4771

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Question 4

1. From the problem we know that:

$$Pr[s = 1|x, y = 0] = 0$$

So using bayes formula we have:

$$Pr[s = 1|x, y = 0] = \frac{Pr[s = 1] \cdot Pr[y = 0|x, s = 1]}{Pr[y = 0|x]} = 0$$

Because $Pr[y = 0|x]$ and $Pr[s = 1]$ should not equal to 0, so

$$Pr[y = 0|x, s = 1] = 0$$

Thus $Pr[y = 1|x, s = 1] = 1$
By bayes:

$$Pr[y = 1|x, s = 1] = \frac{Pr[y = 1|x] \cdot Pr[s = 1|x, y = 1]}{Pr[s = 1|x]} = 1$$

As given y, s and x are conditionally independent, $Pr[s = 1|x, y = 1] = Pr[s = 1|y = 1]$
So,

$$\frac{Pr[y = 1|x] \cdot Pr[s = 1|y = 1]}{Pr[s = 1|x]} = 1$$

$$Pr[y = 1|x] = \frac{Pr[s = 1|x]}{Pr[s = 1|y = 1]}$$

Here is another way of doing this:

$$Pr[y=1|x] \cdot Pr[s=1|y=1] = Pr[s=1, y=1|x]$$

Also we have

$$Pr[s=1|x, y=1] = \frac{Pr[s=1, y=0|x]}{Pr[s=1|x]} = 0 \Rightarrow Pr[s=1, y=0|x] = 0$$

So

$$Pr[y=1|x] \cdot Pr[s=1|y=1] = Pr[s=1, y=1|x] + Pr[s=1, y=0|x] = Pr[s=1|x]$$

$$\Rightarrow Pr[y=1|x] = \frac{Pr[s=1|x]}{Pr[s=1|y=1]}$$

2. From problem 1 we know that

$$Pr[y=1|x, s=0] = \frac{Pr[y=1|x] \cdot Pr[s=0|x, y=1]}{Pr[s=0|x]} = \frac{Pr[s=1|x]}{Pr[s=1|y=1]} \cdot \frac{Pr[s=0|x, y=1]}{Pr[s=0|x]}$$

Also

$$Pr[s=0|x] = 1 - Pr[s=1|x]$$
$$Pr[s=0|x, y=1] = 1 - Pr[s=1|x, y=1]$$

As given y, s and x are conditionally independent, $Pr[s=1|x, y=1] = Pr[s=1|y=1]$
So the origin formula can be written as:

$$Pr[y=1|x, s=0] = \frac{1 - Pr[s=1|y=1]}{Pr[s=1|y=1]} \cdot \frac{Pr[s=1|x]}{1 - Pr[s=1|x]}$$

3.

$$\begin{aligned}
\mathbb{E}_{(x,y)\sim D}[\mathbf{1}[f(x) \neq y]] &= \int_x Pr[f(x) \neq y] dx \\
&= \int_x Pr[f(x)=1, y=0] + Pr[f(x)=0, y=1] dx \\
&= \int_x Pr[f(x)=0] \cdot Pr[y=1|x] + Pr[f(x)=1] \cdot Pr[y=0|x] dx \\
&= \int_x Pr[f(x)=0](Pr[y=1, s=0|x] + Pr[y=1, s=1|x]) \\
&\qquad + Pr[f(x)=1](Pr[y=0, s=0|x] + Pr[y=0, s=1|x]) dx \\
&= \int_x \mathbf{1}[f(x) \neq 1](Pr[y=1, s=0|x] + Pr[y=1, s=1|x]) \\
&\qquad + \mathbf{1}[f(x) \neq 0] \cdot Pr[y=0, s=0|x] dx
\end{aligned}$$

We have proved that

$$Pr[y=1|x,s=1] = \frac{Pr[y=1,s=1|x]}{p(x,s=1)} = 1$$

So

$$Pr[y=1,s=1|x] = p(x,s=1)$$

$$\begin{aligned}\mathbb{E}_{(x,y)\sim D}[\mathbf{1}[f(x)\neq y]] = \int_x & \mathbf{1}[f(x)\neq 1]\cdot p(x,s=1) + \mathbf{1}[f(x)\neq 1]\cdot Pr[y=1,s=0|x] \\ & +\mathbf{1}[f(x)\neq 0]\cdot Pr[y=0,s=0|x]dx\end{aligned}$$

Because

$$Pr[y=1,s=0|x] = p(x,s=0)\cdot Pr[y=1|x,s=0]$$

and

$$Pr[y=0,s=0|x] = p(x,s=0)\cdot Pr[y=0|x,s=0]$$

So

$$\begin{aligned}\mathbb{E}_{(x,y)\sim D}[\mathbf{1}[f(x)\neq y]] = \int_x & \mathbf{1}[f(x)\neq 1]\cdot p(x,s=1) \\ & + p(x,s=0)(Pr[y=1|s=0,x]\cdot\mathbf{1}[f(x)\neq 1] \\ & \quad + Pr[y=0|s=0,x]\cdot\mathbf{1}[f(x)\neq 0])dx\end{aligned}$$

4. Under the assumption that there exists $x\in X$ such that $Pr[Y=1|X=x]=1$ then $max_{x\in X}g(x) = Pr[S=1|Y=1]$ where $g(x)=Pr[S=1|X=x]$. Note that g(x) (and hence its max) can be estimated from (x, s) data only.
Alternatively, $Pr[S=1|Y=1]$ is just a single number (does not depend on X)
So suppose $c = Pr[s=1|y=1]$, Here, c is the constant probability that a positive example is labeled.
Let $w(x) = Pr[y=1|x,s=0]$,

$$\begin{aligned}Pr[y=1|x,s=0] &= \frac{1-Pr[s=1|y=1]}{Pr[s=1|y=1]}\cdot\frac{Pr[s=1|x]}{1-Pr[s=1|x]} \\ &= \frac{1-c}{c}\cdot\frac{Pr[s=1|x]}{1-Pr[s=1|x]}\end{aligned}$$

And E can be written as:

$$
\begin{aligned}
\mathbb{E}_{(x,y)\sim D}[\mathbf{1}[f(x) \neq y]] = \int_x & \mathbf{1}[f(x) \neq 1] \cdot p(x, s=1) \\
& + p(x, s=0)(Pr[y=1|s=0,x] \cdot \mathbf{1}[f(x) \neq 1] \\
& \qquad + Pr[y=0|s=0,x] \cdot \mathbf{1}[f(x) \neq 0])dx \\
= \int_x & \mathbf{1}[f(x) \neq 1] \cdot p(x, s=1) \\
& + p(x, s=0)(w(x) \cdot \mathbf{1}[f(x) \neq 1] + (1-w(x))\mathbf{1}[f(x) \neq 0])dx
\end{aligned}
$$

As the number of samples is limited,

$$
\mathbb{E}_{(x,y)\sim D}[\mathbf{1}[f(x) \neq y]] = \frac{1}{|S|}\left[\sum_{x,s=1} \mathbf{1}[f(x) \neq 1] + \sum_{x,s=0} (w(x) \cdot \mathbf{1}[f(x) \neq 1] + (1-w(x))\mathbf{1}[f(x) \neq 0])\right]
$$