Exploring Chaotic Dynamics in the Damped Driven Pendulum, the Kicked Harmonic Oscillator and the Kicked Pendulum.

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1 Abstract

The damped driven pendulum and kicked harmonic oscillators are systems with many applications in physics and as a result, their chaotic behaviour is of great interest. Numerical and analytic methods are used to investigate these systems. Bifurcation diagrams are used to locate chaotic conditions for the damped driven pendulum and chaotic behaviour is demonstrated through the use of Poincaré sections. The kicked harmonic oscillator is analytically solved and the effects of changing the initial conditions for the system are then explored. The stochastic web - characteristic for this system, is produced and discussed. The kicked pendulum is then modelled and comparisons are made between this system and the kicked harmonic oscillator. Unpredicted behaviour is found for higher value initial conditions for the kicked pendulum and further study is recommended.

2 Introduction

2.1 What is chaos?

Chaos theory has a wide and diverse range of applications, from more physics based areas such as meteorology [1] and quantum chaos [2], to more unexpected areas such as traffic management [3] and economics [4]. The study of chaos has produced profound concepts like the butterfly effect [5] - the notion that a small action can cascade and effect huge changes in the future. It also brings into question how far we can accurately predict the future of systems and explains the difficulty of practices such as weather forecasting [1].

Interestingly, there is no universally accepted definition for chaotic behaviour. Despite this, Robert Devaney produced a widely accepted set of criteria [6] for chaotic behaviour. The criteria state that in order for a system to exhibit chaotic behaviour, it must be sensitive to initial conditions, it must be topologically transitive and it must have dense periodic orbits. Stephen Strogatz developed a simpler set of conditions necessary for a system to be described as chaotic [7]. According to Strogatz, a chaotic system must have aperiodic longer term behaviour such that trajectories will not settle down to periodic orbits or fixed points, the behaviour must be deterministic, so there are no random inputs to the system and the behaviour is in theory entirely predictable, and finally, it must be sensitive to initial conditions, so that the behaviour of two close sets of initial conditions will deviate rapidly, or even exponentially.

2.2 Dynamical systems and chaotic behaviour

The study of chaos centres around dynamical systems. These systems obey differential equations that describe the system's evolution through time. This means they are deterministic - they contain no random elements, meaning that for a particular set of initial conditions, the dynamical system will behave the same way. These equations are not always analytically solvable and in the cases where they are not, approximations have to be calculated iteratively over time. Modern advances in computing technology have made studying these types of systems more viable due to the speed with which huge numbers of approximations can be calculated in a short amount of time.

These approximations are not perfect, however. There are limits to the accuracy with which computers can approximate the solutions to systems of differential equations and this is especially prevalent in the context of chaotic systems as their high sensitivity to tiny changes, producing wildly differing results, means that it becomes practically infeasible to simulate even simple chaotic systems for longer time-frames. This is especially prevalent in applications such as meteorology [1]. In solving chaotic systems, any minuscule error in the values calculated by the computer will compound, with the approximation becoming exponentially more inaccurate over time. This is described quite poetically through the butterfly effect [5] - the notion that small changes, over time, can effect huge consequences.

2.3 Attractors

Attractors are sets of points in phase space towards which dynamical systems gravitate, usually as a result of dissipation of energy from an action such as damping. Dynamical systems can have infinitely many attractors, corresponding to the infinite set of initial conditions that a system may have, however not all are unique. For simple systems, an attractor may just be a point, such as for the damped pendulum settling down to hanging vertically below its pivot. Some systems however have *strange attractors* for some initial conditions. Indicative of chaos, these structures are fractals: sets of points with dimensions that are irrational or fractional. A system that tends towards a strange attractor will never repeat the same motions twice. This is due to the fractal geometry of the strange attractor - there are infinitely many trajectories within it, each unique to a set of initial conditions [8].

Another structure in phase space that satisfies Strogatz' criteria for chaos is the stochastic web [9]. Stochastic webs are similar to strange attractors, however the systems that produce them lack the energy dissipation required to settle down onto a fractal set of points over a finite area. Instead, these structures stretch infinitely throughout phase space as meshes of seemingly random points surrounding tiled, empty patches where the non-chaotic tori would be located for non-chaotic parameters.

2.4 Chaotic behaviour in pendula and other oscillators

The pendulum is one such example of a dynamical system that can exhibit chaotic behaviour. When left to freely swing, the pendulum is predictable even with the nonlinear form. When damping and driving is added however, with the right parameters, chaotic behaviour arises. This report shall demonstrate the chaotic nature of the nonlinear damped driven pendulum in section 3.2, then go on to examine the kicked harmonic oscillator in section 3.3. The effects of adding the periodic kicking from the kicked harmonic oscillator to the nonlinear pendulum is explored in section 3.4. Results are then discussed in section 4, where an interesting disparity is found between the kicked harmonic oscillator and the kicked pendulum for higher value initial conditions.

3 Methodology

3.1 Phase portraits and Poincaré sections

Phase portraits are a common and useful tool for studying dynamical systems. They are constructed by plotting velocity against position and they provide more information than directly plotting the position and velocity over time, as they can more clearly show periodic behaviour.

Phase portraits show the general behaviour of a dynamical system, however they often provide too much superfluous information. When used in a periodic system, a Poincaré section eliminates unnecessary information and only shows the changes in a system after each period - allowing the reader to see the behaviour of the dynamics more clearly. Poincaré sections have been used extensively to demonstrate chaotic behaviour in dynamical systems.

A Poincaré section is effectively a stroboscopic set of snapshots of a dynamical system over time. There is no single way to produce a Poincaré section for a system, as behaviours can be studied through using different point sampling methods. Typically, for a dynamical system, points from the phase portrait are sampled each time a plane is intersected. This plane is perpendicular to the direction of travel on the phase portrait. This provides a series of points demonstrating the change in the system over time.

Conveniently, for the system of the driven, damped pendulum, a Poincaré section can be produced by sampling points every driving period. The mathematics of this are explained in detail in [10], however, simply put, this uses the fact the system's phase portrait can be displayed in three dimensions, of $x = \theta$, $y = \dot{\theta}$ and $z = \omega t$ where ω is the driving frequency and $0 \leq z < 2\pi$. Joining the boundaries of $z = 0$ and $z = 2\pi$ to give a continuous representation of the system forms a toroidal shape. Taking a cross section of this toroidal structure provides a Poincaré section for the system. A visualization is provided in figure 1.

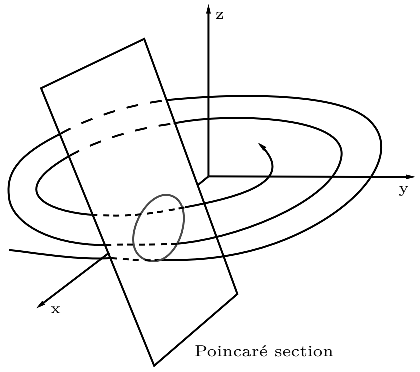


Figure 1: A visualization of the construction of a Poincaré section.

3.2 The Damped, Driven Pendulum

3.2.1 The equations of motion for the damped, driven pendulum

The non-dimensional equations of motion for a pendulum freely swinging pendulum [11] are:

$$\dot{\theta} = \frac{d\theta}{dt}, \tag{1}$$

$$\ddot{\theta} = -\sin\theta. \tag{2}$$

Adding in damping and periodic driving terms, the equations of motion become:

$$\dot{\theta} = \frac{d\theta}{dt}, \tag{3}$$

$$\ddot{\theta} = -\frac{\dot{\theta}}{Q} - \sin\theta + A\cos\omega t, \tag{4}$$

where Q is the damping coefficient - inversely proportional to the strength of the damping, ω is the driving frequency, and A is the driving strength.

3.2.2 Numerical solution to the nonlinear driven damped pendulum

The equations of motion for the driven damped pendulum when using the nonlinear term $\sin\theta$, have no simple analytical solution. They can however be solved quite simply through numerical methods. MATLAB was used for this task, through use of the ODE45 function for solving differential equations. This function is simple to use and is versatile. It also adapts the time-step appropriately, having larger time-steps for regions of less change and smaller time-steps for regions of more rapid change. This allows better performance with minimal impact on accuracy.

A MATLAB script, listed in appendix B.5.1 was written to solve these equations for a given length of time, with a relative tolerance of 1×10^{-10} configured in the ODE45 function.

3.2.3 Locating chaos in the damped, driven pendulum

Not all systems are chaotic and not all initial conditions for chaotic systems produce chaotic behaviour. In order to find chaotic conditions and to study the effects of changing parameters on the chaotic nature of the system, bifurcation diagrams can be used.

Bifurcation diagrams are a way of visualizing the change in behaviour of a dynamical system while changing a parameter. Figure 2 shows a bifurcation diagram for the damped driven pendulum, with initial conditions $\theta_0 = \dot{\theta}_0 = 0$ and parameters: driving frequency, $\omega = 0.67$ and damping factor, $Q = 2$. On the x-axis, the driving strength is increased from 0 to 1.2, and on the y-axis, the velocity is plotted. This diagram was produced by calculating the points for Poincaré sections for varying values of driving strength, A, collapsing each section down to a single dimension in $\dot{\theta}$, and plotting these collapsed sections against the driving strength. It is helpful to think of this as looking at the series of Poincaré sections edge-on for a range of different driving strengths. From figure 2, it was clear that chaotic behaviour could be found at a range of values for driving strength. A value of $A = 1.15$ was chosen.

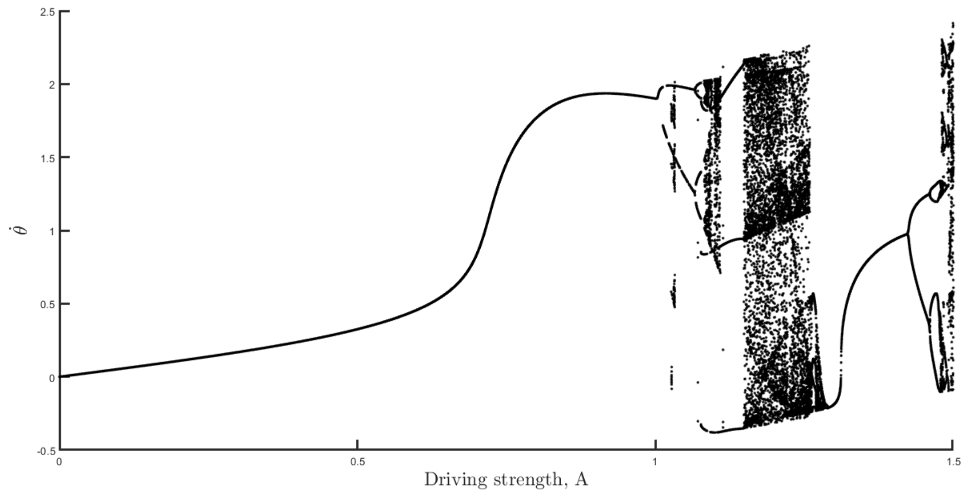


Figure 2: Bifurcation diagram for the driven damped pendulum with dimensionless parameters: damping factor, $Q = 2$ and driving frequency, $\omega = 0.67$.

3.3 The kicked harmonic oscillator

3.3.1 The equations of motion for the kicked harmonic oscillator

The kicked harmonic oscillator (KHO) system [12] has been a popular system for analysis, with many applications in physics. The KHO is a non-damped, non-driven oscillator that is periodically kicked - imparting instantaneous velocity changes. The periodicity and strength of the kicking are parameters that can be adjusted to study the behaviour of the system.

The KHO is a relatively simple system and can be solved analytically, resulting in a kick-to-kick mapping, where given a state at a particular kick, the position and momentum at the next kick can be calculated via the simple mapping - equations 5 and 6. These non-dimensional equations are derived in appendix A:

$$x_{n+1} = x_n \cos(\tau) + [p_n + \bar{K} sin(\sqrt{2} x_n)] \sin(\tau), \tag{5}$$

$$p_{n+1} = [p_n + \bar{K} sin(\sqrt{2} x_n)] cos(\tau) - x_n \sin(\tau), \tag{6}$$

where x is the position, p is the momentum, $\bar{K}$ is the kicking strength and τ is the interval between kicks.

3.3.2 Locating Chaos in the kicked harmonic oscillator

Given the simplicity of this mapping, a large number of iterations can be completed relatively quickly. Again using a Poincaré section to simplify the dynamics of the system, the position was plotted against the velocity at intervals of the kicking period, which conveniently is the time between points obtained from the mapping. Only kicking frequencies that are rational fractions of the natural frequency of the oscillator were considered - since this is where the interesting dynamics reside [12]. Plotting Poincaré sections for different initial conditions, we get a sense of how the system behaves. Figure 3 shows the results.

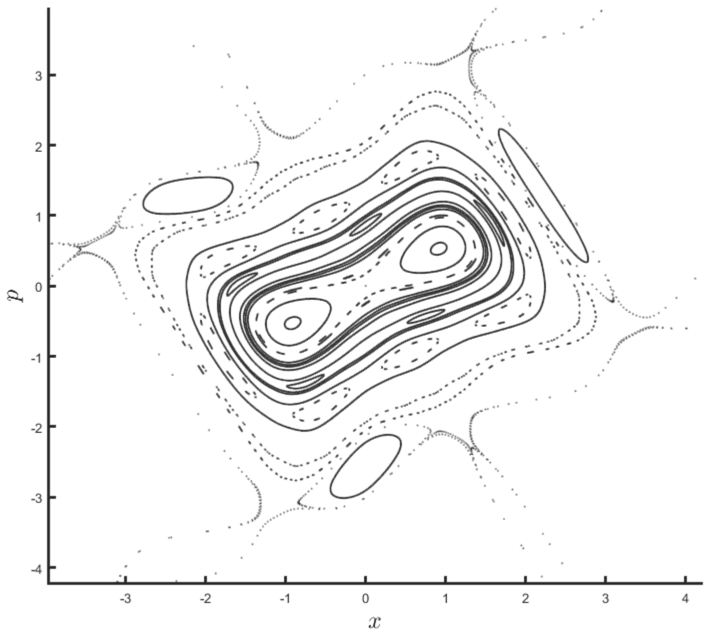


Figure 3: Poincaré section for the KHO for a range of different initial conditions. Kicking strength: -1.1, kicks per period: 3, x_0, $p_0 = 0 \to 1.8$

From figure 3, it can be seen that the outermost Poincaré section is behaving strangely; its points seem to be scattered far further than the other sections' points, hinting at a stochastic web. The initial conditions that demonstrated this best were $x_0 = p_0 = 1.7679$.

3.4 The kicked pendulum

3.4.1 The equations of motion for the kicked pendulum

The system of the periodically kicked nonlinear pendulum was then investigated, for which the equations of motion can be found by modifying the non-dimensional equations of motion of the kicked harmonic oscillator to be nonlinear,

$$\frac{dx}{dt} = p \qquad \to \qquad \frac{dx}{dt} = p, \tag{7}$$

$$\frac{dp}{dt} = -x \qquad \to \qquad \frac{dp}{dt} = -\sin x. \tag{8}$$

The periodic kicking can be added, as with the kicked harmonic oscillator, through altering the momentum of the system at specific time intervals.

3.4.2 Numerical solution to the periodically kicked nonlinear pendulum

MATLAB's ODE45 function is convenient for solving the nonlinear ODEs in the case of the damped, driven pendulum, however it becomes infeasible to use such a function for the kicked pendulum, since the momentum of the system cannot be altered while the ODE45 function is running. The ODE45 function would therefore need to be repeatedly called between kicks, and the computational overhead would severely hinder performance. The solution is to manually implement computational methods to solve the ODE and make the kicking built-in. For this purpose, the Runge-Kutta 4th order method [13] (RK4) for solving ODEs was utilised. The method was chosen due to its relative simplicity whilst maintaining a high degree of accuracy.

The Runge-Kutta 4th order method makes an approximation to a solution by utilising four local evaluations of the slope of the function and weighting them appropriately - derived from the Taylor expansion. With this manual implementation, it was possible to add kicking to the system with little extra computation. The code for this is listed in appendix B.5.2. The RK4 method was then used to solve the periodically kicked nonlinear pendulum and the Poincaré sections for this system were plotted for a set of initial conditions of increasing magnitude. These were plotted against the Poincaré sections for the kicked harmonic oscillator for comparison.

4 Results and discussion

4.1 The damped driven pendulum

4.1.1 Bifurcation of the driven, damped pendulum

Figure 2 shows the bifurcation of the damped, driven pendulum for driving strength, A, varying from $A = 0$ to 1.5, allowing us to study the effects of changing the driving strength on the attractor for the initial conditions $\theta = 0$, $\dot{\theta} = 0$. Increasing the driving strength from 0 through 1, the Poincaré sections are single points. This is known as period one motion, since the system repeats itself with each driving period. The momentum of these points increases for larger A, indicating that there is more energy in the system due to the stronger driving force, however no new dynamics present themselves. Just past a strength of 1 however, the period one line splits into two - bifurcating, indicating a period two attractor, where the system repeats itself with every two driving periods. A little further on, these two lines bifurcate again, each into two more lines, hence period four behaviour, with an attractor that is comprised of four points. Just after this second bifurcation, the system devolves into chaos, shown by the seemingly random dense scattering of points. It is this area that we are interested in. We can visualize this chaos further using a Poincaré section from the chaotic regions.

4.1.2 Poincaré section for the chaotic damped driven pendulum

The Poincaré sections for the chaotic regions do not show a finite number of points, unlike the previously observed period n behaviour. Instead, the system never repeats itself. On the Poincaré section this manifests as a strange attractor - an attractor with fractal structure and non-integer dimensions. Two points that may seem to be on the same fold on the attractor will always in fact be on different folds, one just has to iterate further to see it, as described in [8].

Figure 4 shows the Poincaré map produced for the damped, driven pendulum for $A = 1.15$. The fractal nature of the attractor can clearly be seen, with the curve folding in on itself repeatedly. No matter how far the image is scaled up, there are yet more folds and each line turns into several more, densely packed lines. This is a visual demonstration of the criteria of 'dense periodic orbits' as described by Devaney [6]. Any point on the Poincaré section, because of the section's fractal nature, is arbitrarily close to another point, one just has to iterate for long enough and to zoom in far enough.

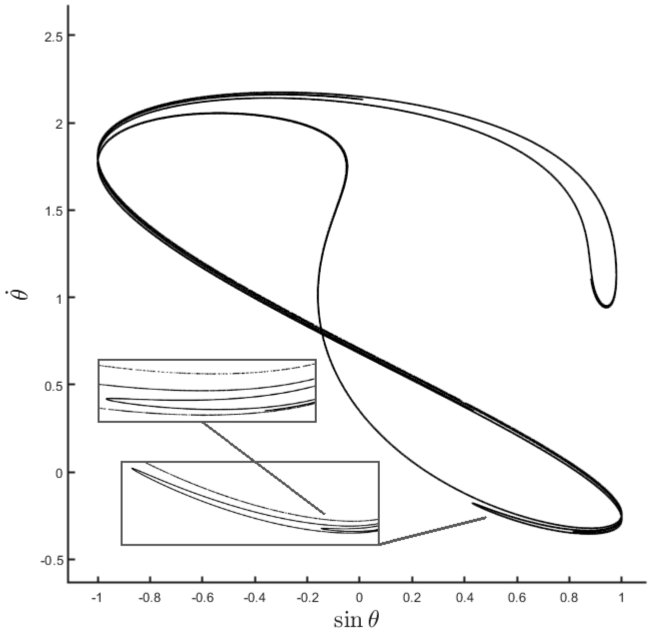


Figure 4: Poincaré section for the damped driven pendulum for parameters: $A = 1.15$, $\omega = 0.67$, $Q = 2$.

4.2 The kicked harmonic oscillator

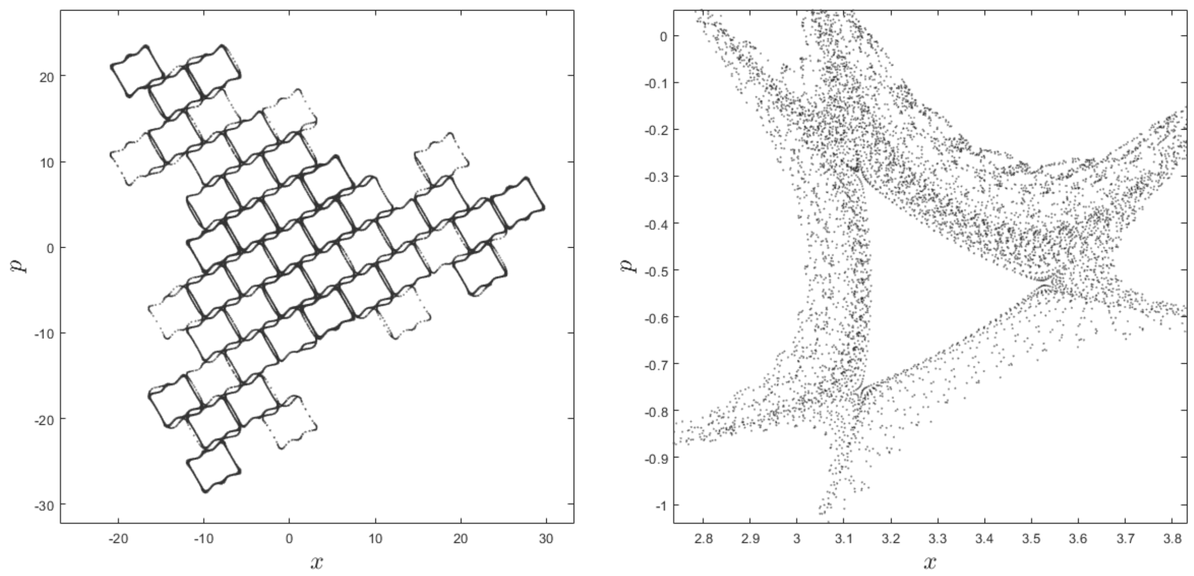
Figure 3 shows the Poincaré sections of the kicked harmonic oscillator system for different initial conditions, with kicking strength of -1.1. The Poincaré section for this system is found by sampling points every kicking period, similar to how the points were sampled every driving period for the damped, driven pendulum. The figure starts with stable initial conditions closer to the origin, resulting in the two innermost oval shapes, then increasing the initial conditions such that $x_0 = p_0$, producing Poincaré sections further out, demonstrating the increase in energy provided by the initial conditions. Approaching unstable, chaotic conditions near $x = p = 1.7679$, the outermost plot hints at the characteristic stochastic web structure. It is this web structure which depicts the chaotic behaviour of the system.

The initial conditions of $x = p = 1.7679$ provide a better picture of the stochastic web, shown in figure 5a. In this plot, the web stretches quite far and the repeating lattice is easy to discern. In a perfect plot, this web would stretch all throughout phase space, however the initial conditions used are not perfect. Taking a closer look at the stochastic web, figure 5b shows the chaotic dynamics, with the areas of chaos covered densely with seemingly random points. This is a sharp contrast to the ordered shapes produced by similar initial conditions in figure 3 where there is clear structure and the shapes are unbroken.

This sudden tiling across the entire phase-plane implies that given a small perturbation and some time, a particle is able to move to a wide region of the phase plane, unlike for the non chaotic sections that are bound to single tori [12]. This demonstrates the fact that the system does not repeat itself for these parameters, satisfying the first of Strogatz' criteria for chaotic behaviour. This also seems to be at odds with the KAM theory, where small perturbations in kicking strength should not break the tori, however as figure 3 shows, they clearly are in this case. This is discussed further in [14].

4.3 Nonlinear KHO - The kicked pendulum and looking forward

In order to study the behaviour of more complex systems such as the kicked pendulum and the kicked, damped, driven pendulum, the kicked harmonic oscillator needs to be able to be



(a) (b)

Figure 5: Stochastic web structure for the KHO. Kicking strength: -1.1, kicks per period: 3, $x_0 = p_0 = 1.7679$. (b) is a zoomed-in view.

solved numerically, as the aforementioned systems have no simple analytic solution and so a simple mapping cannot be produced. The Runge-Kutta 4th order iterative method (RK4) was used due to its high degree of accuracy for relatively low computational cost compared to other methods such as Euler's method. A MATLAB script was produced implementing this method for numerically solving systems of differential equations. The script is listed in appendix B.5.2.

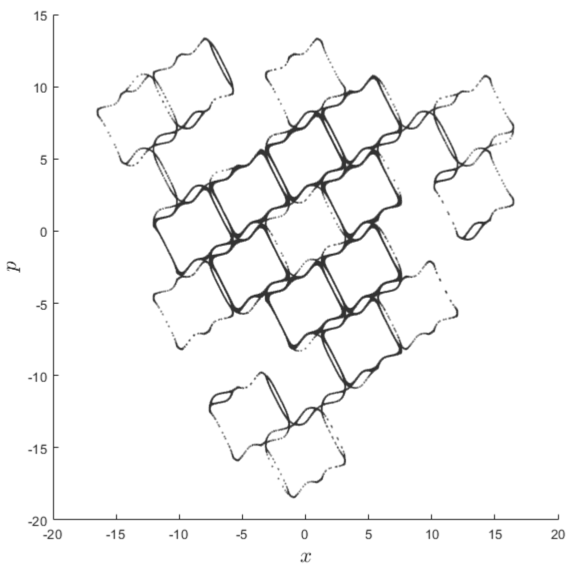
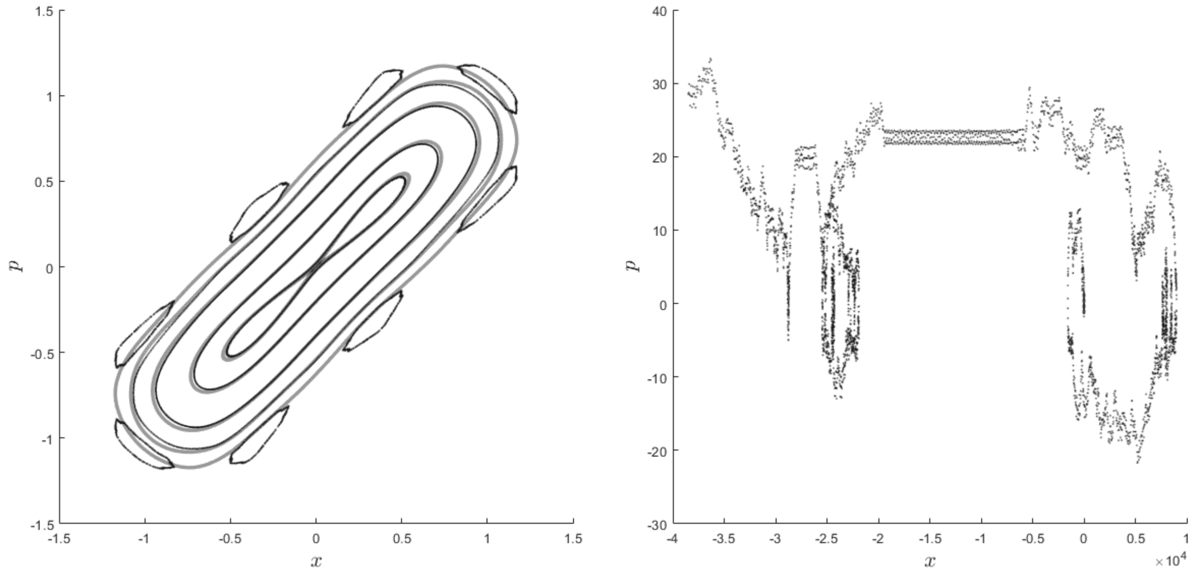


Figure 6: Stochastic web structure for the KHO calculated numerically using an RK4 implementation, with parameters equal to those used in figure 5.

Figure 6 shows the Poincaré section for $x = p = 1.7679$, calculated using the RK4 MATLAB script. The plot compares very well to figure 5a however due to the amount of calculation required for this solution compared to the simple analytical mapping from before, the result



(a)

(b)

Figure 7: Poincaré section for the KHO using RK4 over a range of initial conditions. Kicks per period: 3, kicking strength: -1.48445. In (a), $x_0 = p_0 = [0.0001, 0.01, 0.1, 0.2, 0.3]$, grey points are linear, black points are nonlinear. In (b), $x_0 = p_0 = 0.389$.

takes a lot longer to produce, resulting in fewer iterations than in figure 5a.

Using the nonlinear equations of motion for the kicked harmonic oscillator to model the kicked pendulum, it is clear from figure 7a that discrepancies in the behaviour of the KHO and the kicked pendulum arise as the initial conditions of the system are increased. This is to be expected given that for small angles, a pendulum will behave as a simple oscillator with linear equations of motion due to the approximation $\sin\theta = \theta$ for $\theta \approx 0$. For higher value initial conditions, this approximation breaks down however, and the nonlinear system of the pendulum becomes erratic, as seen in the outermost plot in figure 7a. Continuing on from these fairly small initial conditions, the difference grows large and the Poincaré section for the nonlinear pendulum loses all apparent structure, as seen in figure 7b. In figure 7b, the Poincaré section becomes unpredictable and no longer follows the linear plot, even loosely. Interestingly, there seems to be some structure to the plot, with a tubular shape forming near $x = 1.5$, $p = 23$, raising the question that there could be structure to this seemingly random trajectory. The nature of this structure would be interesting to study in further research on this system.

Following on from this, adding this kicking to the driven, damped pendulum would also be interesting to study in future research. This could easily be done by adding the driving and damping terms from the damped, driven pendulum, into the equations of motion for the kicked pendulum:

$$\frac{dx}{dt} = p, \tag{9}$$

$$\frac{dp}{dt} = -\frac{p}{Q} - \sin x + A\cos\omega t. \tag{10}$$

There is a problem with this however; it becomes tricky to plot Poincaré sections for this system as there no longer is just one periodicity to consider when sampling points. This could perhaps be overcome by setting the driving frequency equal to the kicking frequency.

5 Conclusions

A numerical method for solving the driven damped pendulum was developed and chaotic behaviour was located with the use of a bifurcation diagram and was subsequently demonstrated with a Poincaré section. The strange attractor for the system in chaotic conditions was then shown. The kicked harmonic oscillator (KHO) was then modelled using an analytic mapping, its chaotic behaviour was demonstrated in the stochastic web produced. A script was then developed using the Runge-Kutta 4th order method, with the added ability to periodically alter the system's velocity, resembling instantaneous kicking. The script was used to model the kicked pendulum and it was found that the this system follows the KHO closely for lower value conditions, however this similarity breaks down for higher value conditions. This result was expected and is explained by the small angle approximation. This exemplifies the similarity between the two systems for a range of initial conditions. For initial conditions with larger values, the behaviour of the pendulum becomes erratic. A strange structure was formed at $\theta = -1.5$, $\dot{\theta} = 23$, resembling a tube-like shape. It is unknown what could have caused this and further investigation is recommended. The use of MATLAB is a double-edged sword. While it is simple to use for prototyping, it is slower than other languages since it is an interpreted language as opposed to compiled languages like Fortran, the use of which could reduce the computation time and make studying these systems easier.

A natural extension to the work produced, is to apply damping and driving to this system. This could be done quite simply via changes to the equations of motion, however there is a potential issue in that there are two periodicities - the driving frequency and the kicking frequency. This may be able to be overcome by setting these to be equal.

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Appendices

A Analytic solution to the kicked harmonic oscillator

A.1 Equations of motion

The system of the kicked harmonic oscillator can be analytically solved, providing a mapping from one time step to the next. To begin with, the Hamiltonian for the harmonic oscillator is considered:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \tag{11}$$

Where ω is the natural frequency of the oscillator and m is the mass.

Adding a term for periodic kicking with kicking strength $\bar{K}$, time interval between kicks, τ and wave number k,

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \bar{K}\cos kx \sum_{n=-\infty}^{\infty} \delta\left(t - n\tau\right) \tag{12}$$

However, it will be simpler in the long term to work with dimensionless variables. Introducing dimensionless forms of the kicking strength and time interval,

$$\bar{K}' = \frac{k^2 \bar{K}}{\sqrt{2}\omega m} \tag{13}$$

$$\tau' = \omega\tau \tag{14}$$

$$\tag{15}$$

This changes the other variables accordingly,

$$t' = \omega t \tag{16}$$

$$p' = \frac{k}{\sqrt{2}\omega m} p \tag{17}$$

$$x' = \frac{k}{\sqrt{2}} x \tag{18}$$

Applying these changes to the Hamiltonian,

$$H = \frac{p'^2}{2} + \frac{x'^2}{2} + \frac{\bar{K}'}{\sqrt{2}} \cos\sqrt{2}x' \sum_{n=-\infty}^{\infty} \delta(t' - n\tau') \tag{19}$$

For simplicity, from now on, the primes will be omitted.

Deconstructing the Hamiltonian, we produce the equations of motion,

For a general Hamiltonian,

$$H(q,p) = T(p,q) + U(q) \tag{20}$$

Where T is kinetic energy and U is potential energy, q is position and p is momentum,

$$\dot{q} = \frac{\partial H}{\partial p} \tag{21}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{22}$$

Therefore, from equation (19),

$$\frac{dp}{dt} = -x + \bar{K}\sin\sqrt{2}x \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \tag{23}$$

$$\frac{dx}{dt} = p \tag{24}$$

Now, considering only the time between kicks, so ignoring the kicking term in equation (23),

$$\frac{d^2x}{dt^2} = -x \tag{25}$$

$$\therefore x = x(0)\cos t + p(0)\sin t, \tag{26}$$

$$p = -x(0)\sin t + p(0)\cos t \tag{27}$$

Considering the momentum of the oscillator before and after a kick, the change in momentum can be determined by integrating from just before, to just after a given kick. Introducing ϵ, the infinitesimal change in time, with $\epsilon < \tau$ as a requirement, we integrate from $\tau - \epsilon$ to $\tau + \epsilon$ and then having $\epsilon \to 0$

$$\Delta p = \int_{n'\tau-\epsilon}^{n'\tau+\epsilon} \frac{dp}{dt} dt \tag{28}$$

$$= -\int_{n'\tau-\epsilon}^{n'\tau+\epsilon} x(t)dt + \bar{K}\int_{n'\tau-\epsilon}^{n'\tau+\epsilon} \sin\left(\sqrt{2}x\right) \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \tag{29}$$

$$= -\int_{n'\tau-\epsilon}^{n'\tau+\epsilon} x(t)dt + \bar{K}\sin\left(\sqrt{2}x(n'\tau)\right) \tag{30}$$

Letting $t \to 0$,

$$\Delta p = \bar{K}sin(\sqrt{2}x(n'\tau)) \tag{31}$$

A.2 Kick to kick mapping

By combining equations (26), (27) and (31), a kick-to-kick mapping can be produced. This is done by adding the change in momentum at each iteration, n to the momentum in the equations of motion already derived.

$$p(0) \to p(0) + \Delta p \tag{32}$$

$$x_{n+1} = x_n cos(\tau) + [p_n + \bar{K}sin(\sqrt{2}x_n)]sin(\tau) \tag{33}$$

$$p_{n+1} = [p_n + \bar{K}sin(\sqrt{2}x_n)]cos(\tau) - x_n\sin(\tau) \tag{34}$$

B Matlab Code

B.1 Bifurcation for the driven damped pendulum

```

clear
bkgcol1 = [240, 240, 240]./255;
axescol1 = [50, 50, 50]./255;
set(gca,'Color',bkgcol1)
set(gcf,'color',bkgcol1)
axeslinewidth = 2;
set(gca,'linewidth',axeslinewidth)

Q = 2;
w = 2/3;
theta0 = 0;
v0 = 0;
endtime = 1000;
times = 0:2*pi/w:endtime;
theta_ic = [theta0; v0];

% Array of driving strengths
AArray = 1.49:0.0001:1.5;

% Preallocate data array
bfdata = zeros(length(times)-1,length(AArray));

for i=1:length(AArray)

    % Set driving strength
    A = AArray(i);

    % Create Poincare plot data
    [~, theta] = ODE45Numerical(times, theta_ic, Q, A, w);

    % Convert the 2d Poincare map to 1d and add to the dataset
    bfdata(:,i) = (theta(2:end,2));
    disp("Processed " + string(AArray(i)) + " out of " ...
    ... + string(AArray(end)) + ". " ...
    ... + string((AArray(i)-AArray(1))*100/(AArray(end) ...
    ... - AArray(1))) + "%")
end

hold on
for i=1:length(AArray)
    temp_xs = ones(length(bfdata(end-5:end,1)),1).*AArray(i);
    scatter(temp_xs,bfdata(end-5:end,i),'.','k');
end

xlabel("Driving strength, A",'Interpreter','latex','FontSize',18)
ylabel('$\dot{\theta}$','Interpreter','latex','FontSize',18)
title('Bifurcation of the nonlinear driven, damped pendulum', ...
... 'Interpreter','latex','FontSize',18)
hold off


```

B.2 Poincaré section for the damped driven pendulum

```

```

```
%% Iterative Poincare - produce data iteratively
while(true)
    % Check for existing data
    if(isfile('data.mat'))
        clear
        % Load constants
        load('data.mat','Q');
        load('data.mat','A');
        load('data.mat','w');
        % Load and assign new times
        load('data.mat','endtime')
        times = endtime:2*pi/w:(endtime + 20000);
        % Load new initial conditions
        load('data.mat','theta');
        theta_ic = [theta(end,1),theta(end,2)];
        % Load iterations
        load('data.mat','iterations')
    else
        % If data does not already exist, instantiate
        Q = 2;
        A = 1.15;
        w = 0.67;
        theta0 = 0;
        v0 = 0;
        endtime = 1000;
        % Only calculate for points on the Poincare section
        times = 0:2*pi/w:endtime;
        theta = [];
        theta_ic = [theta0; v0];
        iterations = 0;
    end

    % Calculate Poincare Points
    [t, newTheta] = ODE45Numerical(times, theta_ic, Q, A, w);
    theta = vertcat(theta, newTheta);
    iterations = iterations + length(times);

    disp("Iterations: " + string(iterations))

    % Save to file
    endtime = times(end);
    save('data.mat','theta','endtime','Q','A','w','iterations')
end
```

```
%% Plotting the data from the iterative Poincare

% Colours
axescol1 = [0.1,0.1,0.1];
axeslinewidth = 1.5;

% Load from file
load('data.mat','theta','iterations')

ax1= subplot(1,1,1);

hold on

plotdata_x = sin(theta(664:end,1));
```

```
plotdata_y = theta(664:end,2);

p1 = plot(ax1, plotdata_x, plotdata_y,'.','Color','k','MarkerSize',2);
xlabel(ax1,'$\sin\theta$','Interpreter','latex','FontSize',18)
ylabel(ax1,'$\dot{\theta}$','Interpreter','latex','FontSize',18)
%title(ax1,'Poincar\`e map of the nonlinear damped, driven pendulum', ...
... 'Interpreter','latex','FontSize',18,'Color',axescol1)
pbaspect(ax1,[1,1,1])
p1.Color(4) = 0.01;

hold off

% Editing
ax1.YColor = axescol1;
ax1.XColor = axescol1;
ax1.LineWidth = axeslinewidth;

disp('Iterations: ' + string(iterations));


```

B.3 The kicked harmonic oscillator - analytic solutions

```
%% Poincare section for the KHO

axescol1 = [50, 50, 50]./255;
axeslinewidth = 2;
set(gca,'linewidth',axeslinewidth)

% Initial conditions
x0 = 1.7679;
p0 = 1.7679;

% Parameters
r = 1;
% Frequency
q = 3;
% Strength
%k = -1.48445;
k = -1.1;
%k = -1.4;

% Kick frequency
tau = 2 * pi * r/q;

% End time
endtime = 800000;

% Create arrays
x = zeros(1,endtime);
p = zeros(1,endtime);

% Set initial points
x(1) = x0;
p(1) = p0;

% Kicks - Kick to kick mapping, this is a Poincare section
for i=2:endtime
    x(i) = x(i-1) * cos(tau) + (p(i-1) + k*sin(sqrt(2)*x(i-1))) * sin(tau);
    p(i) = (p(i-1) + k*sin(sqrt(2)*x(i-1))) * cos(tau) - x(i-1) * sin(tau);
```

```
end

% Plot
plot(x,p,'.','Color','[0.2,0.2,0.2]','MarkerSize',0.2)
xlabel('$x$','Interpreter','latex','FontSize',18)
ylabel('$p$','Interpreter','latex','FontSize',18)
pbaspect([1 1 1])
```

B.4 Testing the RK4 implementation for solving the KHO

```
%% Testing the RK4 method for solving the KHO

theta0 = 1.7679;
v0 = 1.7679;

% Kicking
kickstrength = -1.1;
kicksperperiod = 3;

% Time start and end
timestart = 0;
timeend = 2 * pi * 60000;

% Size of step
timesperkick = 50;
timestep = (2 * pi / kicksperperiod) / timesperkick;
period = 2*pi;

% Pass to RK4
theta_ic = [theta0; v0];
[t, theta, nonlinTheta] = RK4Numerical(theta_ic, timestart, ...
... timeend, timestep, kickstrength, kicksperperiod, period, ...
... drivingStrength, drivingFrequency, dampingFactor);

theta = theta';
nonlinTheta = nonlinTheta';

% Extract times at kicks
PCtheta = theta(1:timesperkick:end,:);
PCnonlinTheta = nonlinTheta(1:timesperkick:end,:);

% Plotting

hold on
disp('Plotting')

plot(-(PCtheta(1:end-1,1)),PCtheta(1:end-1,2),'.','displayname', ...
... 'linear','Color','[0.2,0.2,0.2]','MarkerSize',0.5); %NON WRAPPED
%plot((PCnonlinTheta(:,1)),PCnonlinTheta(:,2),'.','displayname', ...
... 'nonlinear'); %NON WRAPPED
%plot((wrapToPi(PCtheta(:,1))),PCtheta(:,2),'.','Color','k'); % WRAPPED

xlabel('$x$','Interpreter','latex','FontSize',16)
ylabel('$p$','Interpreter','latex','FontSize',16)
%title('Poincar\'e Map. KPP: ' + string(kicksperperiod) + ". KS: " ...
... + string(kickstrength) + ".",'Interpreter','latex','FontSize',12)
%subtitle("Endtime: " + string(timeend/(2*pi))+" TPK: "+ string(timesperkick))
%legend

pbaspect([1 1 1])
```

```

hold off
```

B.5 Functions used

B.5.1 Function "ODE45Numerical" for solving the damped driven pendulum using MATLAB's ODE45

```

function [t_, theta_] = ODE45Numerical(times, theta_ic, QQ, AA, ww)
    % Uses MATLAB's ODE45 function to solve the driven, damped pendulum
    % equations of motion

    Q = QQ;
    A = AA;
    w = ww;
    options = odeset('AbsTol',1e-10,'RelTol',1e-9);
    [t_, theta_] = ode45(@pendulum, times, theta_ic, options);
    function dtheta = pendulum(t, theta)
        % theta(1) = theta, theta(2) = dtheta


        dtheta = zeros(2, 1);
        dtheta(1) = theta(2);
        dtheta(2) = -dtheta(1)/Q - sin(theta(1)) + A*cos(w * t);
    end
end
```

B.5.2 Function "RK4Numerical" for solving the damped driven pendulum or the KHO for both linear and nonlinear terms

```
%% Runge-Kutta 4th order implementation.
% Can be used to solve both damped driven pendulum or the KHO.
% Equations of motion can be edited as needed.
% Currently returns both linear and nonlinear forms for the KHO.
function [t_, theta_, nonlinearTheta] = RK4Numerical(theta_ic, timestart, ...
... timeend, timestep, kickStrength, kicksPerPeriod, period, drivingStrength, ...
... drivingFrequency, dampingFactor)

    [t_, theta_] = RK4(@linearPendulum,timestep,timestart,timeend,theta_ic(1) ...
    ... ,theta_ic(2));

    [~, nonlinearTheta] = RK4(@nonlinearPendulum,timestep,timestart,timeend, ...
    ... theta_ic(1),theta_ic(2));

    function [t, y] = RK4(F,h,t0,t1,thetaIC,velocityIC)
        % F      Function
        % h      Step size
        % t0,t1  Time bounds
        % y0,y1

        % Kicking variables
        stepsPerPeriod = period/timestep;
            disp("Steps per period: " + string(stepsPerPeriod))
        stepsPerKick = stepsPerPeriod/kicksPerPeriod;
            disp("Steps per kick: " + string(stepsPerKick))
        if(stepsPerKick ~= int8(stepsPerKick))
```

```
            disp("STEPS PER KICK IS NOT INTEGER")
        end

        % Create time array
        t = t0:h:t1;
        % Create theta and velocity arrays
        y = zeros(2,length(t));
        % Set initial conditions
        y(1,1) = thetaIC;
        y(2,1) = velocityIC;


        % TO AVOID KICKING AT TIME 0
            k1 = F( t(1)           , y(:,1)             );
            k2 = F( t(1) + 0.5*h , y(:,1) + 0.5*h*k1 );
            k3 = F( t(1) + 0.5*h , y(:,1) + 0.5*h*k2 );
            k4 = F( t(1) +      h , y(:,1) +      h*k3 );
            % Main equation for RK4
            y(:,1+1) = y(:,1) + (1/6)*( k1 + 2*k2 + 2*k3 + k4 )*h;

        % Iterate over times
        for i=2:(length(t)-2)
            % Kicking
            if(mod(i-1,stepsPerKick) == 0)
                y(2,i) = y(2,i) + kickStrength * sin(sqrt(2) * (y(1,i)));
            end

            % RK4
            k1 = F( t(i)           , y(:,i)             );
            k2 = F( t(i) + 0.5*h , y(:,i) + 0.5*h*k1 );
            k3 = F( t(i) + 0.5*h , y(:,i) + 0.5*h*k2 );
            k4 = F( t(i) +      h , y(:,i) +      h*k3 );
            % Main equation for RK4
            y(:,i+1) = y(:,i) + (1/6)*( k1 + 2*k2 + 2*k3 + k4 )*h;
        end

    end

    % Equations of motion
    function dtheta = linearPendulum(t, theta)
        % theta(1) = theta, theta(2) = dtheta


        dtheta = zeros(2, 1);

         % PENDULUM
         A = drivingStrength;
         w = drivingFrequency;
         Q = dampingFactor;

        % DAMPED DRIVEN
        %dtheta(1) = theta(2);
        %dtheta(2) = -dtheta(1)/Q - sin(theta(1)) + A*cos(w * t); % nonlinear
        %dtheta(2) = -dtheta(1)/Q - (theta(1)) + A*cos(w * t); % linear

        % REGULAR PENDULUM
        dtheta(1) = theta(2);
        dtheta(2) = -(theta(1));

        % OSCILLATOR
        %dtheta(1) = theta(2);
        %dtheta(2) = - theta(1);

```

```
    end

    function dtheta = nonlinearPendulum(t, theta)

        dtheta = zeros(2, 1);

        % Nonlinear Pendulum
        dtheta(1) = theta(2);
        dtheta(2) = -sin(theta(1));

    end

end
```