SUN Yilin
520030910361

Homework 2

1

This problem requires us to show that $f^{-1}(C)$ is convex. We want to show that if $\boldsymbol{x_1}$, $\boldsymbol{x_2} \in f^{-1}(C)$, then $\theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{x_2} \in f^{-1}(C)$ for any $\theta \in [0,1]$. To show whether $\theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{x_2} \in C$ or not, we can check $f(\theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{x_2})$.
$f(\theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{x_2}) = \boldsymbol{A}(\theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{x_2}) + \boldsymbol{b} = \boldsymbol{A}\theta\boldsymbol{x_1} + \boldsymbol{A}\overline{\theta}\boldsymbol{x_2} + \theta\boldsymbol{b} + \overline{\theta}\boldsymbol{b} = \theta(\boldsymbol{A}\boldsymbol{x_1} + \boldsymbol{b}) + \overline{\theta}(\boldsymbol{A}\boldsymbol{x_2} + \boldsymbol{b}) = \theta f(\boldsymbol{x_1}) + \overline{\theta} f(\boldsymbol{x_2})$. Since $\boldsymbol{x_1}$, $\boldsymbol{x_2} \in f^{-1}(C)$, $f(\boldsymbol{x_1})$, $f(\boldsymbol{x_2}) \in C$, then by the convexity of C, $\theta f(\boldsymbol{x_1}) + \overline{\theta} f(\boldsymbol{x_2}) \in C$, *i.e.* $f(\theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{x_2}) \in C$. Thus $\theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{x_2} \in f^{-1}(C)$ for any $\theta \in [0,1]$. Thus $f^{-1}(C)$ is convex.

2

Since C_1 and C_2 are nonempty, $\exists \boldsymbol{x_1} \in C_1$, $\exists \boldsymbol{x_2} \in C_2$. Thus by definition of C, $\boldsymbol{x_1} - \boldsymbol{x_2} \in C$. Thus C is nonempty.
If $\boldsymbol{0} \in C$, then $\exists \boldsymbol{x_1} \in C_1$, $\exists \boldsymbol{x_2} \in C_2$ such that $\boldsymbol{x_1} = \boldsymbol{x_2}$, then $C_1 \cap C_2 \neq \varnothing$, which contradicts the fact that $C_1 \cap C_2 = \varnothing$. Thus $\boldsymbol{0} \notin C$.
Assume $\boldsymbol{x}, \boldsymbol{y} \in C$, then $\exists \boldsymbol{x_1}, \boldsymbol{y_1} \in C_1$, $\exists \boldsymbol{x_2}, \boldsymbol{y_2} \in C_2$ such that $\boldsymbol{x} = \boldsymbol{x_1} - \boldsymbol{x_2}, \boldsymbol{y} = \boldsymbol{y_1} - \boldsymbol{y_2}$. Consider $\theta\boldsymbol{x} + \overline{\theta}\boldsymbol{y}$ where $\theta \in [0,1]$, $\theta\boldsymbol{x} + \overline{\theta}\boldsymbol{y} = \theta(\boldsymbol{x_1} - \boldsymbol{x_2}) + \overline{\theta}(\boldsymbol{y_1} - \boldsymbol{y_2}) = (\theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{y_1}) - (\theta\boldsymbol{x_2} + \overline{\theta}\boldsymbol{y_2})$. By convexity of C_1 and C_2 we know $\theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{y_1} \in C_1$ and $\theta\boldsymbol{x_2} + \overline{\theta}\boldsymbol{y_2} \in C_2$, thus $\exists \theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{y_1} \in C_1$, $\exists \theta\boldsymbol{x_2} + \overline{\theta}\boldsymbol{y_2} \in C_2$ such that $\theta\boldsymbol{x} + \overline{\theta}\boldsymbol{y} = (\theta\boldsymbol{x_1} + \overline{\theta}\boldsymbol{y_1}) - (\theta\boldsymbol{x_2} + \overline{\theta}\boldsymbol{y_2})$, which means $\theta\boldsymbol{x} + \overline{\theta}\boldsymbol{y} \in C$. So C is convex.

3

(a)

We prove this by contradiction.
Assume $intC$ is not convex. Then $\exists \boldsymbol{x}_0, \boldsymbol{y}_0 \in intC$, $\exists \theta_0 \in (0,1)$ such that $\boldsymbol{z}_0 = \theta_0\boldsymbol{x}_0 + \overline{\theta_0}\boldsymbol{y}_0 \notin intC$. Then if we fix θ_0 and $\boldsymbol{x}_0$ and treat $\boldsymbol{z} = \theta_0\boldsymbol{x}_0 + \overline{\theta_0}\boldsymbol{y}$ as a continuous function, given a neighbourhood $Y \subset C$ of $\boldsymbol{y}_0$, there exists a neighbourhood Z of $\boldsymbol{z}_0$ such that for all points in Z the corresponding y is in Y. Since $\boldsymbol{z}_0 \notin intC$, there exists $\boldsymbol{z}_1 \in Z$ but $\boldsymbol{z}_1 \notin C$. Let $\boldsymbol{y}_1$ be the corresponding $\boldsymbol{y}$ for that $\boldsymbol{z}_1$, then $\boldsymbol{z}_1 = \theta_0\boldsymbol{x}_0 + \overline{\theta_0}\boldsymbol{y}_1$ where $\boldsymbol{x}_0, \boldsymbol{y}_1 \in C$. But by our assumption we conclude that their convex combination $\boldsymbol{z}_1 \notin C$, this means that our assumption that $intC$ is not convex contradicts with the fact that C is convex. Thus $intC$ is also convex.

(b)

We prove this directly.
$\forall \boldsymbol{x}, \boldsymbol{y} \in \overline{C}$, for any $r > 0$, $\exists \boldsymbol{x}_0, \boldsymbol{y}_0 \in C$ such that $||\boldsymbol{x} - \boldsymbol{x}_0|| < r, ||\boldsymbol{y} - \boldsymbol{y}_0|| < r$. Thus for any convex combination $\theta\boldsymbol{x} + \overline{\theta}\boldsymbol{y}, \theta \in (0,1)$ and any $r > 0$, $||(\theta\boldsymbol{x} + \overline{\theta}\boldsymbol{y}) - (\theta\boldsymbol{x}_0 + \overline{\theta}\boldsymbol{y}_0)|| = ||(\theta\boldsymbol{x} - \theta\boldsymbol{x}_0) + (\overline{\theta}\boldsymbol{y} - \overline{\theta}\boldsymbol{y}_0)|| \le ||\theta\boldsymbol{x} - \theta\boldsymbol{x}_0|| + ||\overline{\theta}\boldsymbol{y} - \overline{\theta}\boldsymbol{y}_0|| < r$. By convexity of C we know $\theta\boldsymbol{x}_0 + \overline{\theta}\boldsymbol{y}_0 \in C$, thus any $\theta\boldsymbol{x} + \overline{\theta}\boldsymbol{y}, \theta \in (0,1)$ has a $\theta\boldsymbol{x}_0 + \overline{\theta}\boldsymbol{y}_0 \in C$ in any of its neighbourhood. This means that $\theta\boldsymbol{x} + \overline{\theta}\boldsymbol{y} \in \overline{C}$. So $\overline{C}$ is convex.

4

(a)

Let $\boldsymbol{x}, \boldsymbol{y} \in C$, where $\boldsymbol{x} = \sum_{i=1}^m \theta_i \boldsymbol{x}_i$; $\boldsymbol{x}_i \in S, \theta_i \ge 0, i = 1, \dots, m; \sum_{i=1}^m \theta_i = 1$ and $\boldsymbol{y} = \sum_{i=1}^n \phi_i \boldsymbol{y}_i$; $\boldsymbol{y}_i \in S, \phi_i \ge 0, i = 1, \dots, n; \sum_{i=1}^n \phi_i = 1$. Then we check $\alpha\boldsymbol{x} + \overline{\alpha}\boldsymbol{y}$ where $\alpha \in [0,1]$ $\alpha\boldsymbol{x} + \overline{\alpha}\boldsymbol{y} = \alpha\sum_{i=1}^m \theta_i\boldsymbol{x}_i + \overline{\alpha}\sum_{j=1}^n \phi_j\boldsymbol{y}_j = \sum_{i=1}^m \alpha\theta_i\boldsymbol{x}_i + \sum_{j=1}^n \overline{\alpha}\phi_j\boldsymbol{y}_j$. In this expression $m + n \in \mathbb{N}; \boldsymbol{x}_i, \boldsymbol{y}_j \in S, \alpha\theta_i \ge 0, \overline{\alpha}\phi_j \ge 0, i = 1, \dots, m, j = 1, \dots, n$. Also $\sum_{i=1}^m \alpha\theta_i + \sum_{j=1}^n \overline{\alpha}\phi_j = \alpha\sum_{i=1}^m \theta_i + \overline{\alpha}\sum_{j=1}^n \phi_j = \alpha + \overline{\alpha} = 1$. Thus $\alpha\boldsymbol{x} + \overline{\alpha}\boldsymbol{y} \in C$, so C is convex.

(b)

$convS$ is the smallest convex set containing S, thus $S \subset convS$. Thus $\forall \boldsymbol{x} \in S$, $\boldsymbol{x} \in convS$. Since $convS$ is convex, $\forall \boldsymbol{x}_i \in convS, i = 1, \dots, m$, their convex combination $\sum_{i=1}^m \theta_i\boldsymbol{x}_i \in convS$ by the theorem we have proved in class. By definition of C we know $\forall \boldsymbol{x} \in C$, $\boldsymbol{x} = \sum_{i=1}^m \theta_i\boldsymbol{x}_i$ where $\boldsymbol{x}_i \in convS$ for $i = 1 \dots, m$ and $\sum_{i=1}^m \theta_i = 1$. Thus $\boldsymbol{x}$ is a convex combination of elements in $convS$. This means that any element of C is a convex combination of elements in $convS$. We have shown above that the convex combination of elements in $convS$ must also be an element in $convS$(by its convexity), thus we can say $\forall \boldsymbol{x} \in C$, $\boldsymbol{x} \in convS$, *i.e.* $C \subset convS$. And by definition of $convS$ we know $convS \subset C$, thus $C = convS$.

5

$\forall \boldsymbol{x} \in \boldsymbol{V}$, $||\boldsymbol{x} - \boldsymbol{x_0}||_2 \le ||\boldsymbol{x} - \boldsymbol{x_i}||_2, i = 1, 2, \dots, K$. In $\mathbb{R}^n$ this can be written as

$$(\boldsymbol{x} - \boldsymbol{x}_0)^T(\boldsymbol{x} - \boldsymbol{x}_0) \le (\boldsymbol{x} - \boldsymbol{x}_i)^T(\boldsymbol{x} - \boldsymbol{x}_i)$$

$$(\boldsymbol{x}^T - \boldsymbol{x}_0^T)(\boldsymbol{x} - \boldsymbol{x}_0) \le (\boldsymbol{x}^T - \boldsymbol{x}_i^T)(\boldsymbol{x} - \boldsymbol{x}_i)$$

$$\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}_0^T\boldsymbol{x} + \boldsymbol{x}_0^T\boldsymbol{x}_0 \leq \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}_i^T\boldsymbol{x} + \boldsymbol{x}_i^T\boldsymbol{x}_i$$
$$2(\boldsymbol{x}_i^T - \boldsymbol{x}_0^T)\boldsymbol{x} \leq \boldsymbol{x}_i^T\boldsymbol{x}_i - \boldsymbol{x}_0^T\boldsymbol{x}_0$$

That inequality holds for any $i = 1, 2, \ldots, K$. Then by writing the inequalities in matrix form we get

$$\begin{pmatrix} 2\boldsymbol{x}_1^T - 2\boldsymbol{x}_0^T \\ 2\boldsymbol{x}_2^T - 2\boldsymbol{x}_0^T \\ \cdots \\ 2\boldsymbol{x}_K^T - 2\boldsymbol{x}_0^T \end{pmatrix} \boldsymbol{x} \leq \begin{pmatrix} \boldsymbol{x}_1^T\boldsymbol{x}_1 - \boldsymbol{x}_0^T\boldsymbol{x}_0 \\ \boldsymbol{x}_2^T\boldsymbol{x}_2 - \boldsymbol{x}_0^T\boldsymbol{x}_0 \\ \cdots \\ \boldsymbol{x}_K^T\boldsymbol{x}_K - \boldsymbol{x}_0^T\boldsymbol{x}_0 \end{pmatrix}$$

Let

$$\boldsymbol{A} = \begin{pmatrix} 2\boldsymbol{x}_1^T - 2\boldsymbol{x}_0^T \\ 2\boldsymbol{x}_2^T - 2\boldsymbol{x}_0^T \\ \cdots \\ 2\boldsymbol{x}_K^T - 2\boldsymbol{x}_0^T \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} \boldsymbol{x}_1^T\boldsymbol{x}_1 - \boldsymbol{x}_0^T\boldsymbol{x}_0 \\ \boldsymbol{x}_2^T\boldsymbol{x}_2 - \boldsymbol{x}_0^T\boldsymbol{x}_0 \\ \cdots \\ \boldsymbol{x}_K^T\boldsymbol{x}_K - \boldsymbol{x}_0^T\boldsymbol{x}_0 \end{pmatrix}$$

we know $V = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$. Thus V is a polyhedron.