# Complexity and Second Moment of the Mathematical Theory of Communication

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# Noisy channel

The sender inputs
$X_1^{32} =$ `11001001 00001111 11011010 10100010`.

The channel outputs
$Y_1^{32} =$ `1--01-01 ----1--- -101---0 --0--0-0`.

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# Noisy channel

Sender inputs $X_1^{32} \in \mathbb{F}_q^{32}$, where $\mathbb{F}_q$ is called input alphabet. WLoG, we may assume $\mathbb{F}_q$ is a finite field [new idea].

The channel outputs $Y_1^{32}$ according to transition probabilities $P\{Y_j = y \mid X_j = x\} = W(y|x)$ independently for each $j$.

# Noisy-channel coding

The encoder inputs $X_1^{32} \in \mathscr{B} \subsetneq \mathbb{F}_q^{32}$ into a channel.
$\mathscr{B}$ is a *block code* (sometimes a *codebook*) of block length $N = 32$.

The channel outputs $Y_1^{32}$ according to $W(y|x)$.

The decoder, seeing $Y_1^{32} = y_1^{32}$, maximizes the posterior probability
$\hat{X}_1^{32}(y_1^{32}) := \underset{x_1^{32} \in \mathscr{B}}{\operatorname{argmax}} P\{X_1^{32} = x_1^{32} \mid Y_1^{32} = y_1^{32}\}$.

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# Noisy-channel coding theorem

Channel capacity $C := \sup_{X \sim Q} I(X\,;Y)$ (supremum over input distribution).

Block length is $N$.

Error probability is $P_\mathrm{e} := P\{\hat{X}_1^N \neq X_1^N\}$.

Code rate is $R := \ln|\mathcal{B}| \div \ln|\mathbb{F}_q^N|$. (recall that $\mathcal{B} \subset \mathbb{F}_q^N$)

[Shannon 1948]

*One can find block codes $\mathcal{B}$ such that $P_\mathrm{e} \to 0$ and $R \to C$ as $N \to \infty$.*

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# 2nd-order term of coding

How fast do error probability $P_\mathrm{e}$ and code rate $R$ converge to $0$ and $C$ as block length $N \to \infty$? Characterize functions "$P_\mathrm{e}(N)$" and "$R(N)$".

When $R$ is fixed, $P_\mathrm{e} \approx e^{-N}$; or equivalently, $-\ln P_\mathrm{e} \approx N$.
Fano [Fan61], Gallager [Gal65], Shannon–Gallager–Berlekamp [SGB67], [Gal68], [Gal73], Blahut [Bla74], Barg–Forney [BF02], Fàbregas–Land–Martinez [iFLM11], Domb–Zamir–Feder [DZF16].

When $P_\mathrm{e}$ is fixed, $R \approx C - N^{-1/2}$; or equivalently, $(C - R)^{-2} \approx N$.
Wolfowitz [Wol57], Weiss [WEI60], Dobrushin [Dob61], Strassen [Str62], Baron–Khojastepour–Baraniuk [BKB04], Hayashi [Hay09], Polyanskiy–Poor–Verdu [PPV10].

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# Joint 2nd-order term of coding

When both $R$ and $P_{\mathrm{e}}$ vary, $(P_{\mathrm{e}}, R) \approx (e^{-N^{\pi}}, C - N^{-\rho})$ where $\pi + 2\rho = 1$; or equivalently, $(-\ln P_{\mathrm{e}})(C - R)^{-2} \approx N$.
Altuğ–Wagner [AW10], Polyanskiy–Verdú [PV10], [AW14], Hayashi–Tan [HT15].

This is a two-sided bound:
A code $\mathcal{B}$ exists such that $(-\ln P_{\mathrm{e}})(C - R)^{-2} \approx N$.
No code $\mathcal{B}$ exists such that $(-\ln P_{\mathrm{e}})(C - R)^{-2} \gg N$.

Block length $N$ is your income;
invest in error probability $P_{\mathrm{e}}$ or in code rate $R$ or in both.

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# 2nd-order term analog

| Paradigm | Random variable | Random code |
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| law of large numbers | $\bar{X} \to \mu$ | $(P_e, R) \to (0, C)$ |
| large deviation principle | $\mathbb{P}\{\bar{X} - \mu > x\} \approx e^{-nI(x)}$ | $P_e \approx e^{-N}$ |
| central limit theorem | $\bar{X} - \mu \sim \mathcal{N}(0, \sigma/\sqrt{n})$ | $C - R \approx N^{-1/2}$ |
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Achievability via random coding assumes exponential complexity due to the usage of the maximum a posteriori decoder $\operatorname*{argmax}_{x_1^{32} \in \mathcal{B}}$.

Goal: Comparable performance, but with a low-complexity $\operatorname*{do-my-best}_{x_1^{32} \in \mathcal{B}}$.

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# 2nd-order term goal

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# Polar coding

| Par | BEC | SBDMC | p-ary | q-ary | finite | BDMC | a-finite |
|---|---|---|---|---|---|---|---|
| LLN | [Ari09] | [Ari09] | [ŞTA09] | [ŞTA09] | [ŞTA09] | [SRDR12] | ?? |
| LDP* | [AT09] | [AT09] | [ŞTA09] | [MT10] | [Sas11] | [HY13] | ?? |
| CLT* | [KMTU10] | [HAU14] | [BGN+18] | ?? | ?? | ?? | ?? |
| MDP* | [GX15] | [GX15] | [BGS18] | ?? | ?? | ?? | ?? |
| LDP | [KSU10] | [KSU10] | ?? | ?? | ?? | ?? | ?? |
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# Polar coding road map

Channel transformation manipulates channels.

Channel tree is the result of recursive channel transformation.

Channel parameter measures the channels and keep track of the tree.

Channel process is a syntax candy paraphrasing the tree.

Channel polarization is a phenomenon that channels become extreme.

# Channel transformation

Consider a channel $W = (X \mid Y)$, where input is $X$, output is $Y$.
Make two i.i.d. copies $(X_1 \mid Y_1)$ and $(X_2 \mid Y_2)$.

$W^{(1)} := (X_1 - X_2 \mid Y_1^2)$;
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# Channel transformation (other kernel)

$U_1^2$: two free variables; $G$: a $2 \times 2$ matrix (called kernel);
$X_1^2 := U_1^2 \cdot G$: matrix multiplication; channels generate $Y_1^2$ given $X_1^2$.

$W^{(1)} := (U_1 \mid Y_1^2)$;
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# Channel transformation (larger kernel)

$U_1^\ell$: many free variables; $G$: an $\ell \times \ell$ matrix kernel;
$X_1^\ell := U_1^\ell \cdot G$; channels generate $Y_1^\ell$ given $X_1^\ell$.

$$
\begin{aligned}
W^{(1)} &:= (U_1 \mid Y_1^\ell);\\
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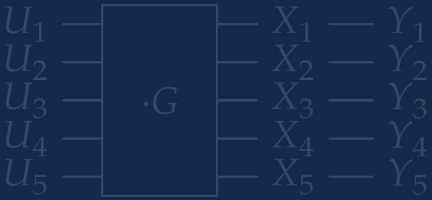

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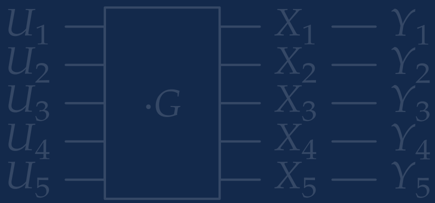

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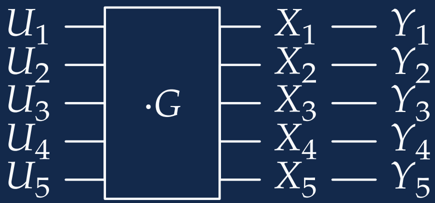

# Channel tree

Channel $W$ grows $W^{(1)}, W^{(2)}, \ldots, W^{(\ell)}$.

For each $i$, channel $W^{(i)}$ grows $(W^{(i)})^{(1)}, \ldots, (W^{(i)})^{(\ell)}$.

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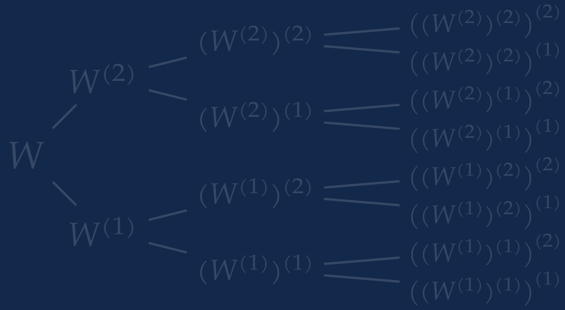

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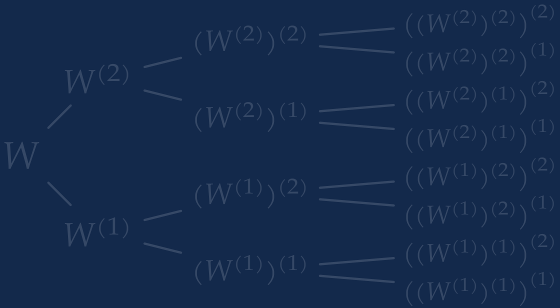

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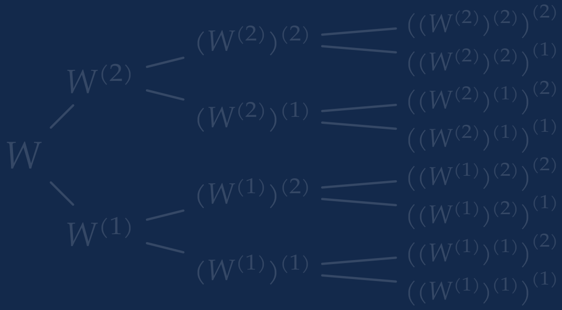

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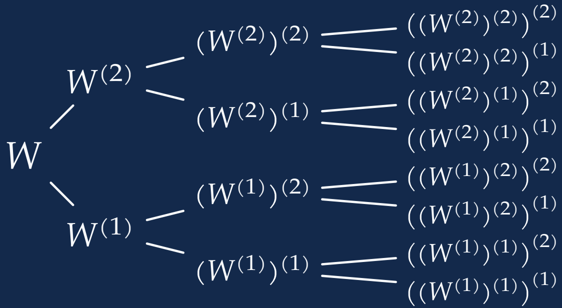

# Dynamic kernel [new idea*]

Channel $W$ grows $W^{(1)}, W^{(2)}, \ldots, W^{(\ell)}$ using $G$.

For each $i$, channel $W^{(i)}$ grows $(W^{(i)})^{(1)}, \ldots, (W^{(i)})^{(\ell)}$ using $G^{(i)}$.

$\forall j$, channel $(W^{(i)})^{(j)}$ grows $((W^{(i)})^{(j)})^{(1)}, \ldots, ((W^{(i)})^{(j)})^{(\ell)}$ using $G^{(ij)}$.

- $W$
  - $W^{(2)}$
    - $(W^{(2)})^{(2)}$
      - $((W^{(2)})^{(2)})^{(2)}$
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Channel $W$ grows $W^{(1)}, W^{(2)}, \ldots, W^{(\ell)}$ using $G$.

For each $i$, channel $W^{(i)}$ grows $(W^{(i)})^{(1)}, \ldots, (W^{(i)})^{(\ell)}$ using $G^{(i)}$.

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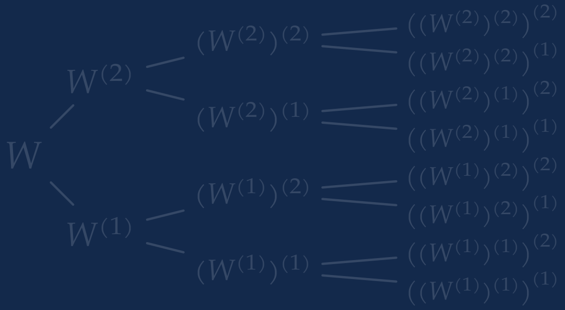

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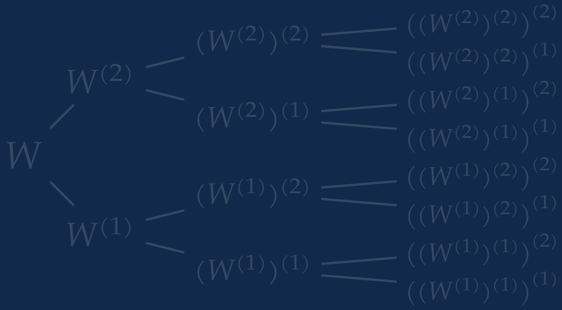

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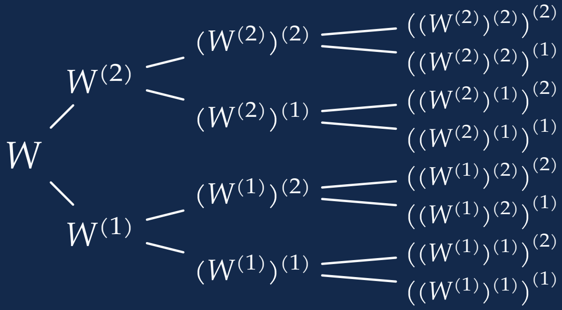

# Channel parameter ($\ell = 2$ and $n = 3$ exmaple)

Block length $N = \ell^n$; for instance $8 = 2^3$.

Select indices $\mathcal{J} \subseteq \{1, \ldots, \ell\}^n$; for instance $\{122, 212, 221, 222\} \subseteq \{1, 2\}^3$.
Code rate $R = |\mathcal{J}|/N = 4/8$ (nontrivial, due to implementation details).

Error probability $P_e \leq \sum_{ijk \in \mathcal{J}} H\Big(\big((W^{(i)})^{(j)}\big)^{(k)}\Big)$ (nontrivial, due to details);

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$$H(W), H(W^{(i)}), H((W^{(i)})^{(j)}), H\Big(((W^{(i)})^{(j)})^{(k)}\Big), H\Big(\Big(((W^{(i)})^{(j)})^{(k)}\Big)^{(l)}\Big), \ldots$$

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$W_0 := W$.
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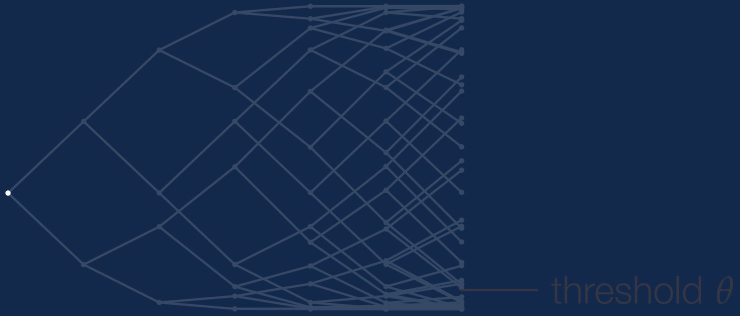
$H_n \to H_\infty$ a.e. as $n \to \infty$; turns out $H_\infty \in \{0, 1\}$ and $P\{H_\infty = 1\} = H_0$.



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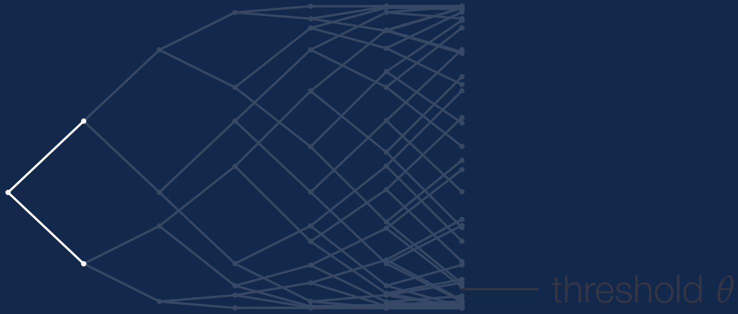

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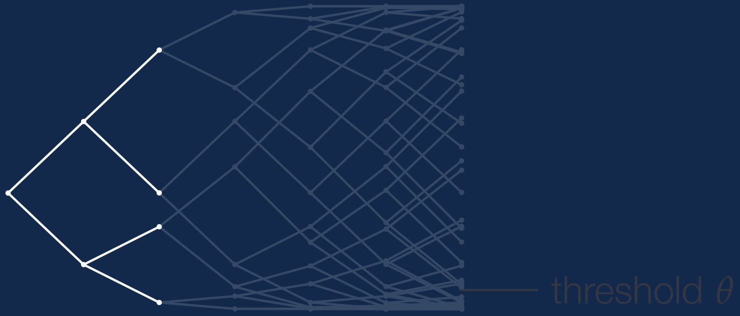

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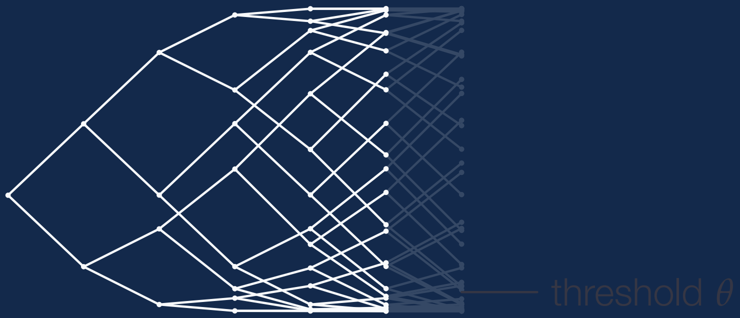

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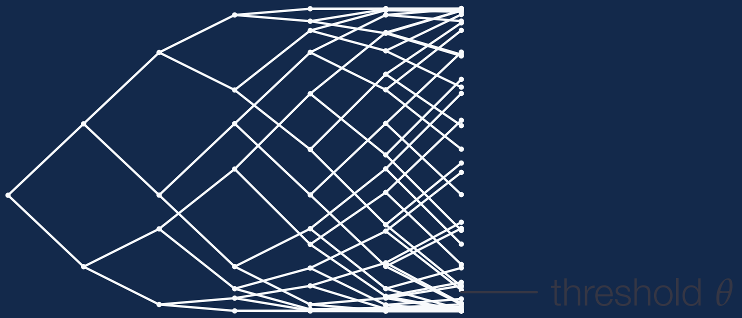

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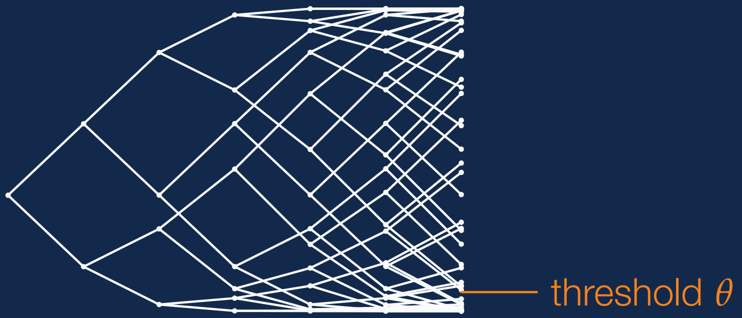

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$$P\{H_n < \text{threshold}\} > C - \text{gap}.$$

Goal: $P\{H_n < e^{-\ell^{\pi n}}\} > C - \ell^{-\rho n}$ for large $n$, where $\pi + 2\rho < 1$.
Then: $N = \ell^n$ and $P_e < Ne^{-N^\pi} \approx e^{-N^\pi}$ and $R > C - N^{-\rho}$.

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# Proof outline

Local LDP behavior: $Z(W^{(j)}) \leq \ell e^{qZ(W)\ell}(qZ(W))^{\lceil j^2/3\ell \rceil}$,
where $Z$ is a parameter such that $Z \leq q^3\sqrt{H}$ and $H \leq q^3\sqrt{Z}$.

Local CLT behavior: $\sum_{j=1}^{\ell} h(H(W^{(j)})) < 4\ell^{1/2+\alpha}$,
where $\alpha = \ln(\ln \ell)/\ln \ell$ and $h(z) := \min(z, 1-z)^{\alpha}$.

Global MDP behavior: $P\{H_n < e^{-\ell^{\pi n}}\} > C - \ell^{-\rho n}$, where $\pi + 2\rho < 1$,
given local LDP and local CLT behaviors.

# Local LDP behavior 1/3

Want to prove $Z(W^{(j)}) \le \ell e^{qz\ell}(qz)^{\lceil j^2/3\ell \rceil}$ where $z := Z(W)$.

Fundamental theorem of polar: $Z(W^{(j)}) \le \sum_{u_{j+1}^{\ell} \in \mathbb{F}_q^{\ell-j}} z^{\operatorname{hwt}(0_1^{j-1} 1_j u_{j+1}^{\ell} \cdot G)}$.

RHS is the weight enumerator of a coset code.

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W^{(1)} &:= (U_1 \mid Y_1^{\ell});\\
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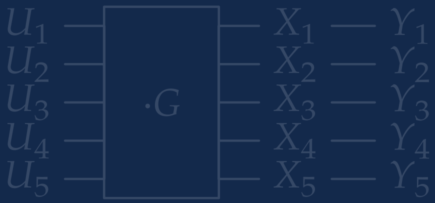

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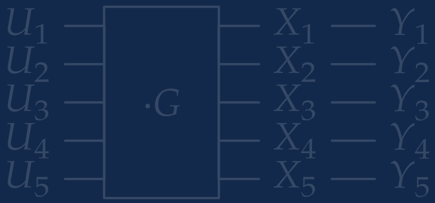

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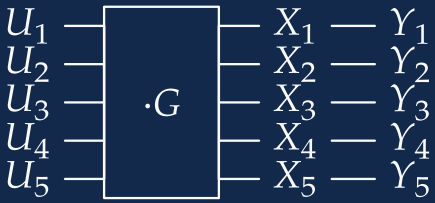

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Compare $(qz)^w$-coefficients: $q^{-j}\binom{\ell}{w}$ vs $\ell \frac{\ell^{w-\lceil j^2/3\ell \rceil}}{(w-\lceil j^2/3\ell \rceil)!}$.

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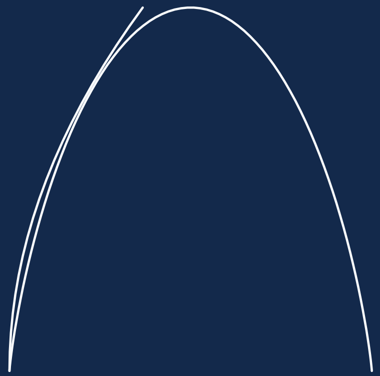

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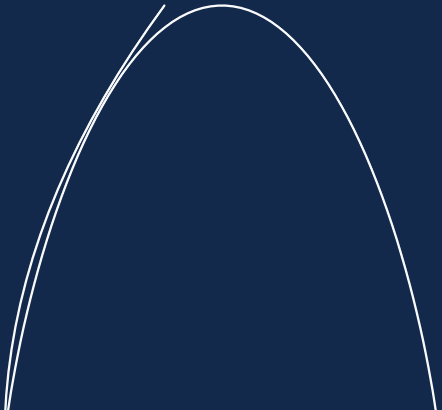

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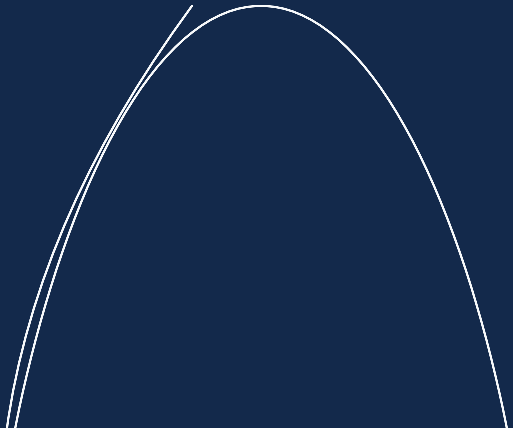

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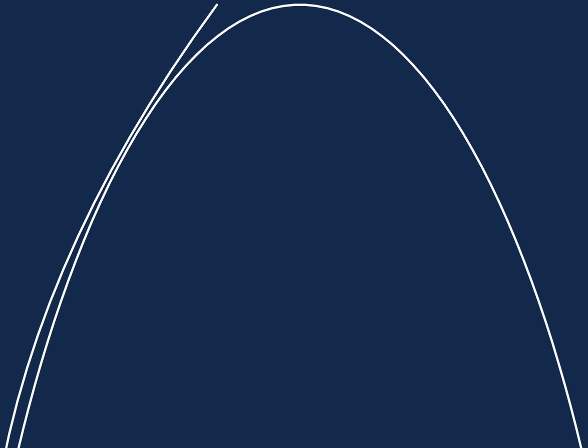

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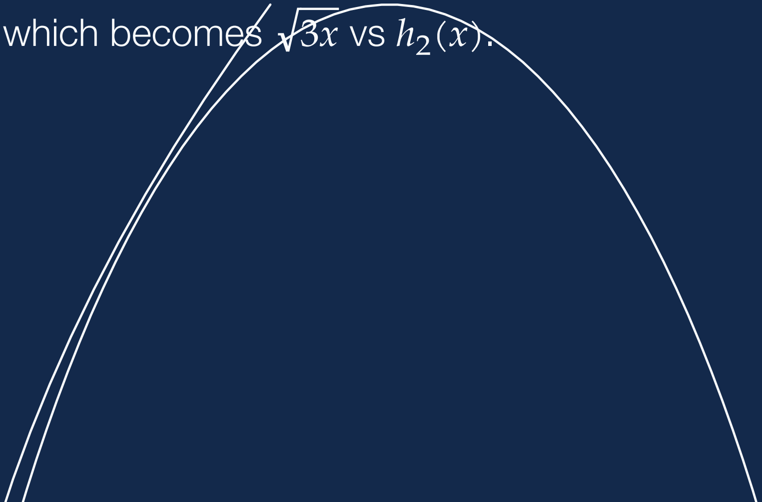

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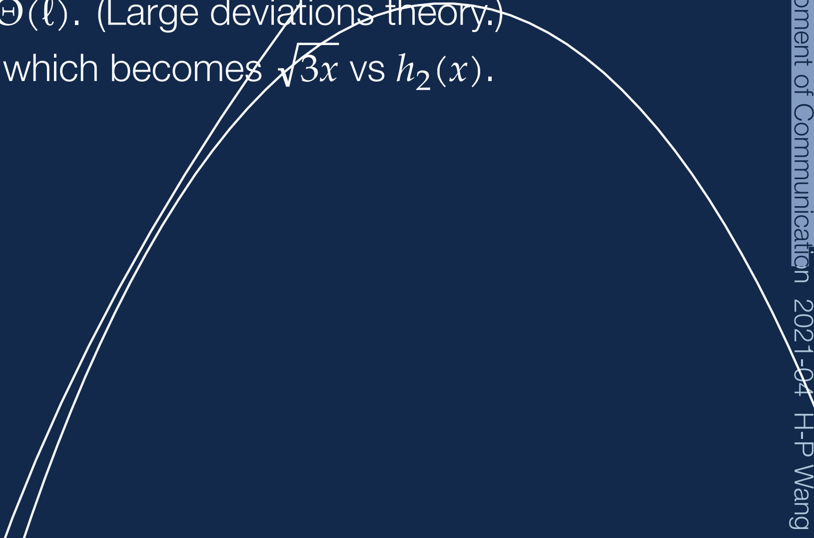

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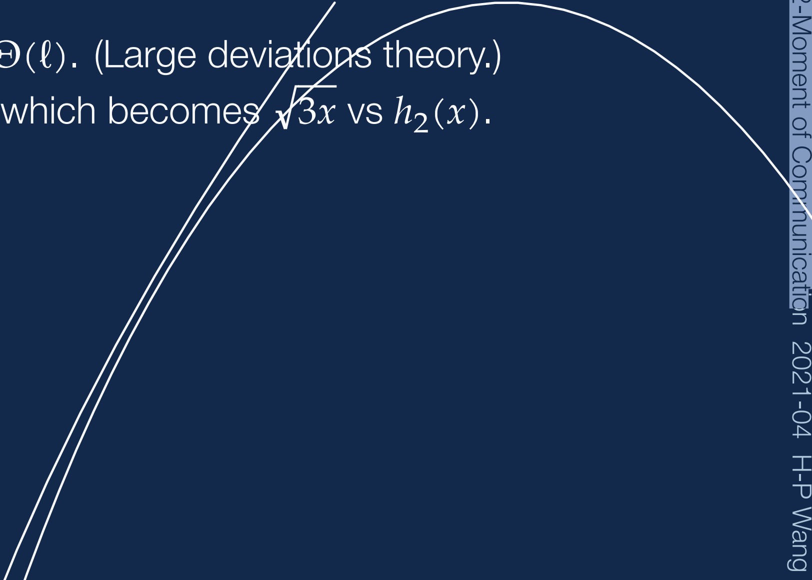

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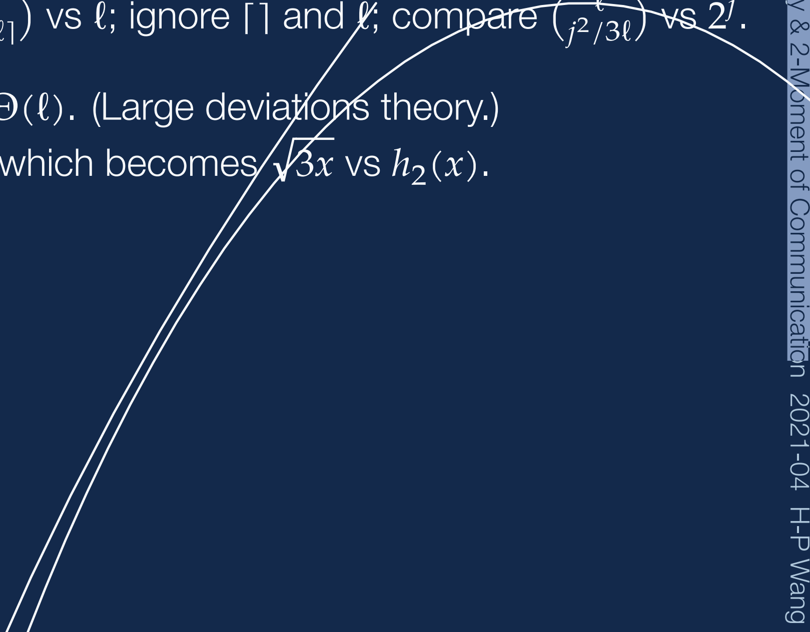

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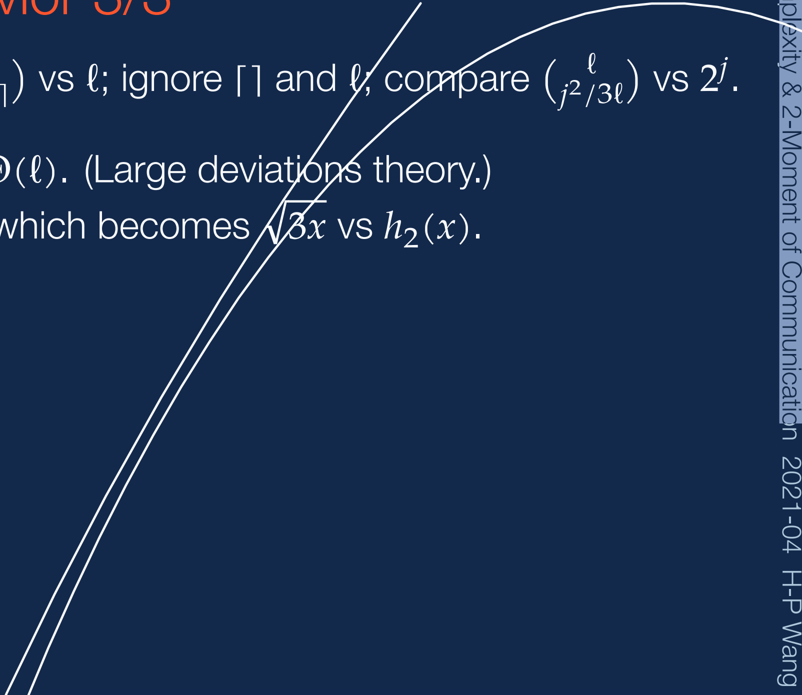

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Want to prove $\sum_{i=1}^{\ell} h(H(W^{(i)})) < 4\ell^{1/2+\alpha}$,
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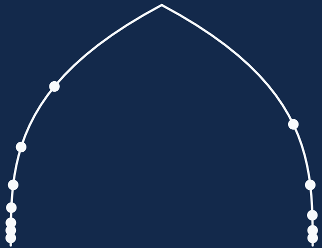

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Want $\sum_{i=j+1}^{\ell} h(H(W^{(i)})) < \ell^{1/2+\alpha}$, where $j := \lceil H(W) + \ell^{-1/2+\alpha} \rceil$.

Jensen LHS; want to show $(\ell - j)h\Big(\frac{1}{\ell - j} \sum_{i=j+1}^{\ell} H(W^{(i)})\Big) < \ell^{1/2+\alpha}$.

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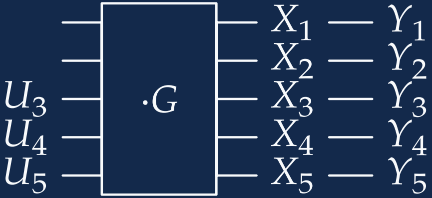


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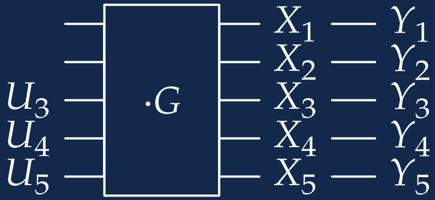


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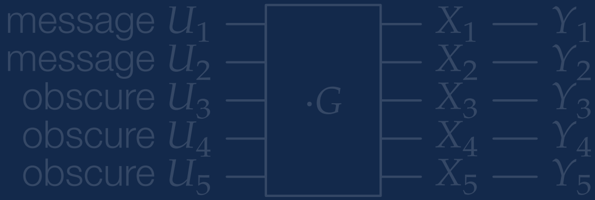

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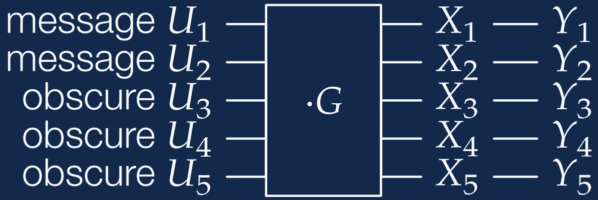

# A calculus machinery [new idea]

Given local LDP behavior: $Z(W^{(j)}) \le \ell e^{qZ(W)\ell}(qZ(W))^{\lceil j^2/3\ell \rceil}$

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For all $\pi + 2\rho < 1$, there exist codes with
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When only $2 \times 2$ kernels are allowed, at least $\pi, \rho > 0$.

It happens that they have complexity $O(\log N)$ per bit.

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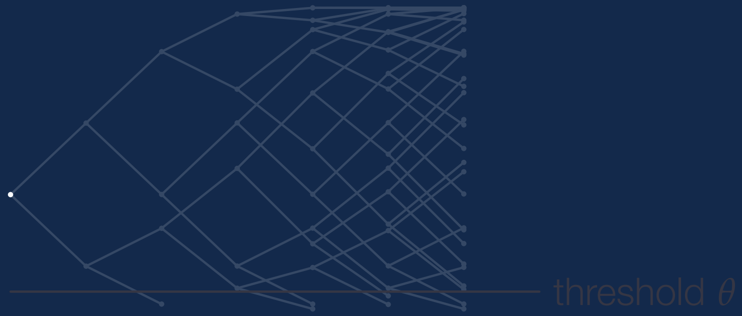


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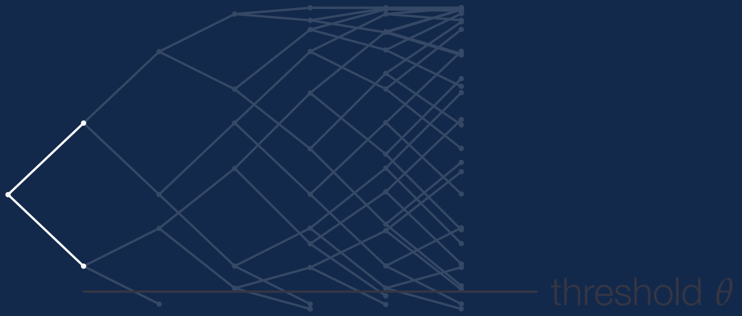


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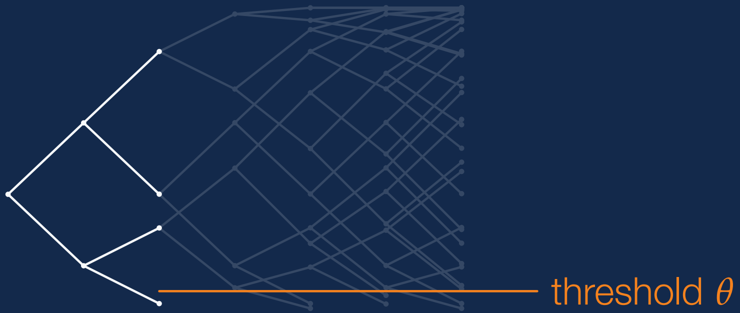


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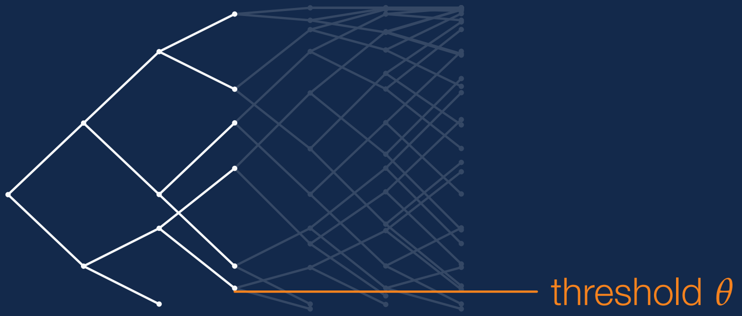


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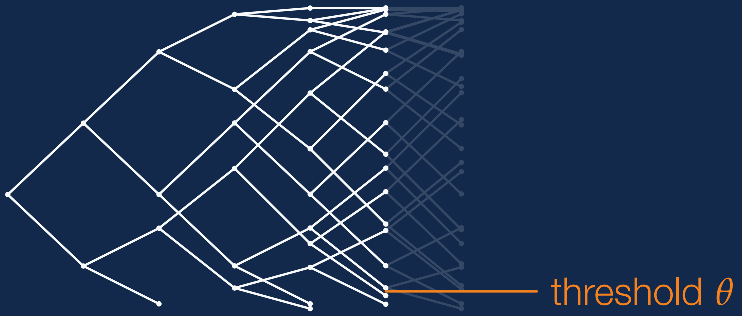


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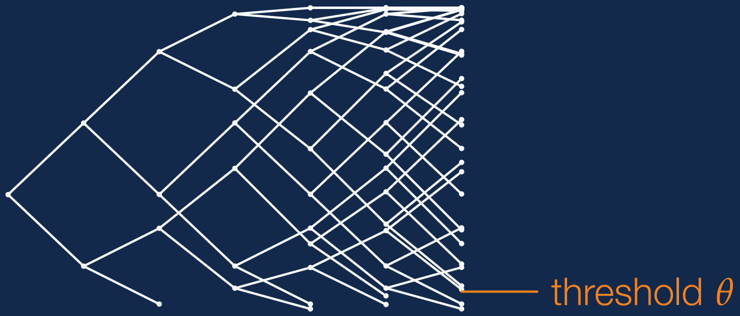


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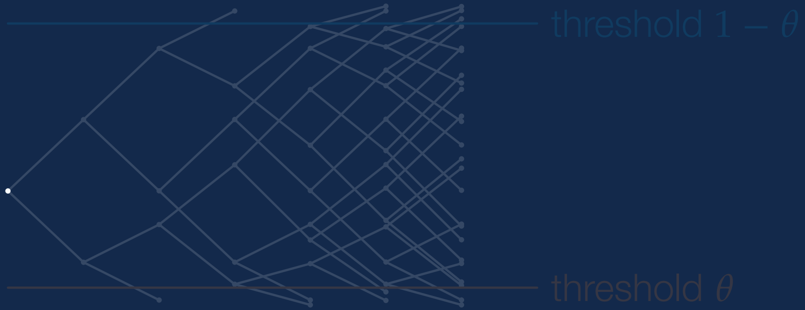


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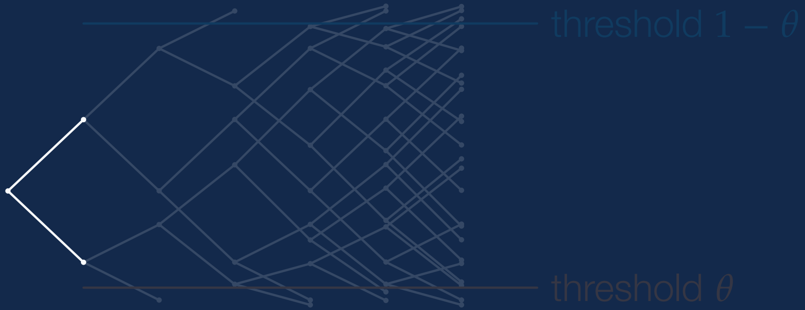


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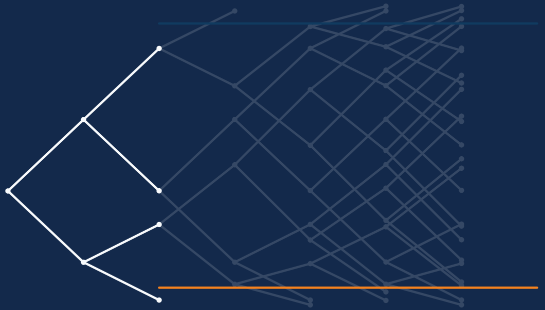


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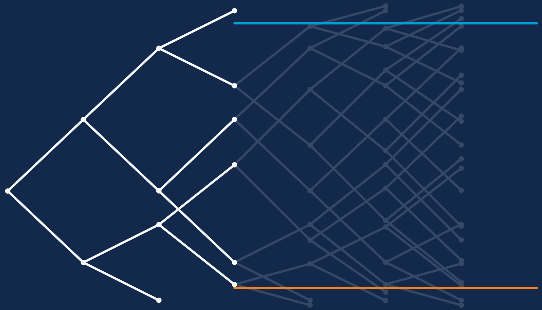


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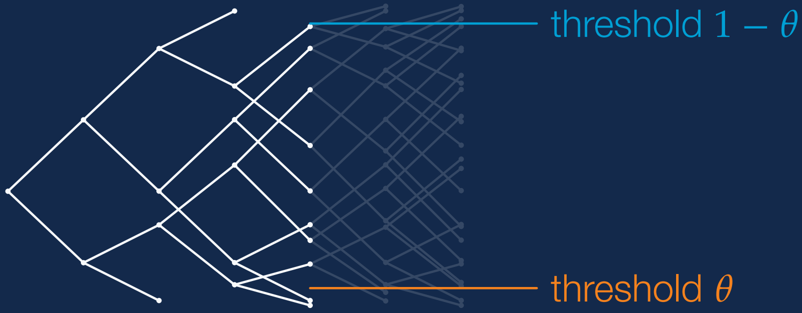


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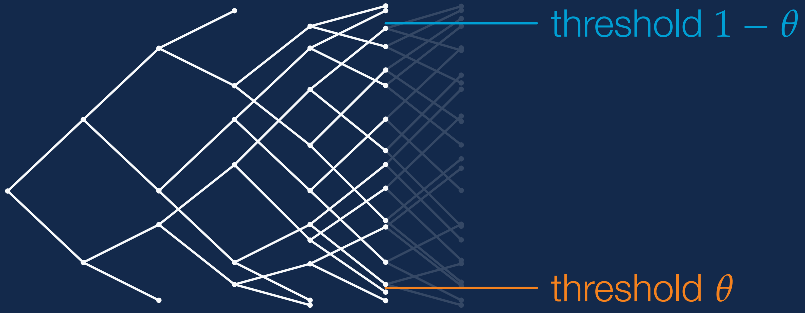


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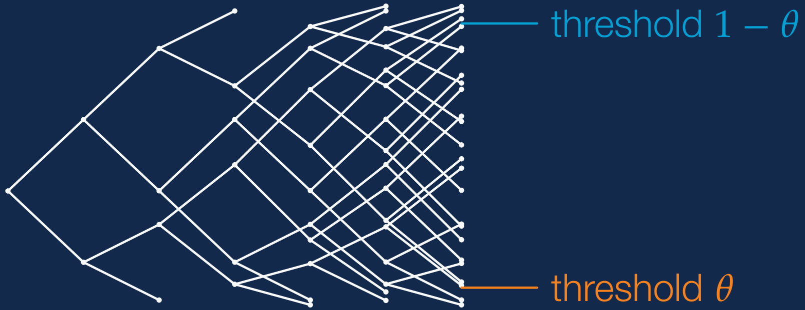


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# Stopping time analysis

$W_n$ has children/needs further transformation if $\theta < H_n < 1 - \theta$.

Set $\theta = N^{-10}$; assume $m > O(\log(\log N))$, then $e^{-2^{\pi m}} < \theta$.

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Then $P\{H_m < \theta\} > P\{H_m < e^{-2^{\pi m}}\} \geq H(W) - \ell^{-\rho m}$ and
$P\{H_m > 1 - \theta\} > P\{H_m > 1 - e^{-2^{\pi m}}\} \geq 1 - H(W) - \ell^{-\rho m}$

That is to say, $P\{\theta < H_m < 1 - \theta\} \leq 2\ell^{-\rho m}$, hard to stay in the middle.

# Geometric complexity

Complexity = #transformations = $\sum_{m=0}^{n} P\{\theta < H_m < 1-\theta\}$.

$$\begin{cases} \sum_{m=O(\log(\log N))}^{n} P\{\theta < H_m < 1-\theta\} \le \sum_{m=O(\log(\log N))}^{n} 2\ell^{-\rho m} = O(1), \\ \sum_{m=0}^{O(\log(\log N))} P\{\theta < H_m < 1-\theta\} \le \sum_{m=0}^{O(\log(\log N))} 1 = O(\log(\log N)). \end{cases}$$

Complexity is $O(\log(\log N))$ per bit, or $O(N\log(\log N))$ per block.

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# Summary of pruning and whatnot

There exist codes with complexity $O(\log(\log N))$ per bit, error probability $P_{\mathrm{e}} < N^{-9}$, and code rate $R = C - N^{-\rho}$.

(Earlier) we have codes with complexity $O(\log N)$ per bit, error probability $P_{\mathrm{e}} < e^{-N^{\pi}}$, and code rate $R > C - N^{-\rho}$.

Are there codes in between? Yes, continuously.

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# Summary

Log-log code taken from (with Duursma)
Log-logarithmic Time Pruned Polar Coding
https://ieeexplore.ieee.org/document/9274497.

MDP code taken from (with Duursma)
Polar Codes' Simplicity, Random Codes' Durability
https://ieeexplore.ieee.org/document/9274521.

# Question?

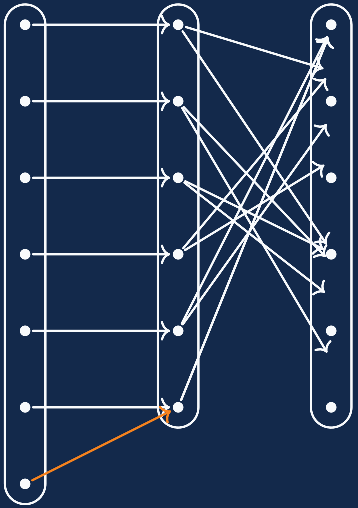
Predefined questions:
What does each chapter in dissertation do?
Why input alphabet is finite field? What is the advantage?
Definition of Bhattacharyya parameter?
References for XYZ?
Your contribution over others?
Future plan?

| Code | Error | Gap | Complexity | Channel |
|---|---|---|---|---|
| random | $e^{-N^\pi}$ | $N^{-\rho}$ | $\exp(N)$ | DMC |
| concatenation | $e^{-N^\pi}$ | $\to 0$ | $\operatorname{poly}(N)$ | DMC |
| RM | $\to 0$ | $\to 0$ | $O(N^2)$ | BEC |
| LDPC | $\to 0$ | $\to 0$ | unclear | SBDMC |
| RA family | $\to 0$ | $\to 0$ | $O(1)$ | BEC |
| old prune | $e^{-N^{1/2}}$ | $O(1)$ | $\Theta(\log N)$ | SBDMC |
| loglog-polar [W.] | $e^{-n^\tau}$ | $N^{-\rho}$ | $O(\log(\log N))$ | DMC |
| MDP-polar [W.] | $e^{-N^\pi}$ | $N^{-\rho}$ | $O(\log N)$ | DMC |

| | Symmetric | | | | | Asymmetric | |
|---|---|---|---|---|---|---|---|
| Par | BEC | SBDMC | p-ary | q-ary | finite | BDMC | a-finite |
| LLN | [Ari09] | [Ari09] | [ŞTA09] | [ŞTA09] | [ŞTA09] | [SRDR12] | [W.] |
| LDP$^\star$ | [AT09] | [AT09] | [ŞTA09] | [MT10] | [Sas11] | [HY13] | [W.] |
| CLT$^\star$ | [KMTU10] | [HAU14] | [BGN$^+$18] | [W.] | [W.] | [W.] | [W.] |
| MDP$^\star$ | [GX15] | [GX15] | [BGS18] | [W.] | [W.] | [W.] | [W.] |
| LDP | [KSU10] | [KSU10] | [W.] | [W.] | [W.] | [W.] | [W.] |
| CLT | [FHMV18] | [GRY20] | [W.] | [W.] | [W.] | [W.] | [W.] |
| MDP | [W.] | [W.] | [W.] | [W.] | [W.] | [W.] | [W.] |

# Input alphabet [new idea]

$$\begin{bmatrix} W(y_1|1) & W(y_2|1) & W(y_3|1) & \cdots \\ W(y_1|2) & W(y_2|2) & W(y_3|2) & \cdots \\ W(y_1|3) & W(y_2|3) & W(y_3|3) & \cdots \\ W(y_1|4) & W(y_2|4) & W(y_3|4) & \cdots \\ W(y_1|5) & W(y_2|5) & W(y_3|5) & \cdots \\ W(y_1|6) & W(y_2|6) & W(y_3|6) & \cdots \\ W(y_1|6) & W(y_2|6) & W(y_3|6) & \cdots \end{bmatrix}$$



# Asymmetric channels [HY13]

Recall $U_j$ is the coordinate as in $X_1^\ell := U_1^\ell \cdot G$.
The difficulty of asymmetric channels is $U_j$ being nonuniform and dependent.

Define synthetic channel $Q^{(i)} := (U_i \mid U_1^{i-1})$.
Define tree $Q^{(i)}, (Q^{(i)})^{(j)}, ((Q^{(i)})^{(j)})^{(k)}, \ldots$; define channel process $\{\boldsymbol{Q}_n\}$.
It polarizes, and at the same pace.

High $H(\boldsymbol{Q}_n)$ low $H(\boldsymbol{W}_n)$ vs both high vs both low.

# Bhattacharyya parameter

Binary $Z(W) := \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}$.

Non-binary $:= \frac{1}{q-1} \sum_{\substack{x,x' \in \mathbb{F}_q \\ x \neq x'}} \sum_{y \in \mathcal{Y}} \sqrt{W(x,y)W(x',y)}$.

[New idea] $:= \max_{0 \neq d \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_q} \sum_{y \in \mathcal{Y}} \sqrt{W(x,y)W(x+d,y)}$.

# A List of Important Contributions (Chronological)

A combinatorial trick to recover scaling exponent (een13 → elpin).

The pruning technique/stopping time analysis/log-log complexity.

Improved combinatorial trick (en23 → een13 → elpin).

Dynamic kernel, later random dynamic kernel.

Alphabet reduction to finite field (trivial but powerful and last mile).

Improve definition of Bhattacharyya parameter, then and FTPCZ.

Reducing local CLT to noisy-channel and wiretap-channel coding.

For wiretap bound, extend the universal bound via continuation.

A topological argument to show positive $\varrho$ (that is, CLT$^\star$).

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