1 Doob's Martingale Inequality

Consider $\tau = \arg\min_{t \le n}\{X_t \ge \alpha\}$ or $\tau = n$ if $\forall 0 \le t \le n, X_t < \alpha$.
Clearly τ is a stopping time because by our definition for any $t \ge 0$, $\mathbb{1}[\tau \le t]$ is $\mathcal{F}_t$-measurable. Then denote event $X_\tau \ge \alpha$ by A, denote event $\max_{0\le t\le n} X_t \ge \alpha$ by B. We have $B \subset A$ because by our definition of stopping time τ, if $X_\tau \ge \alpha$, then if must follows that $\exists k, 0 \le k \le n$ such that $X_t \ge \alpha$, hence $\max_{0\le t\le n} X_t \ge \alpha$. We know that if $B \subset A$, then $\Pr(B) \le \Pr(A)$. This means that

$$\Pr\left[\max_{0\le t\le n} X_t \ge \alpha\right] \le \Pr\left[X_\tau \ge \alpha\right]$$

Since $X_t \ge 0$, by applying the Markov Inequality we have

$$\Pr\left[X_\tau \ge \alpha\right] \le \frac{\mathbb{E}\left[X_\tau\right]}{\alpha}$$

By our definition of τ we can easily see $\Pr[\tau \le n] = 1$, which means that τ is bounded almost surely, satisfying the first condition for Optional Stopping Theorem. Then by OST we have

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$$

Adding up all the inequalities together we get

$$\Pr\left[\max_{0\le t\le n} X_t \ge \alpha\right] \le \frac{\mathbb{E}\left[X_0\right]}{\alpha}$$

which completes our proof.

2 Biased One-dimensional Random Walk

2.1

$$\begin{aligned}
\mathbb{E}(S_{t+1}|\overline{Z_{1,n}}) &= \mathbb{E}(S_t + Z_{t+1} + 2p - 1|\overline{Z_{1,n}}) && (1)\\
&= S_t + 2p - 1 + \mathbb{E}(Z_{t+1}|\overline{Z_{1,n}}) && (2)\\
&= S_t + 2p - 1 + (-1)\cdot p + 1 \cdot (1-p) && (3)\\
&= S_t && (4)
\end{aligned}$$

So $\{S_t\}$ is a martingale.

2.2

$$\begin{aligned}
\mathbb{E}(P_{t+1}|\overline{Z_{1,n}}) &= \mathbb{E}((\frac{p}{1-p})^{X_t+Z_{t+1}}|\overline{Z_{1,n}}) && (5)\\
&= \mathbb{E}((\frac{p}{1-p})^{X_t}\cdot(\frac{p}{1-p})^{Z_{t+1}}|\overline{Z_{1,n}}) && (6)\\
&= P_t\cdot\mathbb{E}((\frac{p}{1-p})^{Z_{t+1}}|\overline{Z_{1,n}}) && (7)\\
&= P_t\cdot(\frac{p}{1-p}\cdot(1-p)+\frac{1-p}{p}\cdot p) && (8)\\
&= P_t && (9)
\end{aligned}$$

So $\{P_t\}$ is a martingale.

2.3

If $p=\frac{1}{2}$, we have shown in class that $\mathbb{E}(\tau)=ab$ in class. Here we only show the case where $p\neq\frac{1}{2}$.

Clearly, $\Pr(\tau<\infty)$ also holds when $p\neq\frac{1}{2}$. $|X_t|$ is bounded, so $|P_t|$ is bounded, indicating that $\{P_t\}$ satisfies the second condition of OST. Also, $\mathbb{E}(|S_{t+1}-S_t||\mathcal{F}_t)\le 2p+1$, indicating that $\{S_t\}$ satisfies the third condition of OST. So we can apply the Optional Stopping theorem on them and thus we have $\mathbb{E}(S_\tau)=\mathbb{E}(S_1)$ and $\mathbb{E}(P_\tau)=\mathbb{E}(P_1)$.

Denote Pr(ending at $-a$) by P_a, Pr(ending at b) by P_b. From $\mathbb{E}(S_\tau)=\mathbb{E}(S_1)=0$ we have

$$\mathbb{E}(\tau)\cdot(2p-1)=aP_a-bP_b$$

From $\mathbb{E}(P_\tau)=\mathbb{E}(P_1)=1$ we have

$$\left(\frac{p}{1-p}\right)^{-a}P_a+\left(\frac{p}{1-p}\right)^{b}P_b=1$$

By the latter equation we can calculate P_a and P_b,

$$P_a=\frac{1-\left(\frac{p}{1-p}\right)^{b}}{\left(\frac{p}{1-p}\right)^{-a}-\left(\frac{p}{1-p}\right)^{b}}\qquad P_b=\frac{\left(\frac{p}{1-p}\right)^{-a}-1}{\left(\frac{p}{1-p}\right)^{-a}-\left(\frac{p}{1-p}\right)^{b}}$$

By putting these two results to the former equation we get

$$\mathbb{E}(\tau)=\frac{(a-b)-a\left(\frac{p}{1-p}\right)^{b}+b\left(\frac{p}{1-p}\right)^{-a}}{(2p-1)\left[\left(\frac{p}{1-p}\right)^{-a}-\left(\frac{p}{1-p}\right)^{b}\right])}\quad(p\neq\frac{1}{2})$$

3 Longest Common Subsequence

3.1

3.2