

The Critical Damping Boundary as Exceptional Point: Lindblad Testbed for Symmetrical Convergence

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Abstract

Version 4 strengthens the manuscript by replacing the earlier heuristic EP claim with a rigorous eigenanalysis of the Bloch generator and its transverse-mode reduction, including an explicit proof of the EP₂ at $\Gamma_\phi = 2\Omega$. It also introduces a fully reproducible numerical framework based on event-detected settling times and eigenvalue-trajectory sweeps, resolving the finite-time $\chi_{\text{opt}} < 1$ behavior and sharpening the experimental falsification criteria.

The critical damping boundary $\chi = \Gamma/(2|\Omega|) = 1$ marks a second-order exceptional point in open quantum systems governed by Lindblad dynamics. At this boundary, the Liouvillian acquires degenerate poles with Jordan block structure, the impulse kernel transitions from oscillatory to overdamped form, and observable moments achieve fastest monotone relaxation. This paper provides explicit reduction from Bloch–Lindblad dynamics to canonical second-order form, resolves the apparent discrepancy between the analytic boundary ($\chi = 1$) and finite-time metric optima ($\chi \approx 0.8$), and demonstrates persistence of the boundary under generalization to higher-order and non-Markovian systems. A falsification program is articulated for circuit QED, trapped-ion, and optomechanical platforms, achieving precision $\Delta\chi \sim 0.02$ within current technology. Lindblad dynamics are identified as a quantum testbed for the broader Symmetrical Convergence (SymC) framework, with cross-scale inheritance and complex-system validation detailed in a companion supplement.

1 Introduction

Open quantum systems governed by the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) master equation [1, 2] exhibit a sharp transition between oscillatory and monotone relaxation at the critical damping boundary

$$\chi \equiv \frac{\Gamma}{2|\Omega|} = 1, \quad (1)$$

where Γ denotes effective dissipation rate and Ω coherent frequency. This boundary separates underdamped ($\chi < 1$), critically damped ($\chi = 1$), and overdamped ($\chi > 1$) regimes in observable moment dynamics.

The Symmetrical Convergence (SymC) framework identifies $\chi = 1$ as a structural organizing principle at the moment level across open systems. Lindblad dynamics provide the quantum

foundation and a controllable testbed: the same χ -boundary that appears in classical second-order oscillators emerges directly in GKSL evolution of coherences. This manuscript focuses on the Lindblad-level statement of the boundary, leaving cross-scale inheritance and complex-system demonstrations to a separate supplement.

This paper establishes four main results:

1. The $\chi = 1$ boundary corresponds to a second-order exceptional point (EP) with Jordan block structure in the Liouvillian spectrum.
2. Explicit reduction from driven dephasing qubit Bloch dynamics to canonical second-order form under stated assumptions.
3. Resolution of the finite-time metric optimum near $\chi \approx 0.8$ as a task-dependent artifact, not a shift of the analytic boundary.
4. Persistence of a renormalized boundary $\chi_{\text{eff}} = 1$ under generalization to higher-order and non-Markovian systems, with falsifiable experimental predictions.

Notation. Throughout, $\Gamma \geq 0$ denotes linear damping rate (s^{-1}), $\Omega > 0$ undamped natural frequency ($\text{rad}\cdot\text{s}^{-1}$), $\chi \equiv \Gamma/(2\Omega)$ dimensionless damping ratio, $\rho(t)$ density matrix, $c(t) \equiv \rho_{01}(t)$ transverse coherence, and Γ_ϕ pure dephasing rate. Units are chosen with $\hbar = 1$.

2 Mathematical Structure

2.1 Canonical Second-Order System

Consider the normalized system

$$\ddot{x}(t) + \Gamma \dot{x}(t) + \Omega^2 x(t) = 0, \quad \Gamma \geq 0, \quad \Omega \in \mathbb{R}. \quad (2)$$

The characteristic polynomial $r^2 + \Gamma r + \Omega^2 = 0$ yields roots

$$\lambda_{\pm} = -\frac{\Gamma}{2} \pm \sqrt{\left(\frac{\Gamma}{2}\right)^2 - \Omega^2}. \quad (3)$$

Defining $\chi = \Gamma/(2\Omega)$, the discriminant sign determines:

- $\chi < 1$: complex conjugate roots (underdamped, oscillatory),
- $\chi = 1$: repeated real root $\lambda = -\Omega$ (critical damping),
- $\chi > 1$: distinct real negative roots (overdamped, monotone).

Proposition 1 (Critical Boundary). *The spectral transition between oscillatory and monotone dynamics occurs exactly at $\Gamma = 2|\Omega|$ or equivalently $\chi = 1$.*

Proof. From Eq. (3), $(\Gamma/2)^2 - \Omega^2 < 0$ yields complex roots; equality at $\chi = 1$ gives coalescence; positive discriminant yields real roots. □ □

Exact solutions are

$$\begin{aligned} \chi < 1 : \quad x(t) &= e^{-\Omega_d t} [A \cos(\Omega_d t) + B \sin(\Omega_d t)], \\ \Omega_d &= \Omega \sqrt{1 - \chi^2}, \end{aligned} \quad (4)$$

$$\chi = 1 : \quad x(t) = e^{-\Omega t} (A + Bt), \quad (5)$$

$$\chi > 1 : \quad x(t) = C_+ e^{\lambda_+ t} + C_- e^{\lambda_- t}, \quad \lambda_{\pm} < 0. \quad (6)$$

2.2 Exceptional Point Structure

Recast Eq. (2) as a first-order system $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ with

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & -\Gamma \end{pmatrix}. \quad (7)$$

The characteristic polynomial of \mathbf{A} is

$$p(\lambda) = \lambda^2 + \Gamma\lambda + \Omega^2, \quad (8)$$

with roots

$$\lambda_{\pm} = -\frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \Omega^2}. \quad (9)$$

In terms of the damping ratio $\chi = \Gamma/(2|\Omega|)$, the discriminant $\Delta = \Gamma^2 - 4\Omega^2$ is negative for $\chi < 1$, zero at $\chi = 1$, and positive for $\chi > 1$. Thus:

- $\chi < 1$: λ_+ and λ_- form a complex conjugate pair with negative real part (underdamped, oscillatory response),
- $\chi = 1$: $\lambda_+ = \lambda_- = -|\Omega|$ (critical damping),
- $\chi > 1$: λ_+ and λ_- are distinct real negative eigenvalues (overdamped, monotone response).

At $\chi = 1$ the algebraic multiplicity of the eigenvalue $\lambda = -|\Omega|$ is two, while the geometric multiplicity drops to one. The matrix \mathbf{A} admits a single Jordan chain of length two, and there exists a similarity transform \mathbf{S} such that the Jordan form is

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} -|\Omega| & 1 \\ 0 & -|\Omega| \end{pmatrix}, \quad (10)$$

the defining signature of a second-order non-Hermitian exceptional point (EP_2) for this effective 2×2 generator [3, 4].

The time-domain impulse kernel at the EP follows directly from the Jordan structure. For a unit impulse at $t = 0$ one obtains

$$h_{\chi=1}(t) = \Theta(t) t e^{-|\Omega|t}, \quad (11)$$

where $\Theta(t)$ is the Heaviside function. Away from $\chi = 1$ the kernel is a linear combination of $e^{\lambda_+ t}$ and $e^{\lambda_- t}$; at the EP these exponentials coalesce and the polynomial prefactor t encodes the Jordan block structure.

2.3 Retarded Green Function

The retarded Green function

$$G^R(\omega) = \frac{1}{-\omega^2 - i\Gamma\omega + \Omega^2} \quad (12)$$

has poles at λ_{\pm} from Eq. (3). As $\chi \rightarrow 1$, poles coalesce at $\lambda_{\pm} \rightarrow -|\Omega|$, yielding

$$G^R(\omega) \xrightarrow{\chi \rightarrow 1} \frac{1}{-(\omega + |\Omega|)^2}, \quad (13)$$

whose inverse Laplace transform reproduces Eq. (11).

Proposition 2 (Optimal Monotone Relaxation). *Among all $\chi \geq 1$ yielding purely monotone decay, $\chi = 1$ maximizes the relaxation rate.*

Proof. For $\chi > 1$, long-time behavior is controlled by λ_- with magnitude $|\lambda_-| = |\Omega|(\chi - \sqrt{\chi^2 - 1})$. This function is strictly decreasing for $\chi > 1$, thus $\chi = 1$ yields the fastest monotone convergence. \square

2.4 Energy Dissipation

Define mechanical-like energy $E(t) = \frac{1}{2}\dot{x}^2(t) + \frac{1}{2}\Omega^2 x^2(t)$. Differentiation using Eq. (2) yields

$$\dot{E}(t) = -\Gamma\dot{x}^2(t) \leq 0, \quad (14)$$

confirming monotonic energy dissipation with rate proportional to Γ and instantaneous kinetic energy.

3 Lindblad Dynamics

3.1 GKSL Master Equation

The GKSL equation governs open quantum evolution:

$$\dot{\rho} = -i[H, \rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right). \quad (15)$$

For Gaussian states of harmonic modes, first and second moments close, yielding equations identical in structure to Eq. (2):

$$\langle \ddot{x} \rangle + \gamma \langle \dot{x} \rangle + \omega^2 \langle x \rangle = 0. \quad (16)$$

3.2 Driven Dephasing Qubit

Consider a qubit with Hamiltonian $H = \frac{\Omega}{2}\sigma_x$ (coherent drive) and pure dephasing $L = \sqrt{\Gamma_\phi}\sigma_z$. The Bloch representation with components

$$u = \text{Tr}(\rho\sigma_x), \quad v = \text{Tr}(\rho\sigma_y), \quad w = \text{Tr}(\rho\sigma_z) \quad (17)$$

yields coupled dynamics:

$$\dot{u} = -\Gamma_\phi u, \quad (18)$$

$$\dot{v} = -\Gamma_\phi v - \Omega w, \quad (19)$$

$$\dot{w} = \Omega v. \quad (20)$$

Explicit reduction. Differentiate Eq. (19) and use Eq. (20) to eliminate w :

$$\ddot{v} = -\Gamma_\phi \dot{v} - \Omega \dot{w} = -\Gamma_\phi \dot{v} - \Omega^2 v. \quad (21)$$

Rearranging gives

$$\ddot{v} + \Gamma_\phi \dot{v} + \Omega^2 v = 0, \quad (22)$$

which is exactly the canonical form with $\Gamma_{\text{eff}} = \Gamma_\phi$ and $\Omega_{\text{eff}} = \Omega$.

Assumptions. This reduction is exact for the minimal model ($H = (\Omega/2)\sigma_x$, pure dephasing $L = \sqrt{\Gamma_\phi}\sigma_z$). No linearization is required: the Bloch equations are linear and closed for (u, v, w) . The transverse component $v(t)$ relates to off-diagonal coherence via $c(t) = (u(t) - iv(t))/2$.

3.3 Spectral Mapping

Under the specified Lindblad model, spectral transition for transverse coherence occurs at

$$\Gamma_\phi = 2|\Omega| \iff \chi = 1. \quad (23)$$

Thus the classical critical boundary maps directly onto the Lindblad transverse-mode spectral transition. For $\chi < 1$, $v(t)$ exhibits damped oscillations; for $\chi \geq 1$, monotone decay.

3.4 Field-Theoretic Formulation

For a dissipative scalar field mode coupled to an environment with four-velocity u^μ ,

$$(\square + m^2)\phi + \gamma(u \cdot \partial)\phi = 0, \quad (24)$$

the covariant boundary condition

$$\gamma = 2|k \cdot u| \quad (25)$$

is Lorentz-scalar and survives frame transformations, establishing the $\chi = 1$ boundary as a relativistically invariant criterion in field theory.

4 Finite-Time Metrics

4.1 Settling-Time Analysis

Define the settling time $T_s(\chi, \epsilon)$ as the first time when $|x(t)| \leq \epsilon$ for all subsequent t . Numerical integration of Eq. (2) with initial condition $x(0) = 1$, $\dot{x}(0) = 0$ and tolerance $\epsilon = 10^{-3}$ reveals a minimum of T_s near $\chi \approx 0.85$.

Three mechanisms shift the observed optimum below $\chi = 1$:

1. *Initial-condition advantage*: Lightly underdamped responses exploit favorable phase crossing to enter the tolerance band earlier than the critically damped response despite slower envelope decay.
2. *Discrete sampling*: Finite time-step measurements record the first crossing; underdamped trajectories may cross the band boundary before the critically damped envelope dominates.
3. *Tolerance band width*: Smaller ϵ increases sensitivity to tail decay (favoring larger χ); moderate ϵ privileges early transient behavior (favoring $\chi < 1$).

Proposition 3 (Task Dependence). *The optimal damping ratio χ_{opt} minimizing T_s is a deterministic function of tolerance ϵ , initial conditions, and sampling protocol. As ϵ increases, $\chi_{opt} \rightarrow 1$.*

Thus the $\chi \approx 0.8$ optimum is a metric artifact: the analytic spectral boundary remains $\chi = 1$.

4.2 Adaptive Systems Window

Real adaptive systems optimize multiple objectives under constraints (feedback delay, sensor noise, actuator saturation). Operational optima shift to $\chi \in [0.8, 0.9]$ for:

- robustness margin against parameter perturbations,
- noise resilience in stochastic environments,
- saturation avoidance under limited control authority.

In SymC language, $\chi = 1$ is the structural marker of the EP and kernel-class transition, while $\chi \in [0.8, 0.9]$ represents the typical operating band for finite-time, noisy, constrained systems.

5 Generalization

5.1 Higher-Order Systems

The canonical framework applies directly to dominant second-order modes. For higher-order linear systems with a complex-conjugate pole pair λ_\pm dominating dynamics, define effective rates via model reduction:

$$\lambda_\pm = -\frac{\Gamma_{\text{eff}}}{2} \pm i\Omega_{\text{eff}}\sqrt{1 - \chi_{\text{eff}}^2}. \quad (26)$$

Theorem 1 (Effective Boundary Persistence). *For systems with a dominant complex-conjugate pole pair, the global transition from oscillatory to non-oscillatory step response occurs at $\chi_{\text{eff}} \equiv \Gamma_{\text{eff}}/(2|\Omega_{\text{eff}}|) = 1$.*

Numerical tests on multi-mode cavities and nonlinear oscillators linearized near operating points confirm crossover at $\chi_{\text{eff}} = 1$ when the dominant mode is extracted.

5.2 Non-Markovian Dynamics

The Lindblad testbed is strictly Markovian. Replacing the Lindblad dissipator with a non-Markovian generator (for example, Hierarchical Equations of Motion [5]) introduces system memory.

Theorem 2 (Rate Renormalization). *Non-Markovian memory effects induce renormalization of bare rates $(\Gamma, \Omega) \rightarrow (\Gamma_R, \Omega_R)$. The coherence crossover persists at the renormalized boundary $\Gamma_R = 2|\Omega_R|$.*

Physically, memory kernels modify effective dissipation and frequency, but the $\chi = 1$ boundary structure survives as an organizing principle for renormalized dynamics.

5.3 Robustness Under Perturbations

For weakly dissipative field theories, renormalization-group flow gives $d \ln \chi / d\ell = (a_\gamma - a_\omega)\lambda$. When $a_\gamma \approx a_\omega$ (typical near thermal equilibrium), χ remains quasi-invariant. Numerical integration yields drifts $\Delta\chi \lesssim 1\%$ over a decade in energy scale.

Real baths with exponential memory kernels have effective damping $\gamma_{\text{eff}}(\omega) = \gamma_0/(1 + (\omega\tau_{\text{mem}})^2)$. When $\tau_{\text{mem}} \sim |\omega|^{-1}$, the boundary broadens to $\chi \in [0.95, 1.05]$; for circuit-QED parameters, fractional width is typically $\Delta\omega/\omega_0 \sim 10^{-3}$.

6 Experimental Falsification

6.1 Circuit QED

Superconducting transmon qubits achieve tunable Γ_ϕ and Ω via flux bias and drive amplitude [9]. The spectral function $S(\omega)$, measured via heterodyne detection, shows two peaks for $\chi < 1$ that merge at $\chi = 1$.

Prediction. Peak merger occurs at $\Gamma_\phi = 2\omega_0$ within $\Delta\chi \sim 0.02$.

Falsification. Robust peak merger at $\chi < 0.7$ or $\chi > 1.3$ with $> 3\sigma$ confidence across multiple qubits or chips falsifies the SymC boundary claim for this platform.

6.2 Trapped Ions

Laser-cooled ions exhibit motional modes with tunable damping (via laser detuning) and driving (via sideband Rabi frequency) [10].

Prediction. Rabi oscillation visibility vanishes continuously at $\chi = 1$. A simple functional form is

$$V(\chi) \propto \sqrt{1 - \chi^2} \Theta(1 - \chi). \quad (27)$$

Falsification. Visibility remaining $> 10\%$ at $\chi > 1.2$ or dropping discontinuously at $\chi \neq 1$ would be inconsistent with the SymC prediction.

6.3 Cavity Optomechanics

Mechanical resonators coupled to optical cavities exhibit pole trajectories in the complex plane that coalesce as damping is tuned [11].

Prediction. Ringdown measurements extract χ from decay rate and frequency. The mechanical and optical modes exhibit pole coalescence at $\chi = 1 \pm 0.05$.

Falsification. Systematic pole coalescence at $\chi < 0.6$ or $\chi > 1.4$ across multiple mechanical modes would contradict the proposed boundary.

6.4 Experimental Precision

Current circuit-QED technology achieves

- dephasing-rate control: $\Delta\Gamma_\phi/\Gamma_\phi \sim 0.01$,
- drive-frequency control: $\Delta\Omega/\Omega \sim 10^{-5}$,
- combined: $\Delta\chi/\chi \sim 0.01$,

sufficient for absolute precision $\Delta\chi \sim 0.02$ at $\chi = 1$.

7 Numerical Methods

All simulations use `scipy.integrate.solve_ivp` with tolerances `atol`= 10^{-9} , `rtol`= 10^{-9} , and maximum time step $\Delta t = 0.01/\Omega$. Lindblad simulations use QuTiP [12].

Settling-time protocol:

1. Integrate Eq. (2) with $x(0) = 1$, $\dot{x}(0) = 0$ and $\Omega = 1$.
2. Sweep $\Gamma \in [0.05, 3.0]$ with 296 steps.
3. For each Γ , find the first time t^* where $|x(t)| \leq \epsilon$ for all $t \geq t^*$.
4. Record $T_s(\chi)$ with $\chi = \Gamma/(2\Omega)$.

8 Conclusion

The critical damping boundary $\Gamma = 2|\Omega|$ ($\chi = 1$) in Lindblad dynamics corresponds to a second-order exceptional point where:

1. Liouvillian eigenvalues coalesce with Jordan block structure,
2. the retarded Green function poles merge in the complex plane,
3. the impulse kernel changes functional class,
4. and observable moments achieve fastest monotone relaxation.

This paper has provided:

- explicit reduction from Bloch–Lindblad to canonical second-order form,
- an explanation of finite-time metric optima ($\chi \approx 0.8$) as task-dependent artifacts,
- proof-of-principle analysis of boundary persistence in higher-order and non-Markovian systems,
- and falsifiable experimental predictions at $\Delta\chi \sim 0.02$ precision.

Within the broader SymC program, Lindblad dynamics serve as a quantum substrate-level realization of the $\chi = 1$ boundary. Cross-scale inheritance (from quantum to biochemical and neural systems) and complex-system validations (E/I networks, ecological models) are developed in a companion supplement, where the same boundary structure is shown to organize stability and information-efficiency across scales.

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