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Homework 1

1 Probability Space of Tossing Coins

1.1

$\forall n \in \mathbb{N}$ and $\forall s \in \{0,1\}^n$, $\exists \omega \in \Omega$ such that $\omega_i = s_i$, $\forall i \in n$. So $\forall n \in \mathbb{N}, \forall s$, $C_s \neq \emptyset$.
$\forall n \in \mathbb{N}$, for any two sets C_{s_i} and C_{s_j} such that $i \neq j$, $\exists k \in [n]$ such that $s_{ik} \neq s_{jk}$. So C_{s_i} and C_{s_j} are disjoint.
$\forall n \in \mathbb{N}$ and $\forall \omega \in \Omega$, $\exists s \in \{0,1\}^n$ such that $\omega_i = s_i$, $\forall i \in n$, which means $\exists s \in \{0,1\}^n$ such that $\omega \in C_s$. So $\cup_{s \in \{0,1\}^n} C_s = \Omega$.
So $\forall n \in \mathbb{N}$, the collection $\{C_s\}$ forms a partition of Ω.

1.2

For every $n \in \mathbb{N}$, since $\mathcal{F}_n$ is generated by $\{C_s\}$, $C_s \in \mathcal{F}_n$. From what we have proven above we know that $\{C_s\}$ forms a partition of Ω, so any of their unions are distinctive to each other and their complements are simply their unions formed by other sets. So the cardinality of $\mathcal{F}_n$ is

$$\sum_{i=0}^{n} C_{2^n}^{i} = 2^{2^n}$$

Also note that the cardinality of $2^{\{0,1\}^n}$ is simply $2^{|\{0,1\}^n|} = 2^{2^n}$, so $\mathcal{F}_n$ and $2^{\{0,1\}^n}$ are equinumerous. So there exists a bijection between them.

1.3

For any $n \in \mathbb{N}$, $\forall C_s \in \mathcal{F}_n$, $\exists s_1, s_2 \in \{0,1\}^{n+1}$ such that $C_s = C_{s_1} \cup C_{s_2}$. By definition, $C_{s_1}, C_{s_2} \in \mathcal{F}_{n+1}$. Since $\mathcal{F}_{n+1}$ is a σ-algebra, $C_{s_1} \cup C_{s_2} \in \mathcal{F}_{n+1}$, meaning that $C_s \in \mathcal{F}_{n+1}$.
Note that $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$ are both σ-algebra, so for the unions and complements of C_s which are in $\mathcal{F}_n$, they must also be in $\mathcal{F}_{n+1}$. So $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Note that $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$ are not equinumerous as we have proven above, meaning that $\mathcal{F}_n \neq \mathcal{F}_{n+1}$. So $\mathcal{F}_n \subsetneq \mathcal{F}_{n+1}$. So the sequence of sets $\mathcal{F}_1, \mathcal{F}_2, \ldots$ is increasing.

1.4

$\forall A \in \mathcal{F}_\infty$, $\exists n \in \mathbb{N}$ such that $A \in \mathcal{F}_n$. Since $\mathcal{F}_n$ is a σ-algebra, $A^C \in \mathcal{F}_n$. So $A^C \in \mathcal{F}_\infty$.
$\forall A, B \in \mathcal{F}_\infty$, $\exists m, n \in \mathbb{N}$ such that $A \in \mathcal{F}_n, B \in \mathcal{F}_m$. without loss of generality, let $m \geq n$. From the last question we know $A \in \mathcal{F}_m$ as well. Since $\mathcal{F}_m$ is a σ-algebra, $A \cup B \in \mathcal{F}_m$, meaning that $A \cup B \in \mathcal{F}_\infty$. So $\mathcal{F}_\infty$ is an algebra.

$\forall \omega \in \Omega$, consider the set $\{\omega\} \in 2^{\Omega}$. It is clear that $\{\omega\} \notin \mathcal{F}_n, \forall n \in \mathbb{N}$. So $\{\omega\} \notin \mathcal{F}_{\infty}$. So $\mathcal{F}_{\infty} \neq 2^{\Omega}$.

1.5

$\forall \omega \in \Omega$, $\{\omega\} = \cap_{i=1}^{\infty} C_{s_i}$ where $s_i \in \{0,1\}^i$. It can also be expressed as $\{\omega\} = (\cup_{i=1}^{\infty} C_{s_i}^C)^C$ by DeMorgan's law. Since $C_{s_i} \in \mathcal{F}_{\infty}$, $C_{s_i} \in \sigma(\mathcal{F}_{\infty})$. Since $\sigma(\mathcal{F}_{\infty})$ is a σ-algebra, $\{\omega\} = (\cup_{i=1}^{\infty} C_{s_i}^C)^C \in \sigma(\mathcal{F}_{\infty})$.
Note that we have proven $\{\omega\} \notin \mathcal{F}_{\infty}$ in the last question, so $\{\omega\} \in \sigma(\mathcal{F}_{\infty}) \backslash \mathcal{F}_{\infty}$.

1.6

$\forall A \in \mathcal{F}_{\infty}$, $\exists n \in \mathbb{N}$ such that $A \in \mathcal{F}_n$. Since $\{C_s\}$ is a partition of Ω, $\exists C_{s_i}, i \in [k]$ such that $A \subset \cup_{i=1}^{k} C_{s_i}$ and $\forall C_{s_i}, \exists a \in A$ such that $a \in C_{s_i}$. If $A \neq \cup_{i=1}^{k} C_{s_i}$, $\exists a \in \cup_{i=1}^{k} C_{s_i}, a \notin A$, so $\exists C_s$ such that $a \in C_s, a \notin A$, i.e. $a \in C_s \backslash A$. That is to say, $\exists C_s \backslash A$ which satisfies that $C_s \backslash A \neq \emptyset, C_s \backslash A \neq C_s$.
Now notice that $A \in \mathcal{F}_n, C_s \in \mathcal{F}_n$, so $A \cup C_s^C \in \mathcal{F}_n$. So $C_s \backslash A = (A \cup C_s^C)^C \in \mathcal{F}_n$, which is in contradiction with the fact that $\mathcal{F}_n$ is the σ-algebra generated by $\{C_s\}$. So the assumption that $A \neq \cup_{i=1}^{k} C_{s_i}$ is not true, it follows that $A = \cup_{i=1}^{k} C_{s_i}$.
Now by the existence of n we know that there exists a smallest n_0 and k_0 accordingly. Suppose we have derived the value $\frac{k_0}{2^{n_0}}$, we now prove that $\forall n > n_0$, $\frac{k}{2^n} = \frac{k_0}{2^{n_0}}$ by induction. Consider $n_0 + 1$, we have shown in problem 1.3 that $\forall C_{s_i}$ such that $A = \cup_{i=1}^{k_0} C_{s_i}$, there exists exactly two C_{s_1}, C_{s_2} where $s_1, s_2 \in \{0,1\}^{n_0+1}$ such that $C_{s_i} = C_{s_1} \cup C_{s_2}$. So the number k for $n_0 + 1$ is actually $2k_0$. So the number for $n_0 + 1$ is actually $\frac{2k_0}{2^{n_0+1}}$, which is equal to $\frac{k_0}{2^{n_0}}$. This holds true for any $n_0 > n$, so we can say the value only depends on A.

1.7

1.7.1 My original trial of proof

Below is my original trial of proof before our teacher stated that we should use Caratheodory's extension Theorem to prove.
Firstly we can construct a probability measure P where $\forall A \in \mathcal{F}_{\infty}$, $P(A) = \frac{k}{2^n}$ as defined above; $\forall \{\omega\} \in \mathcal{B}(\Omega) \backslash \mathcal{F}_{\infty}$, $P(\{\omega\}) = 0$; $P(\Omega) = 1$. As for other events in $\mathcal{B}(\Omega)$, the probability can be calculated by the countable additivity. Surely such a probability measure does exist. Then we claim that a probability measure with respect to $\{\omega\}$ must be $P(\{\omega\}) = 0$, hence P is unique. Consider any $\{\omega\}$, it can be expressed as $\{\omega\} = \cap_{i=1}^{\infty} C_{s_i}$ where $s_i \in \{0,1\}^i$ and

$\forall i > j, C_{s_i} \subset C_{s_j}$. Thus by the continuity of probability measures $P(\{\omega\}) = P(\cap_{i=1}^{\infty} C_{s_i}) = \lim_{i\to\infty} P(C_{s_i}) = \lim_{i\to\infty} \frac{1}{2^i} = 0$. So the probability measure is unique.

1.7.2 Proof by Caratheodory's extension Theorem

Caratheodory's extension Theorem says that any pre-measure defined on a given ring R of subsets of a given set Ω can be extended to a measure on the σ-algebra generated by R, and this extension is unique if the pre-measure is σ-finite. Consequently, any pre-measure on a ring containing all intervals of real numbers can be extended to the Borel algebra of the set of real numbers.
So here our σ-algebra is $\sigma(\mathcal{F}_\infty)$. Our pre-measure defined above on $\mathcal{F}_\infty$ hence can be extended to a measure on the σ-algebra generated by $\mathcal{F}_\infty$, i.e. $\sigma(\mathcal{F}_\infty)$. And clearly our definition of the pre-measure is a finite number, hence σ-finite, so our extension is unique. So by Caratheodory's extension Theorem there exists a unique probability measure satisfying the restrictions given by this problem.

1.8

Toss a fair coin infinitely many times, let X be the number of trials until the first Head(or Tail if you want), then the distribution of X is geometric distribution with parameter $\frac{1}{2}$ in the probability space we have constructed above.

2 Conditional Expectations

2.1

Since f is a measurable function, $\forall a \in \mathbb{R}$, notice that singletons in $\mathbb{R}$ are also Borel Sets, $A = f^{-1}(a)$ must be a Borel Set. The random variable X is $\sigma(X)$-measurable itself, so for any Borel Set A, $X^{-1}(A) \in \sigma(X)$. So $\forall a \in \mathbb{R}, f(X)^{-1}(a) \in \sigma(X)$, which means $f(X)$ is $\sigma(X)$-measurable.

2.2

Since $\sigma(Y) = \sigma(Y')$, $\forall \omega \in \Omega, Y^{-1}(Y(\omega)) = Y'^{-1}(Y'(\omega))$, denoted by A. So $\forall \omega \in \Omega, \mathbb{E}(X|Y = Y(\omega)) = \mathbb{E}(X|Y' = Y'(\omega)) = \mathbb{E}(X|A)$.

2.3

$\mathbb{E}(X|\mathcal{F})$ should be defined as a random variable from Ω to $\mathbb{R}$. $\forall \omega \in \Omega$, $\mathbb{E}(X|\mathcal{F}(\omega)) = \mathbb{E}(X|A)$ where A is the element satisfying $A \in \mathcal{F}, \omega \in A$ with the smallest cardinality.

2.4

Firstly we can find such sets $A_i, i \in [n]$ in $\mathcal{F}_2$ which are not supersets of any other elements in $\mathcal{F}_2$, i.e. $\mathcal{F}_2$ is the σ-algebra generated by $\{A_i\}$. Then $\forall A \in \mathcal{F}_1$, $\exists A_i \in \mathcal{F}_2, i \in [k]$ such that $A = \cup_{i=1}^{k} A_i$.
Now $\forall \omega \in \Omega$, by definition of last problem, $\mathbb{E}(X|\mathcal{F}_1(\omega)) = \mathbb{E}(X|A) = \sum_x xPr(X = x|A)$ where $A \in \mathcal{F}_1$, $\omega \in A$ and A has the smallest cardinality. Now we compute the value of $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)|\mathcal{F}_2)$ and $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_2)|\mathcal{F}_1)$ at ω respectively.

$$
\begin{aligned}
\mathbb{E}(\mathbb{E}(X|\mathcal{F}_2)|\mathcal{F}_1(\omega)) &= \sum_x xPr(\mathbb{E}(X|\mathcal{F}_2) = x|A) && (1)\\
&= \sum_{i=1}^{n} \mathbb{E}(X|A_i)Pr(A_i|A) && (2)\\
&= \sum_{i=1}^{n} \sum_x xPr(X = x|A_i)Pr(Ai|A) && (3)\\
&= \sum_{A_i \subset A} \sum_x xPr(X = x|A_i)Pr(Ai|A) && (4)\\
&= \sum_{A_i \in A} \sum_x xPr(X = x|A_i)\frac{Pr(Ai)}{Pr(A)} && (5)\\
&= \sum_{A_i \in A} \sum_x x\frac{Pr(X = x \wedge A_i)}{Pr(A_i)}\frac{Pr(Ai)}{Pr(A)} && (6)\\
&= \sum_x x \sum_{A_i \subset A} \frac{Pr(X = x \wedge A_i)}{Pr(A)} && (7)\\
&= \sum_x xPr(X = x|A) && (8)\\
&= \mathbb{E}(X|\mathcal{F}_1(\omega)) && (9)
\end{aligned}
$$

$$\begin{aligned}
\mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)|\mathcal{F}_2(\omega)) &= \sum_x xPr(\mathbb{E}(X|\mathcal{F}_1) = x|A_i), \omega \in A_i, A_i \in \mathcal{F}_2 && (10)\\
&= \sum_{A \subset \mathcal{F}_1} \mathbb{E}(X|A)Pr(A|A_i) && (11)\\
&= \mathbb{E}(X|A)Pr(A|A_i), \omega \in A, \omega \in A_i, A \in \mathcal{F}_1, A_i \in \mathcal{F}_2 && (12)\\
&= \mathbb{E}(X|A) * 1 && (13)\\
&= \mathbb{E}(X|\mathcal{F}_1(\omega)) && (14)
\end{aligned}$$

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The idea of constructing the probability measure is from Dr. Krishna Jagannathan's Lecture Notes in course EE5110: Probability Foundations for Electrical Engineers. In Lecture 8, he described the basic ideas of this construction without showing the details. I tried to understand it and express in my own words in problem 1.7