Homework 4

1 Doob's Martingale Inequality

Consider $\tau = \arg\min_{t\leq n}\{X_t \geq \alpha\}$ or $\tau = n$ if $\forall 0 \leq t \leq n, X_t < \alpha$.
Clearly τ is a stopping time because by our definition for any $t \geq 0$, $\mathbb{1}[\tau \leq t]$ is $\mathcal{F}_t$-measurable. Then denote event $X_\tau \geq \alpha$ by A, denote event $\max_{0\leq t\leq n} X_t \geq \alpha$ by B. We have $B \subset A$ because by our definition of stopping time τ, if $\exists k, 0 \leq k \leq n$ such that $X_t \geq \alpha$, hence $\max_{0\leq t\leq n} X_t \geq \alpha$, then it must follows that $X_\tau \geq \alpha$, We know that if $B \subset A$, then $\Pr(B) \leq \Pr(A)$. This means thats

$$\Pr\left[\max_{0\leq t\leq n} X_t \geq \alpha\right] \leq \Pr\left[X_\tau \geq \alpha\right]$$

Since $X_t \geq 0$, by applying the Markov Inequality we have

$$\Pr\left[X_\tau \geq \alpha\right] \leq \frac{\mathbb{E}\left[X_\tau\right]}{\alpha}$$

By our definition of τ we can easily see $\Pr[\tau \leq n] = 1$, which means that τ is bounded almost surely, satisfying the first condition for Optional Stopping Theorem. Then by OST we have

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$$

Adding up all the inequalities together we get

$$\Pr\left[\max_{0\leq t\leq n} X_t \geq \alpha\right] \leq \frac{\mathbb{E}\left[X_0\right]}{\alpha}$$

which completes our proof.

2 Biased One-dimensional Random Walk

2.1

$$\begin{aligned}
\mathbb{E}(S_{t+1}|\overline{Z_{1,n}}) &= \mathbb{E}(S_t + Z_{t+1} + 2p - 1|\overline{Z_{1,n}}) && (1)\\
&= S_t + 2p - 1 + \mathbb{E}(Z_{t+1}|\overline{Z_{1,n}}) && (2)\\
&= S_t + 2p - 1 + (-1)\cdot p + 1\cdot(1-p) && (3)\\
&= S_t && (4)
\end{aligned}$$

So $\{S_t\}$ is a martingale.

2.2

$$
\begin{aligned}
\mathbb{E}(P_{t+1}|\overline{Z_{1,n}}) &= \mathbb{E}((\frac{p}{1-p})^{X_t+Z_{t+1}}|\overline{Z_{1,n}}) && (5)\\
&= \mathbb{E}((\frac{p}{1-p})^{X_t}\cdot(\frac{p}{1-p})^{Z_{t+1}}|\overline{Z_{1,n}}) && (6)\\
&= P_t\cdot\mathbb{E}((\frac{p}{1-p})^{Z_{t+1}}|\overline{Z_{1,n}}) && (7)\\
&= P_t\cdot(\frac{p}{1-p}\cdot(1-p)+\frac{1-p}{p}\cdot p) && (8)\\
&= P_t && (9)
\end{aligned}
$$

So $\{P_t\}$ is a martingale.

2.3

If $p=\frac{1}{2}$, we have shown in class that $\mathbb{E}(\tau)=ab$ in class. Here we only show the case where $p\neq\frac{1}{2}$.
Clearly, $\Pr(\tau<\infty)$ also holds when $p\neq\frac{1}{2}$. $|X_t|$ is bounded, so $|P_t|$ is bounded, indicating that $\{P_t\}$ satisfies the second condition of OST. Also, $\mathbb{E}(|S_{t+1}-S_t||\mathcal{F}_t)\leq 2p+1$, indicating that $\{S_t\}$ satisfies the third condition of OST. So we can apply the Optional Stopping theorem on them and thus we have $\mathbb{E}(S_\tau)=\mathbb{E}(S_1)$ and $\mathbb{E}(P_\tau)=\mathbb{E}(P_1)$.
Denote $\Pr(\text{ending at } -a)$ by P_a, $\Pr(\text{ending at } b)$ by P_b. From $\mathbb{E}(S_\tau)=\mathbb{E}(S_1)=0$ we have

$$\mathbb{E}(\tau)\cdot(2p-1)=aP_a-bP_b$$

From $\mathbb{E}(P_\tau)=\mathbb{E}(P_1)=1$ we have

$$\left(\frac{p}{1-p}\right)^{-a}P_a+\left(\frac{p}{1-p}\right)^{b}P_b=1$$

By the latter equation we can calculate P_a and P_b,

$$P_a=\frac{1-\left(\frac{p}{1-p}\right)^{b}}{\left(\frac{p}{1-p}\right)^{-a}-\left(\frac{p}{1-p}\right)^{b}}\qquad P_b=\frac{\left(\frac{p}{1-p}\right)^{-a}-1}{\left(\frac{p}{1-p}\right)^{-a}-\left(\frac{p}{1-p}\right)^{b}}$$

By putting these two results to the former equation we get

$$\mathbb{E}(\tau)=\frac{(a+b)-a\left(\frac{p}{1-p}\right)^{b}-b\left(\frac{p}{1-p}\right)^{-a}}{(2p-1)\left[\left(\frac{p}{1-p}\right)^{-a}-\left(\frac{p}{1-p}\right)^{b}\right])}\quad(p\neq\frac{1}{2})$$

3 Longest Common Subsequence

3.1

Existence of c_1:
We first find a c_1 for $n = 2$ and then show that this c_1 works for general n.
With $n = 2$, the string x, y can only take the form of $00, 01, 10, 11$. It is very easy for us to calculate $\mathbb{E}(X)$ directly. The result is $\mathbb{E}(X) = \frac{9}{8}$.
Then clearly there exists $\frac{1}{2} < c_1 < \frac{9}{16}$ such that with $n = 2$, $c_1 n < \mathbb{E}(X)$.
For a general n, for convenience we first assume n is even, we can slways treat two consecutive elements together and use the result with $n = 2$. To be specific, with $x, y \in \{0, 1\}^n$, we can cut them into $x_1x_2, x_3x_4, x_5x_6, \ldots, x_{n-1}x_n$ and $y_1y_2, y_3y_4, y_5y_6, \ldots, y_{n-1}y_n$. For any of those consecutive subsequences with length=2, the expectated length of their longest common subsequence is larger than $2c_1$. Then by linearity of expectations, $\mathbb{E}(X) > \frac{n}{2} \cdot 2c_1$, or equivalently, $\mathbb{E}(X) > c_1 n$ for the same c_1 as with case $n = 2$.
For odd n, the analysis is the same with some trivial adjustment, say, putting the last three consecutive elements together and treat them individually.
Existence of c_2:
Following the hint, we would like to estimate the probability that two sequences with length n has at least $c_2 n$ common (but not necessarily consecutive) elements.
This probability is **smaller than**

$$\frac{\binom{n}{c_2 n}}{2^{c_2 n}}$$

where the numerator is picking such $c_2 n$ elements to form a common subsequence and the denominator is all possible situations for those $c_2 n$ elements.
We want to show there exists a c_2 where such a probability approaches 0 for sufficiently n.
The probability can be expressed as

$$\frac{n!}{(c_2 n)!(n - c_2 n)!} \cdot \frac{1}{2^{c_2 n}}$$

Then by applying the Stirling's Formula we get

$$\frac{1}{\sqrt{2\pi n c_2(1 - c_2)} \left(\frac{1}{2c_2^{c_2}(1-c_2)^{1-c_2}}\right)^n}$$

For a sufficiently larger n, if c_2 approaches 1, clearly

$$\left(\frac{1}{2c_2^{c_2}(1 - c_2)^{1-c_2}}\right)^n$$

approaches 0 with exponential speed (approximately $(\frac{1}{2})^n$).
Also, $\sqrt{2\pi n c_2(1-c_2)} \to +\infty$ with such a constant $c_2 < 1$.
Since out probability that two sequences with length n has at least $c_2 n$ common (but not necessarily consecutive) elements is smaller than such a probability, there exists a $c_2 < 1$ such that for sufficiently larger n, Pr[the length of longest common ubsequence is larger than $c_2 n$]=0. It then follows that the expectated length of the longest common subsequence is less than $c_2 n$. So there exists $c_2 < 1$ such that $\mathbb{E}(X) < c_2 n$.

3.2

We can treat each single bit of x and y as i.i.d. random variables uniformly chosen from 0 or 1, Then the function X is actually $X(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ where X calculates the length of longest common subsequence of two random strings $x = x_1 x_2 \ldots x_n$ and $y = y_1 y_2 \ldots y_n$.
Clearly X is a 2-Lipschitz function because flipping any single bit of a sequence will make the lenght of longest common subsequence change for 2 at most.
Then by McDiarmid's Inequality, for such a function X on $2n$ variables satisfying 2-Lipschitz condition,

$$\Pr(|X - \mathbb{E}(X)| \geq t) \leq e^{-\frac{2t^2}{2n \cdot 2^2}} = e^{-\frac{t^2}{4n}}$$

Thus X is well-concentrated around $\mathbb{E}(X)$.

4 Collaborators and Acknowledgements

For problem 3.1, the hint is too abstract for me. As a result, only reading the hint itself did not provide me the intuition for solving this problem.
I then resorted to Liyuan MAO for help. He taught me in detail the method of estimating such a probability and why this method would work. I cannot say that I fully understood this method, but at least I could perform some basic analysis at last.