Egg dropping puzzle
Analytical solution proof

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Task

There is a building of n floors and you have m identical eggs. The properties of the eggs are such that they will break if you drop them from the floor $n \geq n^*$ and will get no damage if dropped from $n \leq n^* - 1$. The task is to find the floor n^* in the least drops possible.

Proof

1. Let $f(t, m)$ be the numbers of floors we can reach in t tries using m eggs.
   Then we can define the recurrence relation for $f(t, m)$ as follows:

   $$f(t, m) = f(t-1, m-1) + f(t-1, m) + 1$$
   $$\text{where } f(0, m) = 0, \quad f(t, 0) = 0$$

   The logic is simple:

   - With every drop we lose one try: $t - 1$.
   - With every drop an egg may break or survive: $m - 1$ or m.
   - $+1$ for the current floor.

2. Now we will solve the recurrence relation using generation functions.

   2.1. Rewrite the recurrence relation without subtractions:

   $$f(t+1, m+1) = f(t, m) + f(t, m+1) + 1$$

2.2. Define a generation function for equation:

$$\begin{aligned} G(x,y) &= \sum_{t,m\geq 0} f(t,m)x^t y^m \\ &= f(0,0) + f(1,0)x + f(0,1)y + f(1,1)xy + \dots \end{aligned}$$

2.3. Write every piece of the recurrence relation using generating functions:

i. $f(t+1,m+1)$:

$$\begin{aligned} &\sum_{t,m\geq 0} f(t+1,m+1)x^t y^m = \\ &= \frac{G(x,y) - \sum_{m\geq 0} f(0,m)y^m - \sum_{t\geq 0} f(t,0)x^t + f(0,0)}{xy} = \\ &= \frac{G(x,y) - G(0,y)^{=0} - G(x,0)^{=0} + f(0,0)^{=0}}{xy} = \\ &= \frac{G(x,y)}{xy} \end{aligned}$$

ii. $f(t,m)$:

$$\sum_{t,m\geq 0} f(t,m)x^t y^m = G(x,y)$$

iii. $f(t,m+1)$:

$$\begin{aligned} &\sum_{t,m\geq 0} f(t,m+1)x^t y^m = \\ &= \frac{G(x,y) - \sum_{t\geq 0} f(t,0)x^t}{y} = \\ &= \frac{G(x,y) - G(x,0)^{=0}}{y} = \\ &= \frac{G(x,y)}{y} \end{aligned}$$

iv. 1:

$$\frac{1}{1-x}$$

2.4. Now we should solve the equation for $G(x, y)$:

$$\frac{G(x,y)}{xy} = G(x,y) + \frac{G(x,y)}{y} + \frac{1}{1-x}$$
$$\frac{G(x,y) - xyG(x,y) - xG(x,y)}{xy} = \frac{1}{1-x}$$
$$G(x,y)(1-xy-x) = \frac{xy}{1-x}$$
$$G(x,y) = \frac{xy}{(1-x)(1-xy-x)}$$

This is our generating function equation.

2.5. Now we would like to know every coefficient in Taylor series expansion of $G(x, y)$.

i. Simplify the equation using partial fractions:

$$\frac{xy}{(1-x)(1-xy-x)} = \frac{A}{(1-x)} + \frac{B}{(1-xy-x)}$$
$$xy = A(1-xy-x) + B(1-x)$$
$$-A = 1 \quad A = -1$$
$$-A - B = 0 \quad B = 1$$
$$\frac{xy}{(1-x)(1-x-y)} = -\frac{1}{1-x} + \frac{1}{(1-xy-x)}$$

ii. Work with pieces:

$$\frac{d}{d^n x}\left(-\frac{1}{1-x}\right)(0)\frac{1}{n!} = -1$$

$$\frac{1}{1-xy-x} = \frac{1}{1-(xy+x)} =$$
$$= \sum_{t\geq 0}(x+xy)^t =$$
$$= \sum_{t\geq 0}\sum_{m=0}^{t}\binom{t}{m}x^{t-m}(xy)^m =$$
$$= \sum_{t\geq 0}\sum_{m=0}^{t}\binom{t}{m}x^t y^m$$

The last sum $\sum_{m=0}^{t}\binom{t}{m}x^t y^m$ can be thought of as sum of events when we have t tries and m eggs.

iii. The result is:

$$f(t, m) = -1 + \sum_{m=0}^{t} \binom{t}{m} =$$
$$= \sum_{m=1}^{t} \binom{t}{m}$$

Going back to our task of dropping eggs. We have to find the smallest t which satisfies:

$$f(t, m) \geq n$$

As t lies in range of $[1, n]$, the value can be found very fast using binary search in this range.