Edmonds' Blossom Algorithm

1

1.1

Proof of M is a maximum matching $\implies$ no M-augmenting path exists:
If there is an M-augmenting path denoted by $P = e_1e_2e_3 \dots e_n$, where $e_i = (v_{i-1}, v_i), \forall i \in [n]$. Then by definition of augmenting path we know that $e_1 \notin M, e_n \notin M$. It follows that, by definition of M-alternating path, $E_{even} = \{e_2, e_4, \dots, e_{n-1}\} \in M$ and $E_{odd} = \{e_1, e_3, e_5, \dots, e_{n-2}, e_n\} \notin M$. Then simply by switching the edges between E_{even} and E_{odd} and construct another matching $M' = (M \backslash E_{even}) \cup E_{odd}$, we would be able to construct a larger matching M', because obviously $|E_{even}| < |E_{odd}|$.

1.2

Proof of M is a maximum matching $\impliedby$ no M-augmenting path exists:
If M is not a maximum matching, following the hint, let M' be a maximum matching such that $|M \cap M'|$ is maximized, we now consider the subgraph $H = M \cup M'$ of G. Here when constructing the subgraph we do a subtle but important modification that **for any edge that is both in M and M' we count it twice.** We want to analyse the connected components of H. The degree of any vertex in H is at most 2 because any vertex has at most two edges incident to it. Thus the connected components of H is either a single vertex, a path, or a cycle. We now consider the edge distribution of these cases respectively.
Case 1: Single vertex.
Trivial since no edges in these components.
Case 2: Cycle.
Any cycle of H must be even length otherwise there will be two edges adjacent to each other and from the same matching. It follows that any cycle must contain exactly the same number of edges from M and M'.
Case 3: Path.
The paths can be either even length or odd length. For those even length path, like the cycles, they also contain same number of edges from M and M'. For those odd length path denoted by $P = e_1e_2e_3 \dots e_n$, we define $E_{even} = \{e_2, e_4, \dots, e_{n-1}\}$ and $E_{odd} = \{e_1, e_3, e_5, \dots, e_{n-2}, e_n\}$. Then either $E_{even} \in M, E_{odd} \in M'$ or $E_{even} \in M', E_{odd} \in M$.

From the above three cases, the only case where a connected component contains more edges from M' than M is in case 3 where there is a path P with odd length and $E_{odd} \in M', E_{even} \in M$. Since we know M' is a maximum matching, $|M'| > |M|$, then at least one connected component is in the form of such a path P with odd length. But such a path P is actually an $M - augmenting$ path because, as $E_{odd} \in M'$ the endpoints of P are not covered by matching M. Thus we can conclude that if M is not a maximum matching, there will be an M-augmenting path. So if no M-augmenting path exists M must be a maximum matching.

2

First we introduce some notations that will be used in the proof.
Denote $M \backslash C$ by M'.
Denote vertices in a cycle C in G with $2k+1$ vertices by $\{v_1, v_2, \ldots, v_{2k+1}\}$.
Denote the vertex added by the contraction process in the new graph G' by u.

2.1

Proof of M is a maximum matching of $G \implies M'$ is a maximum matching of G':
Suppose M' is not a maximum matching of G', then there is an M'-augmenting path P' in G' by what we have proven in problem 1. Since C contains exactly k edges in M and meets no other edges in M, any edge in G' incident with u must not be in M'. Then u must be an endpoint of P', otherwise P' would also be an M-augmenting path in G. And clearly the other endpoint s of P' must not be in C.
Now consider the path P' in G. Denote the vertex in C which is not covered by edge in M by v^*, clearly v^* is unique. Denote the vertex in C which is an endpoint of P' by v_i, clearly v_i is also unique. (Note that v_i and v^* may be equal but that does not matter.)
For v_i, we starts from the edge that is both incident with v_i and in $C \cap M$ and moves along the cycle till we meet v^*, denote such a path by P^*(P^* is empty when $v_i = v^*$ but that does not affect our proof). Then clearly $P' \cup P^*$ with endpoints s and v^* is an M-augmenting path in G, which means that M cannot be a maximum matching, hence contradicts with our assumption. Thus M' must be a maximum matching of G'.

2.2

Proof of M is a maximum matching of $G \impliedby M'$ is a maximum matching of G':
Suppose M is not a maximum matching of G, then there is an M-augmenting path P in G. If P does not intersect any of the vertices in C, then obviously P is an M'-augmenting path in G'. If P intersects cycle C, we can always find an endpoint of P which is not in C since C contains exactly one vertex that is not covered by M. Denote such an endpoint outside C

by w. In G', we starts from w and moves along the path P till we reach u, as we have shown in part 2.2, u is not covered by M', so we have found a path P' starting from w and ending at u which is an M'-augmenting path in G'. In either case there is an M'-augmenting path in G', which contradicts with our assumption, hence M must be a maximum matching of G.