Homework 3

1 FTMC for countably infinite chains

1.1

To prove this, we only need to show $[F] + [A] + [I]$ implies $[PR] + [A] + [I]$.
To show this, we only need to show $[F] + [I]$ implies $[PR]$, i.e. a finite and irreducible markov chain is positive recurrent.

Firstly, for a finite markov chain, there exists at least one recurrent state. If not, all the states in this markov chain can only be visited finitely many times. But this finite markov chain only contains finite states, so when we visit infinitely many times this cannot be true. Then if the markov chain is also irreducible, since recurrence is a class property, the existence of a single recurrent state means this markov chain is recurrent.

Now we only need to show a finite irreducible recurrent markov chain is positive recurrent. Similar to our proof in lecture notes, we know that positive recurrence and null recurrence are both class properties. Then if the recurrent markov chain contains a null recurrent state, it is null recurrent itself. Since $\sum_{j\in S} P_{ij}^n = 1$ and there are only finite number of states, it is impossible that $\lim_{n\to\infty} P_{ij}^n = 0$ for all $j \in S$. Thus this markov chain cannot be null recurrent. Then it can only be positive recurrent.

To sum up, a finite irreducible markov chain must be positive recurrent. So the proposition in this problem about these two implications is correct.

1.2

Denote $P(i,i')P(j,j')$ by $Q((i,j),(i',j'))$. To prove $\Pr_{(i,j)}[T_{(k,k)} < \infty] = 1$, we only need to prove that Q is positive recurrent.
To prove this, we first show that Q has a stationary distribution. Note that P is positive recurrent and irreducible, from what we have proven in class, we know that P must have a unique stationary distribution. Denote the stationary distribution by π, then we know that the stationary distribution of Q is when it follows the distribution of π coordinate-wisely. So Q has a unique stationary distribution.
Then we show that Q is irreducible. For any i, j, i', j', we want to find a n such that $Q^n((i,j),(i',j')) = P^n(i,i')P^n(j,j') > 0$. Since P is aperiodic and irreducible, for any j, i' and sufficiently large n, $P^n(j,i') > 0$ always hold. So the n such that $Q^n((i,j),(i',j')) > 0$ always exists. Hence Q is irreducible.
Now we show that for any markov chain, irreducible provided, if the chain has a unique

stationary distribution, then it must be positive recurrent. Let $N_i(n)$ be the number of visits of state i in the first n steps. Since the markov chain is irreducible, the strong law of large numbers for markov chain shows that

$$P_j\left[\lim_{n\to\infty}\frac{N_i(n)}{n}=\frac{1}{\mathbb{E}_i(T_i)}\right]=1$$

Denote the stationary distribution by π and let $X_0\sim\pi$, then by the bounded convergence theorem we have

$$\mathbb{E}_{X_0\sim\pi}\left[\frac{N_i(n)}{n}\right]=\sum_{t=1}^{n}\frac{\mathbb{E}_{X_0\sim\pi}\left[1_{[X_t=i]}\right]}{n}=n\frac{\pi(i)}{n}=\frac{1}{\mathbb{E}_i(T_i)}$$

Since the chain is irreducible, $\pi(i)>0$ for every i, thus $\mathbb{E}_i(T_i)<\infty$ for every i, i.e. the markov chain is positive recurrent.
Since Q is such a markov chain irreducible with a unique stationary distribution, it is positive recurrent. Then it follows that $\Pr_{(i,j)}[T_{(k,k)}<\infty]=1$ for any $i,j,k\in\Omega$.

1.3

We already know from class that $[I]+[PR]$ implies $[S]+[U]$. So we only need to prove that the markov chain will converge if it is aperiodic.
We use total variance distance to describe the convergence. We want to show for any starting distribution μ_0, the total variance distance of μ_t and stationary distribution π, $D_{TV}(\mu_t,\pi)$ goes to 0 as t goes to ∞.
Follow the instructions of finite case from class, we will use two markov chains with the same transition matrix P. Suppose these two markov chains are $\{X_t\}$ and $\{Y_t\}$ where $Y_0\sim\pi$ yet $X_0\sim\mu_0$, an arbitrary distribution.
Now we construct a coupling ω_t of μ_t and π. The coupling is the same as finite case in class. Let $(X_t,Y_t)\sim\omega_t$, we construct ω_{t+1} by the following method. If $X_t=Y_t$ for some $t\geq 0$, then let $X_{t'}=Y_{t'}$ for all $t'>t$. If not, we let X_t and Y_t go to X_{t+1} and Y_{t+1} independently.
The coupling lemma tells us that $D_{TV}(\mu_t,\pi)\leq\Pr(X_t\neq Y_t)$. And we have shown in class that with a finite markov chain

$$\lim_{t\to\infty}\Pr(X_t\neq Y_t)=\lim_{t\to\infty}\sum_{x_0,y_0}\mu_0(x_0)\pi(y_0)\Pr_{x_0,y_0}(X_t\neq Y_t)=0$$

Now we consider the countably infinite case. For our coupling the transition function before $X_t=Y_t$ is what we have defined in problem 1.2, denoted by Q. From what we have proven in problem 1.2, $\Pr_{(i,j)}[T_{(k,k)}<\infty]=1$ for any $i,j,k\in\Omega$. This means that the probability of X_t "meets" Y_t within finite time is 1, which is equivalent to saying that $\lim_{t\to\infty}D_{TV}(\mu_t,\pi)=0$.
So such a positive recurrent countably infinite markov chain will converge to its unique stationary distribution.

2 A Randomized Algorithm for 3-SAT

2.1

3 Acknowledgements

Q1.1

The idea of proving a finite irreducible markov chain is from lecture 5 of STAT253/317, University of Chicago.

Q1.2-1.3

Actually from lecture notes previous year.