1 Probability Space of Tossing Coins

1.1

$\forall n \in \mathbb{N}$ and $\forall s \in \{0,1\}^n$, $\exists \omega \in \Omega$ such that $\omega_i = s_i$, $\forall i \in n$. So $\forall n \in \mathbb{N}, \forall s$, $C_s \neq \emptyset$.
$\forall n \in \mathbb{N}$, for any two sets C_{s_i} and C_{s_j} such that $i \neq j$, $\exists k \in [n]$ such that $s_{ik} \neq s_{jk}$. So C_{s_i} and C_{s_j} are disjoint.
$\forall n \in \mathbb{N}$ and $\forall \omega \in \Omega$, $\exists s \in \{0,1\}^n$ such that $\omega_i = s_i$, $\forall i \in n$, which means $\exists s \in \{0,1\}^n$ such that $\omega \in C_s$. So $\cup_{s \in \{0,1\}^n} C_s = \Omega$.
So $\forall n \in \mathbb{N}$, the collection $\{C_s\}$ forms a partition of Ω.

1.2

For every $n \in \mathbb{N}$, since $\mathcal{F}_n$ is generated by $\{C_s\}$, $C_s \in \mathcal{F}_n$. From what we have proven above we know that $\{C_s\}$ forms a partition of Ω, so any of their unions are distinctive to each other and their complements are simply their unions formed by other sets. So the cardinality of $\mathcal{F}_n$ is

$$\sum_{i=0}^{n} C_{2^n}^{i} = 2^{2^n}$$

Also note that the cardinality of $2^{\{0,1\}^n}$ is simply $2^{|\{0,1\}^n|} = 2^{2^n}$, so $\mathcal{F}_n$ and $2^{\{0,1\}^n}$ are equinumerous. So there exists a bijection between them.

1.3

For any $n \in \mathbb{N}$, $\forall C_s \in \mathcal{F}_n$, $\exists s_1, s_2 \in \{0,1\}^{n+1}$ such that $C_s = C_{s_1} \cup C_{s_2}$. By definition, $C_{s_1}, C_{s_2} \in \mathcal{F}_{n+1}$. Since $\mathcal{F}_{n+1}$ is a σ-algebra, $C_{s_1} \cup C_{s_2} \in \mathcal{F}_{n+1}$, meaning that $C_s \in \mathcal{F}_{n+1}$.
Note that $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$ are both σ-algebra, so for the unions and complements of C_s which are in $\mathcal{F}_n$, they must also be in $\mathcal{F}_{n+1}$. So $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Note that $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$ are not equinumerous as we have proven above, meaning that $\mathcal{F}_n \neq \mathcal{F}_{n+1}$. So $\mathcal{F}_n \subsetneq \mathcal{F}_{n+1}$. So the sequence of sets $\mathcal{F}_1, \mathcal{F}_2, \ldots$ is increasing.

1.4

$\forall A \in \mathcal{F}_\infty$, $\exists n \in \mathbb{N}$ such that $A \in \mathcal{F}_n$. Since $\mathcal{F}_n$ is a σ-algebra, $A^C \in \mathcal{F}_n$. So $A^C \in \mathcal{F}_\infty$.
$\forall A, B \in \mathcal{F}_\infty$, $\exists m, n \in \mathbb{N}$ such that $A \in \mathcal{F}_n, B \in \mathcal{F}_m$. without loss of generality, let $m \geq n$. From the last question we know $A \in \mathcal{F}_m$ as well. Since $\mathcal{F}_m$ is a σ-algebra, $A \cup B \in \mathcal{F}_m$, meaning that $A \cup B \in \mathcal{F}_\infty$. So $\mathcal{F}_\infty$ is an algebra.

$\forall \omega \in \Omega$, consider the set $\{\omega\} \in 2^\Omega$. It is clear that $\{\omega\} \notin \mathcal{F}_n, \forall n \in \mathbb{N}$. So $\{\omega\} \notin \mathcal{F}_\infty$. So $\mathcal{F}_\infty \neq 2^\Omega$.

1.5

$\forall \omega \in \Omega$, $\{\omega\} = \cap_{i=1}^\infty C_{s_i}$ where $s_i \in \{0,1\}^i$. It can also be expressed as $\{\omega\} = (\cup_{i=1}^\infty C_{s_i}^C)^C$ by DeMorgan's law. Since $C_{s_i} \in \mathcal{F}_\infty$, $C_{s_i} \in \sigma(\mathcal{F}_\infty)$. Since $\sigma(\mathcal{F}_\infty)$ is a σ-algebra, $\{\omega\} = (\cup_{i=1}^\infty C_{s_i}^C)^C \in \sigma(\mathcal{F}_\infty)$.
Note that we have proven $\{\omega\} \notin \mathcal{F}_\infty$ in the last question, so $\{\omega\} \in \sigma(\mathcal{F}_\infty) \backslash \mathcal{F}_\infty$.

1.6

$\forall A \in \mathcal{F}_\infty$, $\exists n \in \mathbb{N}$ such that $A \in \mathcal{F}_n$. Since $\{C_s\}$ is a partition of Ω, $\exists C_{s_i}, i \in [k]$ such that $A \subset \cup_{i=1}^k C_{s_i}$ and $\forall C_{s_i}, \exists a \in A$ such that $a \in C_{s_i}$. If $A \neq \cup_{i=1}^k C_{s_i}$, $\exists a \in \cup_{i=1}^k C_{s_i}, a \notin A$, so $\exists C_s$ such that $a \in C_s, a \notin A$, i.e. $a \in C_s \backslash A$. That is to say, $\exists C_s \backslash A$ which satisfies that $C_s \backslash A \neq \emptyset, C_s \backslash A \neq C_s$.
Now notice that $A \in \mathcal{F}_n, C_s \in \mathcal{F}_n$, so $A \cup C_s^C \in \mathcal{F}_n$. So $C_s \backslash A = (A \cup C_s^C)^C \in \mathcal{F}_n$, which is in contradiction with the fact that $\mathcal{F}_n$ is the σ-algebra generated by $\{C_s\}$. So the assumption that $A \neq \cup_{i=1}^k C_{s_i}$ is not true, it follows that $A = \cup_{i=1}^k C_{s_i}$.
Now by the existence of n we know that there exists a smallest n_0 and k_0 accordingly. Suppose we have derived the value $\frac{k_0}{2^{n_0}}$, we now prove that $\forall n > n_0$, $\frac{k}{2^n} = \frac{k_0}{2^{n_0}}$ by induction. Consider $n_0 + 1$, we have shown in problem 1.3 that $\forall C_{s_i}$ such that $A = \cup_{i=1}^{k_0} C_{s_i}$, there exists exactly two C_{s_1}, C_{s_2} where $s_1, s_2 \in \{0,1\}^{n_0+1}$ such that $C_{s_i} = C_{s_1} \cup C_{s_2}$. So the number k for $n_0 + 1$ is actually $2k_0$. So the number for $n_0 + 1$ is actually $\frac{2k_0}{2^{n_0+1}}$, which is equal to $\frac{k_0}{2^{n_0}}$. This holds true for any $n_0 > n$, so we can say the value only depends on A.

1.7

Firstly we can construct a probability measure P where $\forall A \in \mathcal{F}_\infty$, $P(A) = \frac{k}{2^n}$ as defined above; $\forall \{\omega\} \in \mathcal{B}(\Omega) \backslash \mathcal{F}_\infty$, $P(\{\omega\}) = 0$; $P(\Omega) = 1$. As for other events in $\mathcal{B}(\Omega)$, the probability can be calculated by the countable additivity. Surely such a probability measure does exist.
Then we claim that a probability measure with respect to $\{\omega\}$ must be $P(\{\omega\}) = 0$, hence P is unique. Consider any $\{\omega\}$, it can be expressed as $\{\omega\} = \cap_{i=1}^\infty C_{s_i}$ where $s_i \in \{0,1\}^i$ and $\forall i > j, C_{s_i} \subset C_{s_j}$. Thus by the continuity of probability measures $P(\{\omega\}) = P(\cap_{i=1}^\infty C_{s_i}) = \lim_{i\to\infty} P(C_{s_i}) = \lim_{i\to\infty} \frac{1}{2^i} = 0$. So the probability measure is unique.

1.8

Toss a fair coin infinitely many times, let X be the number of trials until the first Head(or Tail if you want), then the distribution of X is geometric distribution with parameter $\frac{1}{2}$ in the probability space we have constructed above.

2 Conditional Expectations

2.1

Since f is a measurable function, $\forall a \in \mathbb{R}$, notice that singletons in $\mathbb{R}$ are also Borel Sets, $A = f^{-1}(a)$ must be a Borel Set. The random variable X is $\sigma(X)$-measurable itself, so for any Borel Set A, $X^{-1}(A) \in \sigma(X)$. So $\forall a \in \mathbb{R}, f(X)^{-1}(a) \in \sigma(X)$, which means $f(X)$ is $\sigma(X)$-measurable.

2.2

Since $\sigma(Y) = \sigma(Y'), \forall \omega \in \Omega, Y^{-1}(Y(\omega)) = Y'^{-1}(Y'(\omega))$, denoted by A. So $\forall \omega \in \Omega, \mathbb{E}(X|Y = Y(\omega)) = \mathbb{E}(X|Y' = Y'(\omega)) = \mathbb{E}(X|A)$.

2.3

$\mathbb{E}(X|\mathcal{F})$ should be defined as a random variable from Ω to $\mathbb{R}$. $\forall \omega \in \Omega$, $\mathbb{E}(X|\mathcal{F}(\omega)) = \mathbb{E}(X|A)$ where A is the element satisfying $A \in \mathcal{F}, \omega \in A$ with the smallest cardinality.

2.4

Firstly we can find such sets $A_i, i \in [n]$ in $\mathcal{F}_2$ which are not supersets of any other elements in $\mathcal{F}_2$, i.e. $\mathcal{F}_2$ is the σ-algebra generated by $\{A_i\}$. Then $\forall A \in \mathcal{F}_1$, $\exists A_i \in \mathcal{F}_2, i \in [k]$ such that $A = \cup_{i=1}^k A_i$.
Now $\forall \omega \in \Omega$, by definition of last problem, $\mathbb{E}(X|\mathcal{F}_1(\omega)) = \mathbb{E}(X|A) = \sum_x xPr(X = x|A)$ where $A \in \mathcal{F}_1$, $\omega \in A$ and A has the smallest cardinality. Now we compute the value of $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)|\mathcal{F}_2)$ and $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_2)|\mathcal{F}_1)$ at ω respectively.

$$
\begin{aligned}
\mathbb{E}(\mathbb{E}(X|\mathcal{F}_2)|\mathcal{F}_1(\omega)) &= \sum_x xPr(\mathbb{E}(X|\mathcal{F}_2) = x|A) && (1)\\
&= \sum_{i=1}^{n} \mathbb{E}(X|A_i)Pr(A_i|A) && (2)\\
&= \sum_{i=1}^{n} \sum_x xPr(X = x|A_i)Pr(Ai|A) && (3)\\
&= \sum_{A_i \subset A} \sum_x xPr(X = x|A_i)Pr(Ai|A) && (4)\\
&= \sum_{A_i \in A} \sum_x xPr(X = x|A_i)\frac{Pr(Ai)}{Pr(A)} && (5)\\
&= \sum_{A_i \in A} \sum_x x\frac{Pr(X = x \wedge A_i)}{Pr(A_i)}\frac{Pr(Ai)}{Pr(A)} && (6)\\
&= \sum_x x \sum_{A_i \subset A} \frac{Pr(X = x \wedge A_i)}{Pr(A)} && (7)\\
&= \sum_x xPr(X = x|A) && (8)\\
&= \mathbb{E}(X|\mathcal{F}_1(\omega)) && (9)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)|\mathcal{F}_2(\omega)) &= \sum_x xPr(\mathbb{E}(X|\mathcal{F}_1) = x|A_i), \omega \in A_i, A_i \in \mathcal{F}_2 && (10)\\
&= \sum_{A \subset \mathcal{F}_1} \mathbb{E}(X|A)Pr(A|A_i) && (11)\\
&= \mathbb{E}(X|A)Pr(A|A_i), \omega \in A, \omega \in A_i, A \in \mathcal{F}_1, A_i \in \mathcal{F}_2 && (12)\\
&= \mathbb{E}(X|A) * 1 && (13)\\
&= \mathbb{E}(X|\mathcal{F}_1(\omega)) && (14)
\end{aligned}
$$

Acknowledgements

The idea of constructing the probability measure is from Dr. Krishna Jagannathan's Lecture Notes in course EE5110: Probability Foundations for Electrical Engineers. In Lecture 8, he described the basic ideas of this construction without showing the details. I tried to understand it and express in my own words in problem 1.7