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Homework 4

Algorithms
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2.1

We first construct a weighted directed graph H from G. Then we can find maxflow on graph H to see whether there is a perfect matching in G.
Firstly, we let all the edges in G point from V_1 to V_2 in H, and we assign them a weight of $+\infty$.
Then we add a source vertex s and add an edge with weight 1 from s to all vertices in V_1.
Finally we add a sink vertex t and add an edge with weight 1 from all vertices in V_2 to t.
Now we have constructed the graph H, we can run the Network Flow algorithm to find a maxflow of H. If the maxflow of H, denoted by f, turns out to be $f = |V_1| = |V_2|$, then there exists a perfect matching in G. And the perfect matching contains exactly the edges in G used by us to construct the maxflow of H.

2.2

Proof of Necessity:
This part is very straightforward.
Suppose there is a perfect matching M from V_1 to V_2. Then for any $S \subset V_1$, for every vertex $v \in S$, there is an edge in M connecting v to a vertex in V_2. This means that there are at least as many vertices in V_2 that are neighbors of vertices in V_1 as there are vertices in V_1.
That is to say, for any $S \subset V_1$, $|N(S)| \geq |S|$.
Proof of Sufficiency:
Following the hint, we would like to prove by showing that the mincut of the graph we construct is exactly $|V_1|$(or $|V_2|$ if you like).
Firstly, any mincut can only contain the edges in H which are not in G because we assign edges in G with a weight of $+\infty$.
Then there is a cut with capacity $|V_1|$ if we make our S-T cut to be $S = \{s\}$, the singleton.
So the capacity of mincut of H is at most $|V_1|$.
Suppose there is another minimum S-T cut where $S \backslash V_2 = \{s\} \cup (V_1 \backslash V_1')$ and $T \backslash V_1 = \{t\} \cup (V_2 \backslash V_2')$, since this is a mincut, there are no edges from $V_1 \backslash V_1'$ to $V_2 \backslash V_2'$. This means that all the neighbors of $V_1 \backslash V_1'$ must be in V_2'. Then by the property that $|N(S)| \geq |S|$ for any $S \subset V_1$, $|V_1 \backslash V_1'| \leq |V_2'|$.
Finally the capacity of this cut is $|V_1'| + |V_2'|$, $|V_1'| + |V_2'| \geq |V_1'| + |V_1 \backslash V_1'| = |V_1|$. This means that the capacity of any S-T cut is at least $|V_1|$. Then by our instance where $S = \{s\}$, the mincut is indeed $|V_1|$. Finally by the Maxflow-Mincut Theorem, the maxflow of H equals its mincut $|V_1|$, which means that there is a perfect matching in G.

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We prove by showing that the debt network G can be equivalently transformed to another network H which has $n-1$ edges at most.
In the original debt graph G, for any vertex $v \in V$, we define

$$W(v) = \sum_{(u,v)\in E} w(u,v) - \sum_{(v,p)\in E} w(v,p)$$

$W(v)$ is the amount of money that v owes other roommates or other roommates owe v. We then divide V into a partition $V = V_1 \cup V_2 \cup V_3$, where $\forall v \in V_1$, $W(v) > 0$, $\forall v \in V_2$, $W(v) < 0$, $\forall v \in V_3$, $W(v) = 0$.
To sum up, V_1 is the set of all creditors, V_2 is the set of all debtors and V_3 is the set of all people who do not need to pay of receive money from anyone else. In graph $H = (V', E')$ we do not have to consider vertices in V_3, so $V' = V_1 \cup V_2$.
Then we pick anyone from V_2 and call him the pivot p. We construct the graph H by the following method:
For all vertices v in V_1, we add an edge (p, v) with weight $W(v)$.
For all vertices v in $V_2 \backslash \{p\}$, we add an edge (v, p) with weight $W(v)$.
I.e., all debtors excluding pivot p give money to p, then p gives money to all creditors(on behalf of all debtors). So the total times it takes to carry out person-to-person payments is at most $|V_1| + |V_2| - 1$, which is at most $n-1$.
Proof of Correctness:
The reason why H and G are equivalent when deciding the amount of payments is that graph H guarantees that every single person is receiving or giving out exactly the amount of money that he should collect or in debt. So out construction of H can indeed make sure that the debts are paid off at last.

4 Comments

4.1

4.2

4.3