

## 1

### (a)

We know  $e^x$  is convex and  $f$  is simply summing up affine composition of  $e^x$ , so  $f$  is convex. By letting  $\nabla f(\mathbf{x}) = 0$  we get  $\mathbf{x}^* = (-\frac{\ln 2}{2}, 0) = (-0.3466, 0)$  and  $f(\mathbf{x}^*) = \frac{2^{\frac{3}{2}}}{e^{0.1}} = 2.56$ .

### (b)

The solution given by backtracking line search is  $\mathbf{x}^* = (-0.3466, 0)$  and  $f(\mathbf{x}^*) = 2.56$ . Number of iterations in the outer loop is 29, total number of iterations in the inner loop is 156.

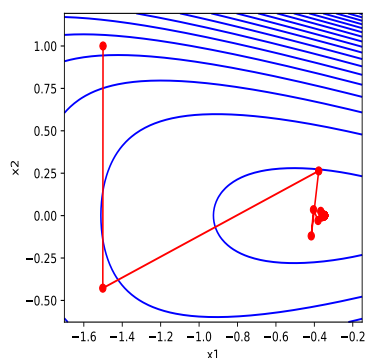


Figure 1: trajectory

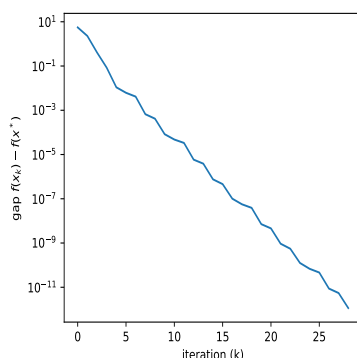


Figure 2: error

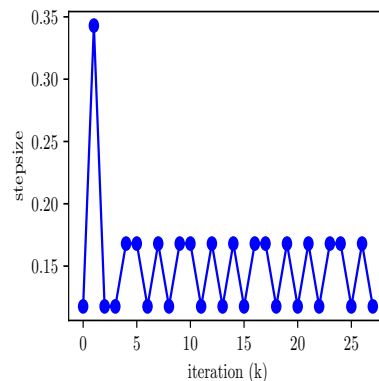
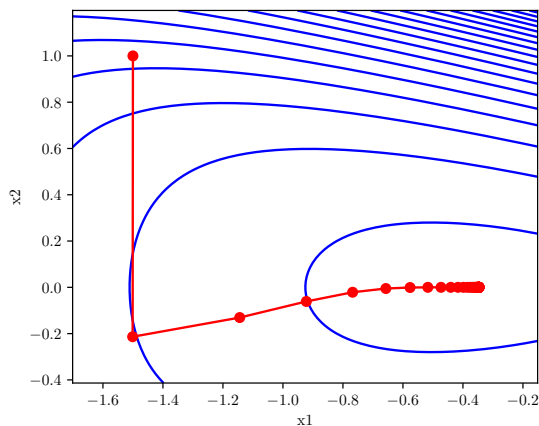


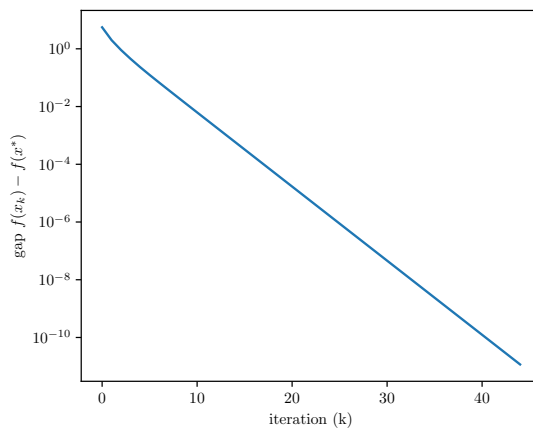
Figure 3: stepsize

(c)

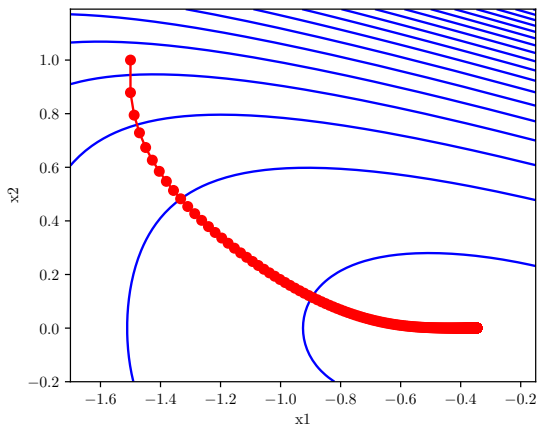
The solution of these two step sizes are both  $\mathbf{x}^* = (-0.3466, 0)$  and  $f(\mathbf{x}^*) = 2.56$ .  
For step size 0.1, we need 44 iterations. And for step size 0.01, we need 489 iterations.  
Below are the trajectories and errors.



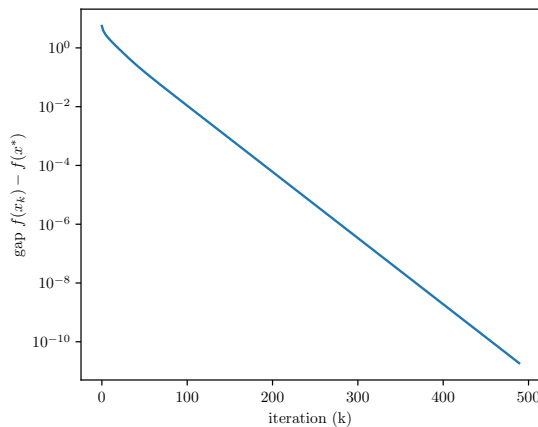
(a) trajectory when stepsize=0.1



(b) error when stepsize=0.1



(c) trajectory when stepsize=0.01



(d) error when stepsize=0.01

Figure 4: trajectories and errors for part (c)

(d)

The solution given by backtracking line search is  $\mathbf{x}^* = (-0.3466, 0)$  and  $f(\mathbf{x}^*) = 2.56$ . Number of iterations in the outer loop is 33, total number of iterations in the inner loop is 202.

The solutions are the same as part (b), but this time we need more iterations.

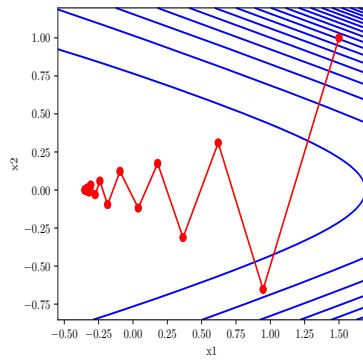


Figure 5: trajectory

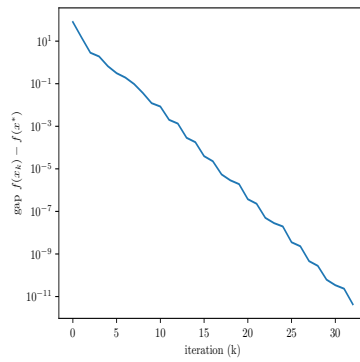


Figure 6: error

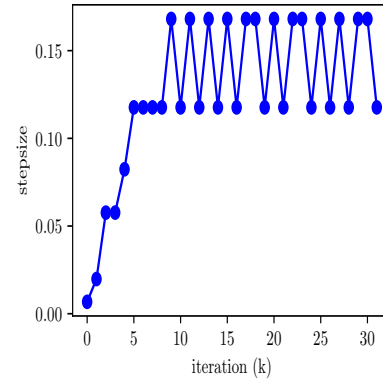
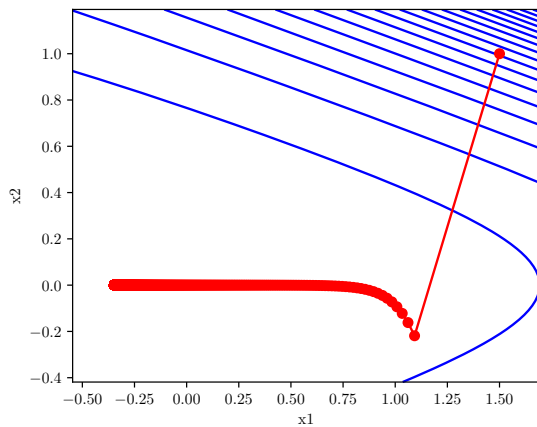
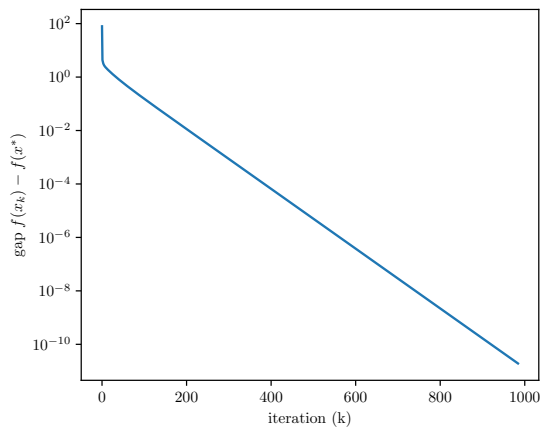


Figure 7: stepsize

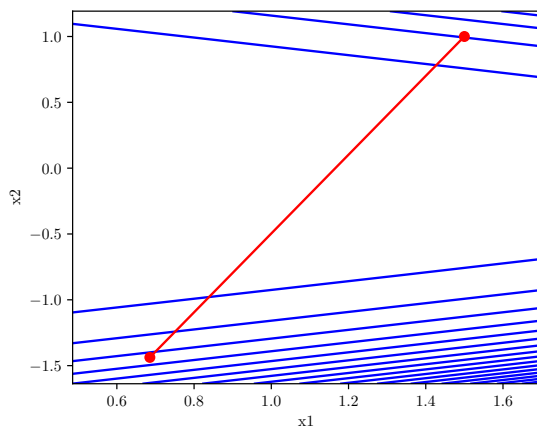
(e)



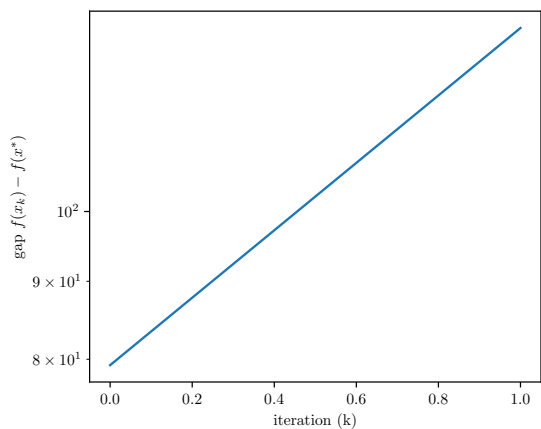
(a) trajectory when stepsize=0.005



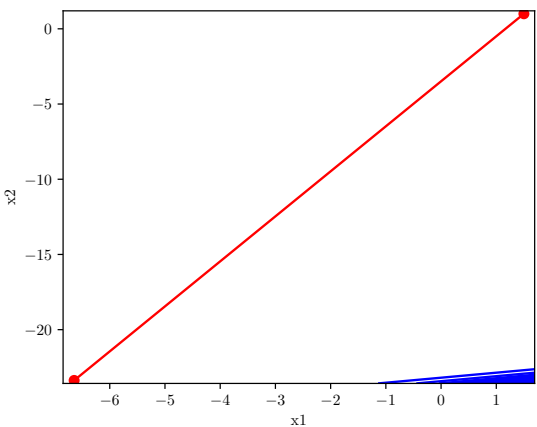
(b) error when stepsize=0.005



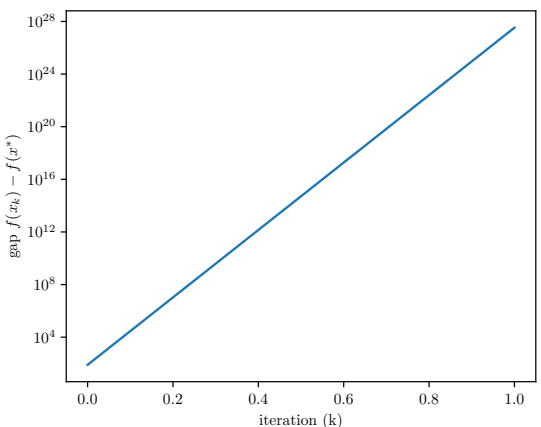
(c) trajectory when stepsize=0.01



(d) error when stepsize=0.01



(e) trajectory when stepsize=0.1



(f) error when stepsize=0.1

Figure 8: trajectories and errors for part (e)

The solution is still  $\mathbf{x}^* = (-0.3466, 0)$  and  $f(\mathbf{x}^*) = 2.56$ . And we need 984 iterations. If we use step sizes in part (c), the step size is so large that it does not converge. We can still try to plot the trajectories but with 2 to 3 iterations the function value is too large, so here I only plot the trajectory of one iteration.

## 2

(a)

$$\begin{aligned}\|\mathbf{x}_{k+1} - \mathbf{x}^*\| &= \|\tilde{\mathbf{x}}_{k+1} - t\boldsymbol{\varepsilon}_k - \mathbf{x}^*\| \leq \|\tilde{\mathbf{x}}_{k+1} - \mathbf{x}^*\| + \|t\boldsymbol{\varepsilon}_k\| \\ \|\mathbf{x}_{k+1} - \mathbf{x}^*\| &\leq \|\tilde{\mathbf{x}}_{k+1} - \mathbf{x}^*\| + tE\end{aligned}$$

(b)

$$\|\tilde{\mathbf{x}}_{k+1} - \mathbf{x}^*\| = \|\mathbf{x}_k - t\nabla f(\mathbf{x}_k) - \mathbf{x}^*\|$$

In convergence analysis of the lecture notes we have proved

$$\|\mathbf{x}_k - t\nabla f(\mathbf{x}_k) - \mathbf{x}^*\|^2 \leq (1 - mt)\|\mathbf{x}_k - \mathbf{x}^*\|^2$$

So

$$\|\mathbf{x}_{k+1} - \mathbf{x}^*\| \leq q\|\mathbf{x}_k - \mathbf{x}^*\| + tE$$

where  $q = \sqrt{1 - mt}$ .

(c)

(i)

When  $k = 1$  obviously the inequality holds.

(ii)

Suppose we already have

$$\|\mathbf{x}_k - \mathbf{x}^*\| \leq q^k\|\mathbf{x}_0 - \mathbf{x}^*\| + \frac{1 - q^k}{1 - q}tE$$

Now we need to show that inequality holds for  $k + 1$ .

From part (b)

$$\|\mathbf{x}_{k+1} - \mathbf{x}^*\| \leq q\|\mathbf{x}_k - \mathbf{x}^*\| + tE$$

By assumption we get

$$\|\mathbf{x}_{k+1} - \mathbf{x}^*\| \leq q(q^k \|\mathbf{x}_0 - \mathbf{x}^*\| + \frac{1 - q^k}{1 - q} tE) + tE$$

Which is equivalent to

$$\|\mathbf{x}_{k+1} - \mathbf{x}^*\| \leq q^{k+1} \|\mathbf{x}_0 - \mathbf{x}^*\| + \frac{1 - q^{k+1}}{1 - q} tE$$

So the inequality holds for any  $k$ .

**(d)**

By taking limits of  $k$  to infinity of the inequality we have proved in part (c) we can get

$$\limsup_{k \rightarrow \infty} \|\mathbf{x}_k - \mathbf{x}^*\| \leq \frac{tE}{1 - q}$$

because  $q < 1$  and  $\|\mathbf{x}_0 - \mathbf{x}^*\|$  is finite. Then by  $(mt)^2 \geq 0$  we know  $4 - 4mt \leq 4 - 4mt + m^2 t^2$ , or equivalently  $2\sqrt{1 - mt} \leq 2 - mt$ . So  $2q \leq 2 - mt$ ,  $\frac{t}{1 - q} \leq \frac{2}{m}$ . So

$$\limsup_{k \rightarrow \infty} \|\mathbf{x}_k - \mathbf{x}^*\| \leq \frac{tE}{1 - q} \leq \frac{2E}{m}$$