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Homework 5

1

1.1

Assume the first customer arrives at time $T - s + t$, $0 \le t \le s$. Since customers arrive according to a Poisson process of rate λ per hour, the time it takes for the first customer to arrive follows Exponential distribution with parameter λ. Then the time it takes for the second customer to arrive also follows a same distribution. So the probability for achieving his goal is

$$\int_0^s \lambda e^{-\lambda t} e^{-\lambda(s-t)} dt$$

which is

$$\int_0^s \lambda e^{-\lambda s} dt$$

which is $\lambda s e^{-\lambda s}$.

1.2

To achieve the best probability, simply take the derivative and the result is

$$(1 - \lambda s)\lambda e^{-\lambda s}$$

So the optimal value of s is $\frac{1}{\lambda}$ and the corresponding success probability is $\frac{1}{e}$.

2

2.1

We prove this by brutal calculations.
To prove that

$$\Pr(X = \lambda + k) \ge \Pr(X = \lambda - k - 1)$$

We only need to show

$$e^{-\lambda} \frac{\lambda^{\lambda+k}}{(\lambda+k)!} \ge e^{-\lambda} \frac{\lambda^{\lambda-k-1}}{(\lambda-k-1)!}$$

This entails

$$\lambda^{2k+1} \ge \frac{(\lambda+k)!}{(\lambda-k-1)!}$$

or equivalently

$$\lambda^{2k+1} \geq (\lambda+k)(\lambda+k-1)\cdots(\lambda-k)$$

But it is obvious that $\lambda^2 \geq \lambda^2 - a^2$ for any a. So that inequality holds for any $k = 0, 1, 2, \ldots, \lambda-1$. So for any $k = 0, 1, \ldots, \lambda-1$, it holds that $\Pr(X = \lambda+k) \geq \Pr(X = \lambda-k-1)$. It then follows that $\Pr(X \geq \lambda) \geq \frac{1}{2}$.

2.2

As shown in the lecture notes,

$$\mathbb{E}\left[f(Y_1, Y_2, \ldots, Y_n)\right] = \sum_{k=0}^{\infty} \mathbb{E}\left[f(Y_1, Y_2, \ldots, Y_n) \middle| \sum_{i=1}^{n} Y_i = k\right] \Pr(\sum_{i=1}^{n} Y_i = k)$$

In the lecture notes, to derive an inequality, we discard all the other terms with only $\sum_{i=1}^{n} Y_i = m$ left. This is too loose. Here to prove this inequality we only discard the terms from $\sum_{i=1}^{n} Y_i = 0$ to $\sum_{i=1}^{n} Y_i = m-1$, this gives us

$$\mathbb{E}\left[f(Y_1, Y_2, \ldots, Y_n)\right] \geq \sum_{k=m}^{\infty} \mathbb{E}\left[f(Y_1, Y_2, \ldots, Y_n) \middle| \sum_{i=1}^{n} Y_i = k\right] \Pr(\sum_{i=1}^{n} Y_i = k)$$

Since $\mathbb{E}\left[f(X_1, X_2, \ldots, X_n)\right]$ is monotonically increasing in m,

$$\mathbb{E}\left[f(Y_1, Y_2, \ldots, Y_n) \middle| \sum_{i=1}^{n} Y_i = k\right] \geq \mathbb{E}\left[f(X_1, X_2, \ldots, X_n)\right], \forall k \geq m$$

Then the inequality becomes

$$\mathbb{E}\left[f(Y_1, Y_2, \ldots, Y_n)\right] \geq \mathbb{E}\left[f(X_1, X_2, \ldots, X_n)\right] \sum_{k=m}^{\infty} \Pr(\sum_{i=1}^{n} Y_i = k)$$

Since $\sum_{i=1}^{n} Y_i \sim \text{Poisson}(m)$, from problem 2.1 we know

$$\sum_{k=m}^{\infty} \Pr(\sum_{i=1}^{n} Y_i = k) = \Pr(\sum_{i=1}^{n} Y_i \geq m) \geq \frac{1}{2}$$

So

$$\mathbb{E}\left[f(Y_1, Y_2, \ldots, Y_n)\right] \geq \mathbb{E}\left[f(X_1, X_2, \ldots, X_n)\right] \frac{1}{2}$$

Or equivalently,

$$\mathbb{E}\left[f(X_1, X_2, \ldots, X_n)\right] \leq 2 \cdot \mathbb{E}\left[f(Y_1, Y_2, \ldots, Y_n)\right]$$

2.3

Here we can simulate this problem by throwing 50 identical balls into 365 bins.
For $i \in [365]$, let X_i be the number of balls in the i-th bin. Then the distribution of $(X_1, X_2, \ldots, X_n)$ is the same as that of $(Y_1, Y_2, \ldots, Y_n$ on condition that $\sum_{i=1}^{n} Y_i = 50$.
We define

$$f(X_1, X_2, \ldots, X_n) = \mathbb{1}\left[\exists i, X_i \geq 4\right]$$

then clearly

$$\mathbb{E}\left[f(X_1, X_2, \ldots, X_n)\right] = \Pr\left[\exists i, X_i \geq 4\right]$$

It is also clear that such an expectation $\mathbb{E}\left[f(X_1, X_2, \ldots, X_n)\right]$ is indeed monotonically increasing in the number of balls, because the more balls there are, the more likely there is going to be a bin which contains at least four balls.
Then by result in problem 2.2 we conclude that

$$\Pr\left[\exists i, X_i \geq 4\right] = \mathbb{E}\left[f(X_1, X_2, \ldots, X_n)\right] \leq 2 \cdot \mathbb{E}\left[f(Y_1, Y_2, \ldots, Y_n)\right] = 2 \cdot \Pr\left[\exists i, Y_i \geq 4\right]$$

Now we can estimate the probability by calculating $\Pr\left[\exists i, Y_i \geq 4\right]$.
Since $Y_i, i \in [n]$ are independent Poisson random variables with rate $\lambda = \frac{50}{365}$, $\Pr\left[\exists i, Y_i \geq 4\right] = 1 - \Pr\left[\forall i, Y_i \leq 3\right]$. Then such a probability can be expressed as

$$1 - \Pr\left[\forall i, Y_i \leq 3\right] = 1 - \left(\frac{\lambda^3 + 3\lambda^2 + 6\lambda + 6}{6} e^{-\lambda}\right)^{365}$$

The results from computer shows that

$$1 - \Pr\left[\forall i, Y_i \leq 3\right] = 0.4789\%$$

So our probability to estimate is

$$\Pr\left[\exists i, X_i \geq 4\right] \leq 0.9578\%$$

which is less than 1%.