

Proof for Lemma 2.1

1 Notations

- N : Number of users
- n : Number of items, $n = \begin{cases} 2m-1, & \text{if } n \text{ is odd} \\ 2m, & \text{otherwise} \end{cases}$
- \mathcal{P}_n : the space of permutation of n items
- $\mathbf{R}^1, \dots, \mathbf{R}^N$: full rankings given by the users, $\mathbf{R}_j \sim \text{Mallows}(\alpha^0, \boldsymbol{\rho}^0)$
- $\{o_1, \dots, o_n\}$: an ordering of n items that corresponds to a ranking $\{i_1, \dots, i_n\}$ s.t. $i_{o_j} = j$

1.1

Lemma 1.1.1. *For any given n and $\alpha^0 > 0$, $\mathbb{E}[R_{o_j}] < \mathbb{E}[R_{o_{j+1}}]$, where $\{\rho_{o_1}, \dots, \rho_{o_n}\} = \{1, 2, \dots, n\}$*

Proof. According to the definition of expectations,

$$\mathbb{E}[R_{o_j}] = \sum_{\mathbf{r} \in \mathcal{P}_n} r_{o_j} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\}, \text{ and } \mathbb{E}[R_{o_{j+1}}] = \sum_{\mathbf{r} \in \mathcal{P}_n} r_{o_{j+1}} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\}$$

Given a permutation r s.t. $r_{o_j} = a$ and $r_{o_{j+1}} = b$, $a < b$, we can find another permutation r' s.t. $r'_{o_{j+1}} = a$ and $r_{o_j} = b$, $r_i = r'_i, \forall i \neq o_j, o_{j+1}$.

If we divide \mathcal{P}_n into

1. $\mathcal{P}_A = \{r : r_{o_j} = a, r_{o_{j+1}} = b\}$, $\mathcal{P}_{A'} = \{r' : r'_{o_j} = b, r'_{o_{j+1}} = a\}$, $a \geq j+1$
2. $\mathcal{P}_B = \{r : r_{o_j} = a, r_{o_{j+1}} = b\}$, $\mathcal{P}_{B'} = \{r' : r'_{o_j} = b, r'_{o_{j+1}} = a\}$, $b \leq j$
3. $\mathcal{P}_C = \{r : r_{o_j} = a, r_{o_{j+1}} = b\}$, $\mathcal{P}_{C'} = \{r' : r'_{o_j} = b, r'_{o_{j+1}} = a\}$, $a \leq j, b \geq j+1$,

we can rewrite the expectations to be:

$$\mathbb{E}[R_{o_j}] = \sum_{\mathbf{r} \in \mathcal{P}_A} r_{o_j} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\} + \sum_{\mathbf{r} \in \mathcal{P}_{A'}} r_{o_j} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\} +$$

$$\sum_{\mathbf{r} \in \mathcal{P}_B} r_{o_j} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\} + \sum_{\mathbf{r} \in \mathcal{P}_{B'}} r_{o_j} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\} + \\ \sum_{\mathbf{r} \in \mathcal{P}_C} r_{o_j} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\} + \sum_{\mathbf{r} \in \mathcal{P}_{C'}} r_{o_j} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\}, \text{ and similarly,}$$

$$\mathbb{E}[R_{o_{j+1}}] = \sum_{\mathbf{r} \in \mathcal{P}_A} r_{o_{j+1}} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\} + \sum_{\mathbf{r} \in \mathcal{P}_{A'}} r_{o_{j+1}} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\} + \\ \sum_{\mathbf{r} \in \mathcal{P}_B} r_{o_{j+1}} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\} + \sum_{\mathbf{r} \in \mathcal{P}_{B'}} r_{o_{j+1}} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\} + \\ \sum_{\mathbf{r} \in \mathcal{P}_C} r_{o_{j+1}} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\} + \sum_{\mathbf{r} \in \mathcal{P}_{C'}} r_{o_{j+1}} \exp\{-\frac{\alpha^0}{n} d(\mathbf{r}, \boldsymbol{\rho}^0)\}.$$

Let us first consider \mathcal{P}_A and $\mathcal{P}_{A'}$'s contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$. For any $r \in \mathcal{P}_A$ and its corresponding $r' \in \mathcal{P}_{A'}$ s.t. $r_{o_j} = r'_{o_{j+1}} = a$, $r_{o_{j+1}} = r'_{o_j} = b$, and $r_i = r'_i \forall i \neq o_j, o_{j+1}$, it can be inferred that $P(\mathbf{r}) = P(\mathbf{r}')$, since

$$d(\mathbf{r}, \boldsymbol{\rho}^0) = \sum_{i=1}^{j-1} |r_{o_i} - \rho_{o_i}^0| + |r_{o_j} - \rho_{o_j}| + |r_{o_{j+1}} - \rho_{o_{j+1}}^0| + \sum_{i=j+2}^n |r_{o_i} - \rho_{o_i}^0| \\ = \sum_{i \neq j, j+1} |r_{o_i} - i| + |a - j| + |b - (j+1)| \\ = \sum_{i \neq j, j+1} |r_{o_i} - i| + a - j + b - (j+1), \text{ similarly,}$$

$$d(\mathbf{r}', \boldsymbol{\rho}^0) = \sum_{i=1}^{j-1} |r'_{o_i} - \rho_{o_i}^0| + |r'_{o_j} - \rho_{o_j}| + |r'_{o_{j+1}} - \rho_{o_{j+1}}^0| + \sum_{i=j+2}^n |r'_{o_i} - \rho_{o_i}^0| \\ = \sum_{i \neq j, j+1} |r'_{o_i} - i| + |b - j| + |a - (j+1)| \\ = \sum_{i \neq j, j+1} |r_{o_i} - i| + b - j + a - (j+1) \\ = d(\mathbf{r}, \boldsymbol{\rho}^0)$$

For each $\{\mathbf{r}, \mathbf{r}'\}$ pair, their contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$ are:

$$\mathbb{E}[R_{o_j}]|_{\mathbf{r} \in \mathcal{P}_A, \mathbf{r}' \in \mathcal{P}_{A'}} = P(\mathbf{r}) \cdot r_{o_j} + P(\mathbf{r}') \cdot r'_{o_j} = P(\mathbf{r}) \cdot (a + b)$$

$$\mathbb{E}[R_{o_{j+1}}]|_{\mathbf{r} \in \mathcal{P}_A, \mathbf{r}' \in \mathcal{P}_{A'}} = P(\mathbf{r}) \cdot r_{o_{j+1}} + P(\mathbf{r}') \cdot r'_{o_{j+1}} = P(\mathbf{r}) \cdot (a + b)$$

Therefore, for all $\mathbf{r} \in \mathcal{P}_A$ and their corresponding $\mathbf{r}' \in \mathcal{P}_{A'}$, we have

$$\sum_{\mathbf{r} \in \mathcal{P}_A, \mathbf{r}' \in \mathcal{P}_{A'}} \mathbb{E}[R_{o_j}]|_{\mathbf{r}, \mathbf{r}'} = \sum_{\mathbf{r} \in \mathcal{P}_A, \mathbf{r}' \in \mathcal{P}_{A'}} \mathbb{E}[R_{o_{j+1}}]|_{\mathbf{r}, \mathbf{r}'}$$

Similarly, let us now consider \mathcal{P}_B and $\mathcal{P}_{B'}$'s contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$.

For any $r \in \mathcal{P}_B$ and its corresponding $r' \in \mathcal{P}_{B'}$ s.t. $r_{o_j} = r'_{o_{j+1}} = a$, $r_{o_{j+1}} = r'_{o_j} = b$, and $r_i = r'_i \forall i \neq o_j, o_{j+1}$, it can be inferred that $P(\mathbf{r}) = P(\mathbf{r}')$, since

$$d(\mathbf{r}, \boldsymbol{\rho}^0) = \sum_{i=1}^{j-1} |r_{o_i} - \rho_{o_i}^0| + |r_{o_j} - \rho_{o_j}| + |r_{o_{j+1}} - \rho_{o_{j+1}}^0| + \sum_{i=j+2}^n |r_{o_i} - \rho_{o_i}^0|$$

$$\begin{aligned}
&= \sum_{i \neq j, j+1} |r_{o_i} - i| + |a - j| + |b - (j + 1)| \\
&= \sum_{i \neq j, j+1} |r_{o_i} - i| + j - a + (j + 1) - b, \text{ and}
\end{aligned}$$

$$\begin{aligned}
d(\mathbf{r}', \boldsymbol{\rho}^0) &= \sum_{i=1}^{j-1} |r'_{o_i} - \rho_{o_i}^0| + |r'_{o_j} - \rho_{o_j}| + |r'_{o_{j+1}} - \rho_{o_{j+1}}^0| + \sum_{i=j+2}^n |r'_{o_i} - \rho_{o_i}^0| \\
&= \sum_{i \neq j, j+1} |r'_{o_i} - i| + |b - j| + |a - (j + 1)| \\
&= \sum_{i \neq j, j+1} |r_{o_i} - i| + j - b + (j + 1) - a \\
&= d(\mathbf{r}, \boldsymbol{\rho}^0)
\end{aligned}$$

For each $\{\mathbf{r}, \mathbf{r}'\}$ pair, their contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$ are:

$$\mathbb{E}[R_{o_j}] | \mathbf{r} \in \mathcal{P}_B, \mathbf{r}' \in \mathcal{P}_{B'} = P(\mathbf{r}) \cdot r_{o_j} + P(\mathbf{r}') \cdot r'_{o_B} = P(\mathbf{r}) \cdot (a + b)$$

$$\mathbb{E}[R_{o_{j+1}}] | \mathbf{r} \in \mathcal{P}_B, \mathbf{r}' \in \mathcal{P}_{B'} = P(\mathbf{r}) \cdot r_{o_{j+1}} + P(\mathbf{r}') \cdot r'_{o_{j+1}} = P(\mathbf{r}) \cdot (a + b)$$

Therefore, for all $\mathbf{r} \in \mathcal{P}_A$ and their corresponding $\mathbf{r}' \in \mathcal{P}_{A'}$, we have

$$\sum_{\mathbf{r} \in \mathcal{P}_B, \mathbf{r}' \in \mathcal{P}_{B'}} \mathbb{E}[R_{o_j}] | \mathbf{r}, \mathbf{r}' = \sum_{\mathbf{r} \in \mathcal{P}_B, \mathbf{r}' \in \mathcal{P}_{B'}} \mathbb{E}[R_{o_{j+1}}] | \mathbf{r}, \mathbf{r}'$$

Up to this point, we have proven that

$$\sum_{\mathbf{r} \in \mathcal{P}_A, \mathbf{r}' \in \mathcal{P}_{A'}} \mathbb{E}[R_{o_j}] | \mathbf{r}, \mathbf{r}' + \sum_{\mathbf{r} \in \mathcal{P}_B, \mathbf{r}' \in \mathcal{P}_{B'}} \mathbb{E}[R_{o_j}] | \mathbf{r}, \mathbf{r}' = \sum_{\mathbf{r} \in \mathcal{P}_A, \mathbf{r}' \in \mathcal{P}_{A'}} \mathbb{E}[R_{o_{j+1}}] | \mathbf{r}, \mathbf{r}' + \sum_{\mathbf{r} \in \mathcal{P}_B, \mathbf{r}' \in \mathcal{P}_{B'}} \mathbb{E}[R_{o_{j+1}}] | \mathbf{r}, \mathbf{r}',$$

for $\alpha > 0$, proving that $\mathbb{E}[R_{o_j}] < \mathbb{E}[R_{o_{j+1}}]$ is equivalent to proving

$$\sum_{\mathbf{r} \in \mathcal{P}_C, \mathbf{r}' \in \mathcal{P}_{C'}} \mathbb{E}[R_{o_j}] | \mathbf{r}, \mathbf{r}' < \sum_{\mathbf{r} \in \mathcal{P}_C, \mathbf{r}' \in \mathcal{P}_{C'}} \mathbb{E}[R_{o_{j+1}}] | \mathbf{r}, \mathbf{r}'$$

Now let us consider \mathcal{P}_C and $\mathcal{P}_{C'}$'s contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$.

For any $r \in \mathcal{P}_C$ and its corresponding $r' \in \mathcal{P}_{C'}$, we have

$$\begin{aligned}
d(\mathbf{r}, \boldsymbol{\rho}^0) &= \sum_{i=1}^{j-1} |r_{o_i} - \rho_{o_i}^0| + |r_{o_j} - \rho_{o_j}| + |r_{o_{j+1}} - \rho_{o_{j+1}}^0| + \sum_{i=j+2}^n |r_{o_i} - \rho_{o_i}^0| \\
&= \sum_{i \neq j, j+1} |r_{o_i} - i| + |a - j| + |b - (j + 1)| \\
&= \sum_{i \neq j, j+1} |r_{o_i} - i| + j - a + b - (j + 1) \\
&= \sum_{i \neq j, j+1} |r_{o_i} - i| + b - a - 1, \text{ and}
\end{aligned}$$

$$\begin{aligned}
d(\mathbf{r}', \boldsymbol{\rho}^0) &= \sum_{i=1}^{j-1} |r'_{o_i} - \rho_{o_i}^0| + |r'_{o_j} - \rho_{o_j}| + |r'_{o_{j+1}} - \rho_{o_{j+1}}^0| + \sum_{i=j+2}^n |r'_{o_i} - \rho_{o_i}^0| \\
&= \sum_{i \neq j, j+1} |r'_{o_i} - i| + |b - j| + |a - (j + 1)| \\
&= \sum_{i \neq j, j+1} |r_{o_i} - i| + b - j + (j + 1) - a
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i \neq j, j+1} |r_{o_i} - i| + b - a + 1 \\
&= d(\mathbf{r}, \boldsymbol{\rho}^0) + 2
\end{aligned}$$

Therefore, for any $\alpha^0 > 0$ and any given $\mathbf{r} \in \mathcal{P}_C$ and its corresponding $\mathbf{r}' \in \mathcal{P}_{C'}$, we have $P(\mathbf{r}) > P(\mathbf{r}')$. For each $\{\mathbf{r}, \mathbf{r}'\}$ pair, their contributions to $\mathbb{E}[R_{o_j}]$ and $\mathbb{E}[R_{o_{j+1}}]$ are:

$$\begin{aligned}
\mathbb{E}[R_{o_j}]|_{\mathbf{r} \in \mathcal{P}_C, \mathbf{r}' \in \mathcal{P}_{C'}} &= P(\mathbf{r}) \cdot r_{o_j} + P(\mathbf{r}') \cdot r'_{o_B} = a \cdot P(\mathbf{r}) + b \cdot P(\mathbf{r}') \\
\mathbb{E}[R_{o_{j+1}}]|_{\mathbf{r} \in \mathcal{P}_C, \mathbf{r}' \in \mathcal{P}_{C'}} &= P(\mathbf{r}) \cdot r_{o_{j+1}} + P(\mathbf{r}') \cdot r'_{o_{j+1}} = b \cdot P(\mathbf{r}) + a \cdot P(\mathbf{r}') \\
\mathbb{E}[R_{o_j}]|_{\mathbf{r} \in \mathcal{P}_C, \mathbf{r}' \in \mathcal{P}_{C'}} - \mathbb{E}[R_{o_{j+1}}]|_{\mathbf{r} \in \mathcal{P}_B, \mathbf{r}' \in \mathcal{P}_{B'}} &= (a - b)P(\mathbf{r}) + (b - a)P(\mathbf{r}') \\
&= (a - b) \cdot (P(\mathbf{r}) - P(\mathbf{r}'))
\end{aligned}$$

Recall that $a < b$ and $P(\mathbf{r}) > P(\mathbf{r}')$ for any $r \in \mathcal{P}_C$ and its corresponding $r' \in \mathcal{P}_{C'}$, it can be obtained that

$$\mathbb{E}[R_{o_j}]|_{\mathbf{r} \in \mathcal{P}_C, \mathbf{r}' \in \mathcal{P}_{C'}} - \mathbb{E}[R_{o_{j+1}}]|_{\mathbf{r} \in \mathcal{P}_B, \mathbf{r}' \in \mathcal{P}_{B'}} < 0, \text{ i.e., } \mathbb{E}[R_{o_j}]|_{\mathbf{r} \in \mathcal{P}_C, \mathbf{r}' \in \mathcal{P}_{C'}} < \mathbb{E}[R_{o_{j+1}}]|_{\mathbf{r} \in \mathcal{P}_B, \mathbf{r}' \in \mathcal{P}_{B'}}$$

It is straight forward to obtain that:

$$\sum_{\mathbf{r} \in \mathcal{P}_C, \mathbf{r}' \in \mathcal{P}_{C'}} \mathbb{E}[R_{o_j}]|_{\mathbf{r}, \mathbf{r}'} < \sum_{\mathbf{r} \in \mathcal{P}_C, \mathbf{r}' \in \mathcal{P}_{C'}} \mathbb{E}[R_{o_{j+1}}]|_{\mathbf{r}, \mathbf{r}'},$$

and therefore, for any $\alpha > 0$, we have $\mathbb{E}[R_{o_j}] < \mathbb{E}[R_{o_{j+1}}]$

□