1 General Notations

- N: Number of users
- n: Number of items, $n = \begin{cases} 2m-1, & \text{if n is odd} \\ 2m, & \text{otherwise} \end{cases}$
- $\mathcal{P}_n$: the space of permutation of n items
- $\boldsymbol{R}^1, ..., \boldsymbol{R}^N$: full rankings given by the users
- $\boldsymbol{R}^j \in \mathcal{P}_n = \{R_1^j, ..., R_n^j\} \sim \text{Mallows}(\boldsymbol{\rho}^0, \alpha^0)$, defined as $P(\boldsymbol{R}^j|\alpha^0, \boldsymbol{\rho}^0) = \frac{\exp\{-\frac{\alpha^0}{n} d(\boldsymbol{R}^j, \boldsymbol{\rho}^0)\}}{\sum_{\boldsymbol{r} \in \mathcal{P}_n} \exp\{-\frac{\alpha^0}{n} d(\boldsymbol{R}^j, \boldsymbol{\rho}^0)\}}$
- $P(\boldsymbol{\rho}|\boldsymbol{R}^1, ..., \boldsymbol{R}^N, \alpha^o)$: Mallows posterior
- $\{i_1, ..., i_n\}$: a ranking of n items that determines the sequence following which the items are to be sampled. i.e. $i_j = k$ indicates that item j is the k-th item is to be sampled
- $\{o_1, ..., o_n\}$: an ordering of n items that corresponds to $\{i_1, ..., i_n\}$ s.t. $i_{o_k} = k$. $\{o_1, ..., o_n\}$ and $\{i_1, ..., i_n\}$ have a one-to-one relationship
- $Q(\tilde{\boldsymbol{\rho}}|\cdot) = \sum_{\{i_1, ..., i_n\} \in \mathcal{P}_n} q(\tilde{\boldsymbol{\rho}}|i_1, ..., i_n, \alpha_0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N) \cdot g(i_1, ..., i_n|\ldots)$: pseudolikelihood that approximates the Mallows posterior
- $q(\tilde{\boldsymbol{\rho}}|i_1, ..., i_n, \alpha^0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N) = q(\tilde{\boldsymbol{\rho}}|o_1, ..., o_n, \alpha^0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N)$
  $= q(\tilde{\rho}_{o_1}|\alpha^0, o_1, R_{o_1}^1, ..., R_{o_1}^N) \cdot q(\tilde{\rho}_{o_2}|\alpha^0, o_2, \tilde{\rho}_{o_1} R_{o_2}^1, ..., R_{o_2}^N) \cdot ... \cdot$
  $q(\tilde{\rho}_{o_{n-1}}|\alpha^0, o_{n-1}, \tilde{\rho}_{o_1}, ..., \tilde{\rho}_{o_{n-2}}, R_{n-1}^1, ..., R_{n-1}^N) \cdot q(\tilde{\rho}_{o_n}|\alpha^0, o_n, \tilde{\rho}_{o_1}, ..., \tilde{\rho}_{o_{n-1}}, R_n^1, ..., R_n^N)$

  - $q(\tilde{\rho}_{o_1}|\alpha^0, o_1, R_{o_1}^1, ..., R_{o_1}^N) = \frac{\exp\{-\frac{\alpha_0}{n} \sum_{j=1}^N d(R_{o_1}^j, \tilde{\rho}_{o_1})\} \mathbb{1}_{\tilde{\rho}_{o_1} \in \{1, ..., n\}}}{\sum_{\tilde{r}_{o_1} \in \{1, .., n\}} \exp\{-\frac{\alpha_0}{n} \sum_{j=1}^N d(R_{o_1}^j, \tilde{r}_{o_1})\}}$

- $q(\tilde{\rho}_{o_k}|\alpha^0, o_k, \tilde{\rho}_{o_1}, ..., \tilde{\rho}_{o_{k-1}}, R^1_{o_k}, ..., R^N_{o_k}) = \frac{\exp\{-\frac{\alpha_0}{n}\sum_{j=1}^{N} d(R^j_{o_k}, \tilde{\rho}_{o_k})\}\mathbb{1}_{\tilde{\rho}_{o_k}\in\{1,..,n\}\setminus\{\tilde{\rho}_{o_1},...,\tilde{\rho}_{o_{k-1}}\}}}{\sum_{\tilde{r}_{o_k}\in\{1,..,n\}\setminus\{\tilde{\rho}_{o_1},...,\tilde{\rho}_{o_{k-1}}\}}\exp\{-\frac{\alpha_0}{n}\sum_{j=1}^{N} d(R^j_{o_k}, \tilde{r}_{o_k})\}}$

  for $k = 2, ..., n$

- $\boldsymbol{o}^0$: a set of ordering that corresponds to $\boldsymbol{\rho}^0$ s.t. ${\rho^0}^{-1}(m) = o^0_m$
- Define the "v-function" $f_v(\cdot)$ such that $f_v(\boldsymbol{\rho}^0) = \mathcal{V}_{\boldsymbol{\rho}^0}$, where
  - $\mathcal{V}_{\boldsymbol{\rho}^o} = \begin{cases} \{\boldsymbol{r} \in \mathcal{P}_n : r_{o^0_m} = 1, r_{o^0_{m\pm k}} \in \{2k, 2k+1\}, k = 1, ..., m-1\}, & \text{if n is odd} \\ \{\boldsymbol{r} \in \mathcal{P}_n : \{r_{o^0_{m-k}}, r_{o^0_{m+k+1}}\} \in \{2k+1, 2k+2\}, k = 0, ..., m\}, & \text{if n is even} \end{cases}$

2 Theorems and Lemmas

2.1

Lemma 2.1.1 *Given there are odd number of items, i.e.* $n = 2m-1$. $\forall \alpha^0 \in (0, \infty)$,

1. $\mathbb{E}(R_{o^0_m}|\boldsymbol{\rho}_0, \alpha^0) = \rho^0_{o_m} = m$
2. $\forall j \in [1, m-2]$, $j < \mathbb{E}[R_{o^0_j}|\boldsymbol{\rho}^0, \alpha^0] < \mathbb{E}[R_{o^0_{j+1}}|\boldsymbol{\rho}^0, \alpha^0] < m$
3. $\forall j \in [m+2, 2m-1]$, $m < \mathbb{E}[R_{o^0_{j-1}}|\boldsymbol{\rho}^0, \alpha^0] < \mathbb{E}[R_{o^0_j}|\boldsymbol{\rho}^0, \alpha^0] < j$

Similarly, if n is even, i.e. $n = 2m$, $\forall \alpha^0 \in (0, \infty)$,

1. $\forall j \in [1, m-1]$, $j < \mathbb{E}[R_{o^0_j}|\boldsymbol{\rho}^0, \alpha^0] < \mathbb{E}[R_{o^0_{j+1}}|\boldsymbol{\rho}^0, \alpha^0]$
2. $\forall j \in [m+2, 2m]$, $\mathbb{E}[R_{o^0_{j-1}}|\boldsymbol{\rho}^0, \alpha^0] < \mathbb{E}[R_{o^0_j}|\boldsymbol{\rho}^0, \alpha^0] < j$

Note that for both cases, it satisfies that $\forall 1 \leq j < k \leq n$ *and* $\forall \alpha > 0$, $\mathbb{E}[R_{o^0_j}|\boldsymbol{\rho}^0, \alpha^0] < \mathbb{E}[R_{o^0_k}|\boldsymbol{\rho}^0, \alpha^0]$

Lemma 2.1.2 *As* $N \to \infty$, $\frac{1}{N}\sum_{j=1}^{N} R^j_i \to \mathbb{E}[R_i|\boldsymbol{\rho}^0, \alpha^0]$, $\forall i = 1, ..., n$

Definition 1 *Given a vector of length n, i.e.* $\{x_1, ..., x_n\}$,*the function* $rank(x_1, ..., x_n)$ *is defined as* $rank(x_1, ..., x_n) = \{r_1, ..., r_n\}$ *such that* $x_{(r_k)} = x_k$, $\forall k = 1, ..., n$

Theorem 2.1.3 *As* $N \to \infty$, *and* $\forall \alpha > 0$,
$rank(\frac{1}{N}\sum_{j=0}^{N} R^j_1, ..., \frac{1}{N}\sum_{j=0}^{N} R^j_n) \to rank(\mathbb{E}[R_1|\boldsymbol{\rho}^0, \alpha_0], ..., \mathbb{E}[R_n|\boldsymbol{\rho}^0, \alpha_0]) = \boldsymbol{\rho}^0$

To rephrase, as N approaches infinity, the Mallows consensus parameter $\boldsymbol{\rho}^0$ can be inferred from the data by taking the marginal mean for each item and then apply the rank function to these marginal means.

2.2

Theorem 2.2.1 *For a function g defined on $\mathcal{P}_n$ which can depend on $\boldsymbol{\rho}^0$, for any n,*

$$\underset{g\in\mathcal{D}_{\boldsymbol{\rho}^0}}{\arg\min}\lim_{N\to\infty} KL(P(\boldsymbol{\rho}|\alpha^0,\boldsymbol{R}^1,...,\boldsymbol{R}^N)|| \sum_{\{i_1,...,i_n\}\in\mathcal{P}_n} q(\tilde{\boldsymbol{\rho}}|\alpha^0,\boldsymbol{R}^1,...,\boldsymbol{R}^N,i_1,...,i_n)g(i_1,...,i_n|\boldsymbol{\rho}^0)$$
$$= g^*(i_1,...,i_n|\mathcal{V}_{\boldsymbol{\rho}^0}),$$

where

- $\mathcal{D}_{\boldsymbol{\rho}^0}$ *is the set of all distrbutions on* $\mathcal{P}_n$, *which can depend on* $\boldsymbol{\rho}^0$
- $g^*(i_1,...,i_n|\mathcal{V}_{\boldsymbol{\rho}^0})$ *is a distribution whose density is concentrated on* $\boldsymbol{\rho}^0$, *defined as* $\begin{cases} g^*(i_1,...,i_n|\mathcal{V}_{\boldsymbol{\rho}^0}) = |\mathcal{V}_{\boldsymbol{\rho}^0}|^{-1} > 0, & \textit{if } \{i_1,...,i_n\}\in\mathcal{V}_{\boldsymbol{\rho}^0} \\ g^*(i_1,...,i_n|\mathcal{V}_{\boldsymbol{\rho}^0}) = 0, & \textit{if } \{i_1,...,i_n\}\notin\mathcal{V}_{\boldsymbol{\rho}^0}\end{cases}$, *where* $|\mathcal{V}_{\boldsymbol{\rho}^0}| = \begin{cases} 2^{m-1}, & \textit{if } n \textit{ is odd} \\ 2^m, & \textit{otherwise}\end{cases}$

That is to say, for a set of distributions g, which are defined on the space of permutation of n items $\mathcal{P}_n$, the distribution g^* that minimizes the KL-divergence betweeeh the Mallows posterior and the pseudolikelihood defined above, is a uniform distribution with its density concentrated on $\mathcal{V}_{\boldsymbol{\rho}^o}$

2.3

For a given $N<\infty$, define $\hat{\boldsymbol{\rho}}^0$ as $rank(\frac{1}{N}\sum_{j=0}^{N}R_1^j,...,\frac{1}{N}\sum_{j=0}^{N}R_n^j)$ and $\mathcal{V}_{\hat{\boldsymbol{\rho}}^0} = f_v(\hat{\boldsymbol{\rho}}^0)$.

Theorem 2.3.1 $\exists\sigma\geq 0$ *and* $g'(i_1,...,i_n|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0},\sigma)$ *such that*

$$KL\ (P(\boldsymbol{\rho}|\alpha^0,\boldsymbol{R}^1,...,\boldsymbol{R}^N)|| \sum_{\{i_1,...,i_n\}\in\mathcal{P}_n} q(\tilde{\boldsymbol{\rho}}|\alpha^0,\boldsymbol{R}^1,...,\boldsymbol{R}^N,i_1,...,i_n)g^*(i_1,...,i_n|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0)} \geq$$
$$KL\ (P(\boldsymbol{\rho}|\alpha^0,\boldsymbol{R}^1,...,\boldsymbol{R}^N)|| \sum_{\{i_1,...,i_n\}\in\mathcal{P}_n} q(\tilde{\boldsymbol{\rho}}|\alpha^0,\boldsymbol{R}^1,...,\boldsymbol{R}^N,i_1,...,i_n)g'(i_1,...,i_n|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0},\sigma)$$

where $g'(i_1,...,i_n|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0},\sigma) = \sum_{\hat{\boldsymbol{v}}\in\mathcal{V}_{\hat{\boldsymbol{\rho}}^0}} \{g^*(\hat{\boldsymbol{v}}|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0})\int_{\boldsymbol{x}}\mathcal{F}_r(i_1,...,i_n|x_1,...,x_n)\prod_{i=1}^{n}\mathcal{N}(x_i|\hat{v}_i,\sigma)d\boldsymbol{x}\}$, *and*

- $\hat{\boldsymbol{v}} \sim g^*(\hat{\boldsymbol{v}}|\mathcal{V}_{\hat{\boldsymbol{\rho}^0}})$
- $x_i \sim \mathcal{N}(x_i|\hat{v}_i,\sigma)$ *for* $i=1,...,n$
- $i_1,...,i_n \sim \mathcal{F}_r(i_1,...,i_n|x_1,...,x_n)$, *where* $\mathcal{F}_r = \begin{cases} 1, & \textit{if } \{i_1,...,i_n\} = rank(x_1,...,x_n) \\ 0, & \textit{otherwise}\end{cases}$

As N is limited, $\boldsymbol{\rho}^0$ and therefore, $\mathcal{V}_{\boldsymbol{\rho}^0}$ usually cannot be accurately inferred from the data. We can however, sample for $i_1,...,i_n$ by sampling for each item i from a univariate Gaussian distribution centeredd on $\hat{v}_i$ with a fixed variance σ for all items, and then obtain a ranking using the rank function. By introducing the variance, a smaller KL divergence from the Mallows posterior can be achieved.

2.4

Theorem 2.4.1 *With the usage of $g'(i_1, ..., i_n|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0}, \sigma)$, the value of σ that minimizes the KL-divergence between the Mallows posterior and the resulted pseudolikelihood is*

$$\sigma = \begin{cases} 0, & \textit{if } \delta(\alpha^0, n, N) \leq \delta^* \\ f(\alpha^0, n, N), & \textit{otherwise} \end{cases}$$

In other words, σ should be 0 when $\delta(\alpha^0, n, N) \geq \delta^*$. Beyond this point, the optimal choice of σ should be greater than 0, and it follows a function $f(\alpha^0, n, N)$.

2.5

Theorem 2.5.1 *As $N \to \infty, \sigma = 0 \ \forall \alpha > 0$ and $n \geq 1$*

3 Evidence and proofs

3.1 Evidence for Theorem 2.3.1

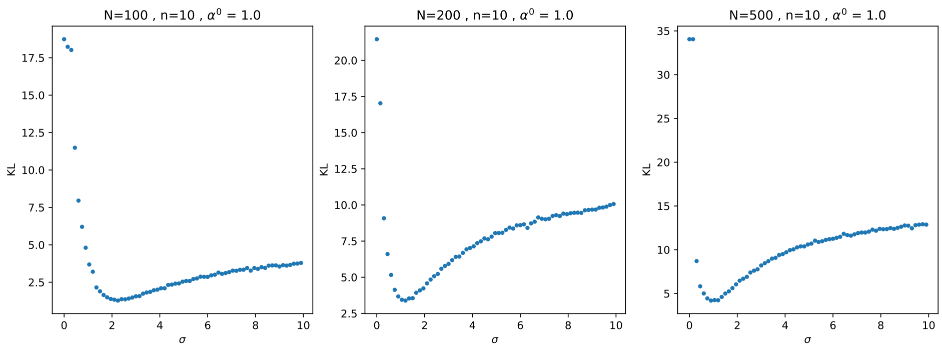
In **Figure** 3.2.1, for each subfigure, we calculate and plot the KL-divergences between the Mallows posterior and the pseudo-likelihood, computed with different choices of σ. The left-most point on each sub-figure corresponds to the KL-divergence when no Gausian variation is introduced, i.e. $\sigma = 0$. It can be observed that for most situations shown in the figure, the lowest KL-divergence is achieved when some level of Gaussian variation is introduced, especially as N and α^0 are relatively small. However, as N and α^0 increase, the optimal σ appears to decrease towards 0.

3.2 Evidence for Theorem 2.4.1

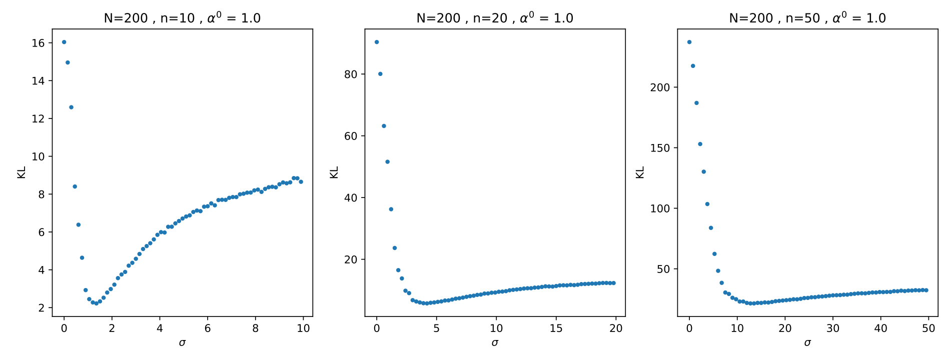
3.2.1 Optimal σ is determined by N, n, α^0

As shown in **Figure** 3.2.1, in each subfigure, the σ value that corresponds to the lowest KL-divergence is the optimal σ for its specific (N, n, α^0) set up. Each row of 3 figures shows a comparison of the optimal σ when one of the variables (N, n, α) changes. It can be observed that all three variables have an impact on the optimal value of σ. More specifically, the optimal choice of σ appear to decrease as N and α^0 increase, and as n decreases.

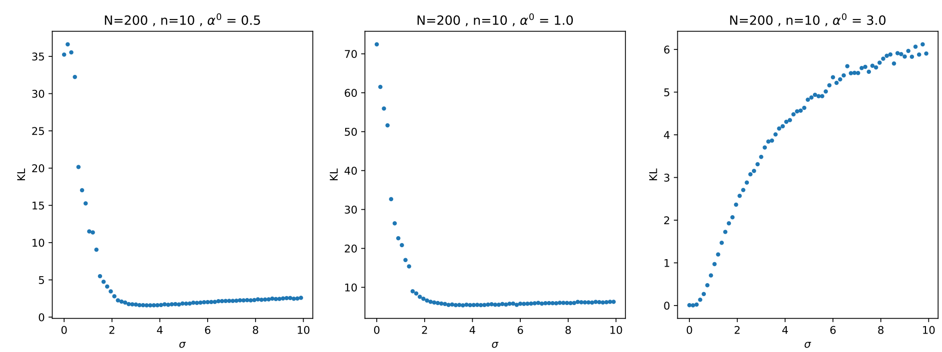
For each N, n and α^0, we simulate 10 datasets, and for each dataset, a grid of σ values are tried out. In **Figure** 3.2.1, we plot α^0 on the x-axis, and its corresponded optimal



(a)



(b)

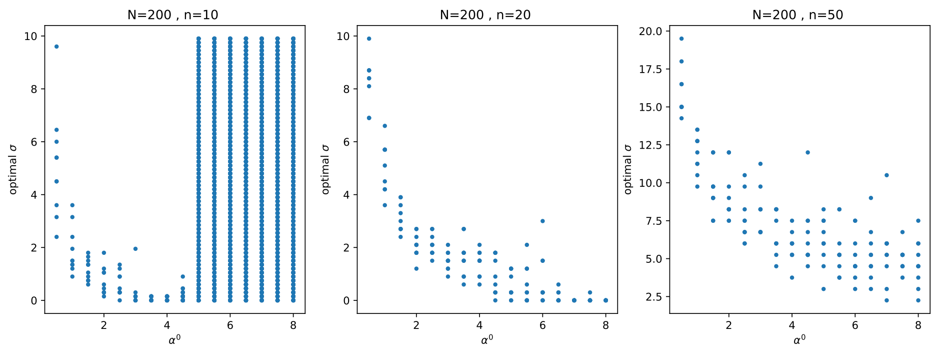


(c)

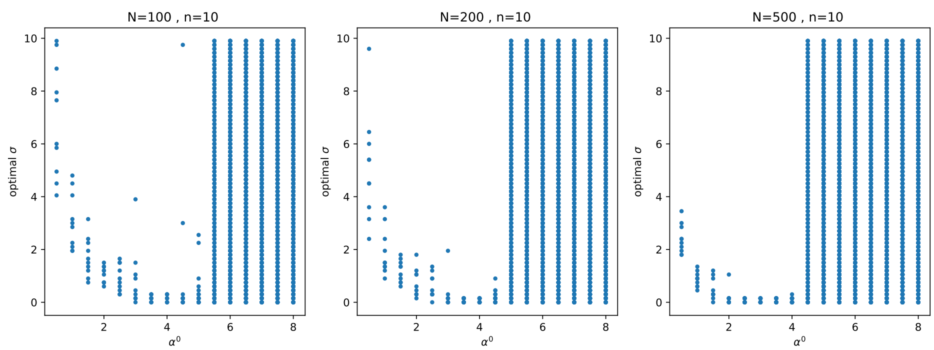
Figure 1: x-axis: σ, y-axis: KL-divergence

σ on the y-axis. It can be observed that when α^0 and N are large, and n is small, all

σ values result in small KL-divergence. To simplify, in these situations, we can safely set $\sigma = 0$ to achieve small KL-divergence. As N and α^0 decreases and as n increases, optimal σ increases.



(a)

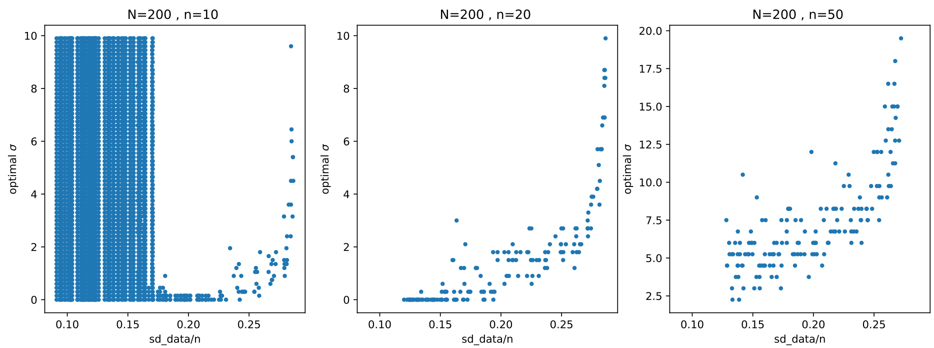


(b)

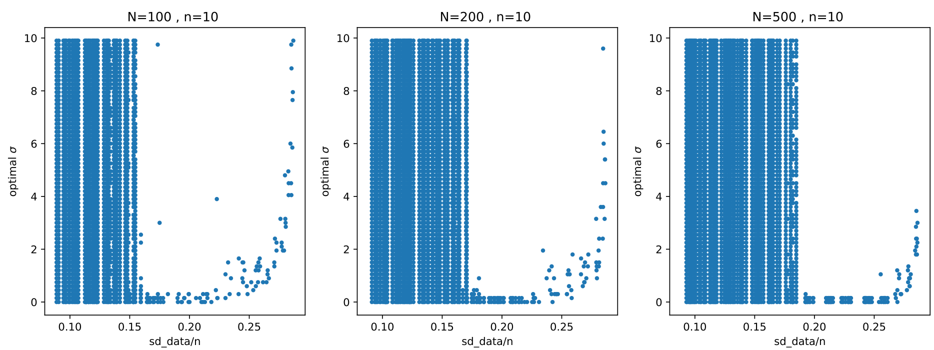
Figure 2: x-axis: α^0, y-axis: optimal σ

3.3 using data standard deviation / n as a proxy for α^0

Under most situations, the value of α^0 is unknown, however, we can compute the marginal standard deviation of each item from the data, and normalize it by deviding it by n. The trend demonstrated in **Figure** 3.2.1 is well-preserved, as shown in **Figure** 3.3.



(a)



(b)

Figure 3: x-axis: α^0, y-axis: optimal σ