Homework 3

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Algorithm:
We sort the customers by ascending order of their service time, then serve the customer with smallest service time each time.
Proof of Correctness:
The total time can be expressed as

$$T = \sum_{i=1}^{n} \sum_{j=1}^{i} t_j$$

which is simply

$$T = \sum_{i=1}^{n} (n-i+1)t_i$$

So if the sequence $\{t_1, t_2, t_3, \ldots, t_n\}$ is in ascending order, i.e. $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_n$, simply by the **rearrangement inequality** we know such a T is the minimum waiting time because it is the so-called "inverse product sum".

Below is the proof of rearrangement inequality for completeness of this problem.
Denote the two sequence that have already sorted by ascending order by $\{a_i\}$ and $\{b_i\}$. Let $p_i = \sum_{j=1}^{i} b_j$, $q_i = \sum_{j=1}^{i} b_{k_j}$. It's easy to see $p_i \leq q_i$ for $i \leq n-1$ and $p_n = q_n$.
Then

$$\begin{aligned}
\sum_{i=1}^{n} a_{n-i+1} b_i &= \sum_{i=1}^{n-1} a_i (p_{n-i+1} - p_{n-i}) + a_n p_1 && (1) \\
&= \sum_{i=1}^{n-1} (a_{n-i+1} - a_{n-i}) p_i + a_1 p_n && (2) \\
&\leq \sum_{i=1}^{n-1} (a_{n-i+1} - a_{n-i}) q_i + a_1 q_n && (3) \\
&= \sum_{i=1}^{n} a_{n-i+1} b_{k_i} && (4)
\end{aligned}$$

Another side of the inequality is not used in our problem, so the proof is omitted here.