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Homework 4

Stochastic Processes
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1 Doob's Martingale Inequality

Consider $\tau = \arg\min_{t \le n}\{X_t \ge \alpha\}$ or $\tau = n$ if $\forall 0 \le t \le n, X_t < \alpha$.
Clearly τ is a stopping time because by our definition for any $t \ge 0$, $\mathbb{1}[\tau \le t]$ is $\mathcal{F}_t$-measurable. Then denote event $X_\tau \ge \alpha$ by A, denote event $\max_{0\le t\le n} X_t \ge \alpha$ by B. We have $B \subset A$ because by our definition of stopping time τ, if $X_\tau \ge \alpha$, then if must follows that $\exists k, 0 \le k \le n$ such that $X_t \ge \alpha$, hence $\max_{0\le t\le n} X_t \ge \alpha$. We know that if $B \subset A$, then $\Pr(B) \le \Pr(A)$. This means that

$$\Pr\left[\max_{0\le t\le n} X_t \ge \alpha\right] \le \Pr\left[X_\tau \ge \alpha\right]$$

Since $X_t \ge 0$, by applying the Markov Inequality we have

$$\Pr\left[X_\tau \ge \alpha\right] \le \frac{\mathbb{E}\left[X_\tau\right]}{\alpha}$$

By our definition of τ we can easily see $\Pr[\tau \le n] = 1$, which means that τ is bounded almost surely, satisfying the first condition for Optional Stopping Theorem. Then by OST we have

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$$

Adding up all the inequalities together we get

$$\Pr\left[\max_{0\le t\le n} X_t \ge \alpha\right] \le \frac{\mathbb{E}\left[X_0\right]}{\alpha}$$

which completes our proof.

2 Biased One-dimensional Random Walk

2.1

$$\begin{aligned}
\mathbb{E}(S_{t+1}|\overline{Z_{1,n}}) &= \mathbb{E}(S_t + Z_{t+1} + 2p - 1|\overline{Z_{1,n}}) && (1)\\
&= S_t + 2p - 1 + \mathbb{E}(Z_{t+1}|\overline{Z_{1,n}}) && (2)\\
&= S_t + 2p - 1 + (-1)\cdot p + 1\cdot(1-p) && (3)\\
&= S_t && (4)
\end{aligned}$$

So $\{S_t\}$ is a martingale.

2.2

$$
\begin{aligned}
\mathbb{E}(P_{t+1}|\overline{Z_{1,n}}) &= \mathbb{E}((\frac{p}{1-p})^{X_t+Z_{t+1}}|\overline{Z_{1,n}}) && (5)\\
&= \mathbb{E}((\frac{p}{1-p})^{X_t} \cdot (\frac{p}{1-p})^{Z_{t+1}}|\overline{Z_{1,n}}) && (6)\\
&= P_t \cdot \mathbb{E}((\frac{p}{1-p})^{Z_{t+1}}|\overline{Z_{1,n}}) && (7)\\
&= P_t \cdot (\frac{p}{1-p} \cdot (1-p) + \frac{1-p}{p} \cdot p) && (8)\\
&= P_t && (9)
\end{aligned}
$$

So $\{P_t\}$ is a martingale.

2.3

3 Longest Common Subsequence