The Fundamental Limits of Communication and How to Achieve Them

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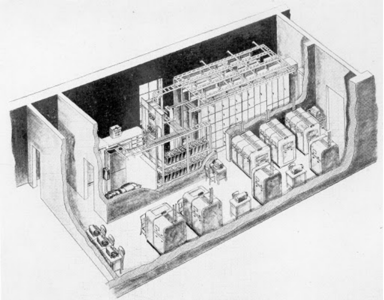
1940s; Bell labs; Model V computer



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A person will come to fix it.
That person does not work on weekends.

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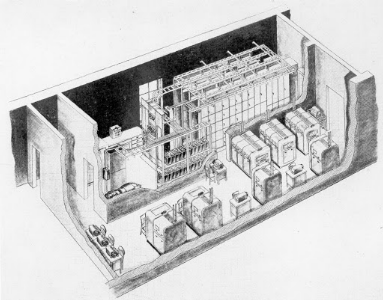

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Hamming code

For every 7 bits, instead of all 128 vectors, allow only the null space of

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Let $\bar{x} \in \mathbb{F}_2^7$ be a valid vector (codeword), that is, $H\bar{x} = 0$.
If $\bar{y} = \bar{x} + \bar{e}$ is the corrupted word, where $\bar{e}$ contains one 1 and six 0's, then $H\bar{y} = H(\bar{x} + \bar{e}) = H\bar{e}$ is the error position; flip that bit back.

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Performance of Hamming code

Block length $N = 7$ (how many bits are grouped together).

Code rate $R = 4/7$ (how efficiently the information is recorded).

What could go wrong? When $\bar{e}$ contains more than one 1.
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Probabilistic model

Input is a random variable $X \in \mathbb{F}_2$; output is another $Y \in \mathbb{F}_2$.
Y is usually X; But Y could be $1 - X$ (flipped) with probability p.

Binary symmetric channel (BSC) with crossover probability p.

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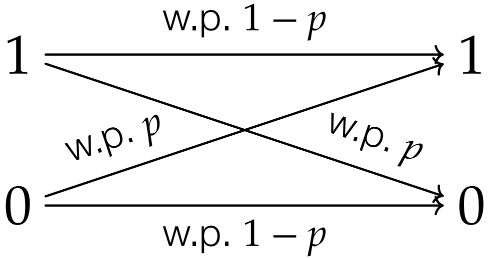

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Binary erasure channel (BEC) with erasure probability p.
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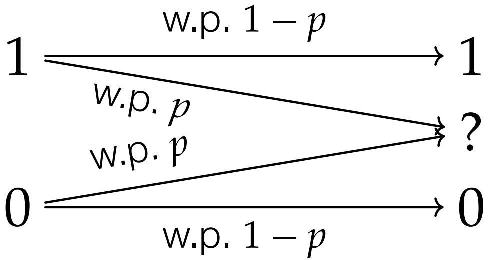

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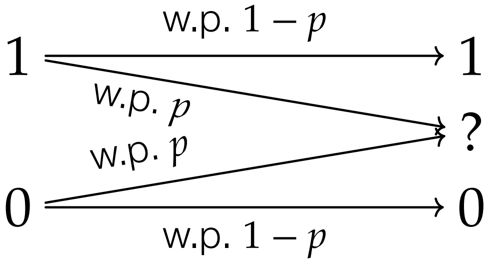

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Coding = encoding + decoding

Choose a codebook $\mathscr{B} \subseteq \mathbb{F}_2^N$, the set of valid codewords.
The inputs are limited to codewords in the codebook $\bar{X}_1^N \in \mathscr{B}$.
$\bar{X}_1^N$ is shorthand for a vector $X_1 X_2 \cdots X_N \in \mathbb{F}_2^N$.

Example: $\mathscr{B} := \{0000, 0111, 1100\} \subseteq \mathbb{F}_2^3$ over BEC.
See 0?00 $\Longrightarrow$ it must be 0000 with one erasure.
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Goal of coding

A good codebook $\mathcal{B}$ should possess the following properties.

Small block length $N \implies$ recall $\mathcal{B} \subseteq \mathbb{F}_2^N$; shorter vectors, easier life.

Big $|\mathcal{B}| \implies$ more codewords mean sending more information at once.
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Fundamental limit of coding

Consider the parameter triple (N, R, P)
(block length, code rate, error probability).

Shannon48: You can find a series of block codes $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots$
such that $N(\mathcal{B}_n) \to \infty$ and $R(\mathcal{B}_n) \to C$ and $P(\mathcal{B}_n) \to 0$ as $n \to \infty$,
where C is a magic number that you cannot beat, provably.

C is called the Shannon capacity. It is the fundamental limit of coding.

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Let N be really large. Let $R := C - \varepsilon$. We preset $|\mathscr{B}| = 2^{RN}$.

Choose a random subset $\mathscr{B} \subseteq \mathbb{F}_2^N$ of that size. (It's just that easy!)

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What Shannon doesn't tell you

Of course we can make P (or R) arbitrarily close to 0 (or C).
But at what cost? If $N \geq 10^{100}$ is needed, the whole theory is useless.

Next step: understand the relation between N and P and $C - R$.

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The trend of P

Fix a code rate $R < C$, error probability P decays to 0 really fast:

$$P \approx e^{-\Theta(N)}.$$

To get a feeling: $P = 10^{-3}$ (once per hour) at $N = 100$,
$P = 10^{-6}$ (once per day) at $N = 200$.
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Fix an error probability P, code rate R approaches C moderately fast:

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$R = C - 1/10$ at $N = 100$ (CPU word).
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The trend of P says $P \approx e^{-\Theta(N)}$; or equivalently $-\log P \approx N$.
The trend of R says $C - R \approx 1/\sqrt{N}$; or equivalently $1/(C - R)^2 \approx N$.

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Analog in probability theory

Probability theory concerns sum of i.i.d.—Z mean μ variance σ.

Large deviations principle:
Fix a z, then $\mathbb{P}\{Z_1 + Z_2 + \cdots + Z_N > N(\mu + z)\} = e^{-\Theta(N)}$.

Central limit theory:
$Z_1 + Z_2 + \cdots + Z_N \approx$ Normal$(\mu, \sigma\sqrt{N})$; or equivalently $\frac{\sum Z}{N} \approx \mu \pm \frac{\Theta(1)}{\sqrt{N}}$.

Generalization: moderate deviations principle:
$\frac{-\log \mathbb{P}\{Z_1+Z_2+\cdots+Z_N>N(\mu+\varepsilon_n z)\}}{\varepsilon_n^2} \approx N$.

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Comparison

	Probability theory	Coding theory
LLN	$\bar{Z} \to \mu$	$(P, R) \to (0, C)$
LDP	$\mathbb{P}\{\bar{Z} - \mu > z\} \approx e^{-NI(z)}$	$P \approx e^{-E_{\mathrm{r}}(R)N}$
CLT	$\bar{Z} - \mu \sim \text{Normal}(0, \frac{\sigma}{\sqrt{N}})$	$C - R \approx \frac{Q^{-1}(P)}{\sqrt{VN}}$
MDP	$\frac{-\log \mathbb{P}\{\bar{Z}-\mu>\varepsilon_N z\}}{\varepsilon_N^2} \approx NI(z)$	$\frac{-\log P}{(C-R)^2} \approx \frac{N}{2V}$

What second-order theory doesn't tell you

Of course we can make P (or R) close to 0 (or C) at the pace of $\frac{-\log P}{(C-R)^2} \approx \frac{N}{2V}$, but at what cost?

Recall that we have to compute likelihoods (assuming i.i.d. channels)
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What second-order theory doesn't tell you

Of course we can make P (or R) close to 0 (or C) at the pace of $\frac{-\log P}{(C-R)^2} \approx \frac{N}{2V}$, but at what cost?

Recall that we have to compute likelihoods (assuming i.i.d. channels)
$\mathbb{P}(Y_1 Y_2 \cdots Y_N \mid x_1 x_2 \cdots x_N) = \mathbb{P}(Y_1 \mid x_1)\mathbb{P}(Y_2 \mid x_2) \cdots \mathbb{P}(Y_N \mid x_N)$
and choose the maximizing $x_1 x_2 x_3 x_4 \in \mathscr{B}$.

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Low complexity codes

Engineers develop “easy codes” regardless of Shannon theory.

Reed–Muller (1954), convolutional (1955), Bose–Chaudhuri–Hocquenghem (1959), Reed–Solomon (1960), trellis modulation (1970s), turbo (1990s), low-density parity-check (1963 and 1996), repeat-accumulate (1998), fountain (1998), and polar (2009).

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Who achieves second-order limits?

	Coding theory	Years
LLN	$(P,R) \to (0,C)$	random 1948, polar 2009, LDPC 2014
LDP	$P \approx e^{-E_{\mathrm{r}}(R)N}$	random 1961, polar 2009–14
CLT	$C - R \approx \frac{Q^{-1}(P)}{\sqrt{VN}}$	random 1950, polar 2010–20
MDP	$\frac{-\log P}{(C-R)^2} \approx \frac{N}{2V}$	random 2014, polar 2016–20

Polar codes key ideas

Consider BEC with erasure probability $0 < p < 1$.

Polar transformation $p \mapsto (p^2, 2p - p^2)$, recursively.
$p^2 \mapsto (p^4, 2p^2 - p^4)$ and $2p - p^2 \mapsto ((2p - p^2)^2, 2(2p - p^2) - (2p - p^2)^2)$.

Arıkan's martingale $Z_{n+1} = Z_n^2$ or $2Z_n - Z_n^2$ with equal probability.

Doob's martingale convergence theorem $Z_n \to Z_\infty \in \{0, 1\}$.
But how fast? (If it takes forever to converge the theory is useless.)

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Polar counterpart of LDP

When p is really really small, $p \mapsto (p^2, p)$. That is, $\log p \mapsto (2\log p, \log p)$, or $\log_2(-\log p) \mapsto (1 + \log_2(-\log p), \log_2(-\log p))$.

With 1/2 probability, $\log_2(-\log Z_n)$ increases by 1, otherwise nothing. For those Z_n that are small, $\log_2(-\log Z_n) \approx n/2$; or $Z_n \approx e^{-2^{n/2}}$.

With more advanced tricks, $Z_n \approx e^{-\ell^{0.99n}}$; this means $P \approx e^{-N^{0.99}}$.

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Polar counterpart of CLT

$\sqrt{Z_n(1-Z_n)}$ is a supermartingale. In fact, $\mathbb{E}[\sqrt{Z_n(1-Z_n)}] \approx 2^{-\rho n}$.

This means that most of the times $\sqrt{Z_n(1-Z_n)}$ is really small, wherein either Z_n is small or $1-Z_n$ is small.
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When $Z_n(1 - Z_n)$ is small, martingale is controlled by the LDP theory.
When $Z_n(1 - Z_n)$ is large, martingale is controlled by the CLT theory.

With positive measure, Z_n goes back and forth between LDP and CLT.

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Future works

Complexity is $O(N \log N)$. With more advanced tricks $O(N \log \log N)$.
$\Longrightarrow$ Can it be better?

Polar codes apply to many-to-one communication,
one-to-many communication, and many others.
$\Longrightarrow$ Generalize polar code to even more scenarios.

In some scenarios, random codes' best performance is still unclear.
$\Longrightarrow$ Use polar to push random.

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Comment and questions

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	Symmetric			Asymmetric	
	binary	prime-ary	finite	binary	finite
LLN	Arikan09	STA09i	STA09i	SRDR12	
LDP*	AT09	MT14	Sasoglu11	HY13	
CLT*	KMTU10,MHU16	BGNRS18			
MDP*	GX15,MHU16	BGS18			
LDP	KSU10,HMTU13				
CLT	FHMV18,GRY20				
MDP					