed with S_{Ai}.

The equations of motion give the accelerations of the two bodies. They have the following structure, either for individual general-frame accelerations or for the COM relative acceleration:

$$\begin{aligned}\boldsymbol{A} = \frac{1}{c^3}\boldsymbol{A}^3_{\text{SO}} + \frac{1}{c^5}\boldsymbol{A}^5_{\text{SO}} + \frac{1}{c^7}\boldsymbol{A}^7_{\text{SO}} + \frac{1}{c^8}\boldsymbol{A}^8_{\text{SO,RR}} \\ + \frac{1}{c^4}\boldsymbol{A}^4_{\text{SS}} + \frac{1}{c^6}\boldsymbol{A}^6_{\text{SS}} + \frac{1}{c^8}\boldsymbol{A}^8_{\text{SS}} \\ + \frac{1}{c^7}\boldsymbol{A}^7_{\text{SSS}} + \frac{1}{c^8}\boldsymbol{A}^8_{\text{SSSS}} + \mathcal{O}(9)\,,\end{aligned} \tag{11}$$

where RR indicates radiation-reaction terms, and tails indicates the presence of tail terms in the equations of motion at the 4PN order. In its current version, the content of the repository is limited to $\mathcal{O}(7)$, but we indicated the order $\mathcal{O}(8)$ in the structure as well to highlight the occurence of SO,RR and SSSS terms.

The equations of motion are completed by the precession equations, that govern the time evolution of the spins. When using conserved-norm spin vectors, the precession equations take the form

$$\dot{\boldsymbol{S}}_A = \boldsymbol{\Omega}_A \times \boldsymbol{S}_A\,, \tag{12}$$

where $\boldsymbol{\Omega}_A$ is the precession vector, with structure

$$\begin{aligned}\boldsymbol{\Omega} = &\frac{1}{c^2}\boldsymbol{\Omega}^2_{\text{NS}} + \frac{1}{c^4}\boldsymbol{\Omega}^4_{\text{NS}} + \frac{1}{c^6}\boldsymbol{\Omega}^6_{\text{NS}} \\ &+ \frac{1}{c^3}\boldsymbol{\Omega}^3_{\text{SO}} + \frac{1}{c^5}\boldsymbol{\Omega}^5_{\text{SO}} \\ &+ \frac{1}{c^6}\boldsymbol{\Omega}^6_{\text{SS}} + \mathcal{O}(7)\,.\end{aligned} \tag{13}$$

The spin order shown here is the explicit spin order in $\boldsymbol{\Omega}$, which differs by 1 from the corresponding spin order in the dynamics. For instance, investigating the dynamics at the SS level requires the SO terms in $\boldsymbol{\Omega}$. Note that the choice of spin variable matters here. In the time derivative of the spin tensor $\dot{S}^{ij}_A$, odd terms at $\mathcal{O}(5)$ would occur at linear order in spin (corresponding to NS in $\boldsymbol{\Omega}$). In the current repository, we do not include the terms SS $\mathcal{O}(6)$ as they were not given in [6]. They would be relevant in particular for the SSS $\mathcal{O}(7)$ terms in $\boldsymbol{J}$.

The conserved energy (obtained as a conserved quantity in the conservative dynamics, ignoring radiation reaction) has the PN structure

$$\begin{aligned}E_S = &\frac{1}{c^3}E^3_{\text{SO}} + \frac{1}{c^5}E^5_{\text{SO}} + \frac{1}{c^7}E^7_{\text{SO}} \\ &+ \frac{1}{c^4}E^4_{\text{SS}} + \frac{1}{c^6}E^6_{\text{SS}} + \frac{1}{c^8}E^8_{\text{SS}} \\ &+ \frac{1}{c^7}E^7_{\text{SSS}} + \frac{1}{c^8}E^8_{\text{SSSS}} + \mathcal{O}(9)\,,\end{aligned} \tag{14}$$

while for the conserved total angular momentum we have

$$\begin{aligned}\boldsymbol{J}_S = &\frac{1}{c}\boldsymbol{J}^1_{\text{SO}} + \frac{1}{c^3}\boldsymbol{J}^3_{\text{SO}} + \frac{1}{c^5}\boldsymbol{J}^5_{\text{SO}} + \frac{1}{c^7}\boldsymbol{J}^7_{\text{SO}} \\ &+ \frac{1}{c^4}\boldsymbol{J}^4_{\text{SS}} + \frac{1}{c^6}\boldsymbol{J}^6_{\text{SS}} + \frac{1}{c^8}\boldsymbol{J}^8_{\text{SS}} \\ &+ \frac{1}{c^7}\boldsymbol{J}^7_{\text{SSS}} + \frac{1}{c^8}\boldsymbol{J}^8_{\text{SSSS}} + \mathcal{O}(9)\,,\end{aligned} \tag{15}$$

with a leading-order spin contribution which is just the total spin,

$$\boldsymbol{J}^1_{SO} = \boldsymbol{S}_1 + \boldsymbol{S}_2\,. \tag{16}$$

Again, we gave the structure up to $\mathcal{O}(8)$ included while our results stop at $\mathcal{O}(7)$. The terms SSS $\mathcal{O}(7)$ in $\boldsymbol{J}$ were not computed in [6], as they do not enter the phasing.

For the energy flux emitted in gravitational waves, we distinguish hereditary contributions (at this order for spin terms, only tail terms appear) from instantaneous ones, see [1]. Factoring out a $\mathcal{O}(c^{-5})$ prefactor, the structure of the spin terms is

$$\begin{aligned}\mathcal{F}_S = \frac{1}{c^5}\Bigg[&\frac{1}{c^3}\mathcal{F}^3_{\text{SO}} + \frac{1}{c^5}\mathcal{F}^5_{\text{SO}} + \frac{1}{c^6}\mathcal{F}^6_{\text{SO,tail}} + \frac{1}{c^7}\mathcal{F}^7_{\text{SO}} + \frac{1}{c^8}\mathcal{F}^8_{\text{SO,tail}} \\ &+ \frac{1}{c^4}\mathcal{F}^4_{\text{SS}} + \frac{1}{c^6}\mathcal{F}^6_{\text{SS}} + \frac{1}{c^7}\mathcal{F}^7_{\text{SS,tail}} + \frac{1}{c^8}\mathcal{F}^8_{\text{SS}} \\ &+ \frac{1}{c^7}\mathcal{F}^7_{\text{SSS}} + \frac{1}{c^8}\mathcal{F}^8_{\text{SSSS}} + \mathcal{O}(9)\Bigg]\,,\end{aligned} \tag{17}$$

Tail contributions were not computed for fully general, non-quasicircular orbits, but only for quasi-circular precessing orbits (leading order in [8] and at next-to-leading order in [5]). They will be only given for spin-aligned, non-precessing circular orbits. We distinguish therefore between instantaneous terms, that will be given in the COM frame for generic orbits, and tail terms that will be given only for circular orbits.

For non-precessing circular orbits, the structure of the modified Kepler relation between separation radius r and orbital frequency ω is the same as hte conservative dynamics:

$$\begin{aligned}\gamma_S = x \Big[& x^{3/2}\gamma^3_{\text{SO}} + x^{5/2}\gamma^5_{\text{SO}} + x^{7/2}\gamma^7_{\text{SO}} \\ &+ x^2\gamma^4_{\text{SS}} + x^3\gamma^6_{\text{SS}} + x^4\gamma^8_{\text{SS}} \\ &+ x^{7/2}\gamma^7_{\text{SSS}} + x^4\gamma^8_{\text{SSSS}} + \mathcal{O}(9)\Big]\,, \end{aligned} \tag{18}$$

while the structure of the obtained phasing is the same as the structure of the flux, with the exception of the occurence of $\ln x$ as a result of the formal term-by-term integration:

$$\begin{aligned}\varphi_S = x^{-5/2} \Big[& x^{3/2}\varphi^3_{\text{SO}} + x^{5/2}\ln x\varphi^5_{\text{SO}} + x^3\varphi^6_{\text{SO,tail}} + x^{7/2}\varphi^7_{\text{SO}} + x^4\varphi^8_{\text{SO,tail}} \\ &+ x^2\varphi^4_{\text{SS}} + x^3\varphi^6_{\text{SS}} + x^{7/2}\varphi^7_{\text{SS,tail}} + x^4\varphi^8_{\text{SS}} \\ &+ x^{7/2}\varphi^7_{\text{SSS}} + x^4\varphi^8_{\text{SSSS}} + \mathcal{O}(9)\Big]\,. \end{aligned} \tag{19}$$

For clarity, let us list what is not included in the present version of this repository:

- NS effects are not summarized here, this is left to a future extension to include recent progress at 4PN (there are subtleties as to how to present these high-order results that were obtained with dimensional regularization and endured shift transformations);
- SS effects at $\mathcal{O}(8)$ are not included, they were not computed in our approach yet (see however recent works by Levi&Steinhoff);
- SS terms at $\mathcal{O}(6)$ in $\boldsymbol{\Omega}$ and SSS terms at $\mathcal{O}(7)$ in $\boldsymbol{J}$ are not included, as they were not computed in [6] (and are not needed for the flux and phasing computation at $\mathcal{O}(7)$);
- SO tail terms were computed for quasi-circular, precessing orbits, but only the results for non-precessing orbits are given here.

IV. NOTATIONS

Tensorial environment

The results presented here are `Mathematica` expressions produced with the help of xTensor [9], as well as the package `PNComBin` developed for post-Newtonian computations by Guillaume Faye. They contain scalar, vector and tensor quantities, the syntax for which is hopefully quite transparent. The (trivial) spatial manifold is three-dimensional and Euclidean. Normal indices are contravariant, while indices with minus signs indicate covariant indices: for a vector $\boldsymbol{a}$, `a[i]` $= a^i$, `a[-i]` $= a_i$, `a[i,-j]` $= {a^i}_j$, and so on. The spatial indices are given by latin letters, with free indices in the range $i, j, \ldots$ and contracted indices in the range $a, b, \ldots$. Repeated indices are summed according to the

Einstein notation. The head `Scalar[]` is used to shield contracted dummy indices against confusion with the rest of the expression. The manifold is equipped with an ordinary Euclidean metric `Metricdelta[-i,-j]` $= \delta_{ij}$ – thus lowering or rising all the indices is trivial, e.g. `a[-i]` $= a_i = \delta_{ij} a^j$. The completely antisymmetric Levi-Civita tensor is `epsilonMetricdelta[-i,-j,-k]` $= \varepsilon_{ijk}$.

All expressions given should be valid xTensor expressions, nonetheless, thus contracted indices should always come in pairs (one upper and one lower index), and multiple occurences of indices should be allowed only when shielded by a `Scalar[]` head.

The content of each file is directly the expression for the relevant quantity, not a `Set` or `IndexSet` instruction. Loading this quantity in Mathematica would therefore take the form `ai = Get[''ai.m'']`, or `IndexSet[ai[i_], Get[''ai.m'']]` when using the xTensor syntax to handle indices. One therefore needs to know the free index (or indices). These should be transparent from the name of the file, as the free indices are always i or i, j.

Notations in files

The gravitational constant G and the speed of light c are kept explicit everywhere. All dynamical quantities are functions of time.

In file names, the convention is the following: a given quantity is given decomposed in spin order (e.g. `a1i_SO7.m` containing only the spin-orbit terms, `a1i_SS6.m` containing only the terms quadratic in spin), along with the grand total (e.g. `a1i_SO7_SS6_SSS7.m`) which is simply the sum of the individual spin orders (excluding, for now, non-spinning terms). The number indicates the *last PN order included* in the file, in the sense of $\mathcal{O}(c^{-n})$.

Note that this does not tell the highest PN order to which a given result is valid. For instance, SSS terms in the dynamics are in fact valid up to $\mathcal{O}(8)$ included, since the first corrections are at the 1PN relative order, at $\mathcal{O}(9)$. For SO terms, on the contrary, there would be $\mathcal{O}(8)$ terms in the dynamics due to the radiation reaction at the 2.5PN relative order.

The additional `c` in the name of spin vectors is here for historical reasons, indicating that these variables are based on the construction of conserved-normed spins.

TABLE I. General-frame quantities - for body index $A = 1, 2$

	Code	Meaning	Notes
m_A	`mA`	Mass of body A	Conserved mass, see [6] and [7]
y_A^i	`yAi[i]`	Position of body A	–
v_A^i	`vAi[i]`	Velocity of body A	–
r_{12}	`r12[]`	Separation $\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|$	–
n_{12}^i	`n12i[i]`	Separation unit vector $y_1^i - y_2^i = r_{12} n_{12}^i$	–
S_A^{ij}	`SAij[i,j]`	Spin tensor of body A	Antisymmetric, spatial components of the spin tensor $S_A^{\mu\nu}$
S_A^i	`SAci[i]`	Conserved-norm spin vector of body A	$S_{Ai} S_A^i = S^2 =$ const, see [7] for the construction
κ_A	`\[Kappa]A`	Coefficient of spin-induced quadrupole for A	Constant, value 1 for a black hole
λ_A	`\[Lambda]A`	Coefficient of spin-induced octupole for A	Constant, value 1 for a black hole

TABLE II. Center-of-mass-frame quantities

	Code	Meaning	Notes
m	`mtot`	Total mass $m = m_1 + m_2$	–
ν	`\[Nu]`	Symmetric mass ratio $\nu = m_1 m_2/m^2$	–
δ	`\[Delta]`	Fractional mass difference $\delta = (m_1 - m_2)/m$	–
r	`rCM[]`	Relative distance $\lvert \boldsymbol{y}_1 - \boldsymbol{y}_2 \rvert$	–
n^i	`nCMi[i]`	Unit separation vector $y_1^i - y_2^i = r n^i$	–
v^i	`vCMi[i]`	Relative velocity $v_1^i - v_2^i$	–
S^i	`Sci[i]`	Spin combination (4a)	built from conserved-norm spin vectors S_A^i
Σ^i	`Sigmaci[i]`	Spin combination (4b)	built from conserved-norm spin vectors S_A^i
$\kappa_\pm$	`\[Kappa]plus`, `\[Kappa]minus`	Combination $\kappa_\pm = \kappa_1 \pm \kappa_2$	No factor 1/2!
$\lambda_\pm$	`\[Lambda]plus`, `\[Lambda]minus`	Combination $\lambda_\pm = \lambda_1 \pm \lambda_2$	No factor 1/2!

TABLE III. Spin-aligned circular orbits quantities

	Code	Meaning	Notes
φ	`\[Phi][]`	Orbital phase	–
ω	`\[Omega][]`	Orbital frequency $\omega = \dot{\varphi}$	–
x	`x[]`	PN parameter $x = (Gm\omega/c^3)^{2/3}$	–
x_0	`x0`	Constant of integration in $\varphi(x)$	See [1]
γ	`\[Gamma][]`	PN parameter $\gamma = Gm/(rc^2)$	–
ℓ^i	`lCMi[i]`	Unit nornal to the orbital plane $\boldsymbol{\ell} \propto \boldsymbol{n} \times \boldsymbol{v}$	Constant in the absence of precession
S_ℓ	`Scl[]`	Spin combination (4a), $\boldsymbol{\ell}$-component	Aligned spins: $(\boldsymbol{S}_A \cdot \boldsymbol{\ell}) = \text{const}$
Σ_ℓ	`Sigmacl[]`	Spin combination (4b), $\boldsymbol{\ell}$-component	Aligned spins: $(\boldsymbol{S}_A \cdot \boldsymbol{\ell}) = \text{const}$

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