Slides available on

https://polar-tutorial.symbol.codes/

Polar Codes Tutorial

Hsin-Po Wang

(Berkeley, EECS)


ChatGPT

Engineering

List decoder

Future

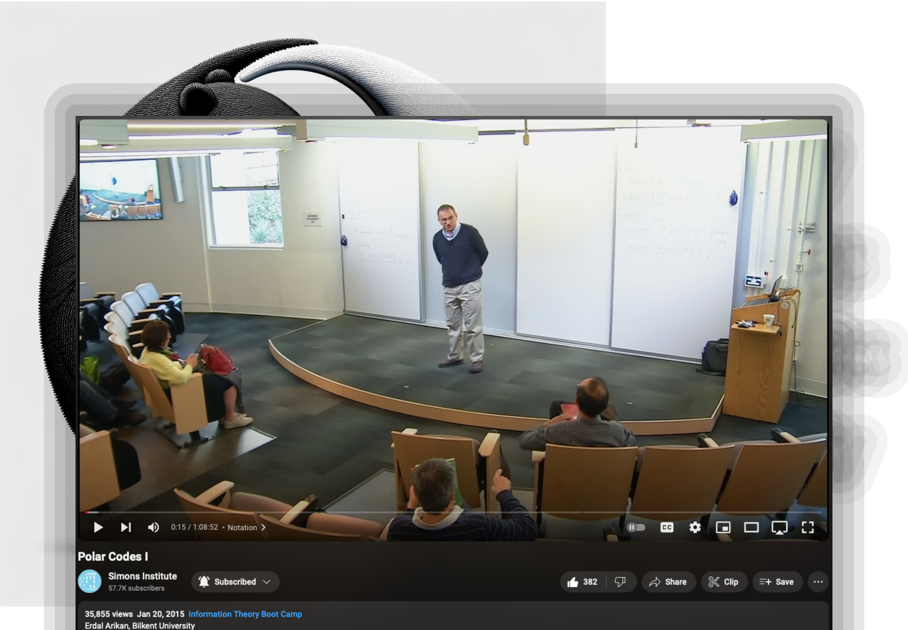
Origin Story

Engineering

List decoder

Future

Origin Story

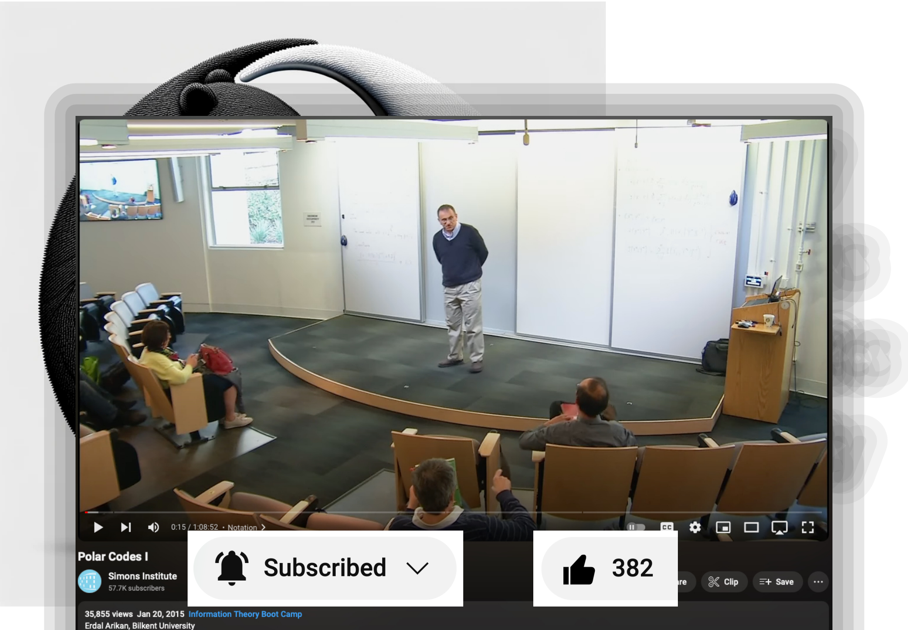


Engineering

List decoder

Future

Origin Story



Engineering

List decoder

Future

Origin Story

Engineering

List decoder

Future

Origin Story

Theory

Math args

Asymptotics

Versatility

What’s so innovative about polar codes?


WHO IS CLAUDE SHANNON?
ENGINEER
INVENTOR
CRYPTOGRAPHER
MATHEMATICIAN
JUGGLER
UNICYCLIST
FATHER OF THE INFORMATION AGE
THE BIT PLAYER
A FILM BY MARK A. LEVINSON
THE IEEE INFORMATION THEORY SOCIETY PRESENTS A MARK A. LEVINSON FILM THE BIT PLAYER JOHN HUTTON JUDITH IVEY KALISWA BREWSTER
EXECUTIVE PRODUCERS MICHELLE EFFROS CHRISTINA FRAGOULI ALON ORLITSKY RUEDIGER URBANKE
CREATIVE PRODUCER SERGIO VERDÚ PRODUCTION DESIGN JEREMY WOODWARD YUKO SOBRIN
COSTUME DESIGN KATJA ANDREIEV JEN TREMBLY GRAPHIC DESIGN AND ANIMATION KIRART ORIGINAL MUSIC ROBERT MILLER
EDITOR TIM STERNBERG DIRECTOR OF PHOTOGRAPHY CLAUDIA RASCHKE WRITTEN, PRODUCED AND DIRECTED BY MARK A. LEVINSON
IMDb

IMDb


ChatGPT

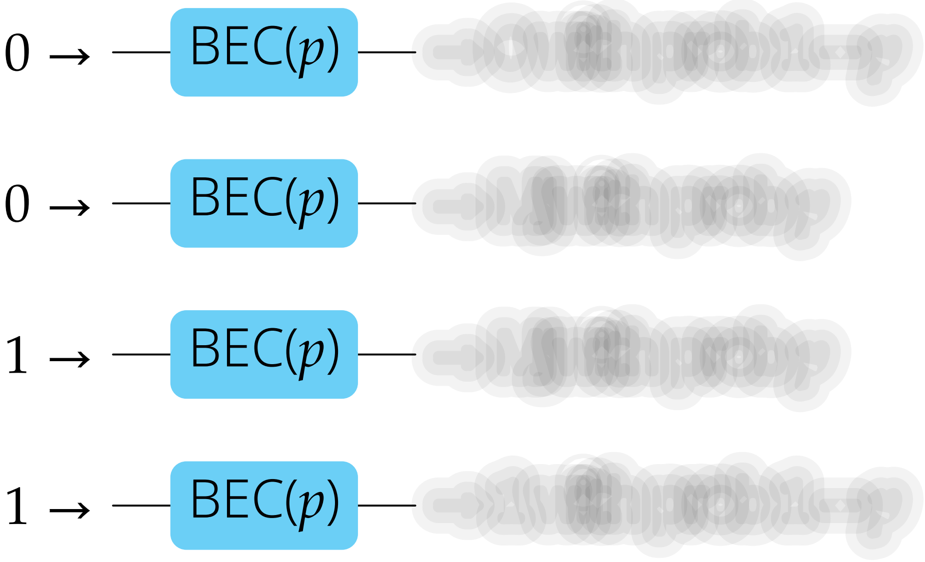
IMDb

ChatGPT

Let’s play with

binary erasure channels

(BECs)


0 →
BEC(p)
0 →
BEC(p)
1 →
BEC(p)
1 →
BEC(p)

$0 \to$ BEC(p) $\to$ 0 with prob $1 - p$

$0 \to$ BEC(p) $\to$ 🤷 with prob p

$1 \to$ BEC(p)

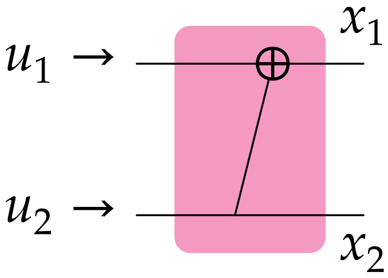
$1 \to$ BEC(p)

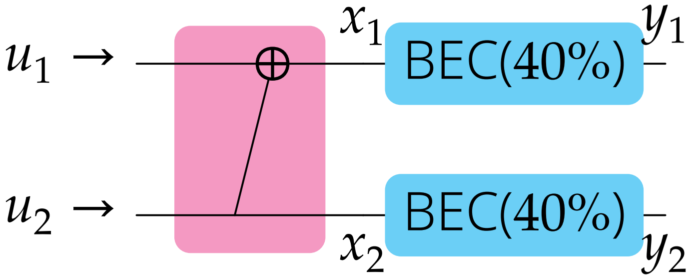
$0 \to$ BEC(p) $\to$ 0 with prob $1-p$

$0 \to$ BEC(p) $\to$ 🤷‍♂️with prob p

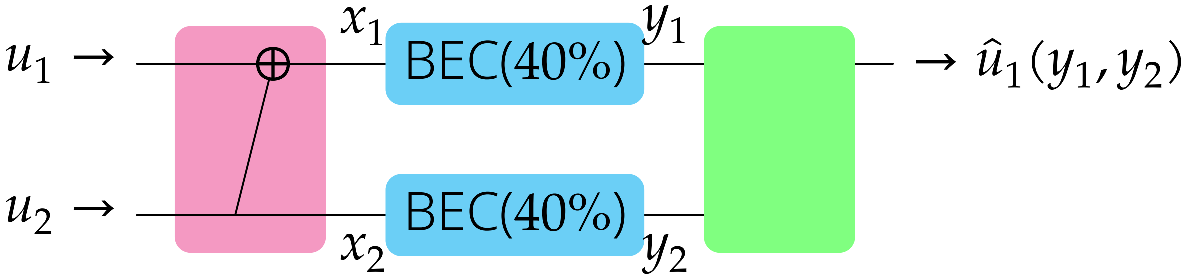
$1 \to$ BEC(p) $\to$ 🤷‍♂️with prob p

$1 \to$ BEC(p) $\to$ 1 with prob $1-p$

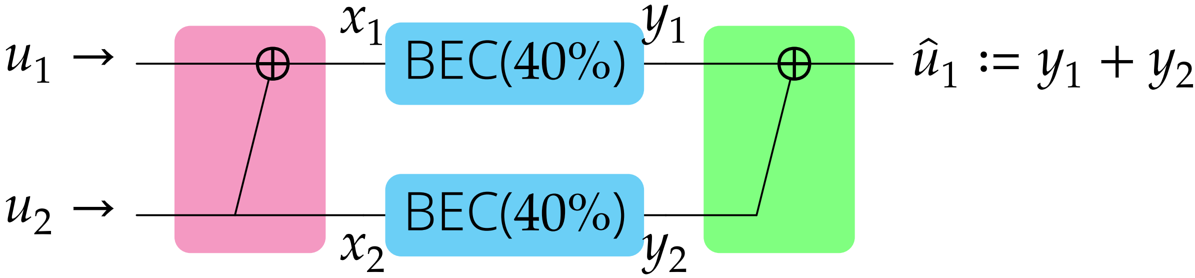

$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2

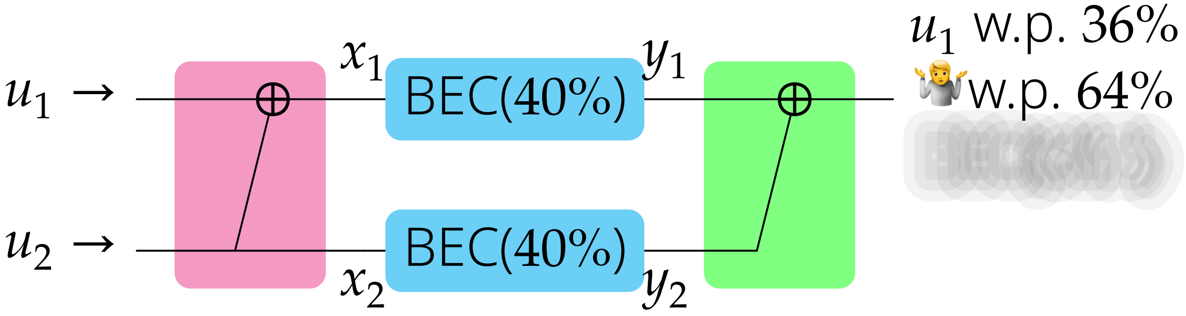


$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2

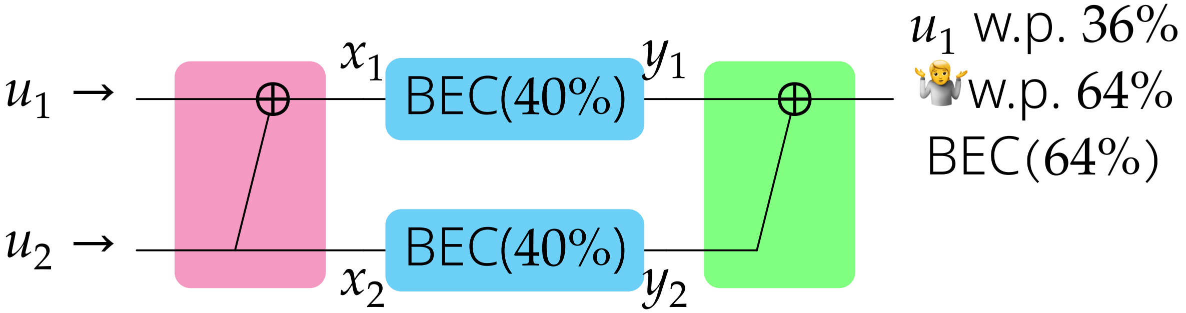


$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\rightarrow \hat{u}_1(y_1, y_2)$


$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\hat{u}_1 := y_1 + y_2$

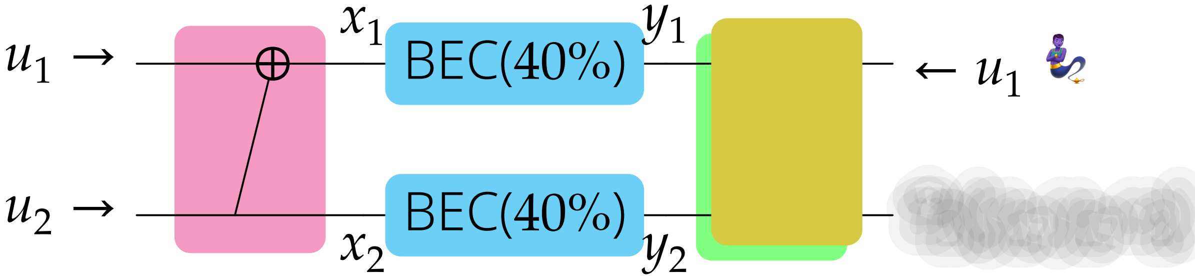


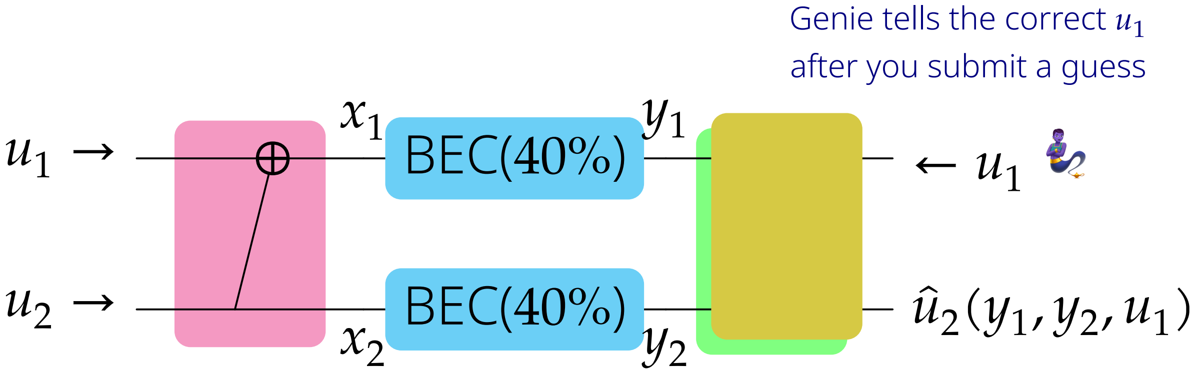
$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
u_1 w.p. 36%
🤷 w.p. 64%

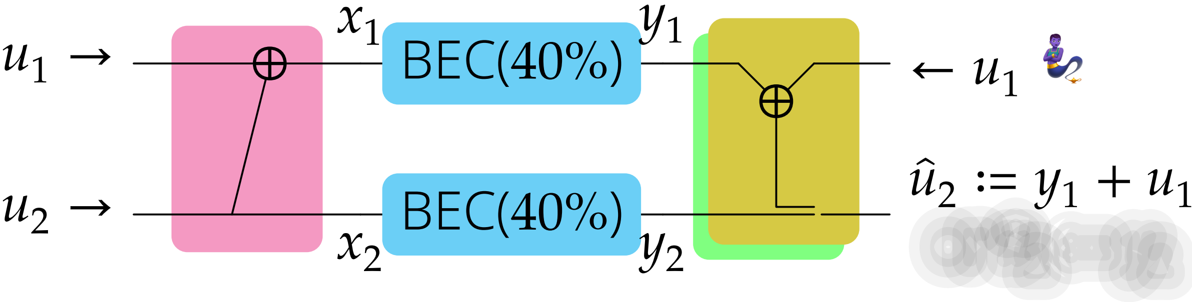


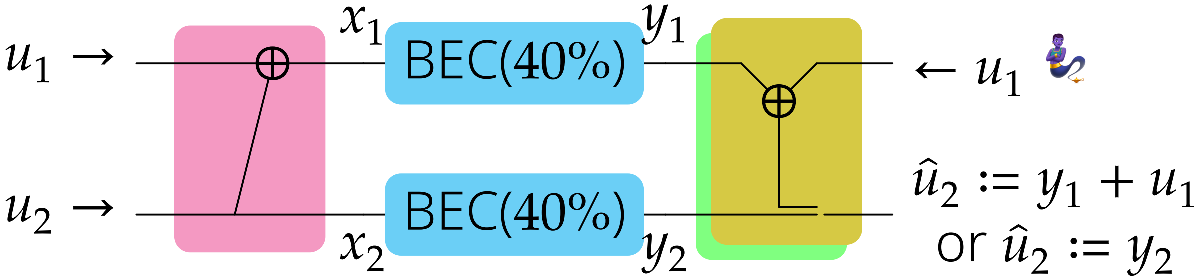
$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
u_1 w.p. 36%
🤷 w.p. 64%
BEC(64%)

Genie tells the correct u_1 after you submit a guess

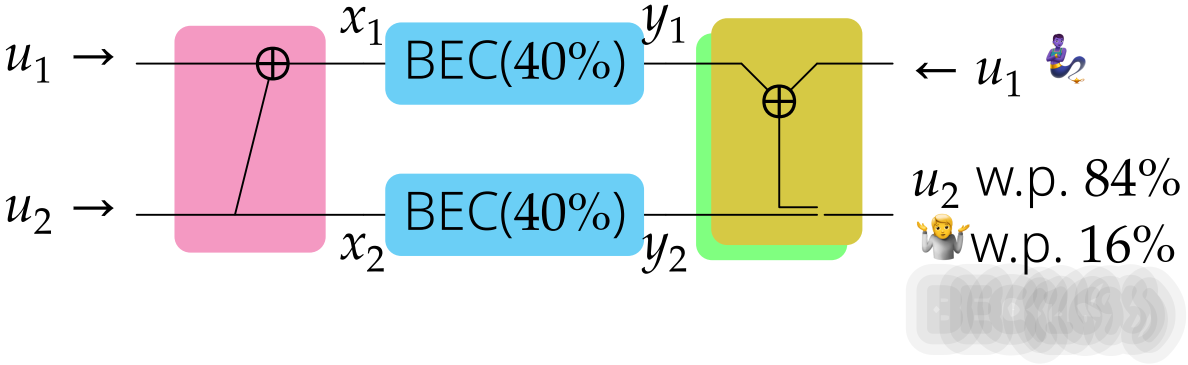



Genie tells the correct u_1
after you submit a guess
$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
$\hat{u}_2(y_1, y_2, u_1)$

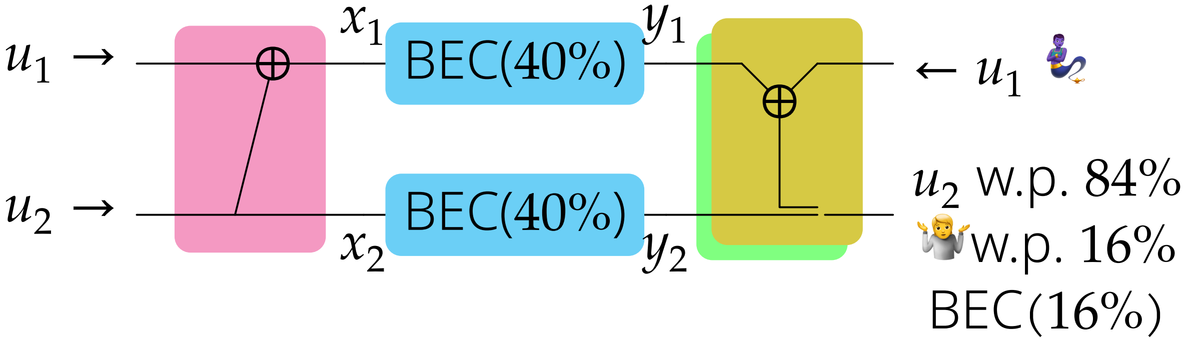

$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
$\hat{u}_2 := y_1 + u_1$

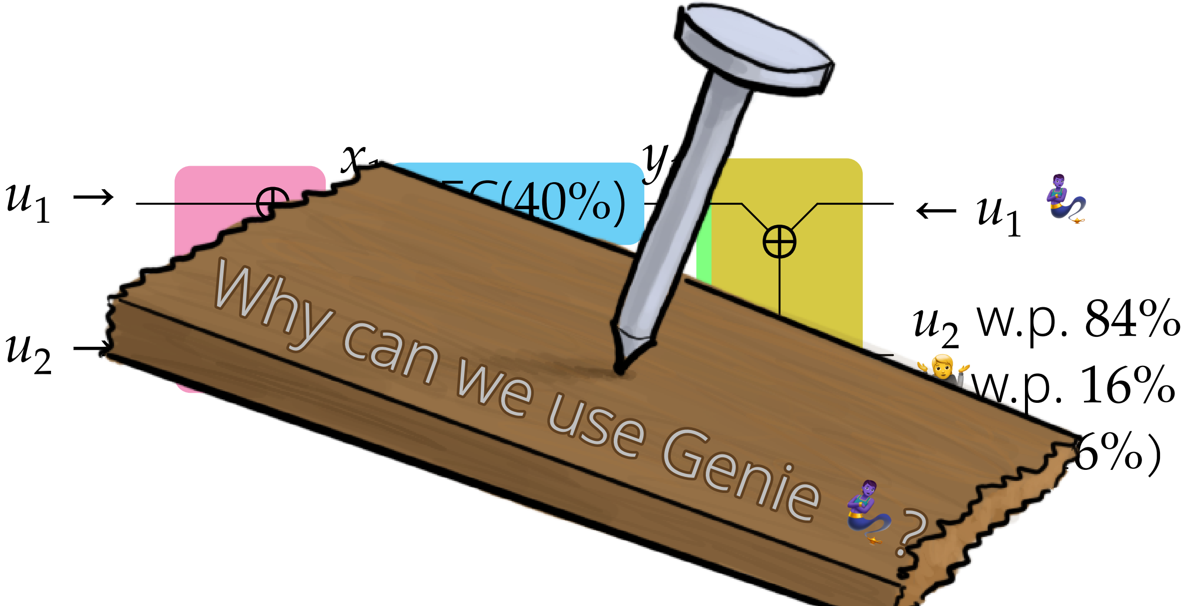


$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
$\hat{u}_2 := y_1 + u_1$
or $\hat{u}_2 := y_2$



$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
u_2 w.p. 84%
w.p. 16%


$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
u_2 w.p. 84%
w.p. 16%
BEC(16%)



$u_1 \rightarrow$
$u_2 \rightarrow$
Why can we use Genie 🧞?
$\leftarrow u_1$ 🧞
u_2 w.p. 84%
🤷 w.p. 16%

- Case 1: $\hat{u}_1 = u_1$
- Case 2:
- Case 3:

- Case 1: $\hat{u}_1 = u_1$
  🧞 does not provide any useful info
- Case 2:
- Case 3:

- Case 1: $\hat{u}_1 = u_1$
  🧞 does not provide any useful info
- Case 2: $\hat{u}_1 \neq u_1$
- Case 3:

- Case 1: $\hat{u}_1 = u_1$
  🧞 does not provide any useful info
- Case 2: $\hat{u}_1 \neq u_1$
  Give up the whole block
- Case 3: [illegible]
  [illegible]
  [illegible]

- Case 1: $\hat{u}_1 = u_1$
  🧞 does not provide any useful info
- Case 2: $\hat{u}_1 \neq u_1$
  Give up the whole block
- Case 3: sender always sends $u_1 \equiv 0$

- Case 1: $\hat{u}_1 = u_1$
  🧞does not provide any useful info

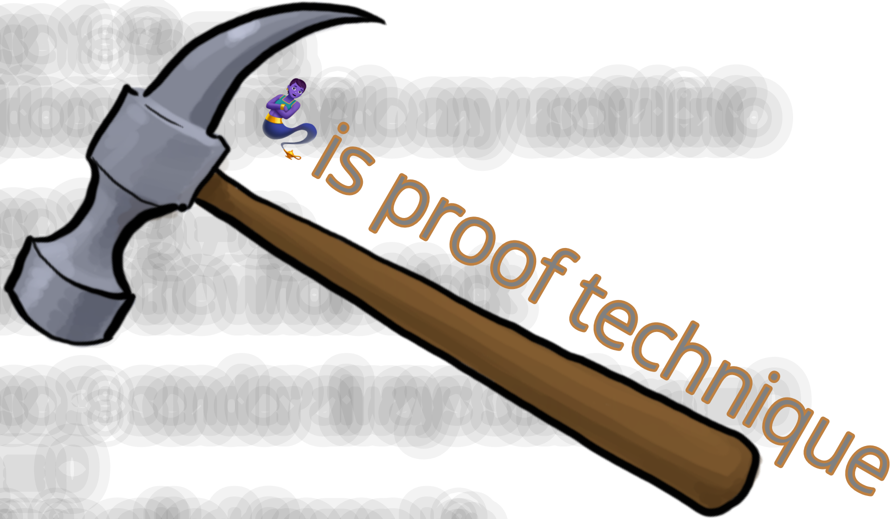
- Case 2: $\hat{u}_1 \neq u_1$
  Give up the whole block

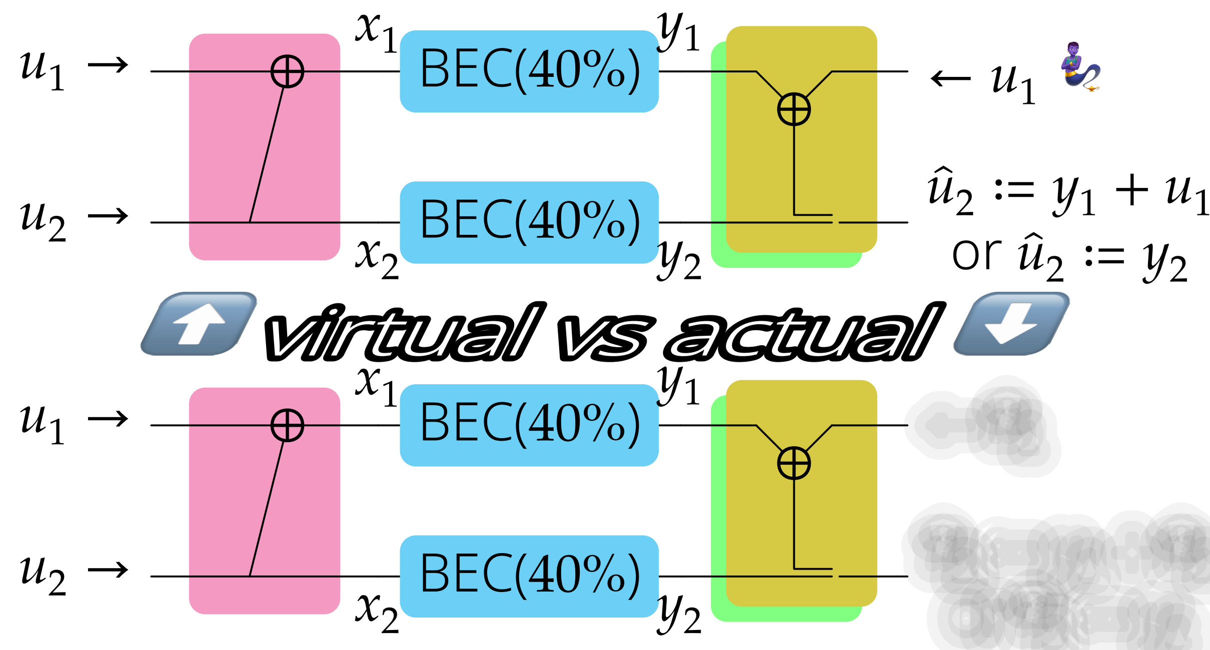
- Case 3: sender always sends $u_1 \equiv 0$
  $\hat{u}_1 \equiv 0$

- Case 1: $\hat{u}_1 = u_1$
  🧞 does not provide any useful info

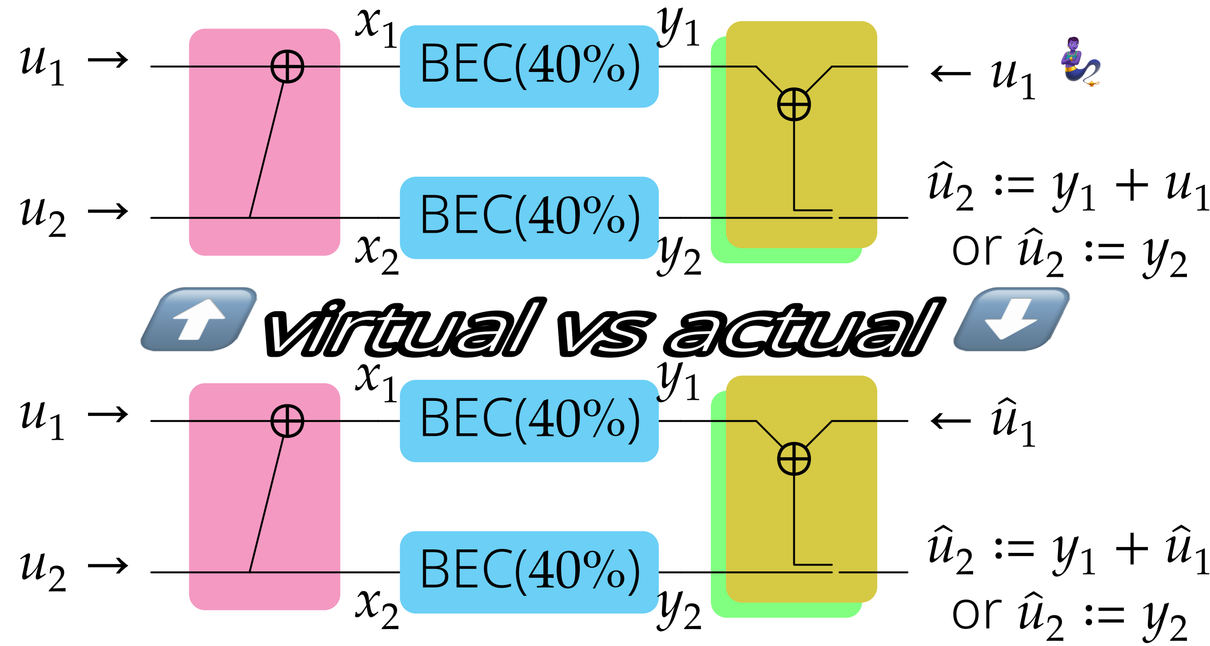
- Case 2: $\hat{u}_1 \neq u_1$
  Give up the whole block

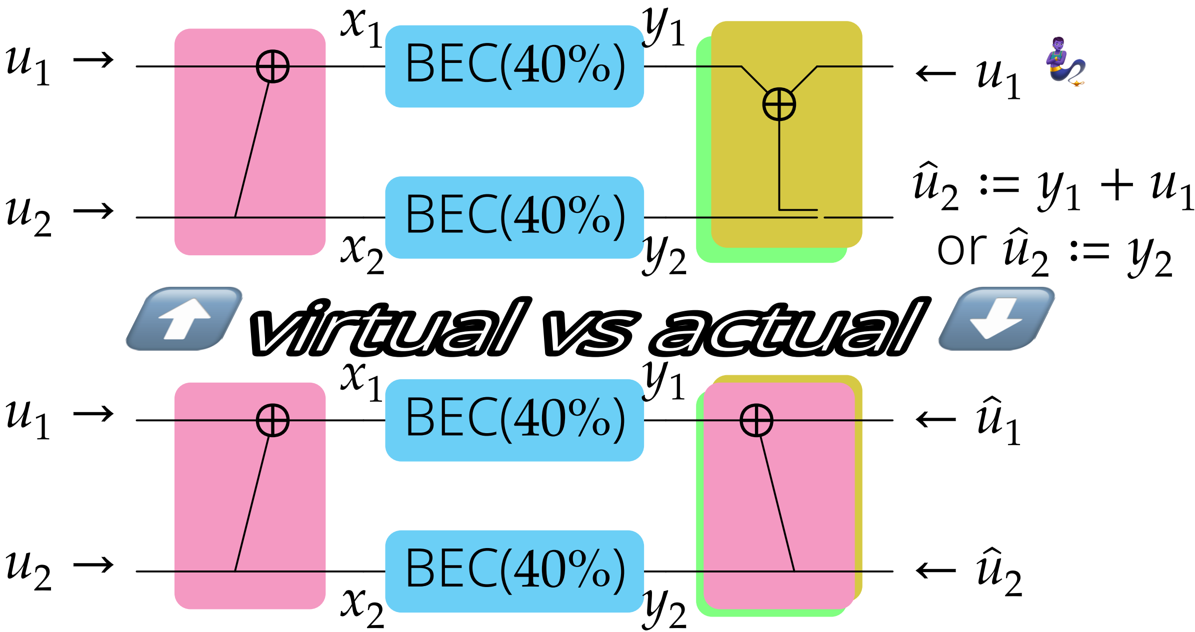
- Case 3: sender always sends $u_1 \equiv 0$
  $\hat{u}_1 \equiv 0$
  This is called **frozen bit**


is proof technique

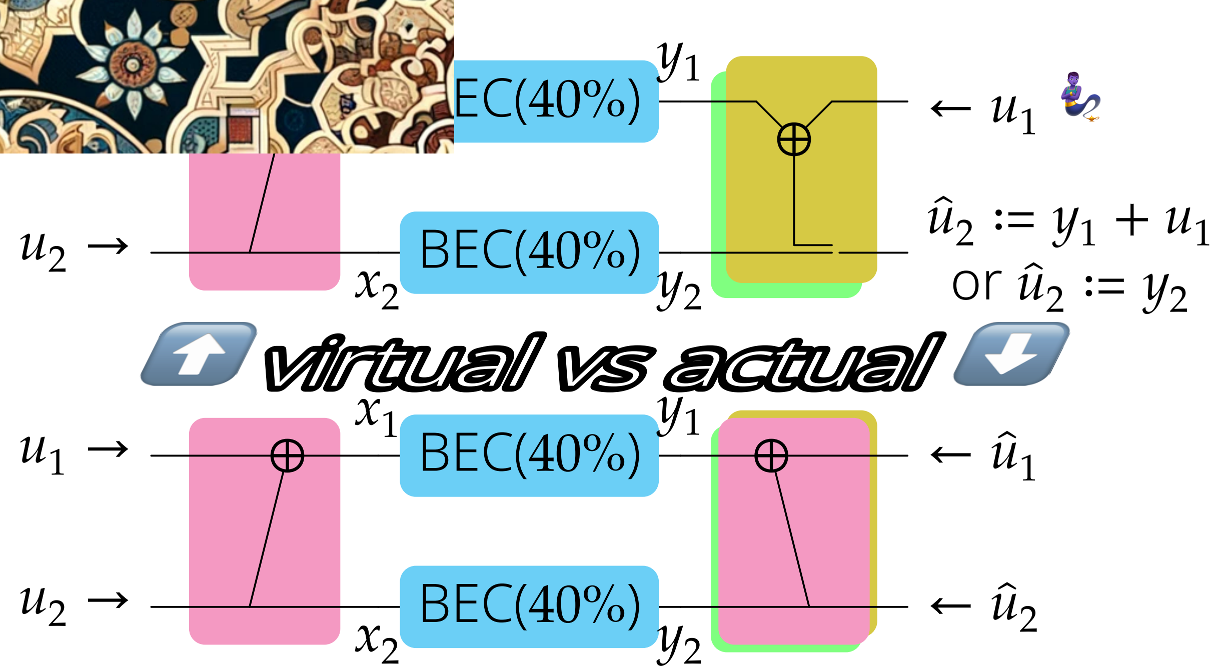


$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
$\hat{u}_2 := y_1 + u_1$
or $\hat{u}_2 := y_2$
virtual vs actual
$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2

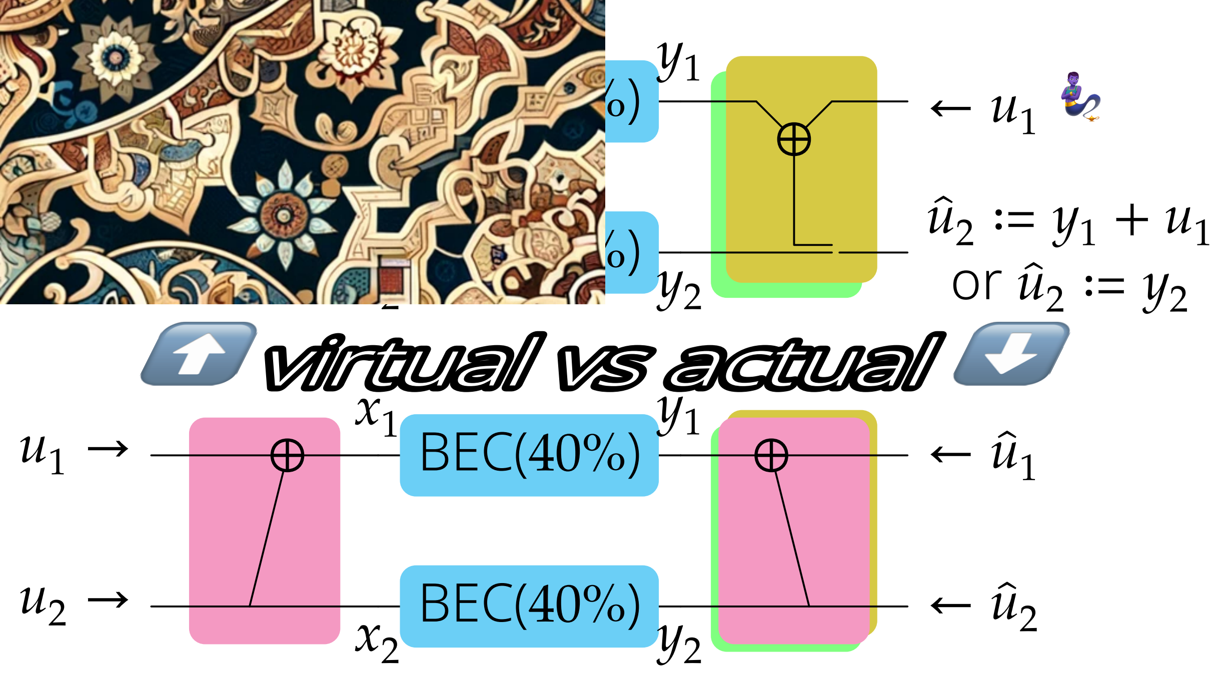

$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
$\hat{u}_2 := y_1 + u_1$
or $\hat{u}_2 := y_2$
virtual vs actual
$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow \hat{u}_1$
$\hat{u}_2 := y_1 + \hat{u}_1$
or $\hat{u}_2 := y_2$



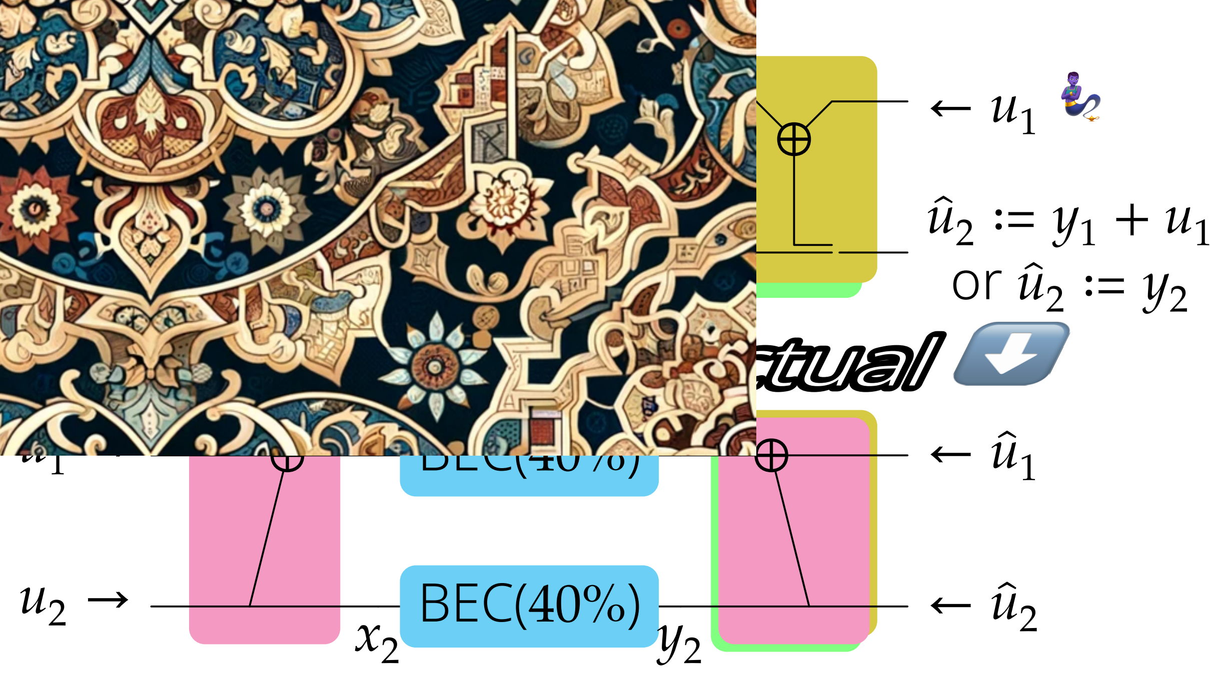
$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
$\hat{u}_2 := y_1 + u_1$
or $\hat{u}_2 := y_2$
virtual vs actual
$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow \hat{u}_1$
$\leftarrow \hat{u}_2$



EC(40%)
y_1
$\leftarrow u_1$
$u_2 \rightarrow$
BEC(40%)
x_2
y_2
$\hat{u}_2 := y_1 + u_1$
or $\hat{u}_2 := y_2$
virtual vs actual
x_1
y_1
$u_1 \rightarrow$
BEC(40%)
$\leftarrow \hat{u}_1$
$u_2 \rightarrow$
BEC(40%)
x_2
y_2
$\leftarrow \hat{u}_2$



y_1
$\leftarrow u_1$
$\hat{u}_2 := y_1 + u_1$
or $\hat{u}_2 := y_2$
y_2
virtual vs actual
$u_1 \rightarrow$
x_1
BEC(40%)
y_1
$\leftarrow \hat{u}_1$
$u_2 \rightarrow$
x_2
BEC(40%)
y_2
$\leftarrow \hat{u}_2$



$\leftarrow u_1$
$\hat{u}_2 := y_1 + u_1$
or $\hat{u}_2 := y_2$
ctual
$\leftarrow \hat{u}_1$
$u_2 \rightarrow$
x_2
BEC(40%)
y_2
$\leftarrow \hat{u}_2$

$\leftarrow u_1$

$\hat{u}_2 := y_1 + u_1$
or $\hat{u}_2 := y_2$

$\leftarrow \hat{u}_1$

$\leftarrow \hat{u}_2$

x_2 y_2

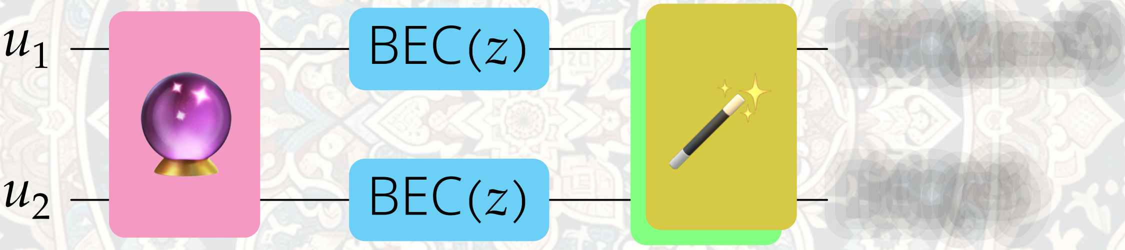
Don't click 💣

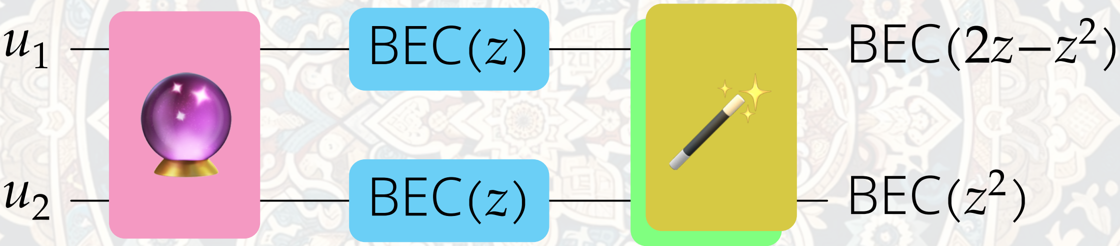
🧞

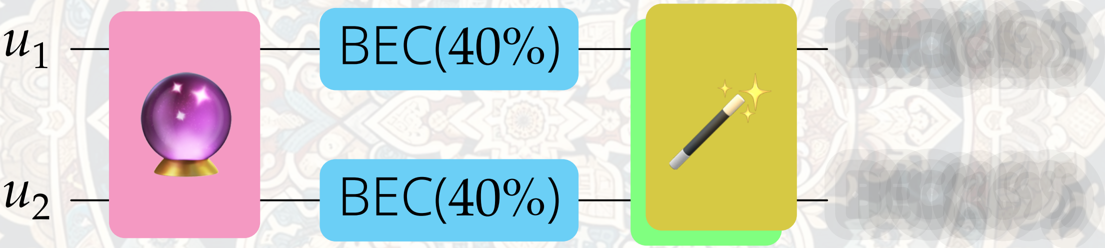
$y_1 + u_1$

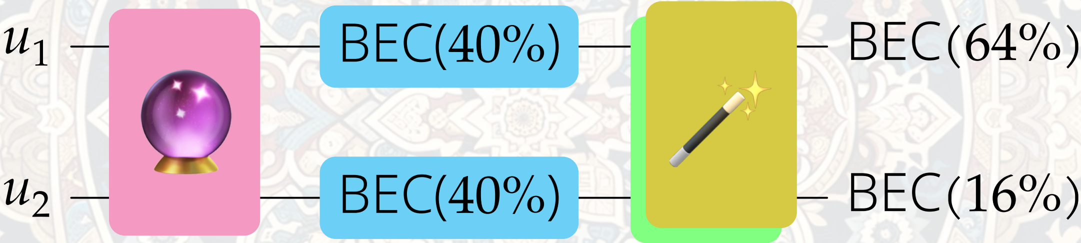
$:= y_2$


ChatGPT

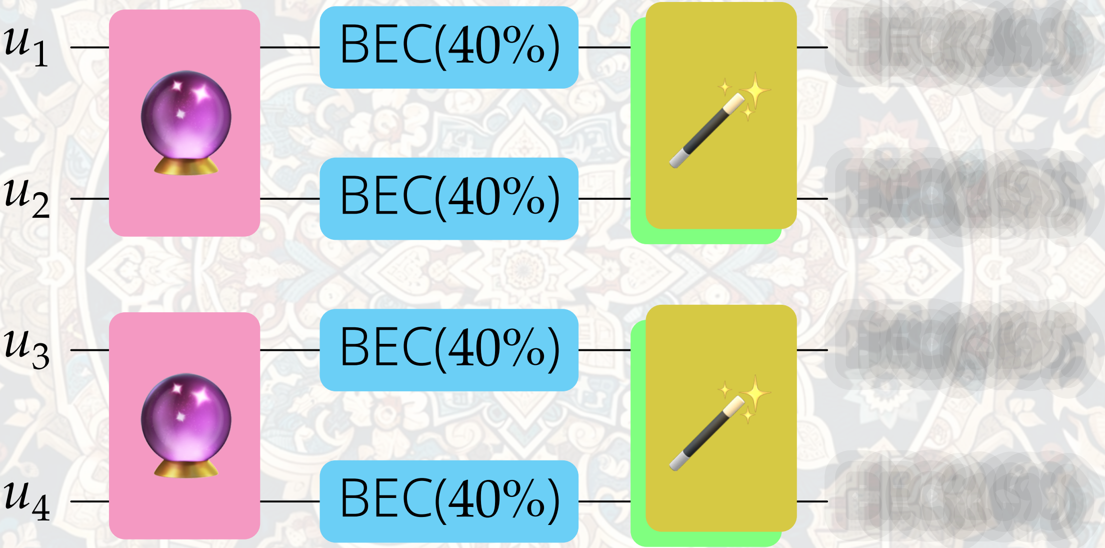

u_1
u_2
BEC(z)
BEC(z)


u_1
u_2
$\mathrm{BEC}(z)$
$\mathrm{BEC}(z)$
$\mathrm{BEC}(2z - z^2)$
$\mathrm{BEC}(z^2)$

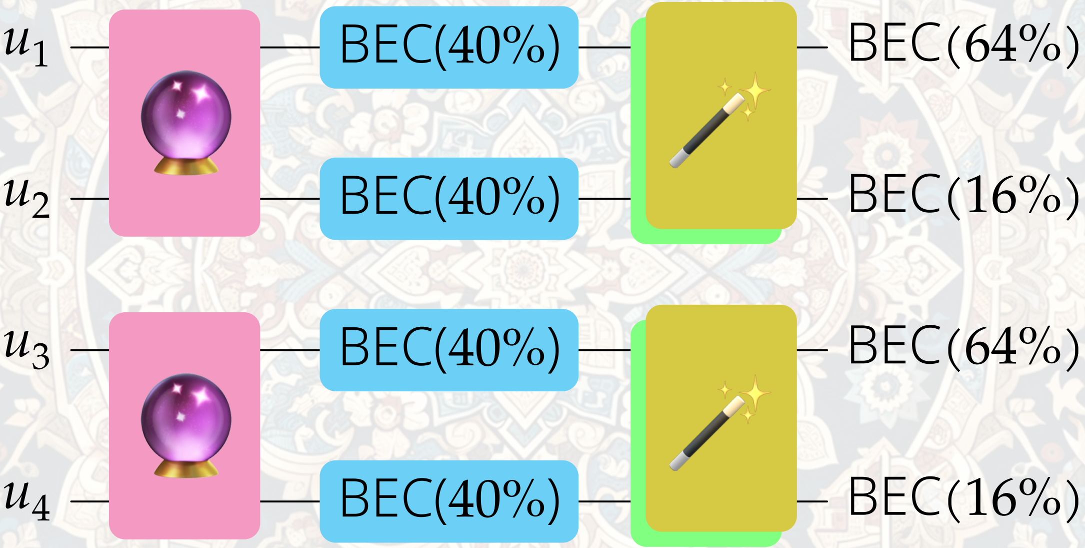

u_1
u_2
BEC(40%)
BEC(40%)

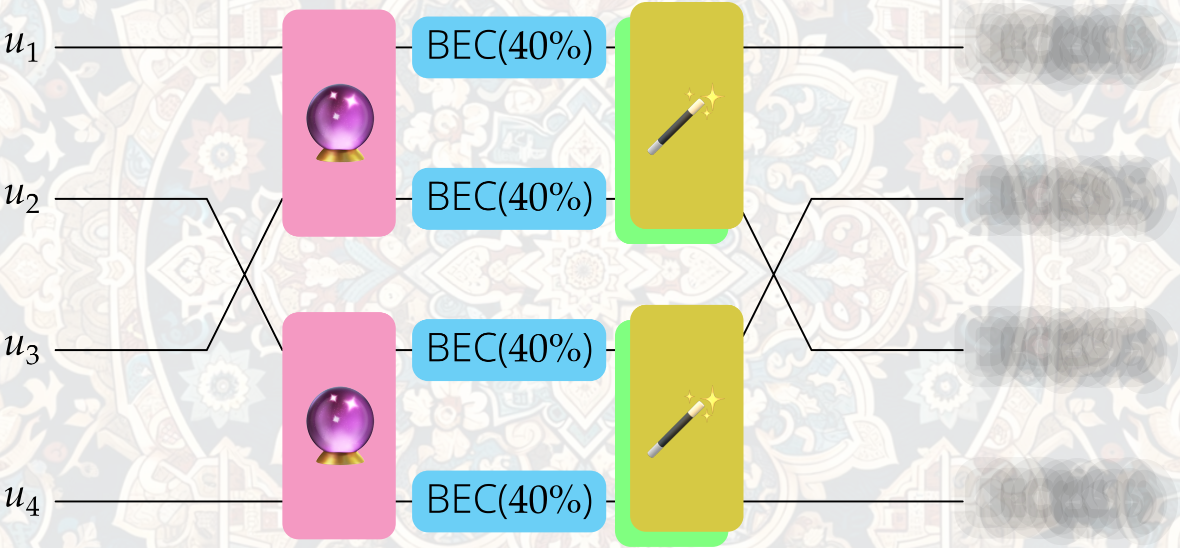


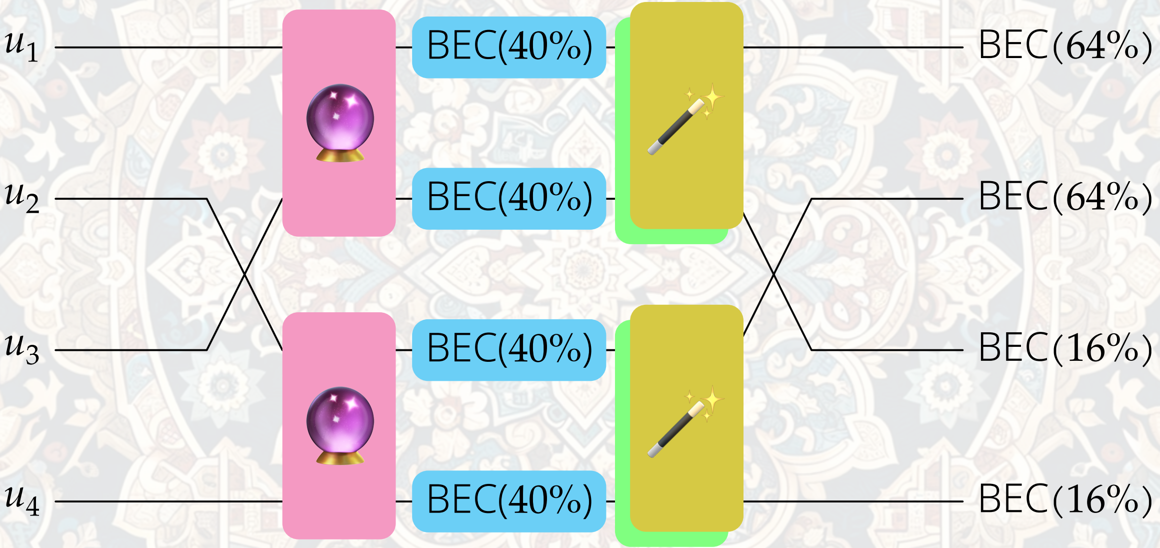
u_1
u_2
BEC(40%)
BEC(40%)
BEC(64%)
BEC(16%)

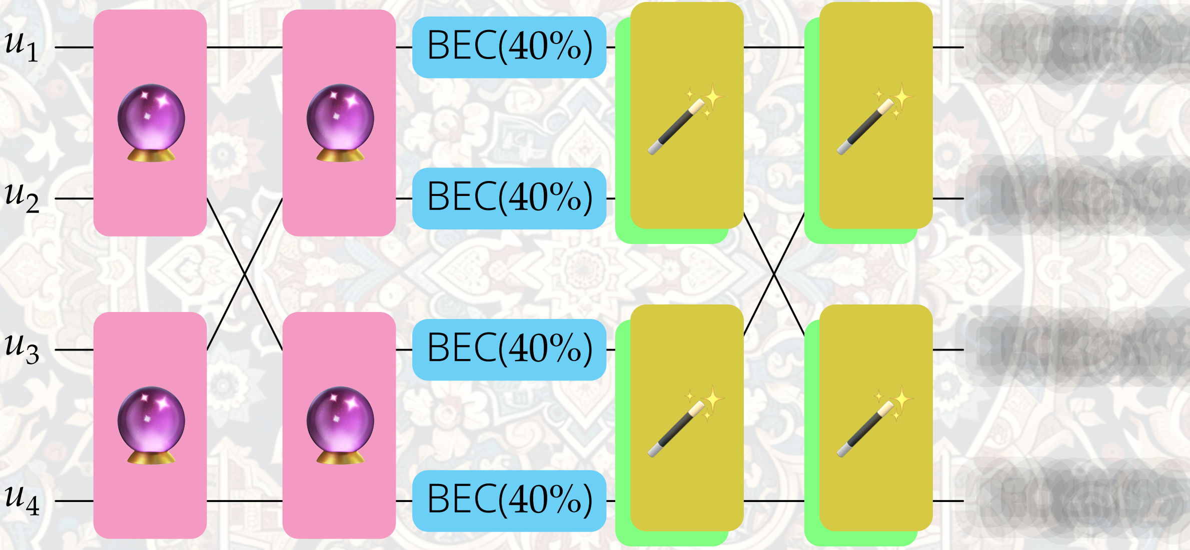


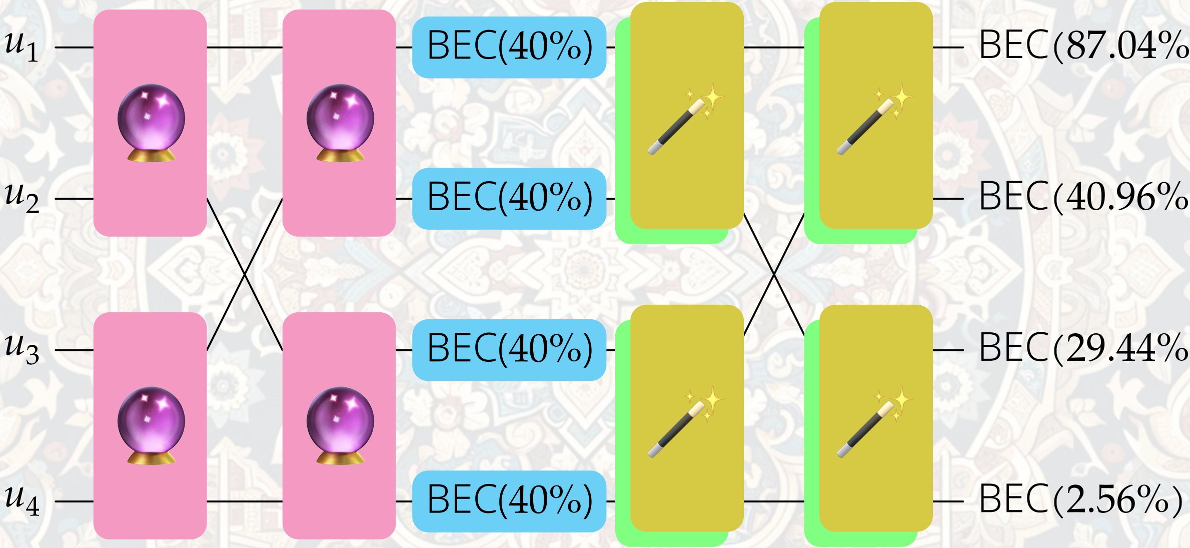
u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)

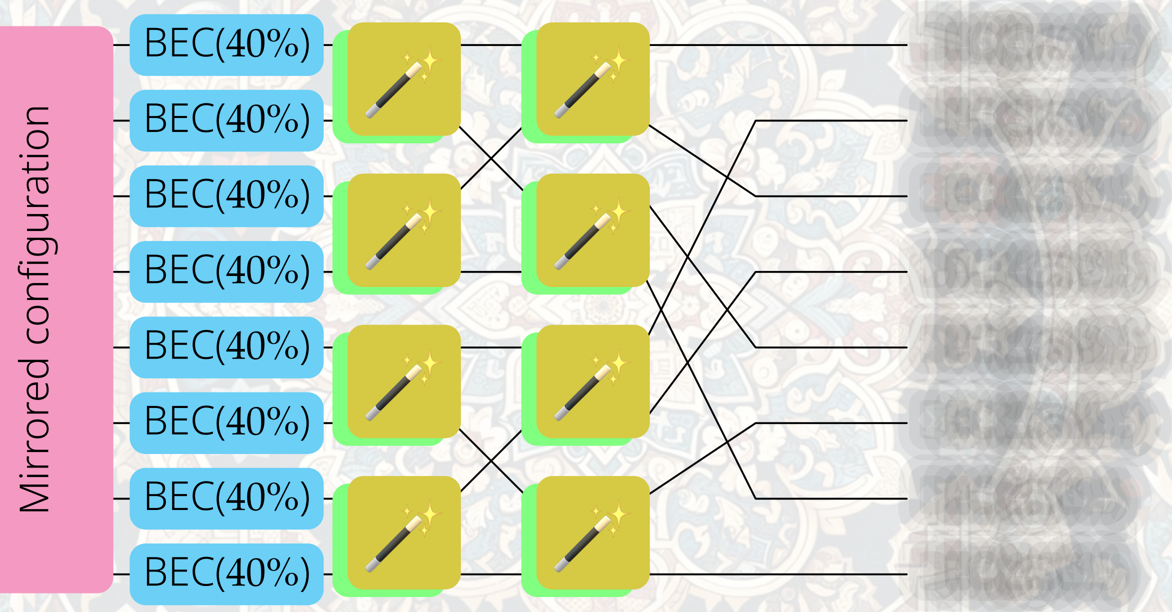

u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(64%)
BEC(16%)
BEC(64%)
BEC(16%)

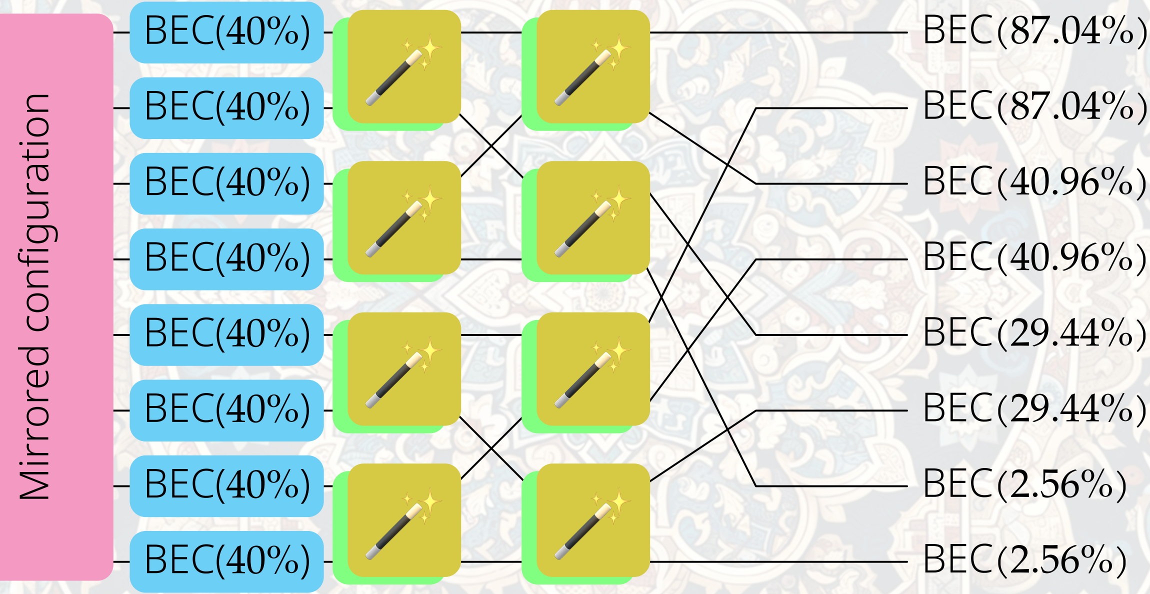

u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)

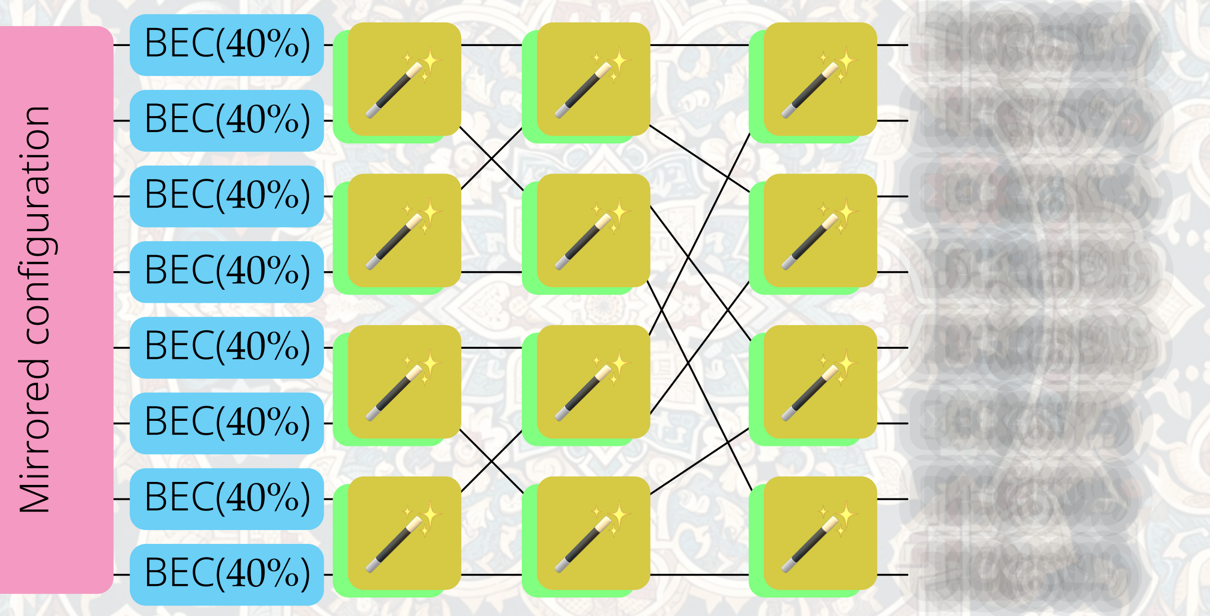

u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(64%)
BEC(64%)
BEC(16%)
BEC(16%)

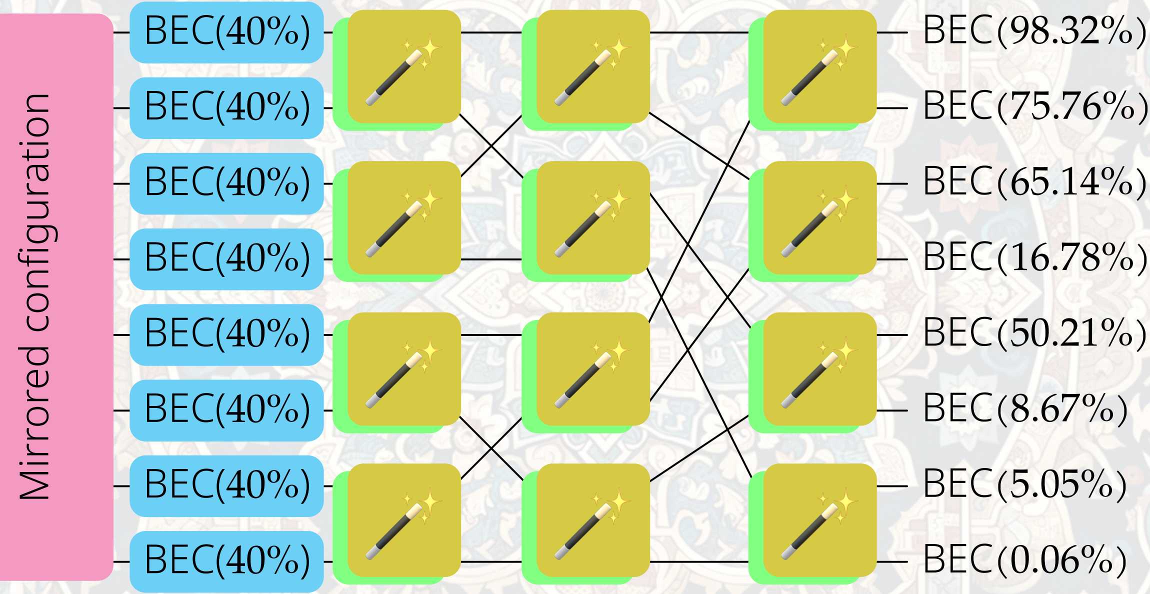

u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)

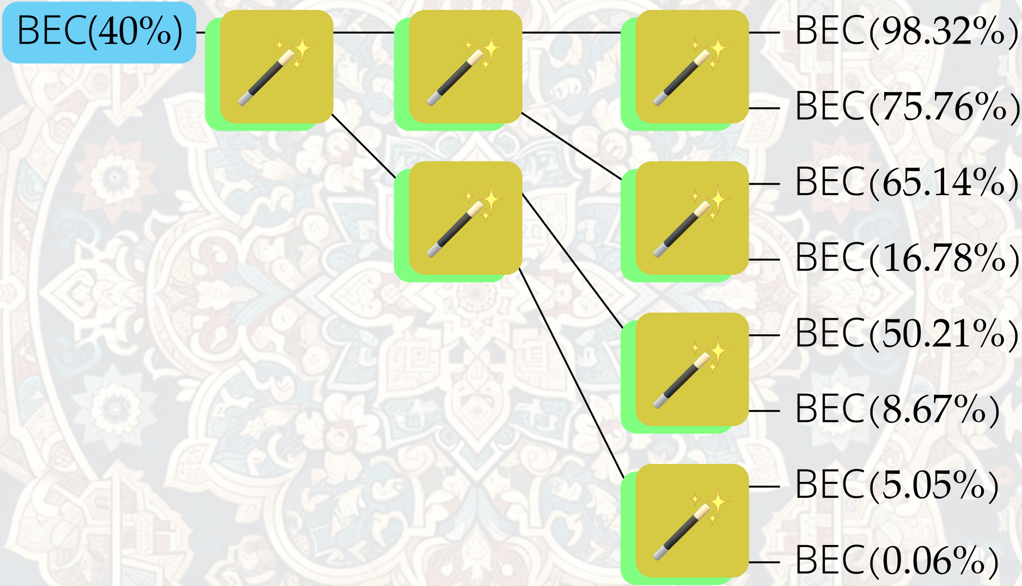

u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(87.04%
BEC(40.96%
BEC(29.44%
BEC(2.56%)


Mirrored configuration
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)


Mirrored configuration
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(87.04%)
BEC(87.04%)
BEC(40.96%)
BEC(40.96%)
BEC(29.44%)
BEC(29.44%)
BEC(2.56%)
BEC(2.56%)


Mirrored configuration
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)

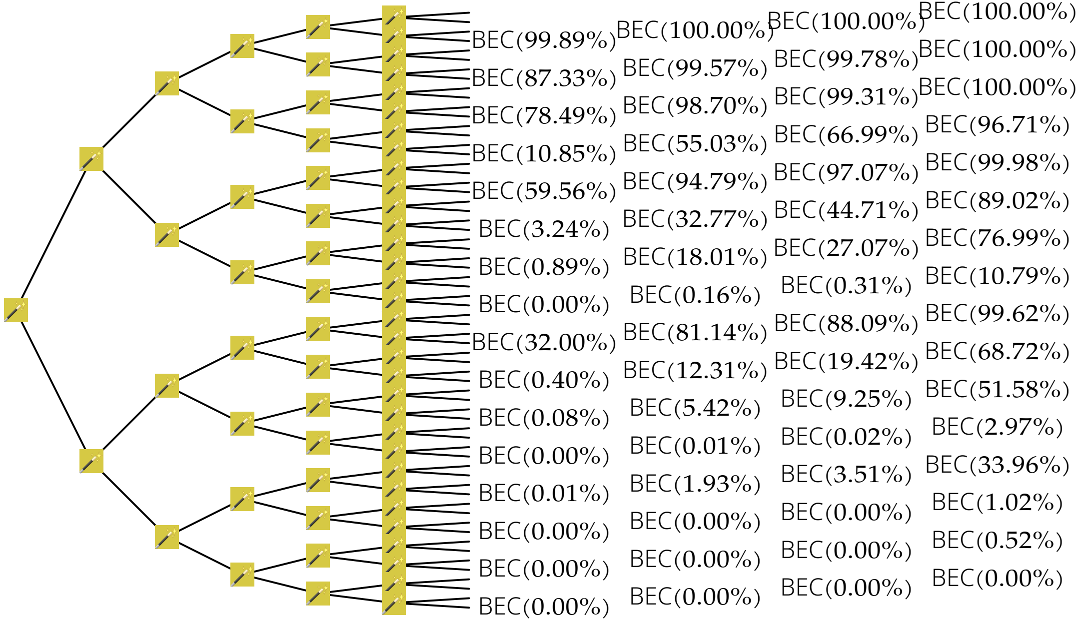

Mirrored configuration
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(98.32%)
BEC(75.76%)
BEC(65.14%)
BEC(16.78%)
BEC(50.21%)
BEC(8.67%)
BEC(5.05%)
BEC(0.06%)

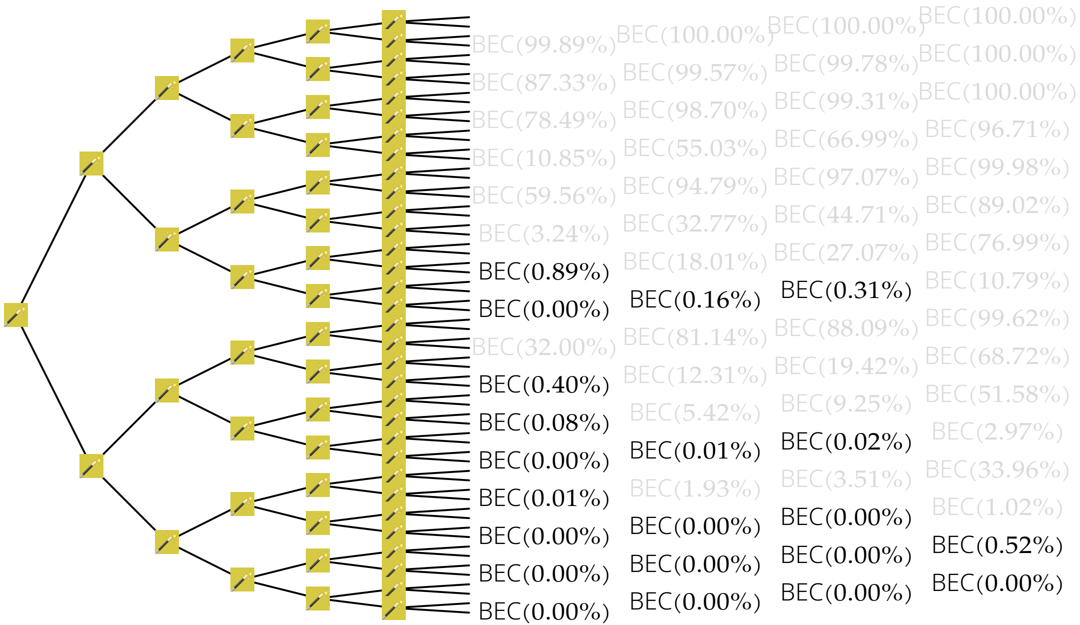

BEC(40%)
BEC(98.32%)
BEC(75.76%)
BEC(65.14%)
BEC(16.78%)
BEC(50.21%)
BEC(8.67%)
BEC(5.05%)
BEC(0.06%)

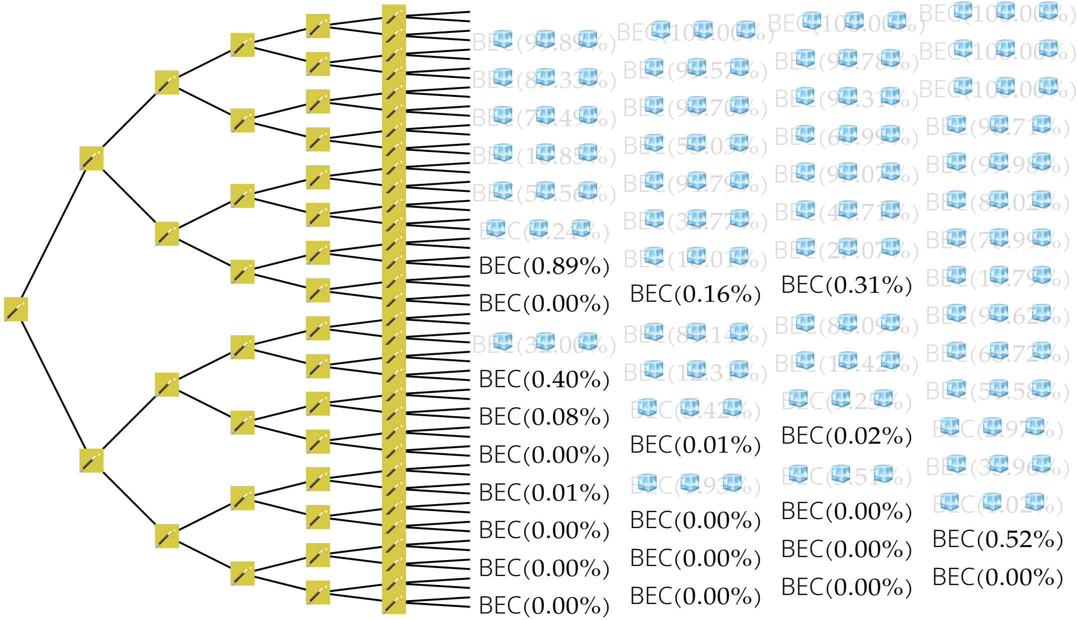
Recap

Recap

Recap


BEC(99.89%)
BEC(87.33%)
BEC(78.49%)
BEC(10.85%)
BEC(59.56%)
BEC(3.24%)
BEC(0.89%)
BEC(0.00%)
BEC(32.00%)
BEC(0.40%)
BEC(0.08%)
BEC(0.00%)
BEC(0.01%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(99.57%)
BEC(98.70%)
BEC(55.03%)
BEC(94.79%)
BEC(32.77%)
BEC(18.01%)
BEC(0.16%)
BEC(81.14%)
BEC(12.31%)
BEC(5.42%)
BEC(0.01%)
BEC(1.93%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(99.78%)
BEC(99.31%)
BEC(66.99%)
BEC(97.07%)
BEC(44.71%)
BEC(27.07%)
BEC(0.31%)
BEC(88.09%)
BEC(19.42%)
BEC(9.25%)
BEC(0.02%)
BEC(3.51%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(96.71%)
BEC(99.98%)
BEC(89.02%)
BEC(76.99%)
BEC(10.79%)
BEC(99.62%)
BEC(68.72%)
BEC(51.58%)
BEC(2.97%)
BEC(33.96%)
BEC(1.02%)
BEC(0.52%)
BEC(0.00%)

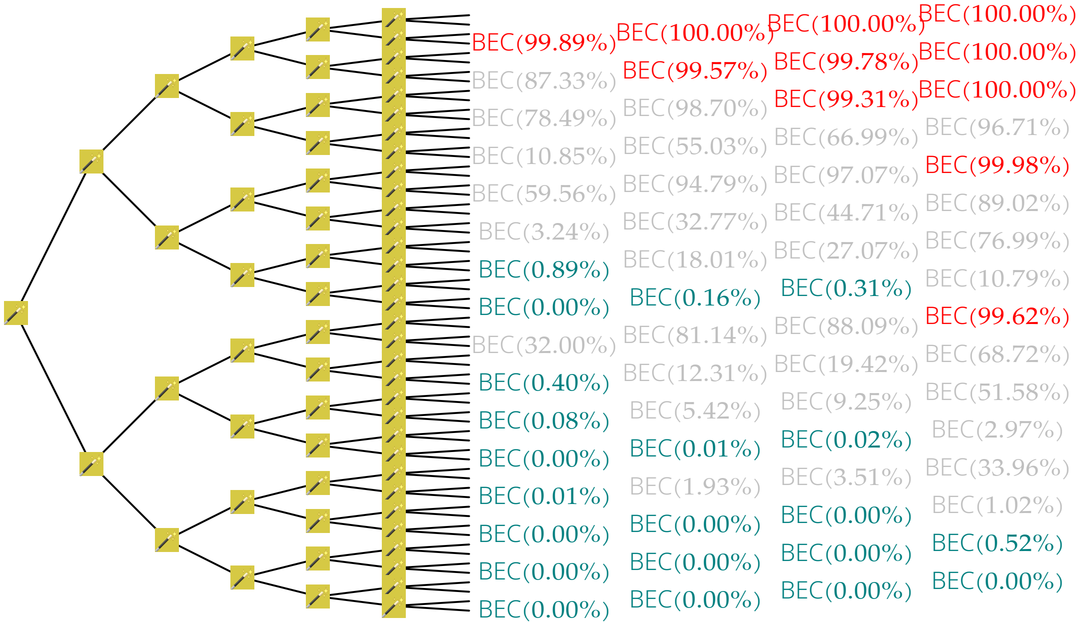

BEC(99.89%)
BEC(87.33%)
BEC(78.49%)
BEC(10.85%)
BEC(59.56%)
BEC(3.24%)
BEC(0.89%)
BEC(0.00%)
BEC(32.00%)
BEC(0.40%)
BEC(0.08%)
BEC(0.00%)
BEC(0.01%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(99.57%)
BEC(98.70%)
BEC(55.03%)
BEC(94.79%)
BEC(32.77%)
BEC(18.01%)
BEC(0.16%)
BEC(81.14%)
BEC(12.31%)
BEC(5.42%)
BEC(0.01%)
BEC(1.93%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(99.78%)
BEC(99.31%)
BEC(66.99%)
BEC(97.07%)
BEC(44.71%)
BEC(27.07%)
BEC(0.31%)
BEC(88.09%)
BEC(19.42%)
BEC(9.25%)
BEC(0.02%)
BEC(3.51%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(96.71%)
BEC(99.98%)
BEC(89.02%)
BEC(76.99%)
BEC(10.79%)
BEC(99.62%)
BEC(68.72%)
BEC(51.58%)
BEC(2.97%)
BEC(33.96%)
BEC(1.02%)
BEC(0.52%)
BEC(0.00%)

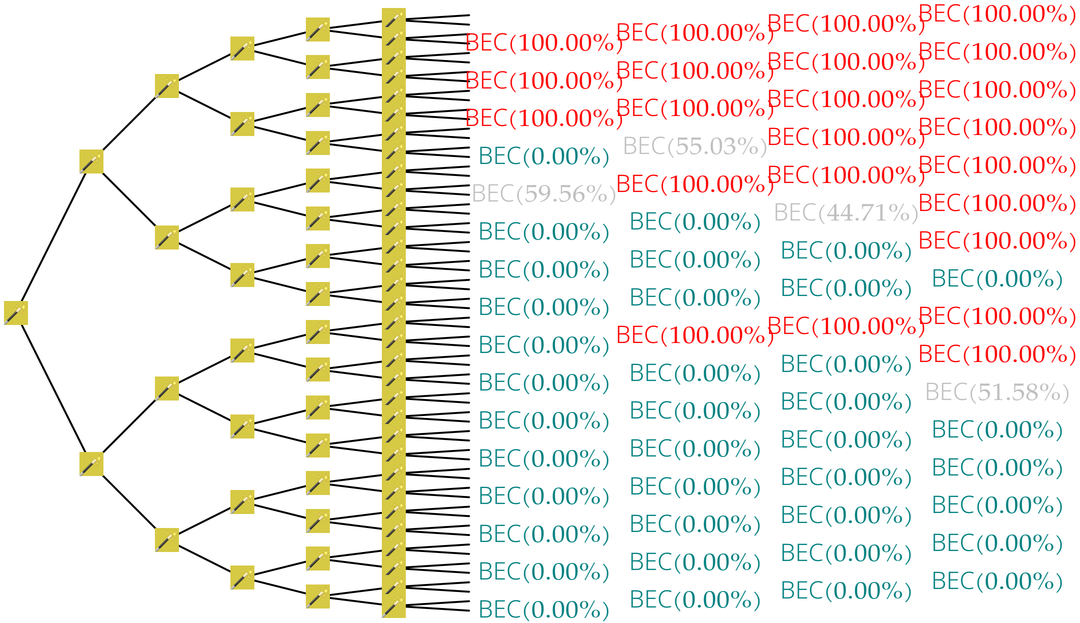

BEC(0.89%)
BEC(0.00%)
BEC(0.40%)
BEC(0.08%)
BEC(0.00%)
BEC(0.01%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.16%)
BEC(0.01%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.31%)
BEC(0.02%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.52%)
BEC(0.00%)

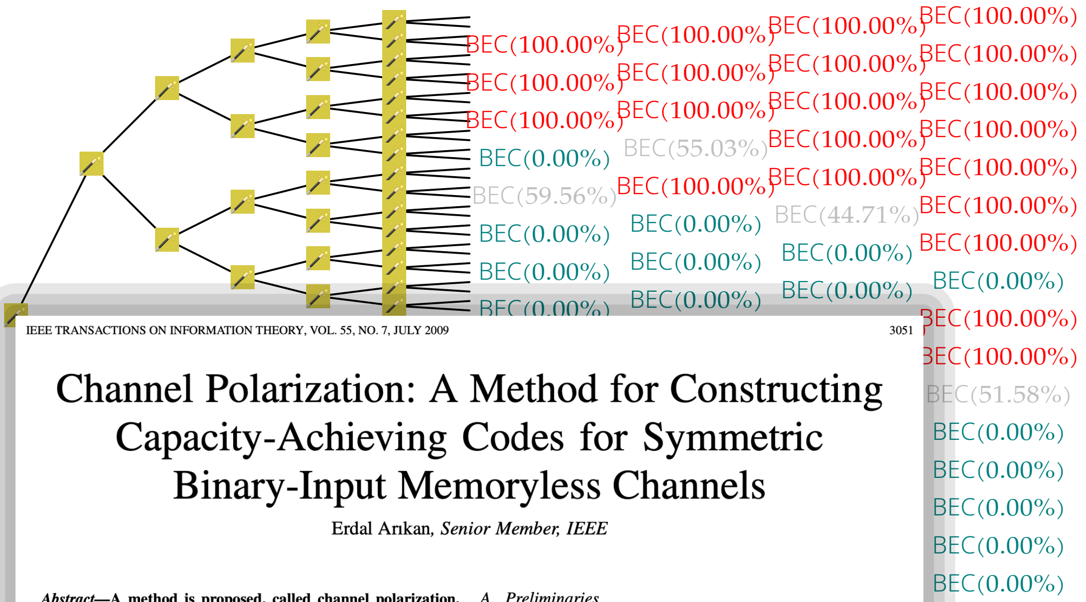
Thm [Arıkan ] If, after synthesizing 2^n channels, k of them are less than p,

polar code has rate $k/2^n$ & block error prob $< kp$

Thm [Arıkan ] If, after synthesizing 2^n channels, k of them are less than p, polar code has rate $k/2^n$ & block error prob $< kp$


BEC(99.89%)
BEC(87.33%)
BEC(78.49%)
BEC(10.85%)
BEC(59.56%)
BEC(3.24%)
BEC(0.89%)
BEC(0.00%)
BEC(32.00%)
BEC(0.40%)
BEC(0.08%)
BEC(0.00%)
BEC(0.01%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(99.57%)
BEC(98.70%)
BEC(55.03%)
BEC(94.79%)
BEC(32.77%)
BEC(18.01%)
BEC(0.16%)
BEC(81.14%)
BEC(12.31%)
BEC(5.42%)
BEC(0.01%)
BEC(1.93%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(99.78%)
BEC(99.31%)
BEC(66.99%)
BEC(97.07%)
BEC(44.71%)
BEC(27.07%)
BEC(0.31%)
BEC(88.09%)
BEC(19.42%)
BEC(9.25%)
BEC(0.02%)
BEC(3.51%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(96.71%)
BEC(99.98%)
BEC(89.02%)
BEC(76.99%)
BEC(10.79%)
BEC(99.62%)
BEC(68.72%)
BEC(51.58%)
BEC(2.97%)
BEC(33.96%)
BEC(1.02%)
BEC(0.52%)
BEC(0.00%)


BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(0.00%)
BEC(59.56%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(55.03%)
BEC(100.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(44.71%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(0.00%)
BEC(100.00%)
BEC(100.00%)
BEC(51.58%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)



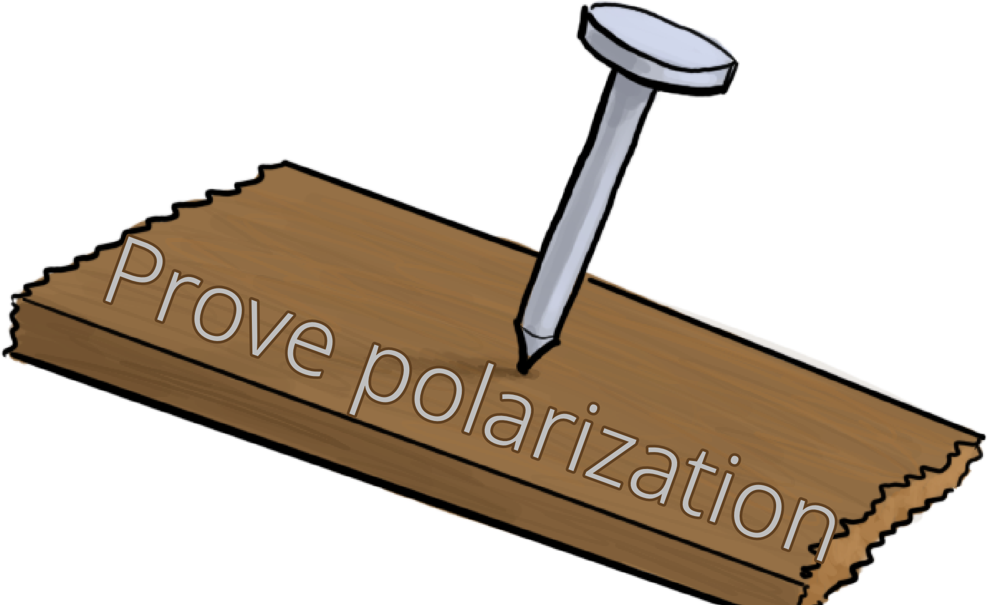
Channel Polarization: A Method for Constructing Capacity-Achieving Codes for Symmetric Binary-Input Memoryless Channels

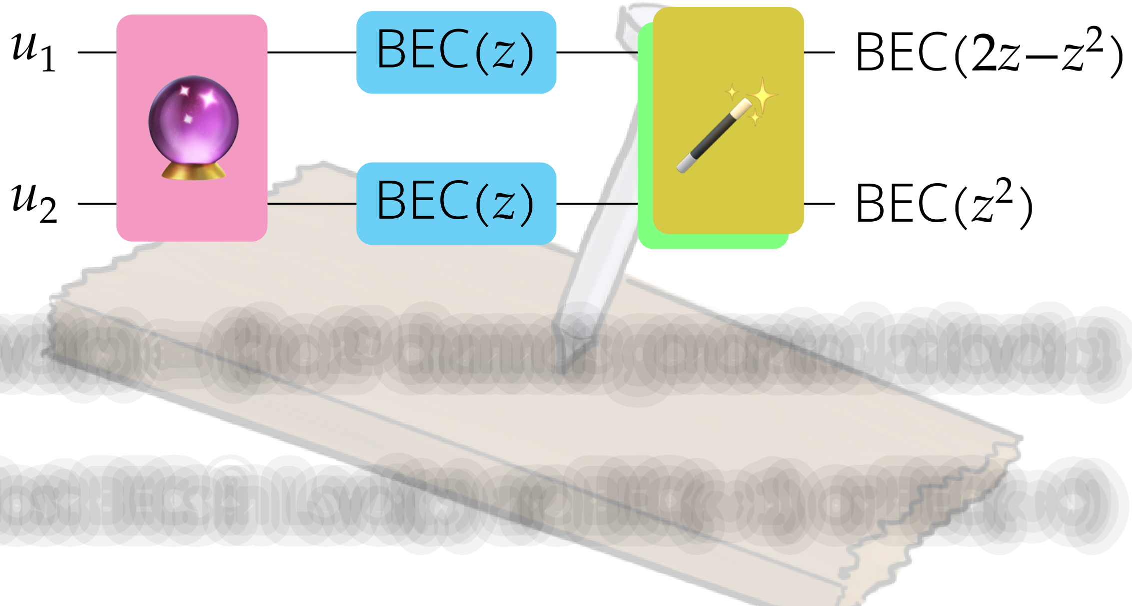
Erdal Arıkan, *Senior Member, IEEE*

***Abstract*—A method is proposed, called channel polarization, to construct code sequences that achieve the symmetric capacity $I(W)$ of any given binary-input discrete memoryless channel (B-DMC) W. The symmetric capacity is the highest rate achiev-**

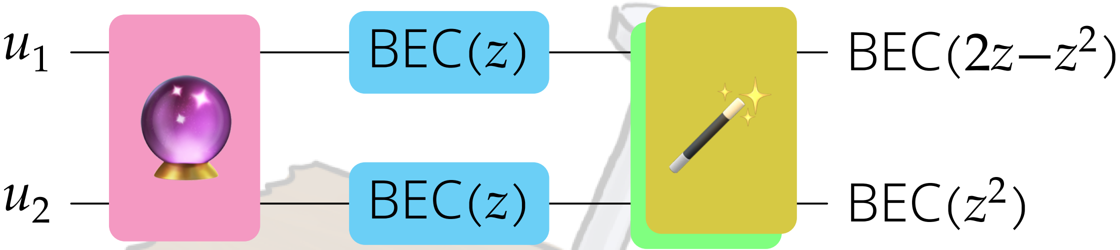
A. Preliminaries

We write $W : \mathcal{X} \to \mathcal{Y}$ to denote a generic B-DMC with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$, and transition probabilities

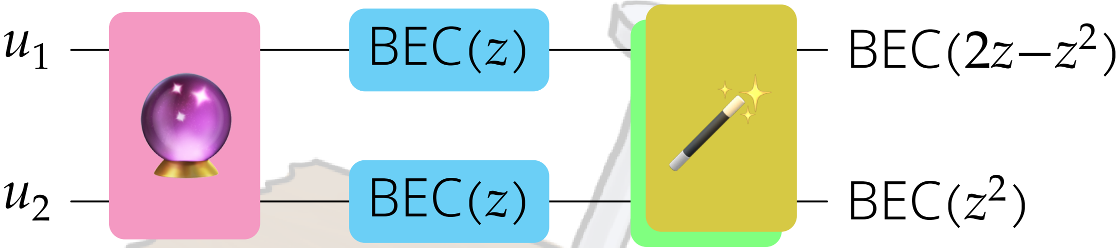

Prove polarization



u_1
u_2
BEC(z)
BEC(z)
BEC($2z-z^2$)
BEC(z^2)

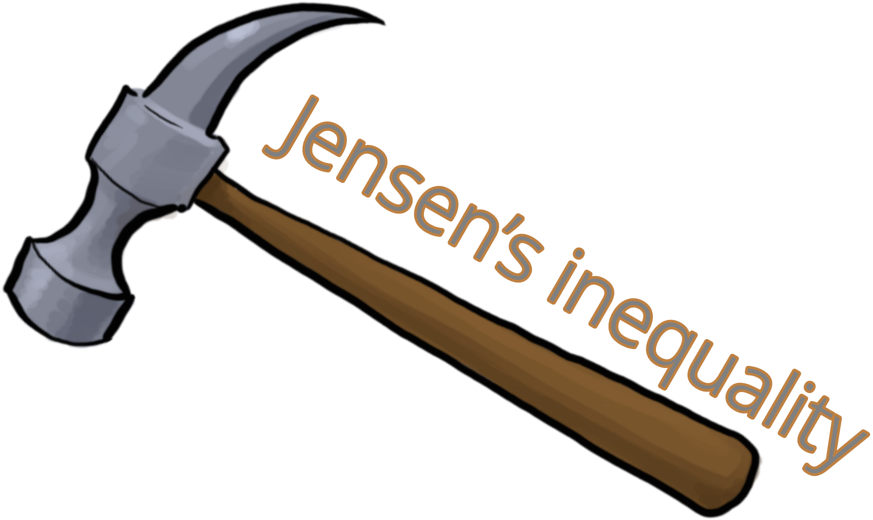


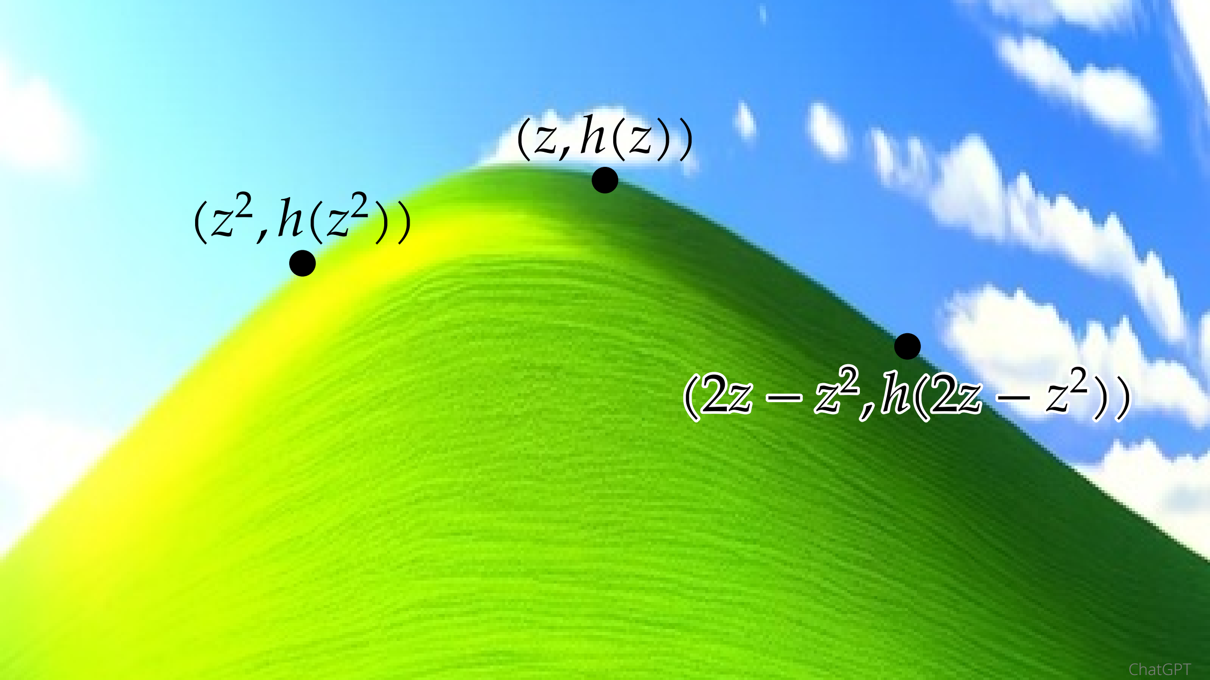
$\text{Level}(n) := \{\text{the } 2^n \text{ channels generated at level } n\}$

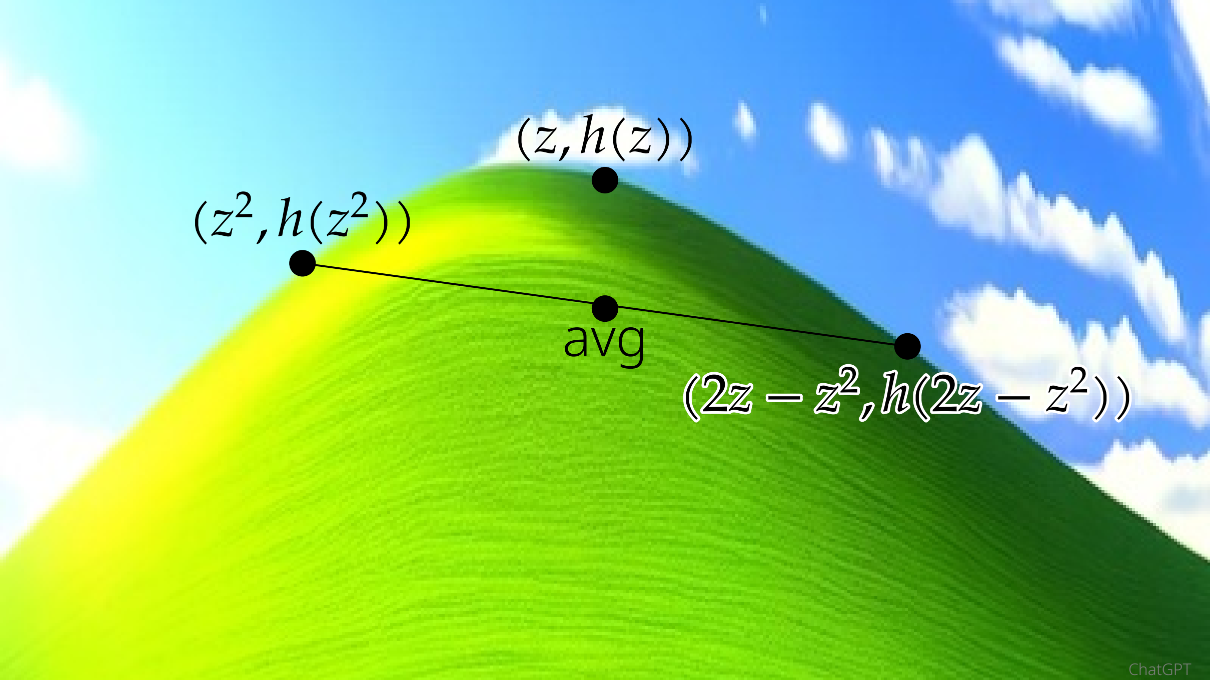


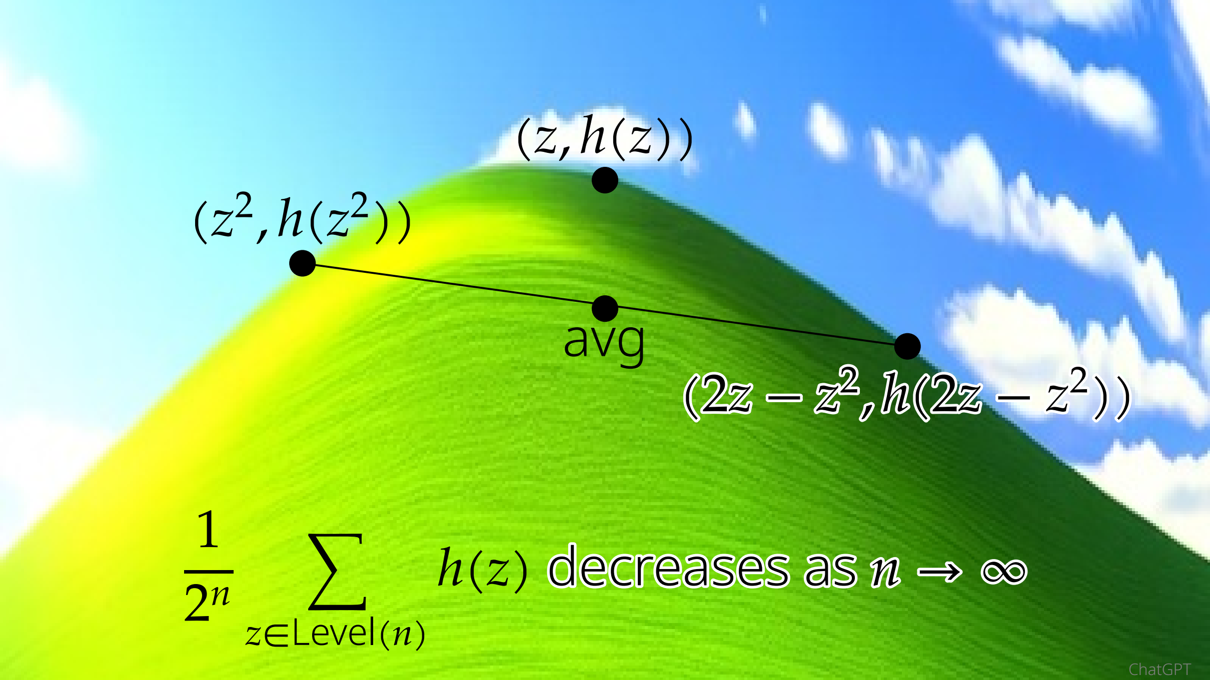
$\mathrm{Level}(n) :=$ {the 2^n channels generated at level n}

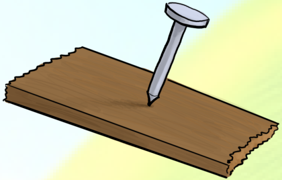
Most BECs in $\mathrm{Level}(n)$ are $\mathrm{BEC}(\approx 1)$ or $\mathrm{BEC}(\approx 0)$


Jensen's inequality

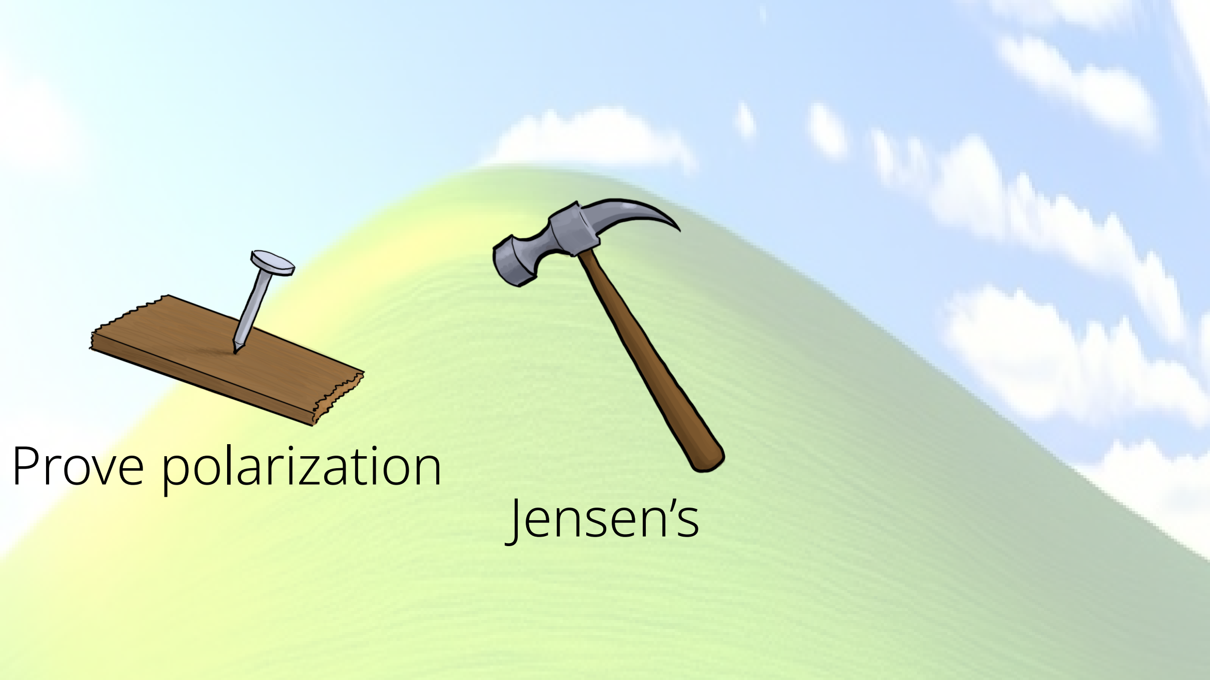

$(z, h(z))$
$(z^2, h(z^2))$
$(2z - z^2, h(2z - z^2))$
ChatGPT

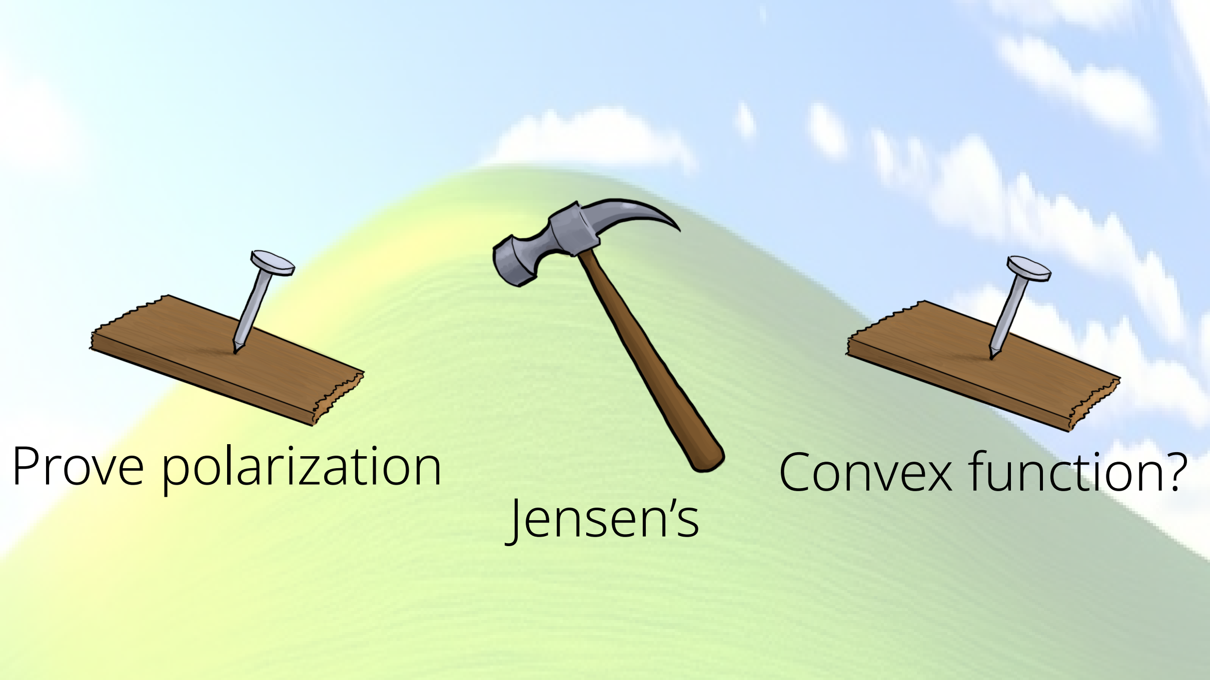

$(z, h(z))$
$(z^2, h(z^2))$
avg
$(2z - z^2, h(2z - z^2))$
ChatGPT


$(z, h(z))$
$(z^2, h(z^2))$
avg
$(2z - z^2, h(2z - z^2))$
$\frac{1}{2^n} \sum_{z \in \mathrm{Level}(n)} h(z)$ decreases as $n \to \infty$
ChatGPT

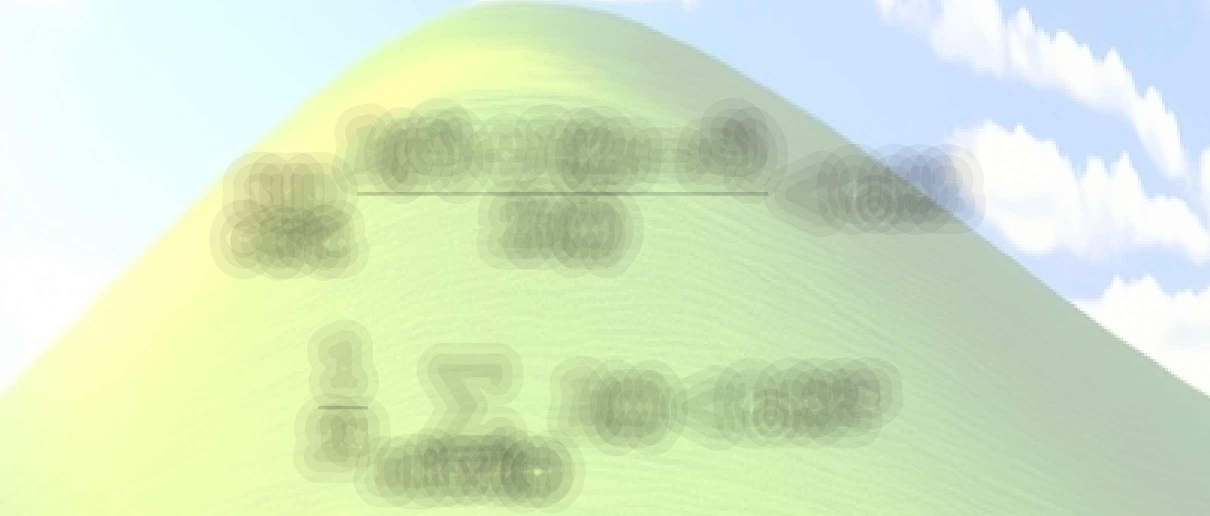


Prove polarization


Prove polarization
Jensen’s


Prove polarization
Jensen's
Convex function?

Hill shape $h(z) := (z(1-z))^{0.663}$



Hill shape $h(z) := (z(1-z))^{0.663}$

$$\sup_{0<z<1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

Hill shape $h(z) := (z(1-z))^{0.663}$

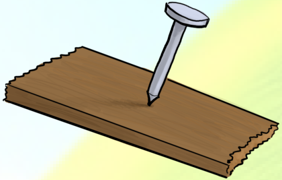
$$\sup_{0<z<1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

$$\frac{1}{2^n} \sum_{z \in \mathrm{Level}(n)} h(z) < 0.832^n$$

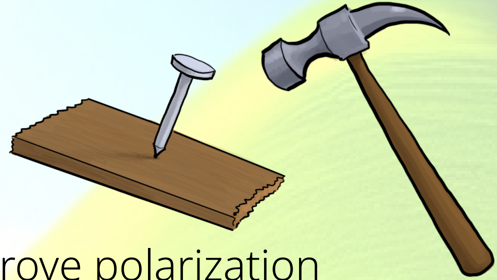
Hill shape $h(z) := \left(z(1-z)\right)^{0.663}$

$$\sup_{0<z<1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

$$\frac{1}{2^n} \sum_{z \in \mathrm{Level}(n)} h(z) < 0.832^n$$

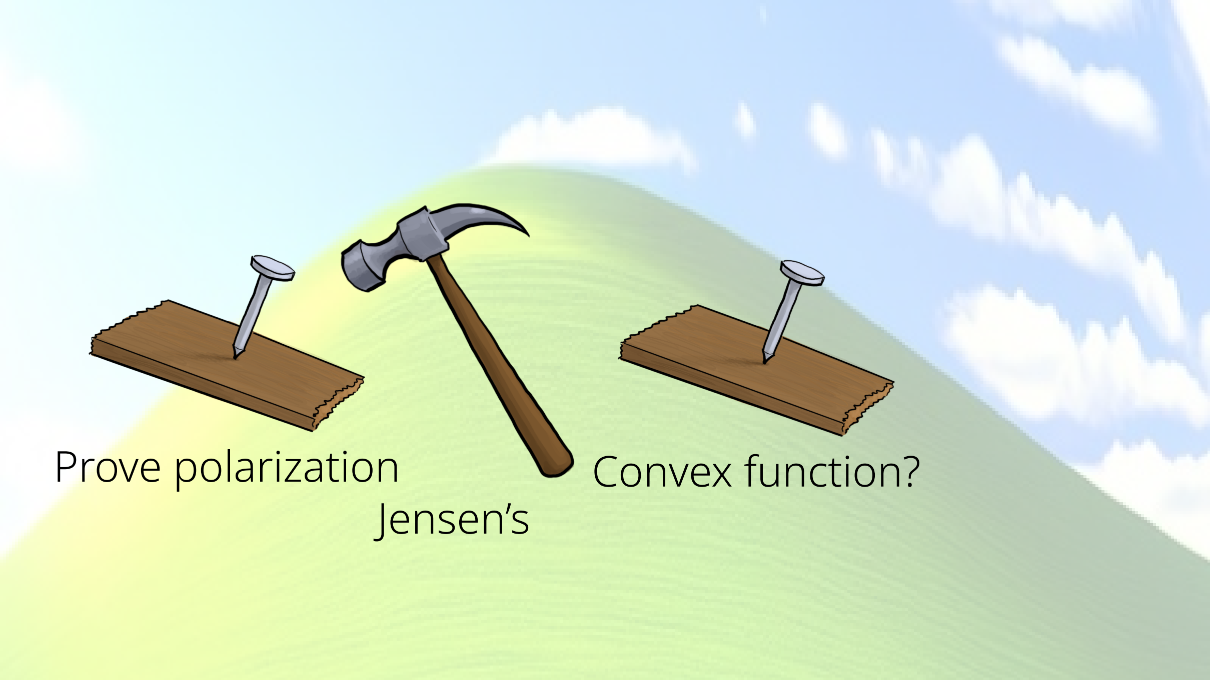


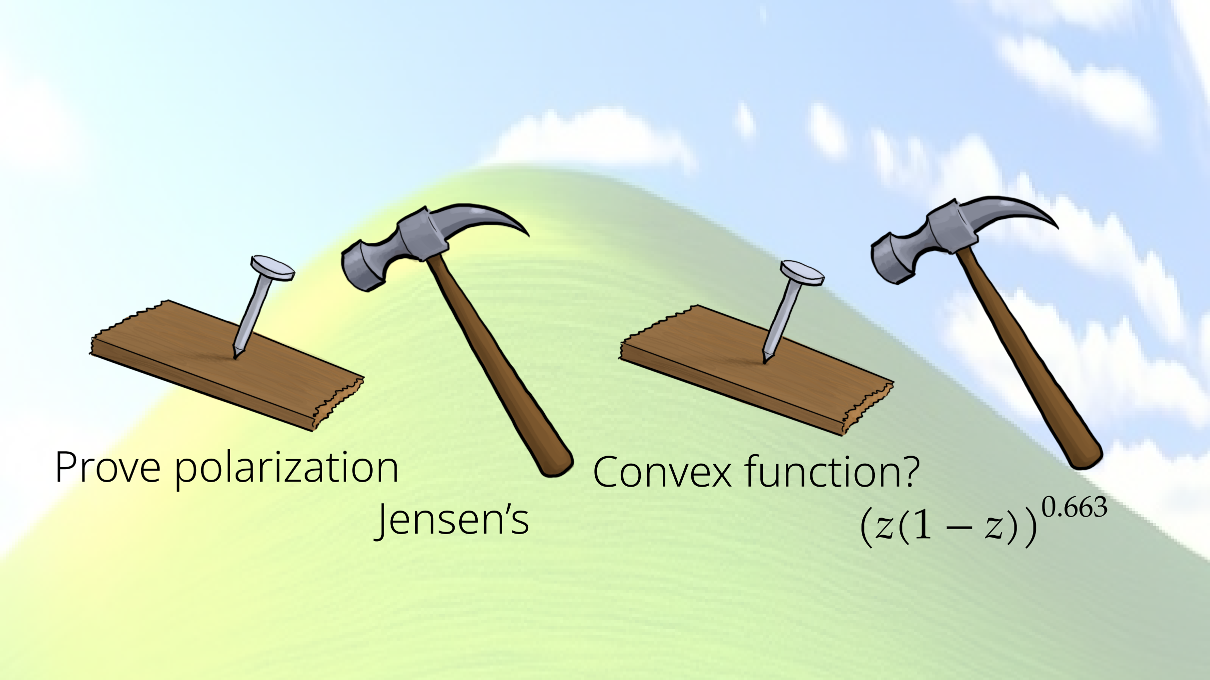
Prove polarization



Prove polarization

Jensen's


Prove polarization
Jensen's
Convex function?


Prove polarization
Jensen's
Convex function?
$(z(1-z))^{0.663}$

If k of 2^n are less than p,
polar code has rate $k/2^n$
& block error prob $< kp$.

$k/2^n = 60\% - 2^{-\Omega(n)}$

$p = 2^{-\Omega(n)}$

If k of 2^n are less than p,
polar code has rate $k/2^n$
& block error prob $< kp$.

$$k/2^n = 60\% - 2^{-\Omega(n)}$$

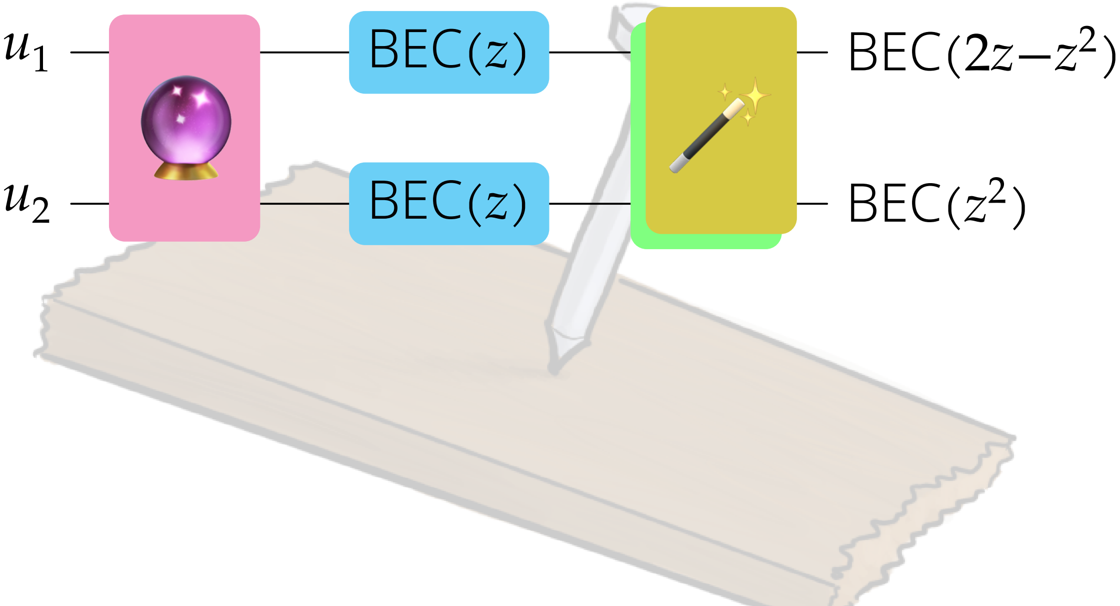
$$p = 2^{-\Omega(n)}$$

If k of 2^n are less than p,
polar code has rate $k/2^n$
& block error prob $< kp$.

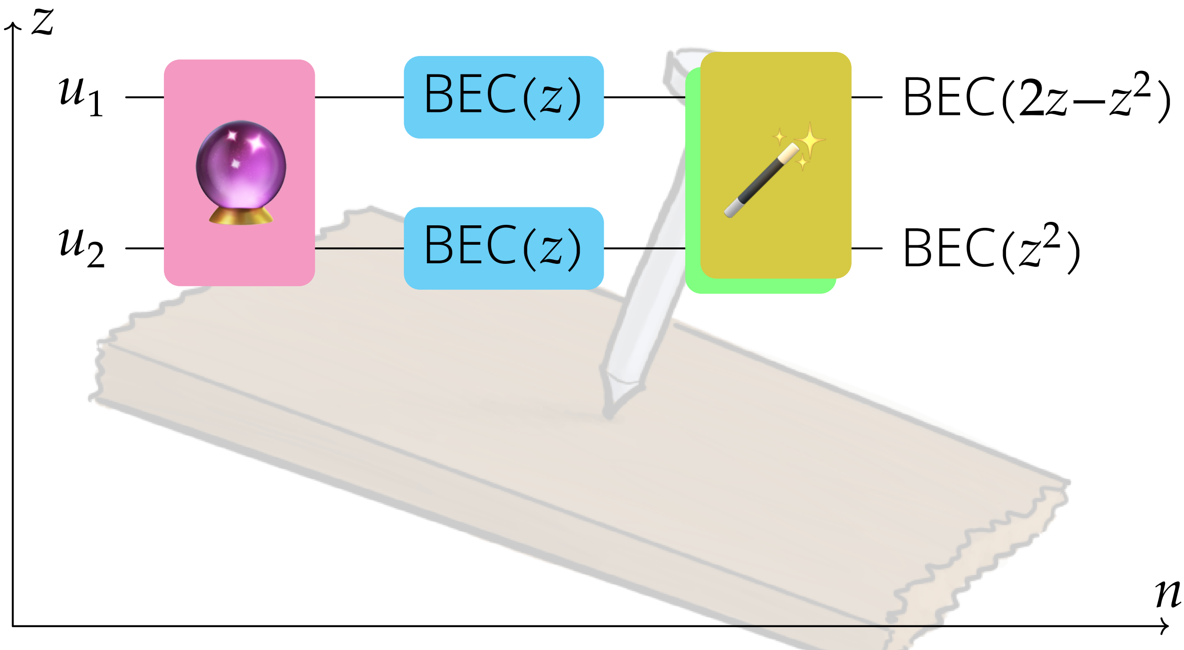
$$k/2^n = 60\% - 2^{-\Omega(n)}$$

$$p = 2^{-\Omega(n)}$$

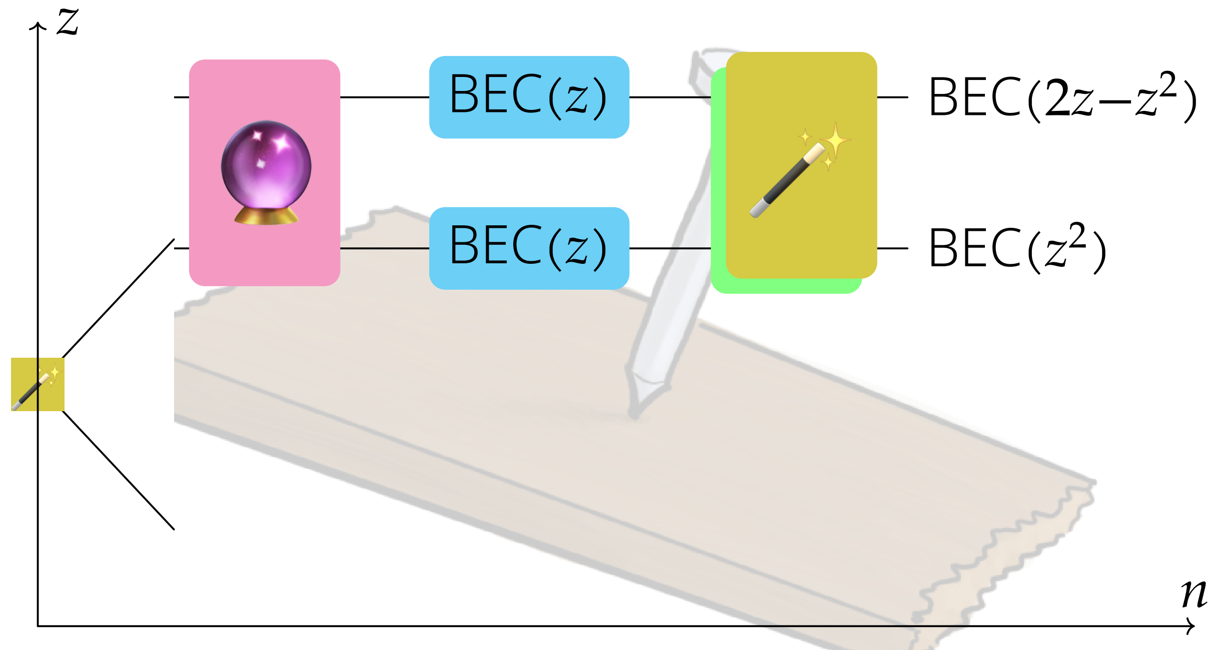

Let’s improve p



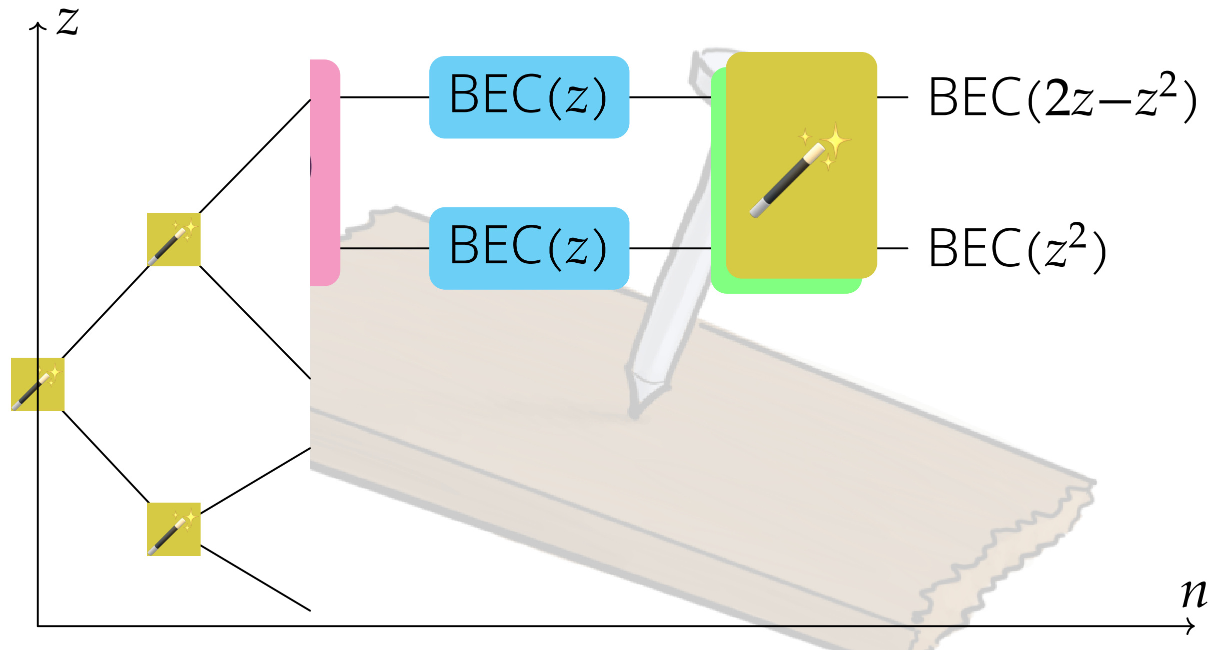
u_1
u_2
BEC(z)
BEC(z)
BEC($2z-z^2$)
BEC(z^2)



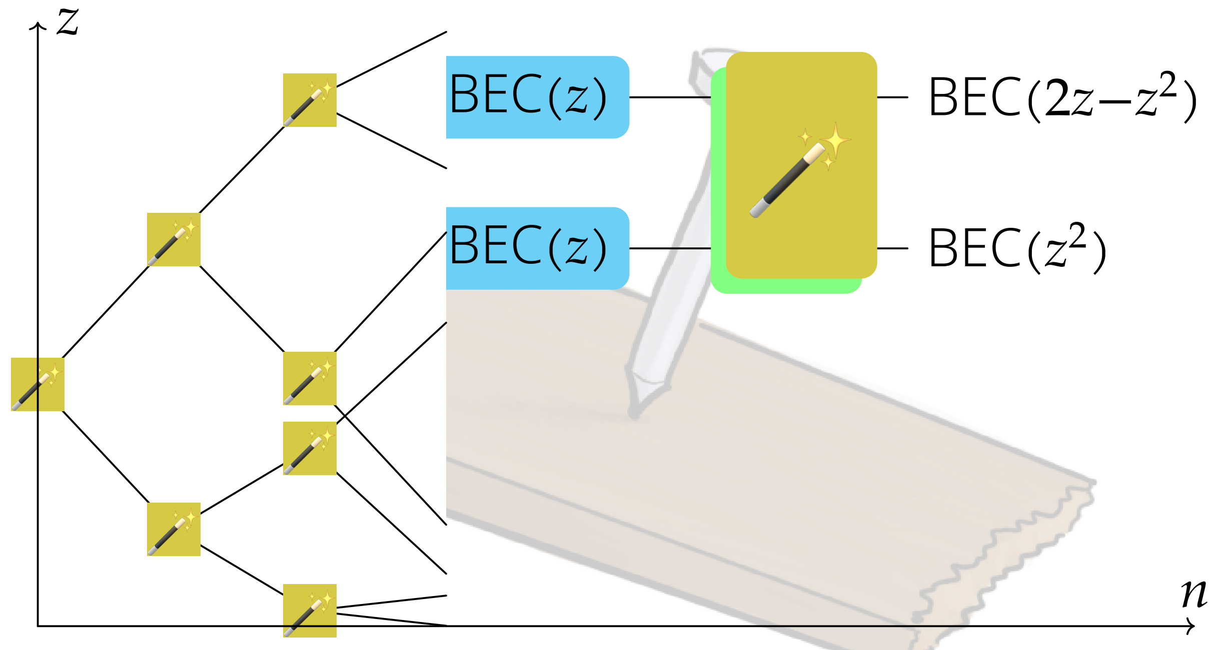
z
u_1
u_2
BEC(z)
BEC(z)
BEC($2z-z^2$)
BEC(z^2)
n



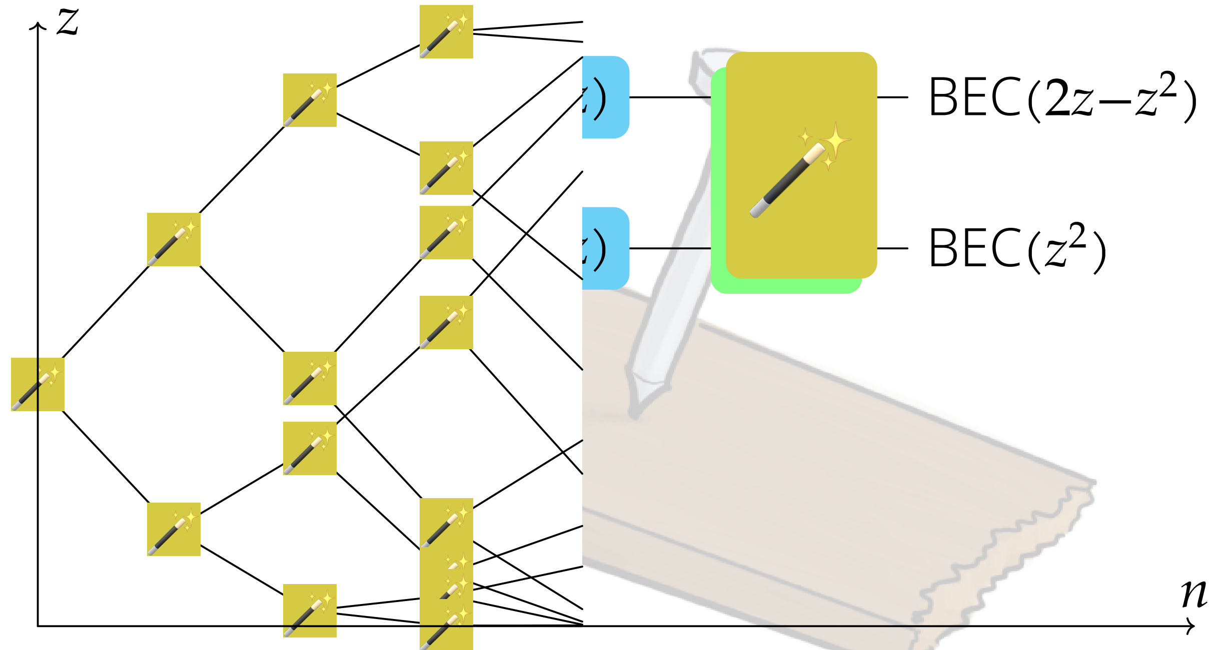
z
BEC(z)
BEC(z)
BEC($2z-z^2$)
BEC(z^2)
n



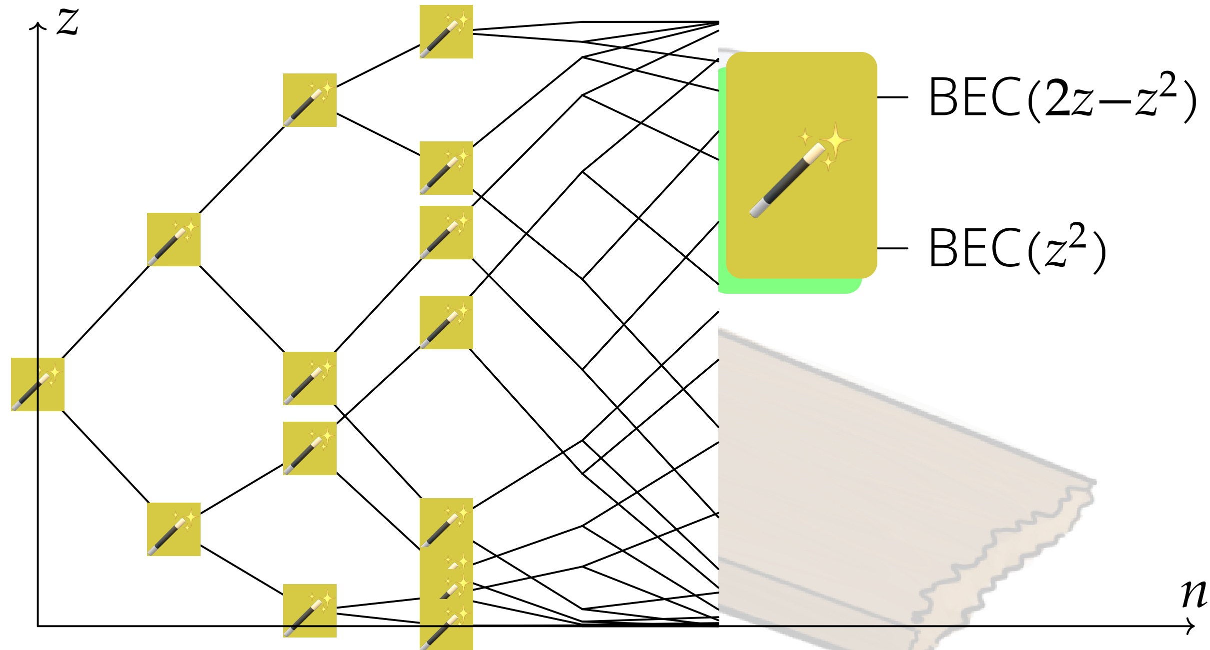
z
BEC(z)
BEC(z)
BEC(2z−z²)
BEC(z²)
n



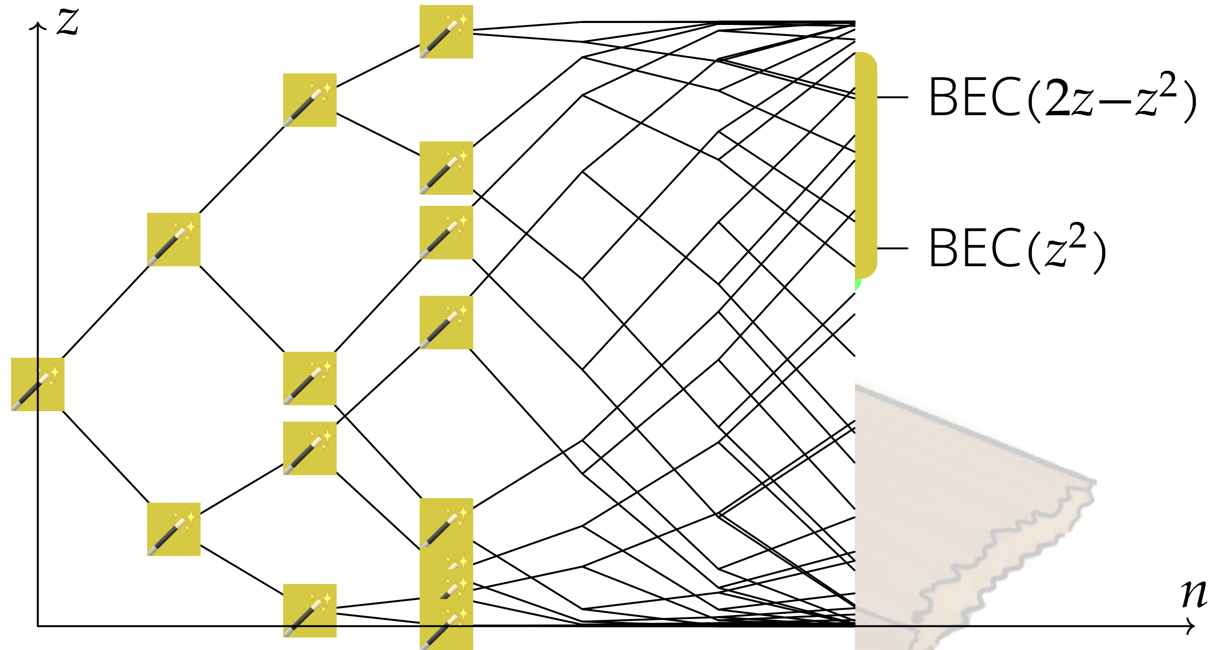
z
BEC(z)
BEC(z)
BEC($2z - z^2$)
BEC(z^2)
n

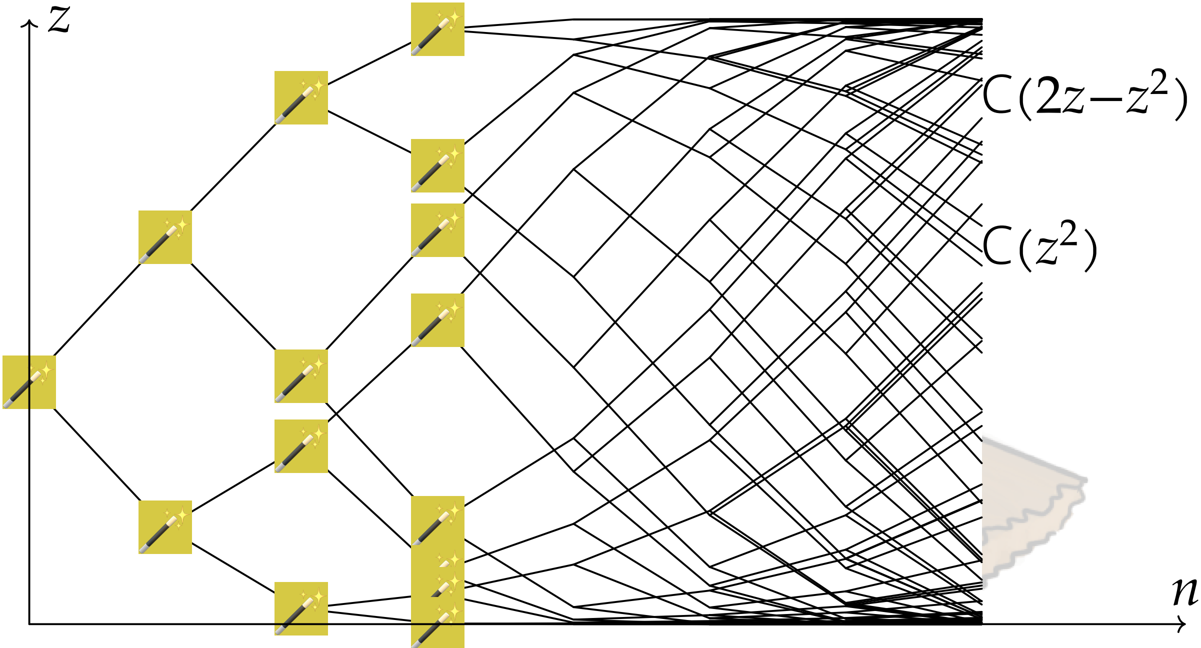


z
n
BEC(2z−z²)
BEC(z²)

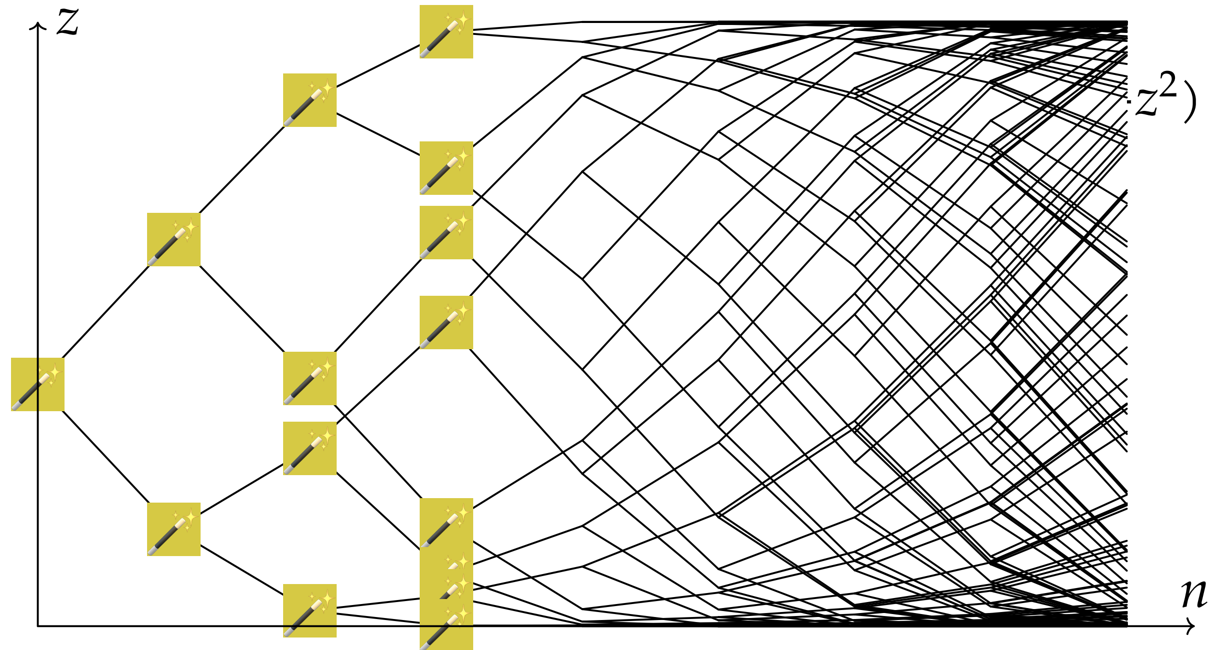


z
BEC$(2z-z^2)$
BEC(z^2)
n

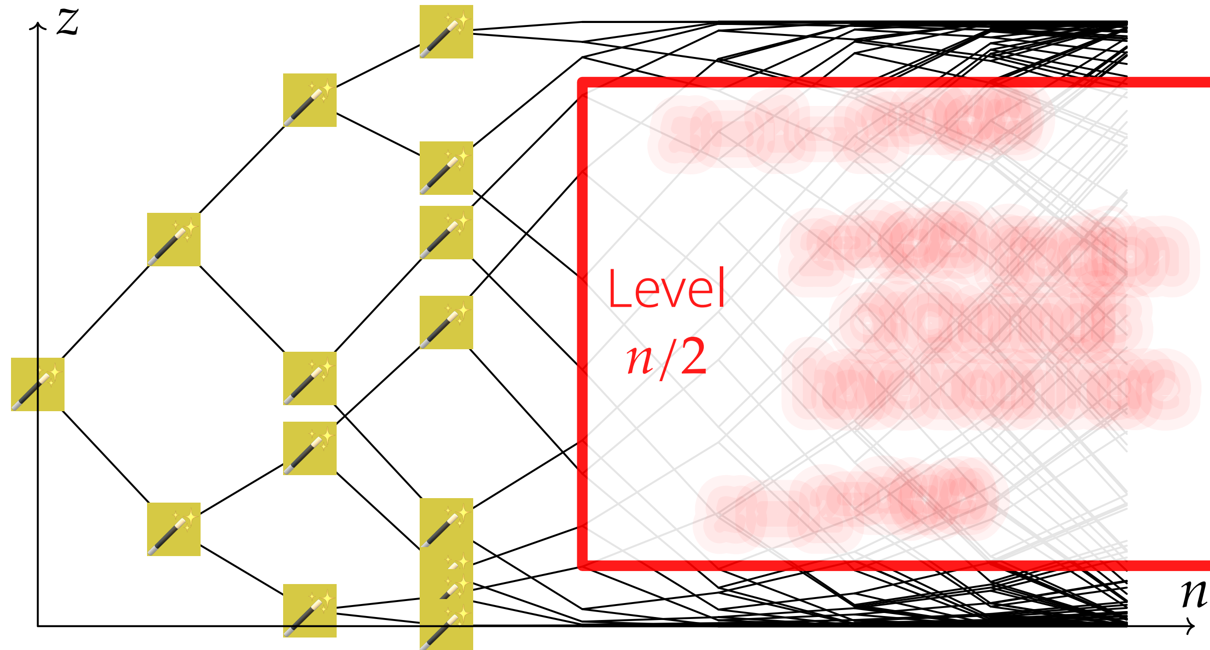

z
n
BEC$(2z-z^2)$
BEC(z^2)

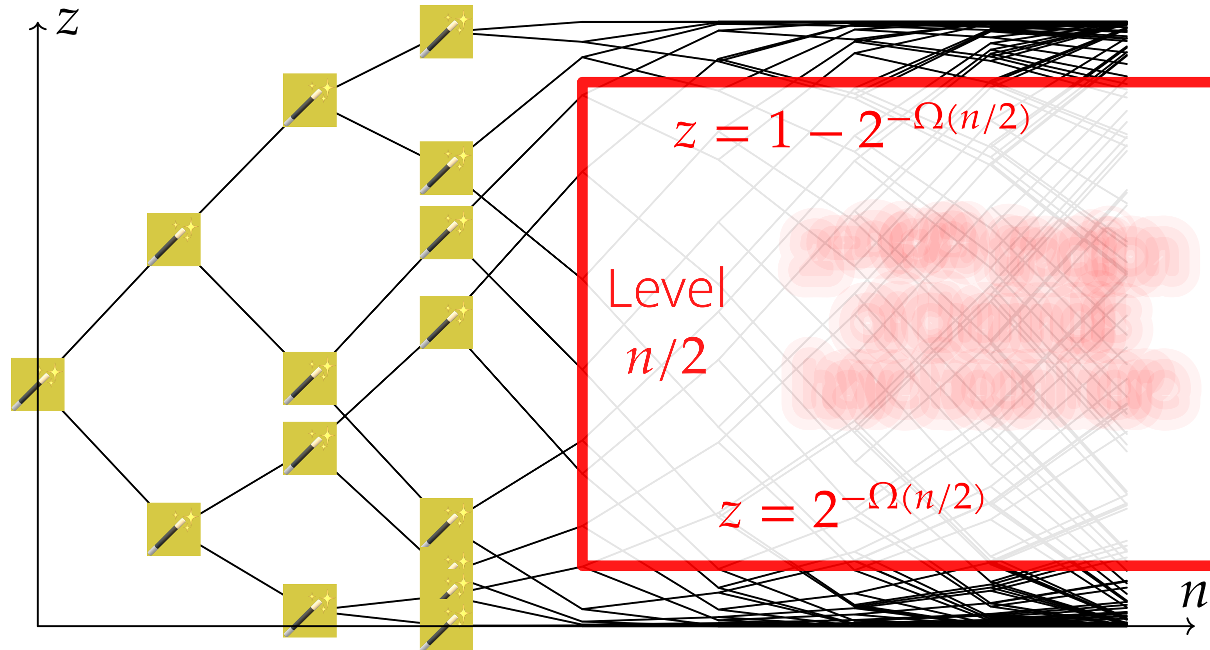


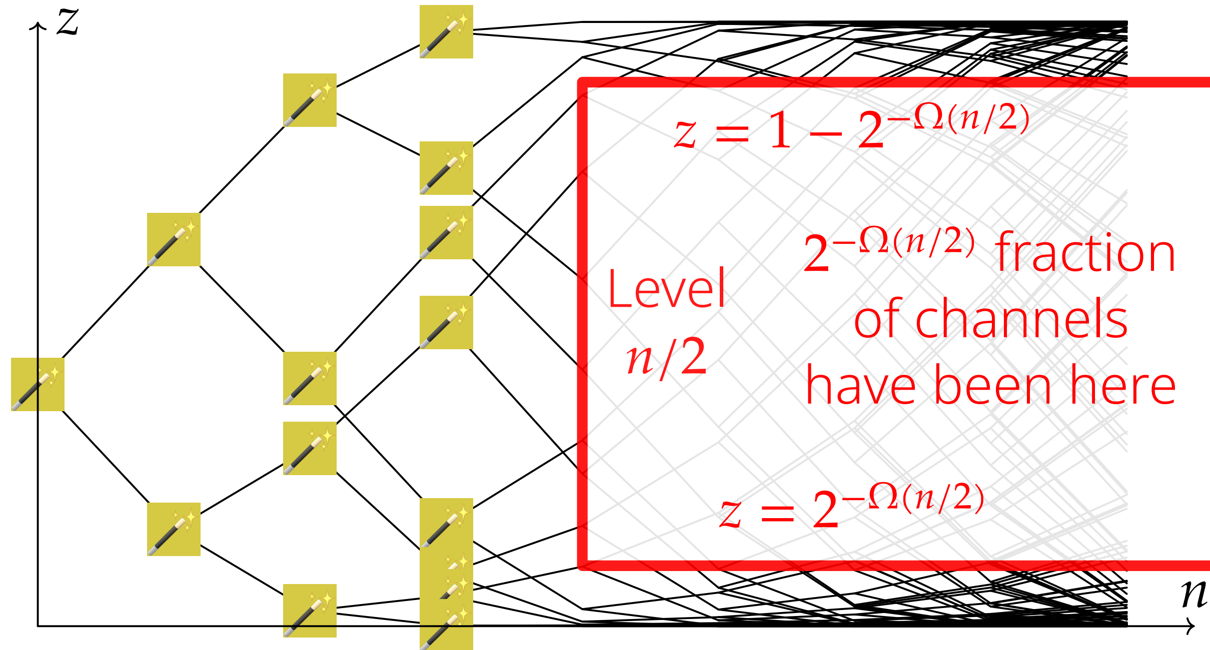
z
n
C(2z−z²)
C(z²)

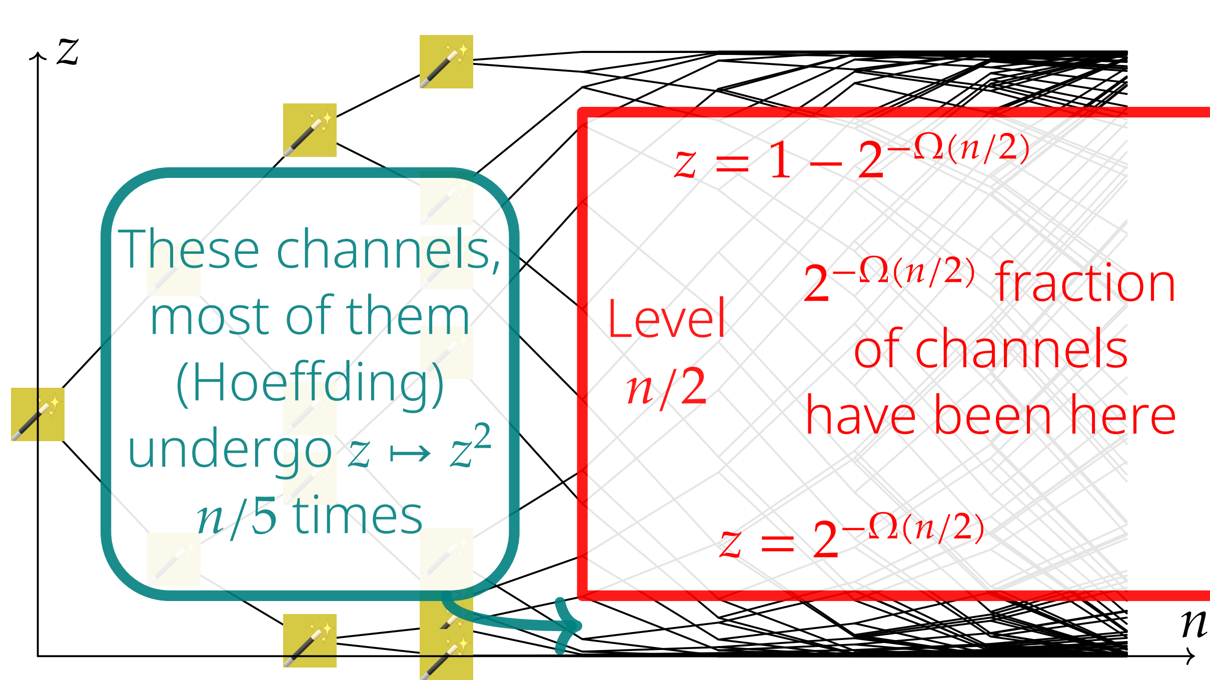


z
$z^2)$
n

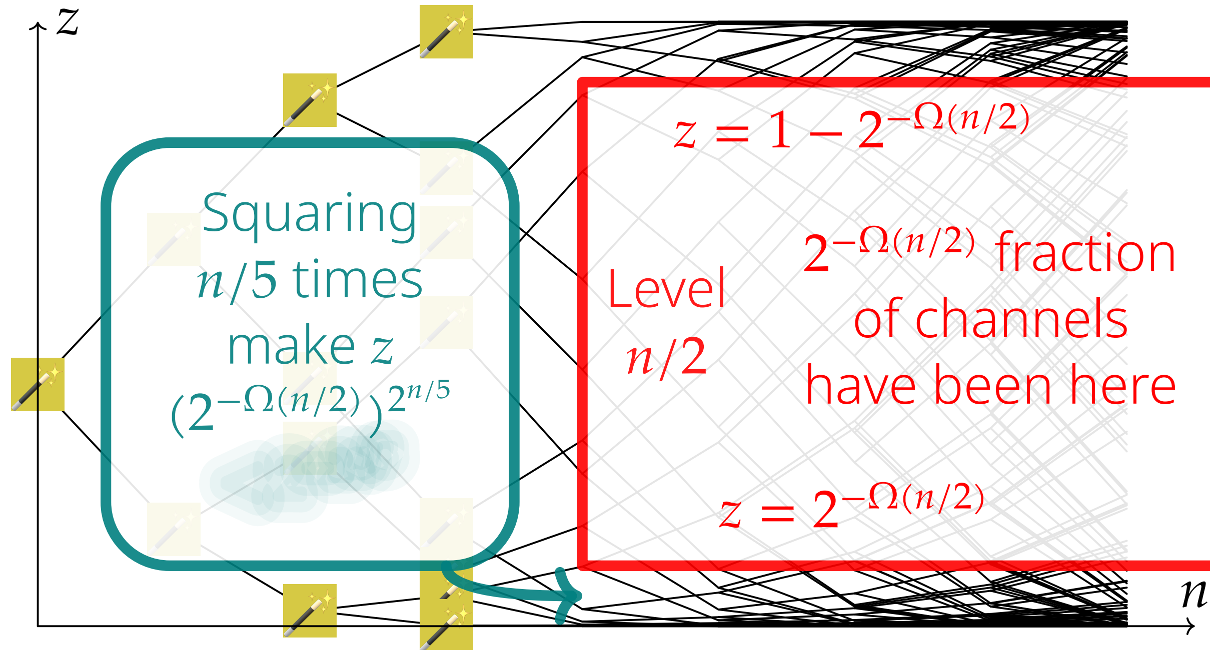

z
Level
n/2
n


z
n
$z = 1 - 2^{-\Omega(n/2)}$
Level
$n/2$
$z = 2^{-\Omega(n/2)}$

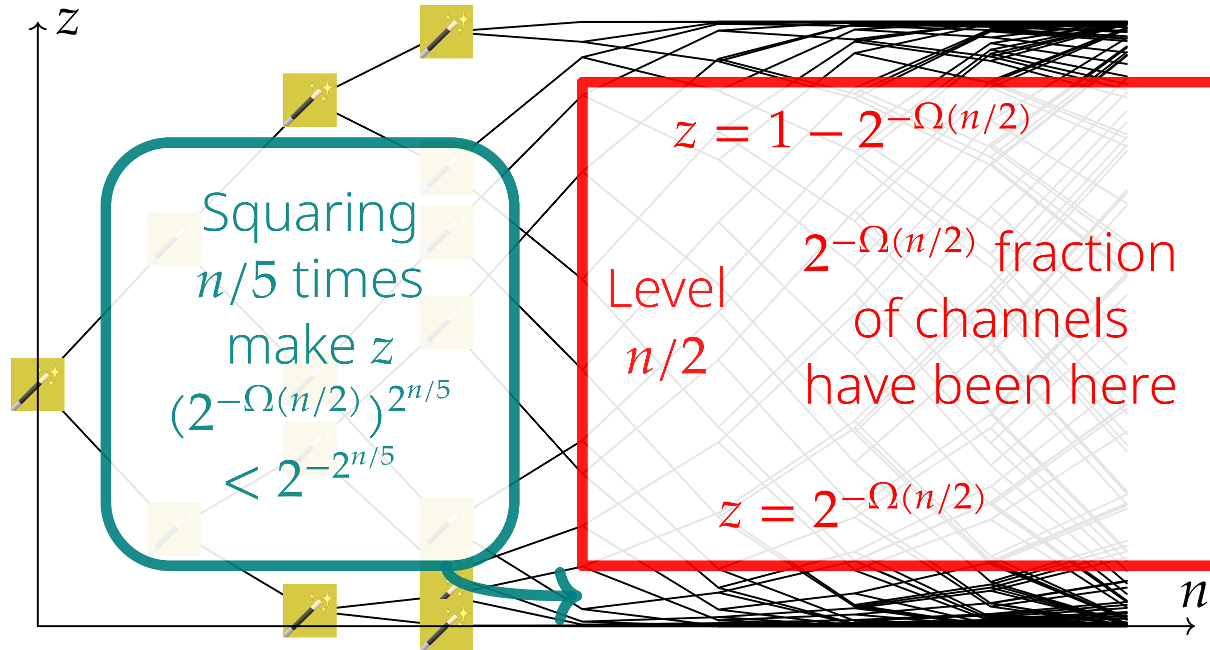

z
n
$z = 1 - 2^{-\Omega(n/2)}$
Level $n/2$
$2^{-\Omega(n/2)}$ fraction of channels have been here
$z = 2^{-\Omega(n/2)}$



z
n
These channels, most of them (Hoeffding) undergo $z \mapsto z^2$ $n/5$ times
$z = 1 - 2^{-\Omega(n/2)}$
Level $n/2$
$2^{-\Omega(n/2)}$ fraction of channels have been here
$z = 2^{-\Omega(n/2)}$



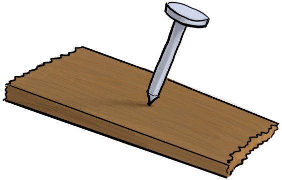
z
n
Squaring $n/5$ times make z
$(2^{-\Omega(n/2)})^{2^{n/5}}$
$z = 1 - 2^{-\Omega(n/2)}$
Level $n/2$
$2^{-\Omega(n/2)}$ fraction of channels have been here
$z = 2^{-\Omega(n/2)}$


z
n
Squaring
$n/5$ times
make z
$(2^{-\Omega(n/2)})^{2^{n/5}}$
$< 2^{-2^{n/5}}$
Level
$n/2$
$z = 1 - 2^{-\Omega(n/2)}$
$2^{-\Omega(n/2)}$ fraction
of channels
have been here
$z = 2^{-\Omega(n/2)}$

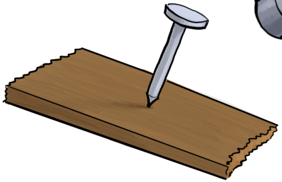
If k of 2^n are less than p,
polar code has rate $k/2^n$
& block error prob $< kp$.

$$k/2^n = 60\% - 2^{-\Omega(n)}$$

$$\cancel{p = 2^{-\Omega(n)}} \quad p = 2^{-2^{\Omega(n)}}$$

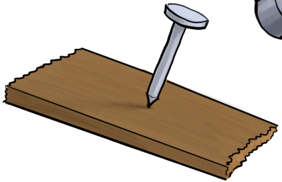


Prove polarization



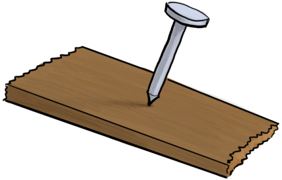
Prove polarization

Jensen's

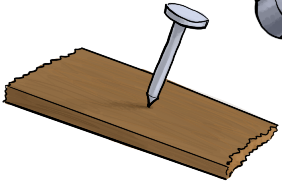


Prove polarization

Jensen’s

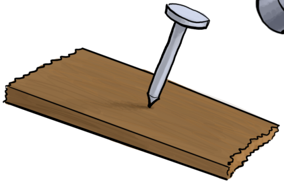


Convex function?



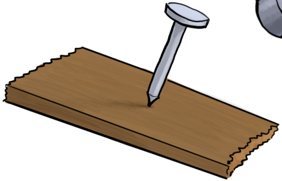
Prove polarization

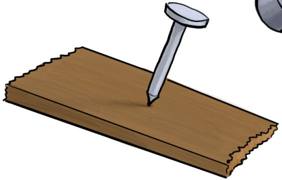
Jensen's



Convex function?

$\left(z(1-z)\right)^{0.663}$



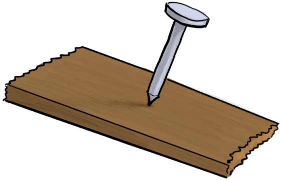


Prove polarization

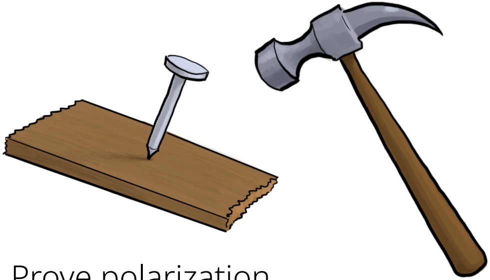
Jensen's

Convex function?

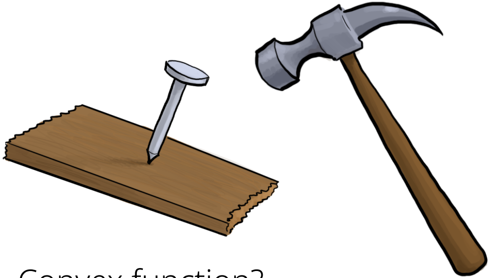
$(z(1-z))^{0.663}$



p too bad



Prove polarization

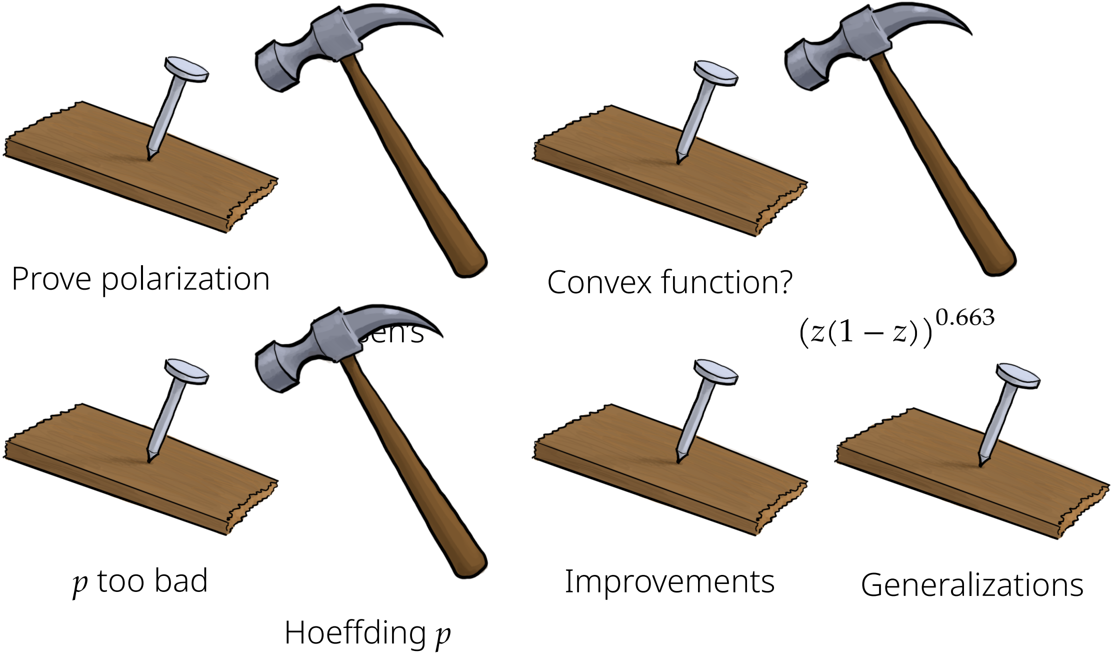


Convex function?

$(z(1-z))^{0.663}$

p too bad

Hoeffding p


Prove polarization
Convex function?
$(z(1-z))^{0.663}$
p too bad
Hoeffding p
Improvements
Generalizations


ChatGPT


successive cancellation
channel polarization
ChatGPT


better gap
to capacity
better
error prob
successive
cancellation
channel
polarization
ChatGPT

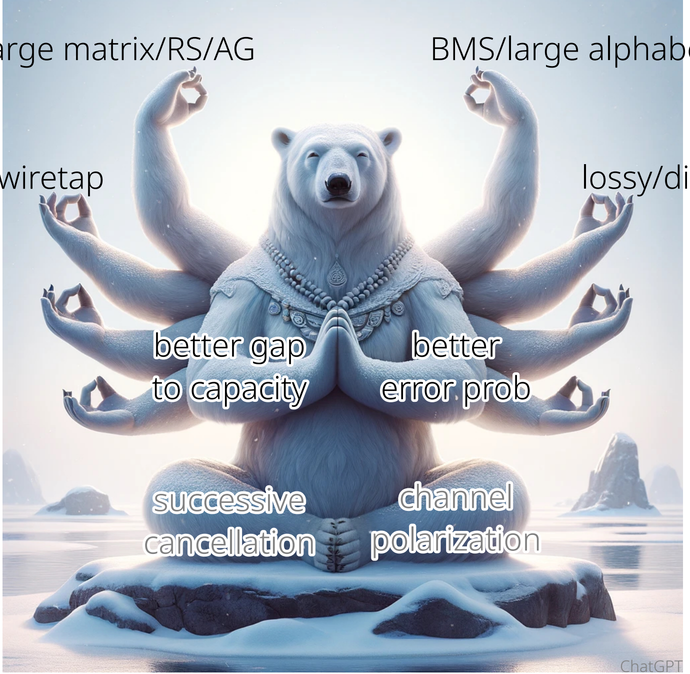

large matrix/RS/AG
BMS/large alphabet
better gap
to capacity
better
error prob
successive
cancellation
channel
polarization
ChatGPT

large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

lossy/distribut compress



large matrix/RS/AG

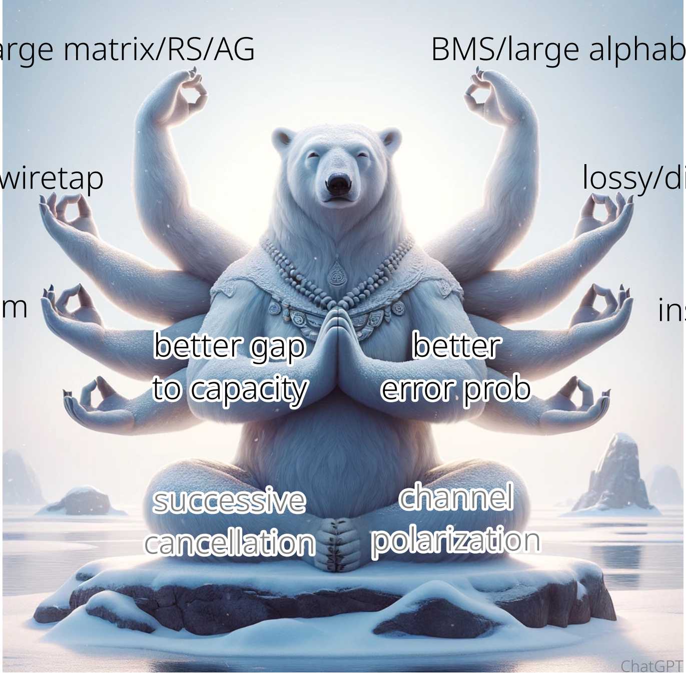
BMS/large alphabet

MAC/broadcast/wiretap

lossy/distribut compress

classical–quantum

insdel channel



large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

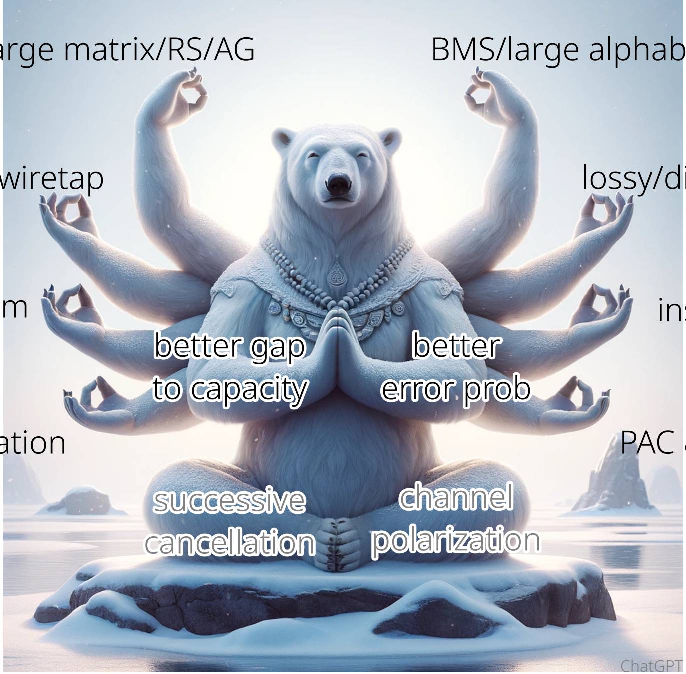
lossy/distribut compress

classical–quantum

insdel channel

coded computation

PAC & list decoding




large matrix/RS/AG
BMS/large alphabet
MAC/broadcast/wiretap
lossy/distribut compress
classical–quantum
insdel channel
better gap
to capacity
better
error prob
coded computation
PAC & list decoding
successive
cancellation
channel
polarization
ChatGPT

Hill shape $h(z) := (z(1-z))^{0.663}$ is not optimal

$$\sup_{0<z<1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

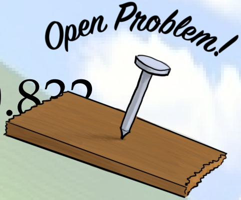
Hill shape $h(z) := (z(1-z))^{0.663}$ is not optimal

$$\sup_{0<z<1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

Use power method $h_{i+1}(z) := \dfrac{h_i(z^2) + h_i(2z - z^2)}{\text{normalize}}$

Hill shape $h(z) := (z(1-z))^{0.663}$ is not optimal

$$\sup_{0<z<1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.8$$



Use power method $h_{i+1}(z) := \dfrac{h_i(z^2) + h_i(2z - z^2)}{\text{normalize}}$

large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

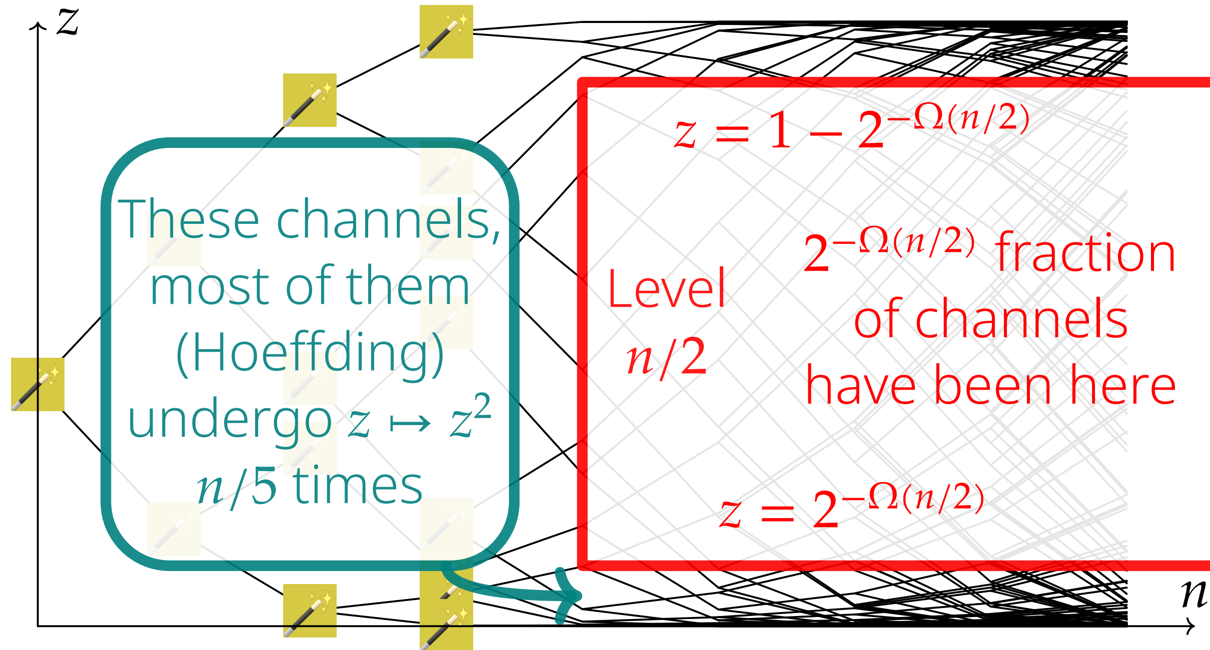
lossy/distribut compress

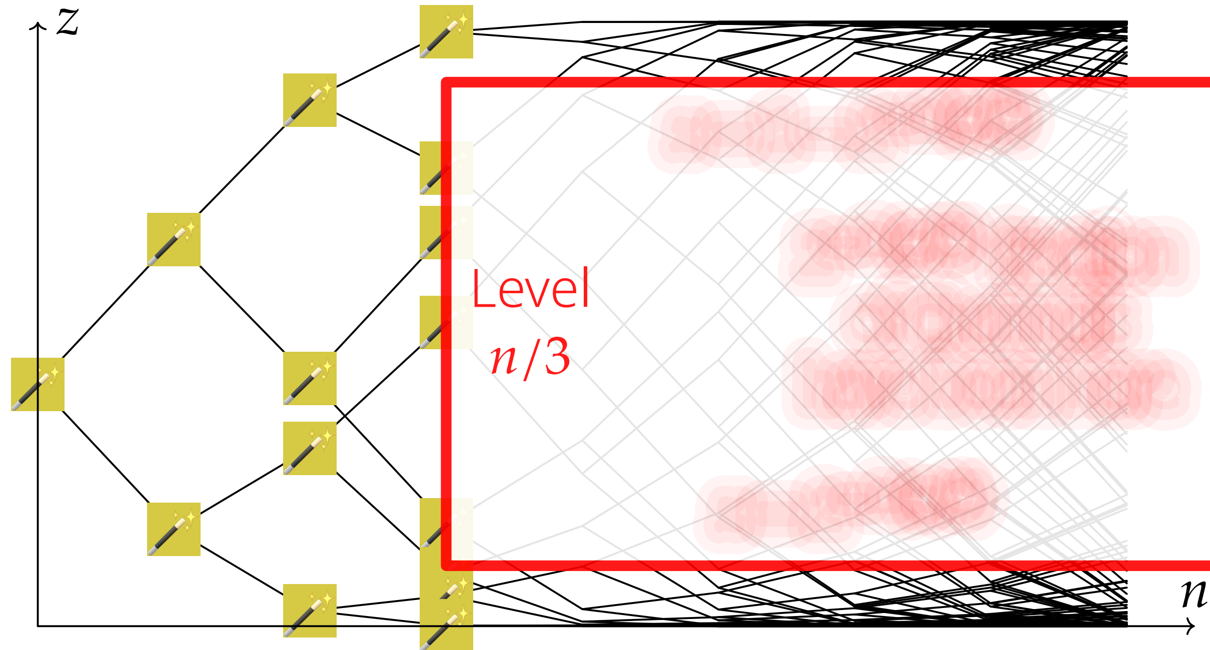
classical–quantum

insdel channel

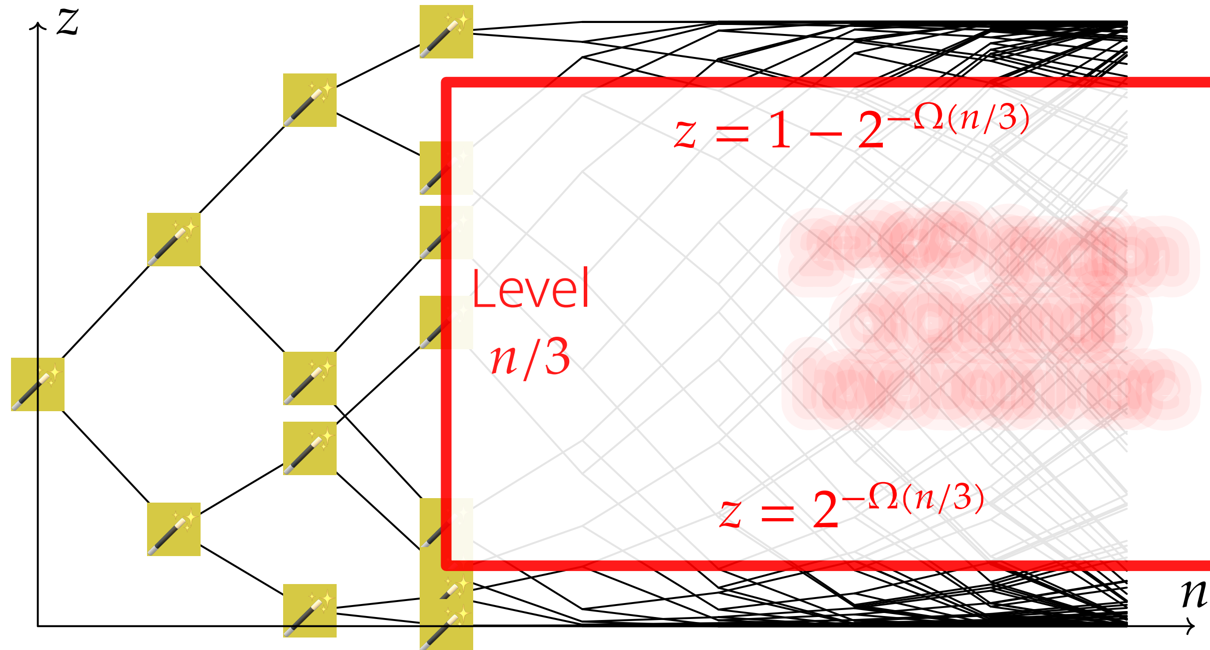
coded computation

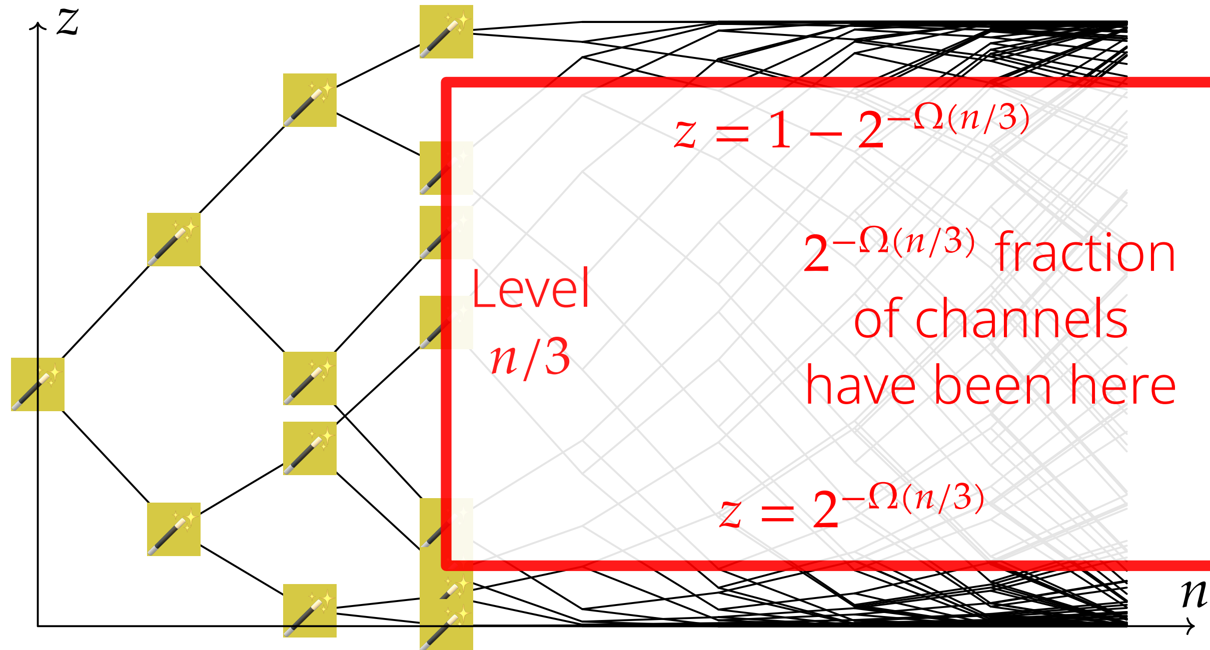
PAC & list decoding


z
n
These channels, most of them (Hoeffding) undergo $z \mapsto z^2$ $n/5$ times
$z = 1 - 2^{-\Omega(n/2)}$
Level $n/2$
$2^{-\Omega(n/2)}$ fraction of channels have been here
$z = 2^{-\Omega(n/2)}$

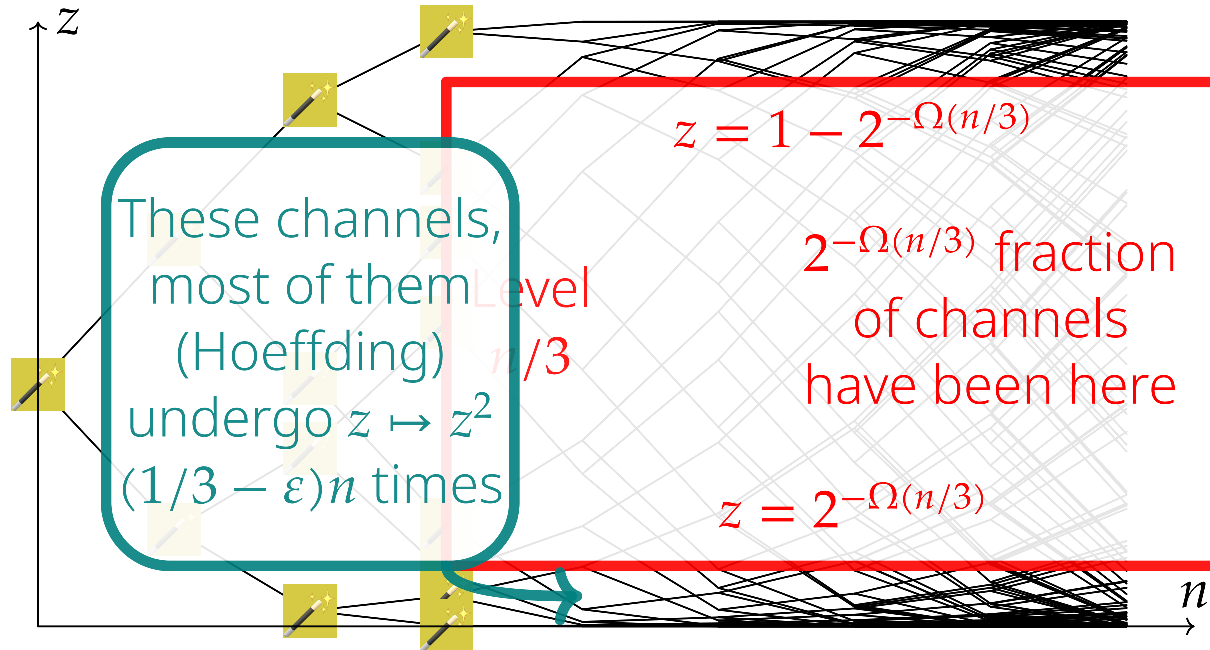


z
n
Level
n/3


z
n
Level
n/3
$z = 1 - 2^{-\Omega(n/3)}$
$z = 2^{-\Omega(n/3)}$



z
$z = 1 - 2^{-\Omega(n/3)}$
$2^{-\Omega(n/3)}$ fraction of channels have been here
Level $n/3$
$z = 2^{-\Omega(n/3)}$
n


z
n
$z = 1 - 2^{-\Omega(n/3)}$
Level
$n/3$
These channels, most of them (Hoeffding) undergo $z \mapsto z^2$ $(1/3 - \varepsilon)n$ times
$2^{-\Omega(n/3)}$ fraction of channels have been here
$z = 2^{-\Omega(n/3)}$

If k of 2^n are less than p,

polar code has rate $k/2^n$

& block error prob $< kp$.

$$k/2^n = 60\% - 2^{-n/3.627}$$

$$p = 2^{-2^{\Omega(n)}}$$

If k of 2^n are less than p,
polar code has rate $k/2^n$
& block error prob $< kp$.

$$k/2^n = 60\% - 2^{-\delta n}$$

$$p = 2^{-2^{(1/2-\varepsilon)n}}$$

large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

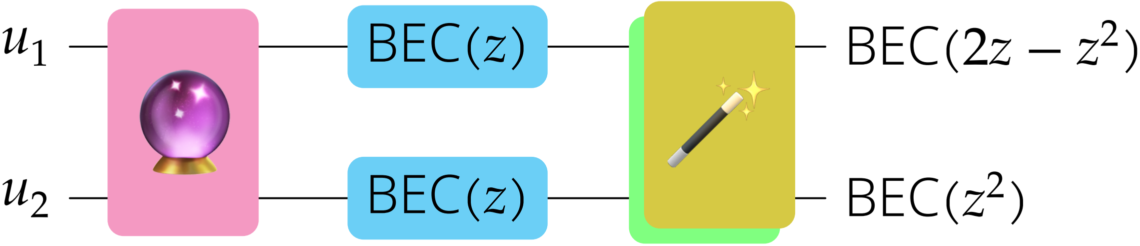
lossy/distribut compress

classical–quantum

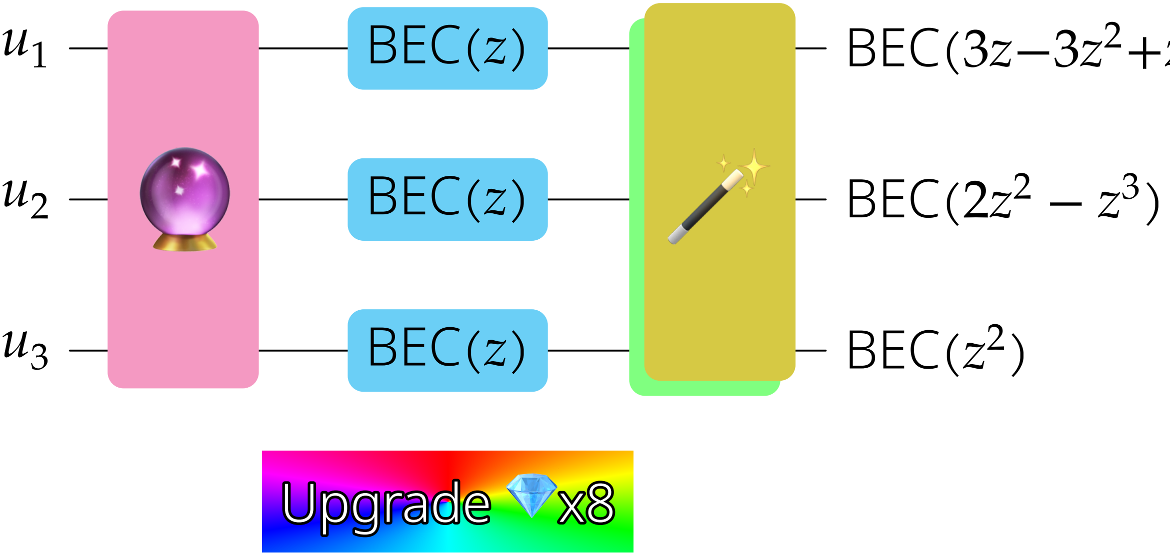
insdel channel

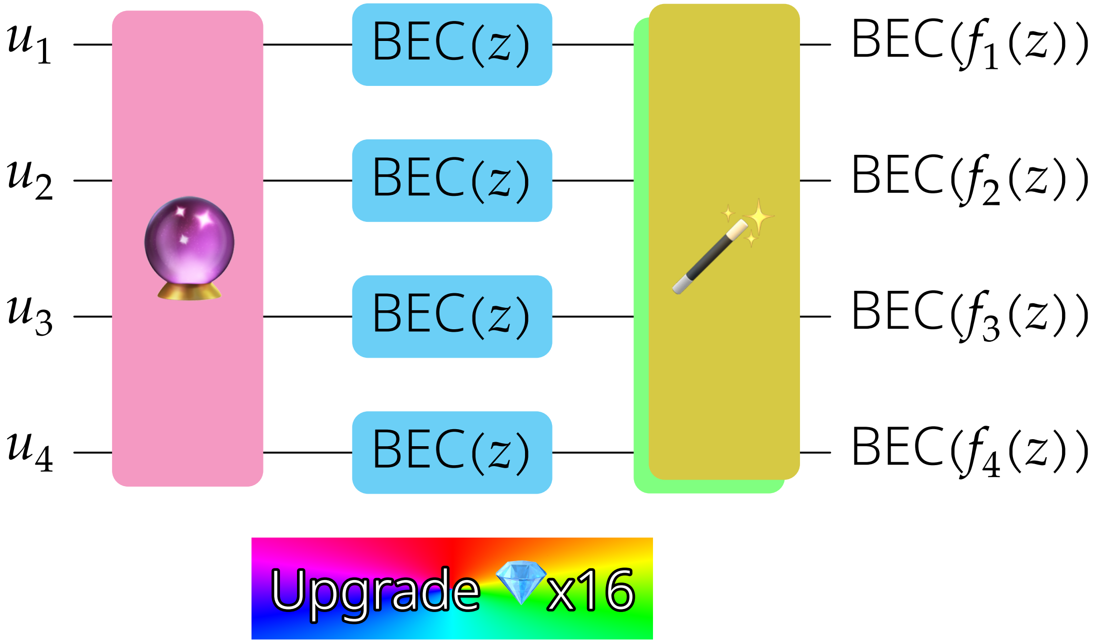
coded computation

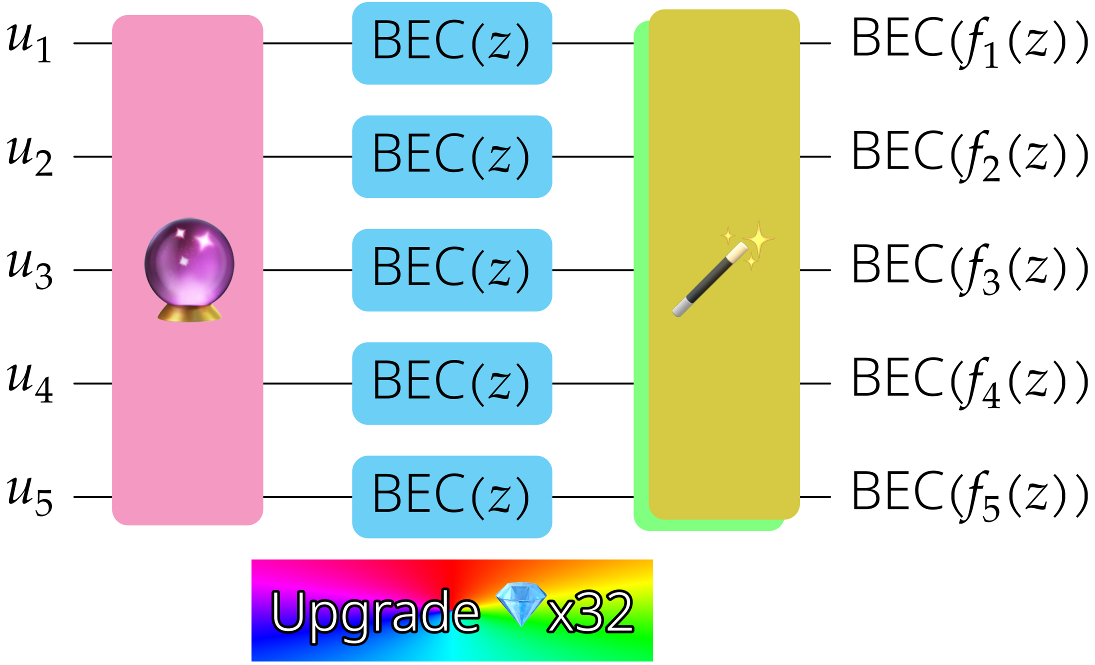
PAC & list decoding


u_1
u_2
BEC(z)
BEC(z)
BEC($2z - z^2$)
BEC(z^2)


Upgrade x4


u_1
u_2
u_3
BEC(z)
BEC(z)
BEC(z)
BEC($3z-3z^2+$
BEC($2z^2 - z^3$)
BEC(z^2)
Upgrade x8


u_1
u_2
u_3
u_4
BEC(z)
BEC(z)
BEC(z)
BEC(z)
BEC($f_1(z)$)
BEC($f_2(z)$)
BEC($f_3(z)$)
BEC($f_4(z)$)
Upgrade x16



u_1
u_2
u_3
u_4
u_5
BEC(z)
BEC(z)
BEC(z)
BEC(z)
BEC(z)
BEC($f_1(z)$)
BEC($f_2(z)$)
BEC($f_3(z)$)
BEC($f_4(z)$)
BEC($f_5(z)$)
Upgrade x32

 $= \ell \times \ell$ matrices such as $\begin{bmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 1 & & 1 & & & & & \\ 1 & & & & 1 & & & \\ 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & & & 1 & 1 & & \\ 1 & & 1 & & 1 & & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

🔮 $= \ell \times \ell$ matrices such as $\begin{bmatrix} 1 &&&&&&& \\ 1 & 1 &&&&&& \\ 1 && 1 &&&&& \\ 1 &&&& 1 &&& \\ 1 & 1 & 1 & 1 &&&& \\ 1 & 1 &&& 1 & 1 && \\ 1 && 1 && 1 && 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$$X_1^\ell := U_1^\ell\,🔮$$

$Y_1^\ell :=$ channel output of X_1^j

The jth new-channel is $(U_j | Y_1^\ell U_1^{j-1})$

$$🔮 = \ell \times \ell \text{ matrices such as } \begin{bmatrix} 1 &&&&&&& \\ 1 & 1 &&&&&& \\ 1 && 1 &&&&& \\ 1 &&&& 1 &&& \\ 1 & 1 & 1 & 1 &&&& \\ 1 & 1 &&& 1 & 1 && \\ 1 && 1 && 1 && 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

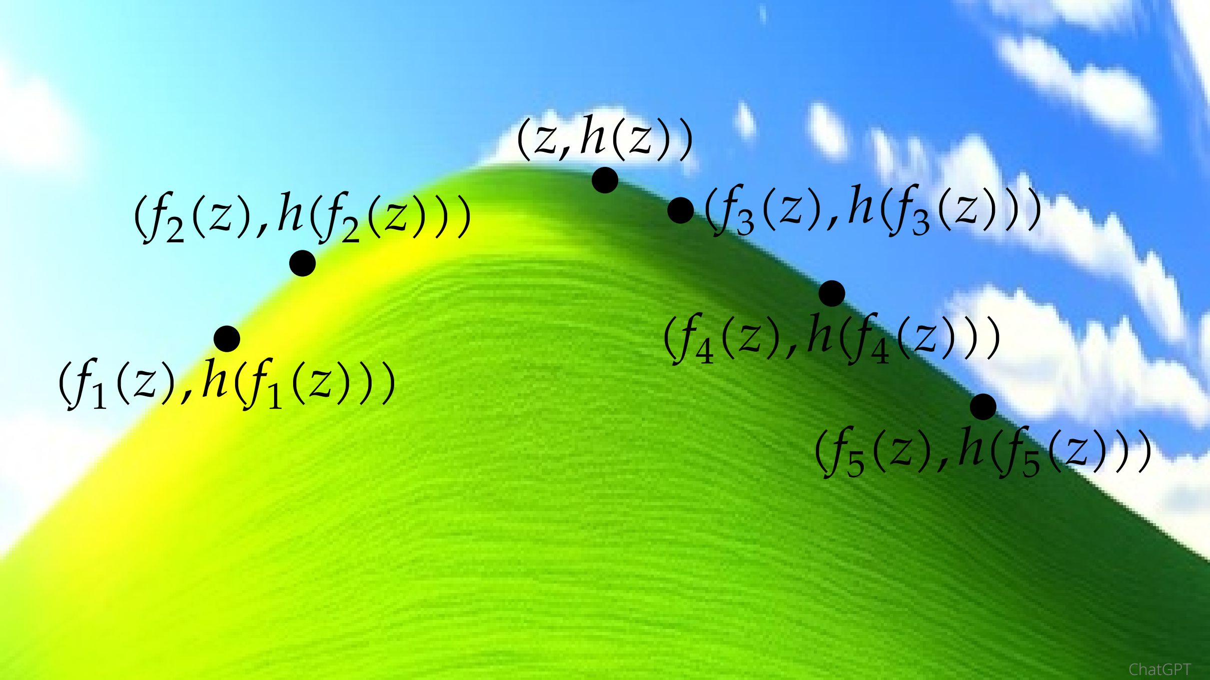
$$X_1^\ell := U_1^\ell 🔮$$

$$Y_1^\ell := \text{channel output of } X_1^j$$

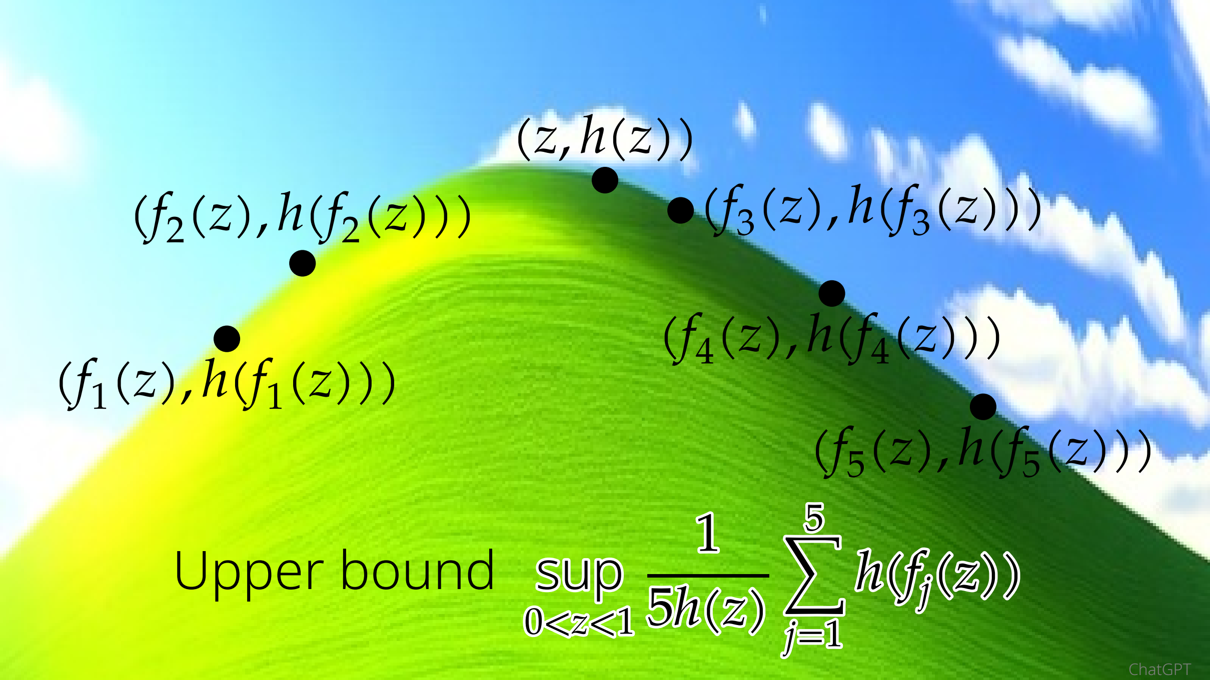
The jth new-channel is $(U_j | Y_1^\ell U_1^{j-1})$

$$f_j(z) := \sum_{C \subset \text{columns}(🔮)} (\text{rk}(C_{j-1}) - \text{rk}(C_j)) z^{\ell - |C|} (1-z)^{|C|}$$

where C_j is C with first j rows removed



$(z, h(z))$
$(f_2(z), h(f_2(z)))$
$(f_3(z), h(f_3(z)))$
$(f_4(z), h(f_4(z)))$
$(f_1(z), h(f_1(z)))$
$(f_5(z), h(f_5(z)))$
ChatGPT

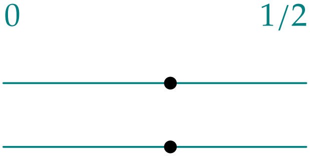

$(z,h(z))$
$(f_2(z),h(f_2(z)))$
$(f_3(z),h(f_3(z)))$
$(f_4(z),h(f_4(z)))$
$(f_1(z),h(f_1(z)))$
$(f_5(z),h(f_5(z)))$
Upper bound $\sup_{0<z<1} \frac{1}{5h(z)} \sum_{j=1}^{5} h(f_j(z))$
ChatGPT

$$\text{Gap to capacity} = \frac{1}{\text{block length}^{???}} \text{ over BEC}$$

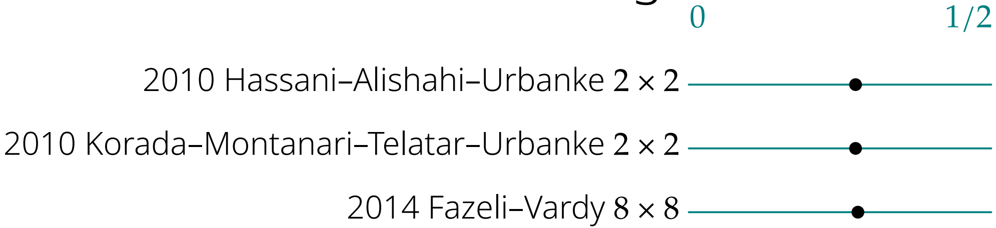
$$\text{Gap to capacity} = \frac{1}{\text{block length}^{???}} \text{ over BEC}$$

2010 Hassani–Alishahi–Urbanke 2×2

2010 Korada–Montanari–Telatar–Urbanke 2×2

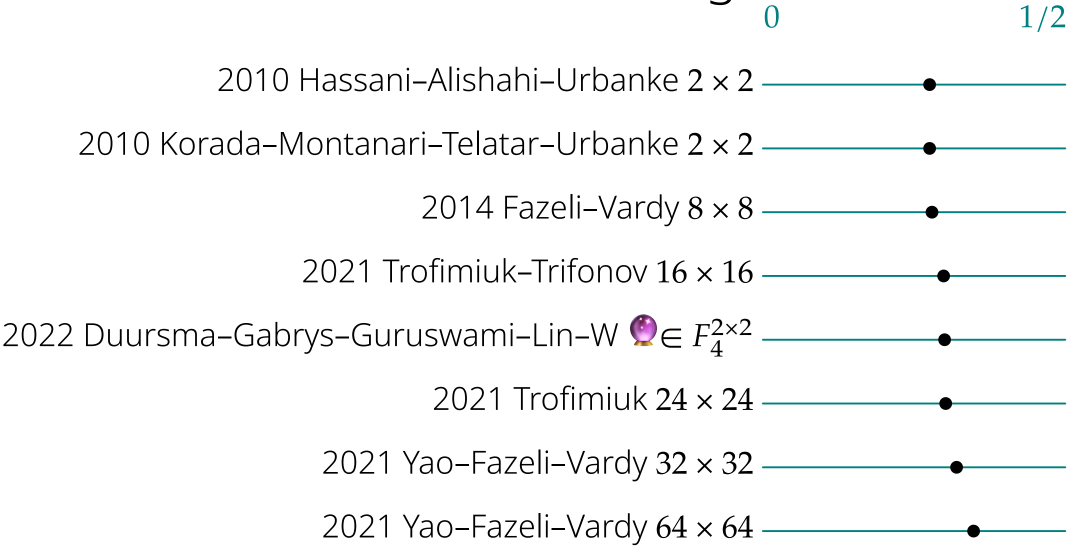


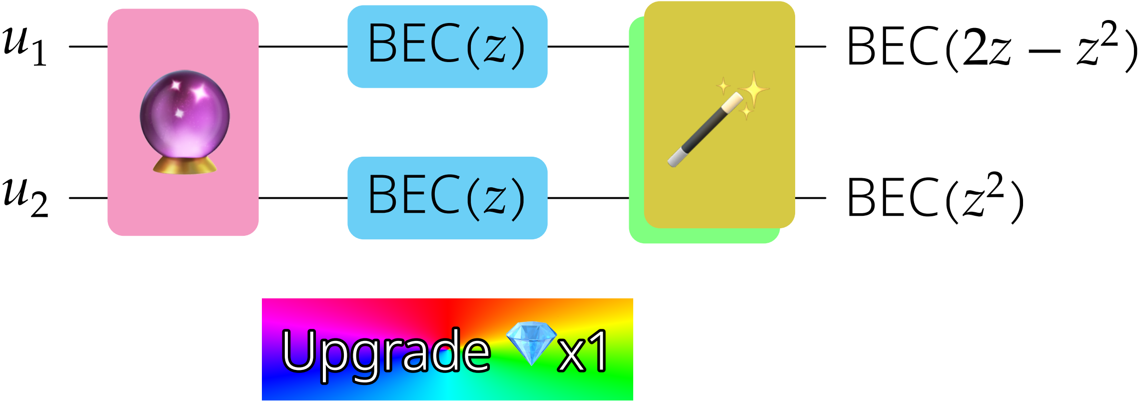
$$\text{Gap to capacity} = \frac{1}{\text{block length}^{???}} \text{ over BEC}$$

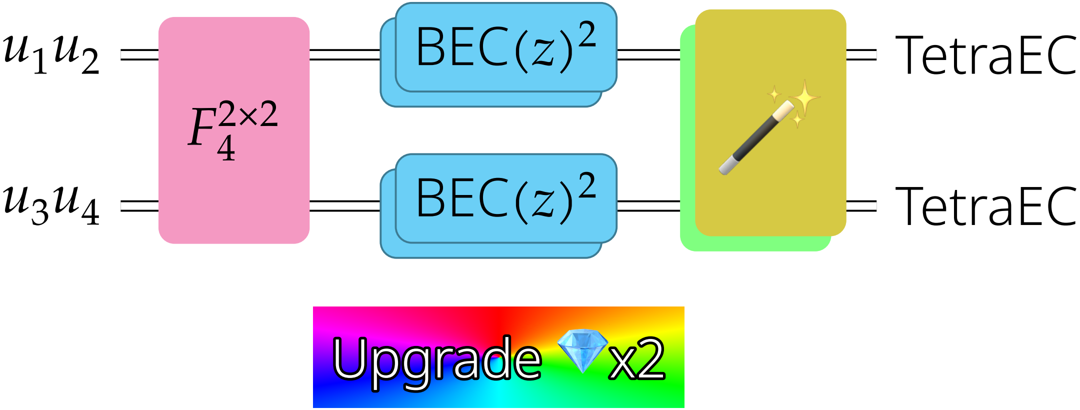


Gap to capacity $= \dfrac{1}{\text{block length}^{???}}$ over BEC

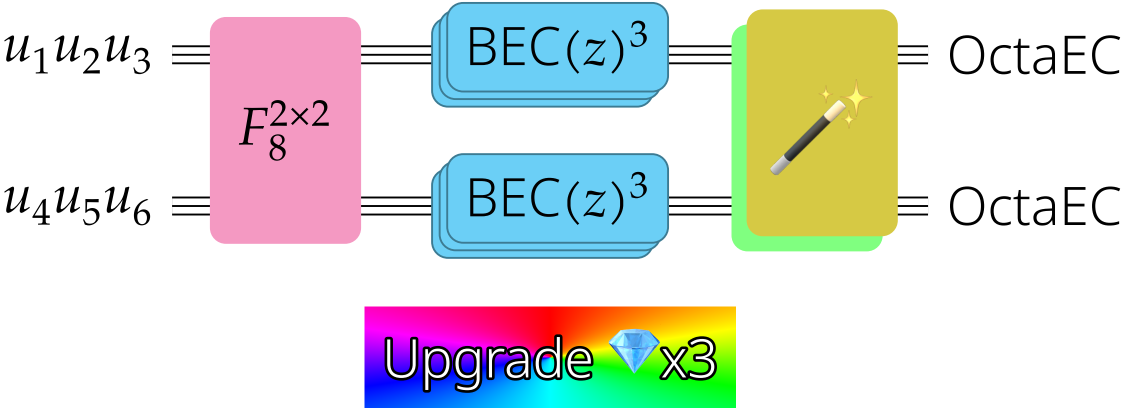
$$\text{Gap to capacity} = \frac{1}{\text{block length}^{???}} \text{ over BEC}$$

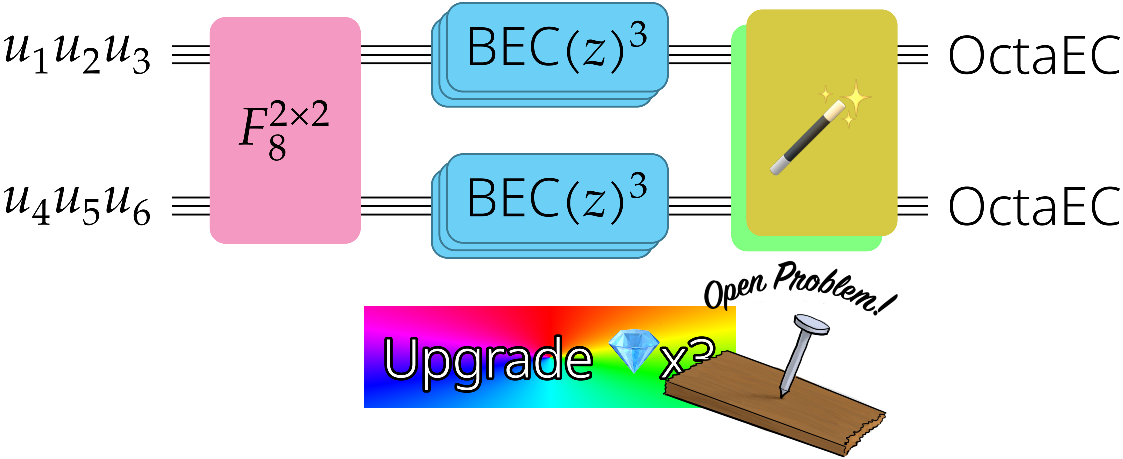



u_1
u_2
$\mathrm{BEC}(z)$
$\mathrm{BEC}(z)$
$\mathrm{BEC}(2z - z^2)$
$\mathrm{BEC}(z^2)$
Upgrade 💎x1



u_1u_2
u_3u_4
$F_4^{2\times2}$
$\mathrm{BEC}(z)^2$
$\mathrm{BEC}(z)^2$
TetraEC
TetraEC
Upgrade 💎x2


$u_1 u_2 u_3$
$u_4 u_5 u_6$
$F_8^{2\times 2}$
$\mathrm{BEC}(z)^3$
$\mathrm{BEC}(z)^3$
OctaEC
OctaEC
Upgrade 💎x3



$u_1u_2u_3$
$u_4u_5u_6$
$F_8^{2\times2}$
$\mathrm{BEC}(z)^3$
$\mathrm{BEC}(z)^3$
OctaEC
OctaEC
Open Problem!
Upgrade 💎 x3

large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

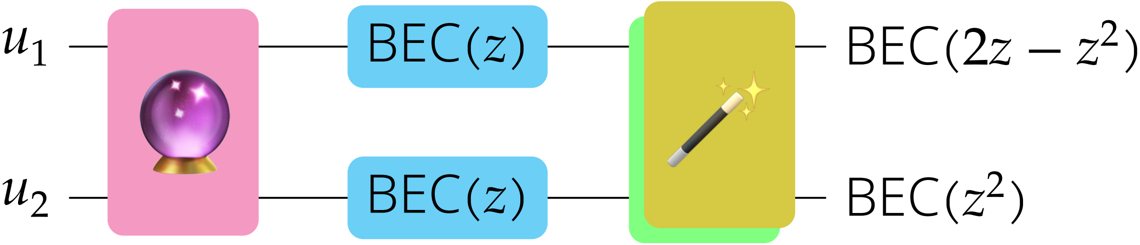
lossy/distribut compress

classical–quantum

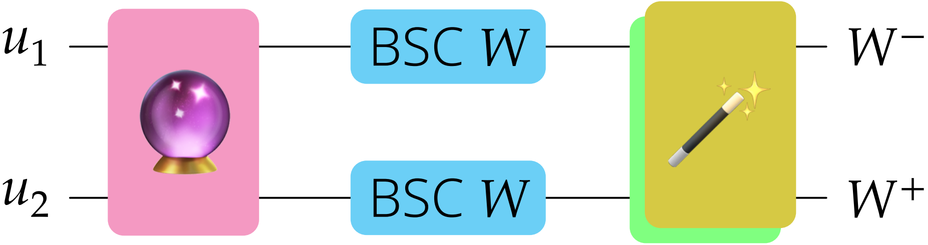
insdel channel

coded computation

PAC & list decoding



Hard Mode 💀



Harder Mode 💀 ☠️

In general, W^{+} and W^{-} are not simple channels

In general, W^+ and W^- are not simple channels

➡️ Invent inequalities of the form

$$Z(W^+) = z^2 \text{ where } z = Z(W)$$

$$z\sqrt{2 - z^2} \leqslant Z(W^-) \leqslant 2z - z^2$$

In general, W^+ and W^- are not simple channels

➡️ Invent inequalities of the form

$$Z(W^+) = z^2 \text{ where } z = Z(W)$$

$$z\sqrt{2 - z^2} \leqslant Z(W^-) \leqslant 2z - z^2$$

➡️ Upper bound $\dfrac{h(Z(W^+)) + h(Z(W^-))}{2h(Z(W))}$

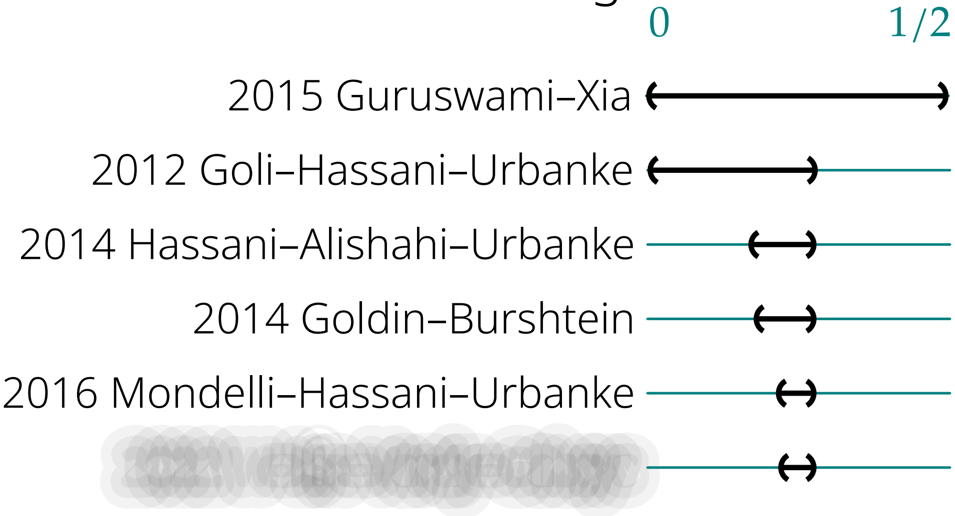
$$\text{Gap to capacity} = \frac{1}{\text{block length}^{???}} \text{ over BSC}$$

$$\text{Gap to capacity} = \frac{1}{\text{block length}^{???}} \text{ over BSC}$$

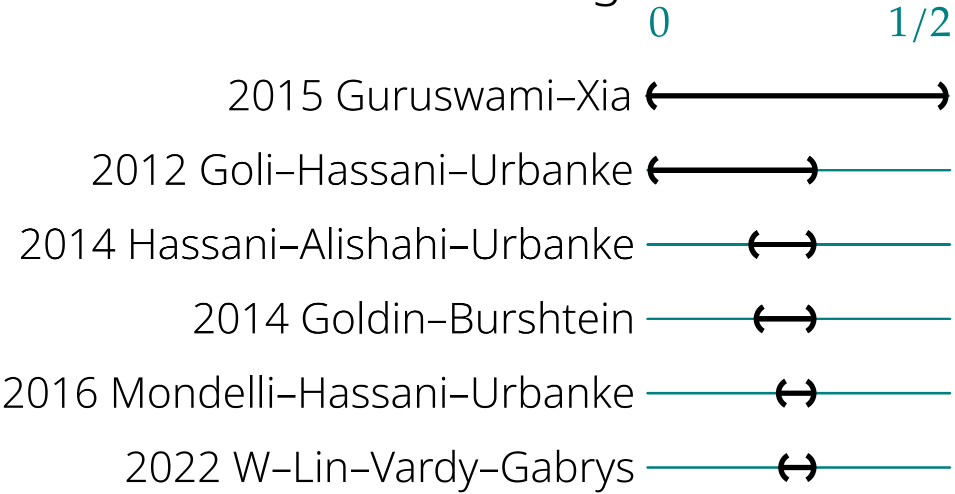
0 $1/2$

2015 Guruswami–Xia

$$\text{Gap to capacity} = \frac{1}{\text{block length}^{???}} \text{ over BSC}$$



$$\text{Gap to capacity} = \frac{1}{\text{block length}^{???}} \text{ over BSC}$$



The last one is special because we bound

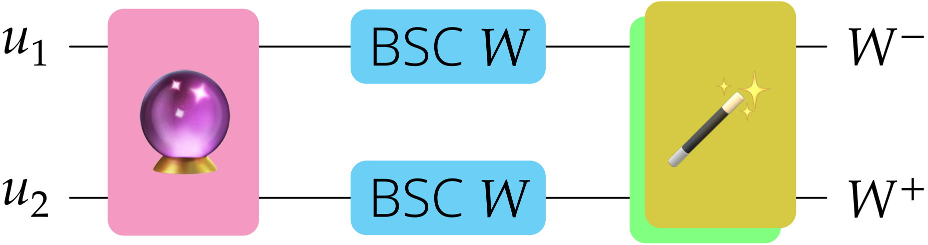
[illegible]

instead of $\dfrac{h(Z(W^{+})) + h(Z(W^{-}))}{2h(W)}$

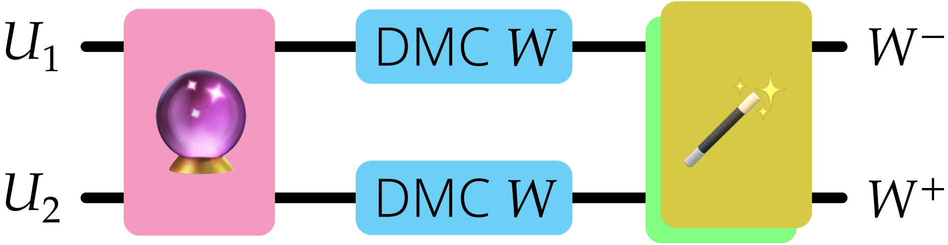
The last one is special because we bound

$$\frac{h(Z(W^{++})) + h(Z(W^{+-})) + h(Z(W^{-+})) + h(Z(W^{--}))}{4h(Z(W))}$$

instead of $\dfrac{h(Z(W^{+})) + h(Z(W^{-}))}{2h(W)}$



Harder Mode 💀 ☠️



Hardest Mode 💀 ☠️ 👻

W^+ and W^- are not simple channels

$Z(W)$ (Bhattacharyya param) does not work well

W^+ and W^- are not simple channels

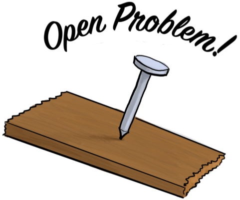
$Z(W)$ (Bhattacharyya param) does not work well

➡️ Invent inequalities of the form
$H(W^-) - H(W^+) > ???$

W^+ and W^- are not simple channels

$Z(W)$ (Bhattacharyya param) does not work well

➡️ Invent inequalities of the form
$H(W^-) - H(W^+) > ???$



Ineq weak when alphabet large

large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

lossy/distribut compress

classical–quantum

insdel channel

better gap to capacity

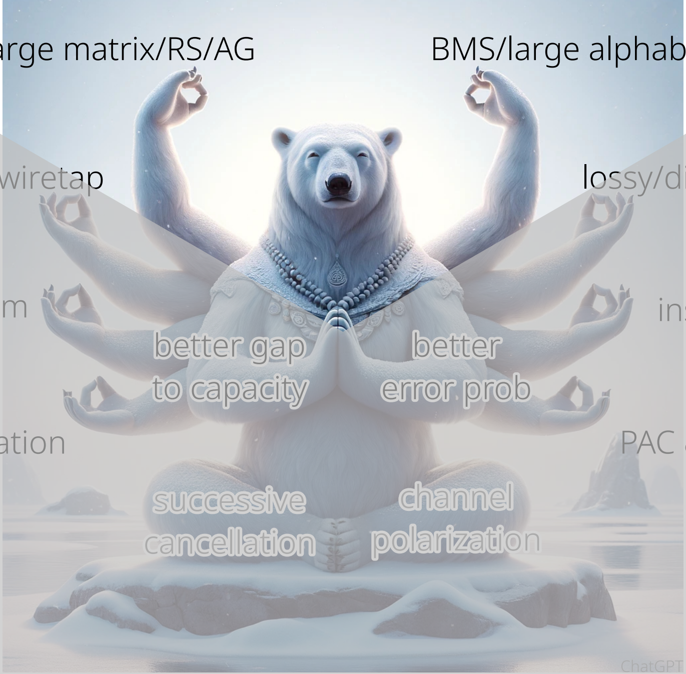
better error prob

coded computation

PAC & list decoding

successive cancellation

channel polarization



$$\text{Optimal Gap} = \frac{1}{\text{block length}^{1/2}} \text{ over channels}$$

$$\text{Optimal Gap} = \frac{1}{\text{block length}^{1/2}} \text{ over channels}$$

0 1/2

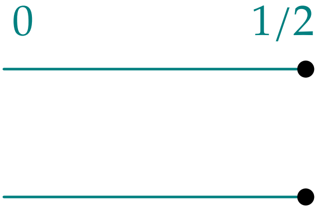
$$\text{Optimal Gap} = \frac{1}{\text{block length}^{1/2}} \text{ over channels}$$

2019 Pfister–Urbanke
q-ary EC, $q \to \infty$

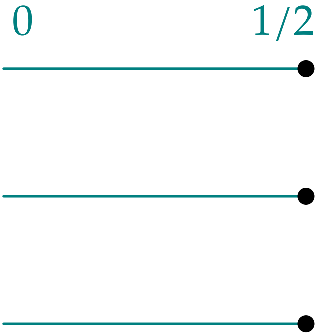
$$\text{Optimal Gap} = \frac{1}{\text{block length}^{1/2}} \text{ over channels}$$

2019 Pfister–Urbanke
q-ary EC, $q \to \infty$

2021 Fazeli–Hassani–Mondelli–Vardy
BEC



$$\text{Optimal Gap} = \frac{1}{\text{block length}^{1/2}} \text{ over channels}$$

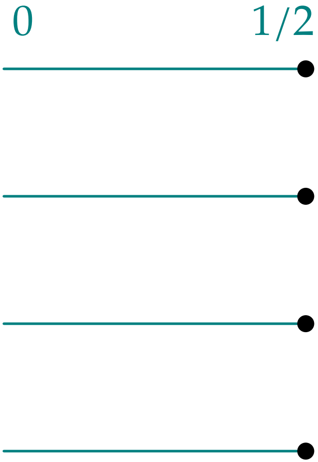


2019 Pfister–Urbanke
q-ary EC, $q \to \infty$

2021 Fazeli–Hassani–Mondelli–Vardy
BEC

2022 Guruswami–Riazanov–Ye
BSC (actually BMSC)

$$\text{Optimal Gap} = \frac{1}{\text{block length}^{1/2}} \text{ over channels}$$

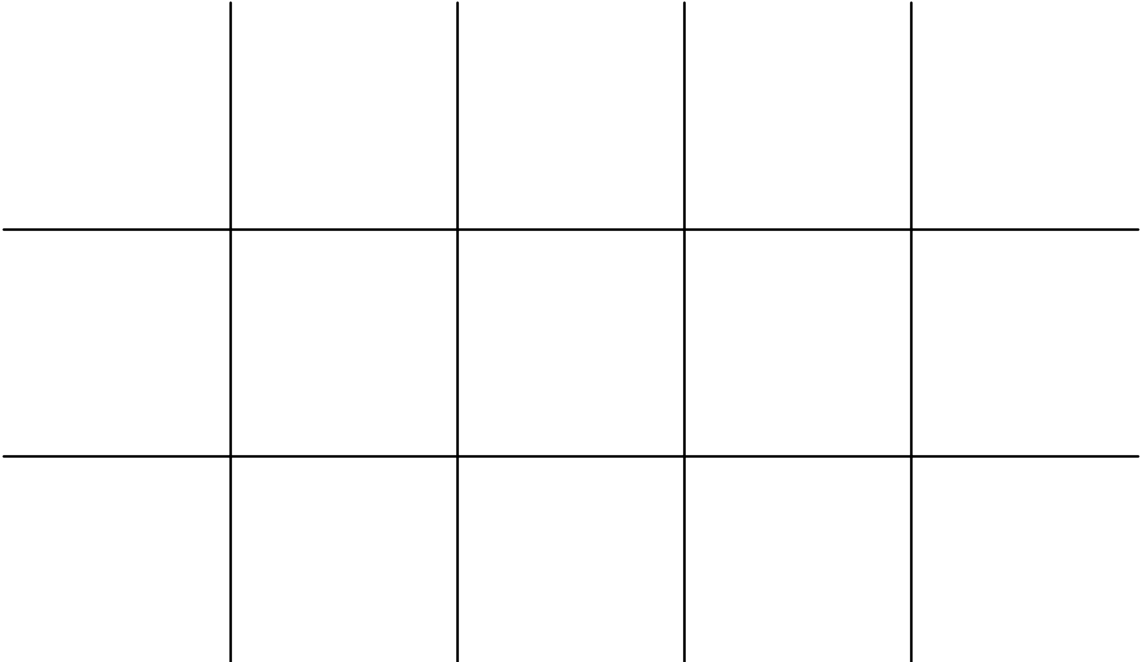


2019 Pfister–Urbanke
q-ary EC, $q \to \infty$

2021 Fazeli–Hassani–Mondelli–Vardy
BEC

2022 Guruswami–Riazanov–Ye
BSC (actually BMSC)

2021 W–Duursma
DMC



Probability Theory				

Probability Theory				
Random Codes				

Probability Theory				
Random Codes				
Polar Codes				

Probability Theory	law of large numbers			
Random Codes				
Polar Codes				

Probability Theory	law of large numbers			
Random Codes	achieve capacity [Shannon]			
Polar Codes				

Probability Theory	law of large numbers			
Random Codes	achieve capacity [Shannon]			
Polar Codes	achieve capacity [Arıkan]			

Probability Theory	law of large numbers	central limit theorem		
Random Codes	achieve capacity [Shannon]			
Polar Codes	achieve capacity [Arıkan]			

Probability Theory	law of large numbers	central limit theorem		
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$		
Polar Codes	achieve capacity [Arıkan]			

Probability Theory	law of large numbers	central limit theorem		
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$		
Polar Codes	achieve capacity [Arıkan]	gap to capacity $1/N^{1/2-\varepsilon}$		

Probability Theory	law of large numbers	central limit theorem	large deviations	
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$		
Polar Codes	achieve capacity [Arıkan]	gap to capacity $1/N^{1/2-\varepsilon}$		

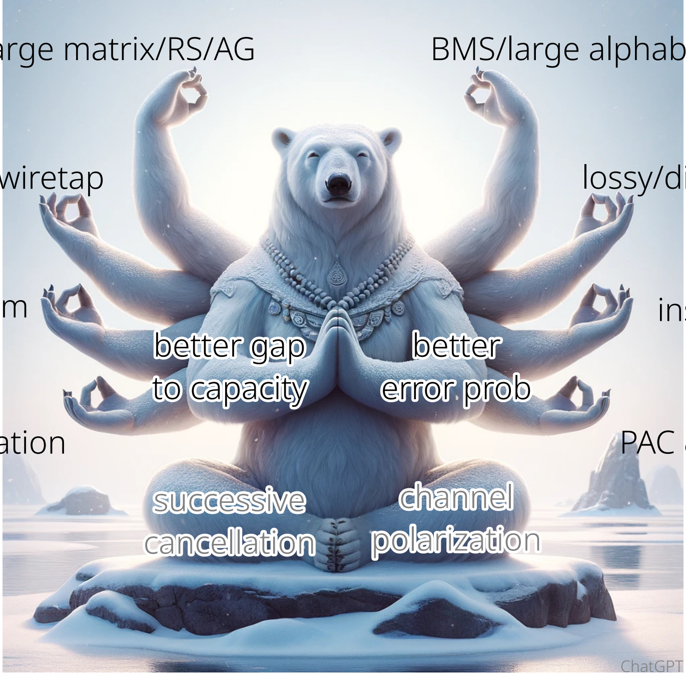
Probability Theory	law of large numbers	central limit theorem	large deviations	
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	error prob $\exp(-N)$	
Polar Codes	achieve capacity [Arıkan]	gap to capacity $1/N^{1/2-\varepsilon}$		

Probability Theory	law of large numbers	central limit theorem	large deviations	
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	error prob $\exp(-N)$	
Polar Codes	achieve capacity [Arıkan]	gap to capacity $1/N^{1/2-\varepsilon}$	error prob $\exp(-N^{1-\varepsilon})$	

Probability Theory	law of large numbers	central limit theorem	large deviations	moderate deviations
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	error prob $\exp(-N)$	
Polar Codes	achieve capacity [Arıkan]	gap to capacity $1/N^{1/2-\varepsilon}$	error prob $\exp(-N^{1-\varepsilon})$	

Probability Theory	law of large numbers	central limit theorem	large deviations	moderate deviations
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	error prob $\exp(-N)$	$\ln(-\ln E)$ $+2\ln(G)$ $=\ln(N)$
Polar Codes	achieve capacity [Arıkan]	gap to capacity $1/N^{1/2-\varepsilon}$	error prob $\exp(-N^{1-\varepsilon})$	

Probability Theory	law of large numbers	central limit theorem	large deviations	moderate deviations
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	error prob $\exp(-N)$	$\ln(-\ln E) + 2\ln(G) = \ln(N)$
Polar Codes	achieve capacity [Arıkan]	gap to capacity $1/N^{1/2-\varepsilon}$	error prob $\exp(-N^{1-\varepsilon})$	$\ln(-\ln E) + 2\ln(G) = (1-\varepsilon)\ln(N)$



large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

lossy/distribut compress

classical–quantum

insdel channel

coded computation

PAC & list decoding

dummy

dummy

Encoding and decoding complexity

Encoding and decoding complexity

is usually $O(N \log N)$ but…

2011 Alamdar-Yazdi–Kschischang:
Prune the tree to reduce complexity.

2017 El-Khamy–Mahdavifar–Feygin–Lee–Kang:
Pruning reduces complexity by a scalar; still $O(N \log N)$.

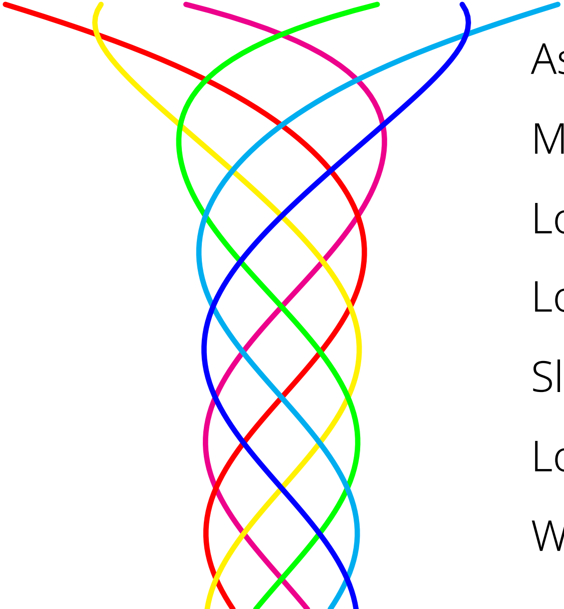
2021 W–Duursma: $O(N \log \log N)$
Trade-off: complexity $\approx O(N \log(-\log(\text{decode error})))$.

2021 Mondelli–Hashemi–Cioffi–Goldsmith,
2021 Hashemi–Mondelli–Fazeli–Vardy–Cioffi–Goldsmith:
Study parallelism vs latency.

Polar code is a mathy code

Polar code is a mathy code

Polar achieves the capacity of



Asymmetric channel

Multiple access channel

Lossless compression

Lossy compression

Slepian–Wolf

Lossless compression w/ helper

Wiretap channel (degradation)

Deletion channel … (good error prob)

Broadcast channel … (good error prob)

Channel with memory … (good error prob)

Wiretap channel (no degradation) … (good error prob)

Hidden Markov chain channel state … (good error prob)

Non-stationary channel … (good gap to capacity)

Classical–Quantum channel … (yes)

Quantum–Quantum channel … (?)