Dynamic Programming Interviews
Tri tiling problem
Recurrence proof

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Task

Determine in how many ways can a 3xn rectangle be completely tiled with 2x1 dominoes. Dominoes can be horizontal or vertical.

Solution

The recurrence relation is:

$$f(n) = f(n-2) + 2g(n-1)$$
$$g(n) = f(n-1) + g(n-2)$$

Let's substitute $g(n-1)$ in equation of $f(n)$.

$$f(n) = f(n-2) + 2g(n-2)$$
$$g(n-1) = \frac{f(n) - f(n-2)}{2}$$

Let's substitute $g(n-1)$ in equation of $g(n)$ to find out what the recurrence relation of $g(n)$ using only $f(n)$.

$$g(n) = f(n-1) + g(n-2)$$
$$g(n) = f(n-1) + \frac{f(n-1)}{2} - \frac{f(n-3)}{2} = \frac{3}{2}f(n-1) - \frac{f(n-3)}{2}$$

Now let's substitute $g(n)$ into $f(n)$.

$$f(n) = f(n-2) + 2 * \left(\frac{3}{2}f(n-2) - \frac{f(n-4)}{2}\right)$$
$$f(n) = 4f(n-2) - f(n-4)$$

We got the recurrence relation which can be solved. Let's solve it. Guess that $f(n) = cx^n$. Then we get.

$$cx^n = 4cx^{n-2} - cx^{n-4}$$
$$x^4 = 4x^2 - 1$$
$$x^4 - 4x^2 + 1 = 0$$
Let $t = x^2$
$$D = 12 \qquad t_{1,2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$
$$x_1 = \sqrt{2+\sqrt{3}} \quad x_2 = \sqrt{2-\sqrt{3}} \quad x_3 = -\sqrt{2+\sqrt{3}} \quad x_4 = -\sqrt{2-\sqrt{3}}$$

So we get new equation for $f(n)$.

$$f(n) = c_1\left(\sqrt{2+\sqrt{3}}\right)^n + c_2\left(\sqrt{2-\sqrt{3}}\right)^n + c_3\left(-\sqrt{2+\sqrt{3}}\right)^n + c_4\left(-\sqrt{2-\sqrt{3}}\right)^n$$

As $f(0) = 1, f(1) = 0, f(2) = 3, f(3) = 0$ we can find coefficients.

$$\begin{pmatrix} \left(\sqrt{2+\sqrt{3}}\right)^0 & \left(\sqrt{2-\sqrt{3}}\right)^0 & \left(-\sqrt{2+\sqrt{3}}\right)^0 & \left(-\sqrt{2-\sqrt{3}}\right)^0 \\ \left(\sqrt{2+\sqrt{3}}\right)^1 & \left(\sqrt{2-\sqrt{3}}\right)^1 & \left(-\sqrt{2+\sqrt{3}}\right)^1 & \left(-\sqrt{2-\sqrt{3}}\right)^1 \\ \left(\sqrt{2+\sqrt{3}}\right)^2 & \left(\sqrt{2-\sqrt{3}}\right)^2 & \left(-\sqrt{2+\sqrt{3}}\right)^2 & \left(-\sqrt{2-\sqrt{3}}\right)^2 \\ \left(\sqrt{2+\sqrt{3}}\right)^3 & \left(\sqrt{2-\sqrt{3}}\right)^3 & \left(-\sqrt{2+\sqrt{3}}\right)^3 & \left(-\sqrt{2-\sqrt{3}}\right)^3 \end{pmatrix} * \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

$$c_1 = \frac{\sqrt{3}+3}{12}$$
$$c_2 = \frac{-\sqrt{3}+3}{12}$$
$$c_3 = \frac{\sqrt{3}+3}{12}$$
$$c_4 = \frac{-\sqrt{3}+3}{12}$$

So the solution to recurrence relation is:

$$f(n) = \left(\frac{\sqrt{3}+3}{12}\right) * \left(\sqrt{2+\sqrt{3}}\right)^n + \left(\frac{-\sqrt{3}+3}{12}\right) * \left(\sqrt{2-\sqrt{3}}\right)^n + \left(\frac{\sqrt{3}+3}{12}\right) * \left(-\sqrt{2+\sqrt{3}}\right)^n + \left(\frac{-\sqrt{3}+3}{12}\right) * \left(-\sqrt{2-\sqrt{3}}\right)^n$$