SUN Yilin
520030910361

Homework 6

Algorithms
June 8, 2022

Problem 1

Proof of NP:
For any possible solution denoted by $G' = (V', E')$, we show that we can check if G' is a clique with size exactly $n/2$ in ploynomial time.
This solution should provide us with a injective mapping f from V' to V where for any $v \in V'$, $f(v) \in V$.
Thus we only need to check whether $|V'| = n/2, |E'| = \frac{n(n-1)}{2}$ and check whether $\forall e = (u, v) \in E', (f(u), f(v)) \in E$. Clearly this verification process can be done in ploynomial time.
Proof of NP-Completeness:
We prove by showing that the k-clique problem is reducible to our probelm here.
For any k-clique problem on given graph $G = (V, E)$ where $|V| = n$, we can construct a new graph G' in ploynomial time by the following rule:
If $k = \frac{n}{2}$, $G' = G$.
If $k < \frac{n}{2}$, we construct $G' = (V', E')$ by adding $n - 2k$ vertices which are mutually connected to each other and are also connected to all vertices in V.
If $k > \frac{n}{2}$, we construct $G' = (V', E')$ by adding $2k - n$ vertices which are not connected to any other vertices in the new graph.

We now show that there is a k-clique in G if and only if there is a clique of size $\frac{|V'|}{2}$ in the new graph G'.
If $k = \frac{n}{2}$, trivally true.
If $k < \frac{n}{2}$, $|V'| = 2n - 2k$. If there is a k-clique in G, clearly there is a clique of size $k + n - 2k = n - k = \frac{|V'|}{2}$ in G', and vice versa.
If $k > \frac{n}{2}$, $|V'| = 2k$. If there is a k-clique in G, clearly there is a clique of size k in the new graph G', and vice versa.
So our k-clique probelm can be reduced to a $\frac{n}{2}$-clique problem in ploynomial time. Since k-clique problem is an NP-Complete problem, our $\frac{n}{2}$-clique probelm is also NP-Complete.

Problem 2

Proof of NP:
For any possible solution denoted by $S = \{s_1, s_2, \dots, s_n\}, s_i \in \{0, 1\}$ where $s_i = 1$ denotes picking item i and $s_i = 0$ denotes not picking, simply check whether $\sum_{i=1}^{n} s_i w_i \leq C$ and $\sum_{i=1}^{n} s_i v_i \geq V$ at the same time. Clearly this verification process can be done in ploynomial time.
Proof of NP-Completeness:
We prove by showing that the Subset Sum problem is reducible to our problem here.
For a set $A = \{a_1, a_2, \dots, a_n\}$ and a sum B, the Subset Sum problem asks us whether there is a solution $S = \{s_1, s_2, \dots, s_n\}, s_i \in \{0, 1\}$ such that $\sum_{i=1}^{n} s_i a_i = B$, we can reduce it to a decision version of Knapsack problem by setting the capacity C and total value V to be $C = V = B$ and set the weights $w_i = a_i$ and values $v_i = a_i$ respectively for all $i \in [n]$.
Clearly this process can be done in ploynomial time.
Also a solution S which satisfies the Subset Sum problem clearly also satisfies the reduced Knapsack problem and vice versa.

Problem 3

Proof of NP:
For any possible solution which provides a subgraph G' of G and a mapping f from G' to H, we can easily check whether G' and H are isomorphic to each other following the mapping f in ploynomial time.
Proof of NP-Completeness:
We prove by showing that the k-clique probelm can be reduced to our problem.
Actually, finding a k-clique of graph G is actually finding a subgraph which is also a **complete graph of size** k. This means that if a graph G has a complete subgraph of size k, this subgraph is also its k-clique and vice versa.
Since k-clique problem is an NP-Complete problem, our subgraph probelm is also NP-Complete.

Comments

I assume the difficulties of these problems are in an ascending order, so for canvas submission of this homework assignment, I only chose the problems $1 \sim 3$.
For these three problems that I submitted, there are no collaborators and I did not spend too much time on them.
But I will spend more time on the rest of the problems(probably).