Proof for Lemma 2.1

1 Notations

- N: Number of users
- n: Number of items, $n = \begin{cases} 2m-1, & \text{if n is odd} \\ 2m, & \text{otherwise} \end{cases}$
- $\mathcal{P}_n$: the space of permutation of n items
- $\boldsymbol{R}^1, ..., \boldsymbol{R}^N$: full rankings given by the users, $\boldsymbol{R}_j \sim \text{Mallows}(\alpha^0, \boldsymbol{\rho}^0)$
- $\{o_1, ..., o_n\}$: an ordering of n items that corresponds to a ranking $\{i_1, ..., i_n\}$ s.t. $i_{o_j} = j$.

1.1

Lemma 1.1.1. *For any given n and $\alpha^0 > 0$, $\mathbb{E}[R_{o_j}] < \mathbb{E}[R_{o_{j+1}}]$, where $\{\rho_{o_1}, ..., \rho_{o_n}\} = \{1, 2, ..., n\}$*

Proof. According to the definition of expectations,
$\mathbb{E}[R_{o_j}] = \sum\limits_{\boldsymbol{r} \in \mathcal{P}_n} r_{o_j} \exp\{-\frac{\alpha^0}{n} d(\boldsymbol{r}, \boldsymbol{\rho^0})\}$, and $\mathbb{E}[R_{o_{j+1}}] = \sum\limits_{\boldsymbol{r} \in \mathcal{P}_n} r_{o_{j+1}} \exp\{-\frac{\alpha^0}{n} d(\boldsymbol{r}, \boldsymbol{\rho^0})\}$
Given a permutation r s.t. $r_{o_j} = a$ and $r_{o_{j+1}} = b$, $a < b$, we can find another permutation r' s.t. $r'_{o_{j+1}} = a$ and $r_{o_j} = b$, $r_i = r'_i$, $\forall i \neq o_j, o_{j+1}$.

If we divide $\mathcal{P}_n$ into

1. $\mathcal{P}_A = \{r : r_{o_j} = a, r_{o_{j+1}=b}\}$, $\mathcal{P}_{A'} = \{r' : r'_{o_j} = b, r'_{o_{j+1}=a}\}$, $a \geq j+1$
2. $\mathcal{P}_B = \{r : r_{o_j} = a, r_{o_{j+1}=b}\}$, $\mathcal{P}_{B'} = \{r' : r'_{o_j} = b, r'_{o_{j+1}=a}\}$, $b \leq j$
3. $\mathcal{P}_C = \{r : r_{o_j} = a, r_{o_{j+1}=b}\}$, $\mathcal{P}_{C'} = \{r' : r'_{o_j} = b, r'_{o_{j+1}=a}\}$, $a \leq j$, $b \geq j+1$,

we can rewrite the expectations to be:
$\mathbb{E}[R_{o_j}] = \sum\limits_{\boldsymbol{r} \in \mathcal{P}_A} r_{o_j} \exp\{-\frac{\alpha^0}{n} d(\boldsymbol{r}, \boldsymbol{\rho^0})\} + \sum\limits_{\boldsymbol{r} \in \mathcal{P}_{A'}} r_{o_j} \exp\{-\frac{\alpha^0}{n} d(\boldsymbol{r}, \boldsymbol{\rho^0})\} +$

$\sum_{\boldsymbol{r}\in\mathcal{P}_B} r_{o_j}\exp\{-\frac{\alpha^0}{n}d(\boldsymbol{r},\boldsymbol{\rho^0})\} + \sum_{\boldsymbol{r}\in\mathcal{P}_{B'}} r_{o_j}\exp\{-\frac{\alpha^0}{n}d(\boldsymbol{r},\boldsymbol{\rho^0})\}+$
$\sum_{\boldsymbol{r}\in\mathcal{P}_C} r_{o_j}\exp\{-\frac{\alpha^0}{n}d(\boldsymbol{r},\boldsymbol{\rho^0})\}+\sum_{\boldsymbol{r}\in\mathcal{P}_{C'}} r_{o_j}\exp\{-\frac{\alpha^0}{n}d(\boldsymbol{r},\boldsymbol{\rho^0})\}$, and similarly,

$\mathbb{E}[R_{o_{j+1}}] = \sum_{\boldsymbol{r}\in\mathcal{P}_A} r_{o_{j+1}}\exp\{-\frac{\alpha^0}{n}d(\boldsymbol{r},\boldsymbol{\rho^0})\} + \sum_{\boldsymbol{r}\in\mathcal{P}_{A'}} r_{o_{j+1}}\exp\{-\frac{\alpha^0}{n}d(\boldsymbol{r},\boldsymbol{\rho^0})\} +$
$\sum_{\boldsymbol{r}\in\mathcal{P}_B} r_{o_{j+1}}\exp\{-\frac{\alpha^0}{n}d(\boldsymbol{r},\boldsymbol{\rho^0})\} + \sum_{\boldsymbol{r}\in\mathcal{P}_{B'}} r_{o_{j+1}}\exp\{-\frac{\alpha^0}{n}d(\boldsymbol{r},\boldsymbol{\rho^0})\}+$
$\sum_{\boldsymbol{r}\in\mathcal{P}_C} r_{o_{j+1}}\exp\{-\frac{\alpha^0}{n}d(\boldsymbol{r},\boldsymbol{\rho^0})\}+\sum_{\boldsymbol{r}\in\mathcal{P}_{C'}} r_{o_{j+1}}\exp\{-\frac{\alpha^0}{n}d(\boldsymbol{r},\boldsymbol{\rho^0})\}$.

Let us first consider $\mathcal{P}_A$ and $\mathcal{P}_{A'}$'s contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$. For any $r\in\mathcal{P}_A$ and its corresponding $r'\in\mathcal{P}_{A'}$ s.t. $r_{o_j}=r'_{o_{j+1}}=a$, $r_{o_{j+1}}=r'_{o_j}=b$, and $r_i=r'_i\forall i\neq o_j,o_{j+1}$, it can be inferred that $P(\boldsymbol{r})=P(\boldsymbol{r'})$, since
$d(\boldsymbol{r},\boldsymbol{\rho^0}) = \sum_{i=1}^{j-1}|r_{o_i}-\rho^0_{o_i}| + |r_{o_j}-\rho_{o_j}| + |r_{o_{j+1}}-\rho^0_{o_{j+1}}| + \sum_{i=j+2}^{n}|r_{o_i}-\rho^0_{o_i}|$
$= \sum_{i\neq j,j+1}|r_{o_i}-i| + |a-j| + |b-(j+1)|$
$= \sum_{i\neq j,j+1}|r_{o_i}-i| + a-j+b-(j+1)$, similarly,

$d(\boldsymbol{r'},\boldsymbol{\rho^0}) = \sum_{i=1}^{j-1}|r'_{o_i}-\rho^0_{o_i}| + |r'_{o_j}-\rho_{o_j}| + |r'_{o_{j+1}}-\rho^0_{o_{j+1}}| + \sum_{i=j+2}^{n}|r'_{o_i}-\rho^0_{o_i}|$
$= \sum_{i\neq j,j+1}|r'_{o_i}-i| + |b-j| + |a-(j+1)|$
$= \sum_{i\neq j,j+1}|r_{o_i}-i| + b-j+a-(j+1)$
$= d(\boldsymbol{r},\boldsymbol{\rho^0})$

For each $\{\boldsymbol{r},\boldsymbol{r'}\}$ pair, their contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$ are:
$\mathbb{E}[R_{o_j}]|_{\boldsymbol{r}\in\mathcal{P}_A,\boldsymbol{r'}\in\mathcal{P}_{A'}} = P(\boldsymbol{r})\cdot r_{o_j} + P(\boldsymbol{r'})\cdot r'_{o_j} = P(\boldsymbol{r})\cdot(a+b)$

$\mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r}\in\mathcal{P}_A,\boldsymbol{r'}\in\mathcal{P}_{A'}} = P(\boldsymbol{r})\cdot r_{o_{j+1}} + P(\boldsymbol{r'})\cdot r'_{o_{j+1}} = P(\boldsymbol{r})\cdot(a+b)$
Therefore, for all $\boldsymbol{r}\in\mathcal{P}_A$ and their corresponding $\boldsymbol{r'}\in\mathcal{P}_{A'}$, we have
$\sum_{\boldsymbol{r}\in\mathcal{P}_A,\boldsymbol{r}\in\mathcal{P}_{A'}}\mathbb{E}[R_{o_j}]|_{\boldsymbol{r},\boldsymbol{r'}} = \sum_{\boldsymbol{r}\in\mathcal{P}_A,\boldsymbol{r}\in\mathcal{P}_{A'}}\mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r},\boldsymbol{r'}}$

Similarly, let us now consider $\mathcal{P}_B$ and $\mathcal{P}_{B'}$'s contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$.

For any $r\in\mathcal{P}_B$ and its corresponding $r'\in\mathcal{P}_{B'}$ s.t. $r_{o_j}=r'_{o_{j+1}}=a$, $r_{o_{j+1}}=r'_{o_j}=b$, and $r_i=r'_i\forall i\neq o_j,o_{j+1}$, it can be inferred that $P(\boldsymbol{r})=P(\boldsymbol{r'})$, since
$d(\boldsymbol{r},\boldsymbol{\rho^0}) = \sum_{i=1}^{j-1}|r_{o_i}-\rho^0_{o_i}| + |r_{o_j}-\rho_{o_j}| + |r_{o_{j+1}}-\rho^0_{o_{j+1}}| + \sum_{i=j+2}^{n}|r_{o_i}-\rho^0_{o_i}|$

$= \sum_{i \neq j, j+1} |r_{o_i} - i| + |a - j| + |b - (j+1)|$
$= \sum_{i \neq j, j+1} |r_{o_i} - i| + j - a + (j+1) - b$, and

$d(\boldsymbol{r'}, \boldsymbol{\rho^0}) = \sum_{i=1}^{j-1} |r'_{o_i} - \rho^0_{o_i}| + |r'_{o_j} - \rho_{o_j}| + |r'_{o_{j+1}} - \rho^0_{o_{j+1}}| + \sum_{i=j+2}^{n} |r'_{o_i} - \rho^0_{o_i}|$
$= \sum_{i \neq j, j+1} |r'_{o_i} - i| + |b - j| + |a - (j+1)|$
$= \sum_{i \neq j, j+1} |r_{o_i} - i| + j - b + (j+1) - a$
$= d(\boldsymbol{r}, \boldsymbol{\rho^0})$

For each $\{\boldsymbol{r}, \boldsymbol{r'}\}$ pair, their contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$ are:
$\mathbb{E}[R_{o_j}]|_{\boldsymbol{r} \in \mathcal{P}_B, \boldsymbol{r'} \in \mathcal{P}_{B'}} = P(\boldsymbol{r}) \cdot r_{o_j} + P(\boldsymbol{r'}) \cdot r'_{o_B} = P(\boldsymbol{r}) \cdot (a + b)$

$\mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r} \in \mathcal{P}_B, \boldsymbol{r'} \in \mathcal{P}_{B'}} = P(\boldsymbol{r}) \cdot r_{o_{j+1}} + P(\boldsymbol{r'}) \cdot r'_{o_{j+1}} = P(\boldsymbol{r}) \cdot (a + b)$

Therefore, for all $\boldsymbol{r} \in \mathcal{P}_A$ and their corresponding $\boldsymbol{r'} \in \mathcal{P}_{A'}$, we have
$\sum_{\boldsymbol{r} \in \mathcal{P}_B, \boldsymbol{r} \in \mathcal{P}_{B'}} \mathbb{E}[R_{o_j}]|_{\boldsymbol{r}, \boldsymbol{r'}} = \sum_{\boldsymbol{r} \in \mathcal{P}_B, \boldsymbol{r} \in \mathcal{P}_{B'}} \mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r}, \boldsymbol{r'}}$

Up to this point, we have proven that
$\sum_{\boldsymbol{r} \in \mathcal{P}_A, \boldsymbol{r} \in \mathcal{P}_{A'}} \mathbb{E}[R_{o_j}]|_{\boldsymbol{r}, \boldsymbol{r'}} + \sum_{\boldsymbol{r} \in \mathcal{P}_B, \boldsymbol{r} \in \mathcal{P}_{B'}} \mathbb{E}[R_{o_j}]|_{\boldsymbol{r}, \boldsymbol{r'}} = \sum_{\boldsymbol{r} \in \mathcal{P}_A, \boldsymbol{r} \in \mathcal{P}_{A'}} \mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r}, \boldsymbol{r'}} + \sum_{\boldsymbol{r} \in \mathcal{P}_B, \boldsymbol{r} \in \mathcal{P}_{B'}} \mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r}, \boldsymbol{r'}}.$

For $\alpha > 0$, to prove that $\mathbb{E}[R_{o_j}] < \mathbb{E}[R_{o_{j+1}}]$ is equivalent to proving
$\sum_{\boldsymbol{r} \in \mathcal{P}_C, \boldsymbol{r} \in \mathcal{P}_{C'}} \mathbb{E}[R_{o_j}]|_{\boldsymbol{r}, \boldsymbol{r'}} < \sum_{\boldsymbol{r} \in \mathcal{P}_C, \boldsymbol{r} \in \mathcal{P}_{C'}} \mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r}, \boldsymbol{r'}}.$

Now let us consider $\mathcal{P}_C$ and $\mathcal{P}_{C'}$'s contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$.

For any $r \in \mathcal{P}_C$ and its corresponding $r' \in \mathcal{P}_{C'}$, we have

$d(\boldsymbol{r}, \boldsymbol{\rho^0}) = \sum_{i=1}^{j-1} |r_{o_i} - \rho^0_{o_i}| + |r_{o_j} - \rho_{o_j}| + |r_{o_{j+1}} - \rho^0_{o_{j+1}}| + \sum_{i=j+2}^{n} |r_{o_i} - \rho^0_{o_i}|$
$= \sum_{i \neq j, j+1} |r_{o_i} - i| + |a - j| + |b - (j+1)|$
$= \sum_{i \neq j, j+1} |r_{o_i} - i| + j - a + b - (j+1)$
$= \sum_{i \neq j, j+1} |r_{o_i} - i| + b - a - 1$, and

$d(\boldsymbol{r'}, \boldsymbol{\rho^0}) = \sum_{i=1}^{j-1} |r'_{o_i} - \rho^0_{o_i}| + |r'_{o_j} - \rho_{o_j}| + |r'_{o_{j+1}} - \rho^0_{o_{j+1}}| + \sum_{i=j+2}^{n} |r'_{o_i} - \rho^0_{o_i}|$
$= \sum_{i \neq j, j+1} |r'_{o_i} - i| + |b - j| + |a - (j+1)|$

$$
\begin{aligned}
&= \sum_{i \neq j, j+1} |r_{o_i} - i| + b - j + (j+1) - a \\
&= \sum_{i \neq j, j+1} |r_{o_i} - i| + b - a + 1 \\
&= d(\boldsymbol{r}, \boldsymbol{\rho^0}) + 2
\end{aligned}
$$

Therefore, for any $\alpha^0 > 0$ and any given $\boldsymbol{r} \in \mathcal{P}_C$ and its corresponding $\boldsymbol{r'} \in \mathcal{P}_{C'}$, we have $P(\boldsymbol{r}) > P(\boldsymbol{r'})$.

For each $\{\boldsymbol{r}, \boldsymbol{r'}\}$ pair, their contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$ are:
$\mathbb{E}[R_{o_j}]|_{\boldsymbol{r} \in \mathcal{P}_C, \boldsymbol{r'} \in \mathcal{P}_{C'}} = P(\boldsymbol{r}) \cdot r_{o_j} + P(\boldsymbol{r'}) \cdot r'_{o_B} = a \cdot P(\boldsymbol{r}) \cdot + b \cdot P(\boldsymbol{r'})$
$\mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r} \in \mathcal{P}_C, \boldsymbol{r'} \in \mathcal{P}_{C'}} = P(\boldsymbol{r}) \cdot r_{o_{j+1}} + P(\boldsymbol{r'}) \cdot r'_{o_{j+1}} = b \cdot P(\boldsymbol{r}) \cdot + a \cdot P(\boldsymbol{r'})$

$$
\begin{aligned}
&\mathbb{E}[R_{o_j}]|_{\boldsymbol{r} \in \mathcal{P}_C, \boldsymbol{r'} \in \mathcal{P}_{C'}} - \mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r} \in \mathcal{P}_B, \boldsymbol{r'} \in \mathcal{P}_{B'}} \\
&= (a-b)P(\boldsymbol{r}) + (b-a)P(\boldsymbol{r'}) \\
&= (a-b) \cdot (P(\boldsymbol{r}) - P(\boldsymbol{r'}))
\end{aligned}
$$

Recall that $a < b$ and $P(\boldsymbol{r}) > P(\boldsymbol{r'})$ for any $r \in \mathcal{P}_C$ and its corresponding $r' \in \mathcal{P}_{C'}$, it can be obtained that:
$\mathbb{E}[R_{o_j}]|_{\boldsymbol{r} \in \mathcal{P}_C, \boldsymbol{r'} \in \mathcal{P}_{C'}} - \mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r} \in \mathcal{P}_B, \boldsymbol{r'} \in \mathcal{P}_{B'}} < 0$, i.e., $\mathbb{E}[R_{o_j}]|_{\boldsymbol{r} \in \mathcal{P}_C, \boldsymbol{r'} \in \mathcal{P}_{C'}} < \mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r} \in \mathcal{P}_B, \boldsymbol{r'} \in \mathcal{P}_{B'}}$

It is straight forward to obtain that:
$$
\sum_{\boldsymbol{r} \in \mathcal{P}_C, \boldsymbol{r} \in \mathcal{P}_{C'}} \mathbb{E}[R_{o_j}]|_{\boldsymbol{r}, \boldsymbol{r'}} < \sum_{\boldsymbol{r} \in \mathcal{P}_C, \boldsymbol{r} \in \mathcal{P}_{C'}} \mathbb{E}[R_{o_{j+1}}]|_{\boldsymbol{r}, \boldsymbol{r'}},
$$

and therefore, for any $\alpha > 0$, we have $\mathbb{E}[R_{o_j}] < \mathbb{E}[R_{o_{j+1}}]$

$\square$