1 General Notations

- N: Number of users
- n: Number of items, $n = \begin{cases} 2m-1, & \text{if n is odd} \\ 2m, & \text{otherwise} \end{cases}$
- $\boldsymbol{R}^1, ..., \boldsymbol{R}^N$: full rankings given by the users
- $\boldsymbol{R}^j = \{R_1^j, ..., R_n^j,\} \sim \text{Mallows}(\boldsymbol{\rho}^0, \alpha^0)$
- $P(\boldsymbol{\rho}|\boldsymbol{R}^1, ..., \boldsymbol{R}^N, \alpha^o)$: Mallows posterior
- $\{i_1, ..., i_n\}$: a ranking of n items that determines the sequence following which the items are to be sampled. i.e. $i_j = k$ indicates that item j is the k-th item is to be sampled
- $\{o_1, ..., o_n\}$: an ordering of n items that corresponds to $\{i_1, ..., i_n\}$ s.t. $i_{o_k} = k$. $\{o_1, ..., o_n\}$ and $\{i_1, ..., i_n\}$ have a one-to-one relationship
- $\sum\limits_{\{i_1,...,i_n\} \in \mathcal{P}_n} q(\tilde{\boldsymbol{\rho}}|i_1, ..., i_n, \alpha_0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N) \cdot g(i_1, ..., i_n| \dots)$: pseudolikelihood that approxiamtes the Mallows posterior
- $q(\tilde{\boldsymbol{\rho}}|i_1, ..., i_n, \alpha^0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N)$
  $= q(\tilde{\rho}_{o_1}|\alpha^0, o_1, R_{o_1}^1, ..., R_{o_1}^N) \cdot q(\tilde{\rho}_{o_2}|\alpha^0, o_2, \tilde{\rho}_{o_1} R_{o_2}^1, ..., R_{o_2}^N) \cdot ... \cdot$
  $q(\tilde{\rho}_{o_{n-1}}|\alpha^0, o_{n-1}, \tilde{\rho}_{o_1}, ..., \tilde{\rho}_{o_{n-2}}, R_{n-1}^1, ..., R_{n-1}^N) \cdot q(\tilde{\rho}_{o_n}|\alpha^0, o_n, \tilde{\rho}_{o_1}, ..., \tilde{\rho}_{o_{n-1}}, R_n^1, ..., R_n^N)$

  – $q(\tilde{\rho}_{o_1}|\alpha^0, o_1, R_{o_1}^1, ..., R_{o_1}^N) = \dfrac{\exp\{-\frac{\alpha_0}{n} \sum\limits_{j=1}^N d(R_{o_1}^j, \tilde{\rho}_{o_1})\} \mathbb{1}_{\tilde{\rho}_{o_1} \in \{1,...,n\}}}{\sum\limits_{\tilde{r}_{o_1} \in \{1,..,n\}} \exp\{-\frac{\alpha_0}{n} \sum\limits_{j=1}^N d(R_{o_1}^j, \tilde{r}_{o_1})\}}$

  – $q(\tilde{\rho}_{o_k}|\alpha^0, o_k, \tilde{\rho}_{o_1}, ..., \tilde{\rho}_{o_{k-1}}, R_{o_k}^1, ..., R_{o_k}^N) = \dfrac{\exp\{-\frac{\alpha_0}{n} \sum\limits_{j=1}^N d(R_{o_k}^j, \tilde{\rho}_{o_k})\} \mathbb{1}_{\tilde{\rho}_{o_k} \in \{1,..,n\} \setminus \{\tilde{\rho}_{o_1}, ..., \tilde{\rho}_{o_{k-1}}\}}}{\sum\limits_{\tilde{r}_{o_k} \in \{1,..,n\} \setminus \{\tilde{\rho}_{o_1}, ..., \tilde{\rho}_{o_{k-1}}\}} \exp\{-\frac{\alpha_0}{n} \sum\limits_{j=1}^N d(R_{o_k}^j, \tilde{r}_{o_k})\}}$

  for $k = 2, ..., n$
- $\boldsymbol{\rho}^0 \leftrightarrow \boldsymbol{o}^0$ s.t. ${\rho^0}^{-1}(m) = o_m^0$

- Define the "v-function" $f_v(\cdot)$ such that $f_v(\boldsymbol{\rho}^0) = \mathcal{V}_{\boldsymbol{\rho}^0}$, where
  - $\mathcal{V}_{\boldsymbol{\rho}^o} = \begin{cases} \{\boldsymbol{r} \in \mathcal{P}_n : r_{o^0_m} = 1, r_{o^0_{m\pm k}} \in \{2k, 2k+1\}, k = 1, ..., m-1\}, & \text{if n is odd} \\ \{\boldsymbol{r} \in \mathcal{P}_n : \{r_{o^0_{m-k}}, r_{o^0_{m+k+1}}\} \in \{2k+1, 2k+2\}, k = 0, ..., m\}, & \text{if n is even} \end{cases}$

2 Theorems and Lemmas

2.1

Lemma 2.1.1 *Given there are odd number of items, i.e.* $n = 2m-1$. $\forall \alpha^0 \in (0, \infty)$,

1. $\mathbb{E}(R_{o^0_m}|\boldsymbol{\rho}_0, \alpha^0) = \rho^0_m = m$
2. $\forall j \in [1, m-2]$, $j < \mathbb{E}[R_{o^0_j}|\boldsymbol{\rho}^0, \alpha^0] < \mathbb{E}[R_{o^0_{j+1}}|\boldsymbol{\rho}^0, \alpha^0] < m$
3. $\forall j \in [m+2, 2m-1]$, $m < \mathbb{E}[R_{o^0_{j-1}}|\boldsymbol{\rho}^0, \alpha^0] < \mathbb{E}[R_{o^0_j}|\boldsymbol{\rho}^0, \alpha^0] < j$

Similarly, if n is even, i.e. $n = 2m$, $\forall \alpha^0 \in (0, \infty)$,

1. $\forall j \in [1, m-1]$, $j < \mathbb{E}[R_{o^0_j}|\boldsymbol{\rho}^0, \alpha^0] < \mathbb{E}[R_{o^0_{j+1}}|\boldsymbol{\rho}^0, \alpha^0]$
2. $\forall j \in [m+2, 2m]$, $\mathbb{E}[R_{o^0_{j-1}}|\boldsymbol{\rho}^0, \alpha^0] < \mathbb{E}[R_{o^0_j}|\boldsymbol{\rho}^0, \alpha^0] < j$

Note that for both cases, it satisfies that $\forall 1 \leq j < k \leq n$ *and* $\forall \alpha > 0$, $\mathbb{E}[R_{o^0_j}|\boldsymbol{\rho}^0, \alpha^0] < \mathbb{E}[R_{o^0_k}|\boldsymbol{\rho}^0, \alpha^0]$

Lemma 2.1.2 *As* $N \to \infty$, $\frac{1}{N}\sum_{j=1}^{N} R_i^j \to \mathbb{E}[R_i|\boldsymbol{\rho}^0, \alpha^0]$, $\forall i = 1, ..., n$

Definition 1 *Given a vector of length n, i.e.* $\{x_1, ..., x_n\}$,*the rank function* $rank(x_1, ..., x_n)$ *is defined as* $rank(x_1, ..., x_n) = \{r_1, ..., r_n\}$ *such that* $x_{(r_k)} = x_k$, $\forall k = 1, ..., n$

Theorem 2.1.3 *As* $N \to \infty$, *and* $\forall \alpha > 0$,
$rank(\frac{1}{N}\sum_{j=0}^{N} R_1^j, ..., \frac{1}{N}\sum_{j=0}^{N} R_n^j) \to rank(\mathbb{E}[R_1|\boldsymbol{\rho}^0, \alpha_0], ..., \mathbb{E}[R_n|\boldsymbol{\rho}^0, \alpha_0]) = \boldsymbol{\rho}^0$

That is to say, as N approaches infinity, the Mallows consensus parameter $\boldsymbol{\rho}^0$ can be inferred from the data by taking the marginal mean for each item from the data and apply the rank functions to these marginal means.

2.2

Theorem 2.2.1 *For a function g defined on the space of permutation* $\mathcal{P}_n$ *which can depend on* $\boldsymbol{\rho}^0$,

$$\arg\min_{g\in\mathcal{D}_{\boldsymbol{\rho}^0}} \lim_{N\to\infty} KL(P(\boldsymbol{\rho}|\alpha^0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N)|| \sum_{\{i_1,...,i_n\}\in\mathcal{P}_n} q(\tilde{\boldsymbol{\rho}}|\alpha^0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N, i_1, ..., i_n) g(i_1, ..., i_n|\boldsymbol{\rho}^0)$$
$$= g^*(i_1, ..., i_n|\mathcal{V}_{\boldsymbol{\rho}^0}),$$

where

- $\mathcal{D}_{\boldsymbol{\rho}^0}$ *is a set of all distrbutions on the space of permutation* $\mathcal{P}_n$, *which depends on* $\boldsymbol{\rho}^0$, *i.e.,* $\mathcal{D}_{\boldsymbol{\rho}^0}$ =*{what is a good notation of this??}*
- $g^*(i_1, ..., i_n|\mathcal{V}_{\boldsymbol{\rho}^0})$ *is a distribution whose density is concentrated on* $\boldsymbol{\rho}^0$, *i.e.*
$$\begin{cases} g^*(i_1, ..., i_n|\mathcal{V}_{\boldsymbol{\rho}^0}) = |\mathcal{V}_{\boldsymbol{\rho}^0}|^{-1} > 0, & if \{i_1, ..., i_n\} \in \mathcal{V}_{\boldsymbol{\rho}^0} \\ g^*(i_1, ..., i_n|\mathcal{V}_{\boldsymbol{\rho}^0}) = 0, & if \{i_1, ..., i_n\} \notin \mathcal{V}_{\boldsymbol{\rho}^0} \end{cases}$$

explanation: for a set of distributions g, which are defined the space of permutation of n items, i.e.$\mathcal{P}_n$, the distribution g^* that minimizes the KL-divergence betweeeh the Mallows posterior and the pseudolikelihood defined above, is a uniform distribution with its density concentrated on $\mathcal{V}_{\boldsymbol{\rho}^o}$

2.3

For a given $N < \infty$, define $\hat{\boldsymbol{\rho}}^0$ as $rank(\frac{1}{N}\sum_{j=0}^{N} R_1^j, ..., \frac{1}{N}\sum_{j=0}^{N} R_n^j)$. $\mathcal{V}_{\hat{\boldsymbol{\rho}}^0} = f_v(\hat{\boldsymbol{\rho}}^0)$

Theorem 2.3.1 $\exists\sigma \geq 0$ *and* $g'(i_1, ..., i_n|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0}, \sigma)$ *such that*

$$KL\ (P(\boldsymbol{\rho}|\alpha^0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N)|| \sum_{\{i_1,...,i_n\}\in\mathcal{P}_n} q(\tilde{\boldsymbol{\rho}}|\alpha^0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N, i_1, ..., i_n) g^*(i_1, ..., i_n|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0)} \geq$$
$$KL\ (P(\boldsymbol{\rho}|\alpha^0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N)|| \sum_{\{i_1,...,i_n\}\in\mathcal{P}_n} q(\tilde{\boldsymbol{\rho}}|\alpha^0, \boldsymbol{R}^1, ..., \boldsymbol{R}^N, i_1, ..., i_n) g'(i_1, ..., i_n|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0}, \sigma)$$

where

- $\hat{\boldsymbol{v}} \sim g^*(\hat{\boldsymbol{v}}|\mathcal{V}_{\hat{\rho}^0})$
- $x_i \sim \mathcal{N}(x_i|\hat{v}_i, \sigma)$ *for* $i = 1, ..., n$
- $i_1, ..., i_n = rank(x_1, ..., x_n)$,

and $g'(i_1, ..., i_n|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0}, \sigma) = \sum_{\hat{\boldsymbol{v}}\in\mathcal{V}_{\hat{\rho}^0}} \{g^*(\hat{\boldsymbol{v}}|\mathcal{V}_{\hat{\boldsymbol{\rho}}^0}) \int_{\boldsymbol{x}} \prod_{i=1}^{n} \mathcal{N}(x_i|\hat{v}_i, \sigma) \cdot 1 d\boldsymbol{x}\}$

Explanation: as N is limited, $\boldsymbol{\rho}^0$, and therefore, $\mathcal{V}_{\boldsymbol{\rho}^0}$ cannot be accurately inferred from the data. We can however, sample for $i_1, ..., i_n$ by sampling for each item i from a univariate Gaussian distribution centred on $\hat{v}_i$ with a fixed variance σ for all items, and then rank the resulting "scores" to convert it back to rankings. By doing so, we can achieve a smaller KL divergence to the Mallows posterior, compared to not introducing the Gaussian noise.

2.4

$$\sigma = \begin{cases} 0, & \text{if } \delta(\alpha, n, N) \leq \delta^* \\ f(\alpha, n, N), & \text{otherwise} \end{cases}$$

Explanation: With the usage of $g'(i_1, ..., i_n | \mathcal{V}_{\hat{\boldsymbol{\rho}}^0})$, the value of σ which minimizes the KL-divergence between the Mallows posterior and the resulted Pseudolikelihood should be 0 when $\delta(\alpha, n, N) \geq \delta^*$. Beyond this point, the optimal choice of σ should be greater than 0, and it follows a function $f(\alpha, n, N)$

2.5

as $N \to \infty, \delta^* \to \max\delta(\alpha, n, N)$, $\forall \alpha > 0, n \geq 1$