Complexity and the 2nd-Order Term of Capacity-Achieving Codes

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Noisy channel

The sender inputs $X_1^{32} =$
`11001001 00001111 11011010 10100010`.

The channel output $Y_1^{32} =$
`1--01-01 ----1--- -101---0 --0--0-0`.

Noisy channel

Sender inputs $X_1^{32} \in \mathbb{F}_q^{32}$, where $\mathbb{F}_q$ is input alphabet.
We may assume $\mathbb{F}_q$ is a finite field [new idea].

Channel outputs Y_1^{32} according to stochastic matrix $\mathbb{P}\{Y_i = y \mid X_i = x\} = W(y|x)$ independently for each i.

Noisy channel coding

The sender inputs $X_1^{32} \in \mathcal{B} \subsetneq \mathbb{F}_q^{32}$.
$\mathcal{B}$ is a block code (codebook) of block length $N = 32$.

The channel output Y_1^{32} according to $W(y|x)$.

The receiver maximize the a posterior probability
$\hat{X}_1^{32} = \underset{x_1^{32} \in \mathcal{B}}{\text{do-my-best}} \, \mathbb{P}\{X_1^{32} = x_1^{32} \mid Y_1^{32}\}$.

Noisy channel coding theorem

Channel capacity $C := \sup_{X \sim Q} I(X\,;Y)$ (mutual information).

Block length is N.
Error probability is $P_\mathrm{e} := \mathbb{P}\{\hat{X}_1^N \neq X_1^N\}$.
Code rate is $R := \log|\mathscr{B}|/N \log q$ (recall that $\mathscr{B} \subset \mathbb{F}_q^N$).

[Shannon 1948] *One can find block code* $\mathscr{B}$
such that $P_\mathrm{e} \to 0$ *and* $R \to C$ *as* $N \to \infty$.
(And C is the greatest number that makes this hold.)

2nd-order term of the theorem

How fast do error probability P_{e} and code rate R converge to 0 and C as block length $N \to \infty$? Characterize them as functions "$P_{\mathrm{e}}(N)$" and "$R(N)$".

When R is fixed, $P_{\mathrm{e}} \approx e^{-N}$, that is, $-\log P_{\mathrm{e}} \approx N$.
When P_{e} is fixed, $R \approx C - N^{-1/2}$, that is, $(C-R)^{-2} \approx N$.

When both R and P_{e} vary, $(-\log P_{\mathrm{e}})(C-R)^{-2} \approx N$.

2nd-order term analysis

This is two-sided bound:
A code $\mathscr{B}$ exists such that $(-\log P_\mathrm{e})(C-R)^{-2} \approx N$.
$\mathscr{B}$ does not exist such that $(-\log P_\mathrm{e})(C-R)^{-2} \gg N$.

Block length N is your income;
invest error probability P_e or code rate R or both.

2nd-order term analog

Paradigm	Random variable
law of large numbers	$\bar{X} \to \mu$
large deviations principle	$\mathbb{P}\{\bar{X} - \mu > x\} \approx e^{-nI(x)}$
central limit theorem	$\bar{X} - \mu \sim \mathcal{N}(0, \sigma\sqrt{n})$
moderate deviations principle	$\frac{-\log \mathbb{P}\{\bar{X}-\mu>\varepsilon_n x\}}{\varepsilon_n^2} \approx nI(x)$

2nd-order term analog

P.	Random variable	Random code
LLN	$\bar{X} \to \mu$	$(P_\mathrm{e}, R) \to (0, C)$
LDP	$\mathbb{P}\{\bar{X} - \mu > x\} \approx e^{-nI(x)}$	$P_\mathrm{e} \approx e^{-N}$
CLT	$\bar{X} - \mu \sim \mathcal{N}(0, \sigma\sqrt{n})$	$C - R \approx N^{-1/2}$
MDP	$\frac{-\log \mathbb{P}\{\bar{X} - \mu > \varepsilon_n x\}}{\varepsilon_n^2} \approx nI(x)$	$\frac{-\log P_\mathrm{e}}{(C-R)^2} \approx N$

However...

The achievability bound for random code $\mathcal{B}$ assumes exponential complexity due to $\operatorname*{argmax}_{x_1^{32} \in \mathcal{B}}$.

Goal: Comparable performance, but with a low-complexity decoder $\operatorname*{do\text{-}my\text{-}best}_{x_1^{32} \in \mathcal{B}}$.

2nd-order term goal

P.	Random code	Low-complexity code
LLN	$(P_{\mathrm{e}}, R) \to (0, C)$	$(P_{\mathrm{e}}, R) \to (0, C)$
LDP	$P_{\mathrm{e}} \approx e^{-N}$	$P_{\mathrm{e}} \approx e^{-N^{\pi}}$
CLT	$C - R \approx N^{-1/2}$	$C - R \approx N^{-\rho}$
MDP	$\frac{-\log P_{\mathrm{e}}}{(C-R)^2} \approx N$	$(P_{\mathrm{e}}, C - R) \approx (e^{-N^{\pi}}, N^{-\rho})$

$(0 < \pi, \rho$ and $\pi + 2\rho < 1)$

Polar coding

[Arıkan 2009] invented polar coding. It produces practical codes with provable bounds on P_e and R.

P.	binary	prime-ary	finite	asymmetric
LDP*	known	known	known	known
MDP*	known	known	???	???
LDP	known	???	???	???
CLT	known	???	???	???

Polar coding road map

Channel transformation manipulates channels.

Channel tree is the result of recursive transformation.

Channel parameter measuress the reliability of channels.

Channel process is syntax candy (very useful).

Channel polarization is a phenomenon.

Channel transformation

Channel $W = (X \mid Y)$; input X; output Y.
Make i.i.d. copies $(X_1 \mid Y_1)$ and $(X_2 \mid Y_2)$.

$W^{(1)} := (X_1 - X_2 \mid Y_1^2)$;
$W^{(2)} := (X_2 \mid (X_1 - X_2)Y_1^2)$ (juxtaposition is tupling).

Channel transformation (other kernel)

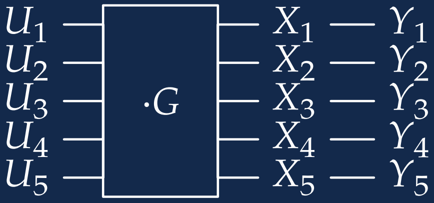
U_1^2 two free variables; G a 2×2 matrix (called kernel);
$X_1^2 := U_1^2 \cdot G$; channels generate Y_1^2.

$W^{(1)} := (U_1 \mid Y_1^2)$;
$W^{(2)} := (U_2 \mid U_1 Y_1^2)$ (juxtaposition is tupling).

Channel transformation (larger kernel)

U_1^ℓ this many free variables; G an $\ell \times \ell$ kernel matrix;
$X_1^\ell := U_1^\ell \cdot G$; channels generate Y_1^ℓ.

$$
\begin{aligned}
W^{(1)} &:= (U_1 \mid Y_1^\ell);\\
W^{(2)} &:= (U_2 \mid U_1 Y_1^\ell);\\
W^{(3)} &:= (U_3 \mid U_1^2 Y_1^\ell);\\
&\vdots\\
W^{(\ell-1)} &:= (U_\ell \mid U_1^{\ell-2} Y_1^\ell);\\
W^{(\ell)} &:= (U_\ell \mid U_1^{\ell-1} Y_1^\ell).
\end{aligned}
$$

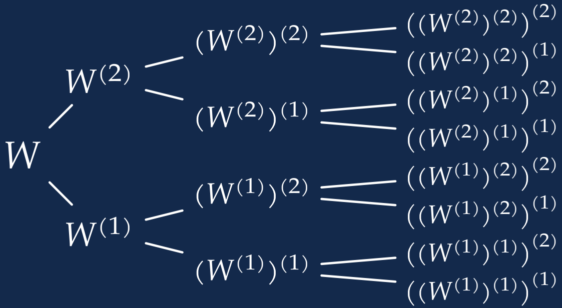


Channel tree

Channel W grows $W^{(1)}, W^{(2)}, \ldots, W^{(\ell)}$.

For each i, channel $W^{(i)}$ grows $(W^{(i)})^{(1)}, \ldots, (W^{(i)})^{(\ell)}$.

For each j, channel $(W^{(i)})^{(j)}$ grows $((W^{(i)})^{(j)})^{(1)}, \ldots$.



Dynamic kernel [new idea*]

Channel W grows $W^{(1)}, W^{(2)}, \ldots, W^{(\ell)}$ using G.

Channel $W^{(i)}$ grows $(W^{(i)})^{(1)}, \ldots, (W^{(i)})^{(\ell)}$ using $G^{(i)}$.

Channel $(W^{(i)})^{(j)}$ grows $((W^{(i)})^{(j)})^{(1)}, \ldots$ using $G^{(ij)}$.

- W
  - $W^{(2)}$
    - $(W^{(2)})^{(2)}$
      - $((W^{(2)})^{(2)})^{(2)}$
      - $((W^{(2)})^{(2)})^{(1)}$
    - $(W^{(2)})^{(1)}$
      - $((W^{(2)})^{(1)})^{(2)}$
      - $((W^{(2)})^{(1)})^{(1)}$
  - $W^{(1)}$
    - $(W^{(1)})^{(2)}$
      - $((W^{(1)})^{(2)})^{(2)}$
      - $((W^{(1)})^{(2)})^{(1)}$
    - $(W^{(1)})^{(1)}$
      - $((W^{(1)})^{(1)})^{(2)}$
      - $((W^{(1)})^{(1)})^{(1)}$

Channel parameter ($\ell = 2$ and $n = 3$)

Block length $N = \ell^n = 2^3 = 8$.

Select indices $\mathcal{I} := \{212, 221, 222\} \in \{1,2\}^3$.
Code rate $R = |\mathcal{I}|/N = 3/8$ (nontrivial).

Error probability $P_\mathrm{e} \leq \sum_{ijk\in\mathcal{I}} H\Big(\big((W^{(i)})^{(j)}\big)^{(k)}\Big)$ (nontrivial);
$H(X \mid Y)$ is conditional entropy (base to be specify).

It suffices to understand

$$H(W), H(W^{(i)}), H((W^{(i)})^{(j)}), H\Big(((W^{(i)})^{(j)})^{(k)}\Big), \ldots.$$

Block length N will be $\ell^{\text{where we stop}}$.

Code rate R will be the fraction of small H-values.

Error probability P_{e} will be $\sum_{\text{those}}$ small H-values.

Channel process (syntax candy)

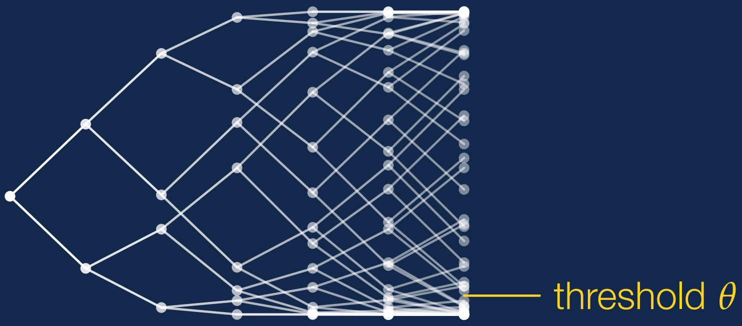
$W_0 := W$.
$W_{n+1} := W_n^{(K_{n+1})}$, where $K_{n+1} \in \{1, 2, \ldots, \ell\}$ i.i.d. uniform.

$H_n := H(W_n)$.

Decide depth n, then block length $N = \ell^n$.
Decide threshold θ, then code rate $R = \mathbb{P}\{H_n < \theta\}$.
Error probability $P_e < \sum$ small $H_n < \sum \theta = RN\theta \leq N\theta$.

Channel polarization

$H_n := H(W_n)$ is a martingale. (Invoke the Doob's.)
$H_n \to H_\infty$ a.e. as $n \to \infty$; it turns out $H_\infty \in \{0,1\}$.



It suffices to understand

$$\mathbb{P}\{H_n < \text{threshold}\} > C - \text{gap}.$$

Goal: $\mathbb{P}\{H_n < e^{-\ell^{\pi n}}\} > C - \ell^{-\rho n}$, where $\pi + 2\rho < 1$.
Then $N = \ell^n$ and $P_\mathrm{e} < Ne^{-N^\pi}$ and $R > C - N^{-\rho}$.

Proof outline

Local LDP behavior: $Z(W^{(k)}) \le \ell e^{qZ(W)\ell}(qZ(W))^{\lceil k^2/3\ell \rceil}$.
(Never heard Bhattacharyya parameter? $Z := H$.)

Local CLT behavior: $\sum_{k=1}^{\ell} f(H(W^{(k)})) < 4\ell^{1/2+\alpha}$,
where $\alpha = \log\log\ell / \log\ell$ and $f(z) := \min(z, 1-z)^{\alpha}$.

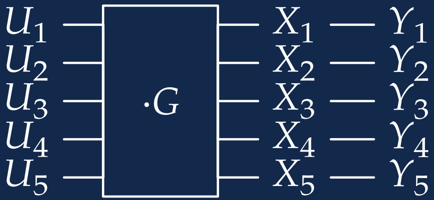
Global MDP behavior: $\mathbb{P}\{H_n < e^{-\ell^{\pi n}}\} > C - \ell^{-\rho n}$, where $\pi + 2\rho < 1$, given local LDP and local CLT behaviors.

Local LDP behavior 1/3

Want to prove $Z(W^{(k)}) \le \ell e^{qZ(W)\ell}(qZ(W))^{\lceil k^2/3\ell \rceil}$.
Let $z := Z(W)$; want $Z(W^{(k)}) \le \ell e^{qz\ell}(qz)^{\lceil k^2/3\ell \rceil}$.

Lemma: $Z(W^{(k)}) \le \sum_{u_{k+1}^{\ell} \in \mathbb{F}_q^{\ell-k}} z^{\mathrm{wt}(0_1^{k-1} 1_k u_{k+1}^{\ell} \cdot G)}$;
RHS is weight enumerator of a coset code.

$W^{(1)} := (U_1 \mid Y_1^{\ell})$;
$W^{(2)} := (U_2 \mid U_1 Y_1^{\ell})$;
$\vdots$
$W^{(\ell)} := (U_\ell \mid U_1^{\ell-1} Y_1^{\ell})$.



Local LDP behavior 2/3

Want $\sum_{u_{k+1}^{\ell}} z^{\mathrm{wt}(0_1^{k-1} 1_k u_{k+1}^{\ell} \cdot G)} \leq \ell e^{qz\ell}(qz)^{\lceil k^2/3\ell \rceil}$ for some G.

G random; $\mathbb{E}\mathrm{LHS} = q^{-k}(1+(q-1)z)^{\ell} \leq q^{-k}(1+qz)^{\ell}$.

Compare $(qz)^w$-coefficients: $q^{-k}\binom{\ell}{w}$ vs $\ell \frac{\ell^{w-\lceil k^2/3\ell \rceil}}{(w-\lceil k^2/3\ell \rceil)!}$.

Simplify: $2^{-k}\binom{\ell}{\lceil k^2/3\ell \rceil}\binom{\ell-\lceil k^2/3\ell \rceil}{w-\lceil k^2/3\ell \rceil}$ vs $\ell\binom{\ell}{w-\lceil k^2/3\ell \rceil}$.

Local LDP behavior 3/3

Boils down to $2^{-k}\binom{\ell}{\lceil k^2/3\ell \rceil}$ vs ℓ; ignore/cancel $\lceil\,\rceil$ and ℓ.

$\binom{\ell}{d}$ is about $2^{\ell h_2(d/\ell)}$ for $d = \Theta(\ell)$. (Large deviations!)
Hence k vs $h_2(k^2/3\ell^2)$, which becomes $\sqrt{3x}$ vs $h_2(x)$.

zoom →

Local CLT behavior 1/4

Want to prove $\sum_{k=1}^{\ell} f(H(W^{(k)})) < 4\ell^{1/2+\alpha}$.

Break into segments
$$\begin{cases} \sum_{k=H(W)+\ell^{-1/2+\alpha}}^{\ell} < \ell^{1/2+\alpha}, \\ \sum_{k=H(W)-\ell^{-1/2+\alpha}}^{H(W)+\ell^{-1/2+\alpha}} < 2\ell^{1/2+\alpha}, \\ \sum_{k=1}^{H(W)-\ell^{-1/2+\alpha}} < \ell^{1/2+\alpha}. \end{cases}$$

Local CLT behavior 2/4

Want to show $\displaystyle\sum_{k=H(W)+\ell^{-1/2+\alpha}}^{\ell} f(H(W^{(k)})) < \ell^{1/2+\alpha}$.

Jensen LHS: $(\ell - m)f\left(\frac{1}{\ell-m}\displaystyle\sum_{k=m+1}^{\ell} H(W^{(k)})\right) < \ell^{1/2+\alpha}$,
where $m = H(W) + \ell^{-1/2+\alpha} - 1$.

$$\vdots$$

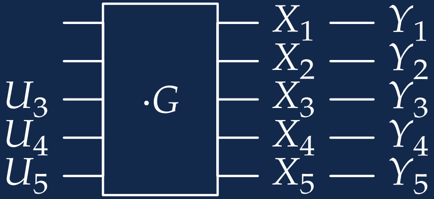
$$W^{(\ell-2)} := (U_{\ell-2} \mid U_1^{\ell-3} Y_1^{\ell}),$$
$$W^{(\ell-1)} := (U_{\ell-1} \mid U_1^{\ell-2} Y_1^{\ell}),$$
$$W^{(\ell)} := (U_{\ell} \mid U_1^{\ell-1} Y_1^{\ell}).$$

$$\sum_{k=m+1}^{\ell} = H(U_{m+1}^{\ell} \mid U_1^{m} Y_1^{\ell}).$$

Local CLT behavior 3/4

$H(U_{m+1}^{\ell} \mid U_1^m Y_1^{\ell})$ is what? $(m = H(W) + \ell^{-1/2+\alpha} - 1)$

The conditional entropy of noisy channel coding.



Gallager has good bounds.

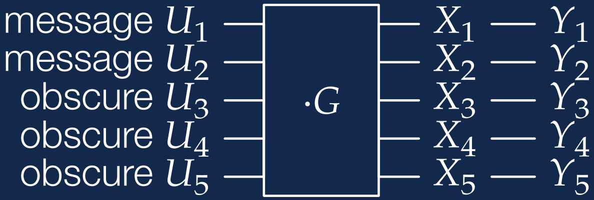
Local CLT behavior 4/4

The other segment: $\displaystyle\sum_{k=1}^{H(W)-\ell^{-1/2+\alpha}} f(H(W^{(k)})) < 4\ell^{1/2+\alpha}$.

Jensen inequality: $mf\left(\frac{1}{m}\sum_{k=1}^{m+1} H(W^{(k)})\right) < 4\ell^{1/2+\alpha}$.

Chain rule: $H(U_1^m \mid Y_1^\ell)$, what is this? Guess?

wiretap channel; Hayashi has good bounds [new idea].



A calculus machinery [new idea]

local LDP behavior: $Z(W^{(k)}) \leq \ell e^{qZ(W)\ell}(qZ(W))^{\lceil k^2/3\ell \rceil}$.

Local CLT behavior: $\sum_{k=1}^{\ell} f(H(W^{(k)})) < 4\ell^{1/2+\alpha}$.

eigen: $\mathbb{E}[f(H_{n+1}) \mid H_0, \ldots, H_n] \leq \ell^{-1/2+3\alpha} f(H_n)$.

en23: $\mathbb{P}\{Z_n < e^{-n^{2/3}}\} > C - \ell^{(-1/2+4\alpha)n}$.

een13: $\mathbb{P}\left\{Z_n < \exp(-e^{n^{1/3}})\right\} > C - \ell^{(-1/2+4\alpha)n}$.

elpin: $\mathbb{P}\{Z_n < e^{-\ell^{\pi n}}\} > C - \ell^{-\rho n}$.

Summary so far

For all $\pi + 2\rho < 1$, there exist codes with error probability $P_\mathrm{e} < e^{-N^\pi}$ and code rate $R > C - N^{-\rho}$.

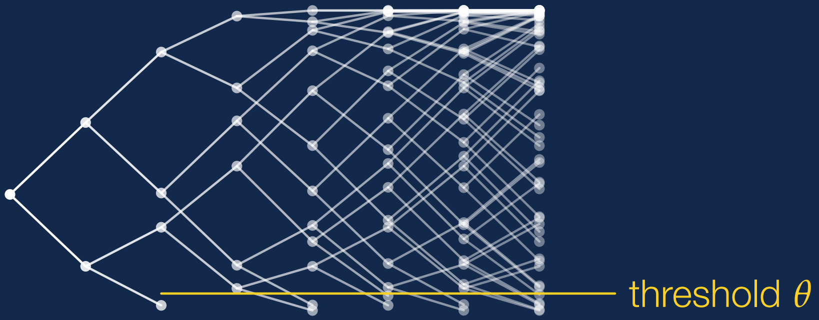
When only 2×2 kernels are allowed, at least $\pi, \rho > 0$.

It happens that they have complexity $O(\log N)$ per bit.

Can we reduce the complexity further
(at the expense of worse performance etc)?

Pruning

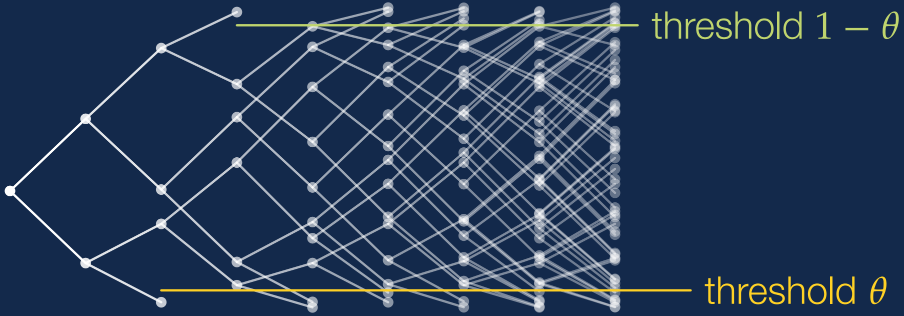
The bottom channel is good enough before we reach our favorite n.



Why do we apply transform any further? (Ans: don't!)

Pruning

The top channel is too bad. Do we expect any of its descendants to be good enough?



We don't.

Stopping time

Channel H_i needs transformation if $\theta < H_i < 1 - \theta$.

Set $\theta = N^{-10}$; assume $i > O(\log \log N)$, then $e^{-2^{\pi i}} < \theta$.

Then $\mathbb{P}\{H_i \leq \theta\} > \mathbb{P}\{H_i \leq e^{-2^{\pi i}}\} \geq C - \ell^{-\rho i}$ and
$\mathbb{P}\{1 - \theta \leq H_i\} > \mathbb{P}\{1 - e^{-2^{\pi i}} \leq H_i\} \geq 1 - C - \ell^{-\rho i}$

That is, $\mathbb{P}\{\theta < H_i < 1 - \theta\} \leq 2\ell^{-\rho i}$.

Geometric complexity

Complexity = #transformations = $\sum_{i=0}^{n} \mathbb{P}\{\theta < H_i < 1-\theta\}$.

$\sum_{i=O(\log\log N)}^{n} \mathbb{P}\{\theta < H_i < 1-\theta\} \leq \sum 2\ell^{-\rho i} = O(1)$;
$\sum_{i=0}^{O(\log\log N)} \mathbb{P}\{\theta < H_i < 1-\theta\} \leq \sum 1 = O(\log\log N)$.

Complexity is $O(\log\log N)$ per bit.

Summary

There exist codes with complexity $O(\log\log N)$ per bit, error probability $P_{\mathrm{e}} < N^{-9}$, and code rate $R = C - N^{-\rho}$.

(Earlier) we have codes with complexity $O(\log N)$ per bit, error probability $P_{\mathrm{e}} < e^{-N^{\pi}}$, and code rate $R > C - N^{-\rho}$.

Are there codes in between?

Summary

Log-log code taken from (with Duursma)
Log-logarithmic Time Pruned Polar Coding
https://arxiv.org/abs/1905.13340.

MDP code taken from (with Duursma)
Polar Codes’ Simplicity, Random Codes’ Durability
https://arxiv.org/abs/1912.08995.

Question?

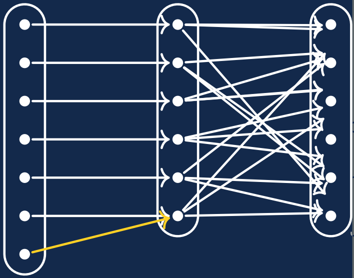
Predefined questions:
Why input alphabet is finite field? What advantage?
Definition of Bhattacharyya parameter?
References for XYZ?
What channels? Your contribution over others?
Future plan?

Code	Error	Gap	Complex	Channel
random	e^{-N^π}	$N^{-\rho}$	$\exp(N)$	DMC
RM	$\to 0$	$\to 0$	$O(N^2)$	BEC
LDPC	$\to 0$	$\to 0$	???	S. BDMC
RA family	$\to 0$	$\to 0$	$O(1)$	BEC
[W. polar]	e^{-N^π}	$N^{-\rho}$	$O(\log N)$	DMC
old prune	$e^{-N^{1/2}}$	$O(1)$	$\Theta(\log N)$	S. BDMC
[W. prune]	N^{-9}	$N^{-\rho}$	$O(\log\log N)$	DMC

	symmetric			asymmetric	
P.	binary	prime-ary	finite	binary	finite
LLN	[3]	[11]	[11]	[35]	[W.]
LDP*	[5]	[29]	[32]	[24]	[W.]
CLT*	[26, 28]	[9]	[W.]	[W.]	[W.]
MDP*	[19, 28]	[10]	[W.]	[W.]	[W.]
LDP	[27, 21]	[W.]	[W.]	[W.]	[W.]
CLT	[15, 20]	[W.]	[W.]	[W.]	[W.]
MDP	[W.]	[W.]	[W.]	[W.]	[W.]

Input alphabet [new idea]

$$\begin{bmatrix} W(y_1|1) & W(y_2|1) & W(y_3|1) & \cdots \\ W(y_1|2) & W(y_2|2) & W(y_3|2) & \cdots \\ W(y_1|3) & W(y_2|3) & W(y_3|3) & \cdots \\ W(y_1|4) & W(y_2|4) & W(y_3|4) & \cdots \\ W(y_1|5) & W(y_2|5) & W(y_3|5) & \cdots \\ W(y_1|6) & W(y_2|6) & W(y_3|6) & \cdots \\ W(y_1|6) & W(y_2|6) & W(y_3|6) & \cdots \end{bmatrix}$$



Asymmetric channels [24]

Recall U_i is the coordinate as in $X_1^\ell := U_1^\ell \cdot G$.
The difficulty of asymmetric channels is nonuniform U_i.

Define synthetic channel $V^{(k)} := (U_i \mid U_1^{i-1})$.
Define $V^{(i)}, (V^{(i)})^{(j)}, ((V^{(i)})^{(j)})^{(k)}, \ldots$; define $\{V_n\}$.
It polarizes, and at the same pace.

High $H(V_n)$ low $H(W_n)$ vs both high vs both low.

Bhattacharyya parameter

Binary $Z(W) := \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}$.

Non-binary $\frac{1}{q-1} \sum_{\substack{x,x' \in \mathbb{F}_q \\ x \neq x'}} \sum_{y \in \mathcal{Y}} \sqrt{W(x,y)W(x',y)}$.

[New idea] $\max_{0 \neq d \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_q} \sum_{y \in \mathcal{Y}} \sqrt{W(x,y)W(x+d,y)}$.

Random codes references

LDP: [14, 16, 33, 18, 17, 8, 6, 25, 13]

CLT: [37, 36, 12, 34, 7, 22, 30]

MDP: [1, 31, 2, 4, 23]

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