SUN Yilin
520030910361

Homework 4

1 Doob's Martingale Inequality

Consider $\tau = \arg\min_{t \leq n}\{X_t \geq \alpha\}$ or $\tau = n$ if $\forall 0 \leq t \leq n, X_t < \alpha$.
Clearly τ is a stopping time because by our definition for any $t \geq 0$, $\mathbb{1}[\tau \leq t]$ is $\mathcal{F}_t$-measurable. Then denote event $X_\tau \geq \alpha$ by A, denote event $\max_{0\leq t\leq n} X_t \geq \alpha$ by B. We have $B \subset A$ because by our definition of stopping time τ, if $X_\tau \geq \alpha$, then if must follows that $\exists k, 0 \leq k \leq n$ such that $X_t \geq \alpha$, hence $\max_{0\leq t\leq n} X_t \geq \alpha$. We know that if $B \subset A$, then $\Pr(B) \leq \Pr(A)$. This means that

$$\Pr\left[\max_{0\leq t\leq n} X_t \geq \alpha\right] \leq \Pr\left[X_\tau \geq \alpha\right]$$

Since $X_t \geq 0$, by applying the Markov Inequality we have

$$\Pr\left[X_\tau \geq \alpha\right] \leq \frac{\mathbb{E}\left[X_\tau\right]}{\alpha}$$

By our definition of τ we can easily see $\Pr[\tau \leq n] = 1$, which means that τ is bounded almost surely, satisfying the first condition for Optional Stopping Theorem. Then by OST we have

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$$

Adding up all the inequalities together we get

$$\Pr\left[\max_{0\leq t\leq n} X_t \geq \alpha\right] \leq \frac{\mathbb{E}\left[X_0\right]}{\alpha}$$

which completes our proof.

2 Biased One-dimensional Random Walk

2.1

2.2

2.3

3 Longest Common Subsequence