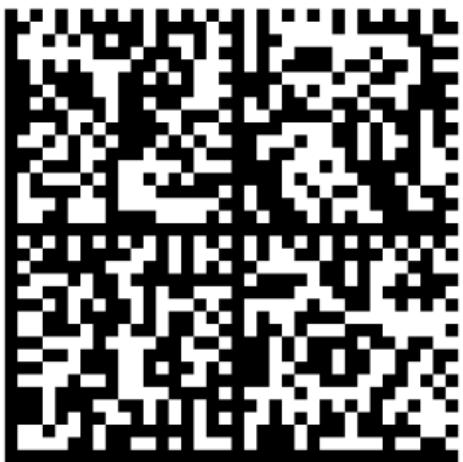


Slides available on



<https://polar-tutorial.symbol.codes/>

# Polar Codes Tutorial

Hsin-Po Wang

(Berkeley, EECS)



ChatGPT

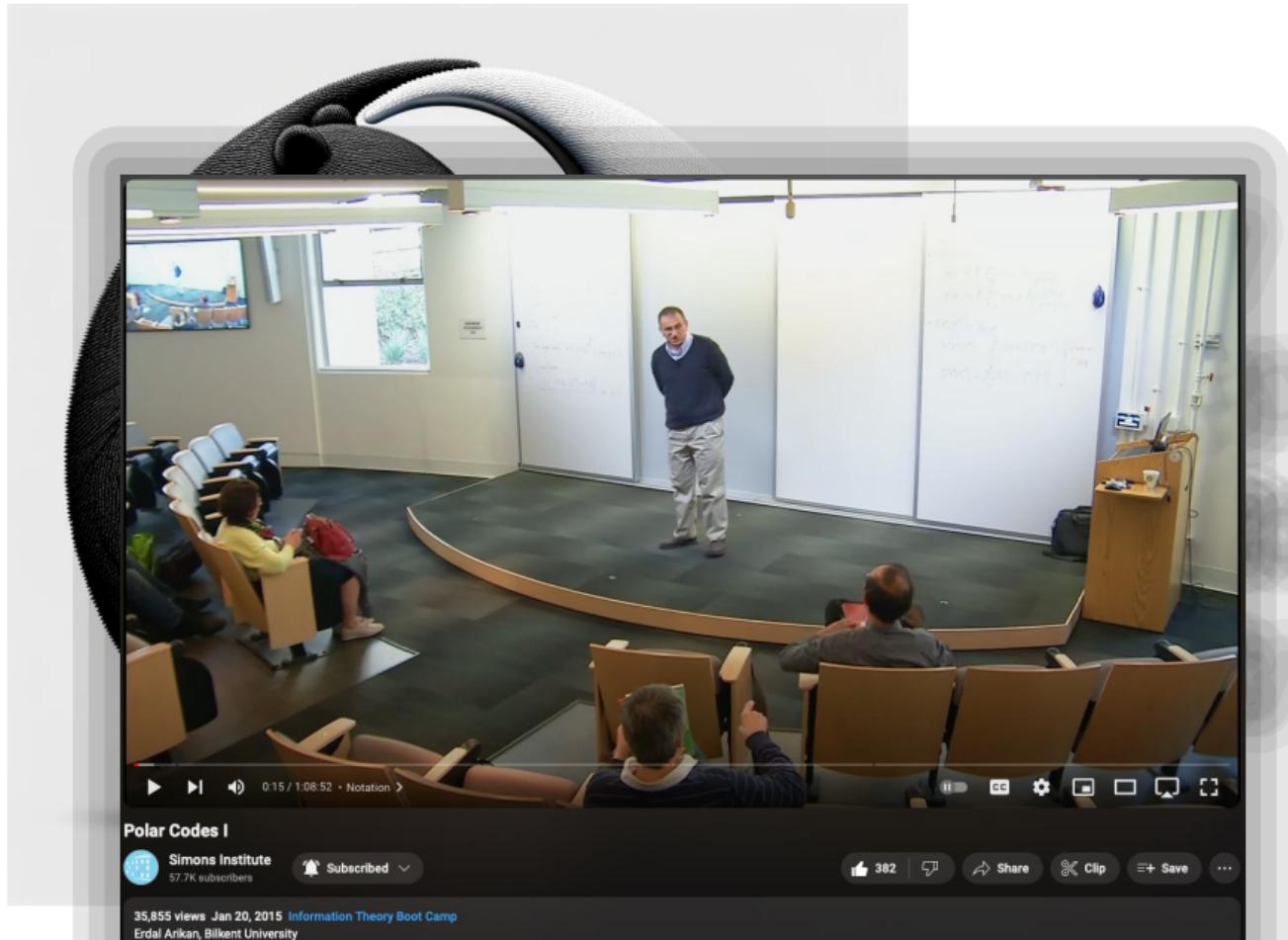
Engineering  
List decoder  
Future  
Origin Story



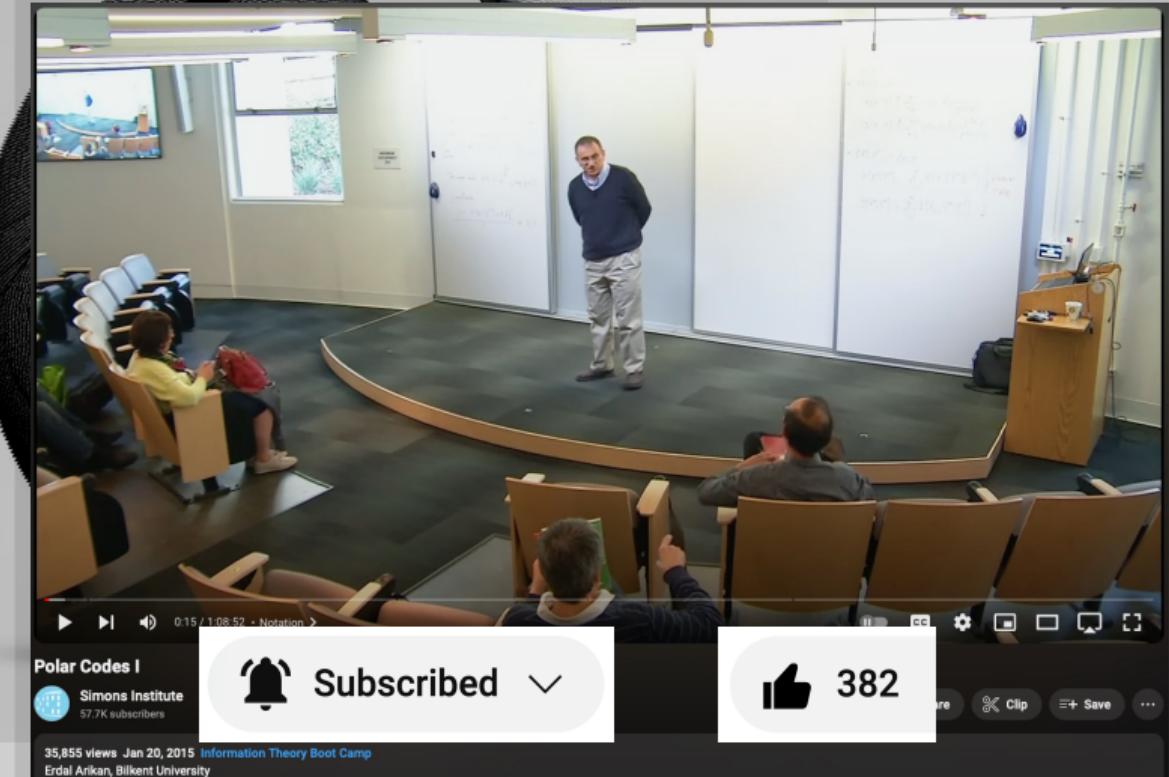
ChatGPT



# Engineering List decoder Future Origin Story



# Engineering List decoder Future Origin Story



Engineering  
List decoder  
Future  
Origin Story



ChatGPT



Engineering  
List decoder  
Future  
Origin Story



Theory  
Math args  
Asymptotics  
Versatility

What's so innovative  
about polar codes?

## WHO IS CLAUDE SHANNON?



# THE BIT PLAYER

A FILM BY MARK A. LEVINSON

IMDb

# WHO IS CLAUDE SHANNON?

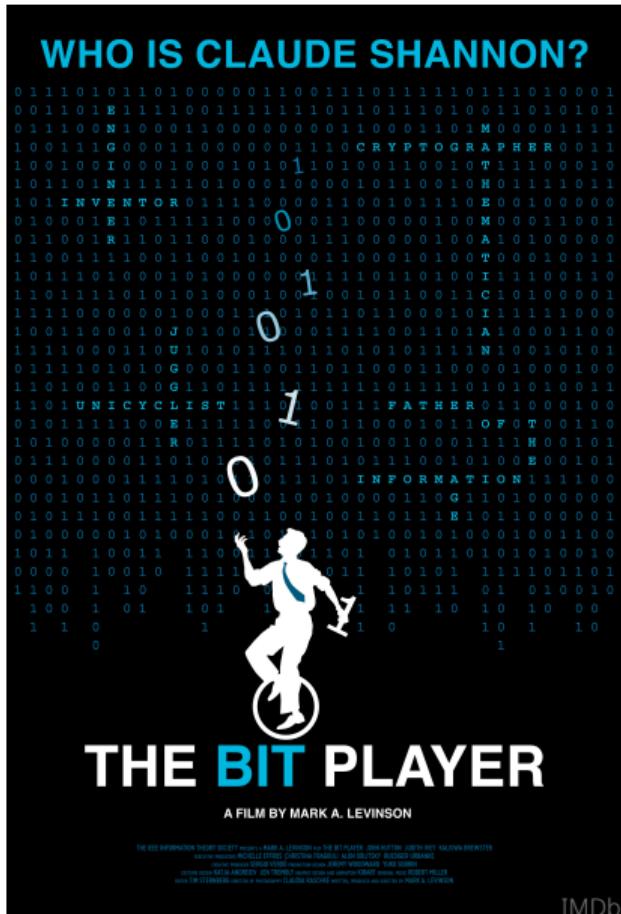


# THE BIT PLAYER

A FILM BY MARK A. LEVINSON

IMDb

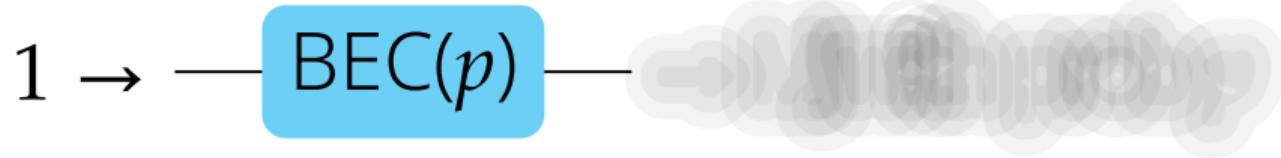




Let's play with

**binary erasure channels**

(BECs)



$0 \rightarrow$  BEC( $p$ )  $\rightarrow 0$  with prob  $1 - p$

$0 \rightarrow$  BEC( $p$ )  $\rightarrow$  🤔 with prob  $p$

$1 \rightarrow$  BEC( $p$ )



$1 \rightarrow$  BEC( $p$ )

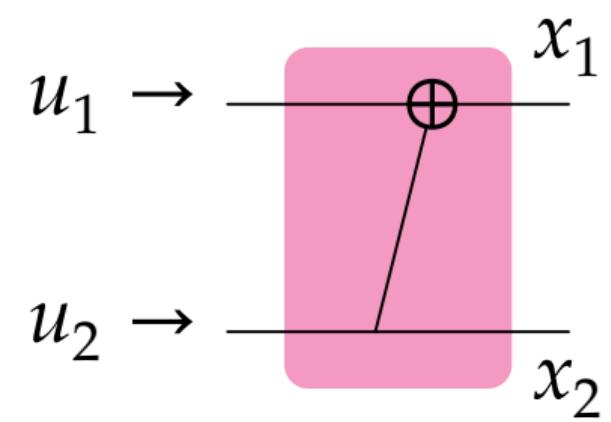


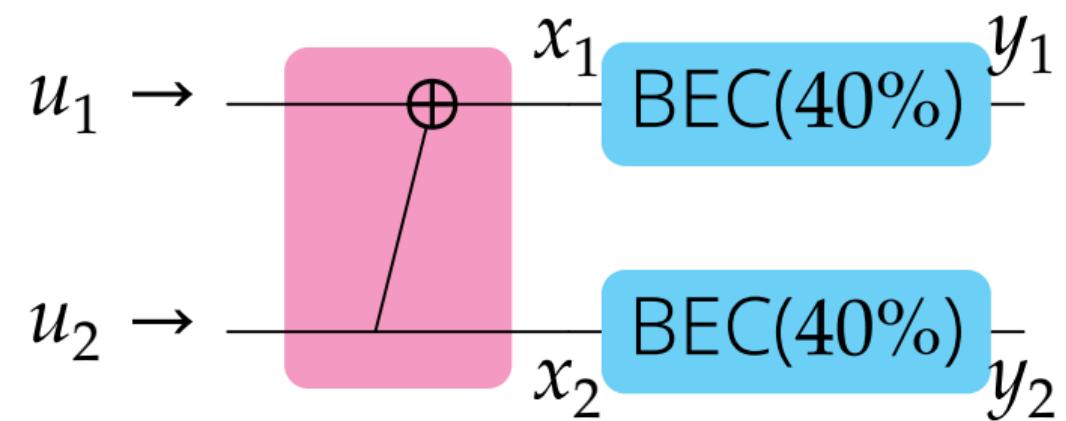
$0 \rightarrow$  BEC( $p$ )  $\rightarrow 0$  with prob  $1 - p$

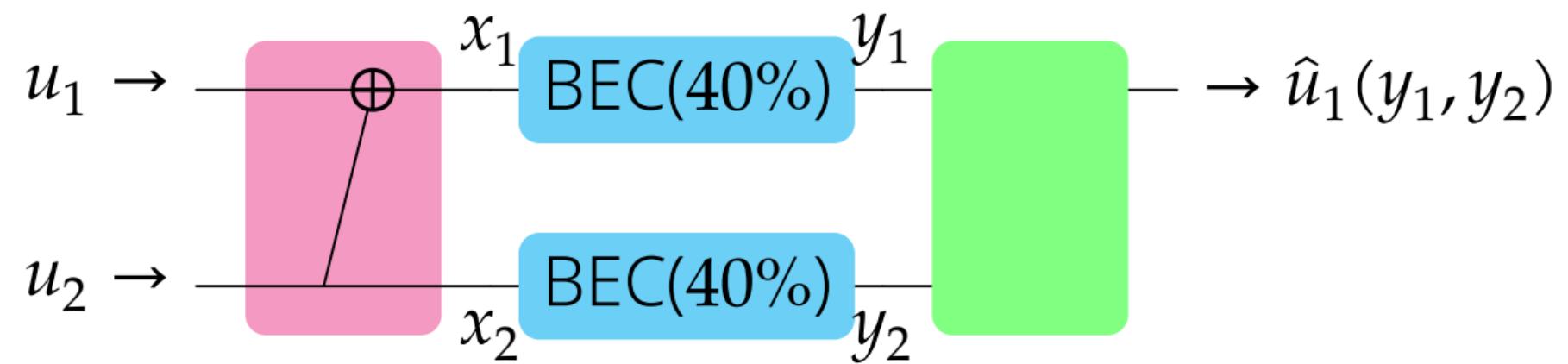
$0 \rightarrow$  BEC( $p$ )  $\rightarrow$  🤔 with prob  $p$

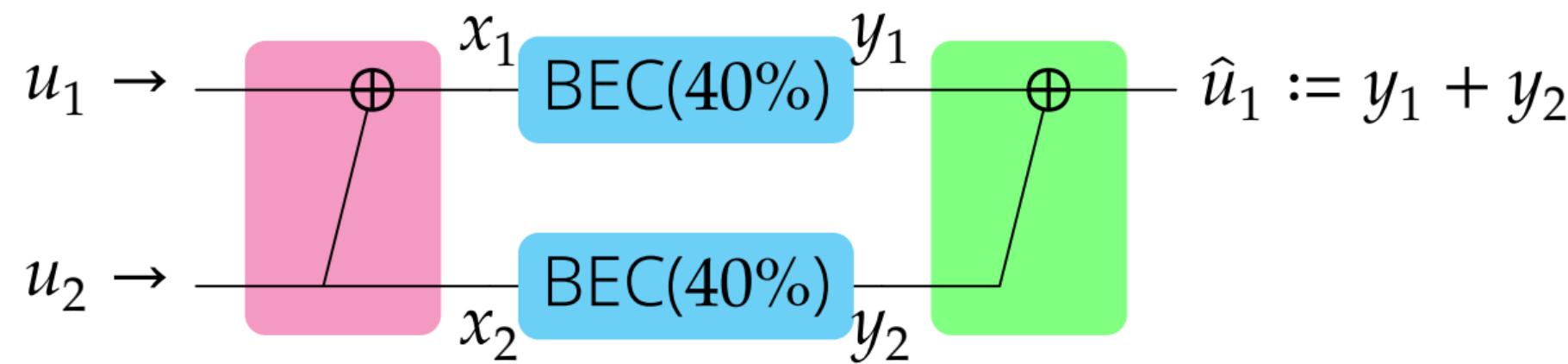
$1 \rightarrow$  BEC( $p$ )  $\rightarrow$  🤔 with prob  $p$

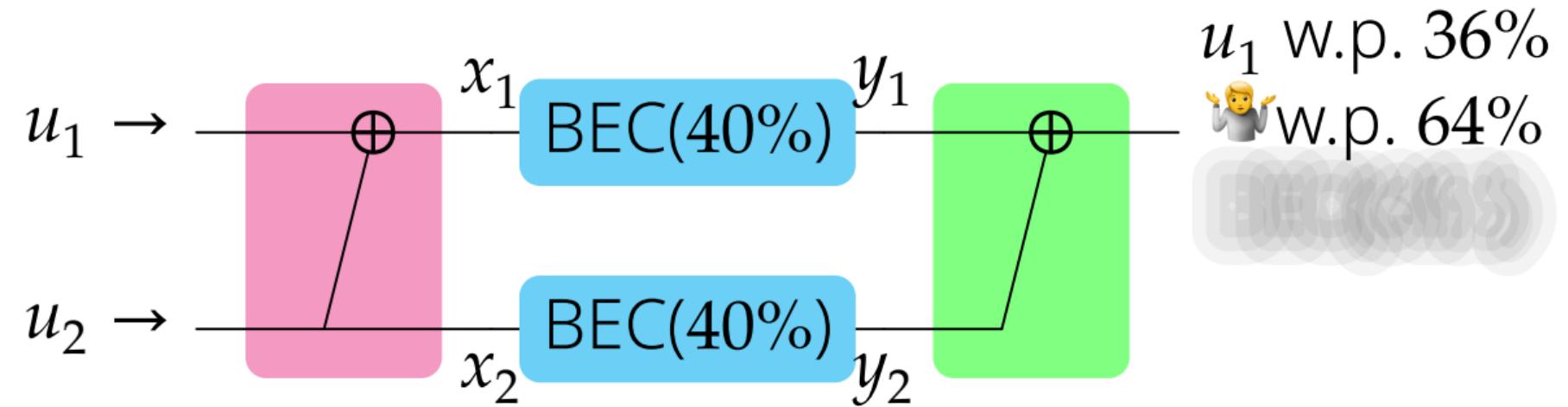
$1 \rightarrow$  BEC( $p$ )  $\rightarrow 1$  with prob  $1 - p$



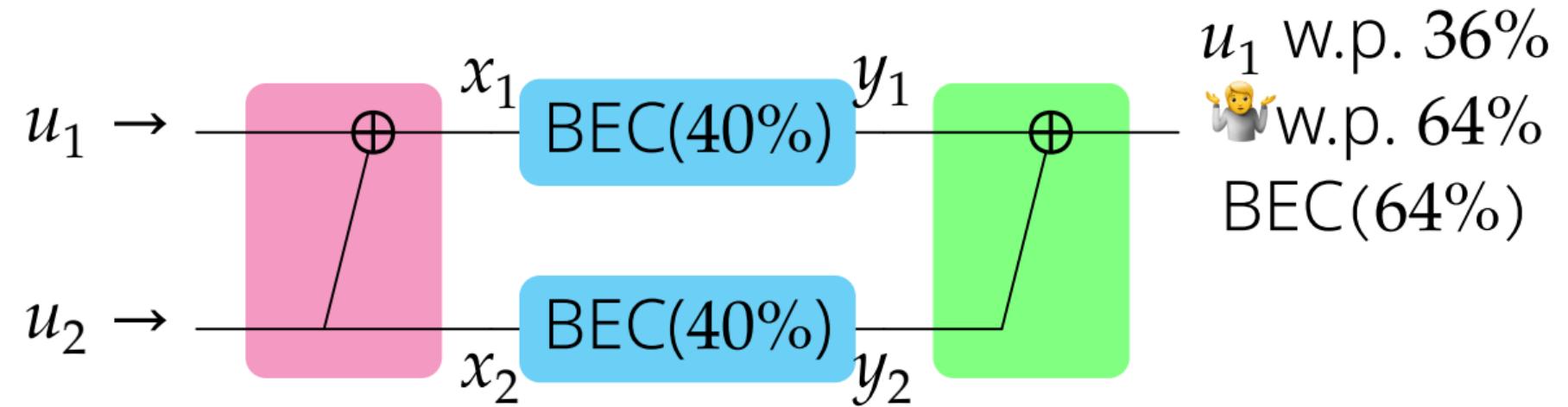




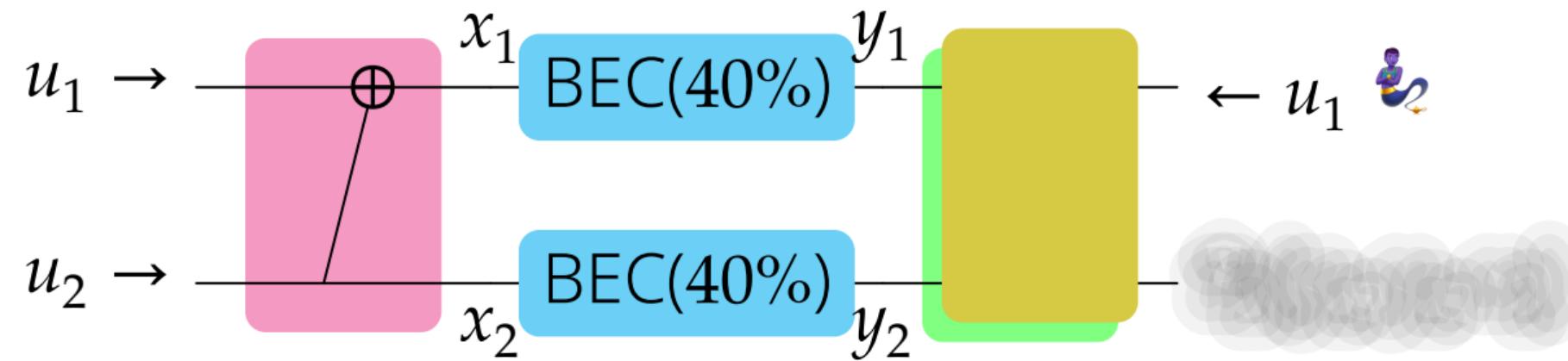




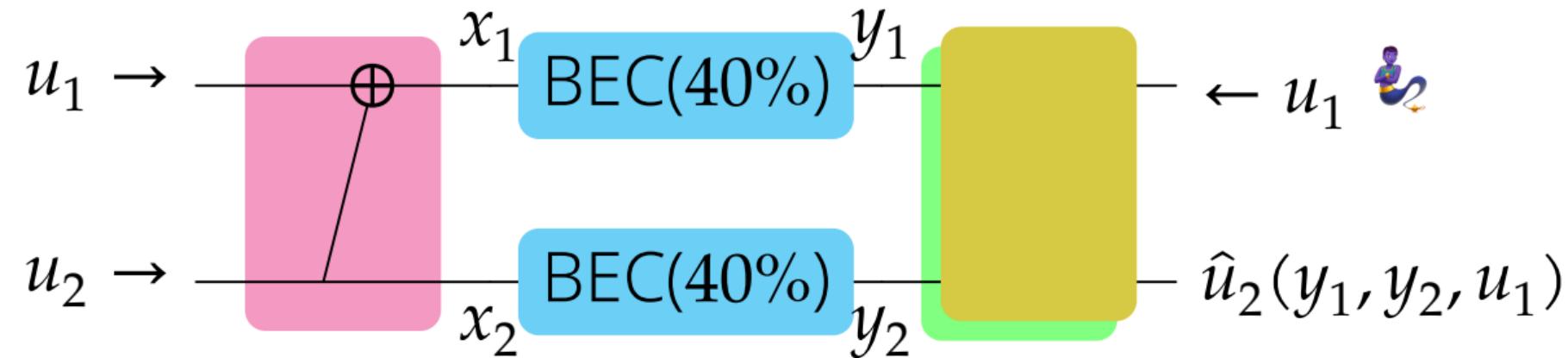
$u_1$  w.p. 36%  
🤷 w.p. 64%

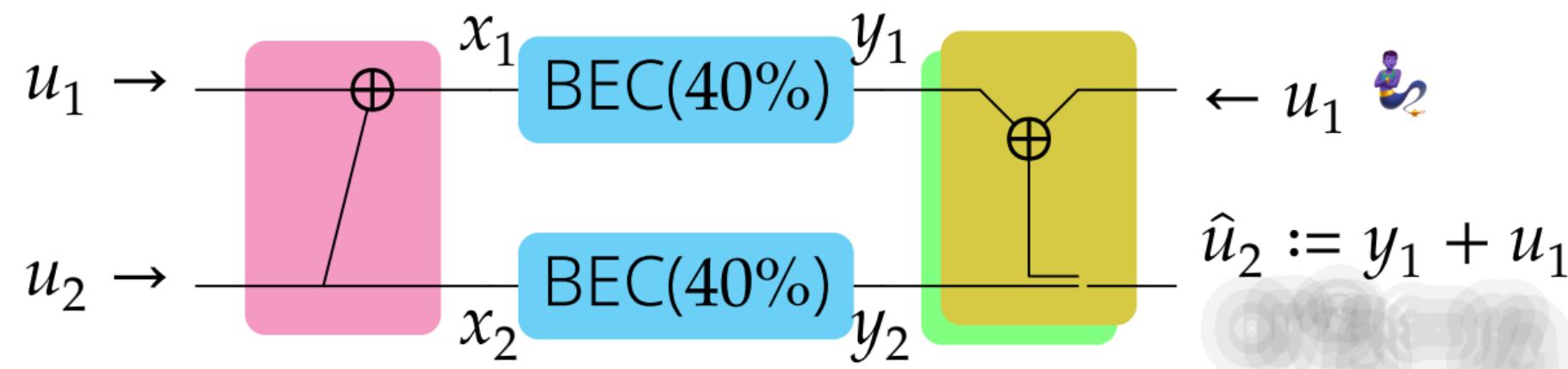


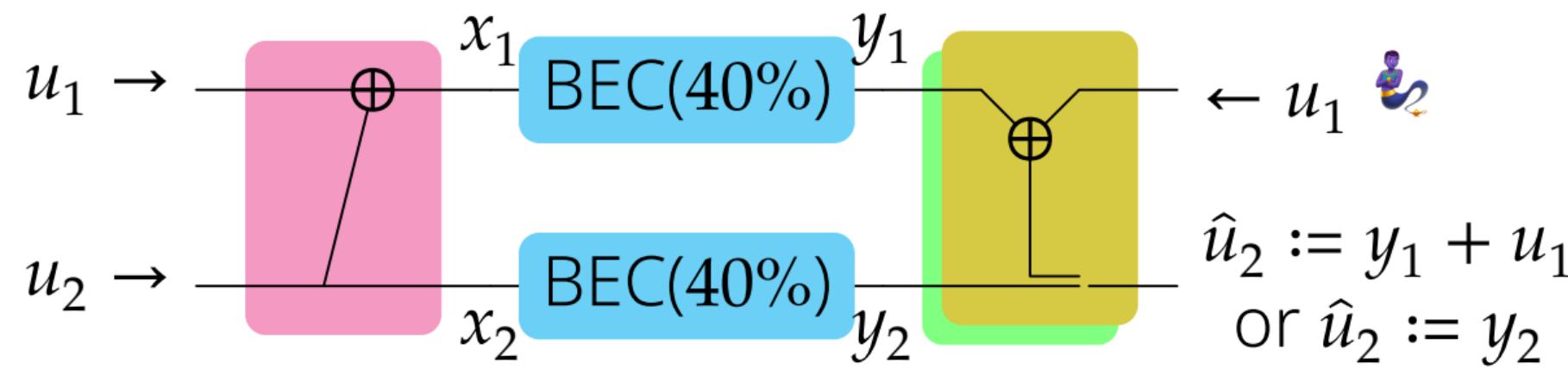
Genie tells the correct  $u_1$   
after you submit a guess

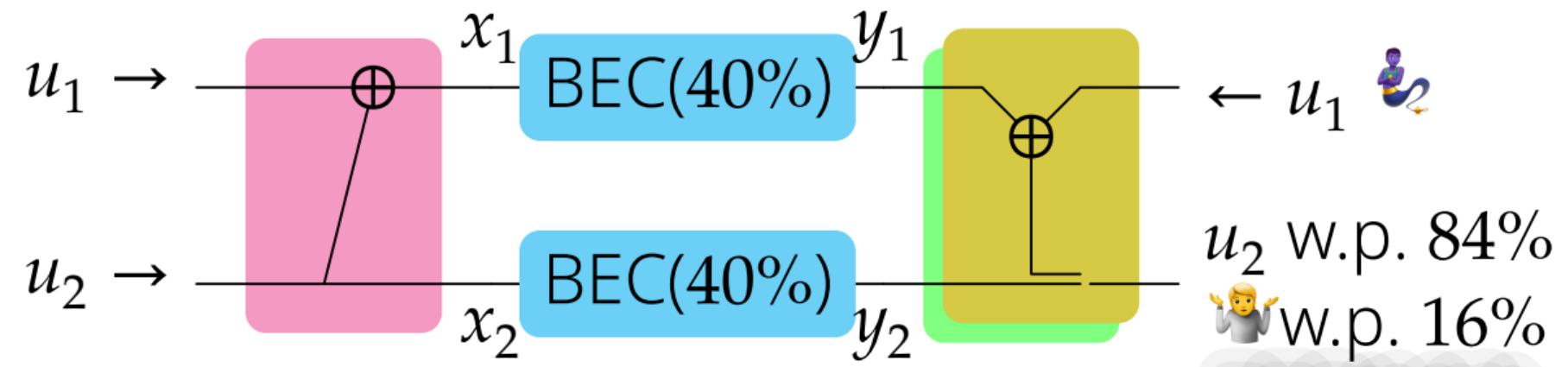


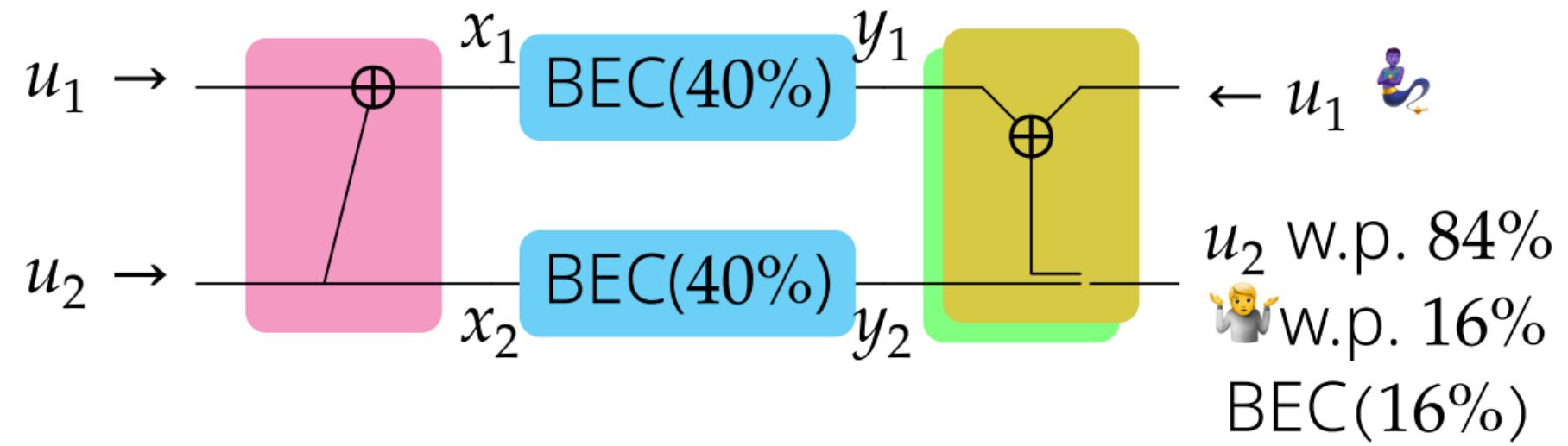
Genie tells the correct  $u_1$   
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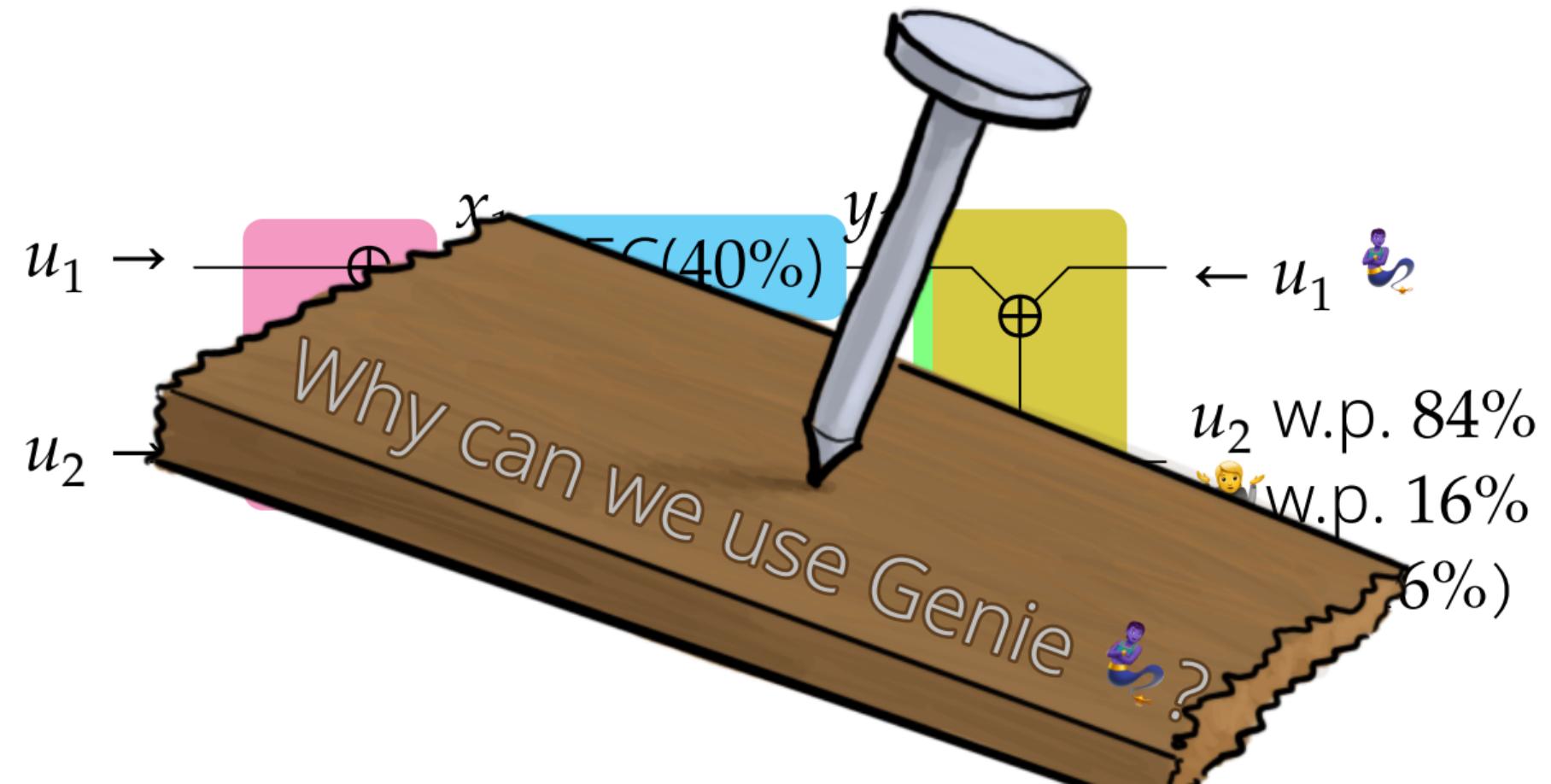












- ▶ Case 1:  $\hat{u}_1 = u_1$

- ▶ Case 2:

- ▶ Case 3:

- ▶ Case 1:  $\hat{u}_1 = u_1$   
does not provide any useful info

- ▶ Case 2:

- ▶ Case 3:

- ▶ Case 1:  $\hat{u}_1 = u_1$   
does not provide any useful info

- ▶ Case 2:  $\hat{u}_1 \neq u_1$

- ▶ Case 3:

- ▶ Case 1:  $\hat{u}_1 = u_1$   
does not provide any useful info
- ▶ Case 2:  $\hat{u}_1 \neq u_1$   
Give up the whole block
- ▶ Case 3:

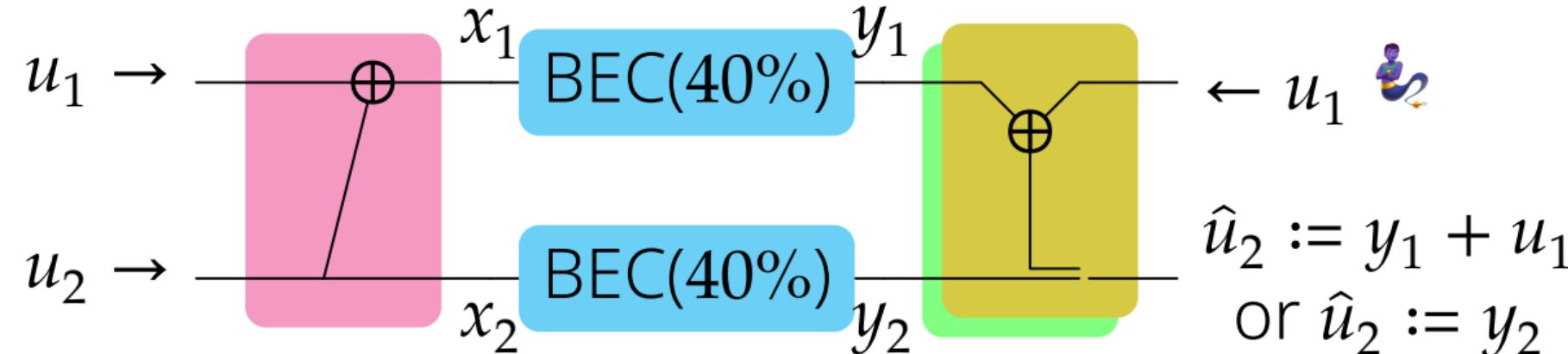
- ▶ Case 1:  $\hat{u}_1 = u_1$   
does not provide any useful info
- ▶ Case 2:  $\hat{u}_1 \neq u_1$   
Give up the whole block
- ▶ Case 3: sender always sends  $u_1 \equiv 0$



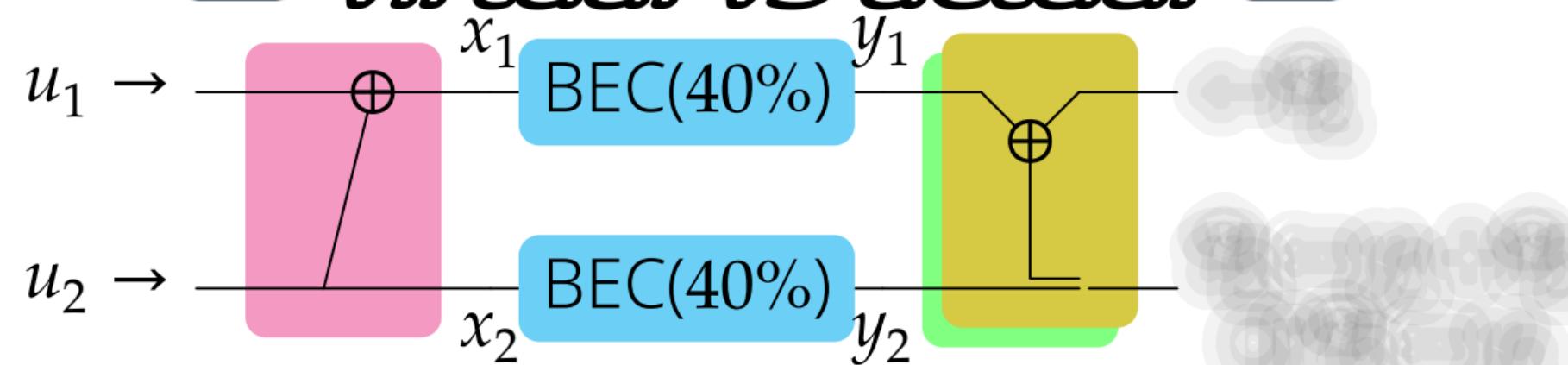
- ▶ Case 1:  $\hat{u}_1 = u_1$   
does not provide any useful info
- ▶ Case 2:  $\hat{u}_1 \neq u_1$   
Give up the whole block
- ▶ Case 3: sender always sends  $u_1 \equiv 0$   
 $\hat{u}_1 \equiv 0$

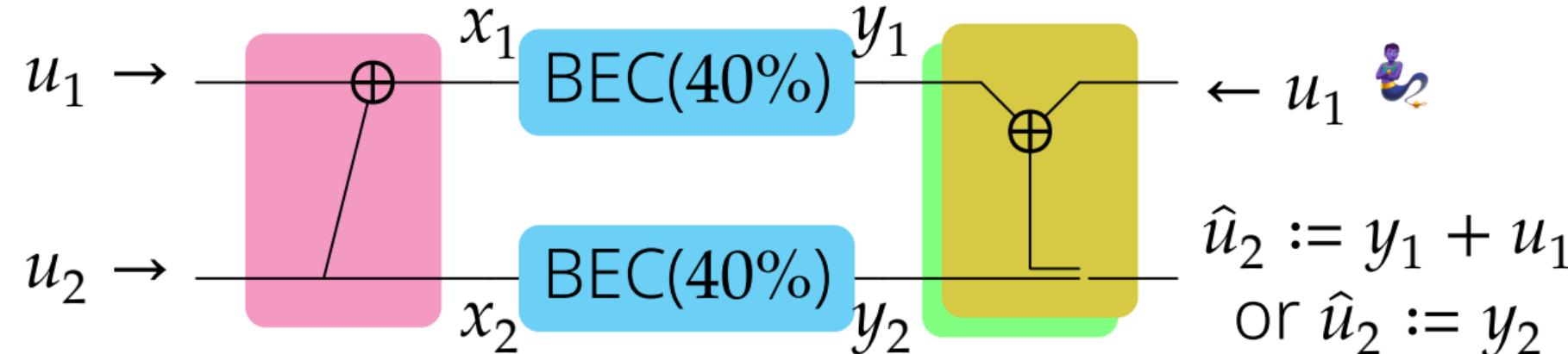
- ▶ Case 1:  $\hat{u}_1 = u_1$   
does not provide any useful info
- ▶ Case 2:  $\hat{u}_1 \neq u_1$   
Give up the whole block
- ▶ Case 3: sender always sends  $u_1 \equiv 0$   
 $\hat{u}_1 \equiv 0$   
This is called **frozen bit**



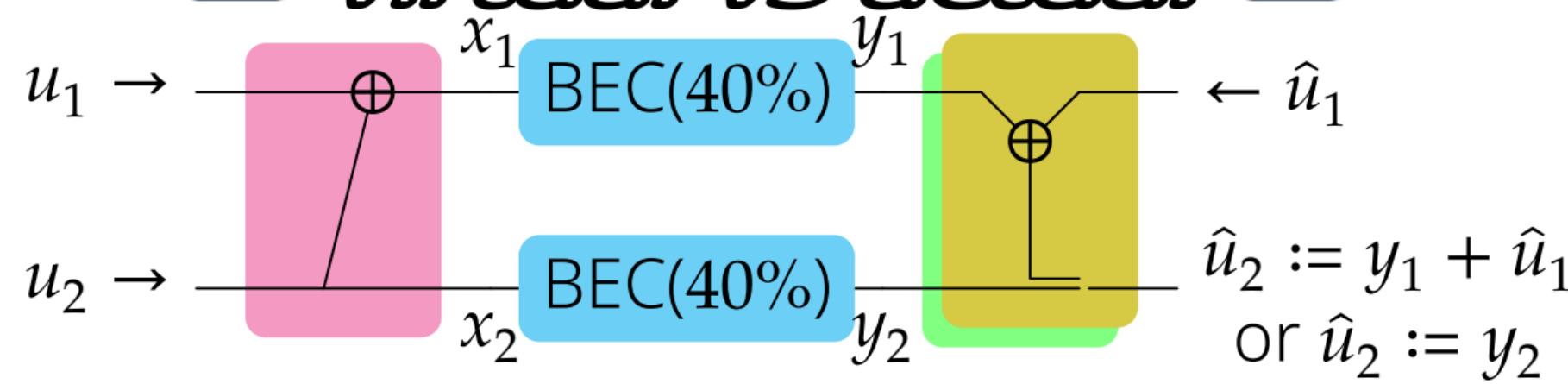


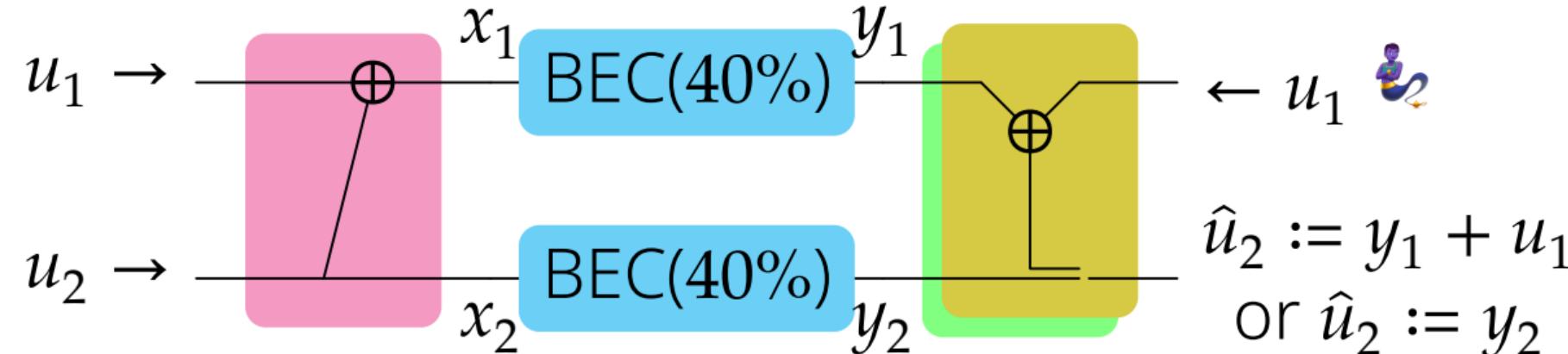
*virtual vs actual*



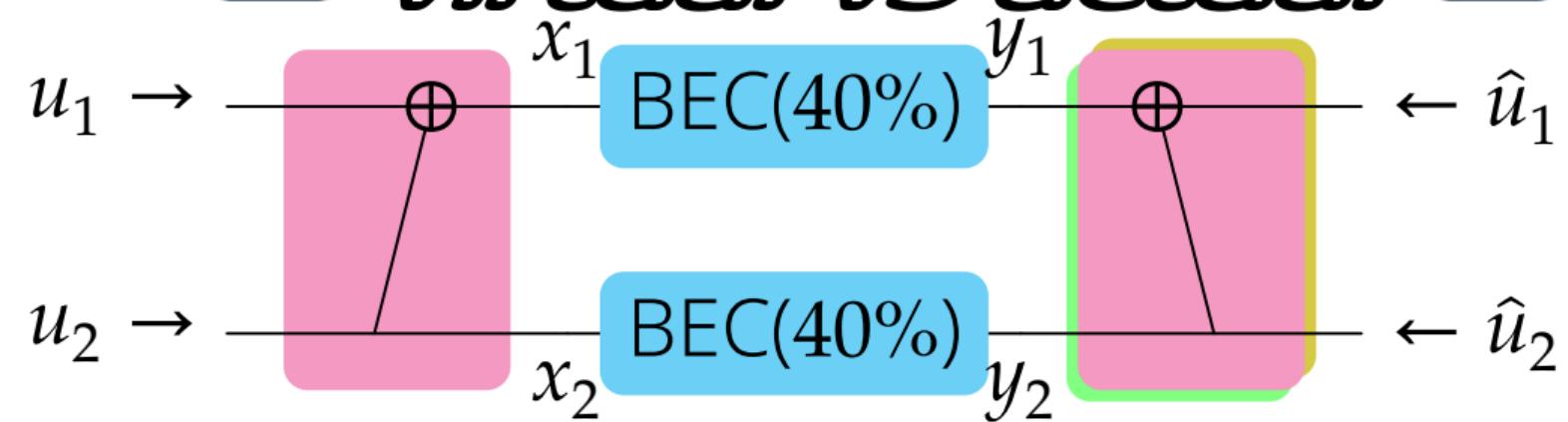


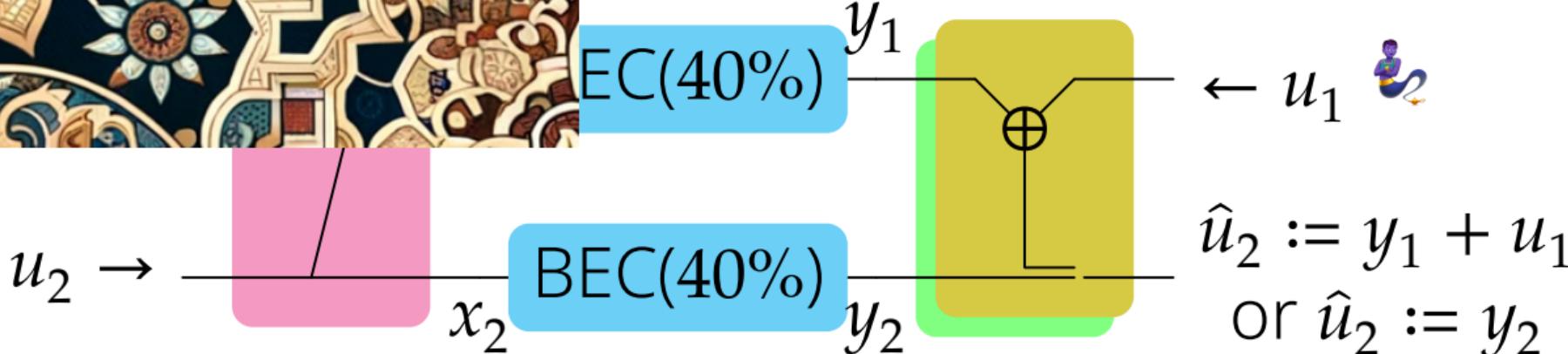
*virtual vs actual*



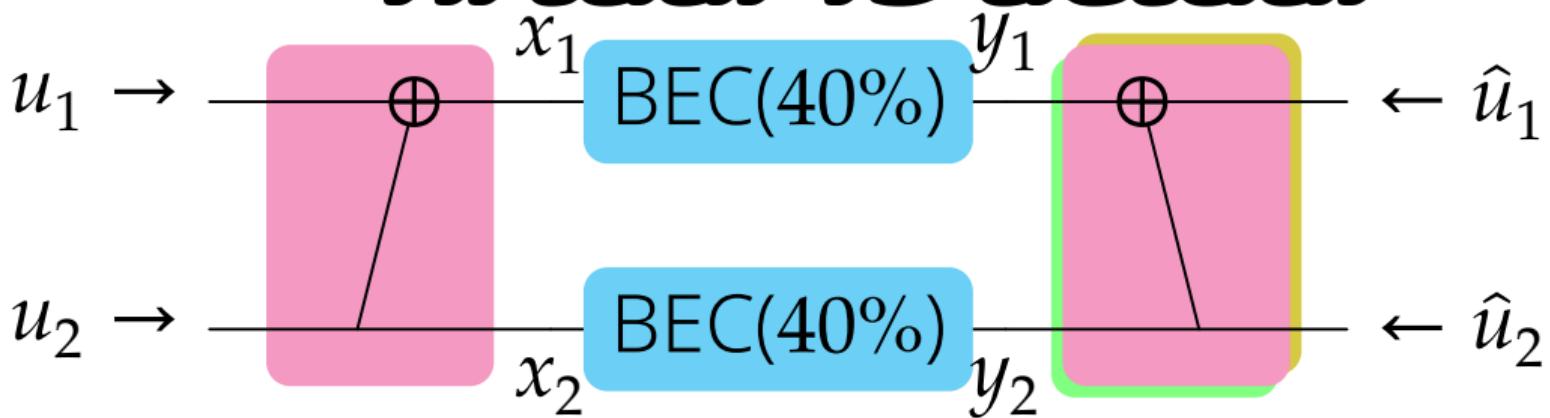


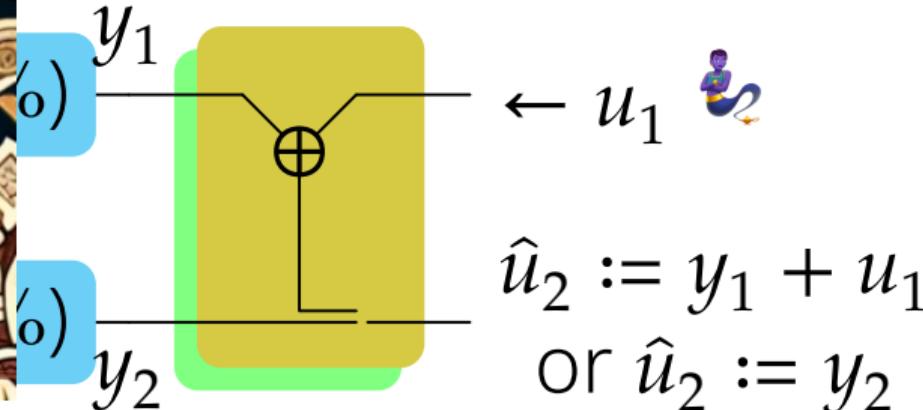
*virtual vs actual*



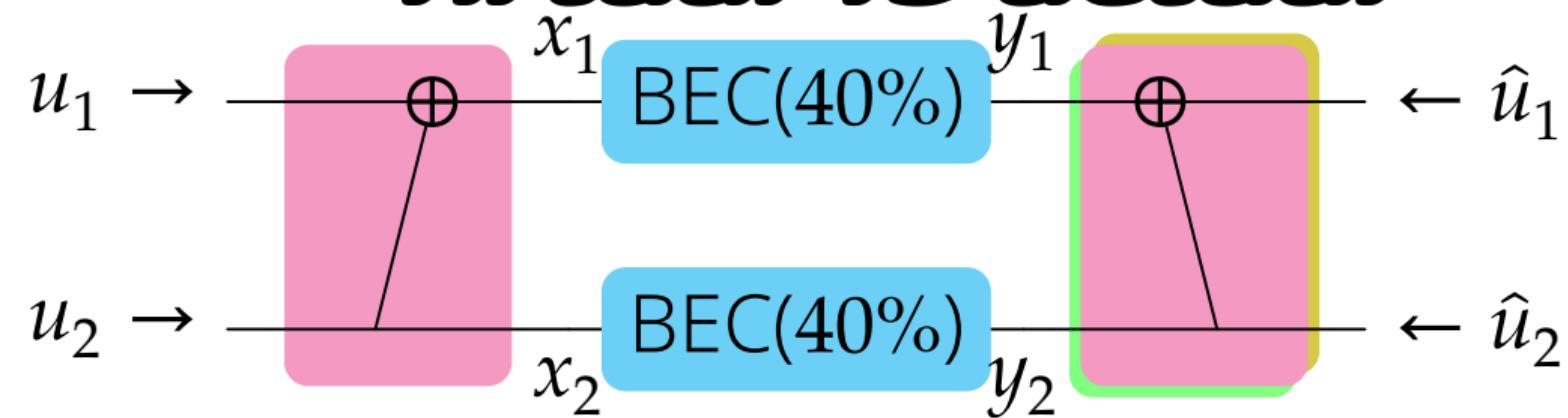


 *virtual vs actual* 



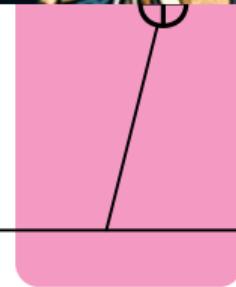


## virtual vs actual





$u_1$



BEC(40%)

$u_2 \rightarrow$

$x_2$  BEC(40%)

$y_2$



$\leftarrow u_1$

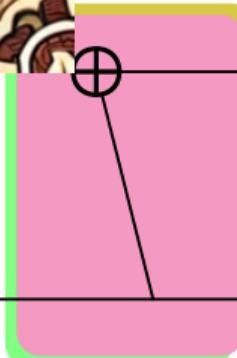


$\hat{u}_2 := y_1 + u_1$   
or  $\hat{u}_2 := y_2$

*virtual*



$\leftarrow \hat{u}_1$



$\leftarrow \hat{u}_2$



$\leftarrow u_1$



$\hat{u}_2 := y_1 + u_1$   
or  $\hat{u}_2 := y_2$



$\leftarrow \hat{u}_1$

$\leftarrow \hat{u}_2$

$u_2$

$x_2$   $\text{DEC}(40\%)$   $y_2$

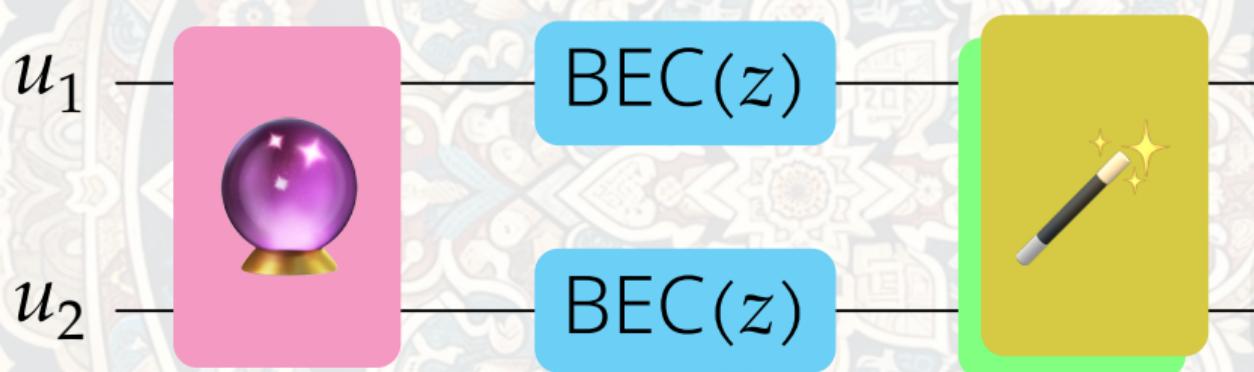
Don't click

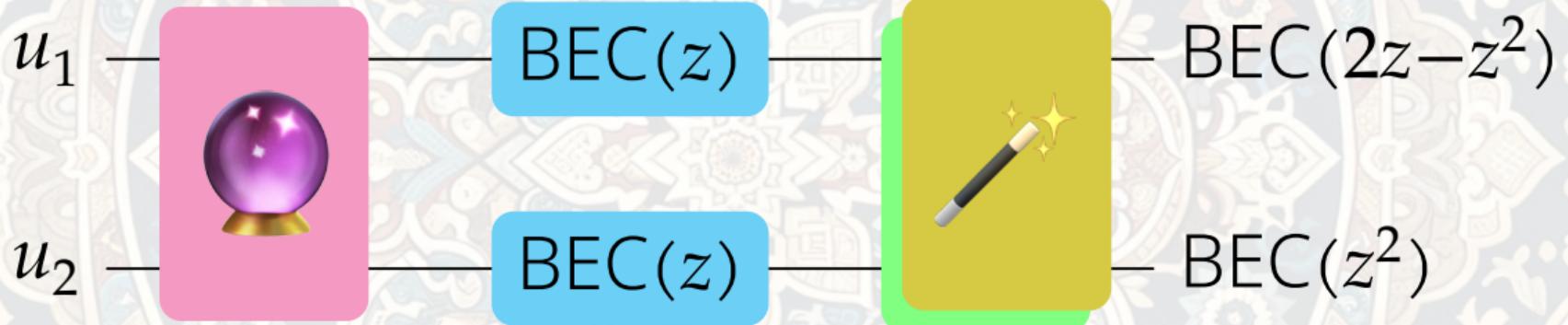


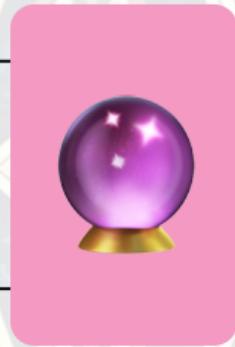
$$y_1 + u_1 \\ := y_2$$









$u_1$  $u_2$ 

BEC(40%)

BEC(40%)





$u_1$



BEC(40%)

$u_2$

BEC(40%)



$u_3$

BEC(40%)

$u_4$

BEC(40%)



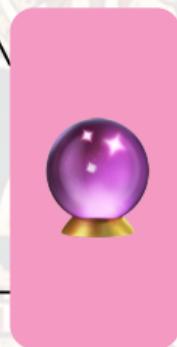


$u_1$  $u_2$  $u_3$  $u_4$ 

BEC(40%)



BEC(40%)

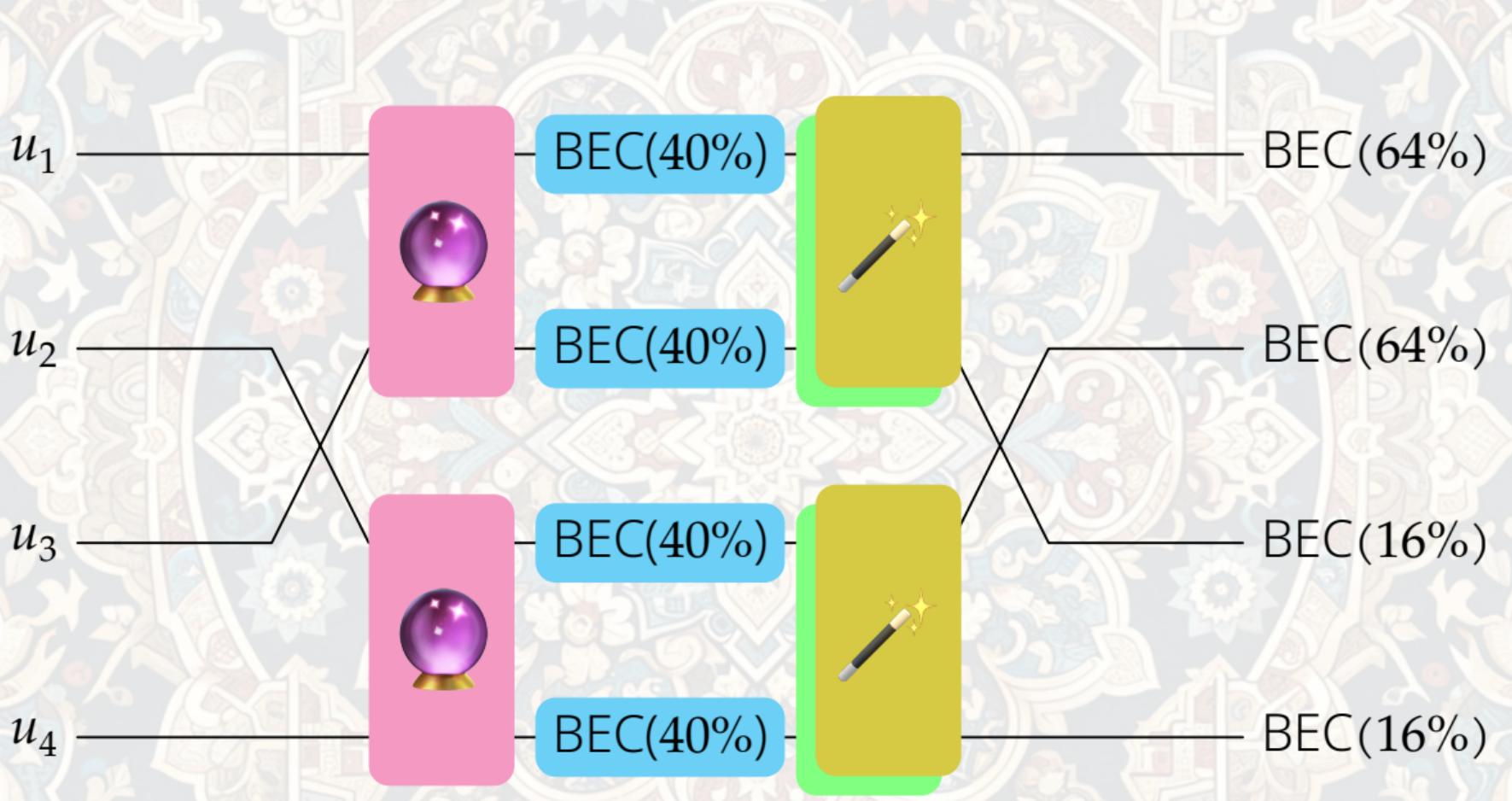


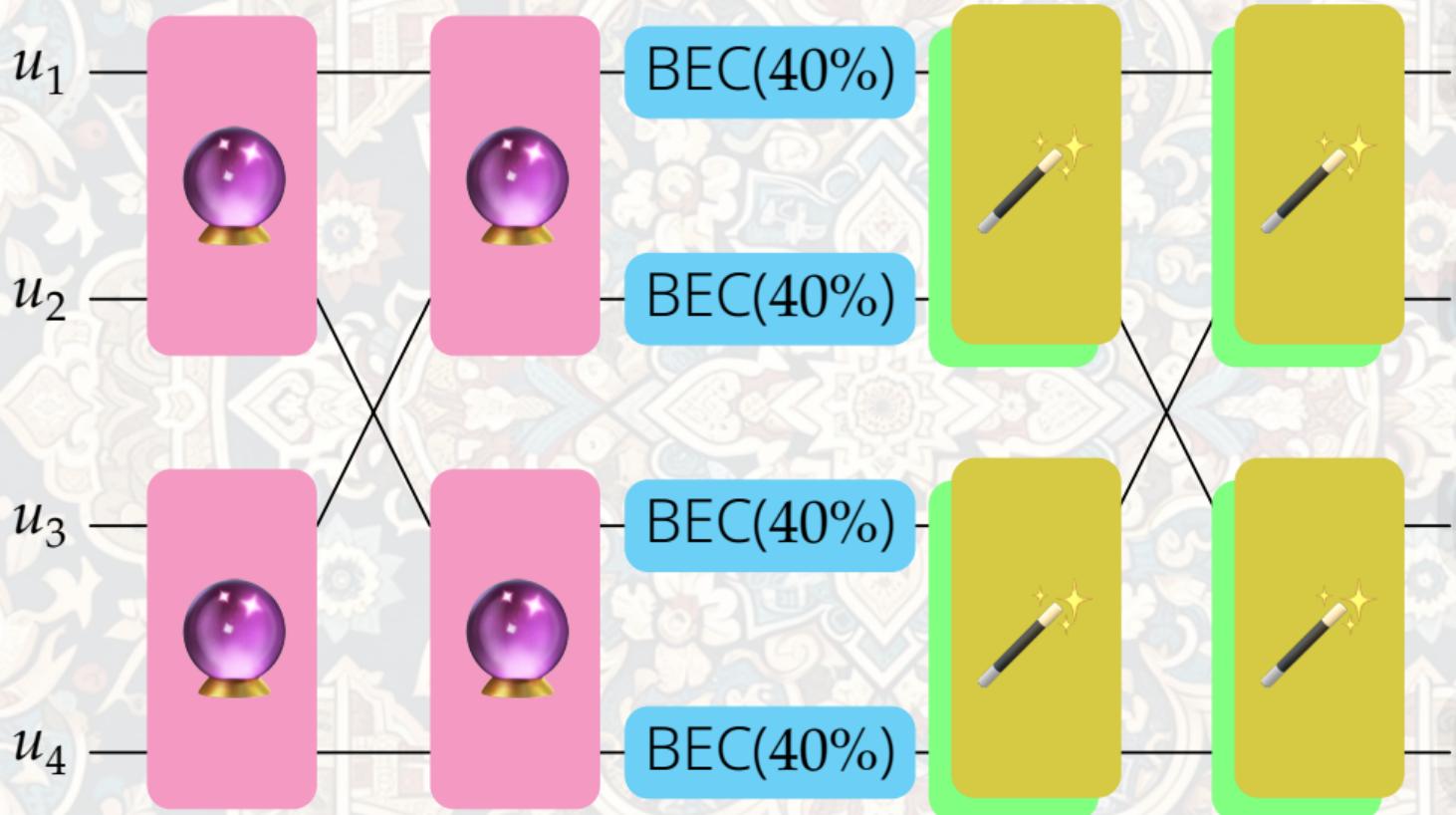
BEC(40%)

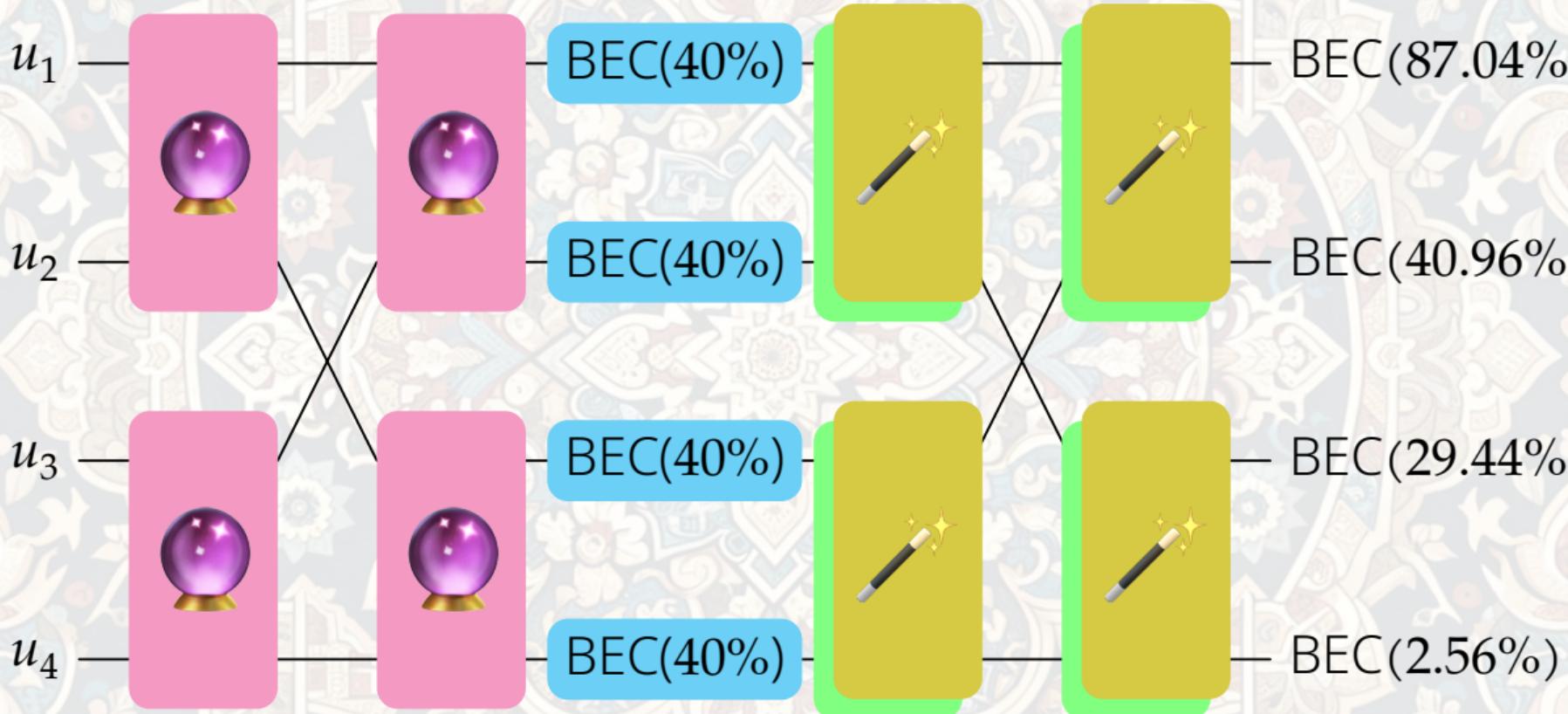


BEC(40%)

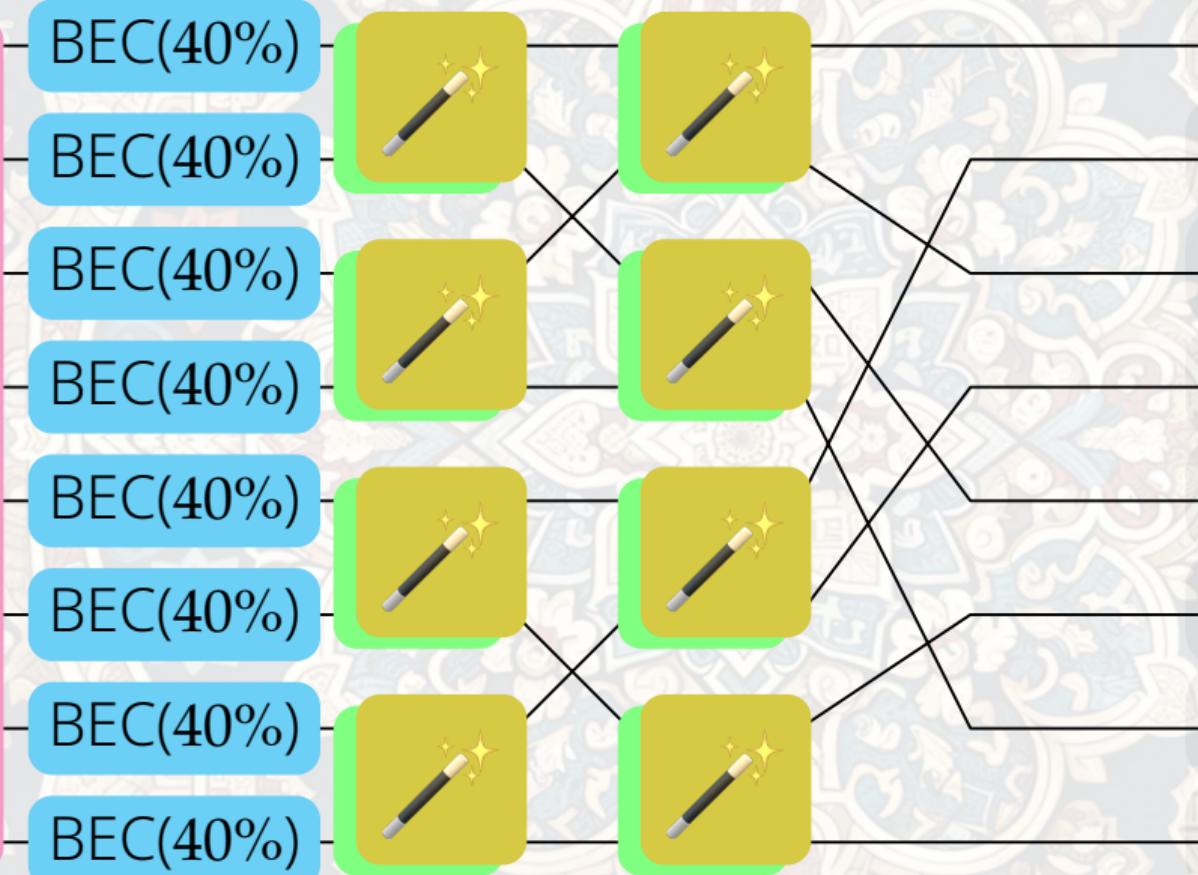




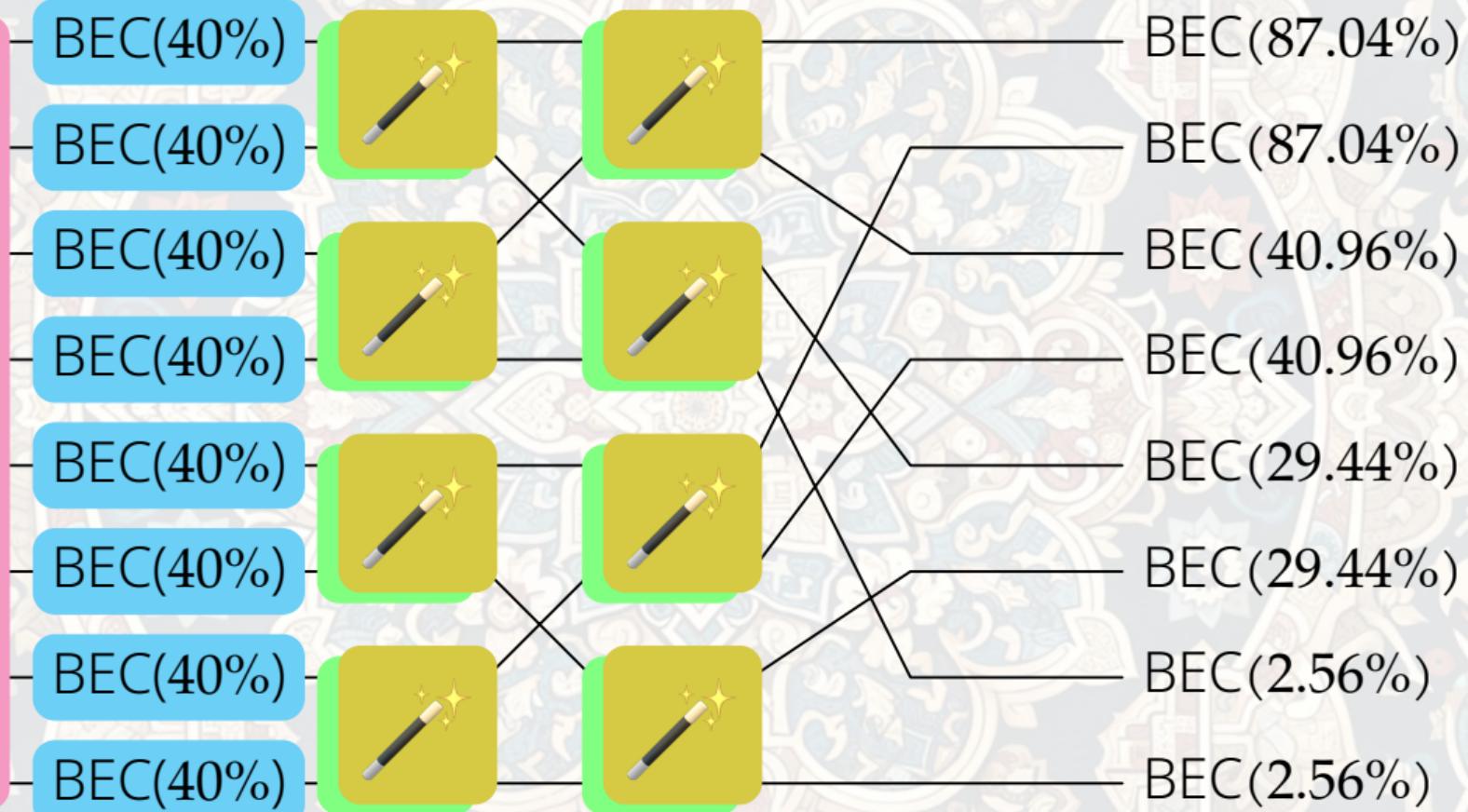




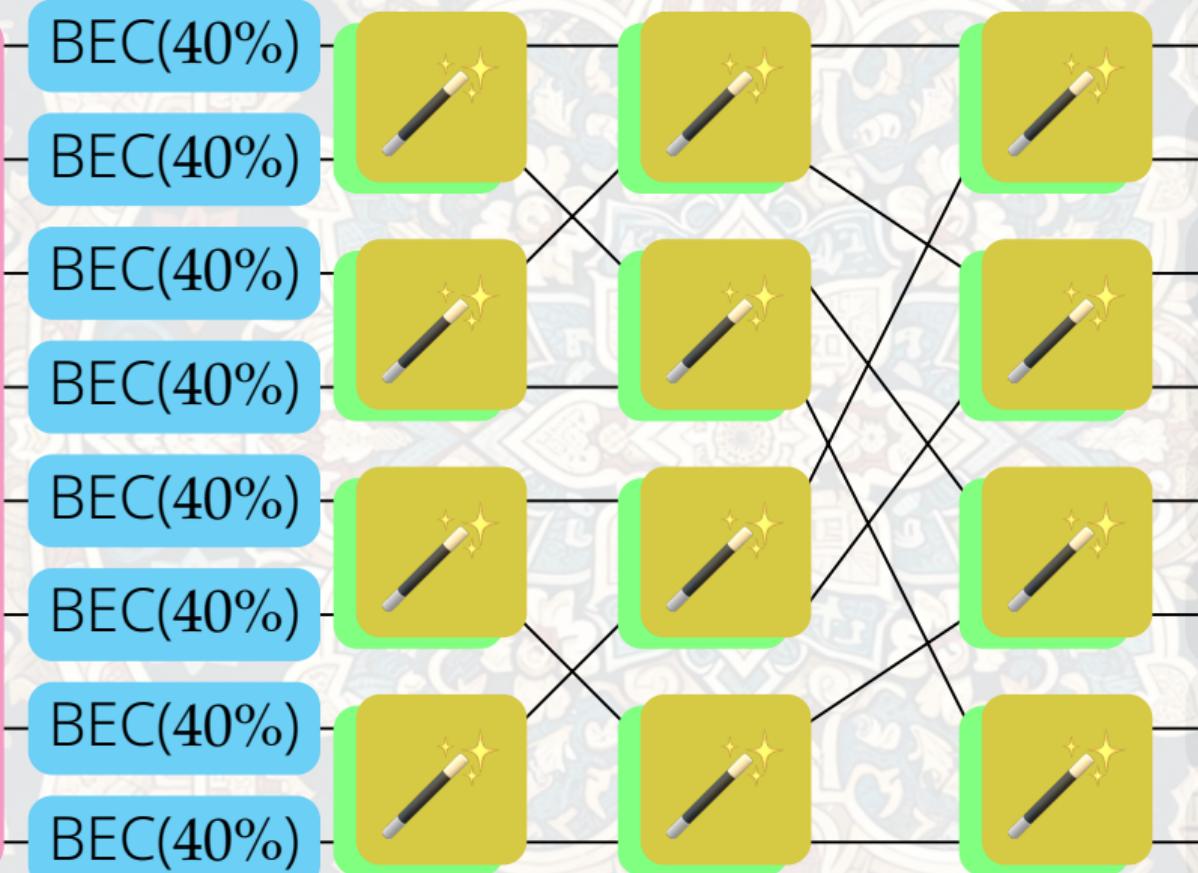
## Mirrored configuration



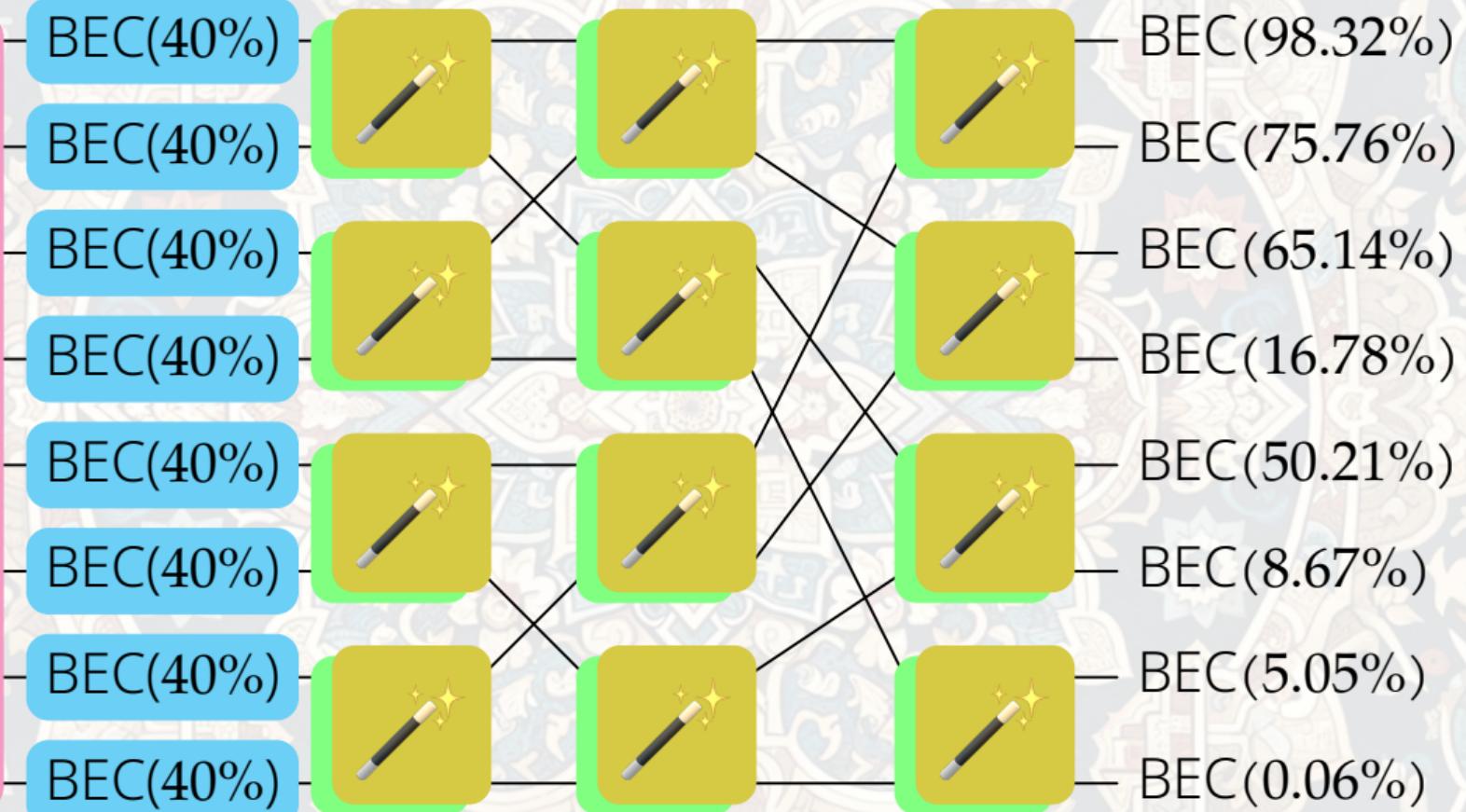
## Mirrored configuration



## Mirrored configuration



## Mirrored configuration



BEC(40%)



BEC(98.32%)

BEC(75.76%)

BEC(65.14%)

BEC(16.78%)

BEC(50.21%)

BEC(8.67%)

BEC(5.05%)

BEC(0.06%)

# Recap



ChatGPT

# Recap

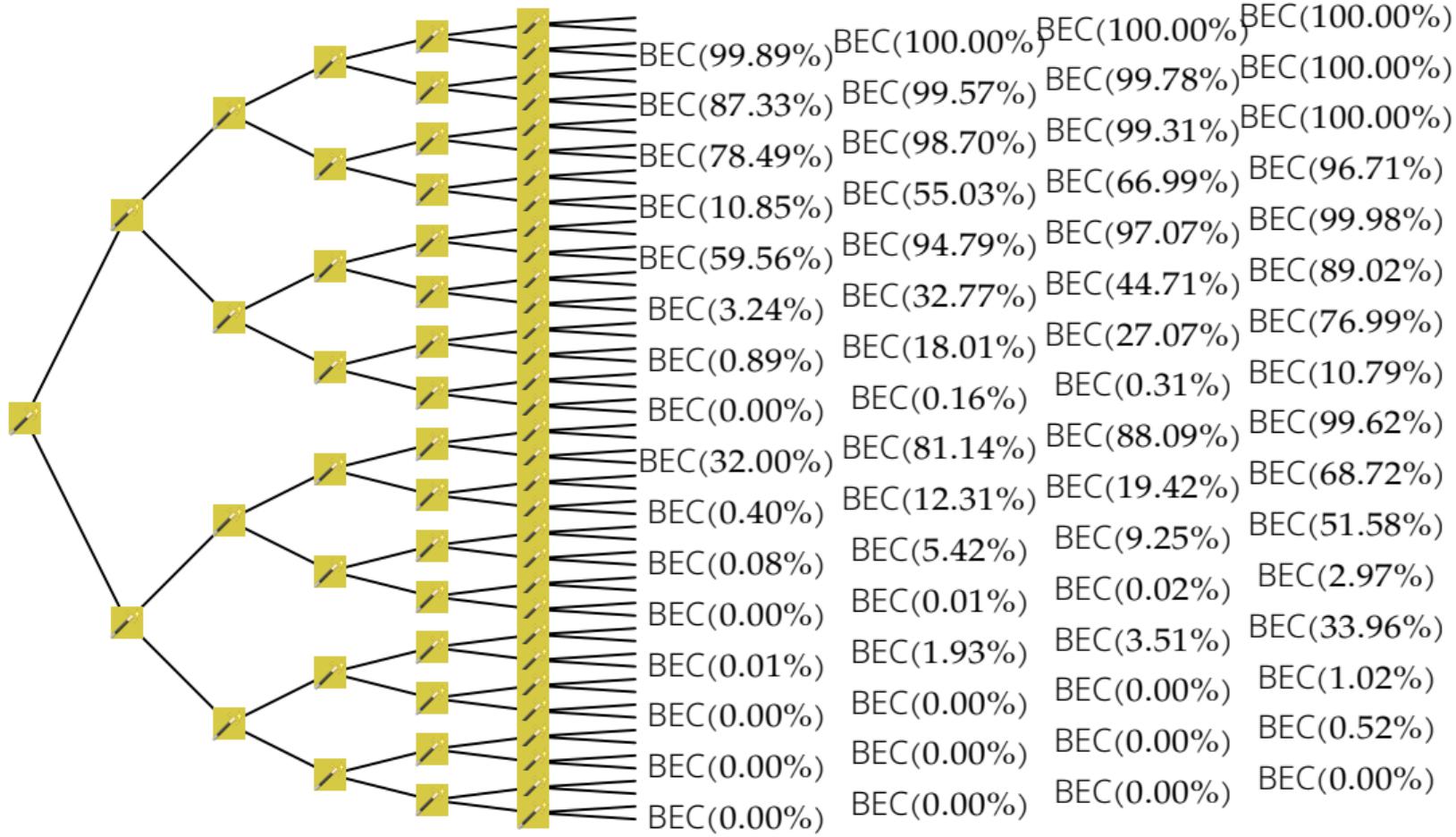


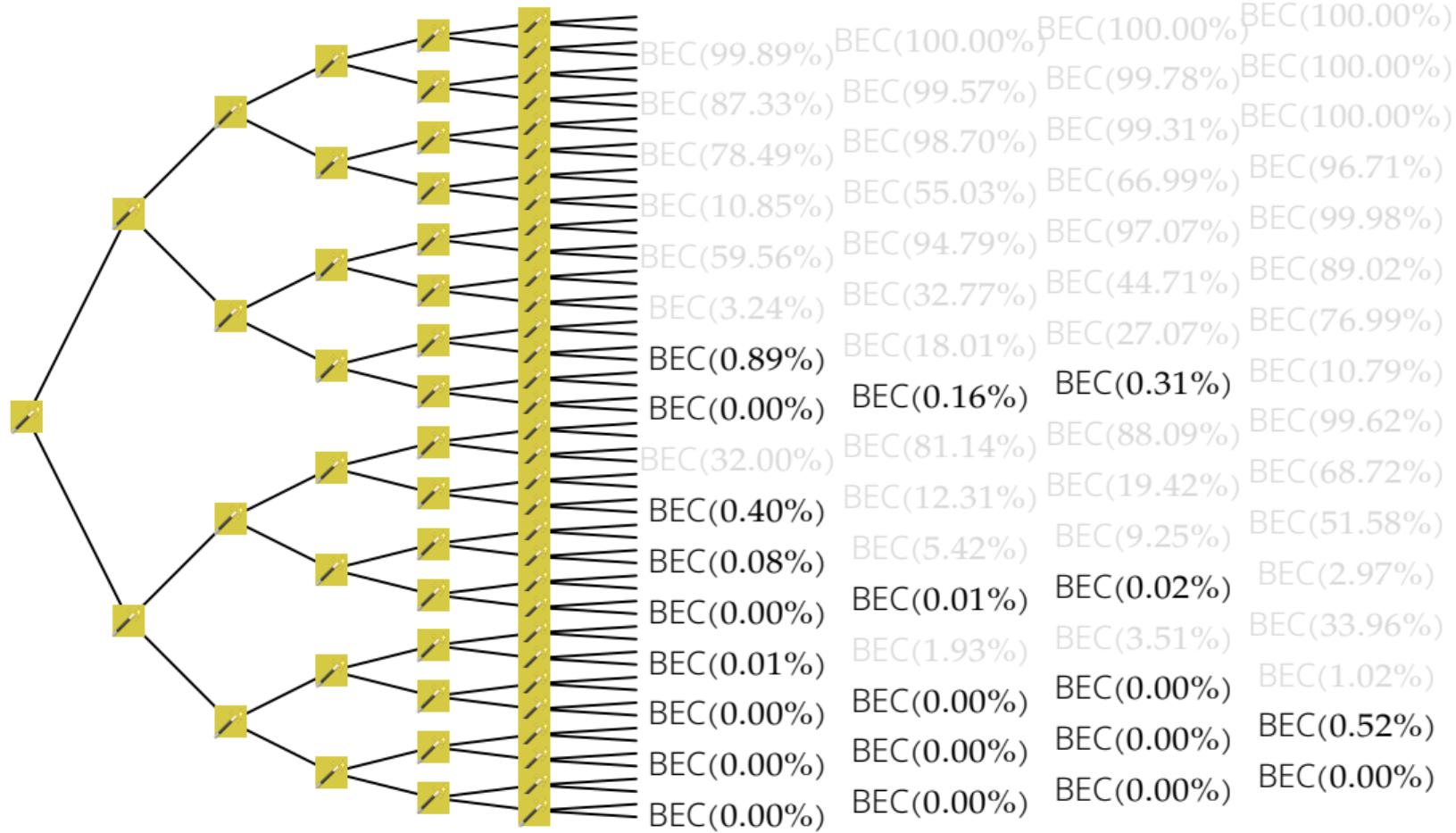
# Recap

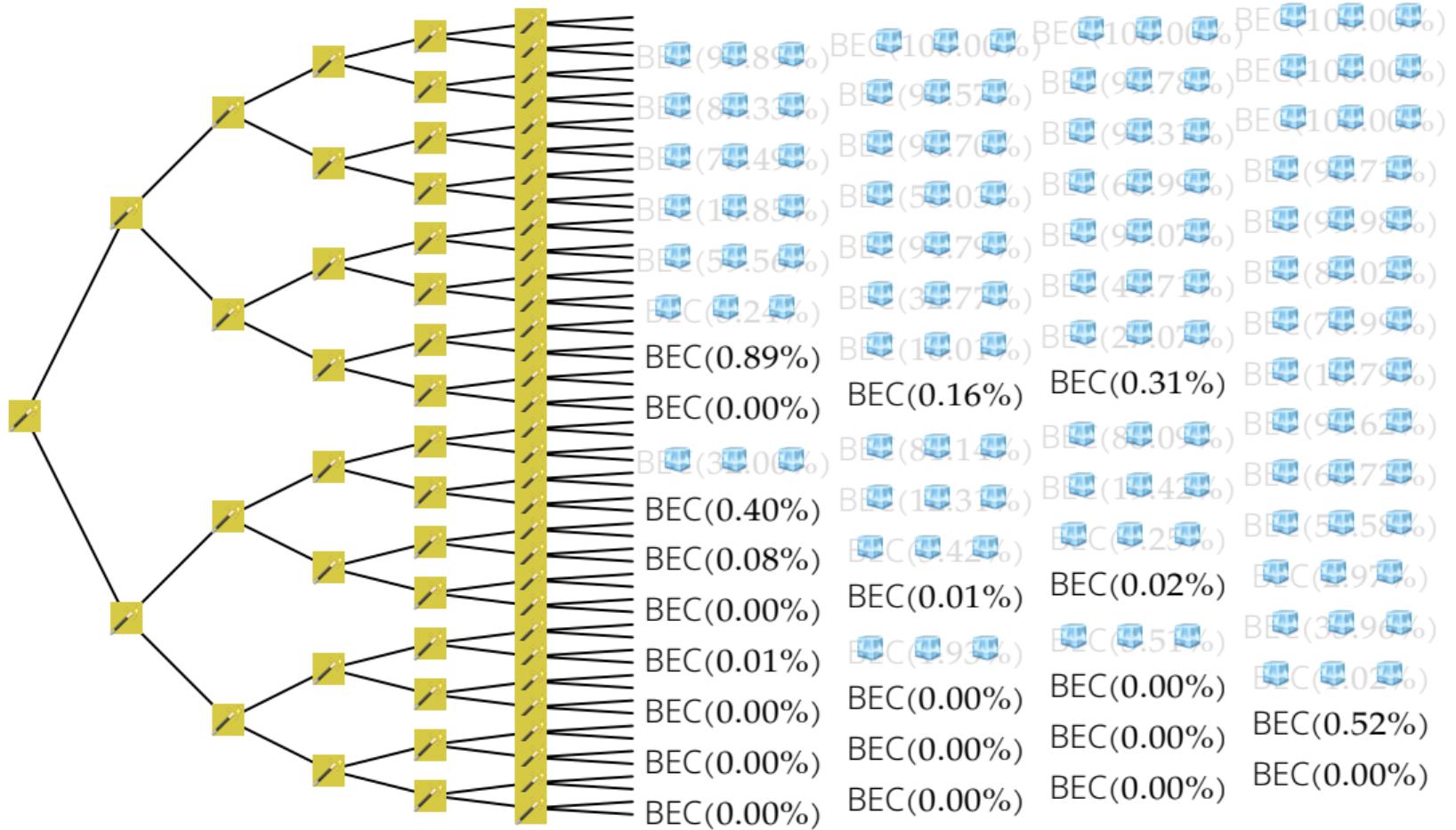


The Channel Gardener

ChatGPT







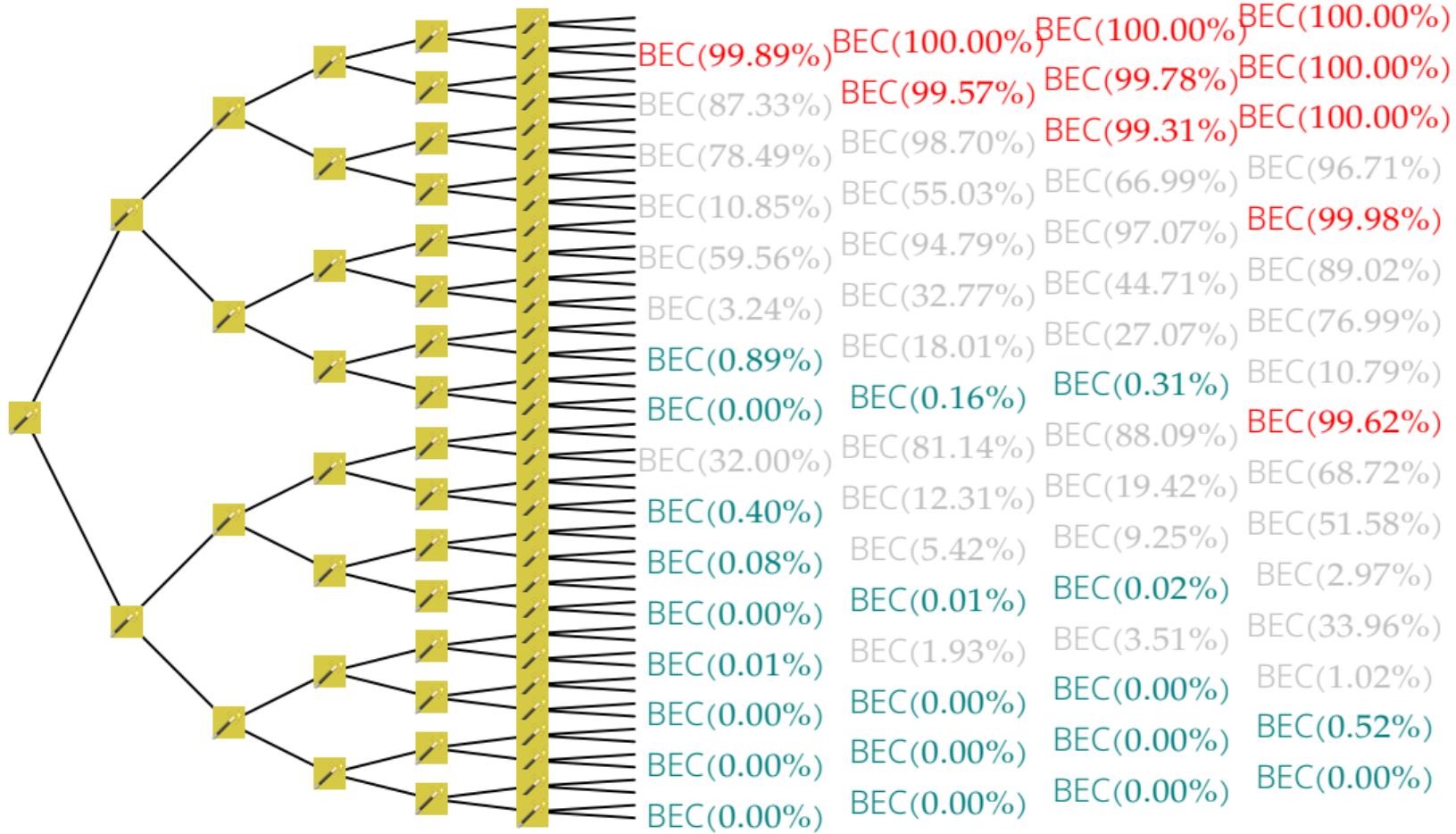
Thm [Arikan] If, after synthesizing  $2^n$  channels,  $k$  of them are less than  $p$ ,

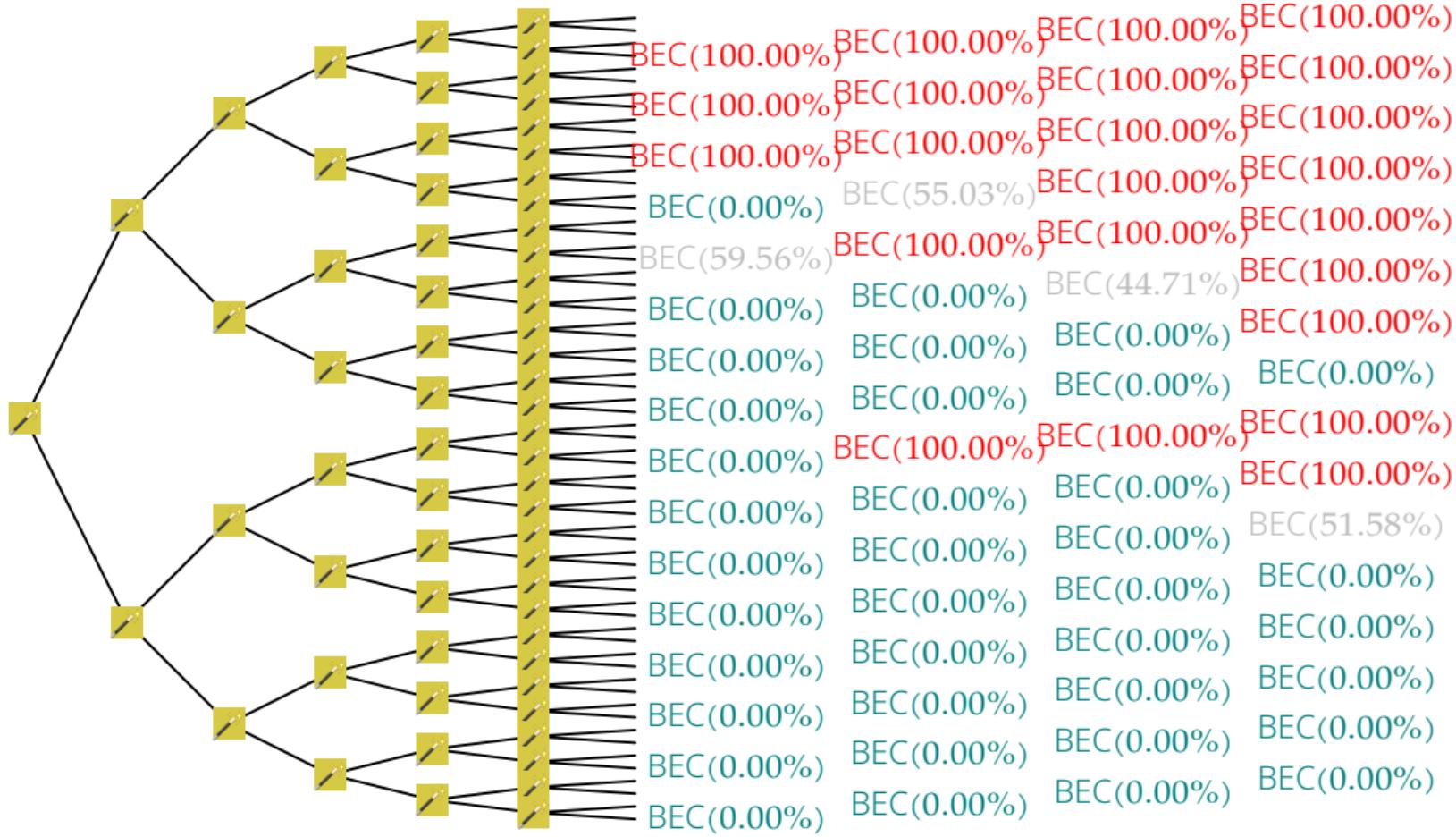


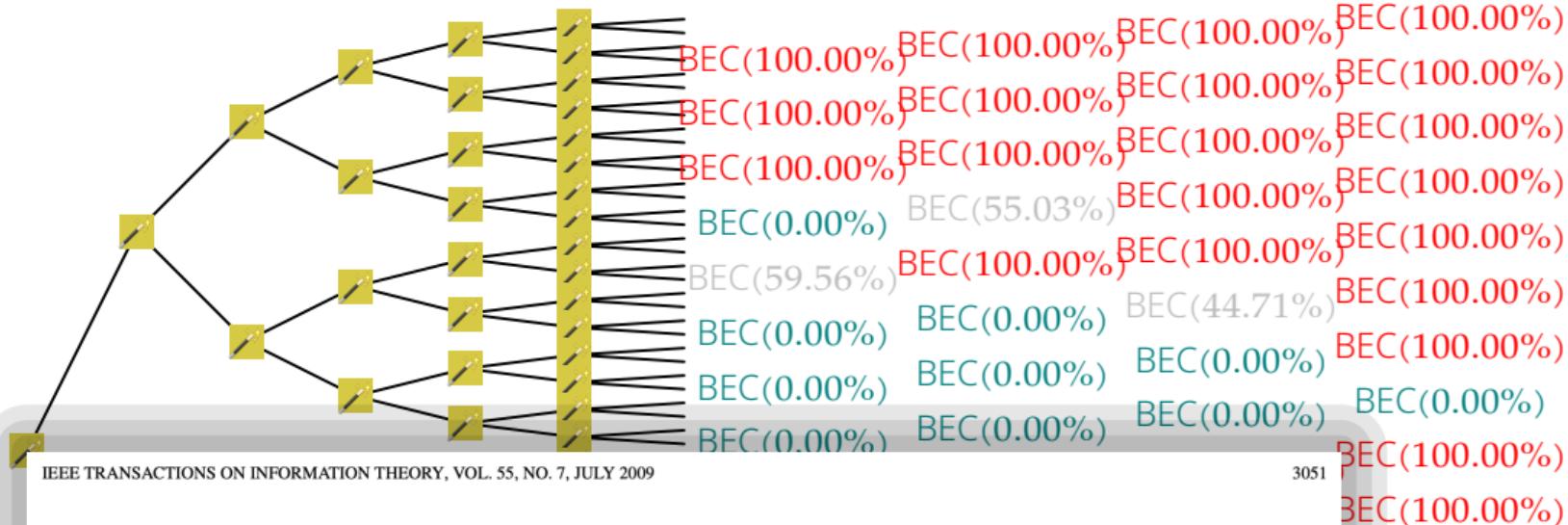
polar code has rate  $k/2^n$   
& block error prob <  $k/2^n$

Thm [Arikan] If, after synthesizing  $2^n$  channels,  $k$  of them are less than  $p$ , polar code has rate  $k/2^n$  & block error prob  $< kp$









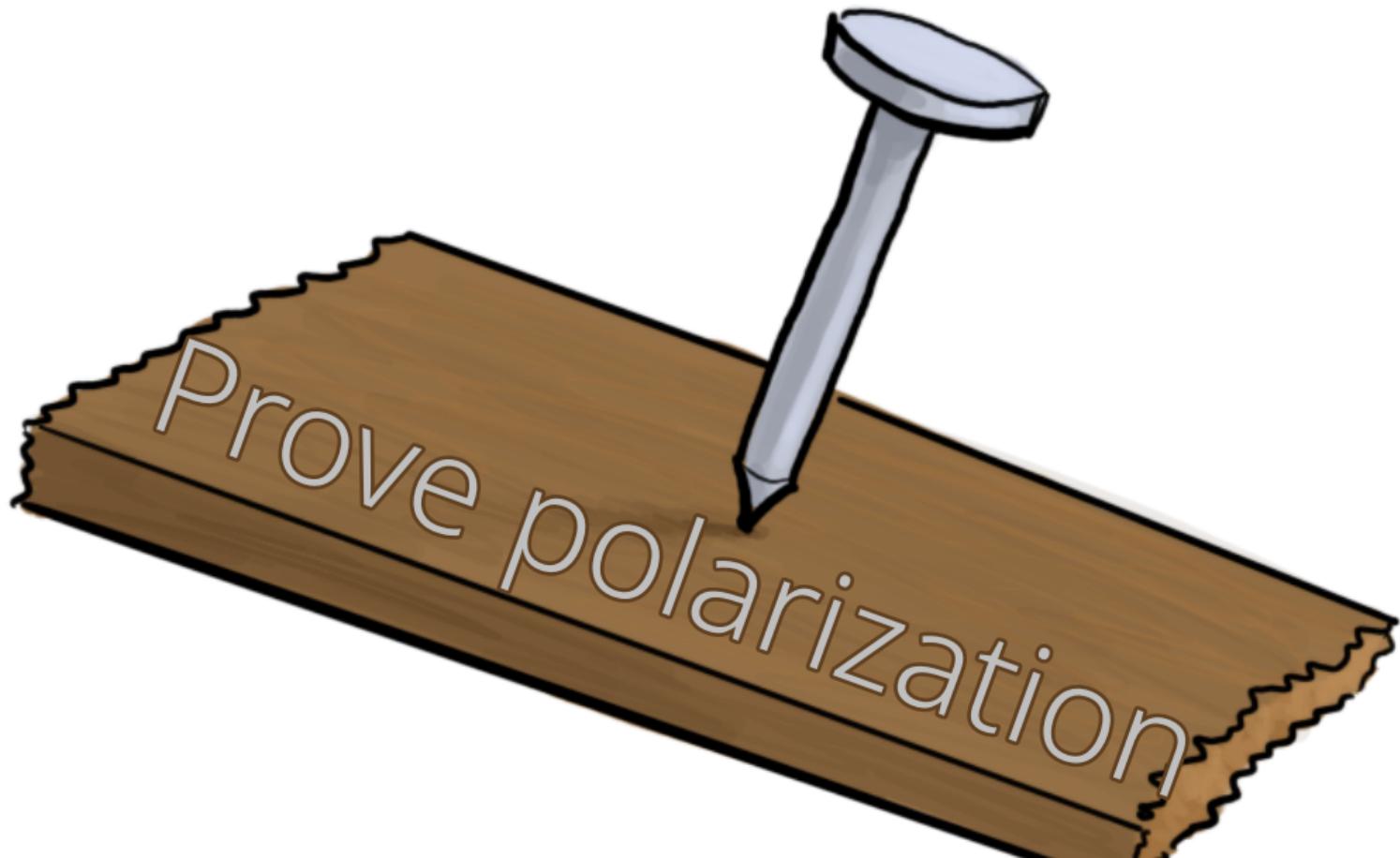
# Channel Polarization: A Method for Constructing Capacity-Achieving Codes for Symmetric Binary-Input Memoryless Channels

Erdal Arıkan, *Senior Member, IEEE*

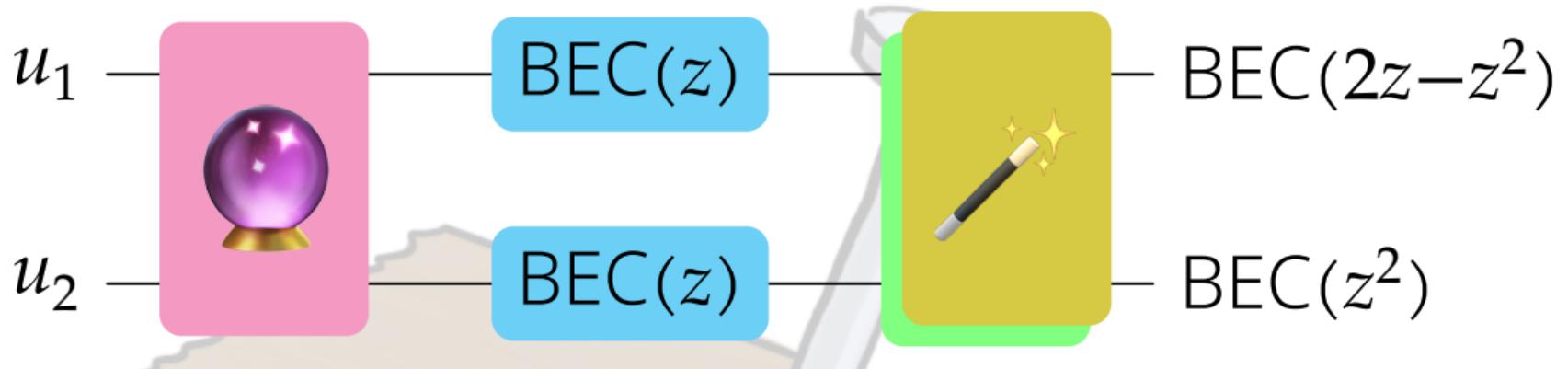
*Abstract*—A method is proposed, called channel polarization, to construct code sequences that achieve the symmetric capacity  $I(W)$  of any given binary-input discrete memoryless channel (B-DMC)  $W$ . The symmetric capacity is the highest rate achiev-

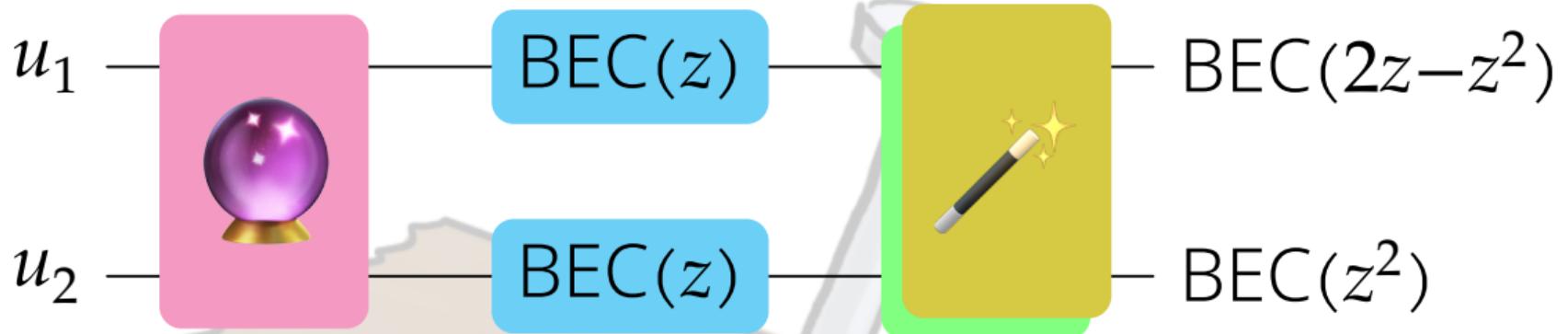
### A. Preliminaries

We write  $W : \mathcal{X} \rightarrow \mathcal{Y}$  to denote a generic B-DMC with input alphabet  $\mathcal{X}$ , output alphabet  $\mathcal{Y}$ , and transition probabilities

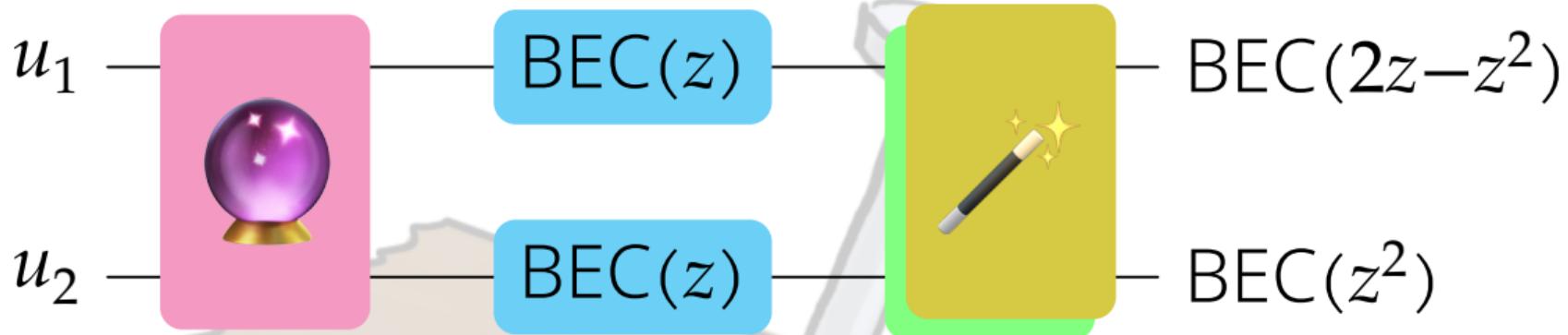


Prove polarization



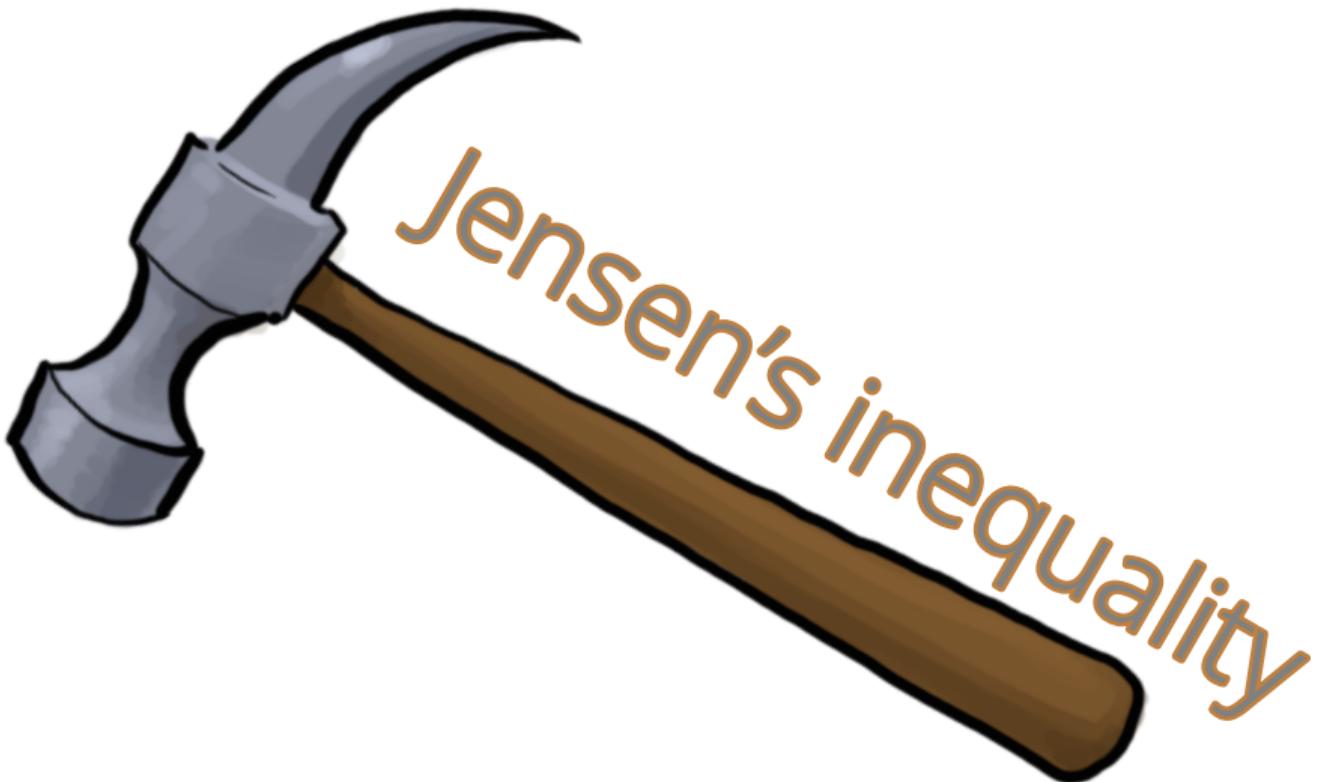


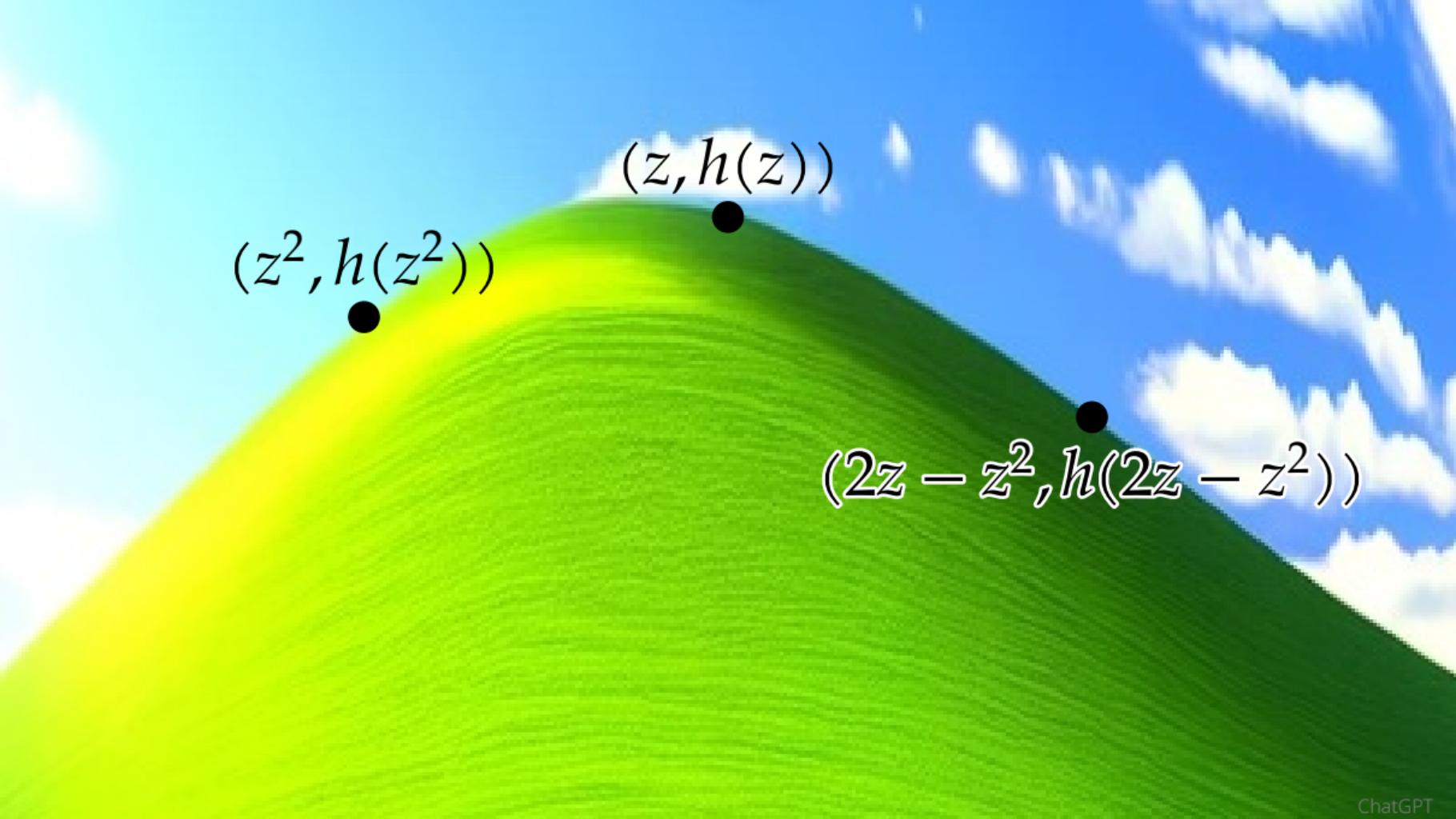
Level( $n$ ) := {the  $2^n$  channels generated at level  $n$ }

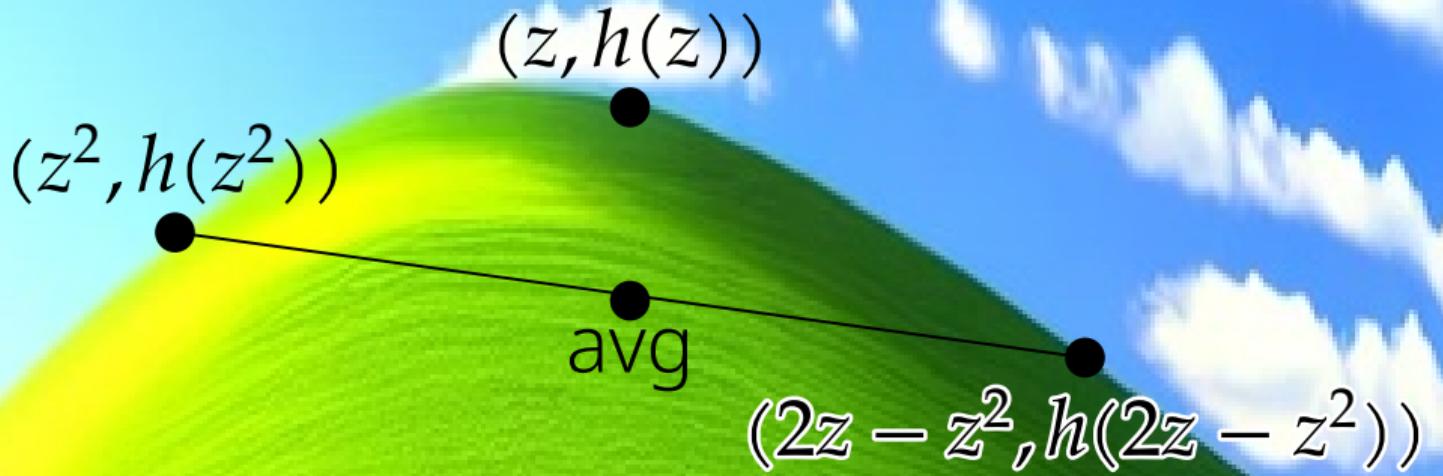


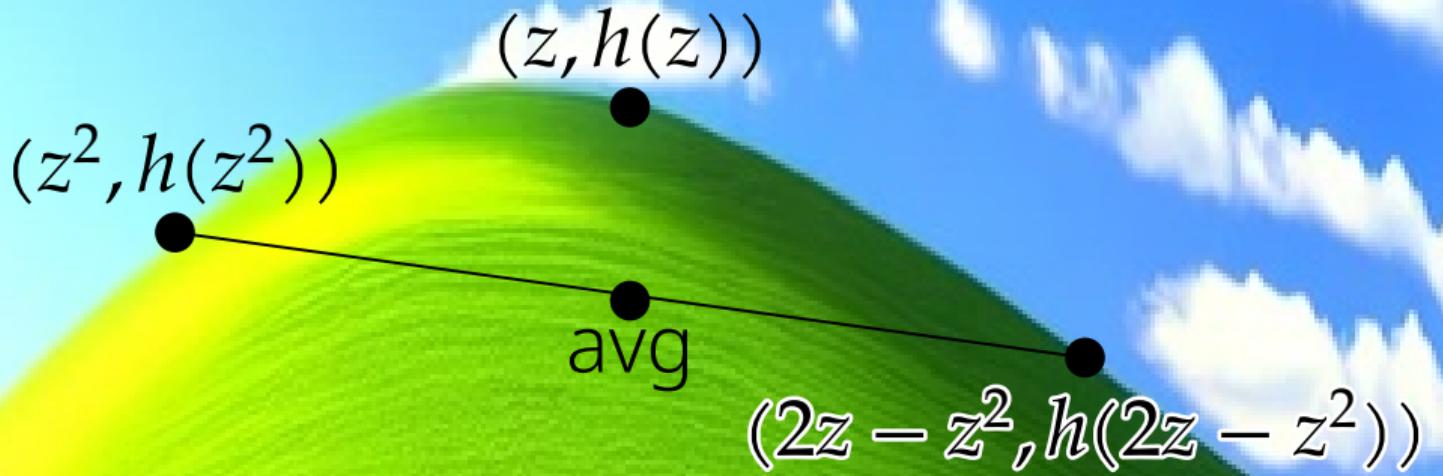
$\text{Level}(n) := \{\text{the } 2^n \text{ channels generated at level } n\}$

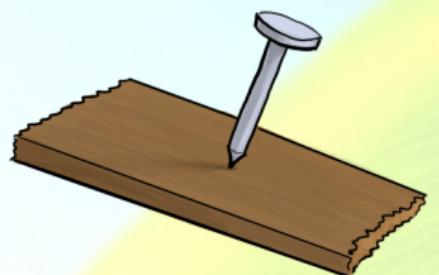
Most BECs in  $\text{Level}(n)$  are  $\text{BEC}(\approx 1)$  or  $\text{BEC}(\approx 0)$



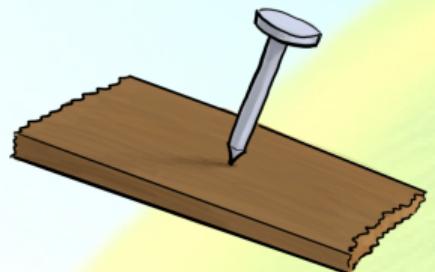





$$\frac{1}{2^n} \sum_{z \in \text{Level}(n)} h(z) \text{ decreases as } n \rightarrow \infty$$



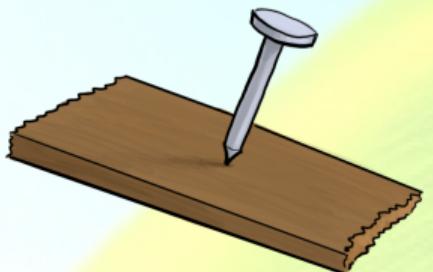
Prove polarization



Prove polarization



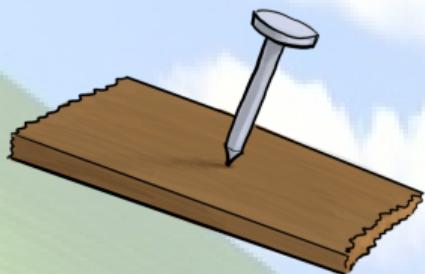
Jensen's



Prove polarization



Jensen's



Convex function?

Hill shape  $h(z) := (z(1 - z))^{0.663}$



Hill shape  $h(z) := (z(1 - z))^{0.663}$

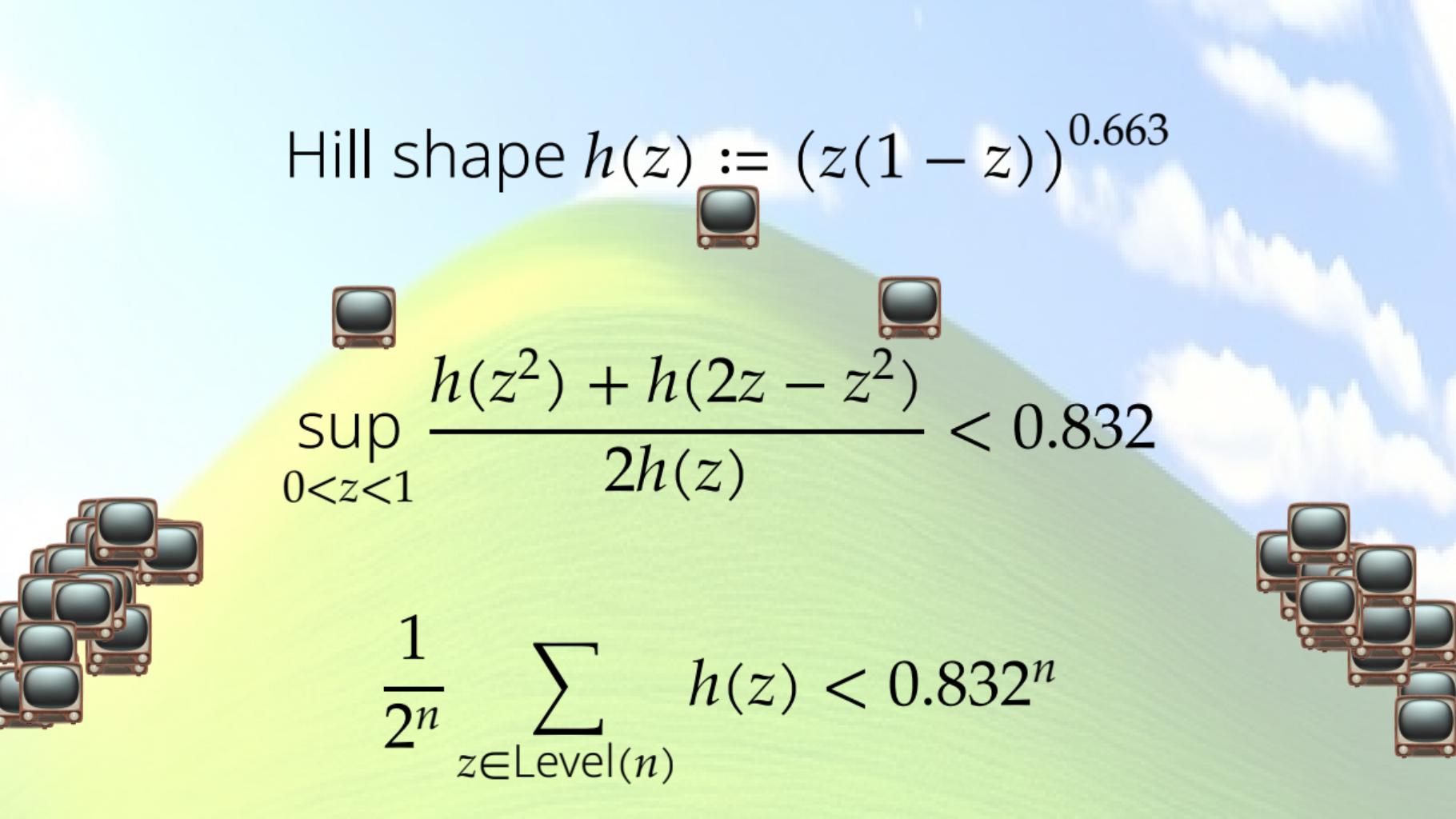
$$\sup_{0 < z < 1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

Hill shape  $h(z) := (z(1 - z))^{0.663}$

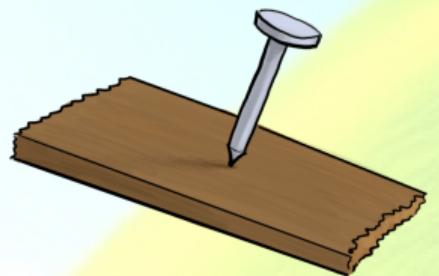
$$\sup_{0 < z < 1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

$$\frac{1}{2^n} \sum_{z \in \text{Level}(n)} h(z) < 0.832^n$$

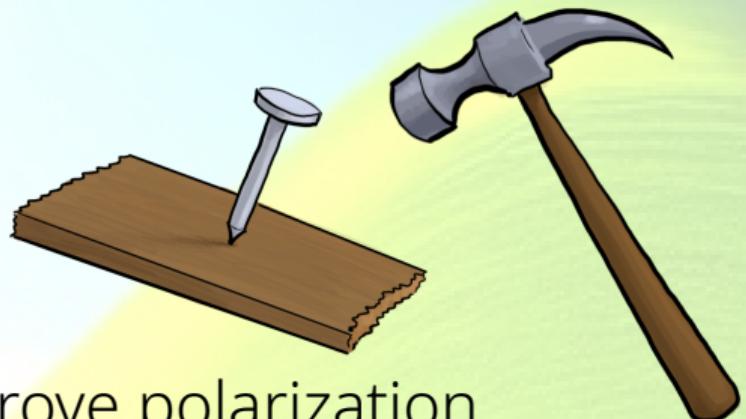
Hill shape  $h(z) := (z(1 - z))^{0.663}$


$$\sup_{0 < z < 1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

$$\frac{1}{2^n} \sum_{z \in \text{Level}(n)} h(z) < 0.832^n$$



Prove polarization



Prove polarization  
Jensen's



Prove polarization

Jensen's

Convex function?



Prove polarization  
Jensen's

Convex function?  
 $(z(1 - z))^{0.663}$

If  $k$  of  $2^n$  are less than  $p$ ,  
polar code has rate  $k/2^n$   
& block error prob  $< kp$ .

If  $k$  of  $2^n$  are less than  $p$ ,  
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& block error prob  $< kp$ .

$$k/2^n = 60\% - 2^{-\Omega(n)}$$

$$p = 2^{-\Omega(n)}$$



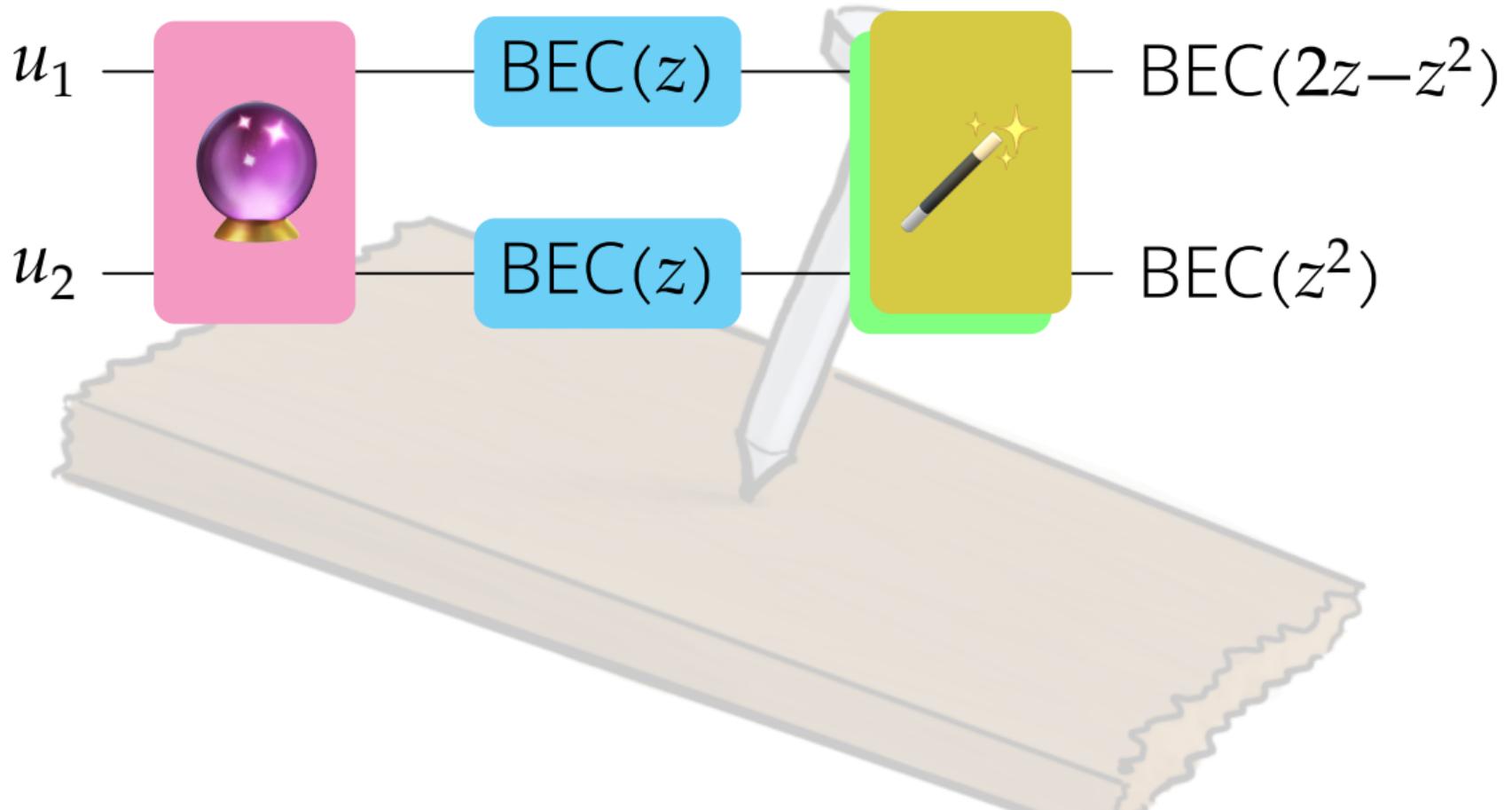
If  $k$  of  $2^n$  are less than  $p$ ,  
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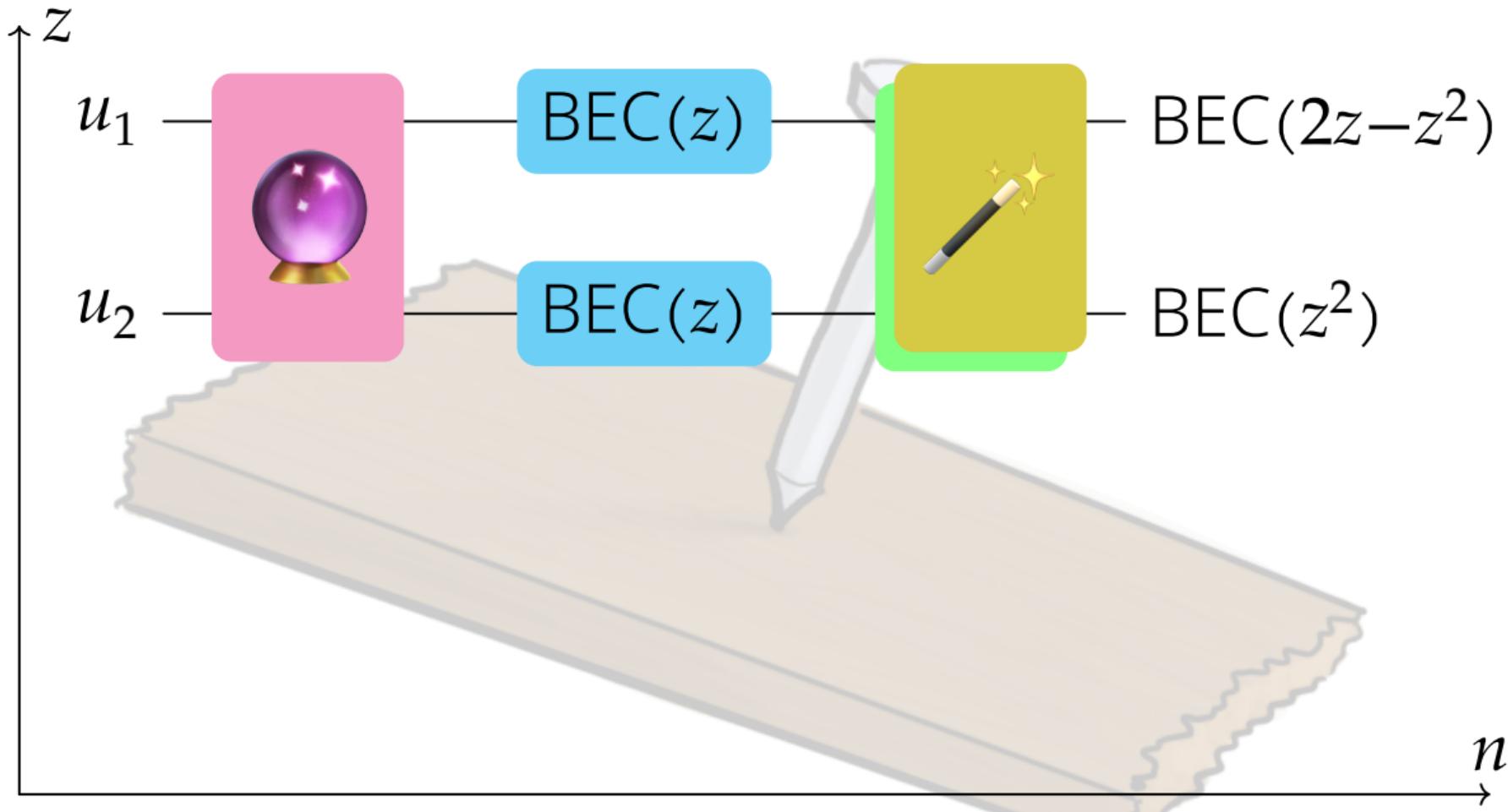
$$k/2^n = 60\% - 2^{-\Omega(n)}$$

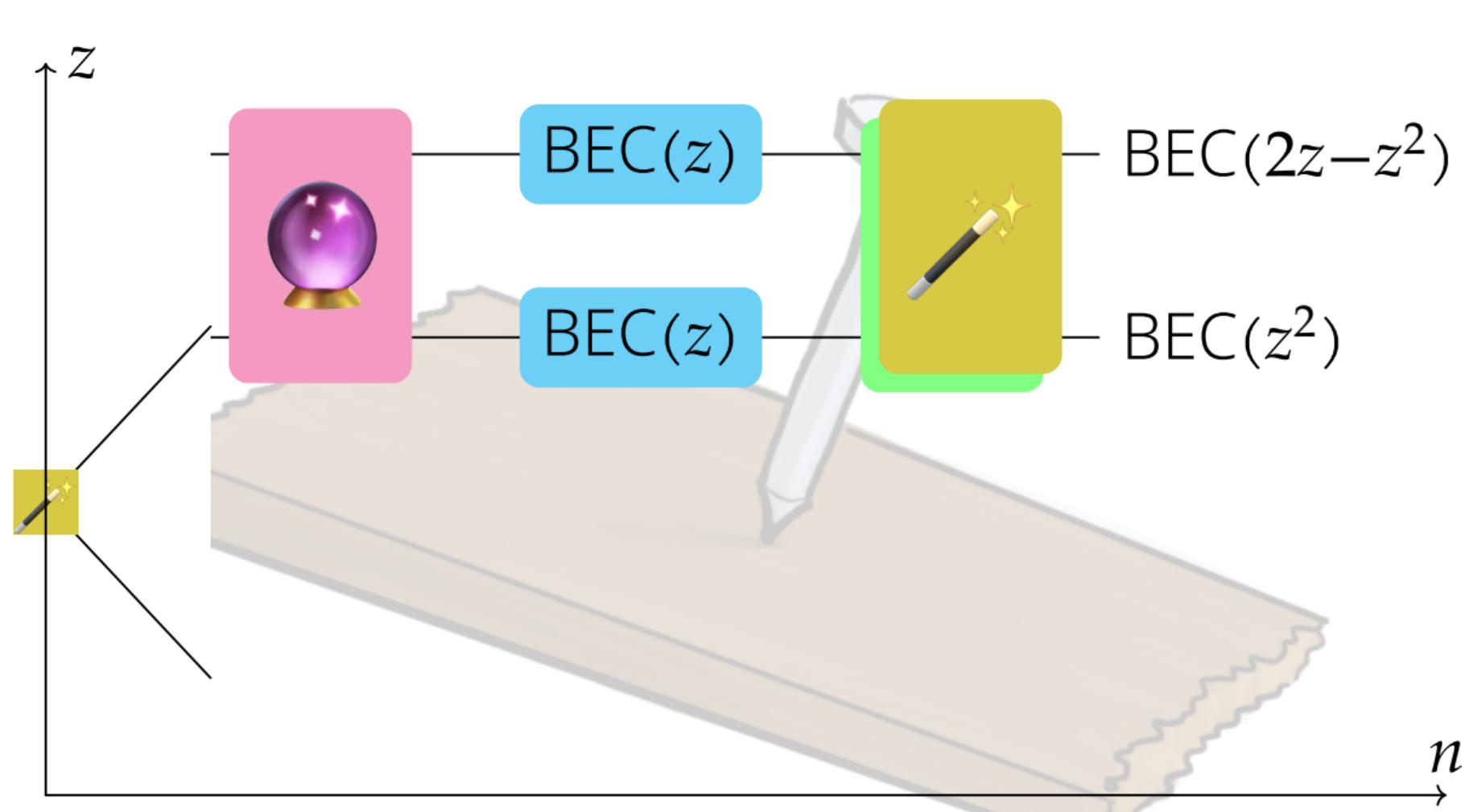
$$p = 2^{-\Omega(n)}$$

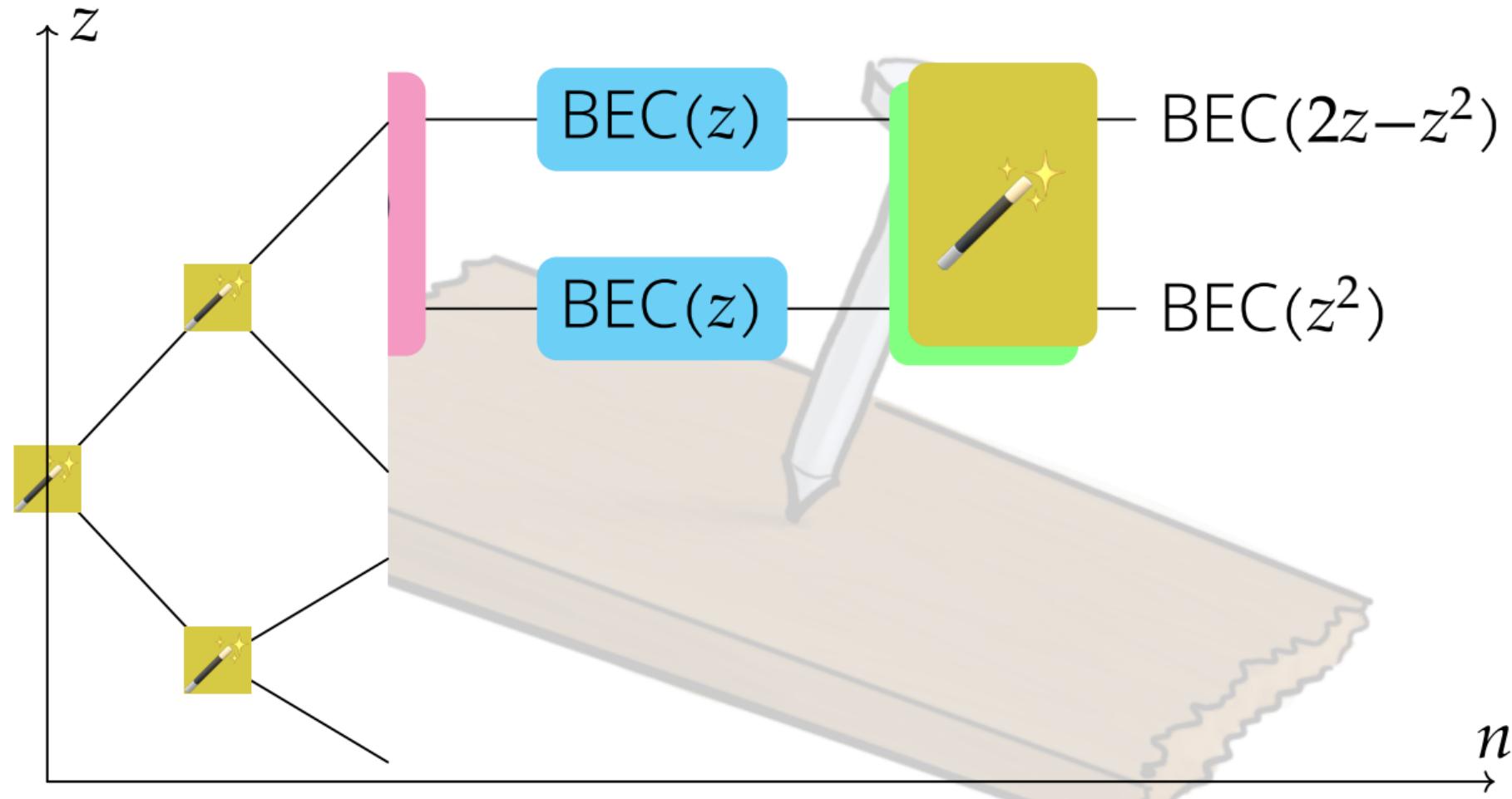


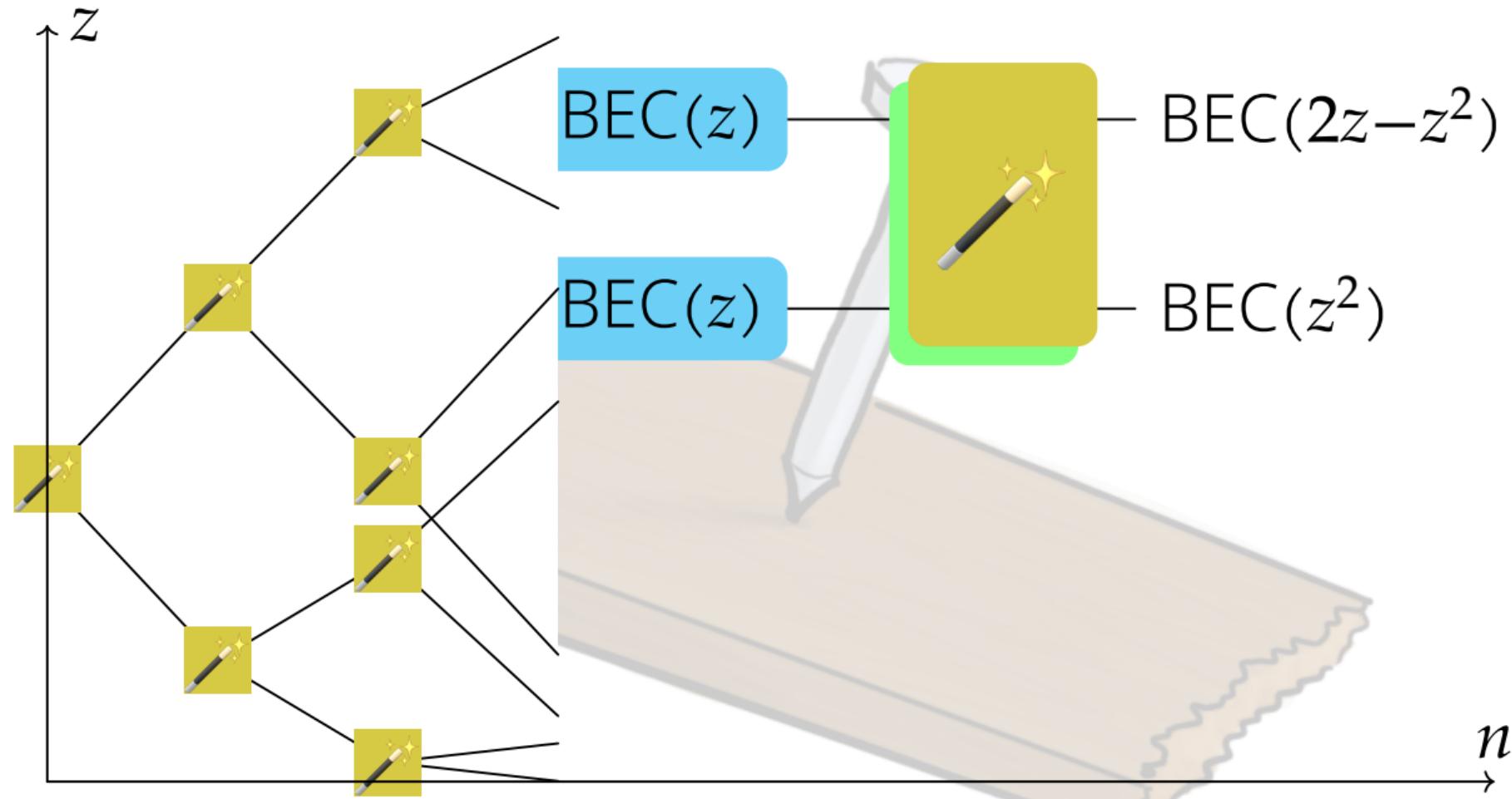
Let's improve p

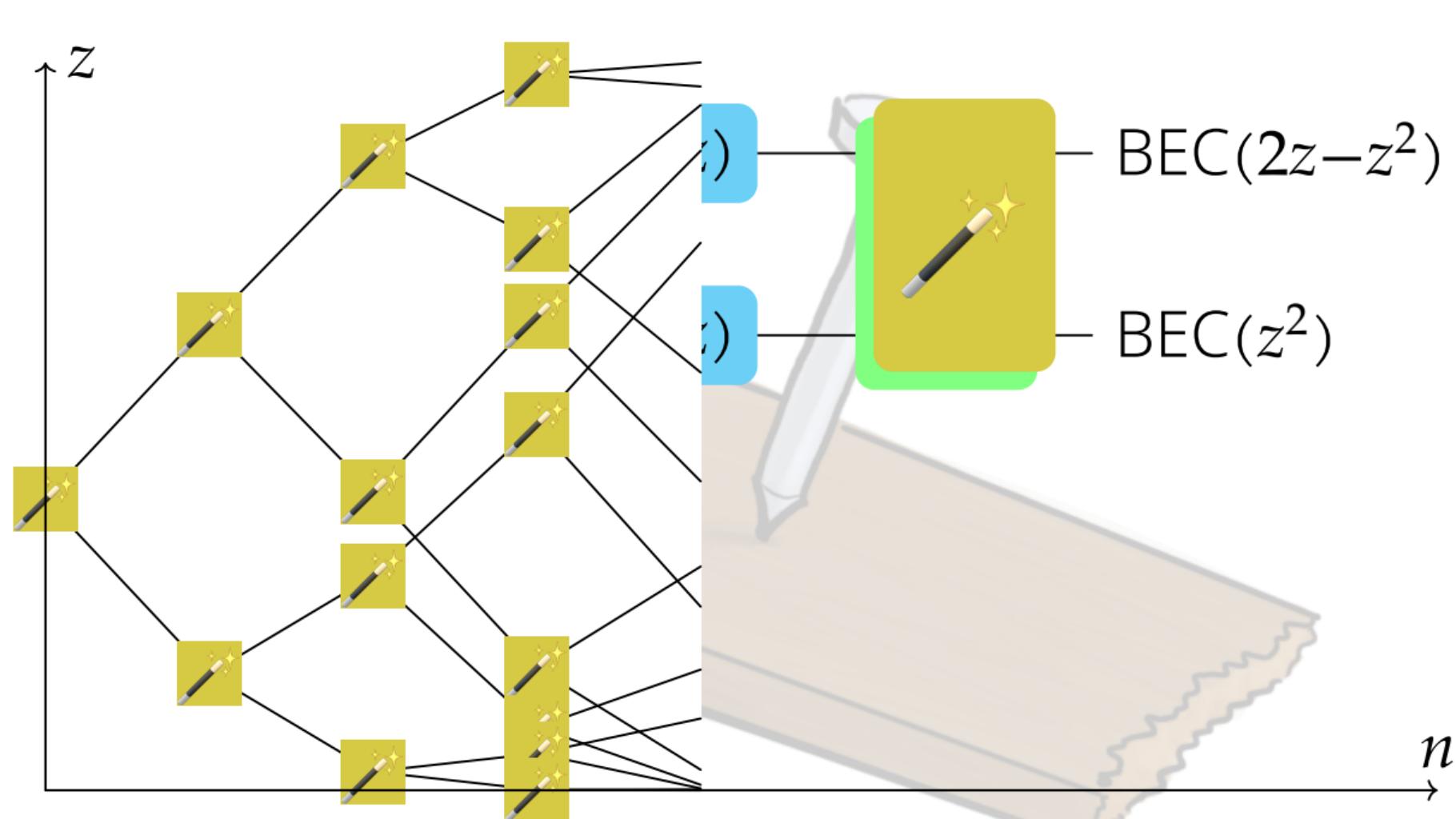


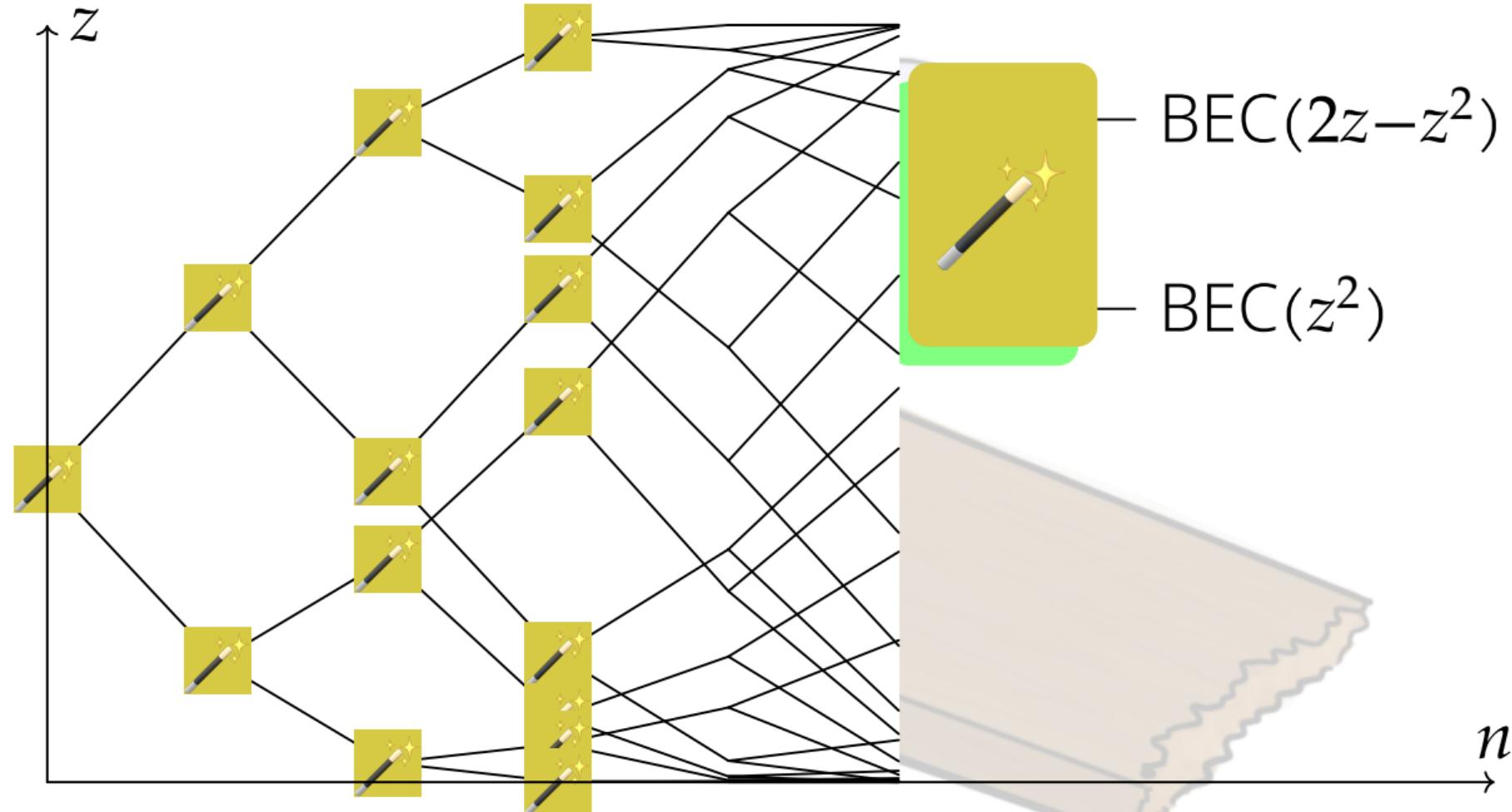


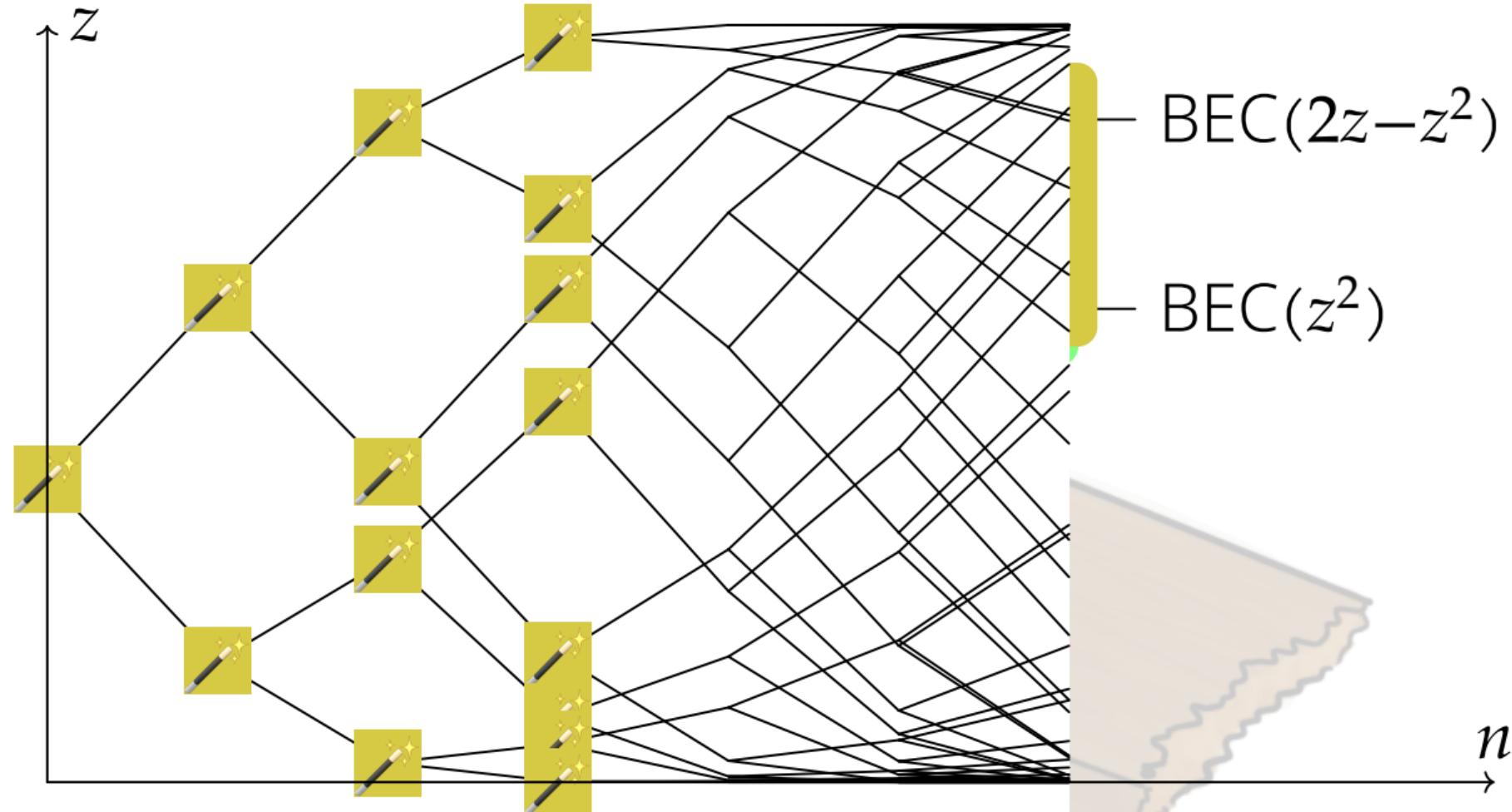


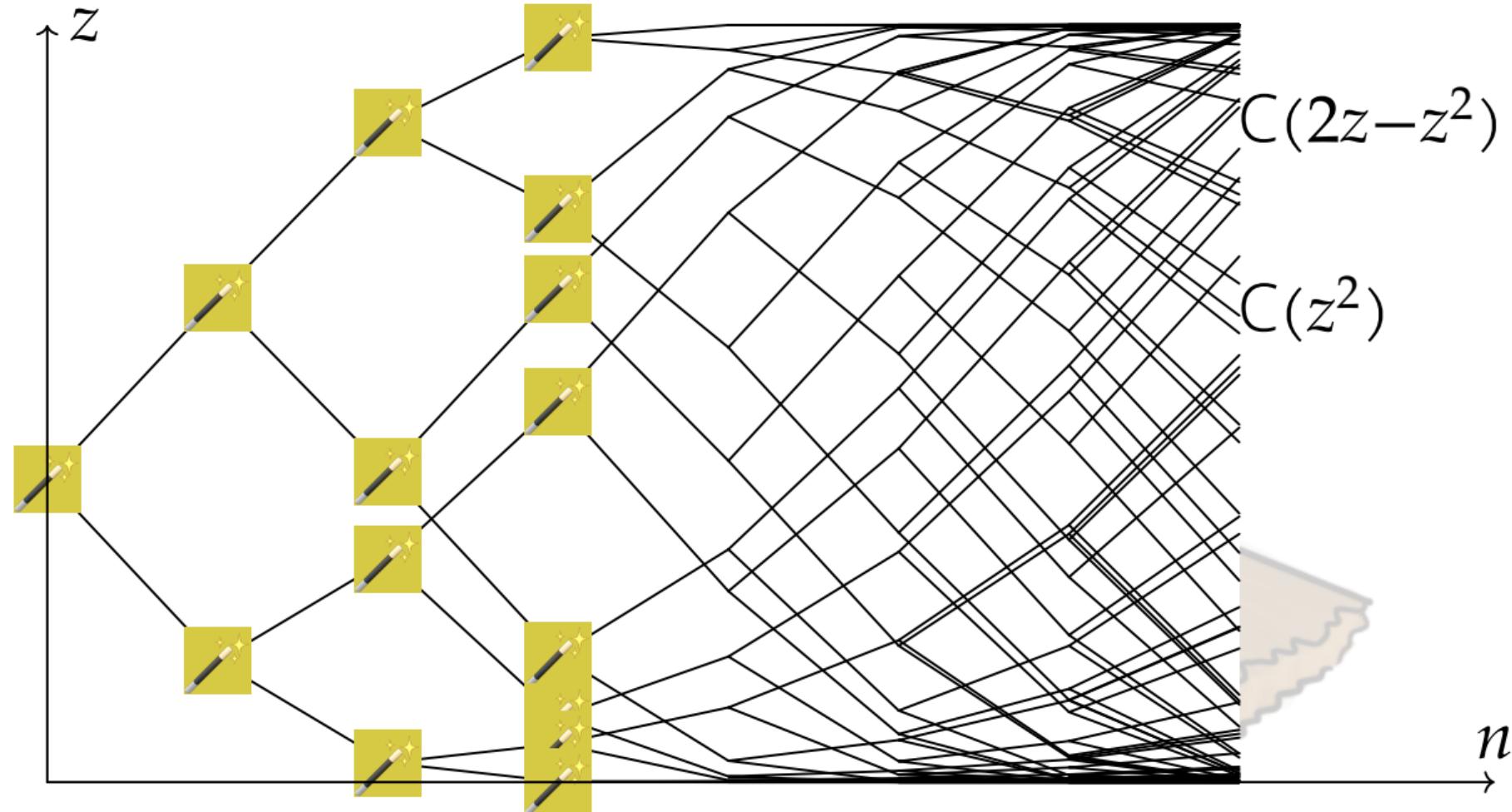


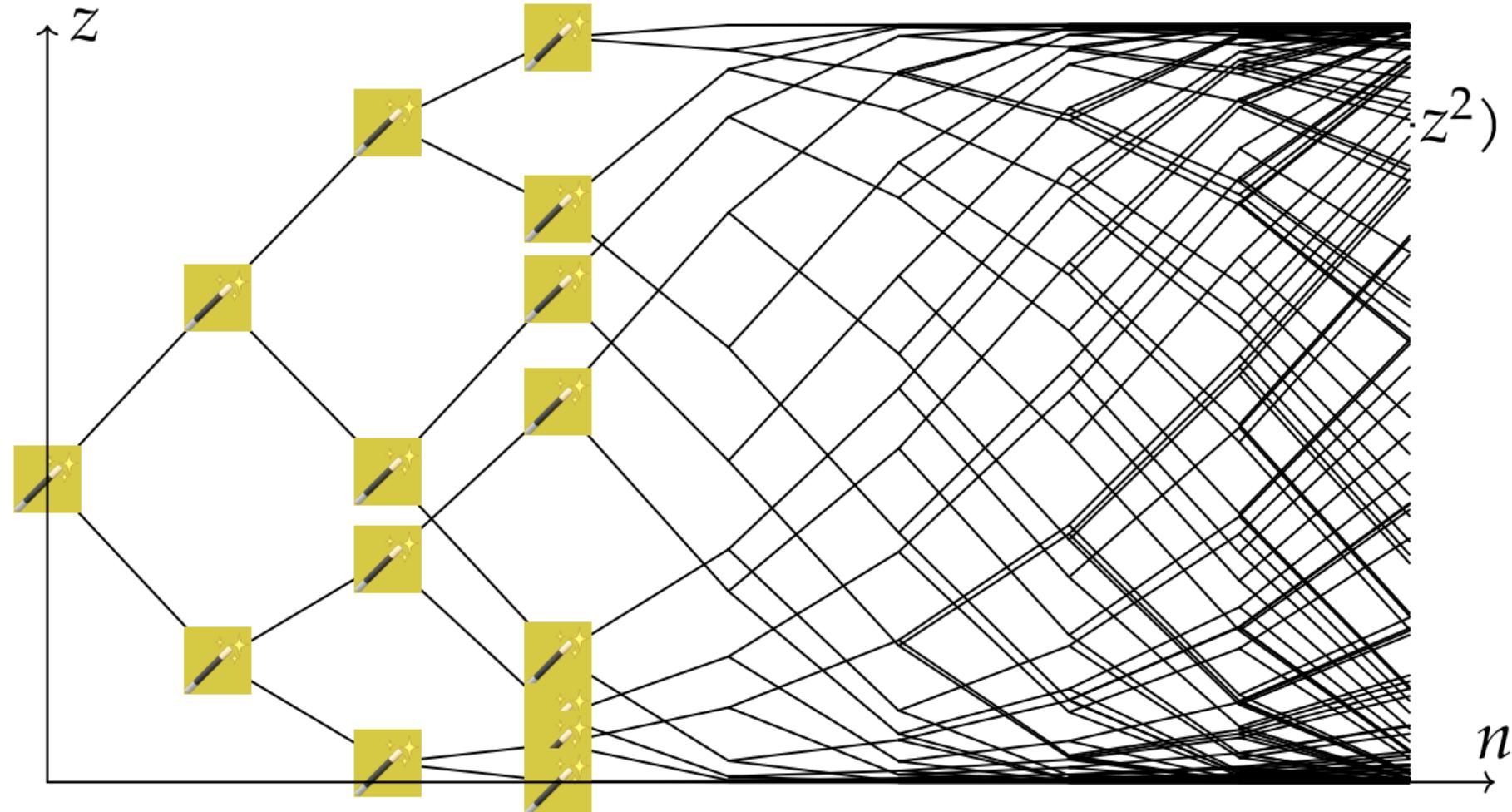


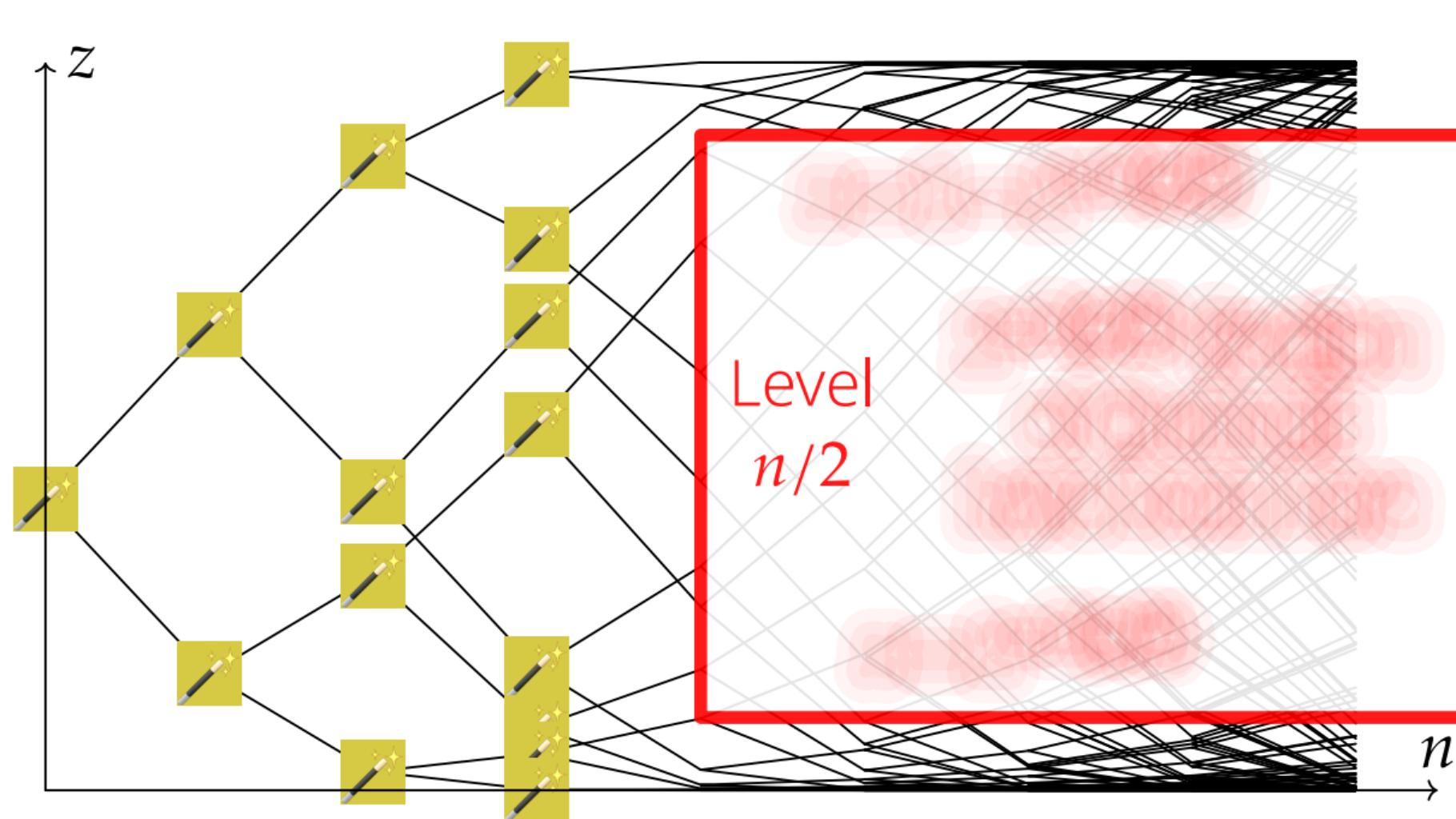


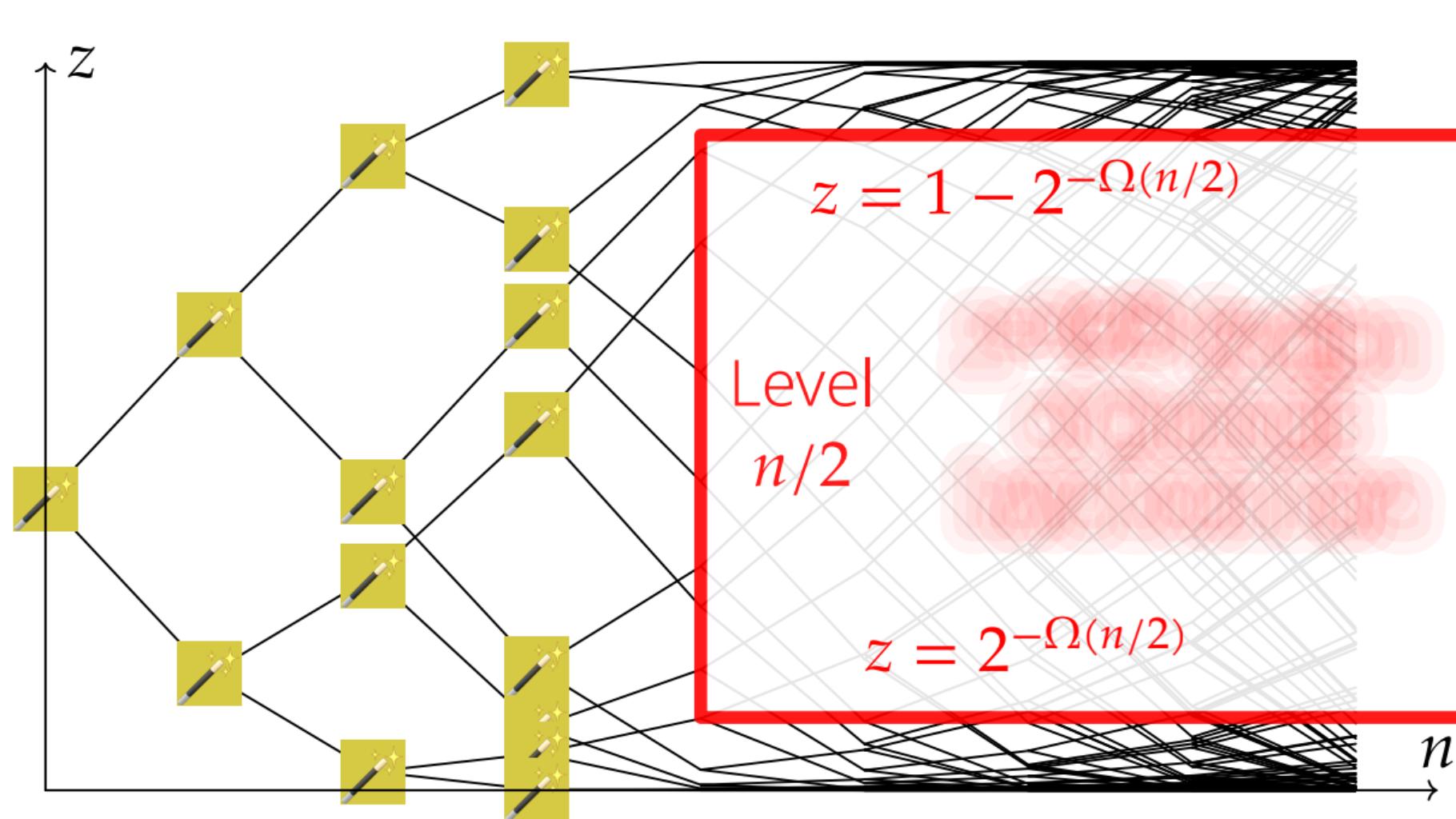


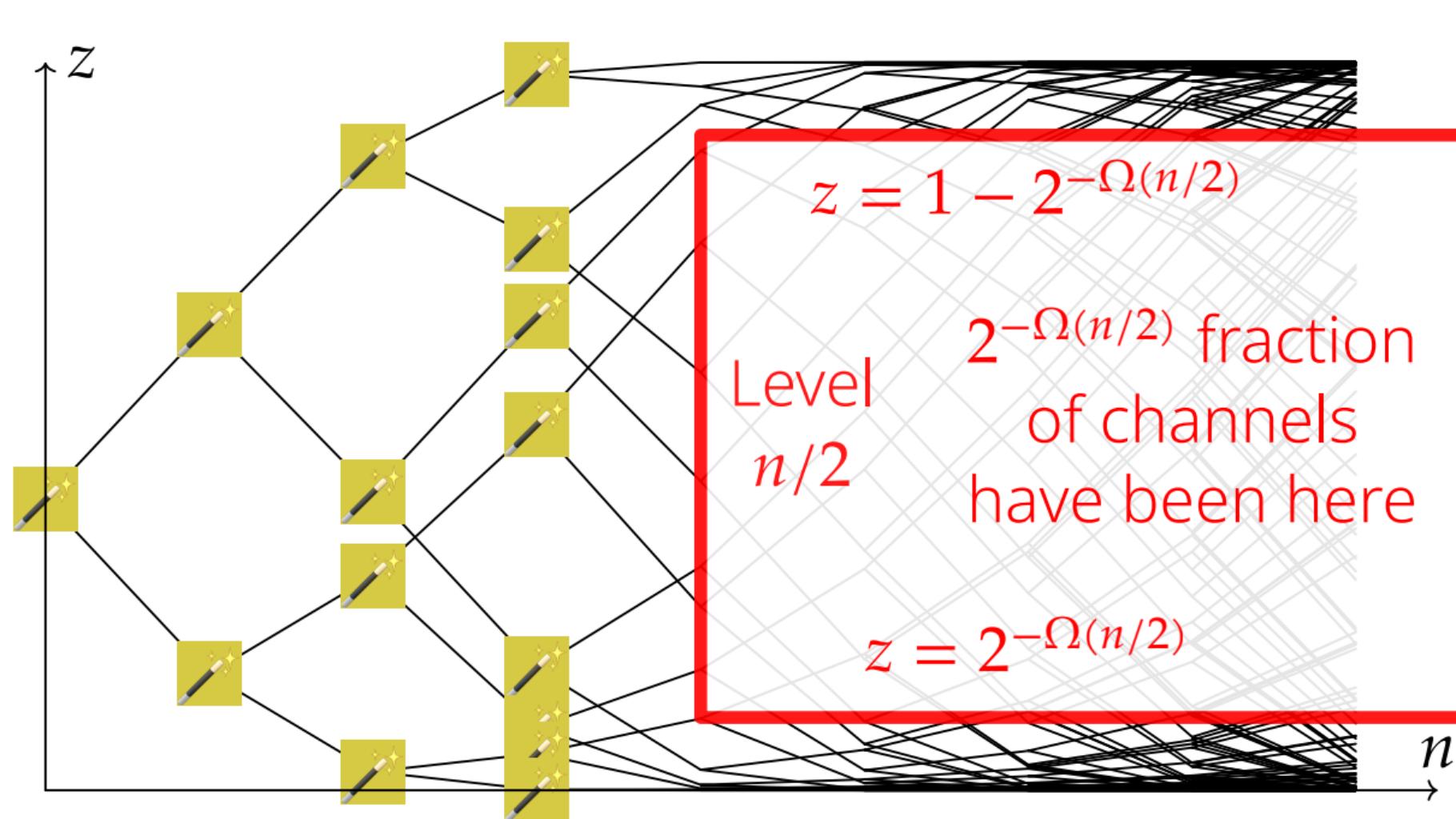












$z$

These channels,  
most of them  
(Chernoff)  
undergo  $z \mapsto z^2$   
 $n/5$  times

$$z = 1 - 2^{-\Omega(n/2)}$$

Level  
 $n/2$

$2^{-\Omega(n/2)}$  fraction  
of channels  
have been here

$$z = 2^{-\Omega(n/2)}$$

$n$

$z$

Squaring  
 $n/5$  times  
make  $z$   
 $(2^{-\Omega(n/2)})^{2^{n/5}}$

$$z = 1 - 2^{-\Omega(n/2)}$$

Level  
 $n/2$

$2^{-\Omega(n/2)}$  fraction  
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$$z = 2^{-\Omega(n/2)}$$

$n$

$z$

Squaring  
 $n/5$  times  
make  $z$   
 $(2^{-\Omega(n/2)})^{2^{n/5}} < 2^{-2^{n/5}}$

$$z = 1 - 2^{-\Omega(n/2)}$$

Level  
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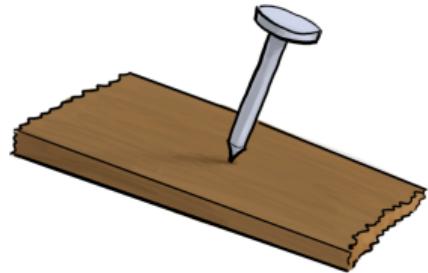
$$z = 2^{-\Omega(n/2)}$$

$n$

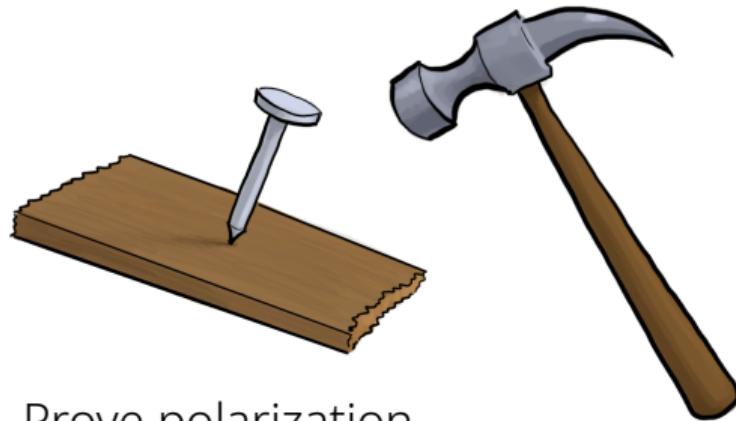
If  $k$  of  $2^n$  are less than  $p$ ,  
polar code has rate  $k/2^n$   
& block error prob  $< kp$ .

$$k/2^n = 60\% - 2^{-\Omega(n)}$$

~~$$p = 2^{-\Omega(n)}$$~~
$$p = 2^{-2^{\Omega(n)}}$$

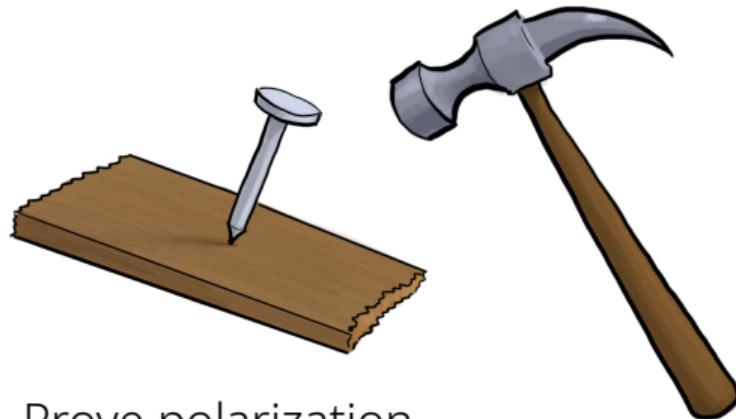


Prove polarization



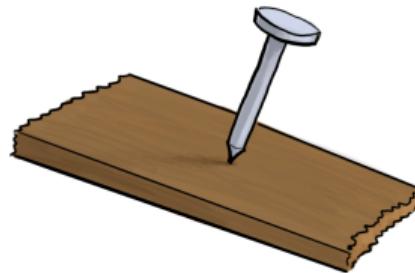
Prove polarization

Jensen's

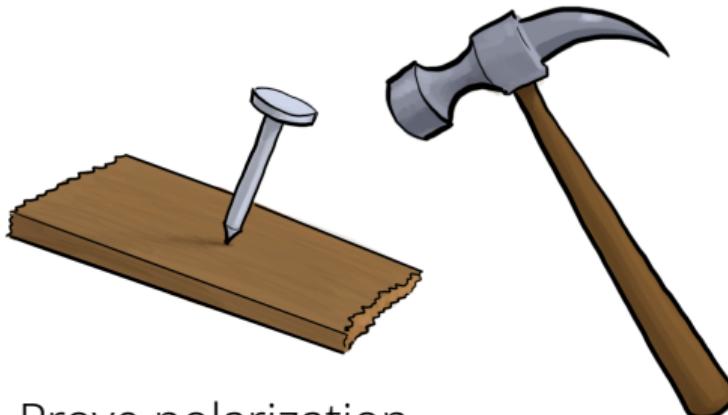


Prove polarization

Jensen's

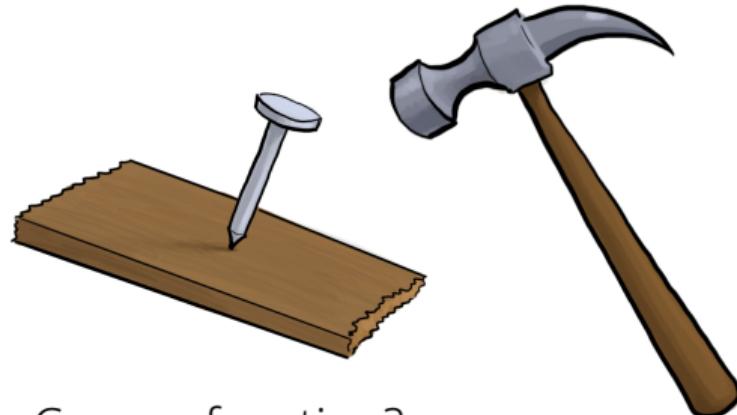


Convex function?



Prove polarization

Jensen's

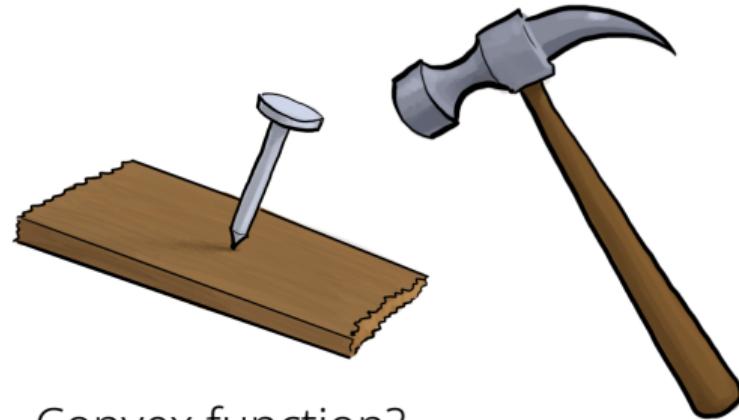


Convex function?

$$(z(1 - z))^{0.663}$$

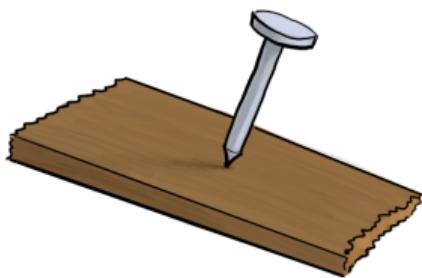


Prove polarization

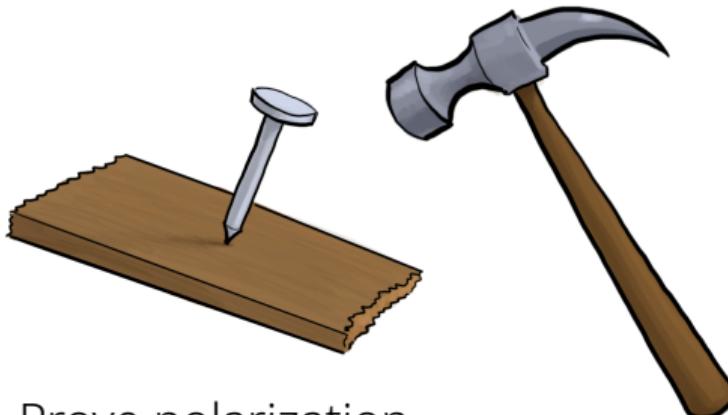


Convex function?

$$(z(1 - z))^{0.663}$$



$p$  too bad

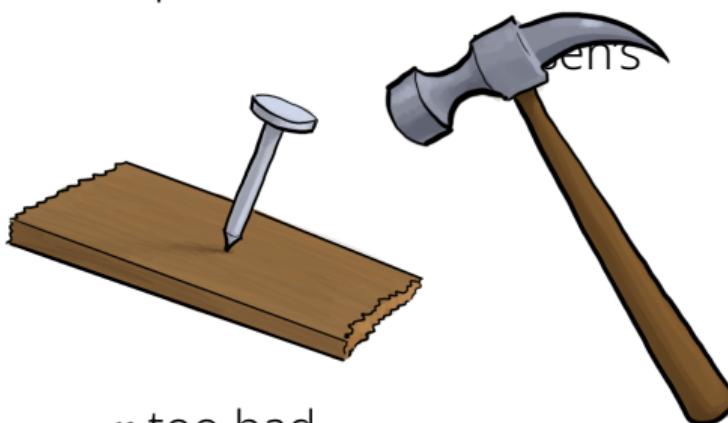


Prove polarization



Convex function?

$$(z(1 - z))^{0.663}$$



$p$  too bad

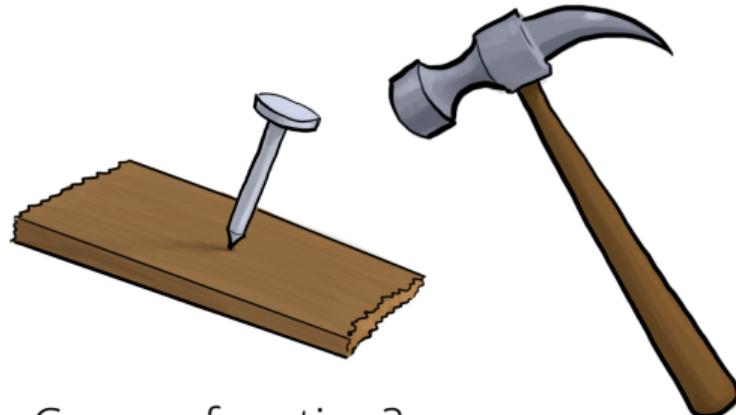
Chernoff  $p$



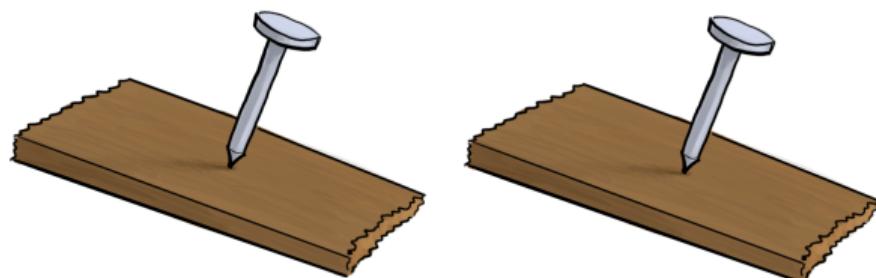
Prove polarization



$p$  too bad



Convex function?



$$(z(1 - z))^{0.663}$$

Improvements

Generalizations



ChatGPT



successive  
cancellation      channel  
polarization



better gap  
to capacity      better  
error prob

successive      channel  
cancellation      polarization

large matrix/RS/AG

BMS/large alphabet



large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

lossy/distribut compress



large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

lossy/distribut compress

classical-quantum

insdel channel

better gap  
to capacity

successive  
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large matrix/RS/AG

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coded computation

PAC & list decoding

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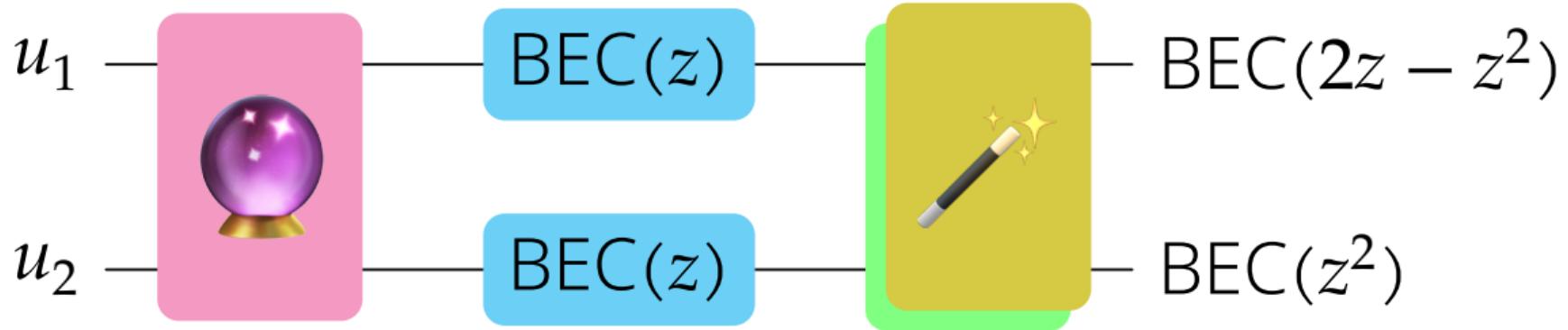
better gap  
to capacity

better  
error prob

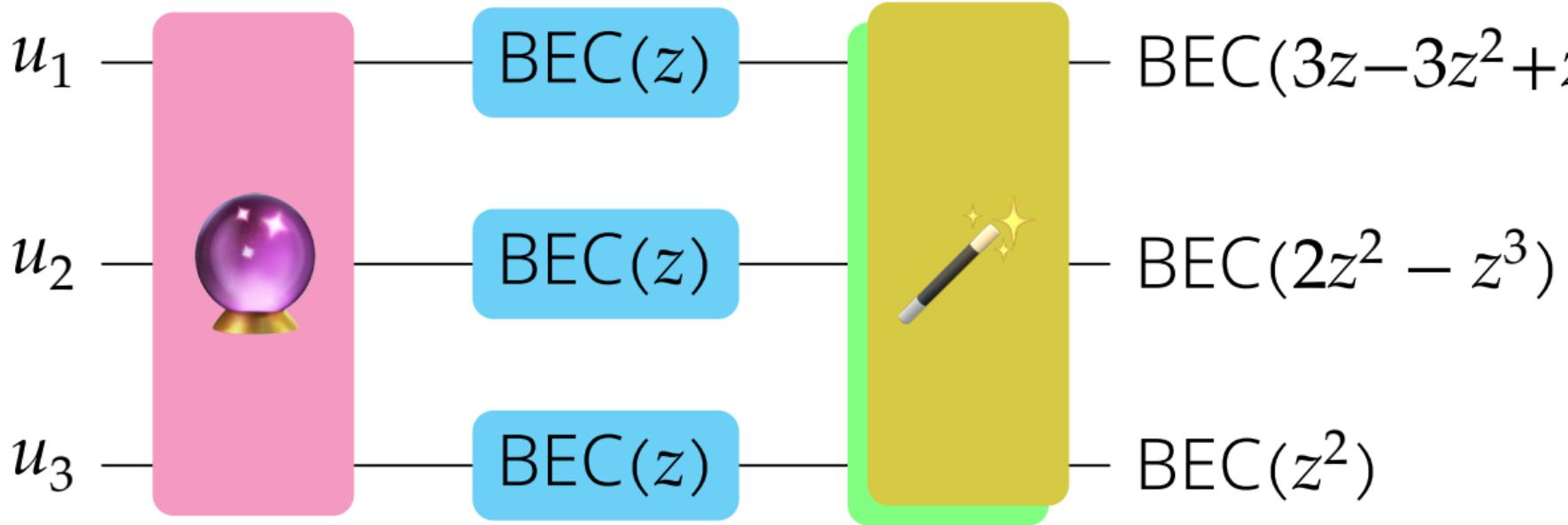
successive  
cancellation

channel  
polarization

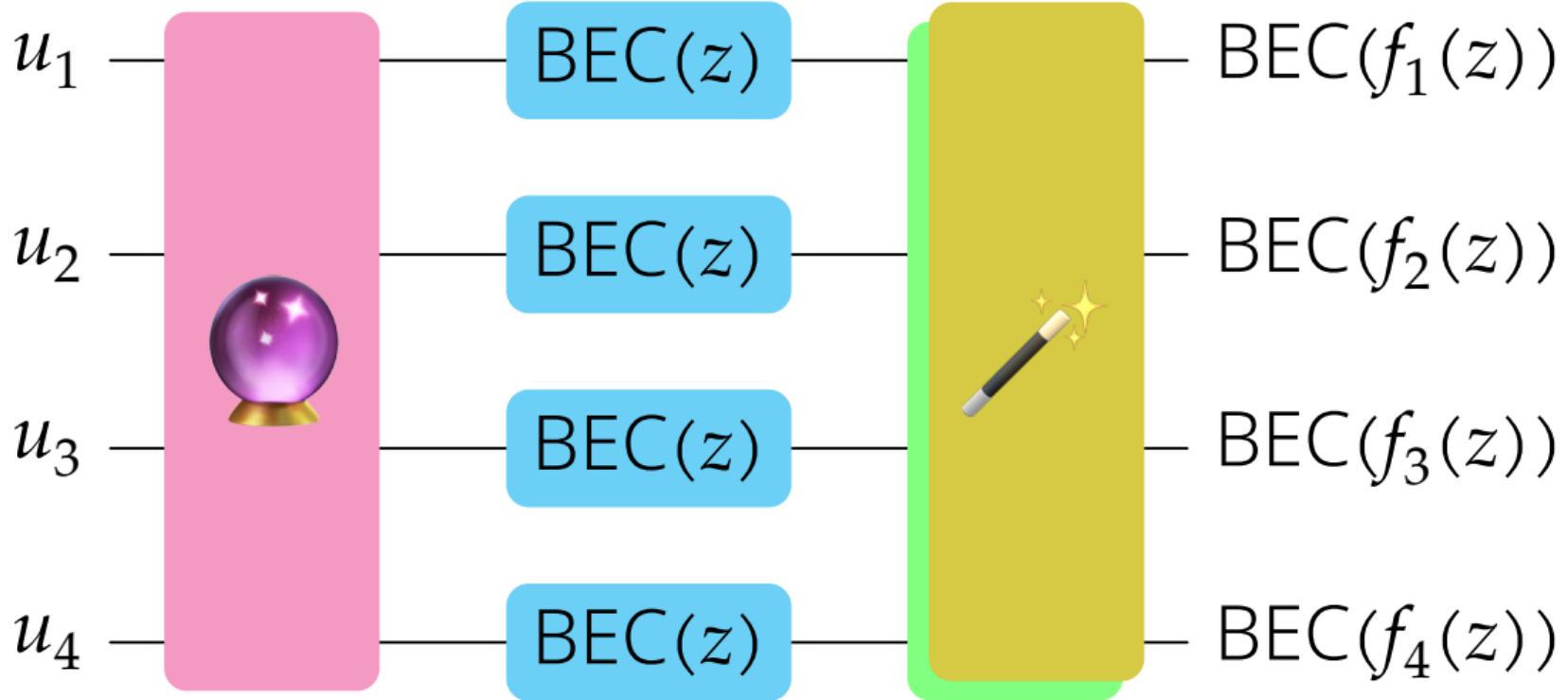




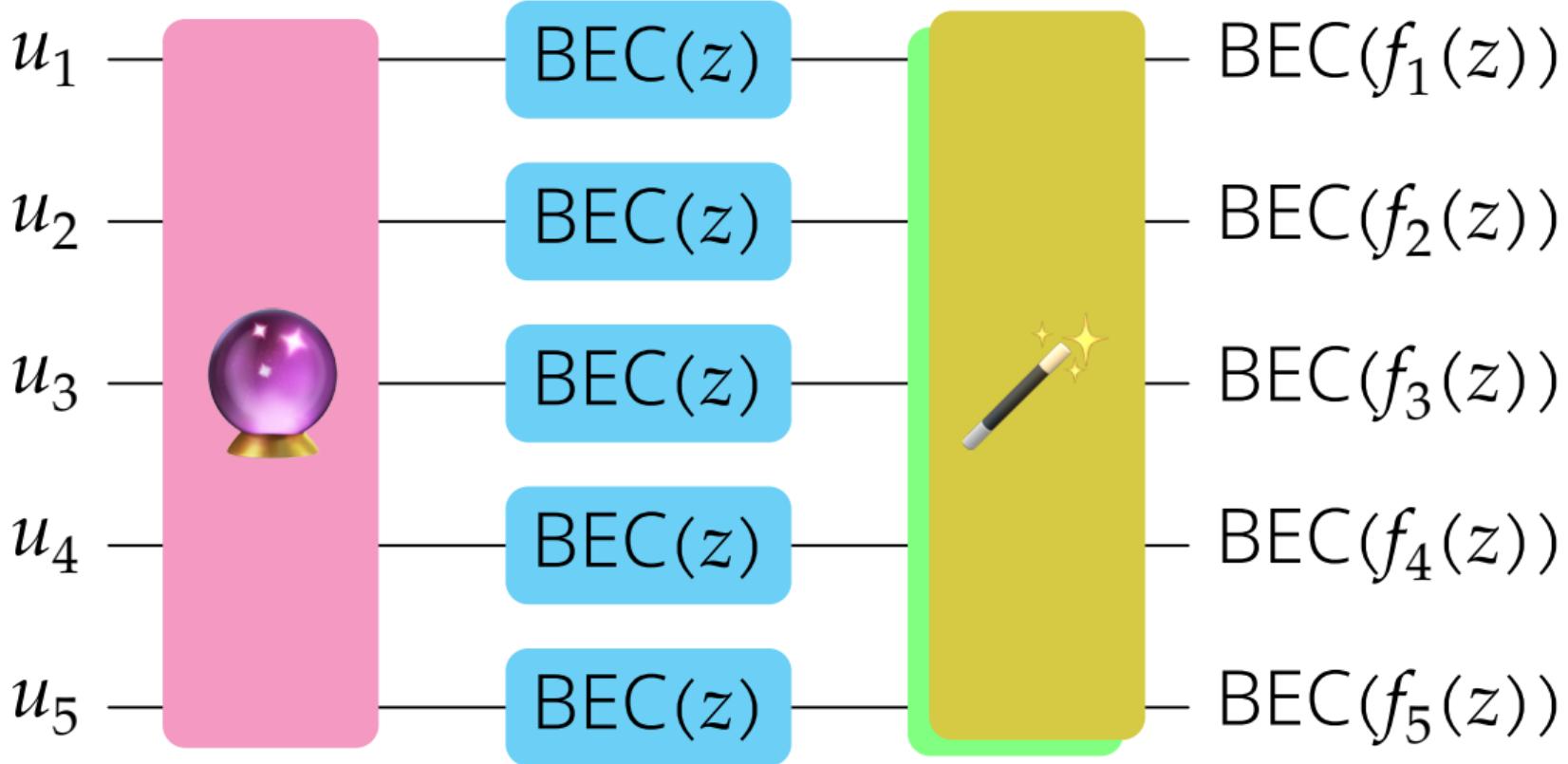
Upgrade  x4



Upgrade  x8



Upgrade x16



Upgrade x32



$\ell \times \ell$  matrices such as

$$\begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



$\ell \times \ell$  matrices such as

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$X_1^\ell := U_1^\ell \bullet$$

$Y_1^\ell :=$  channel output of  $X_1^j$

The  $j$ th new-channel is  $(U_j | Y_1^\ell U_1^{j-1})$



$\ell \times \ell$  matrices such as

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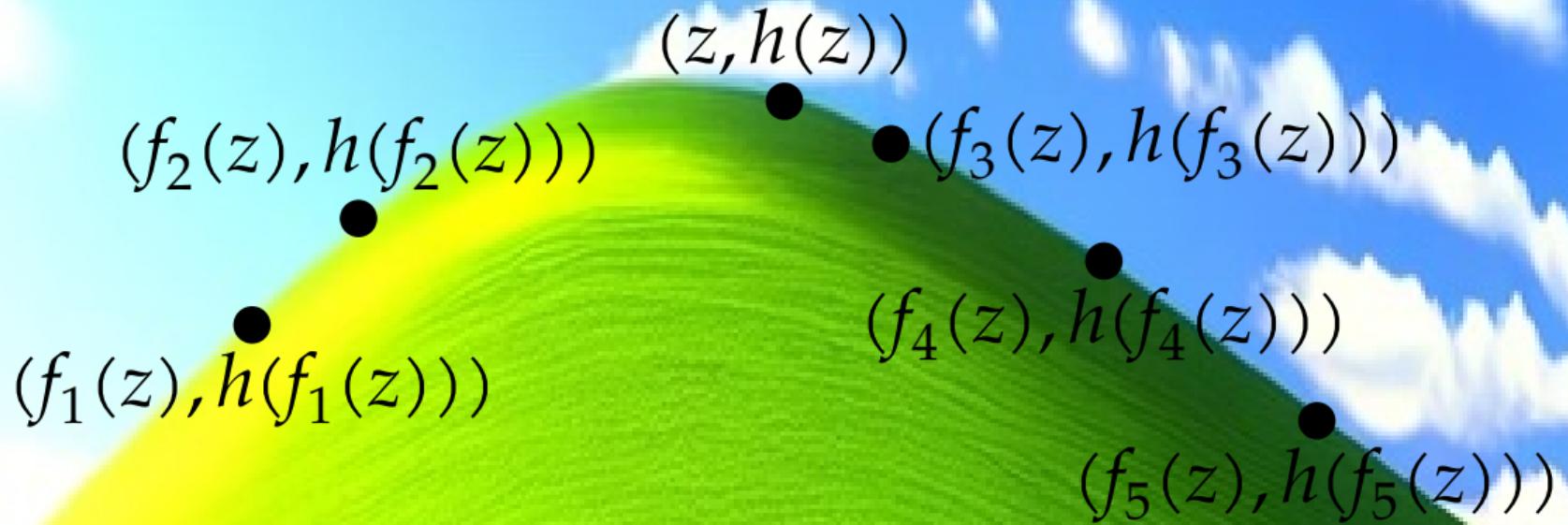
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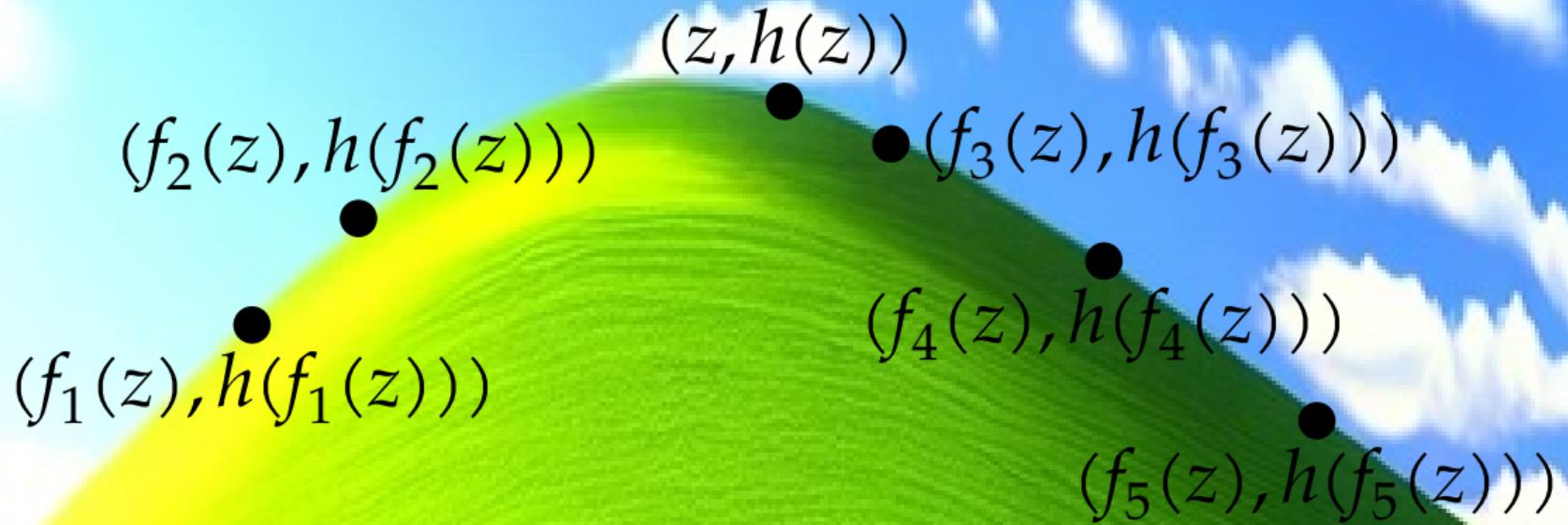
$Y_1^\ell :=$  channel output of  $X_1^j$

The  $j$ th new-channel is  $(U_j | Y_1^\ell U_1^{j-1})$

$$f_j(z) := \sum_{C \subset \text{columns}(\bullet)} (\text{rk}(C_{j-1}) - \text{rk}(C_j)) z^{\ell - |C|} (1-z)^{|C|}$$

where  $C_j$  is  $C$  with first  $j$  rows removed





Upper bound  $\sup_{0 < z < 1} \frac{1}{5h(z)} \sum_{j=1}^5 h(f_j(z))$

Gap to capacity =  $\frac{1}{\text{block length}^{??}}$  over BEC

Gap to capacity =  $\frac{1}{\text{block length}^{??}}$  over BEC

0

1/2

2010 Hassani–Alishahi–Urbanke  $2 \times 2$  

2010 Korada–Montanari–Telatar–Urbanke  $2 \times 2$  

Gap to capacity =  $\frac{1}{\text{block length}^{??}}$  over BEC

0

1/2

2010 Hassani–Alishahi–Urbanke  $2 \times 2$  

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2014 Fazeli–Vardy  $8 \times 8$  

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1/2

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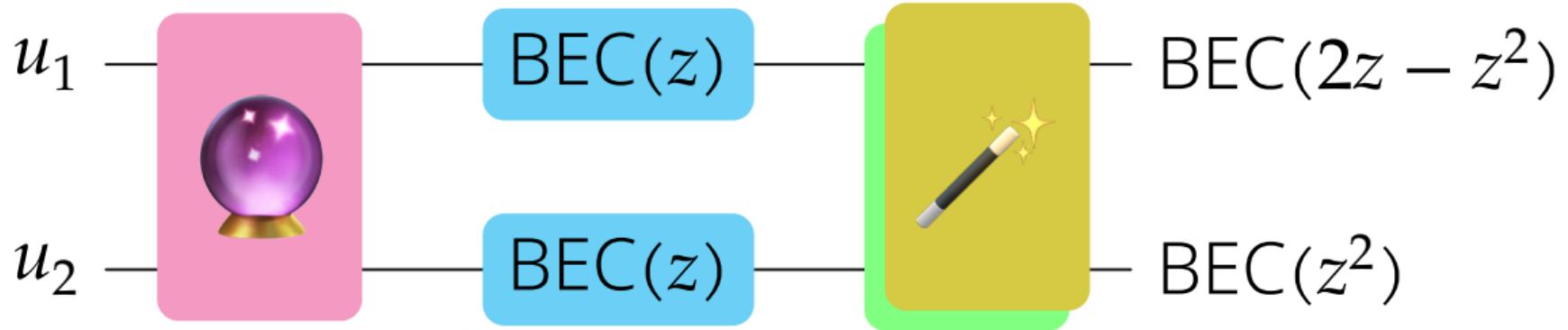
2021 Trofimiuk–Trifonov  $16 \times 16$  

2022 Duursma–Gabrys–Guruswami–Lin–W   $\in F_4^{2 \times 2}$  

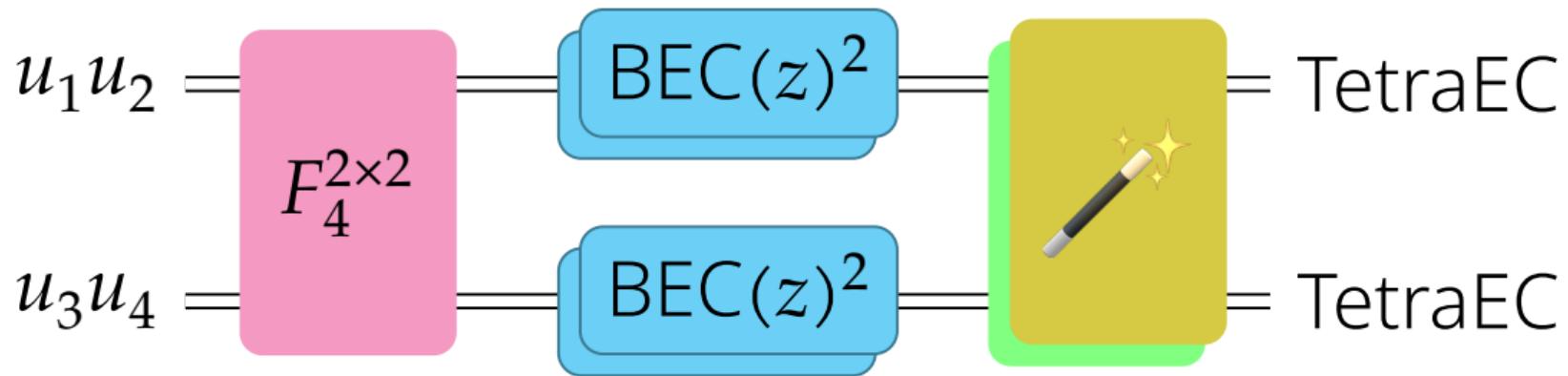
2021 Trofimiuk  $24 \times 24$  

2021 Yao–Fazeli–Vardy  $32 \times 32$  

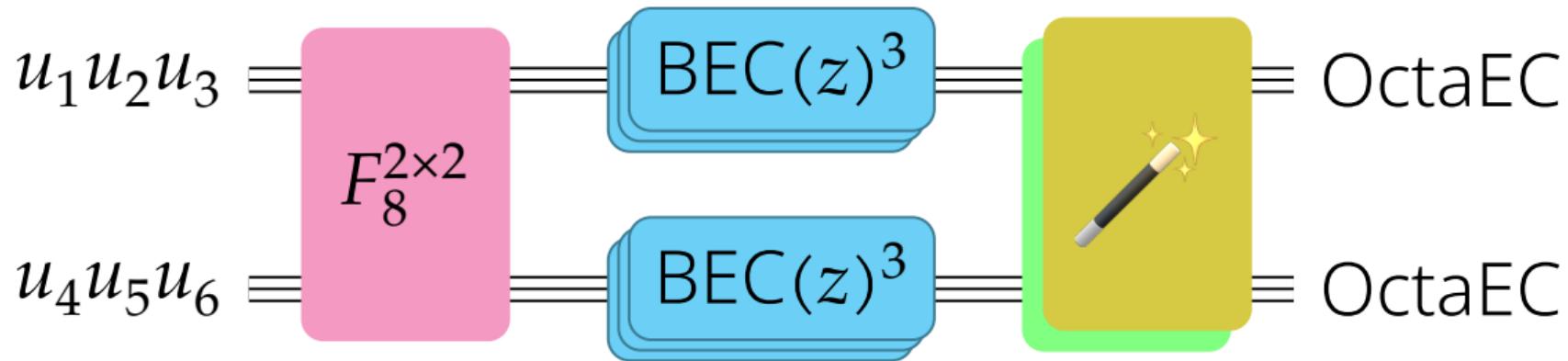
2021 Yao–Fazeli–Vardy  $64 \times 64$  



Upgrade  x1



Upgrade x2



Upgrade x3

$$u_1 u_2 u_3 = \text{F}_8^{2 \times 2} \xrightarrow{\text{BEC}(z)^3} \text{OctaEC}$$

$$u_4 u_5 u_6 = \text{F}_8^{2 \times 2} \xrightarrow{\text{BEC}(z)^3} \text{OctaEC}$$



Insert AI picture; large kernel vs large alphabet

large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

lossy/distribut compress

classical-quantum

insdel channel

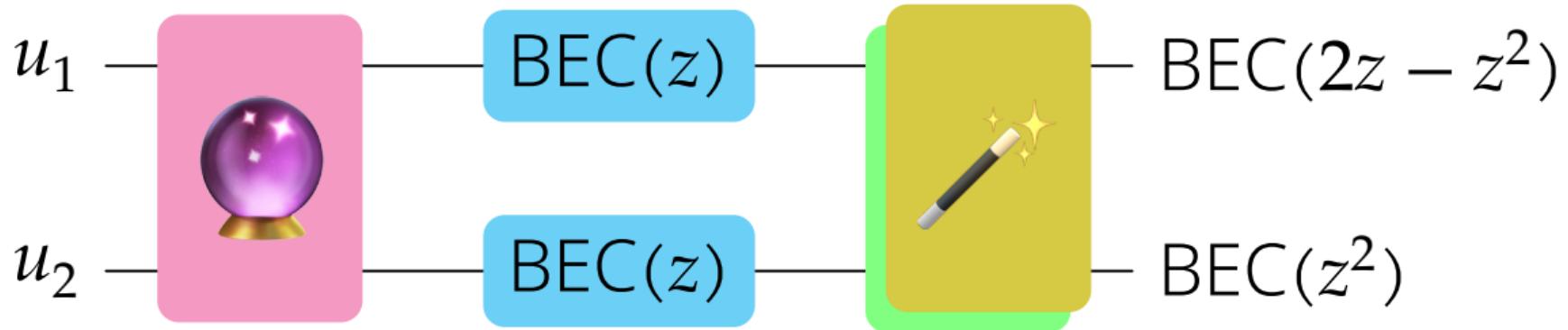
coded computation

PAC & list decoding

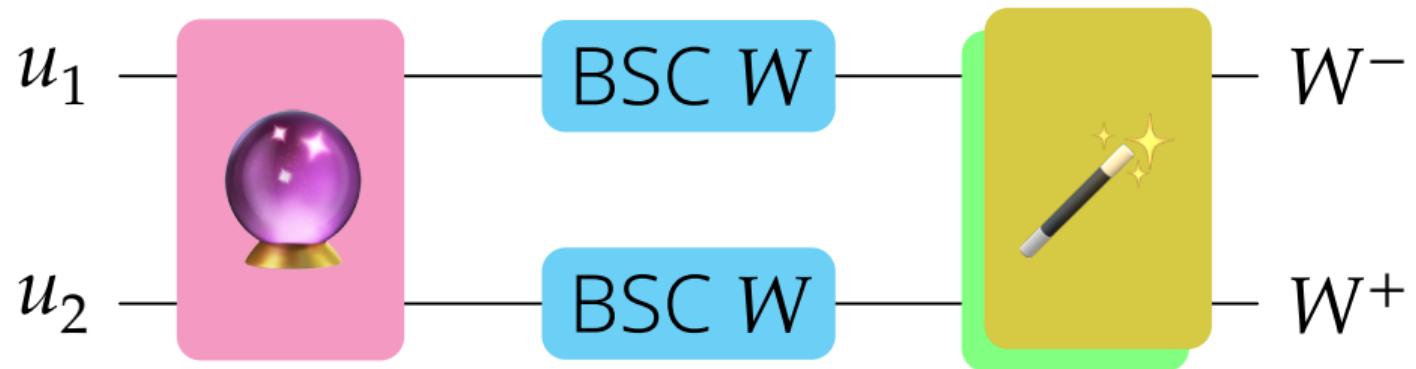
better gap  
to capacity      better  
error prob

successive  
cancellation      channel  
polarization





Hard Mode 💀



Harder Mode 💀💀

In general,  $W^+$  and  $W^-$  are not simple channels



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Invent inequalities of the form

$$Z(W^+) = z^2 \text{ where } z = Z(W)$$

$$z\sqrt{2 - z^2} \leq Z(W^-) \leq 2z - z^2$$

In general,  $W^+$  and  $W^-$  are not simple channels

→ Invent inequalities of the form

$$Z(W^+) = z^2 \text{ where } z = Z(W)$$

$$z\sqrt{2 - z^2} \leq Z(W^-) \leq 2z - z^2$$

→ Upper bound  $\frac{h(Z(W^+)) + h(Z(W^-))}{2h(Z(W))}$

Gap to capacity =  $\frac{1}{\text{block length}^{??}}$  over BSC

Gap to capacity =  $\frac{1}{\text{block length}^{\text{???}}}$  over BSC

0

1/2

2015 Guruswami-Xia



Gap to capacity =  $\frac{1}{\text{block length}^{??}}$  over BSC

0

1/2

2015 Guruswami-Xia  $\longleftrightarrow$

2012 Goli-Hassani-Urbanke  $\longleftrightarrow$

2014 Hassani-Alishahi-Urbanke  $\longleftrightarrow$

2014 Goldin-Burshtein  $\longleftrightarrow$

2016 Mondelli-Hassani-Urbanke  $\longleftrightarrow$



$\longleftrightarrow$

$$\text{Gap to capacity} = \frac{1}{\text{block length}^{???$$

0                            1/2

2015 Guruswami-Xia 

2012 Goli-Hassani-Urbanke 

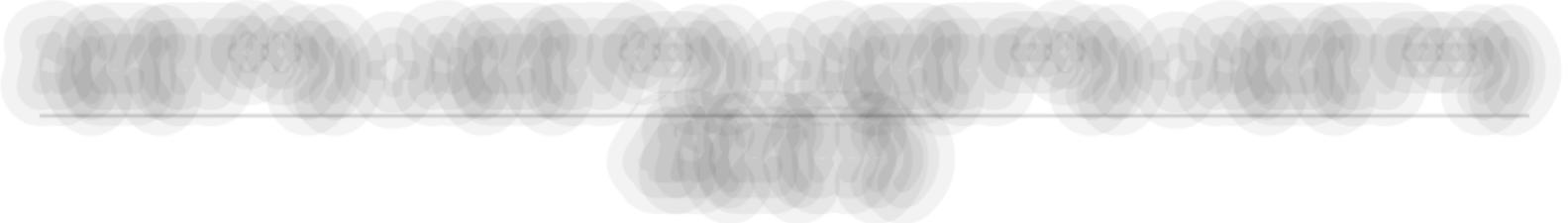
2014 Hassani-Alishahi-Urbanke 

2014 Goldin-Burshtein 

2016 Mondelli-Hassani-Urbanke 

2022 W-Lin-Vardy-Gabrys 

The last one is special because we bound

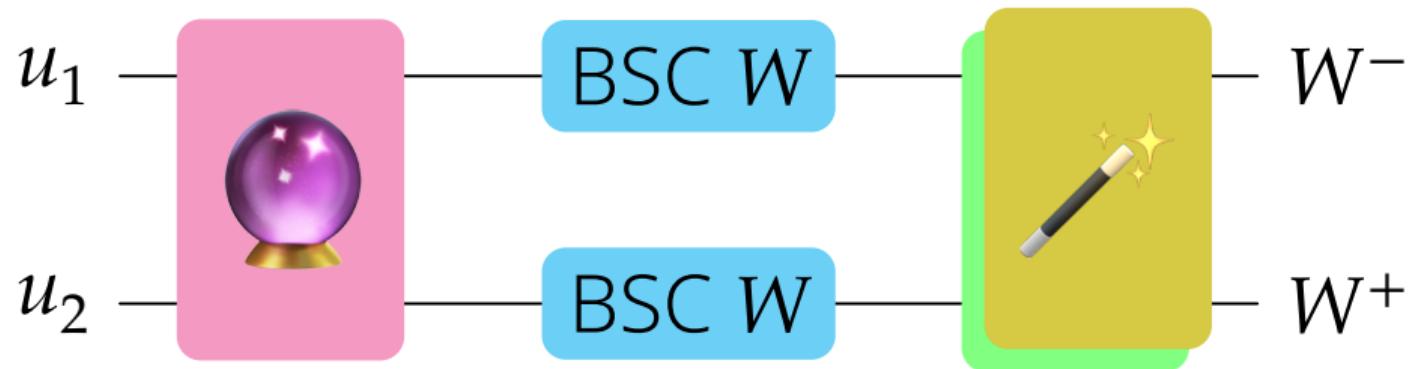


instead of  $\frac{h(Z(W^+)) + h(Z(W^-))}{2h(W)}$

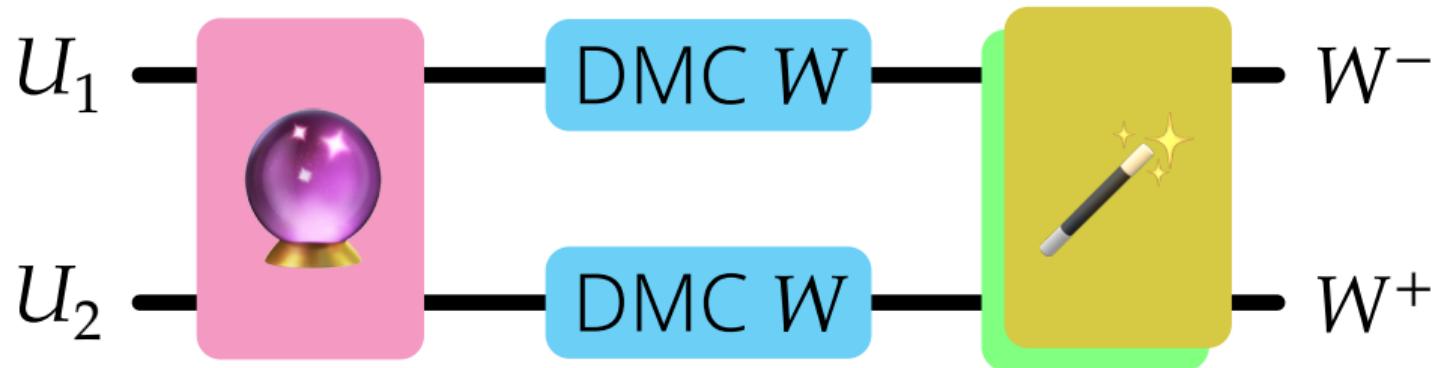
The last one is special because we bound

$$\frac{h(Z(W^{++})) + h(Z(W^{+-})) + h(Z(W^{-+})) + h(Z(W^{--}))}{4h(Z(W))}$$

instead of  $\frac{h(Z(W^+)) + h(Z(W^-))}{2h(W)}$



Harder Mode 💀💀



Hardest Mode



$W^+$  and  $W^-$  are not simple channels

$Z(W)$  (Bhattacharyya param) does not work well

$W^+$  and  $W^-$  are not simple channels

$Z(W)$  (Bhattacharyya param) does not work well

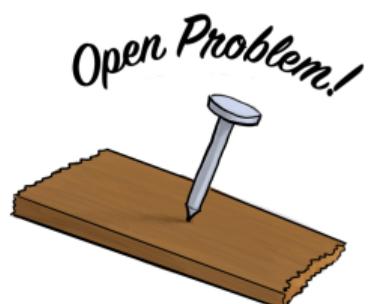
→ Invent inequalities of the form  
 $H(W^-) - H(W^+) > ???$

$W^+$  and  $W^-$  are not simple channels

$Z(W)$  (Bhattacharyya param) does not work well

➡ Invent inequalities of the form  
 $H(W^-) - H(W^+) > ???$

Ineq weak when alphabet large



large matrix/RS/AG

BMS/large alphabet

MAC/broadcast/wiretap

lossy/distribut compress

classical-quantum

insdel channel

coded computation

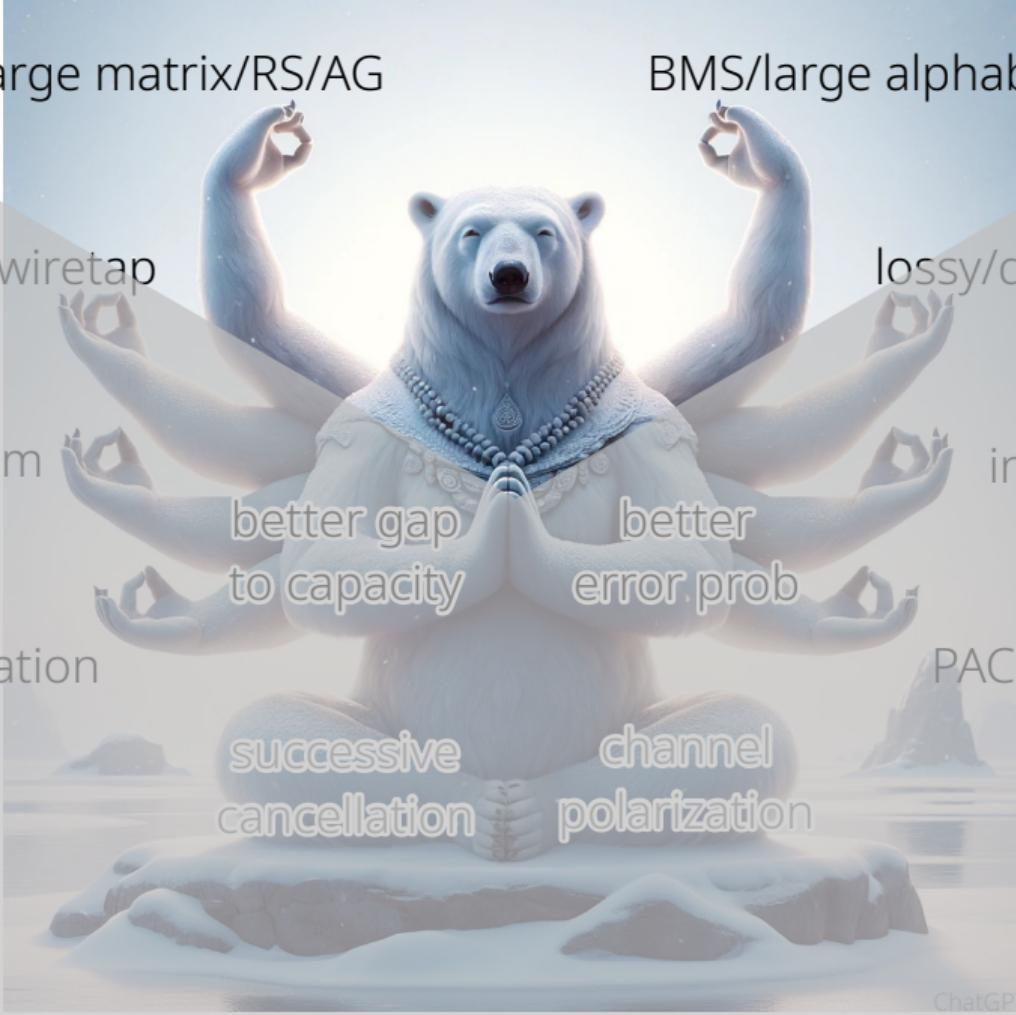
PAC & list decoding

better gap  
to capacity

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successive  
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channel  
polarization



Optimal Gap =  $\frac{1}{\text{block length}^{1/2}}$  over channels

$$\text{Optimal Gap} = \frac{1}{\text{block length}^{1/2}} \text{ over channels}$$

0 1/2

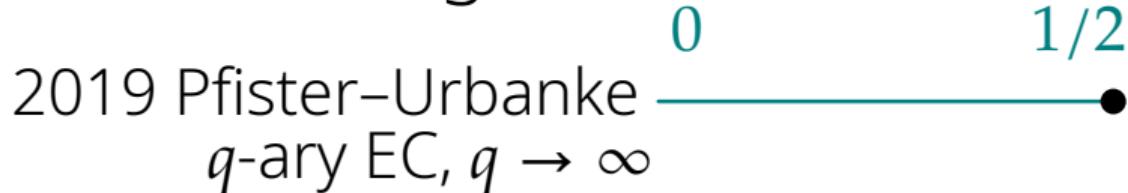
Optimal Gap =  $\frac{1}{\text{block length}^{1/2}}$  over channels

2019 Pfister–Urbanke  
 $q$ -ary EC,  $q \rightarrow \infty$

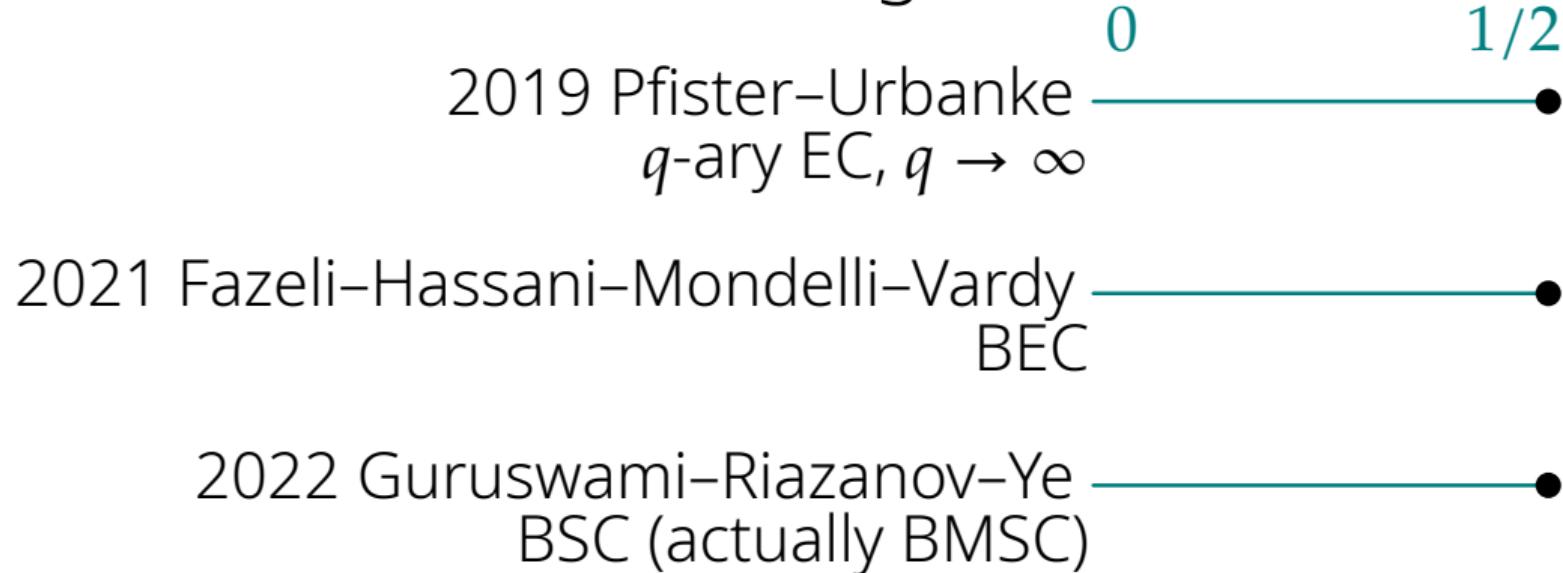


A horizontal number line with a black dot at  $1/2$ . The line starts at  $0$  and ends at  $1/2$ , with a black dot placed exactly halfway along it.

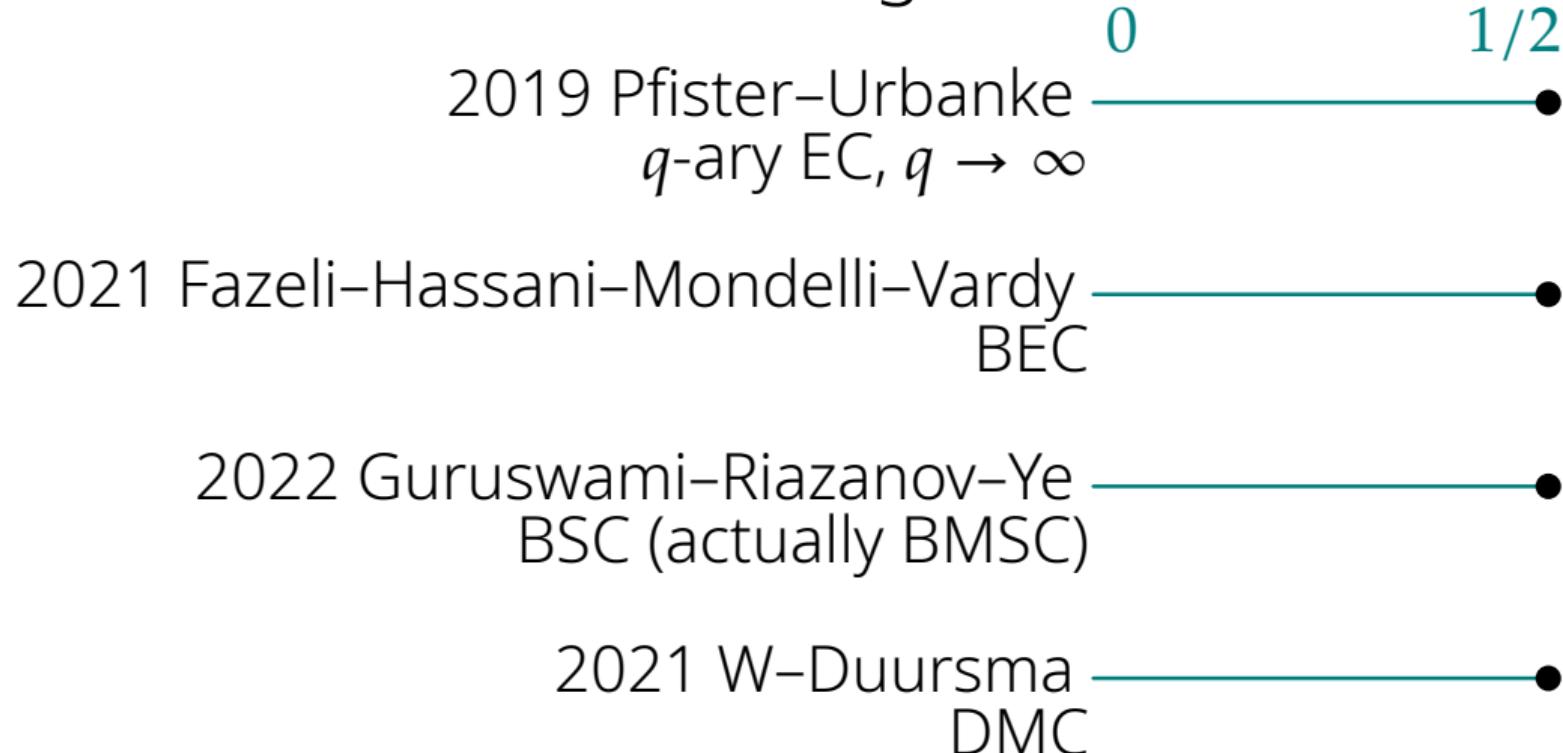
Optimal Gap =  $\frac{1}{\text{block length}^{1/2}}$  over channels

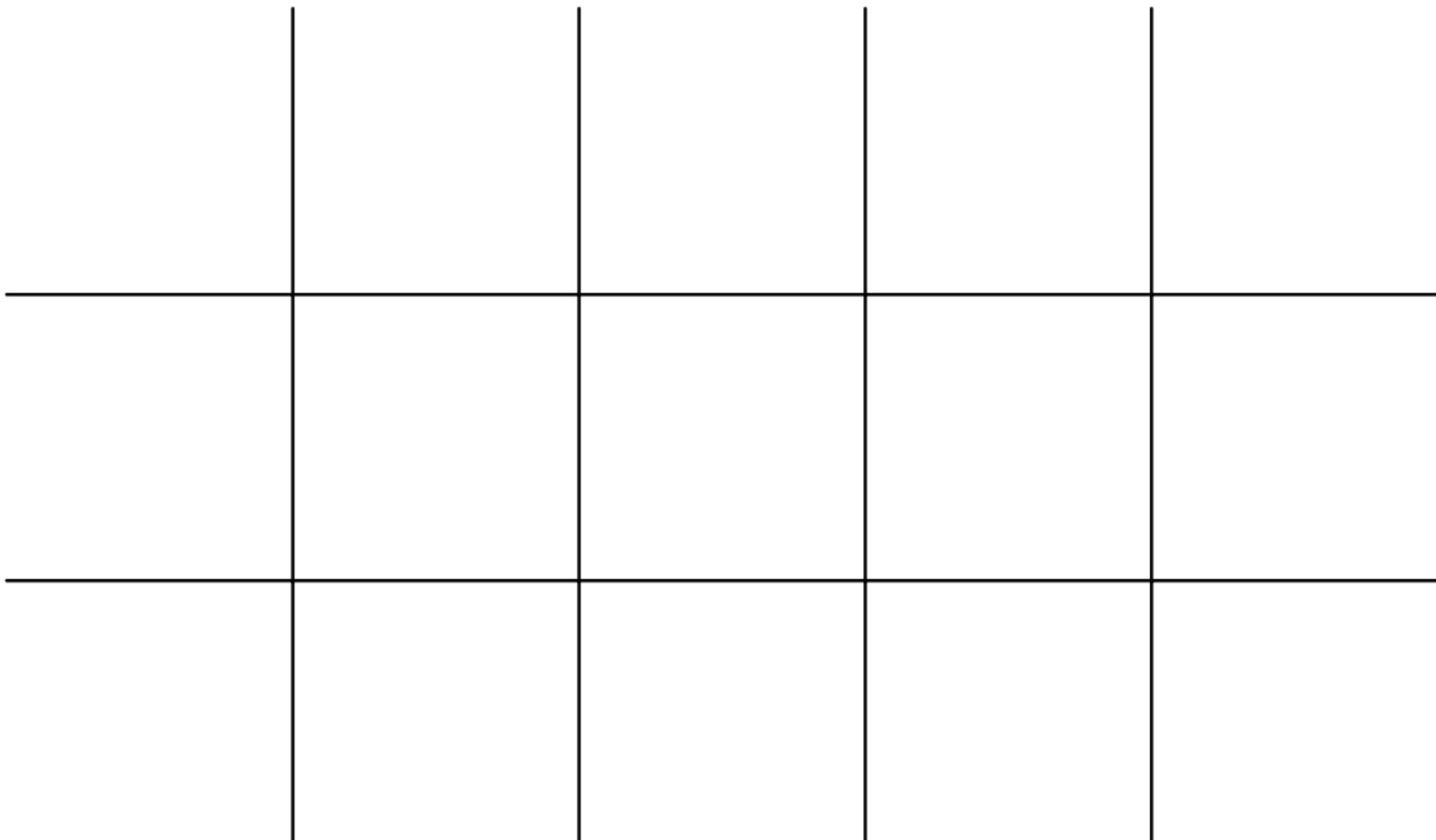


Optimal Gap =  $\frac{1}{\text{block length}^{1/2}}$  over channels



Optimal Gap =  $\frac{1}{\text{block length}^{1/2}}$  over channels





Probability  
Theory


Probability  
Theory

Random  
Codes

Probability  
Theory

Random  
Codes

Polar  
Codes

Probability  
Theory

law of  
large  
numbers

Random  
Codes

Polar  
Codes

Probability  
Theory

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Random  
Codes

achieve  
capacity  
[Shannon]

Polar  
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Polar  
Codes

achieve  
capacity  
[Arıkan]

Probability Theory	law of large numbers	central limit theorem		
Random Codes	achieve capacity [Shannon]			
Polar Codes	achieve capacity [Arıkan]			

Probability Theory	law of large numbers	central limit theorem		
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$		
Polar Codes	achieve capacity [Arıkan]			

Probability Theory	law of large numbers	central limit theorem		
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$		
Polar Codes	achieve capacity [Arikan]	gap to capacity $1/N^{1/2-\varepsilon}$		

Probability Theory	law of large numbers	central limit theorem	large deviations
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	
Polar Codes	achieve capacity [Arıkan]	gap to capacity $1/N^{1/2-\varepsilon}$	

Probability Theory	law of large numbers	central limit theorem	large deviations
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	error prob $\exp(-N)$
Polar Codes	achieve capacity [Arıkan]	gap to capacity $1/N^{1/2-\varepsilon}$	

Probability Theory	law of large numbers	central limit theorem	large deviations
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	error prob $\exp(-N)$
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Probability Theory	law of large numbers	central limit theorem	large deviations	moderate deviations
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	error prob $\exp(-N)$	
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Probability Theory	law of large numbers	central limit theorem	large deviations	moderate deviations
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	error prob $\exp(-N)$	$\ln(-\ln E) + 2 \ln(G) = \ln(N)$
Polar Codes	achieve capacity [Arikan]	gap to capacity $1/N^{1/2-\varepsilon}$	error prob $\exp(-N^{1-\varepsilon})$	

Probability Theory	law of large numbers	central limit theorem	large deviations	moderate deviations
Random Codes	achieve capacity [Shannon]	gap to capacity $1/\sqrt{N}$	error prob $\exp(-N)$	$\ln(-\ln E) + 2 \ln(G) = \ln(N)$
Polar Codes	achieve capacity [Arikan]	gap to capacity $1/N^{1/2-\varepsilon}$	error prob $\exp(-N^{1-\varepsilon})$	$\ln(-\ln E) + 2 \ln(G) = (1-\varepsilon) \ln(N)$

large matrix/RS/AG

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dummy

dummy

2011 Alamdar-Yazdi-Kschischang:  
Prune the tree to reduce complexity.

2017 El-Khamy-Mahdavifar-Feygin-Lee-Kang:  
Pruning reduces complexity by a scalar; still  $O(N \log N)$ .

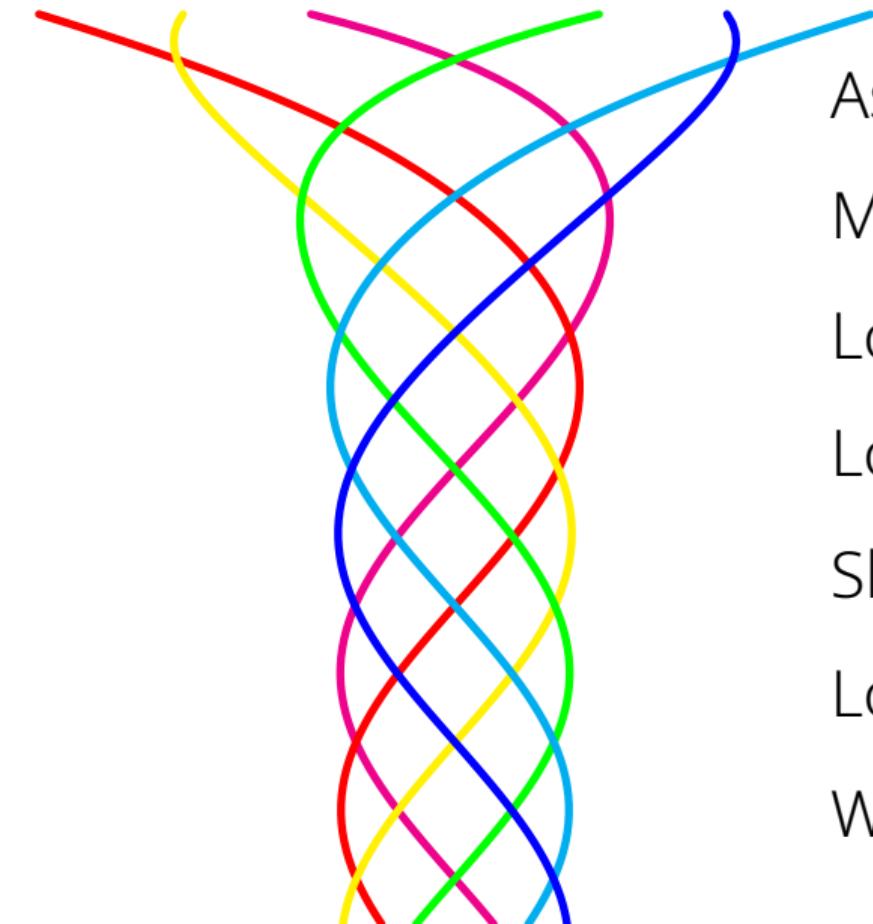
2021 W-Duursma:  $O(N \log \log N)$   
Trade-off: complexity  $\approx O(N \log(-\log(\text{decode error})))$ .

2021 Mondelli-Hashemi-Cioffi-Goldsmith,  
2021 Hashemi-Mondelli-Fazeli-Vardy-Cioffi-Goldsmith:  
Study parallelism vs latency.

Polar code is a mathy code

Polar code is a mathy code

Polar achieves the capacity of



Asymmetric channel

Multiple access channel

Lossless compression

Lossy compression

Slepian–Wolf

Lossless compression w/ helper

Wiretap channel (degradation)

Deletion channel ... (good error prob)

Broadcast channel ... (good error prob)

Channel with memory ... (good error prob)

Wiretap channel (no degradation) ... (good error prob)

Hidden Markov chain channel state ... (good error prob)

Non-stationary channel ... (good gap to capacity)

Classical-Quantum channel ... (yes)

Quantum-Quantum channel ... (?)

