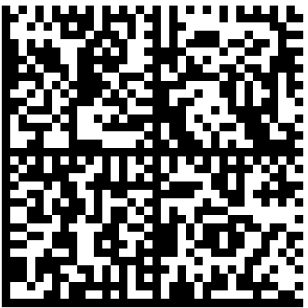
Slides available on



https://polar-tutorial.symbol.codes/

Polar Codes Tutorial

Hsin-Po Wang

(Berkeley, EECS)

ChatGPT

Engineering

List decoder

Future

Origin Story

ChatGPT

Engineering

List decoder

Future

Origin Story

Engineering

List decoder

Future

Origin Story

ChatGPT

Engineering

List decoder

Future

Origin Story

Theory

Math args

Asymptotics

Versatility

What’s so innovative about polar codes?


WHO IS CLAUDE SHANNON?
THE BIT PLAYER
A FILM BY MARK A. LEVINSON
IMDb


WHO IS CLAUDE SHANNON?
THE BIT PLAYER
A FILM BY MARK A. LEVINSON
IMDb


ChatGPT

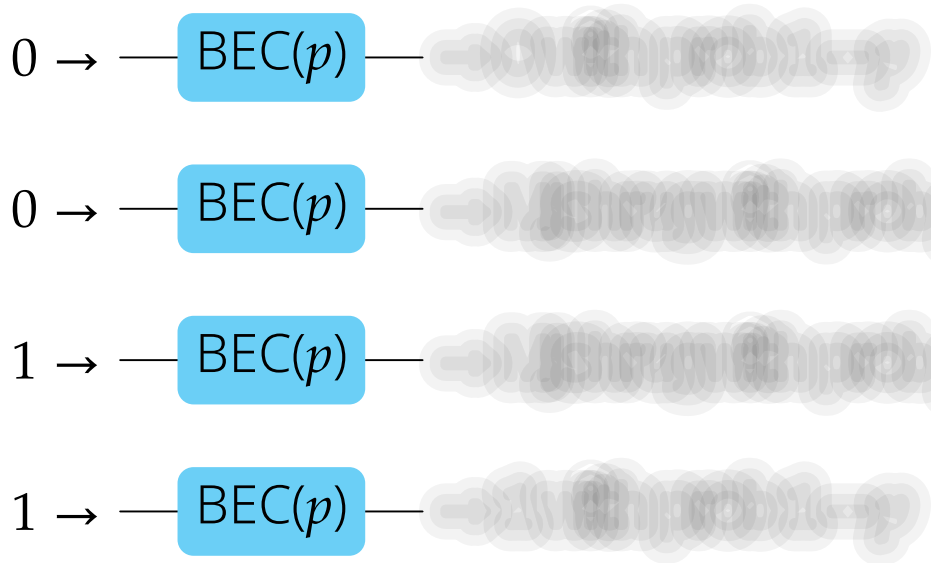

WHO IS CLAUDE SHANNON?
THE BIT PLAYER
A FILM BY MARK A. LEVINSON
IMDb

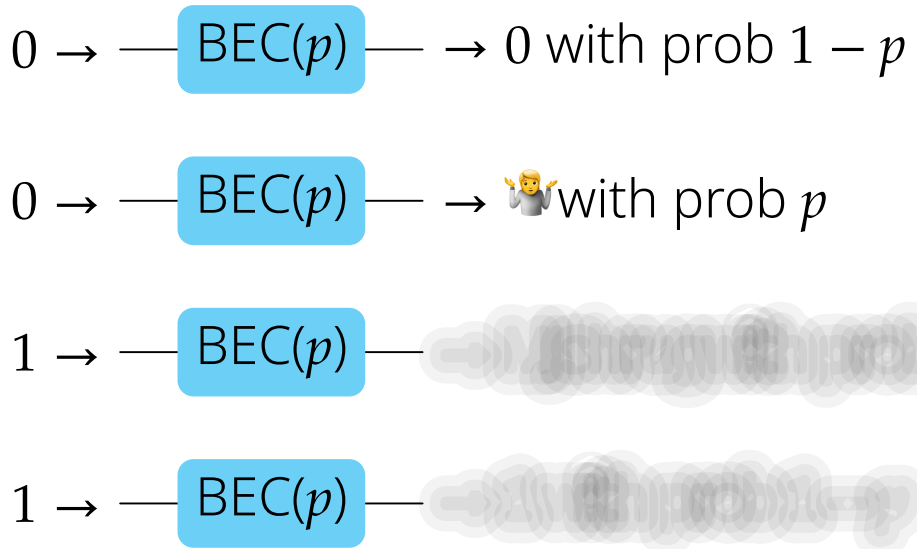

The Channel Player
ChatGPT

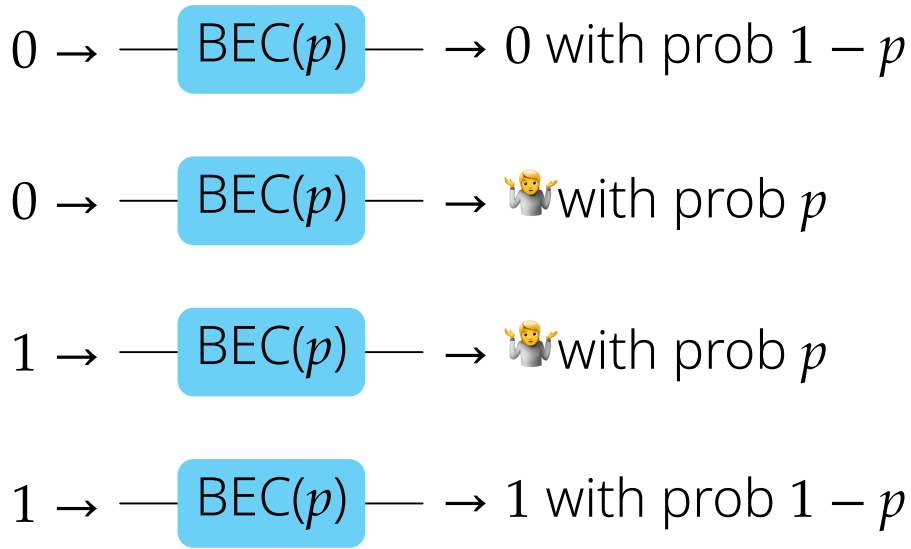
Let’s play with

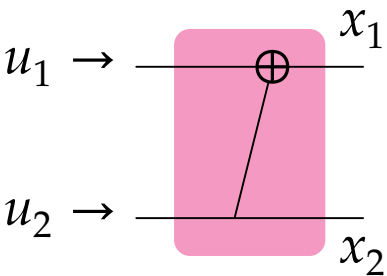
binary erasure channels

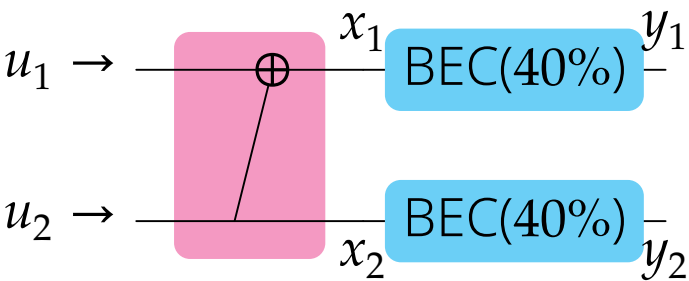
(BECs)

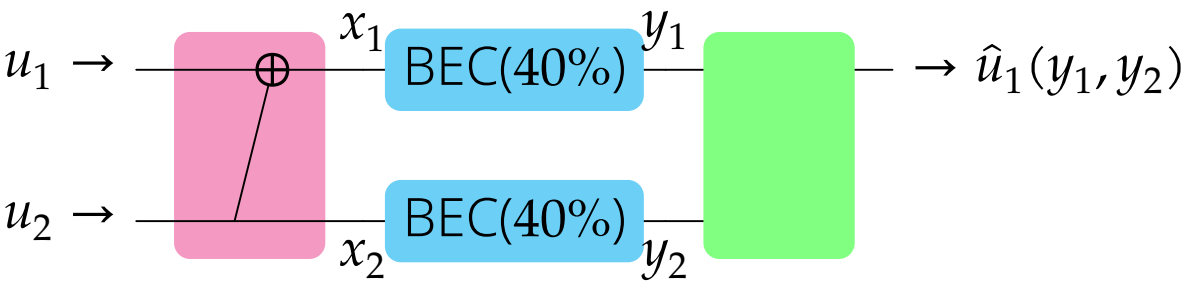

0 →
BEC(p)
0 →
BEC(p)
1 →
BEC(p)
1 →
BEC(p)

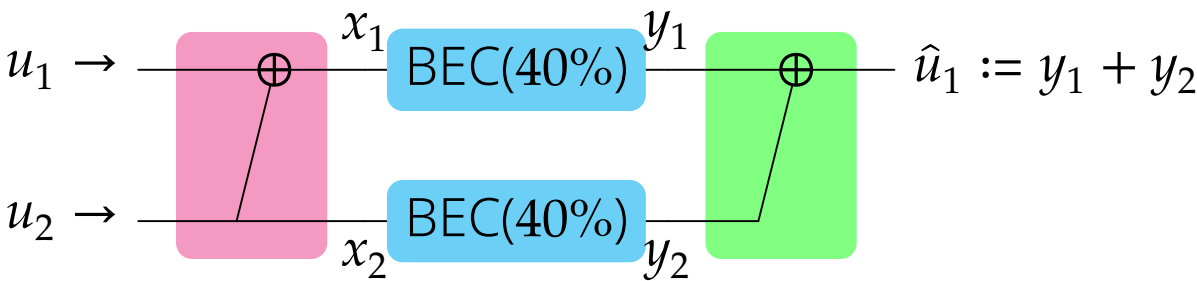

0 →
BEC(p)
→ 0 with prob 1 – p
0 →
BEC(p)
→
with prob p
1 →
BEC(p)
1 →
BEC(p)

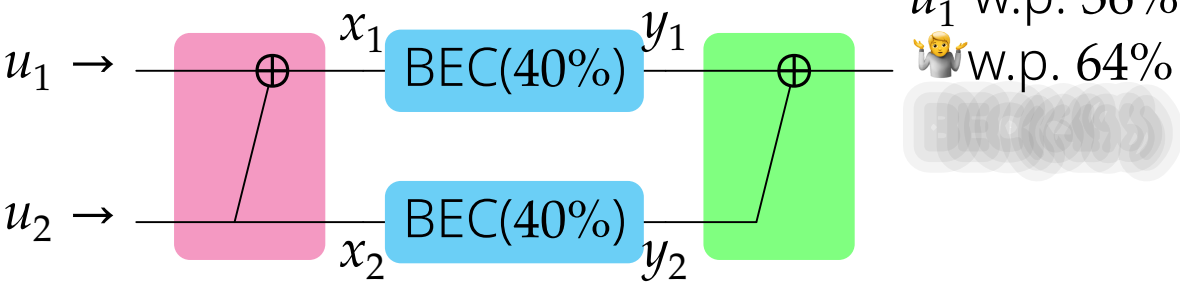

0 →
BEC(p)
→ 0 with prob 1 – p
0 →
BEC(p)
→
with prob p
1 →
BEC(p)
→
with prob p
1 →
BEC(p)
→ 1 with prob 1 – p

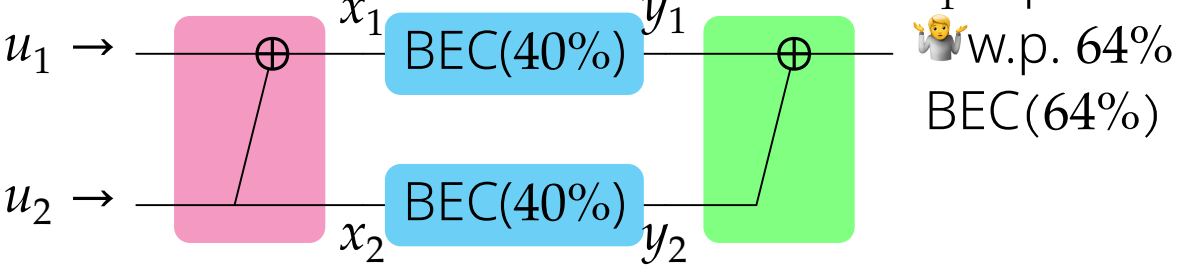

u_1
u_2
x_1
x_2

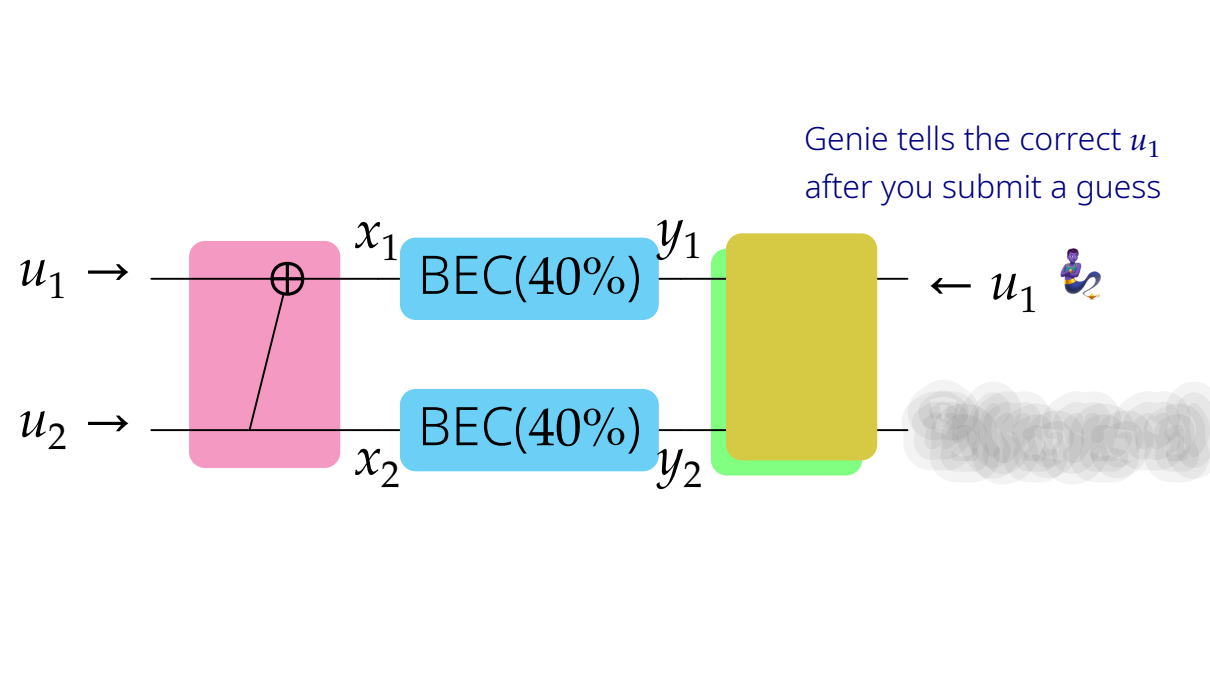

u_1
u_2
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2

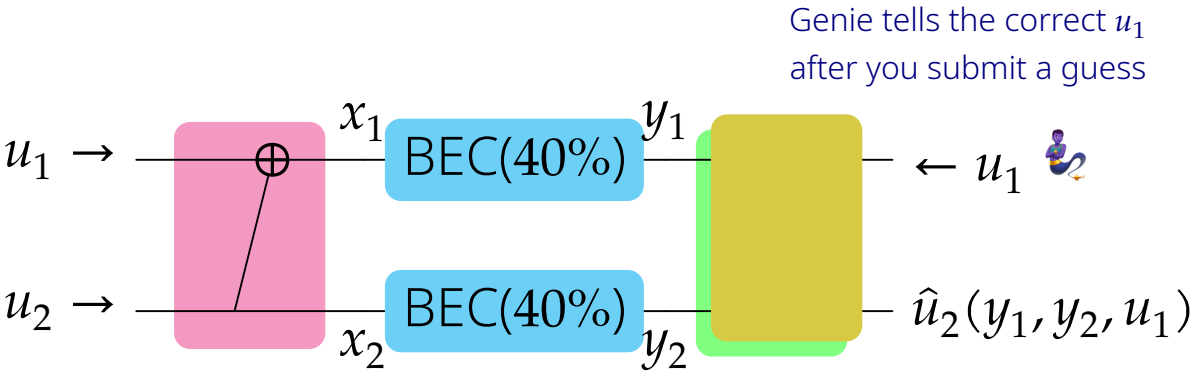

$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\rightarrow \hat{u}_1(y_1, y_2)$

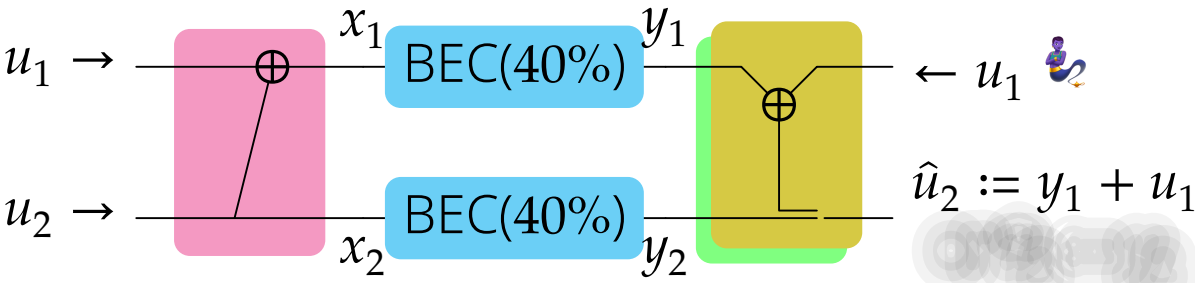

$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\hat{u}_1 := y_1 + y_2$

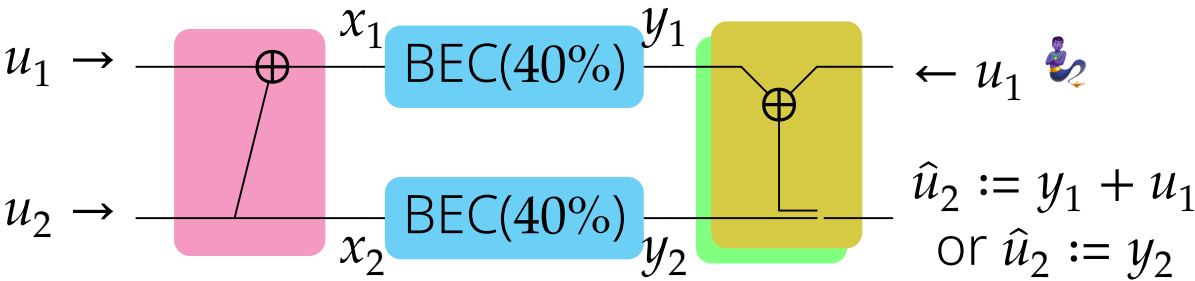

u_1 →
u_2 →
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
u_1 w.p. 36%
w.p. 64%

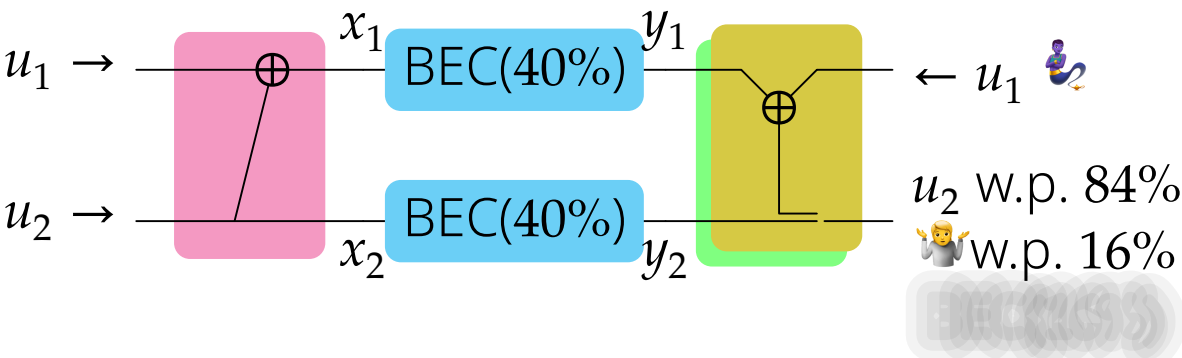

$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
u_1 w.p. 36%
🤷 w.p. 64%
BEC(64%)

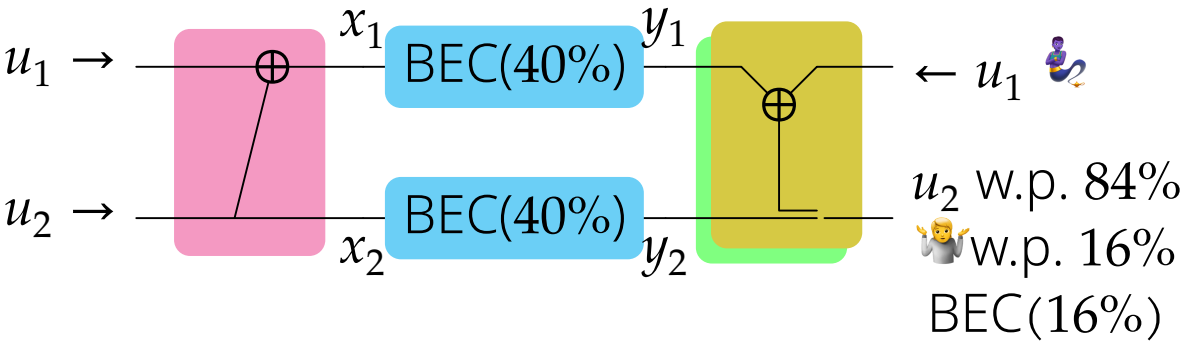

Genie tells the correct u_1
after you submit a guess
$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$

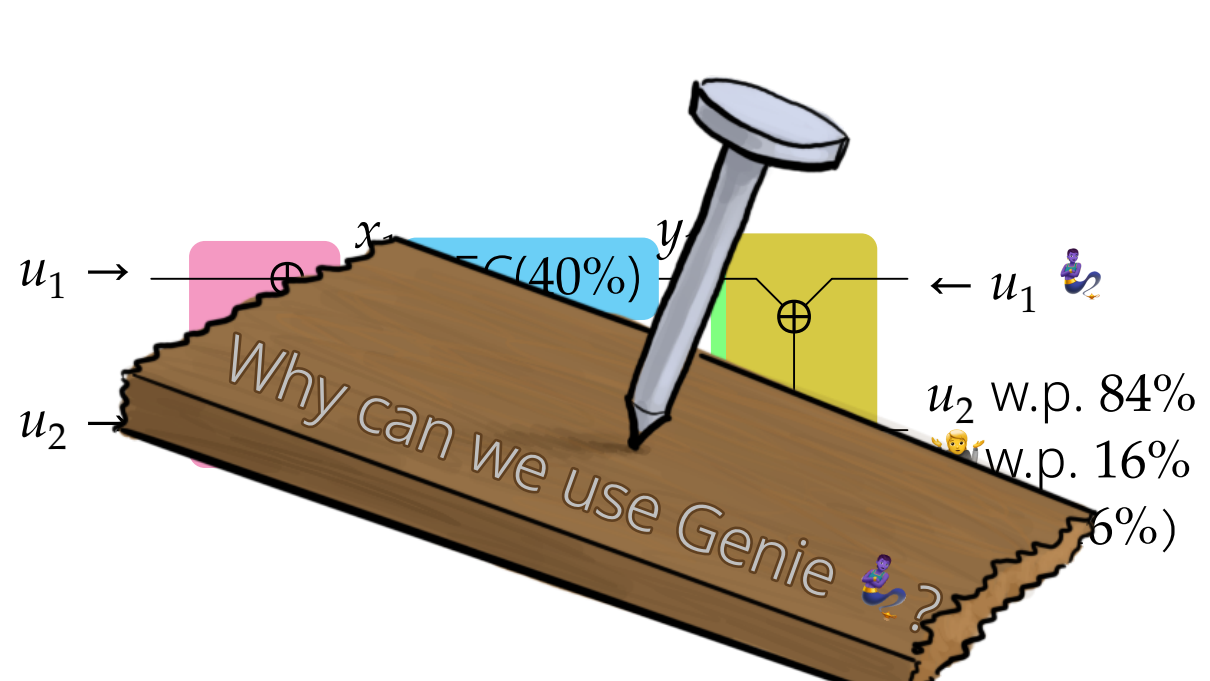

Genie tells the correct u_1
after you submit a guess
$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
$\hat{u}_2(y_1, y_2, u_1)$


$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
$\hat{u}_2 := y_1 + u_1$


$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
$\hat{u}_2 := y_1 + u_1$
or $\hat{u}_2 := y_2$


$u_1 \rightarrow$
$u_2 \rightarrow$
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
$\leftarrow u_1$
u_2 w.p. 84%
w.p. 16%


u_1 →
u_2 →
x_1
x_2
BEC(40%)
BEC(40%)
y_1
y_2
← u_1
u_2 w.p. 84%
w.p. 16%
BEC(16%)


u_1 →
u_2 →
x
y
$\in C(40\%)$
← u_1
u_2 w.p. 84%
w.p. 16%
6%)
Why can we use Genie ?

- Case 1: $\hat{u}_1 = u_1$
- Case 2:
- Case 3:

- Case 1: $\hat{u}_1 = u_1$
  does not provide any useful info
- Case 2:
- Case 3:

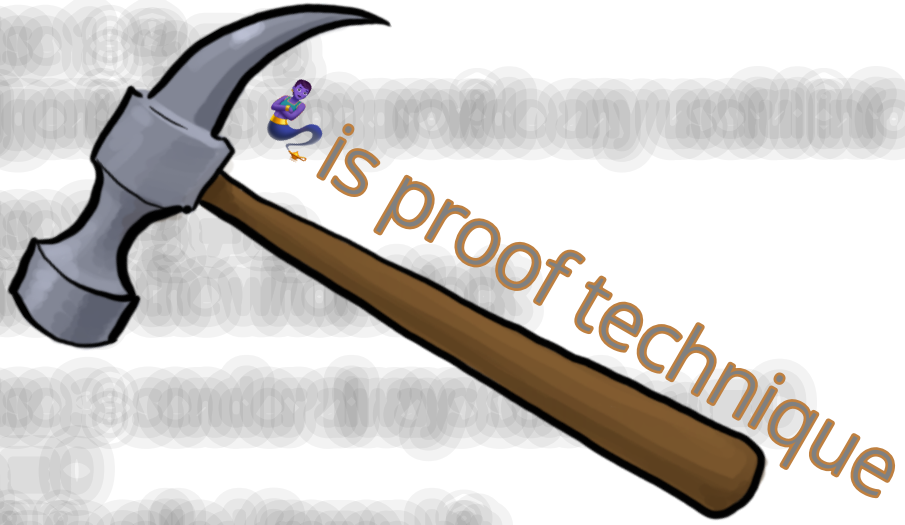
- Case 1: $\hat{u}_1 = u_1$
  does not provide any useful info
- Case 2: $\hat{u}_1 \neq u_1$
- Case 3:

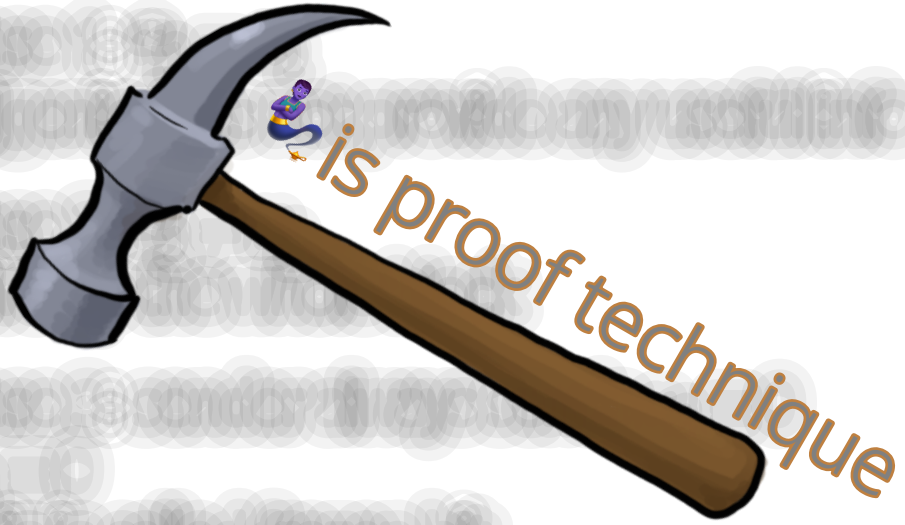
- Case 1: $\hat{u}_1 = u_1$
  does not provide any useful info
- Case 2: $\hat{u}_1 \neq u_1$
  Give up the whole block
- Case 3:

- Case 1: $\hat{u}_1 = u_1$
  does not provide any useful info
- Case 2: $\hat{u}_1 \neq u_1$
  Give up the whole block
- Case 3: sender always sends $u_1 \equiv 0$

- Case 1: $\hat{u}_1 = u_1$
  does not provide any useful info
- Case 2: $\hat{u}_1 \neq u_1$
  Give up the whole block
- Case 3: sender always sends $u_1 \equiv 0$
  $\hat{u}_1 \equiv 0$

- Case 1: $\hat{u}_1 = u_1$
  does not provide any useful info
- Case 2: $\hat{u}_1 \neq u_1$
  Give up the whole block
- Case 3: sender always sends $u_1 \equiv 0$
  $\hat{u}_1 \equiv 0$
  This is called frozen bit


is proof technique

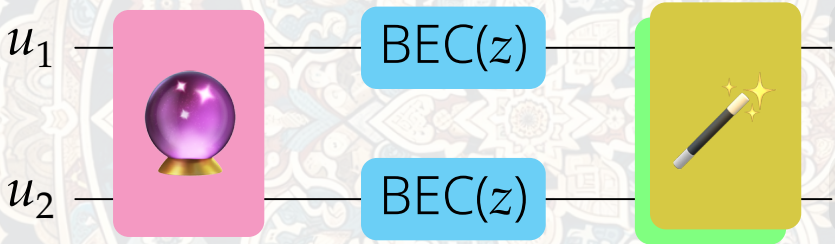

is proof technique

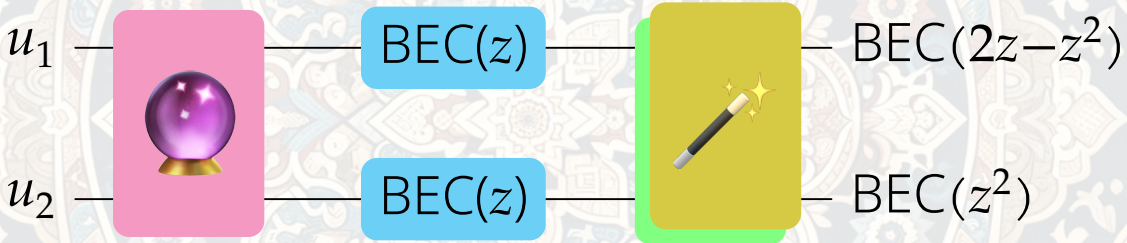
is proof technique

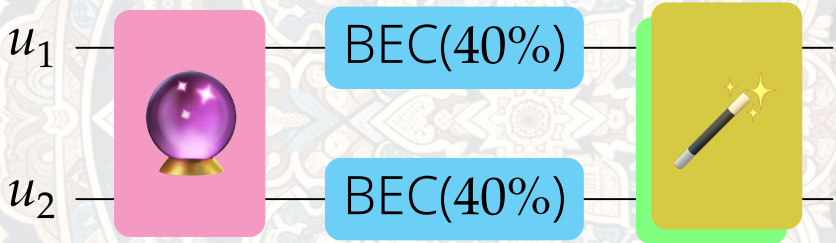
s proof technique

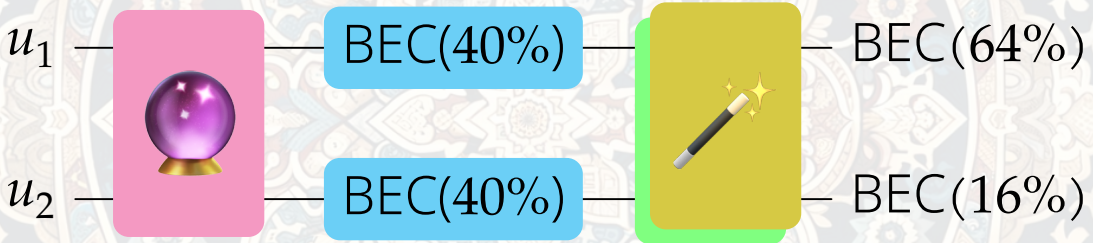
of technique

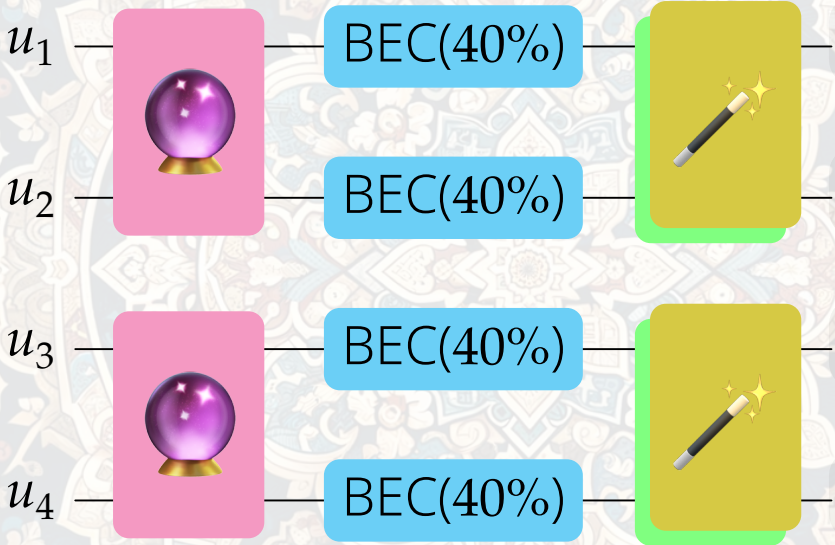
chnique

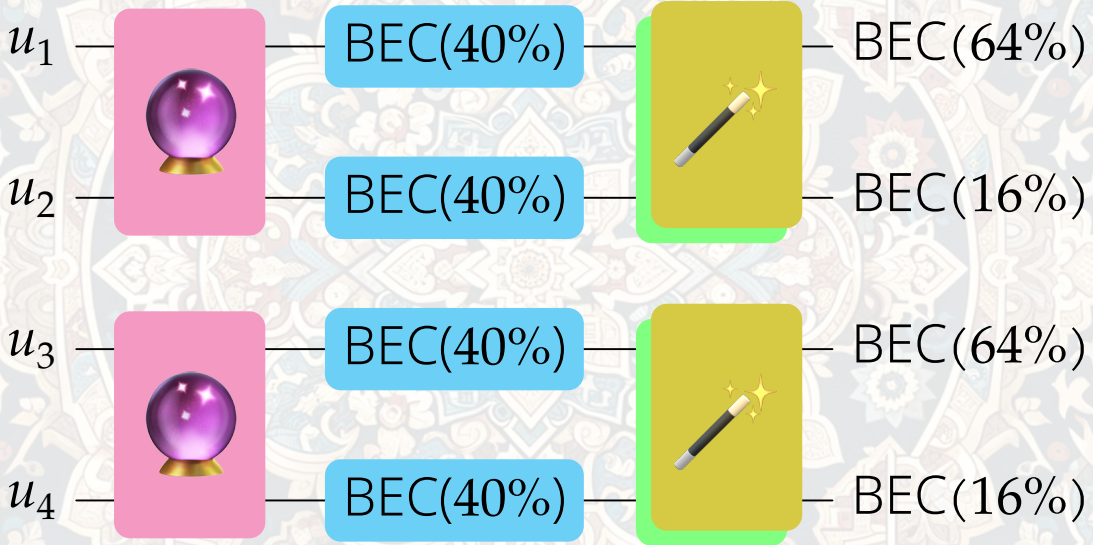

u_1
u_2
BEC(z)
BEC(z)

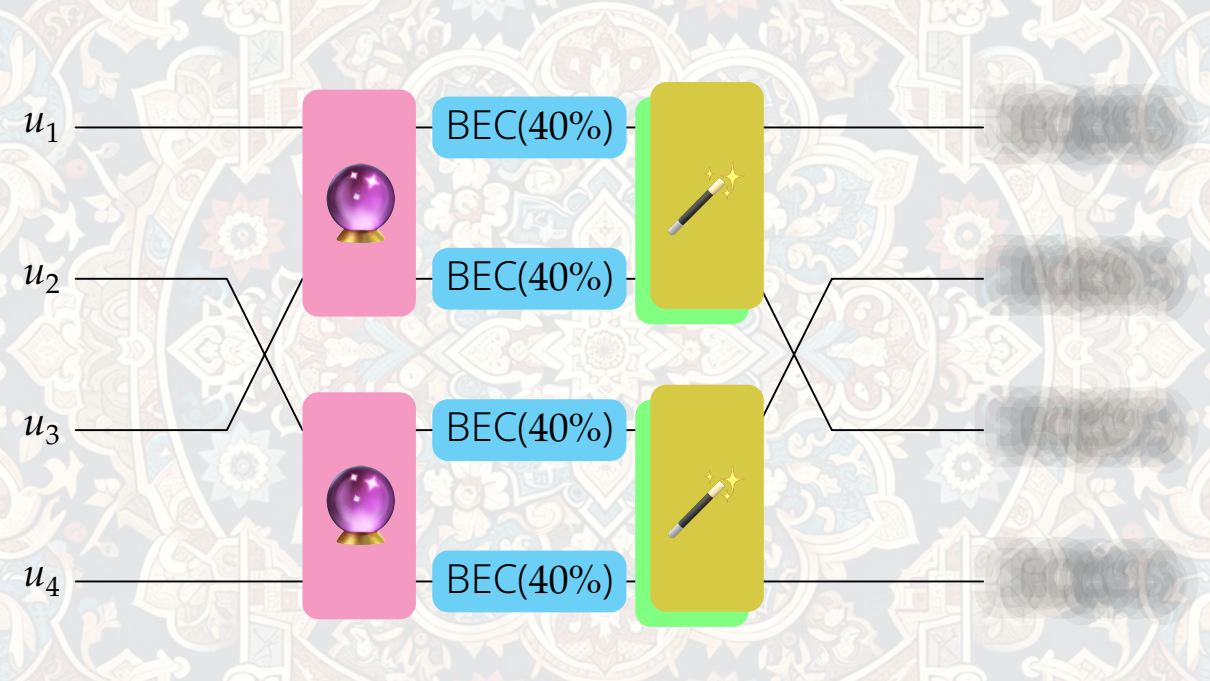

u_1
u_2
BEC(z)
BEC(z)
BEC($2z-z^2$)
BEC(z^2)

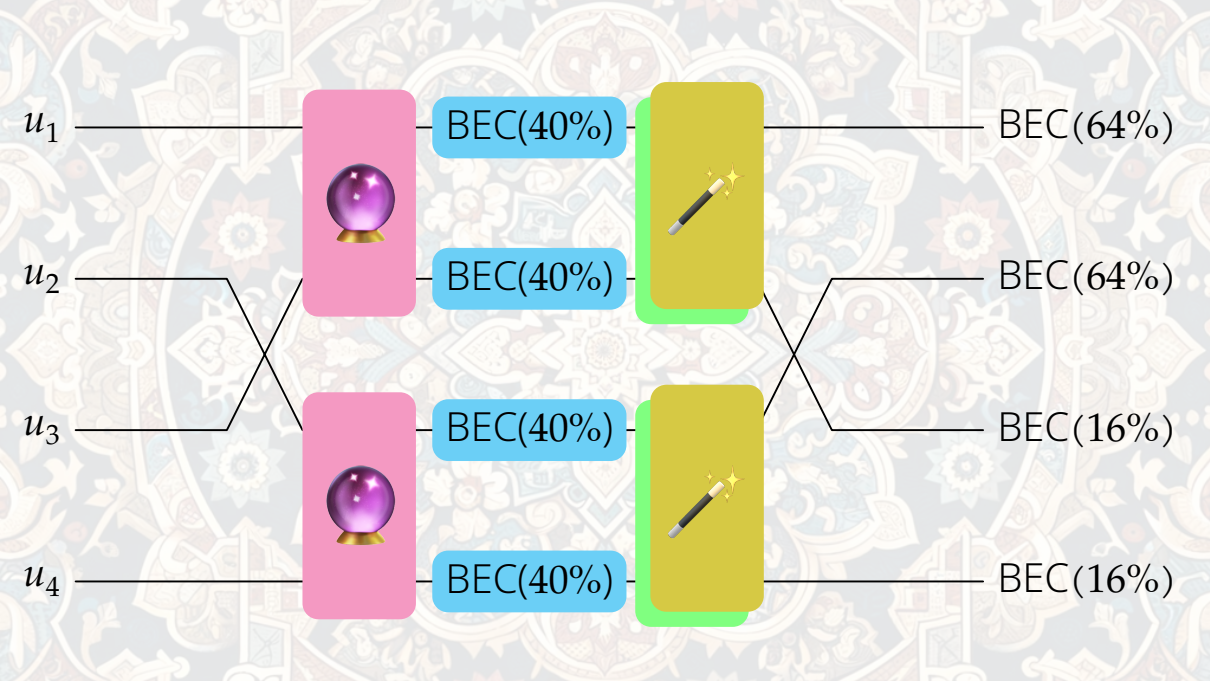

u_1
u_2
BEC(40%)
BEC(40%)

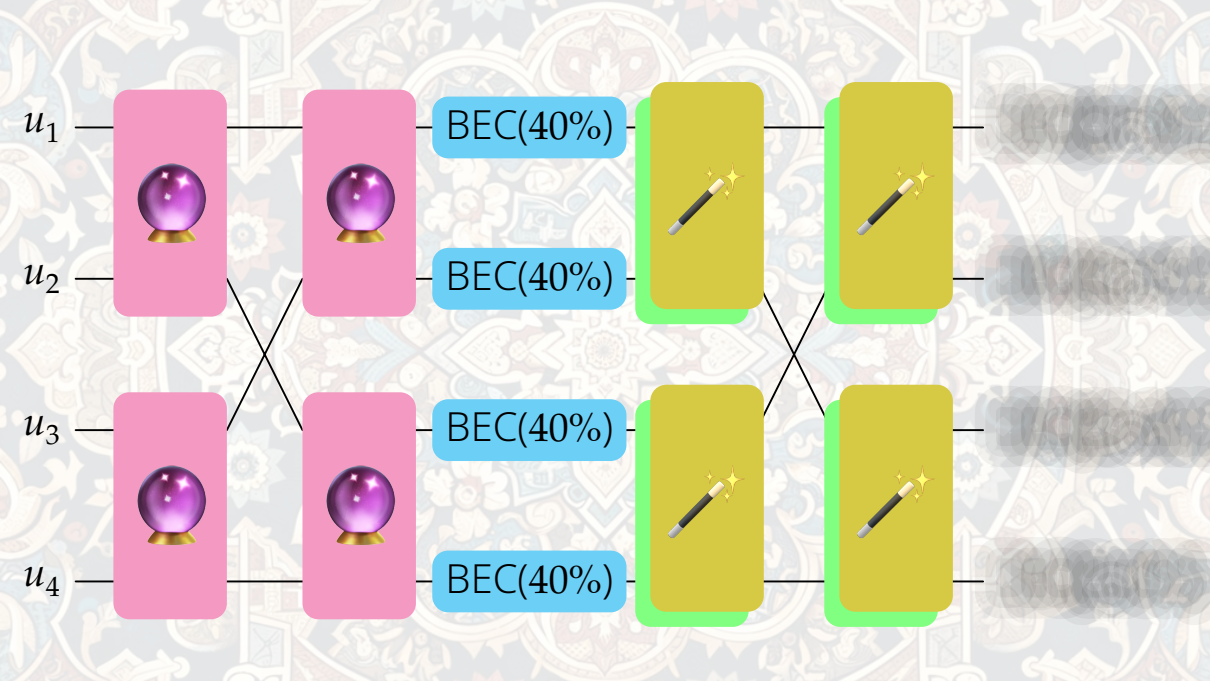

u_1
u_2
BEC(40%)
BEC(40%)
BEC(64%)
BEC(16%)

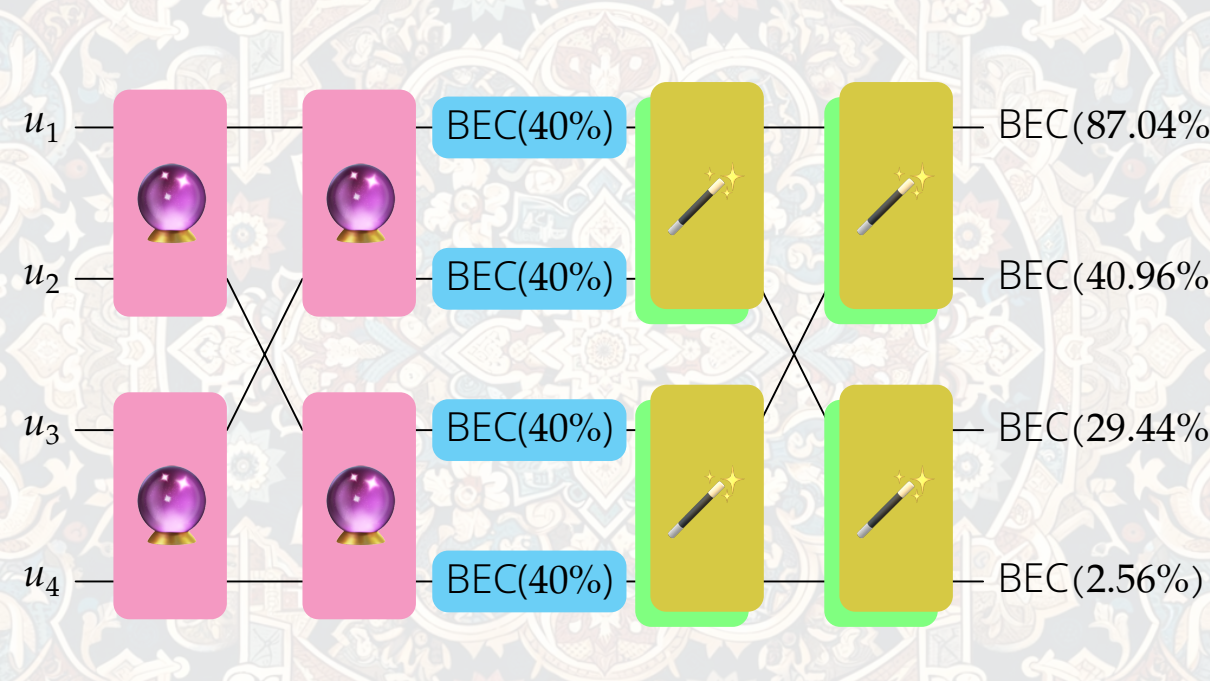

u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)

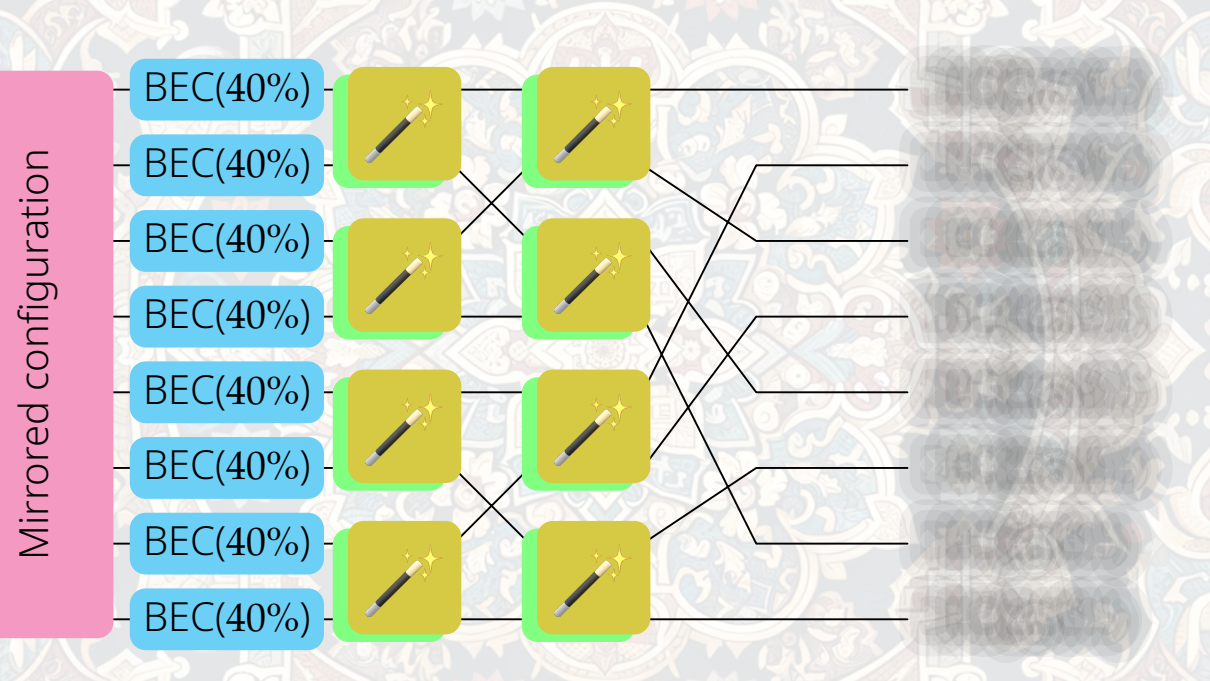

u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(64%)
BEC(16%)
BEC(64%)
BEC(16%)

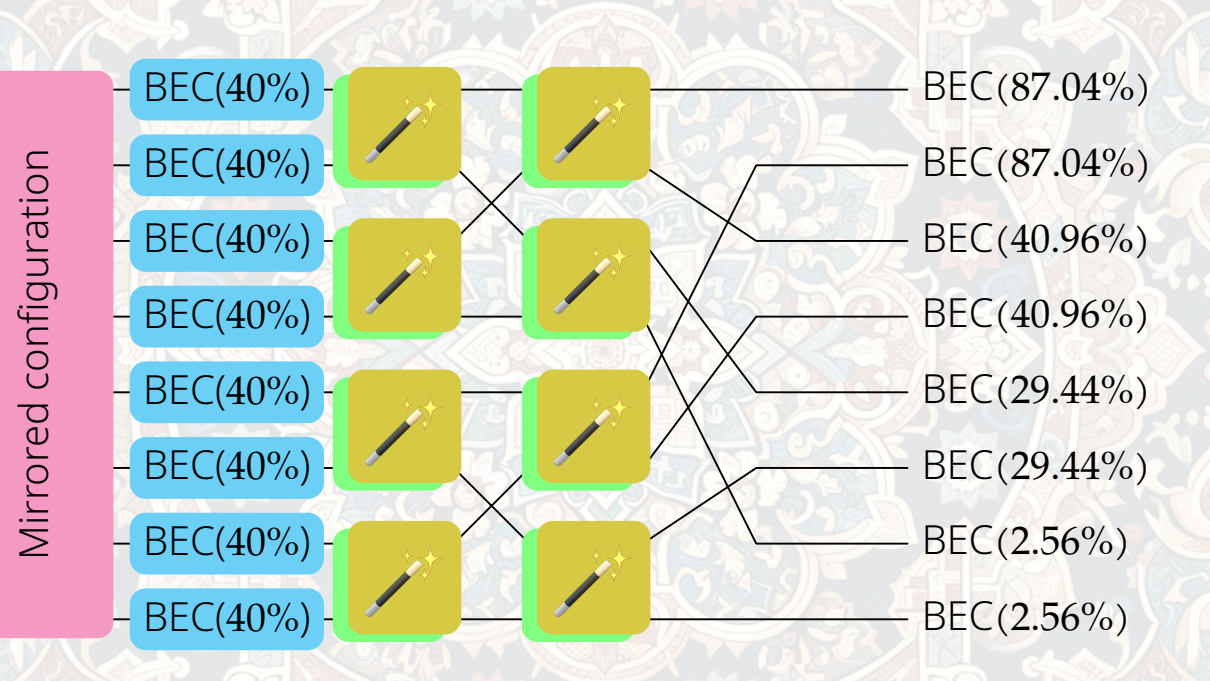

u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)


u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(64%)
BEC(64%)
BEC(16%)
BEC(16%)

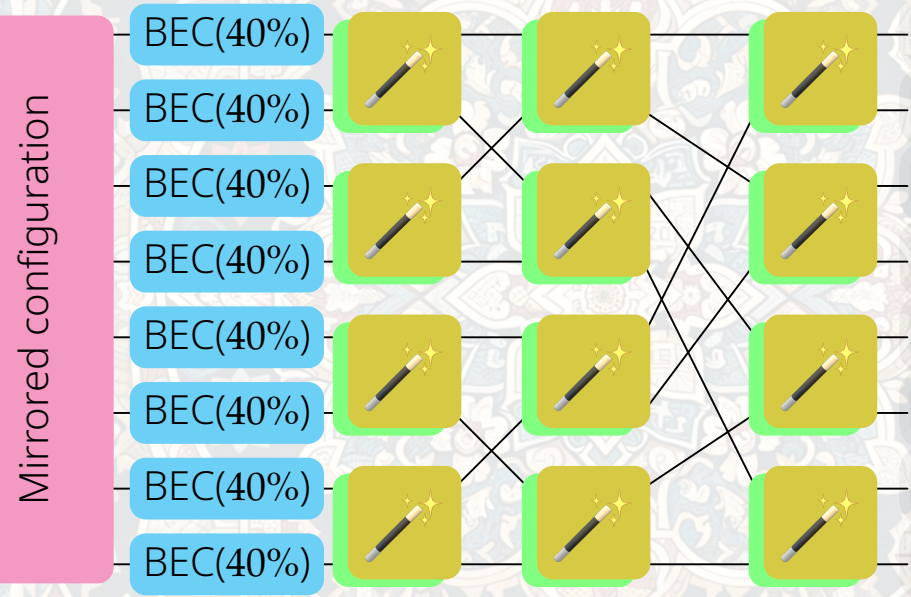

u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)

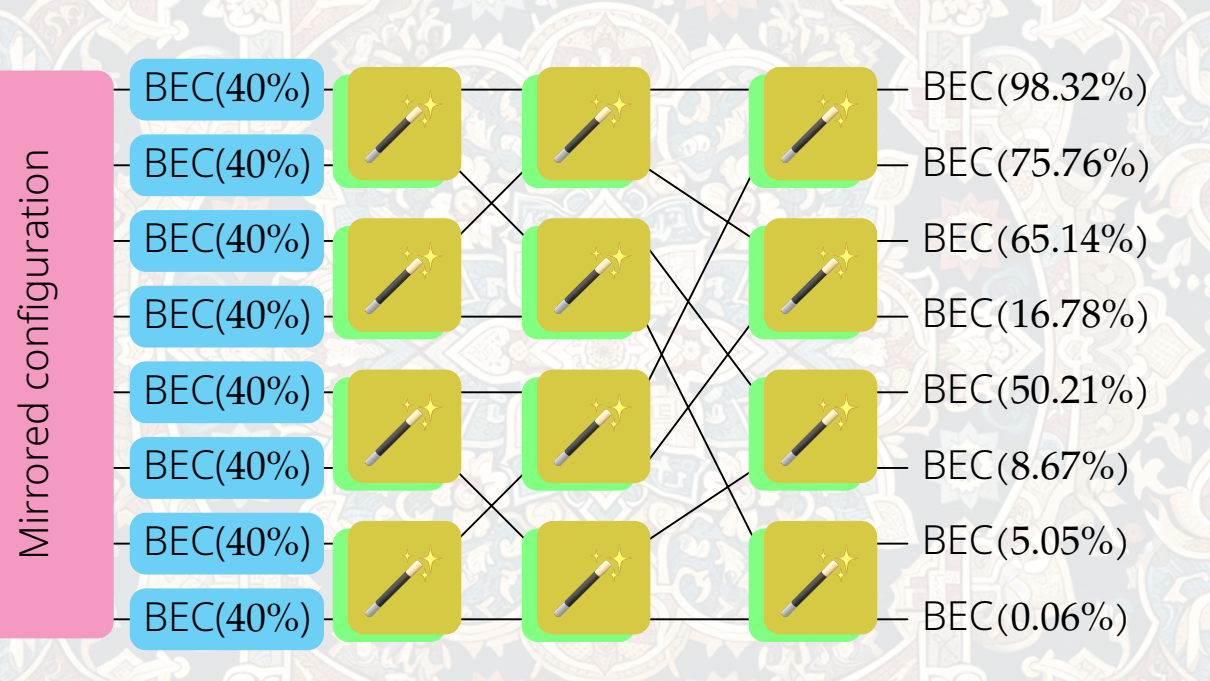

u_1
u_2
u_3
u_4
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(87.04%
BEC(40.96%
BEC(29.44%
BEC(2.56%)

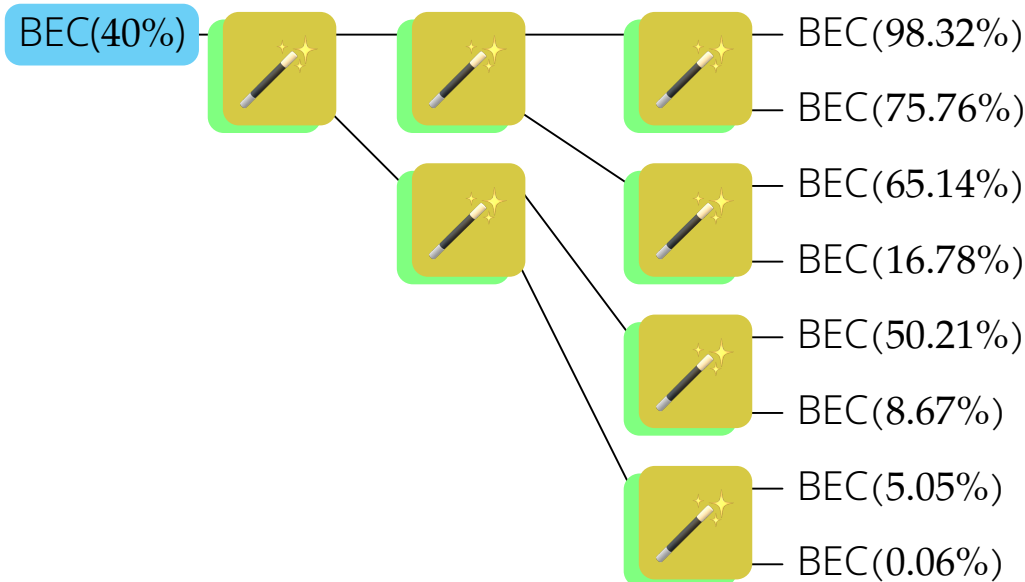

Mirrored configuration
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)


Mirrored configuration
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(87.04%)
BEC(87.04%)
BEC(40.96%)
BEC(40.96%)
BEC(29.44%)
BEC(29.44%)
BEC(2.56%)
BEC(2.56%)

Mirrored configuration



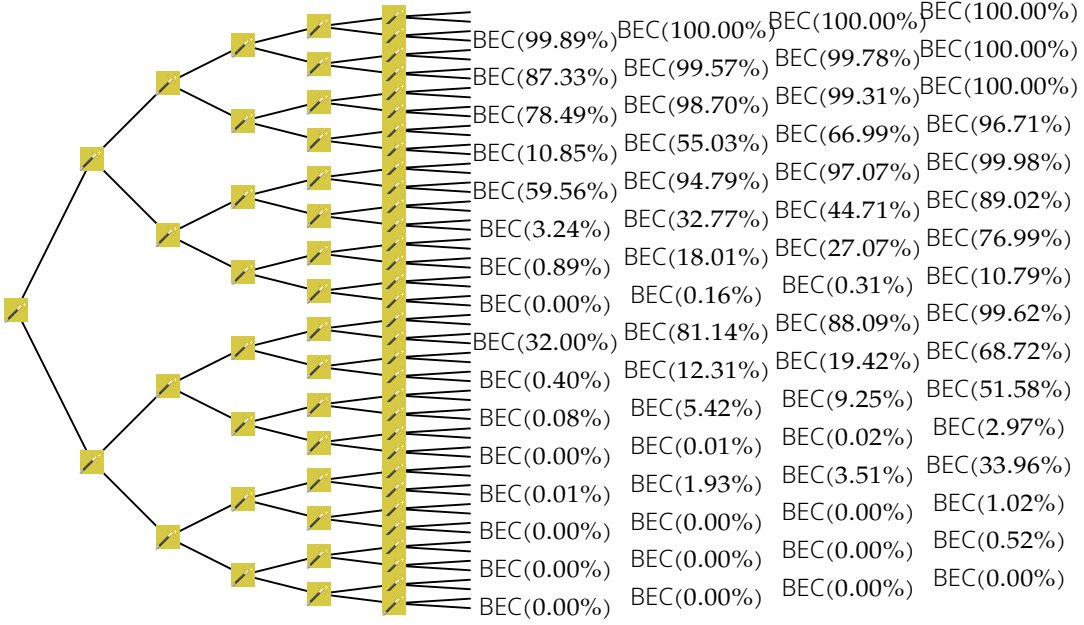

Mirrored configuration
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(40%)
BEC(98.32%)
BEC(75.76%)
BEC(65.14%)
BEC(16.78%)
BEC(50.21%)
BEC(8.67%)
BEC(5.05%)
BEC(0.06%)

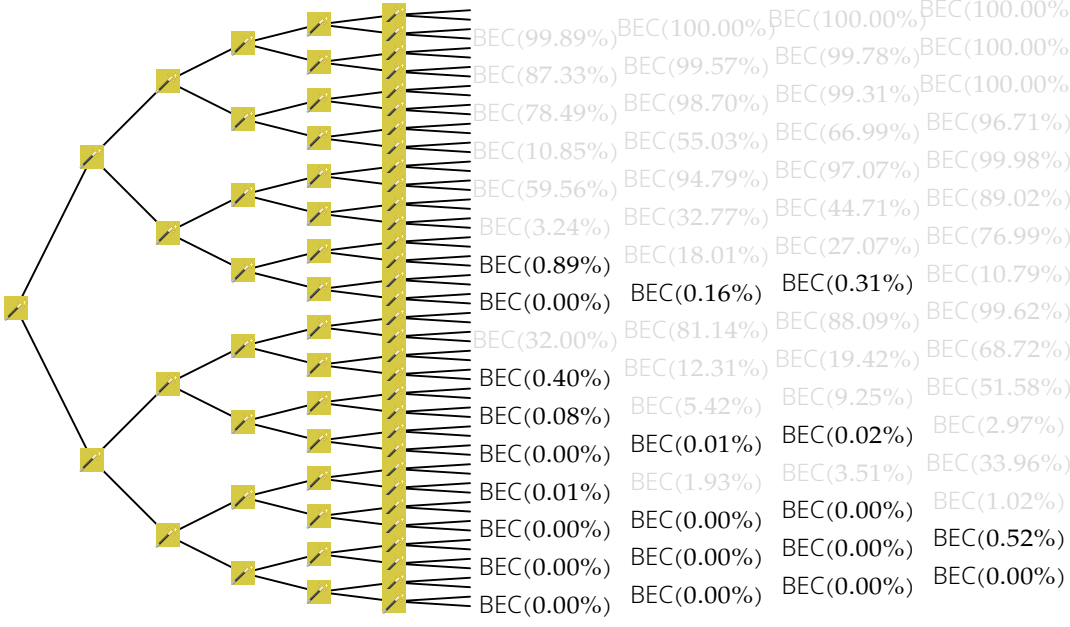

BEC(40%)
BEC(98.32%)
BEC(75.76%)
BEC(65.14%)
BEC(16.78%)
BEC(50.21%)
BEC(8.67%)
BEC(5.05%)
BEC(0.06%)

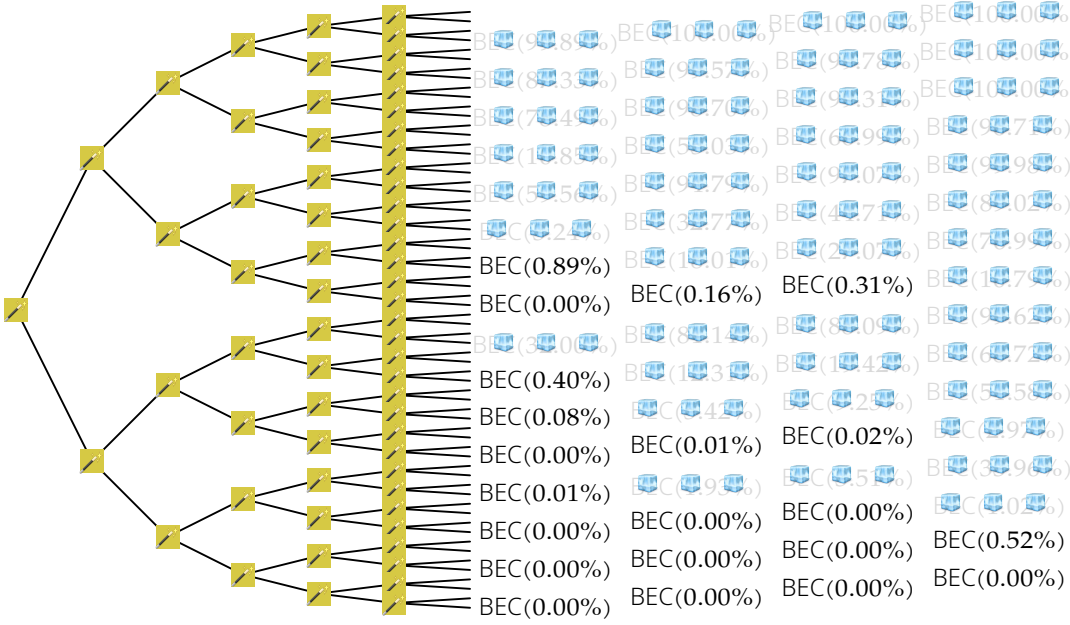
Recap

Recap

Recap


BEC(99.89%) BEC(100.00%) BEC(100.00%) BEC(100.00%)
BEC(87.33%) BEC(99.57%) BEC(99.78%) BEC(100.00%)
BEC(78.49%) BEC(98.70%) BEC(99.31%) BEC(100.00%)
BEC(10.85%) BEC(55.03%) BEC(66.99%) BEC(96.71%)
BEC(59.56%) BEC(94.79%) BEC(97.07%) BEC(99.98%)
BEC(3.24%) BEC(32.77%) BEC(44.71%) BEC(89.02%)
BEC(0.89%) BEC(18.01%) BEC(27.07%) BEC(76.99%)
BEC(0.00%) BEC(0.16%) BEC(0.31%) BEC(10.79%)
BEC(32.00%) BEC(81.14%) BEC(88.09%) BEC(99.62%)
BEC(0.40%) BEC(12.31%) BEC(19.42%) BEC(68.72%)
BEC(0.08%) BEC(5.42%) BEC(9.25%) BEC(51.58%)
BEC(0.00%) BEC(0.01%) BEC(0.02%) BEC(2.97%)
BEC(0.01%) BEC(1.93%) BEC(3.51%) BEC(33.96%)
BEC(0.00%) BEC(0.00%) BEC(0.00%) BEC(1.02%)
BEC(0.00%) BEC(0.00%) BEC(0.00%) BEC(0.52%)
BEC(0.00%) BEC(0.00%) BEC(0.00%) BEC(0.00%)

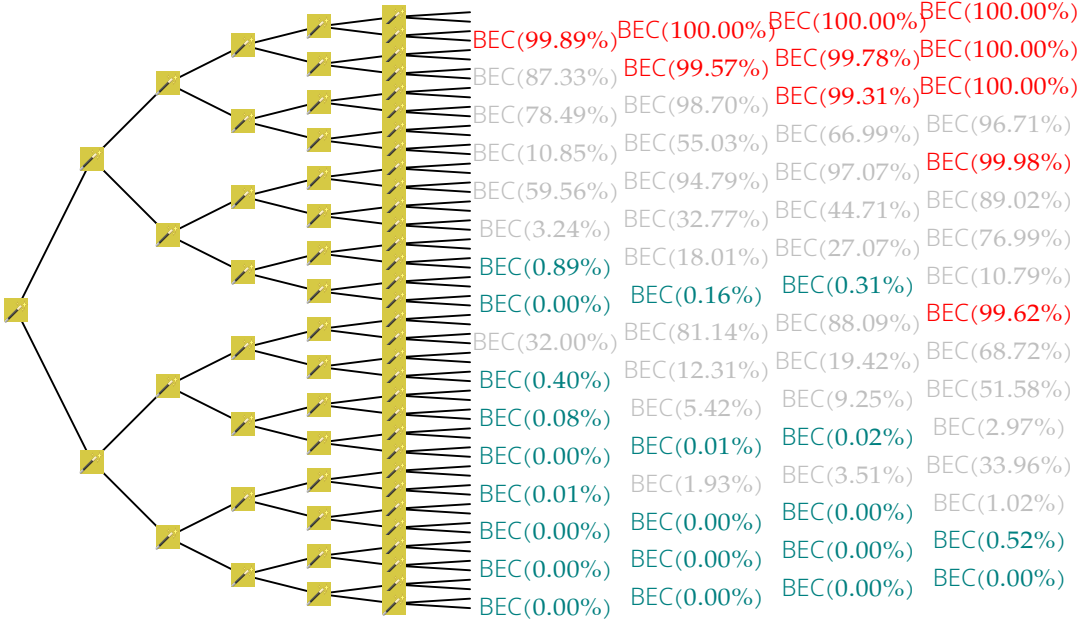

BEC(99.89%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(87.33%)
BEC(99.57%)
BEC(99.78%)
BEC(100.00%)
BEC(78.49%)
BEC(98.70%)
BEC(99.31%)
BEC(100.00%)
BEC(10.85%)
BEC(55.03%)
BEC(66.99%)
BEC(96.71%)
BEC(59.56%)
BEC(94.79%)
BEC(97.07%)
BEC(99.98%)
BEC(3.24%)
BEC(32.77%)
BEC(44.71%)
BEC(89.02%)
BEC(18.01%)
BEC(27.07%)
BEC(76.99%)
BEC(0.89%)
BEC(0.16%)
BEC(0.31%)
BEC(10.79%)
BEC(0.00%)
BEC(32.00%)
BEC(81.14%)
BEC(88.09%)
BEC(99.62%)
BEC(12.31%)
BEC(19.42%)
BEC(68.72%)
BEC(0.40%)
BEC(5.42%)
BEC(9.25%)
BEC(51.58%)
BEC(0.08%)
BEC(0.01%)
BEC(0.02%)
BEC(2.97%)
BEC(0.00%)
BEC(1.93%)
BEC(3.51%)
BEC(33.96%)
BEC(0.01%)
BEC(0.00%)
BEC(0.00%)
BEC(1.02%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.52%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)

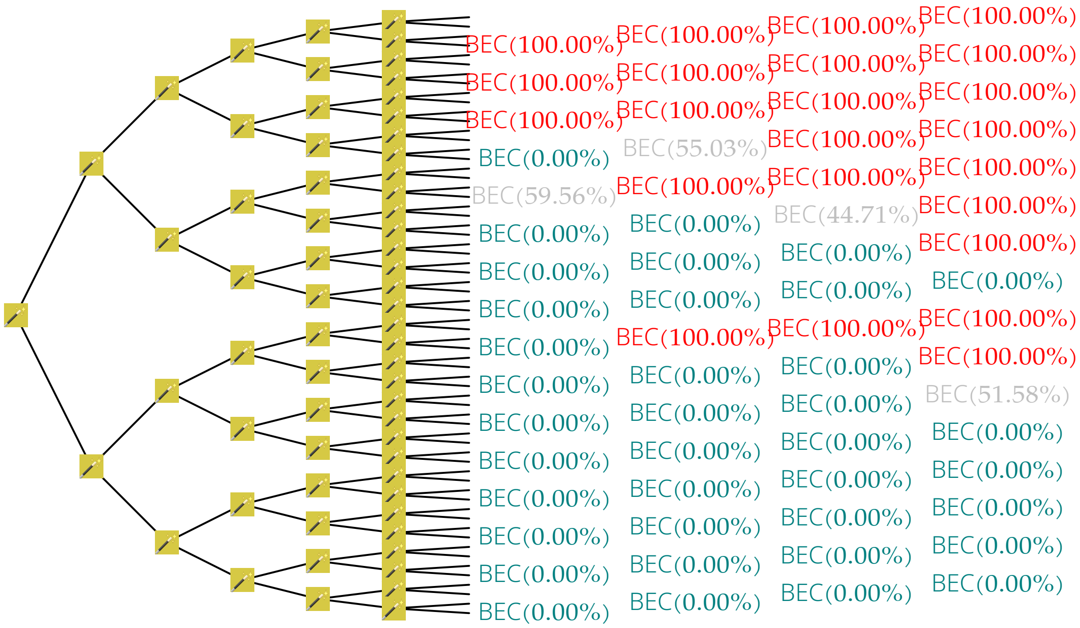

BEC(0.89%)
BEC(0.00%)
BEC(0.40%)
BEC(0.08%)
BEC(0.00%)
BEC(0.01%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.16%)
BEC(0.01%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.31%)
BEC(0.02%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(0.52%)
BEC(0.00%)

Thm [Arıkan ] If, after synthesizing 2^n channels, k of them are less than p,

polar code has rate $k/2^n$
& block error prob $< 1/p$

Thm [Arıkan ] If, after synthesizing 2^n channels, k of them are less than p, polar code has rate $k/2^n$ & block error prob $< kp$.


BEC(99.89%)
BEC(87.33%)
BEC(78.49%)
BEC(10.85%)
BEC(59.56%)
BEC(3.24%)
BEC(0.89%)
BEC(0.00%)
BEC(32.00%)
BEC(0.40%)
BEC(0.08%)
BEC(0.00%)
BEC(0.01%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(99.57%)
BEC(98.70%)
BEC(55.03%)
BEC(94.79%)
BEC(32.77%)
BEC(18.01%)
BEC(0.16%)
BEC(81.14%)
BEC(12.31%)
BEC(5.42%)
BEC(0.01%)
BEC(1.93%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(99.78%)
BEC(99.31%)
BEC(66.99%)
BEC(97.07%)
BEC(44.71%)
BEC(27.07%)
BEC(0.31%)
BEC(88.09%)
BEC(19.42%)
BEC(9.25%)
BEC(0.02%)
BEC(3.51%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(96.71%)
BEC(99.98%)
BEC(89.02%)
BEC(76.99%)
BEC(10.79%)
BEC(99.62%)
BEC(68.72%)
BEC(51.58%)
BEC(2.97%)
BEC(33.96%)
BEC(1.02%)
BEC(0.52%)
BEC(0.00%)


BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(0.00%)
BEC(59.56%)
BEC(0.00%)
BEC(0.00%)
BEC(0.00%)
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BEC(0.00%)
BEC(100.00%)
BEC(100.00%)
BEC(100.00%)
BEC(55.03%)
BEC(100.00%)
BEC(0.00%)
BEC(0.00%)
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BEC(100.00%)
BEC(100.00%)
BEC(44.71%)
BEC(0.00%)
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BEC(100.00%)
BEC(100.00%)
BEC(51.58%)
BEC(0.00%)
BEC(0.00%)
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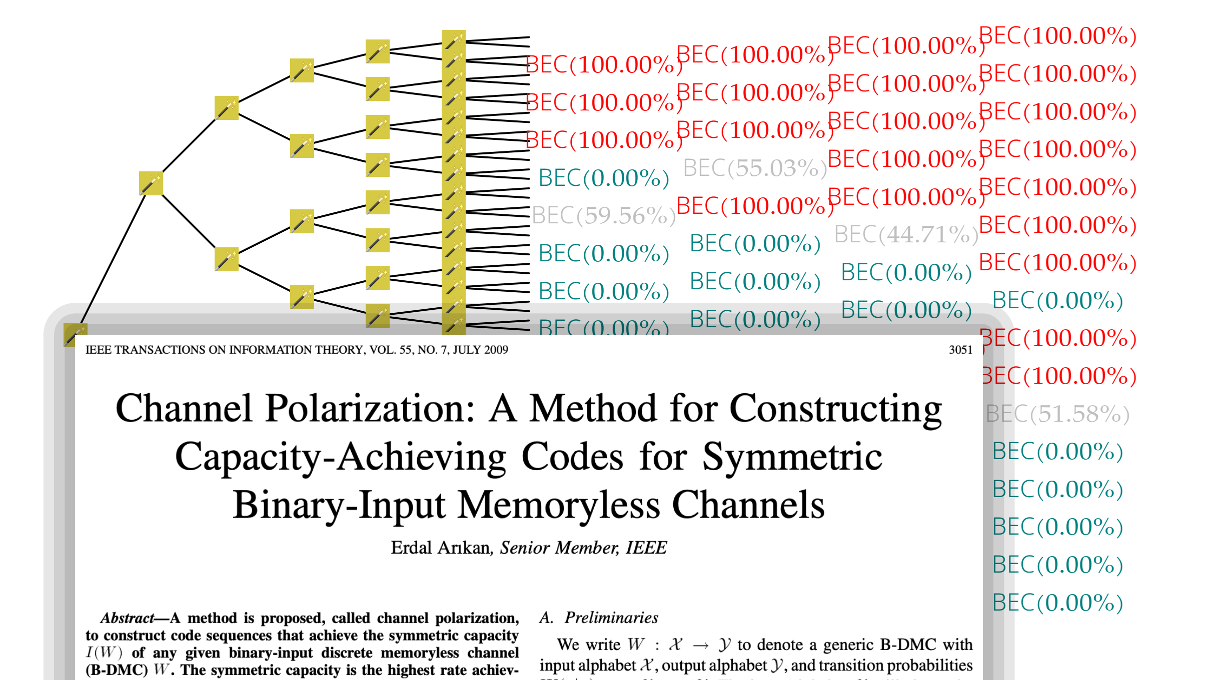
BEC(0.00%)
BEC(0.00%)



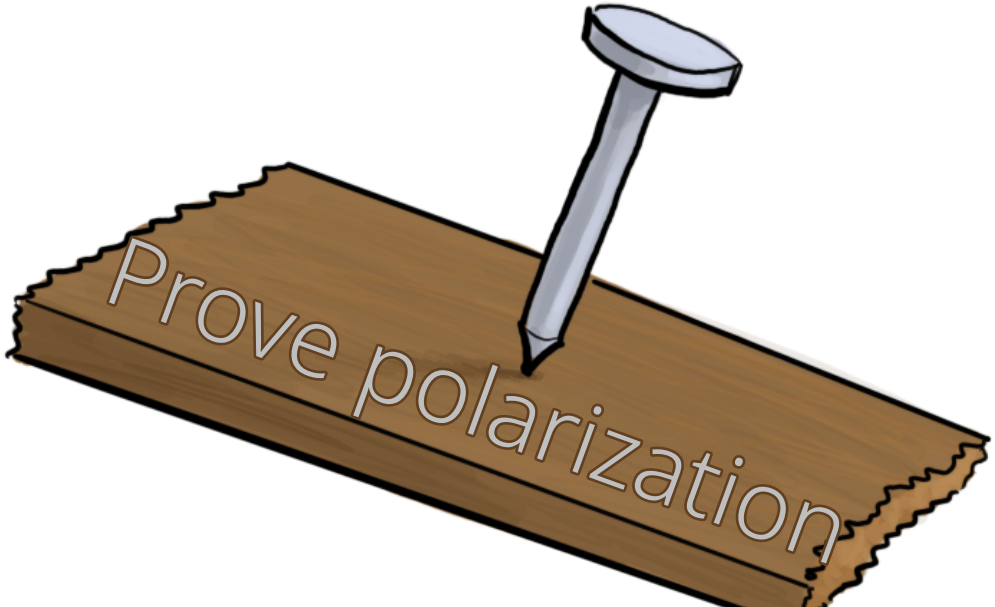
Channel Polarization: A Method for Constructing Capacity-Achieving Codes for Symmetric Binary-Input Memoryless Channels

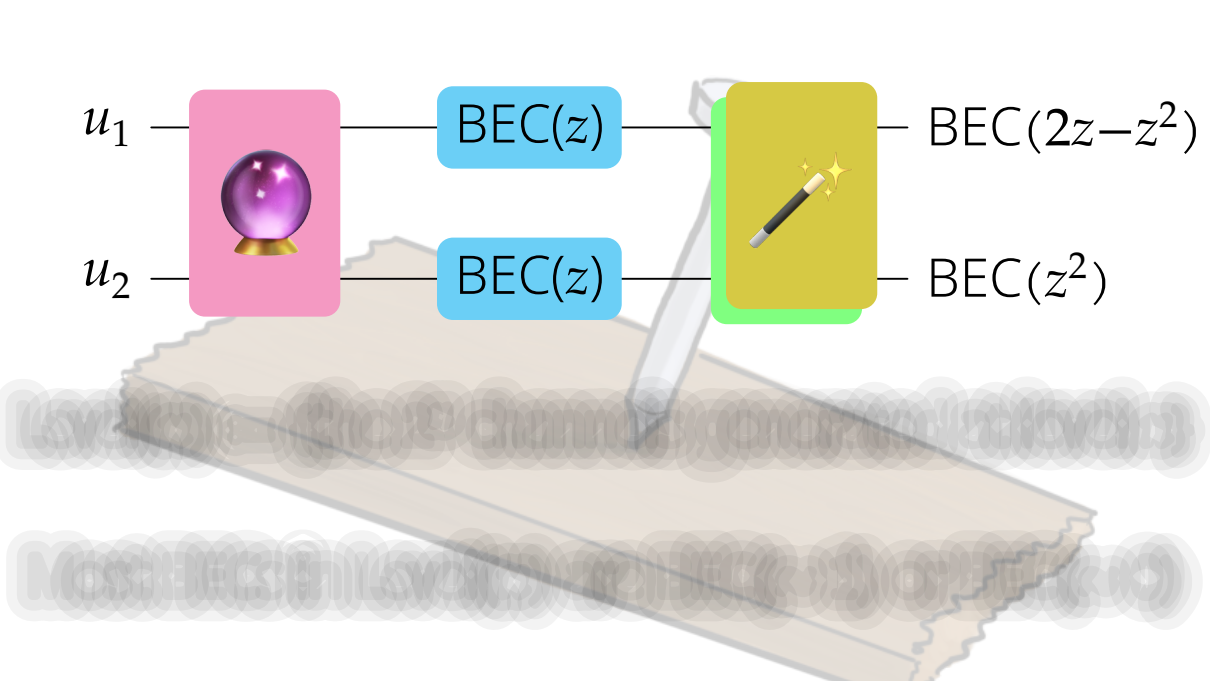
Erdal Arıkan, *Senior Member, IEEE*

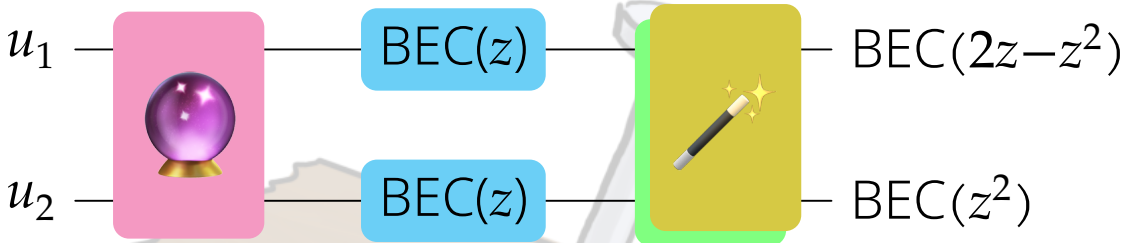
***Abstract*—A method is proposed, called channel polarization, to construct code sequences that achieve the symmetric capacity $I(W)$ of any given binary-input discrete memoryless channel (B-DMC) W. The symmetric capacity is the highest rate achiev-**

A. *Preliminaries*

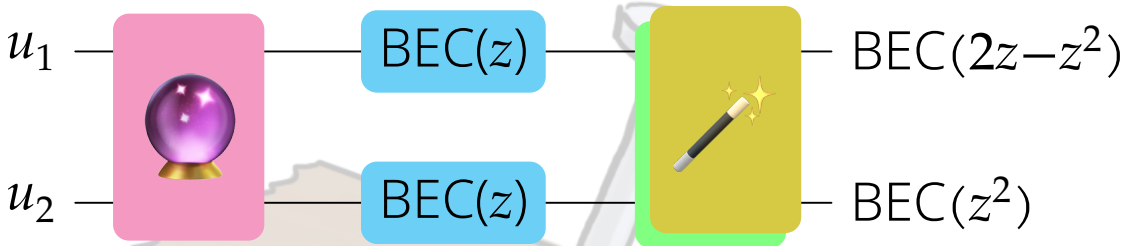
We write $W : \mathcal{X} \to \mathcal{Y}$ to denote a generic B-DMC with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$, and transition probabilities


Prove polarization


u_1
u_2
BEC(z)
BEC(z)
BEC($2z-z^2$)
BEC(z^2)

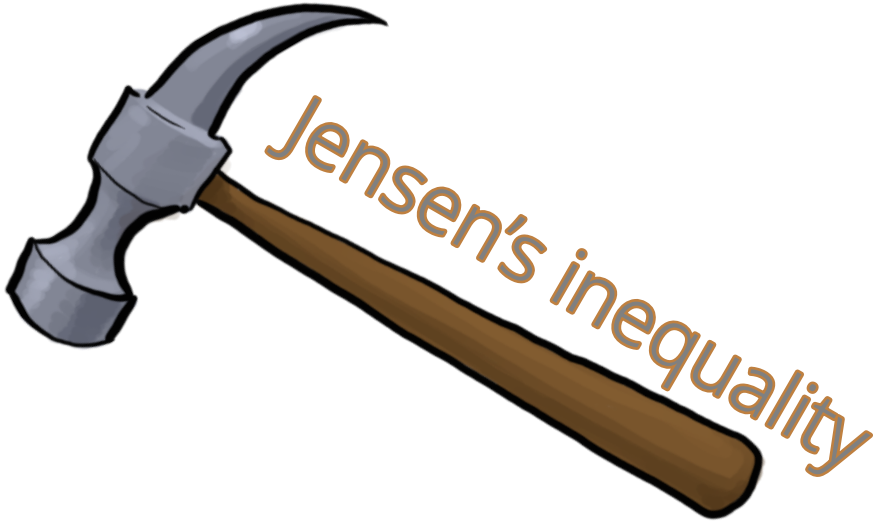


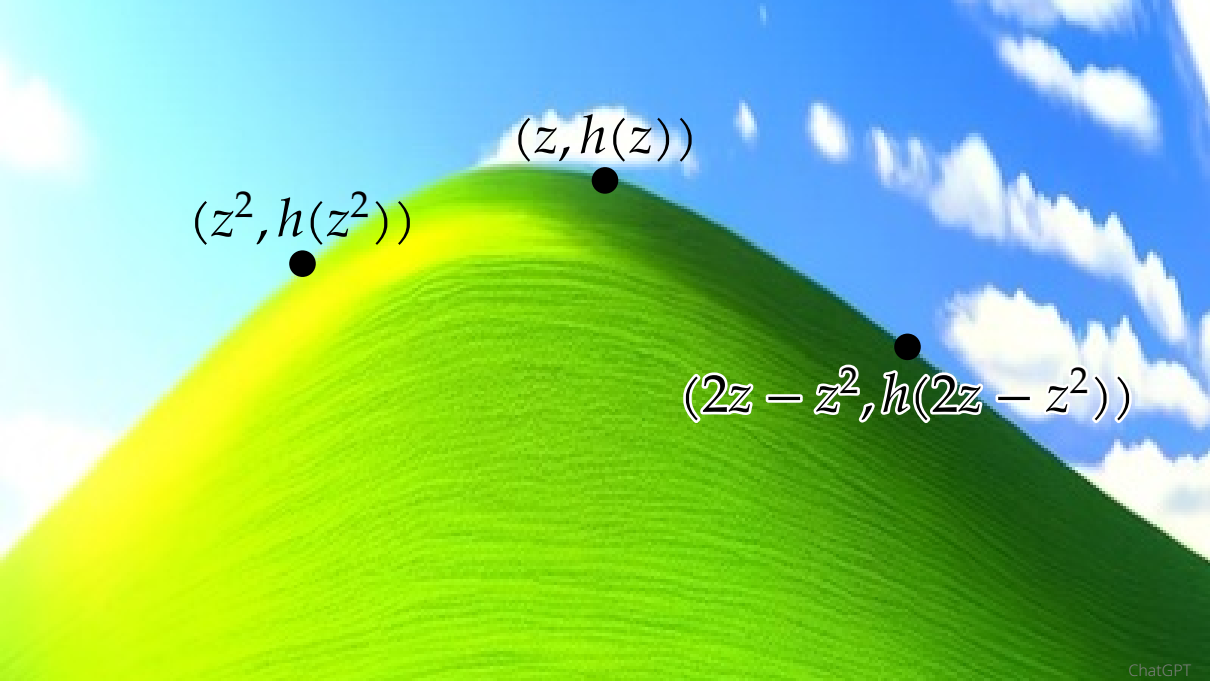
Level(n) := {the 2^n channels generated at level n}

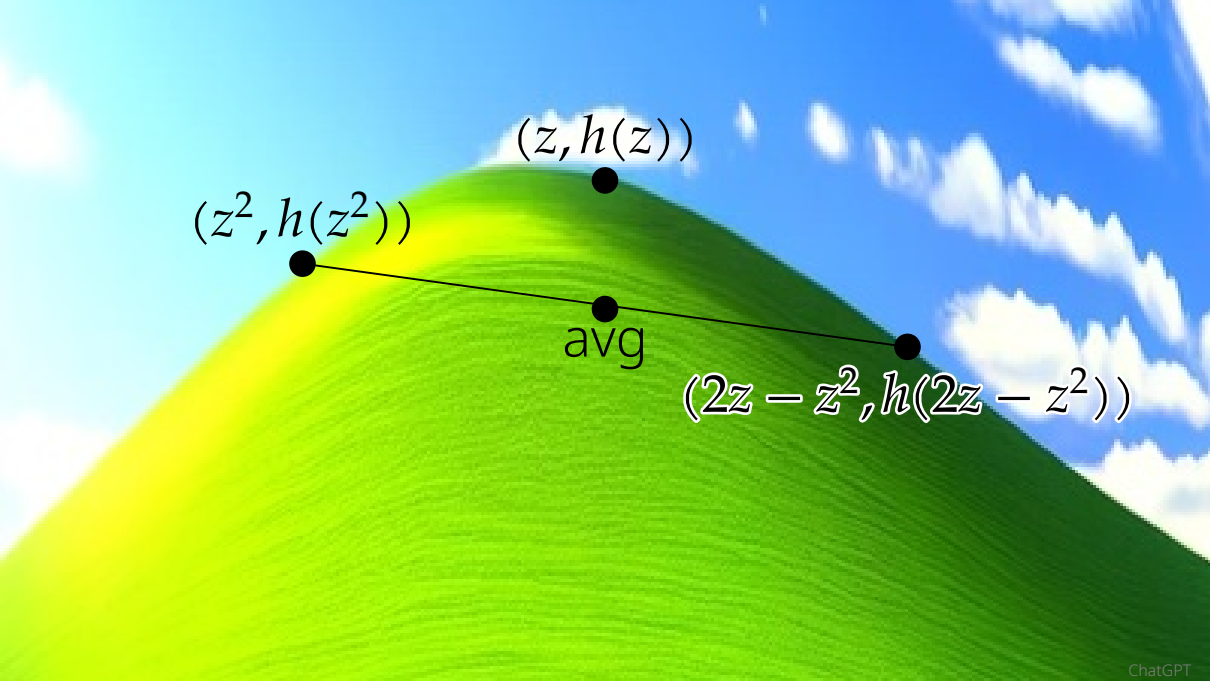


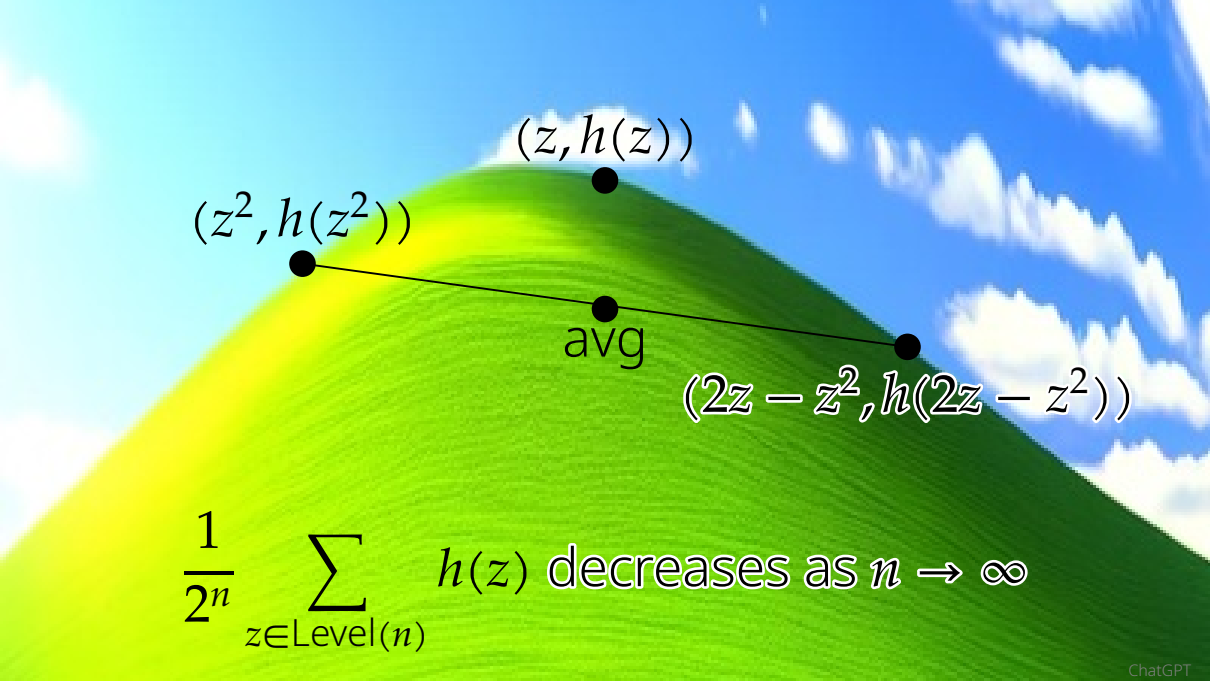
Level(n) := {the 2^n channels generated at level n}

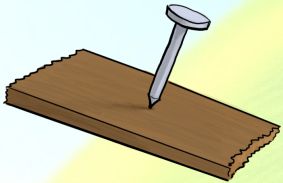
Most BECs in Level(n) are BEC(≈ 1) or BEC(≈ 0)


Jensen's inequality

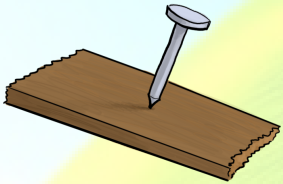



$(z, h(z))$
$(z^2, h(z^2))$
avg
$(2z - z^2, h(2z - z^2))$


$(z, h(z))$
$(z^2, h(z^2))$
avg
$(2z - z^2, h(2z - z^2))$
$\frac{1}{2^n} \sum_{z \in \text{Level}(n)} h(z)$ decreases as $n \to \infty$

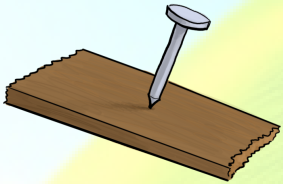


Prove polarization



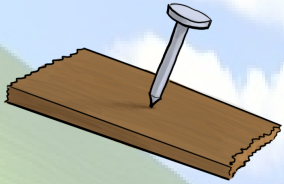
Prove polarization

Jensen’s



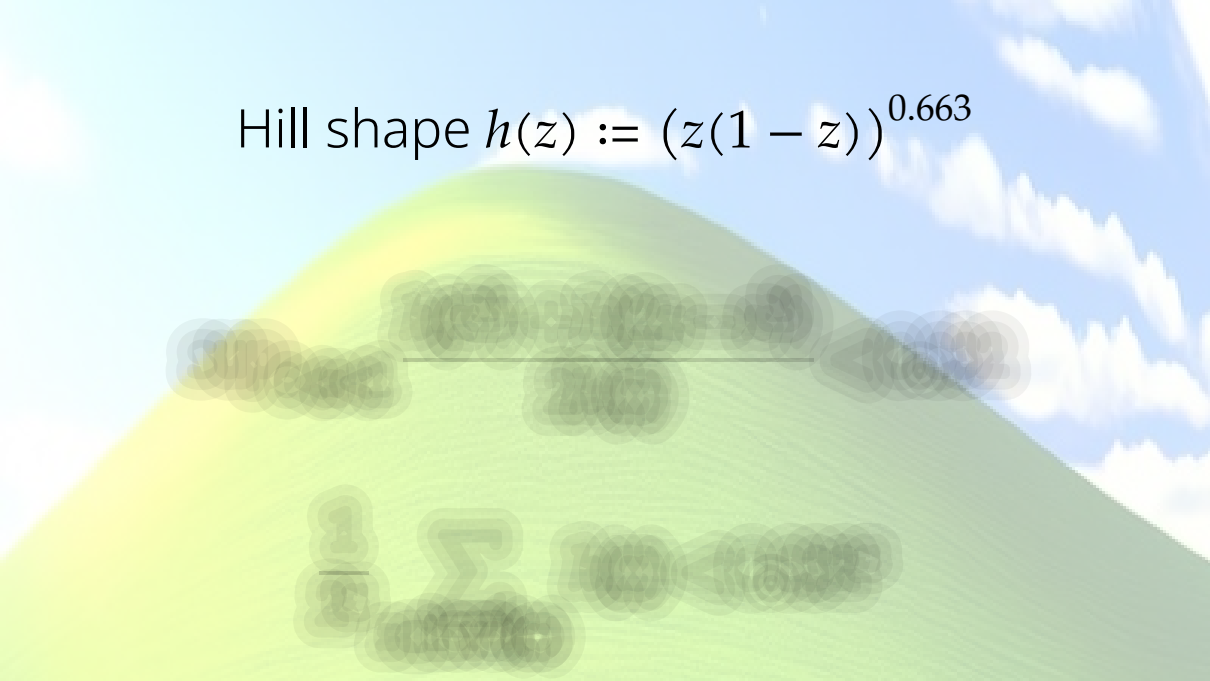
Prove polarization

Jensen's



Convex function?

Hill shape $h(z) := (z(1-z))^{0.663}$



Hill shape $h(z) := (z(1-z))^{0.663}$

$$\sup_{0<z<1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

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$$\sup_{0<z<1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

$$\frac{1}{2^n} \sum_{z \in \text{Level}(n)} h(z) < 0.832^n$$

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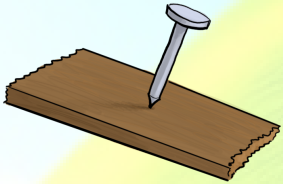
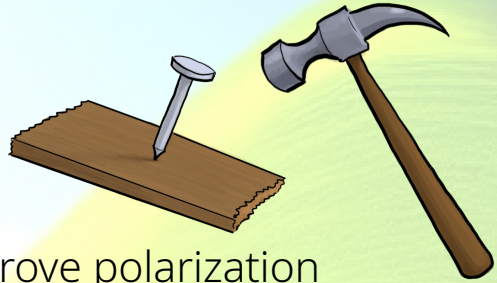
$$\sup_{0<z<1} \frac{h(z^2) + h(2z - z^2)}{2h(z)} < 0.832$$

$$\frac{1}{2^n} \sum_{z \in \text{Level}(n)} h(z) < 0.832^n$$

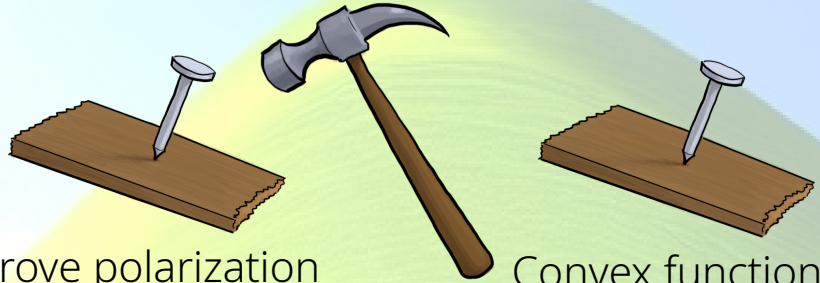


Prove polarization



Prove polarization

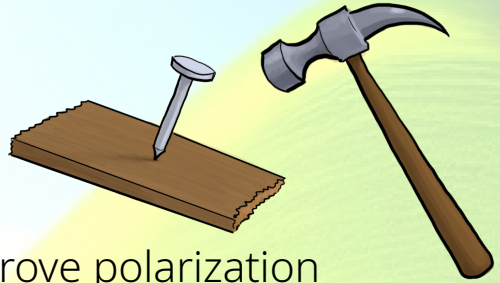
Jensen's



Prove polarization

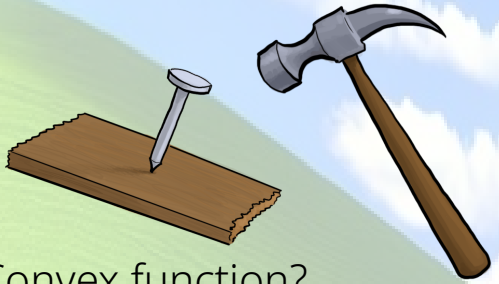
Jensen's

Convex function?



Prove polarization

Jensen's



Convex function?

$(z(1-z))^{0.663}$

If k of 2^n are less than p, polar code has rate $k/2^n$ & block error prob $< kp$

If k of 2^n are less than p,
polar code has rate $k/2^n$
& block error prob $< kp$

$$k/2^n = 60\% - 2^{-\Omega(n)}$$

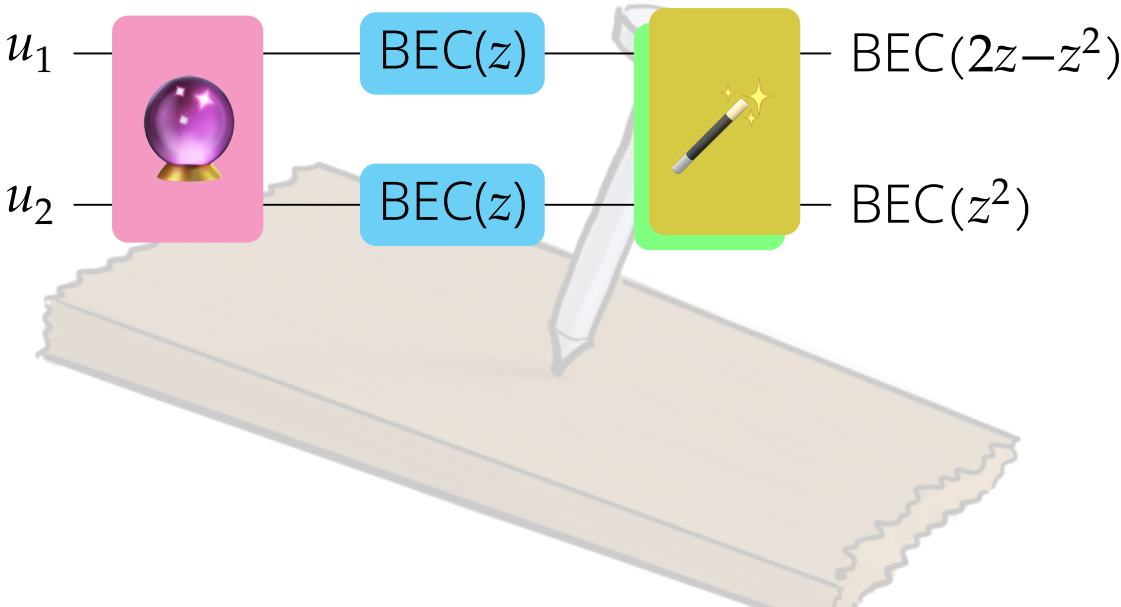
$$p = 2^{-\Omega(n)}$$

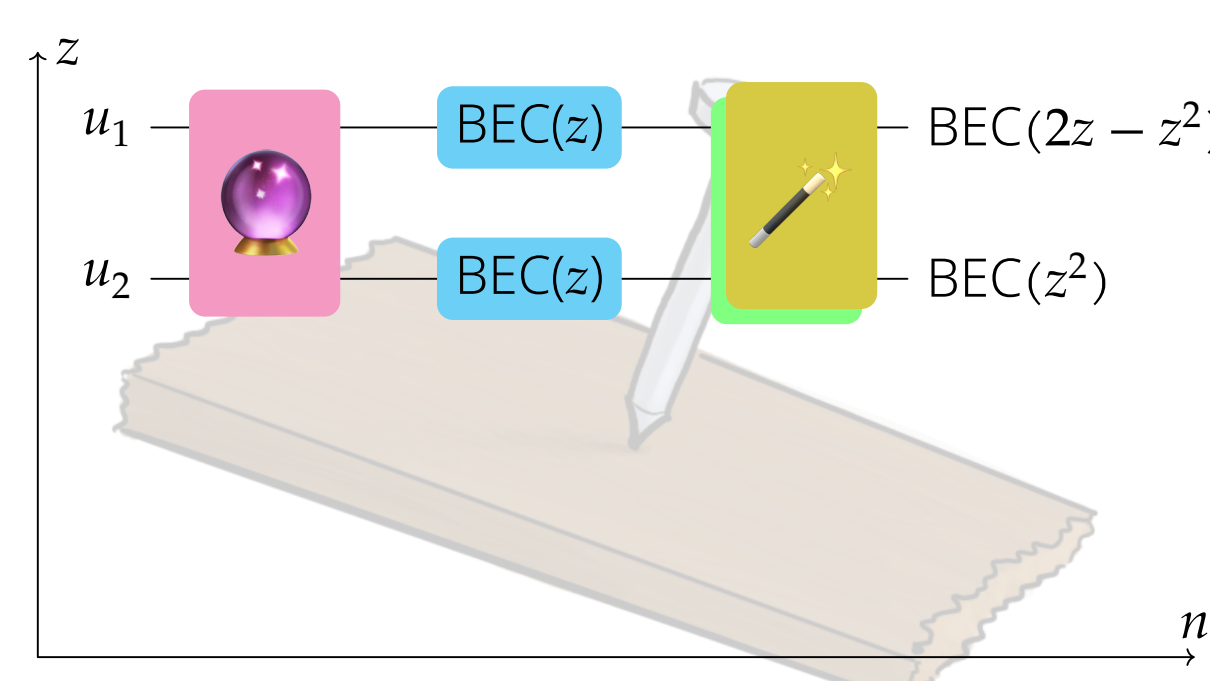
If k of 2^n are less than p,
polar code has rate $k/2^n$
& block error prob $< kp$

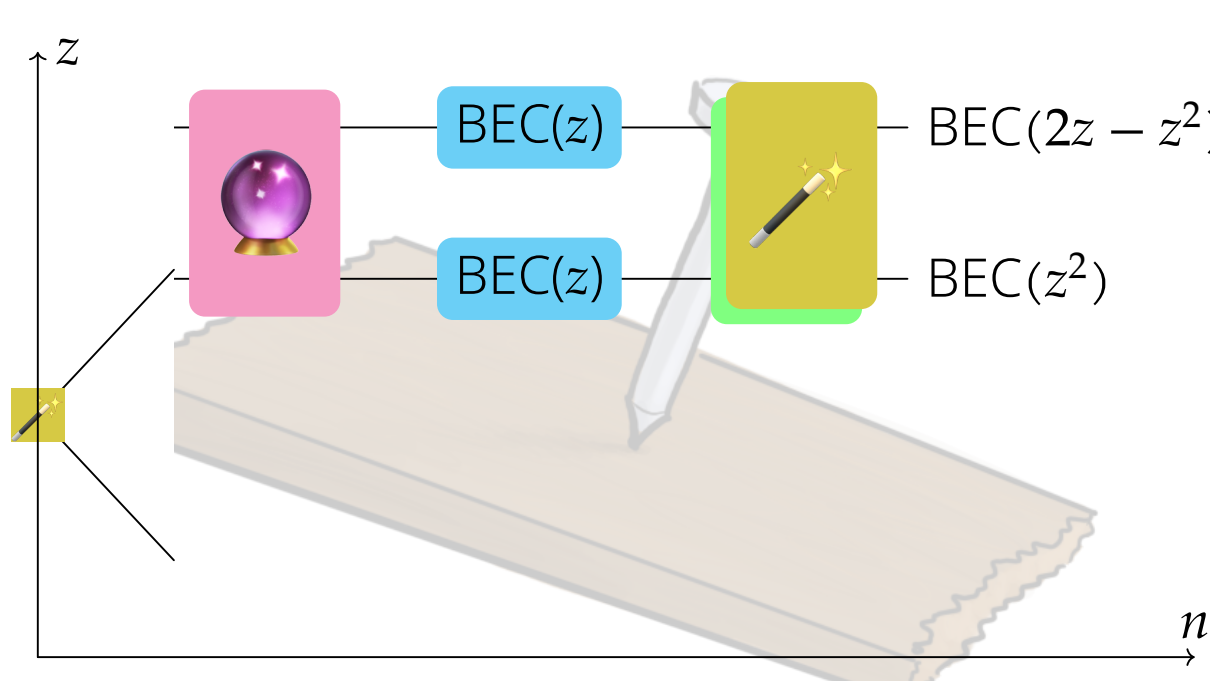
$$k/2^n = 60\% - 2^{-\Omega(n)}$$

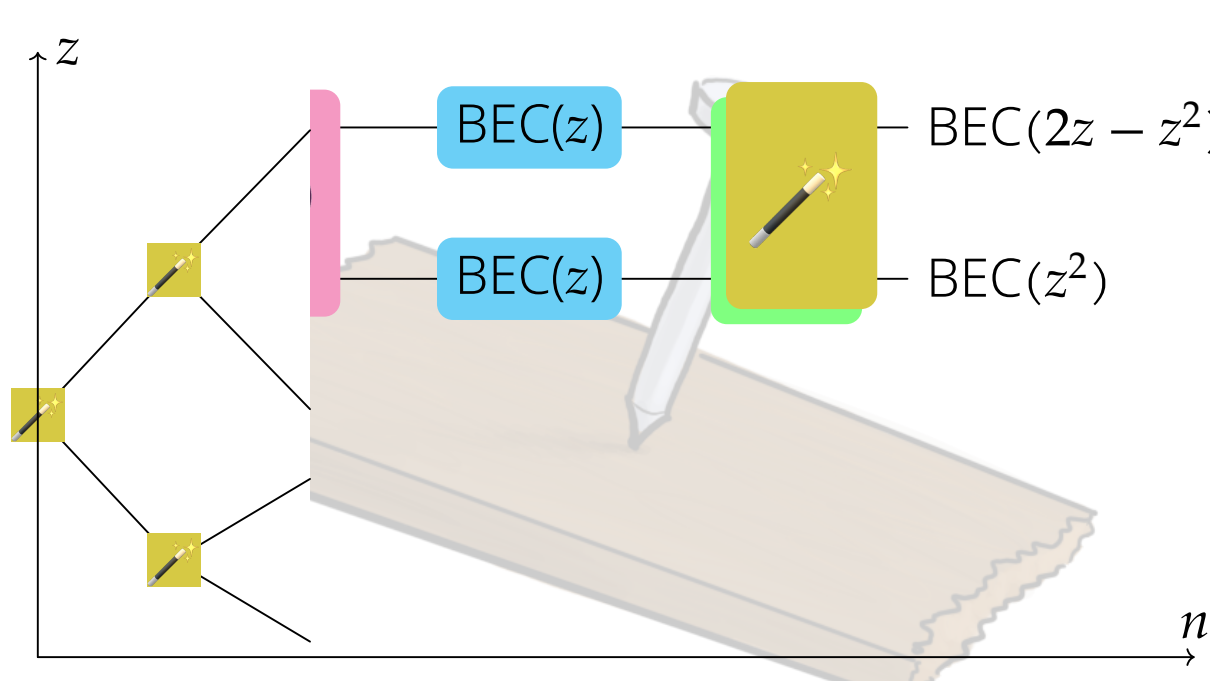
$$p = 2^{-\Omega(n)}$$

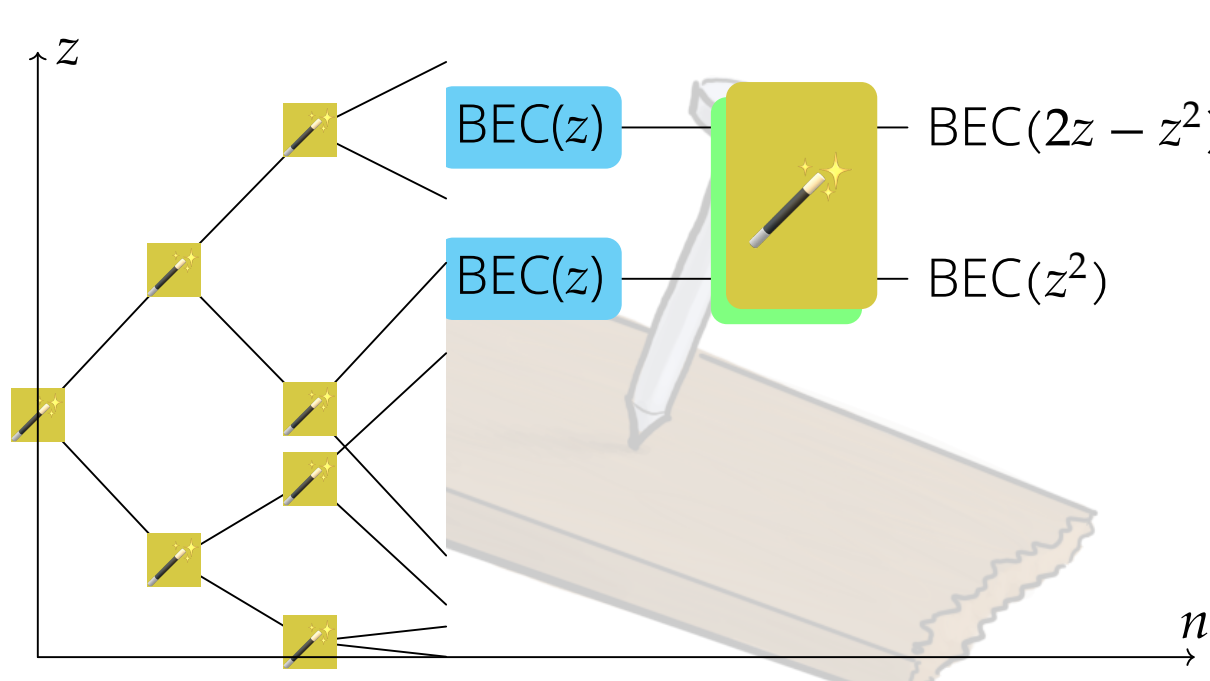

Let's improve p

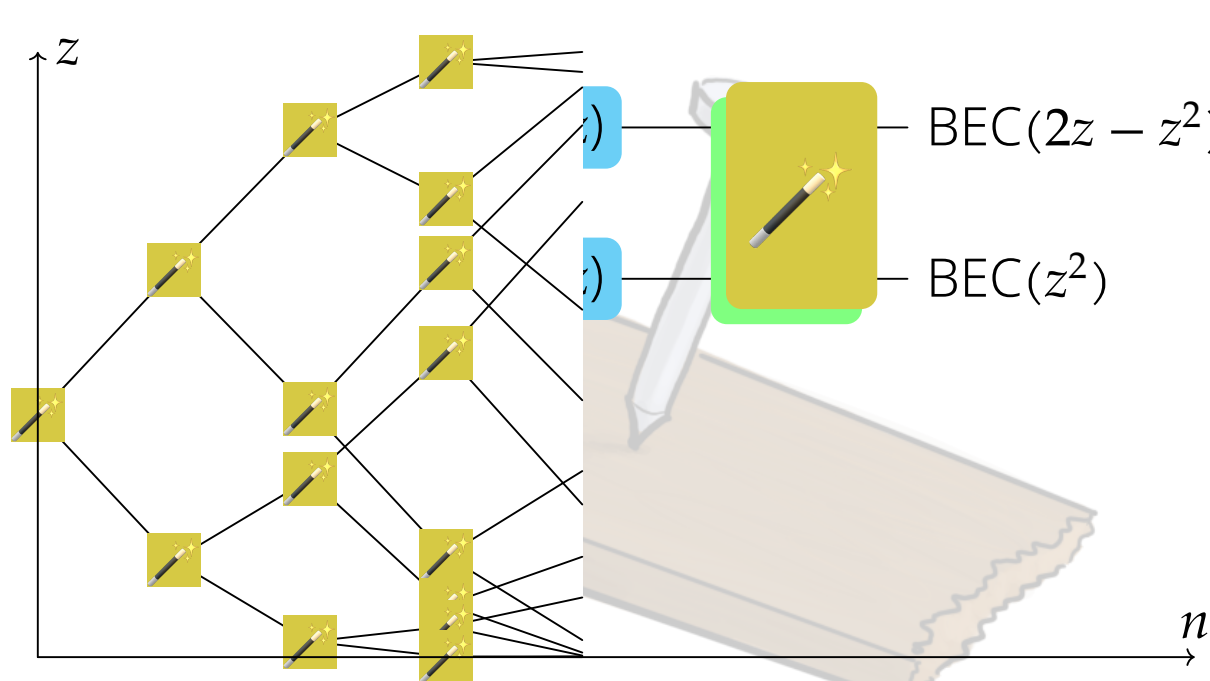

u_1
u_2
BEC(z)
BEC(z)
BEC($2z-z^2$)
BEC(z^2)

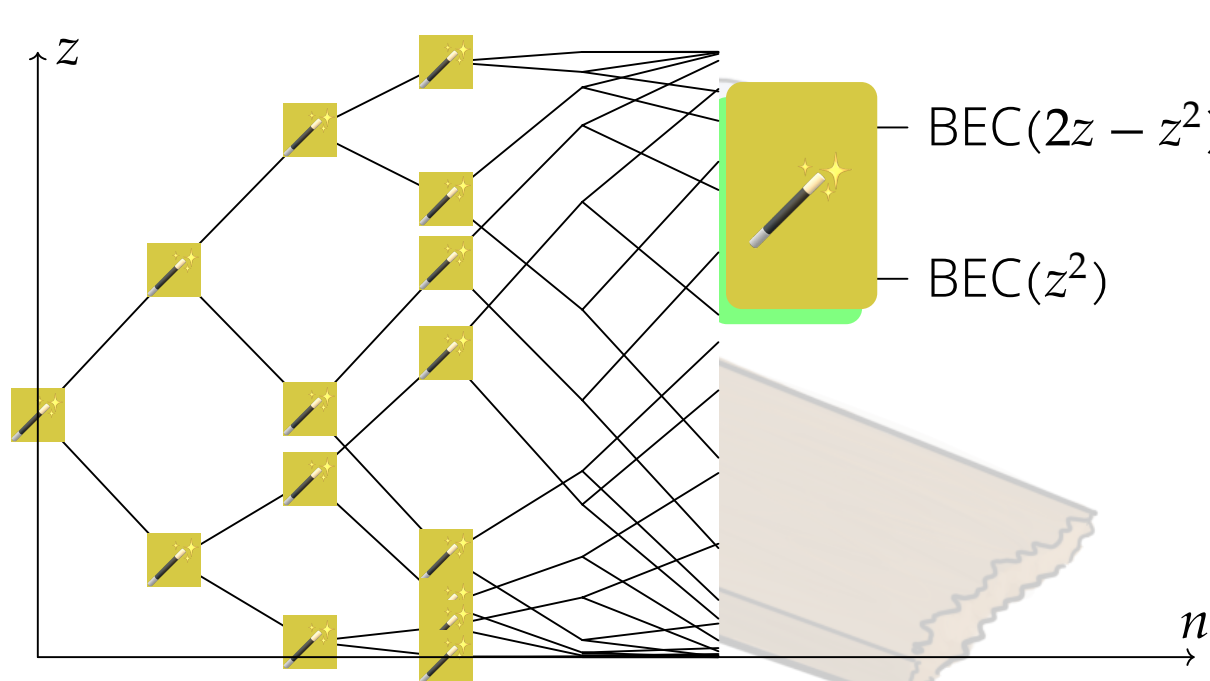

z
u_1
u_2
BEC(z)
BEC(z)
BEC(2z − z²)
BEC(z²)
n

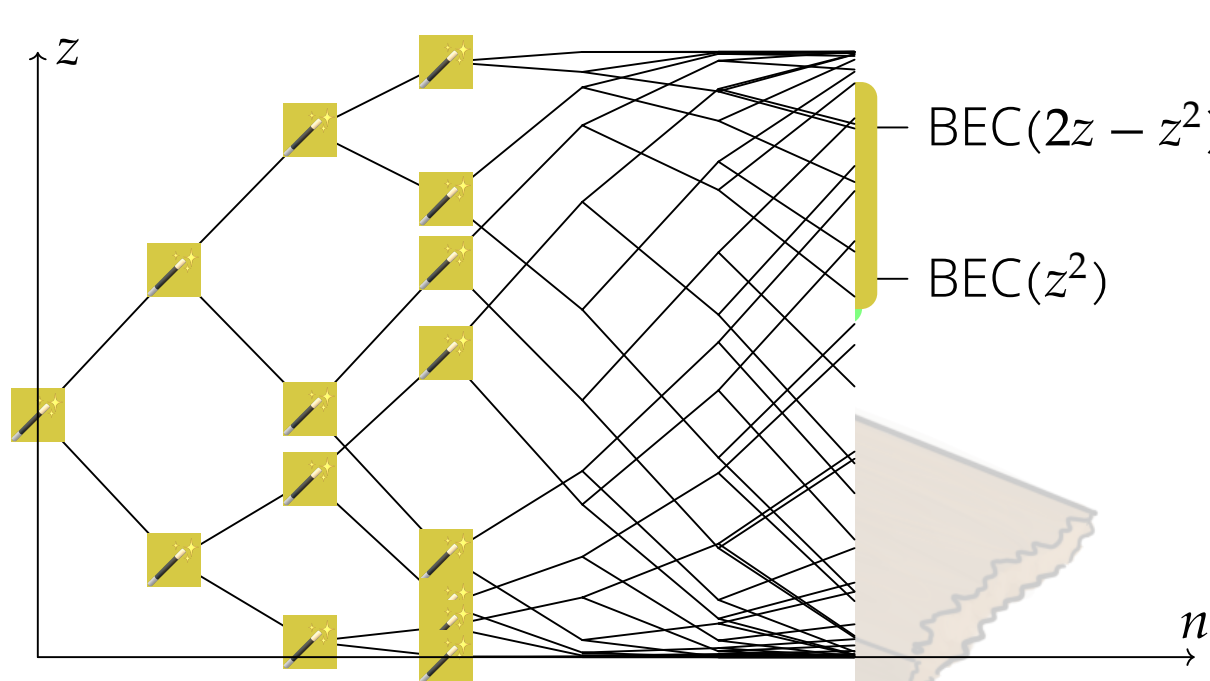

z
BEC(z)
BEC(z)
BEC(2z − z²)
BEC(z²)
n

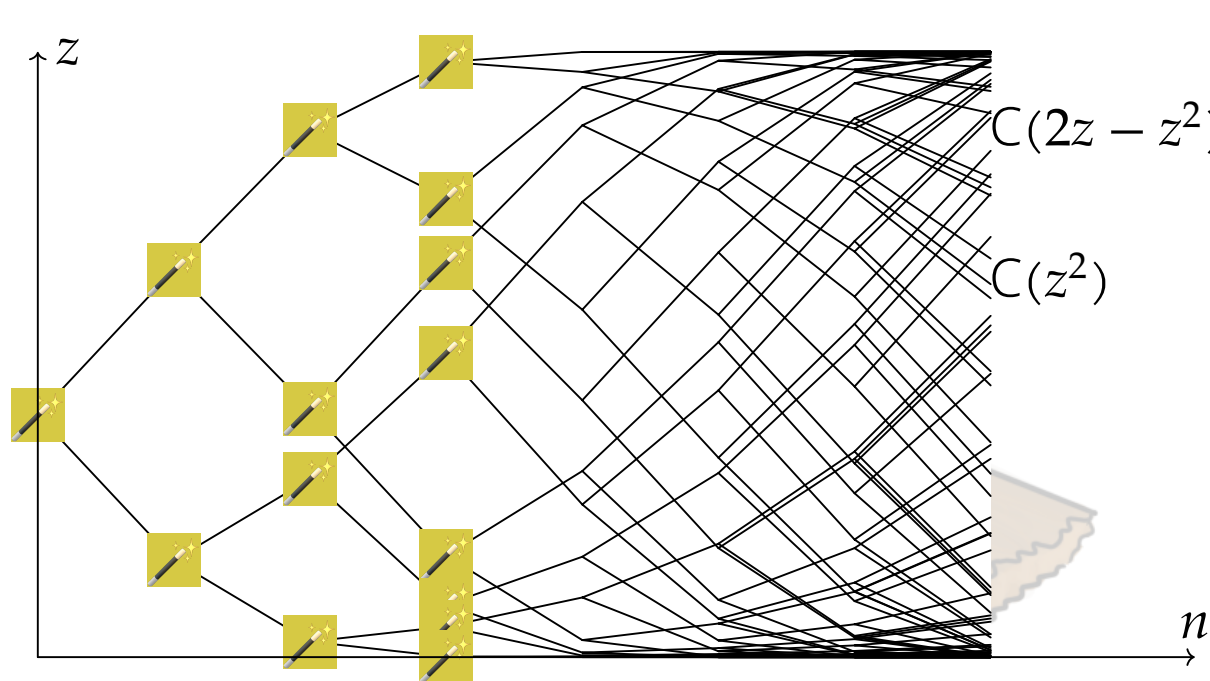

z
BEC(z)
BEC(z)
BEC(2z − z²)
BEC(z²)
n

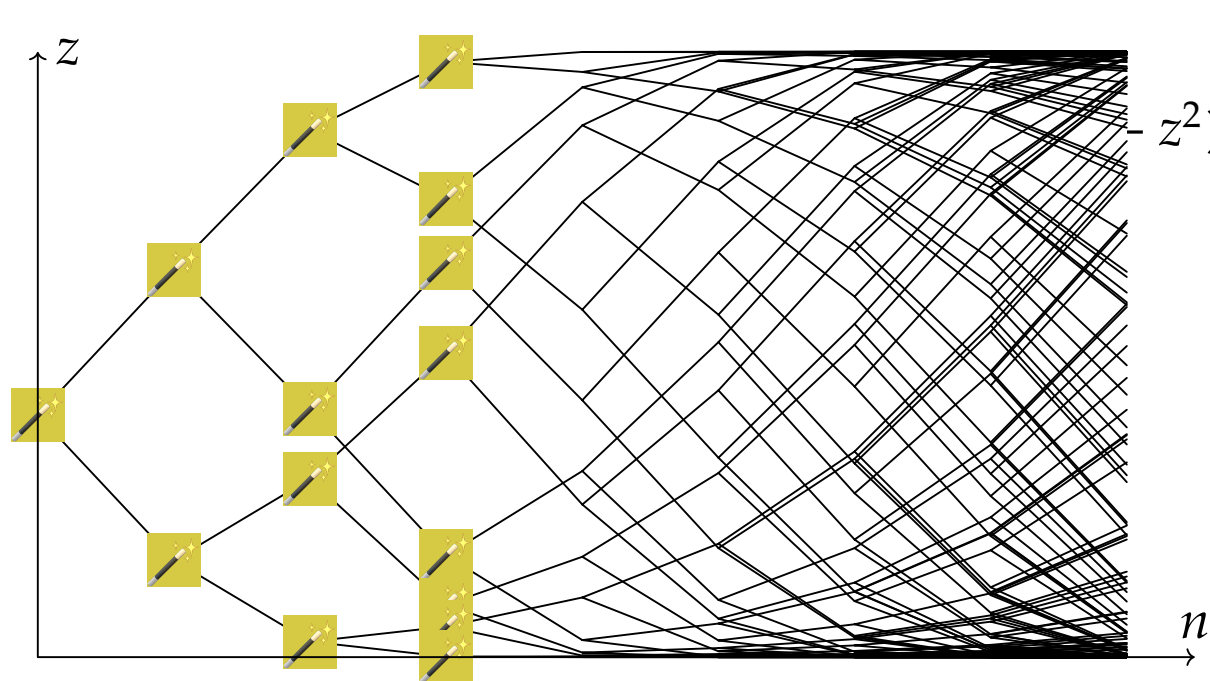

z
BEC(z)
BEC(z)
BEC(2z − z²)
BEC(z²)
n

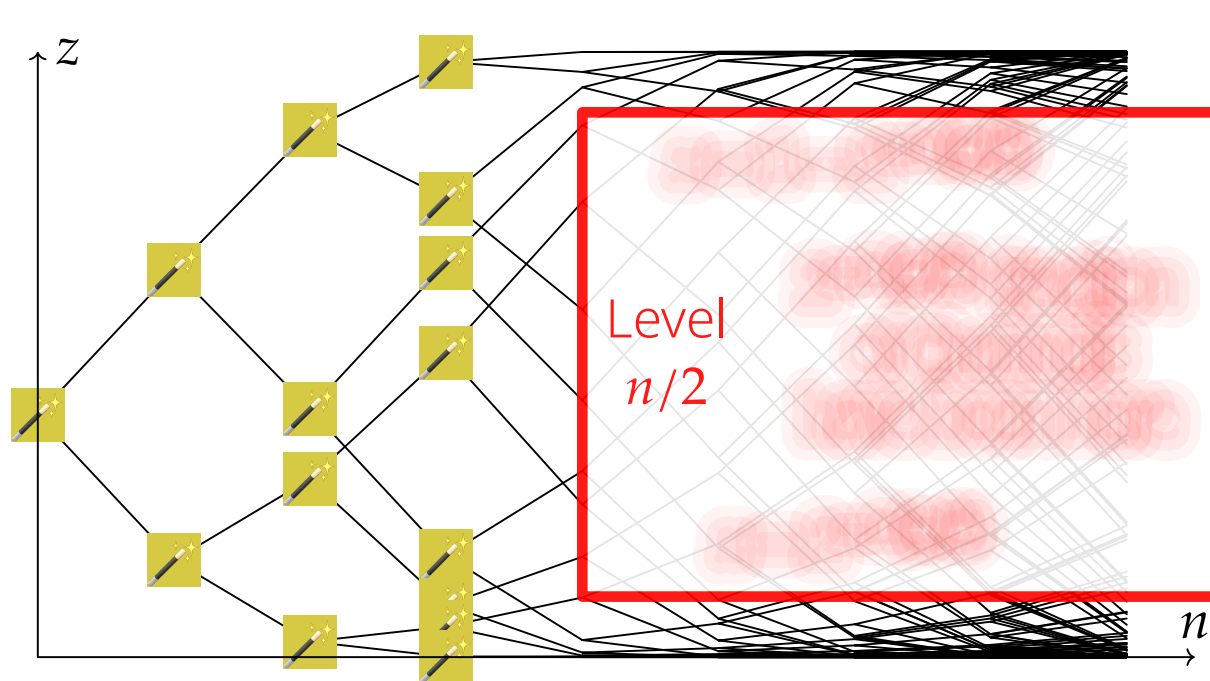

z
BEC(2z − z²)
BEC(z²)
n

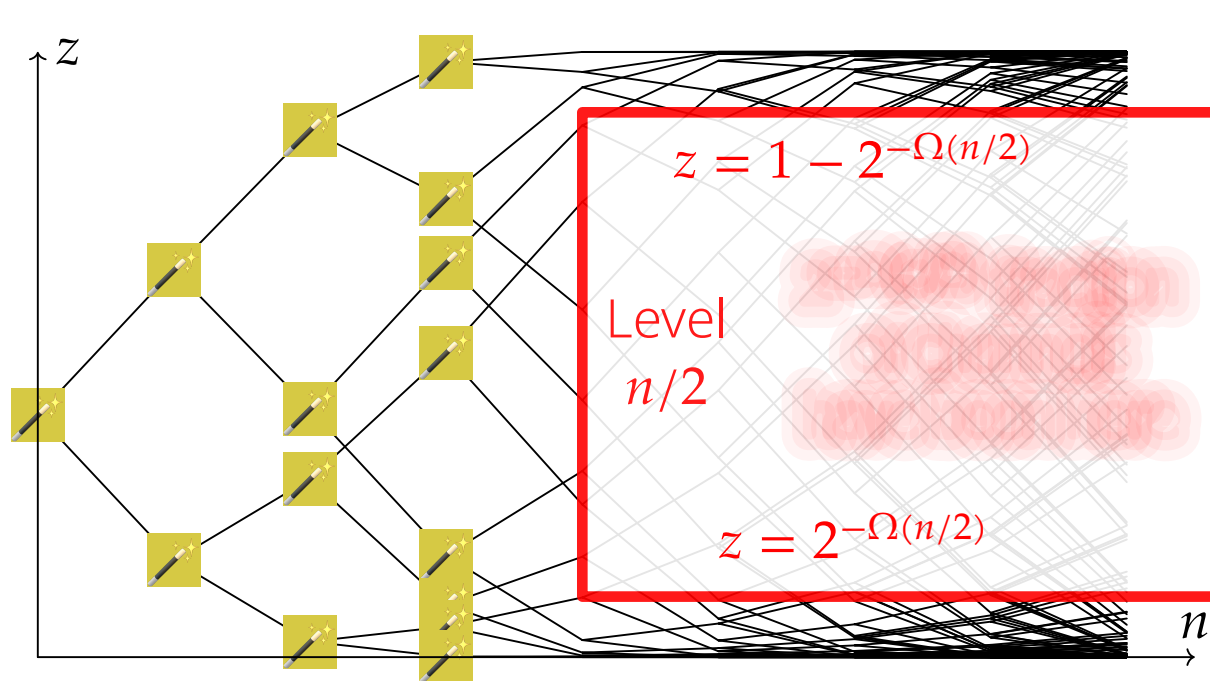

z
BEC(2z − z²)
BEC(z²)
n

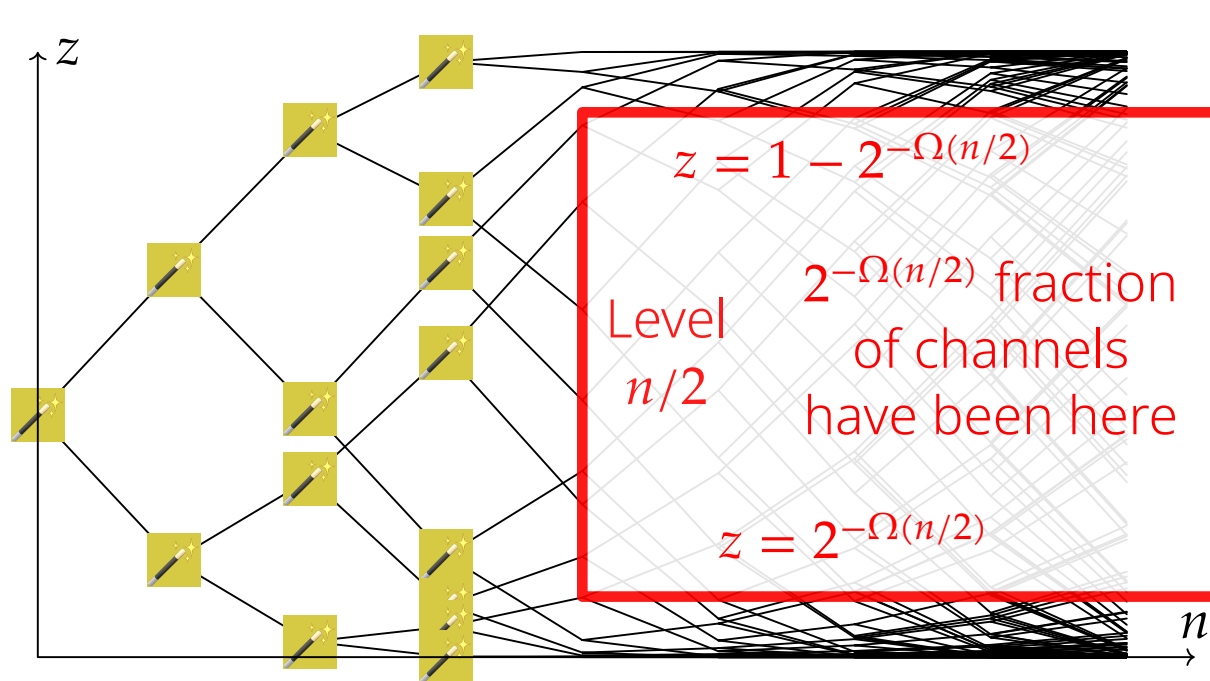

z
BEC(2z − z²)
BEC(z²)
n

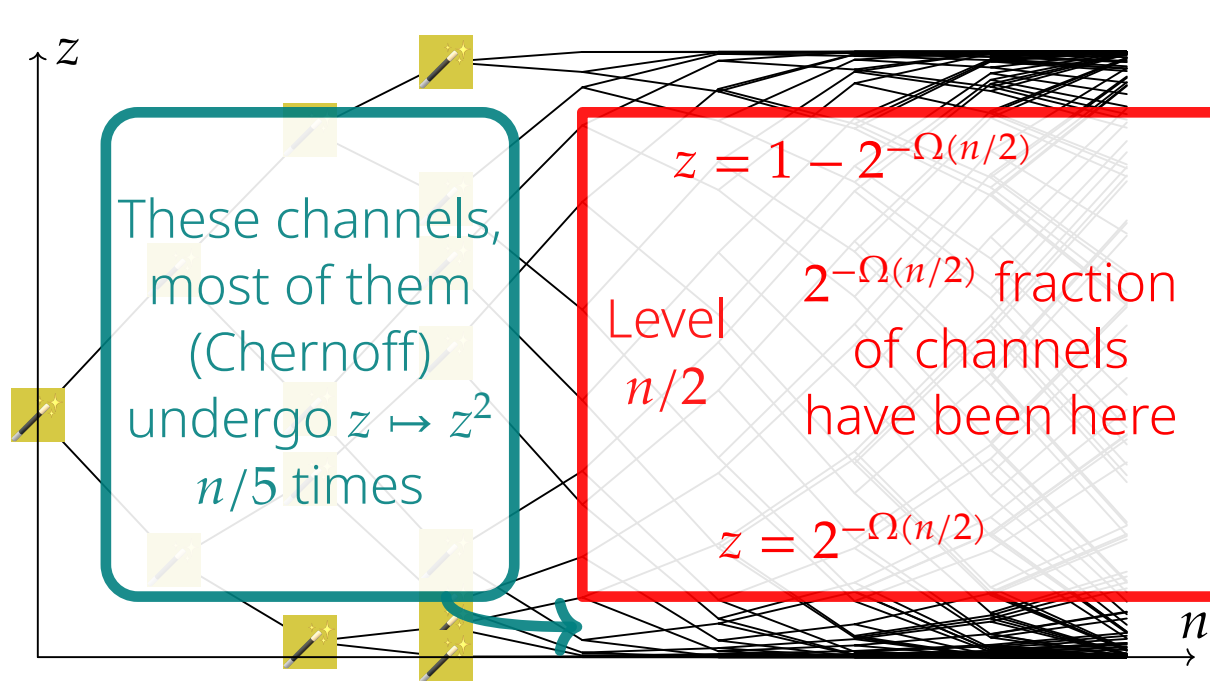

z
C(2z − z²)
C(z²)
n

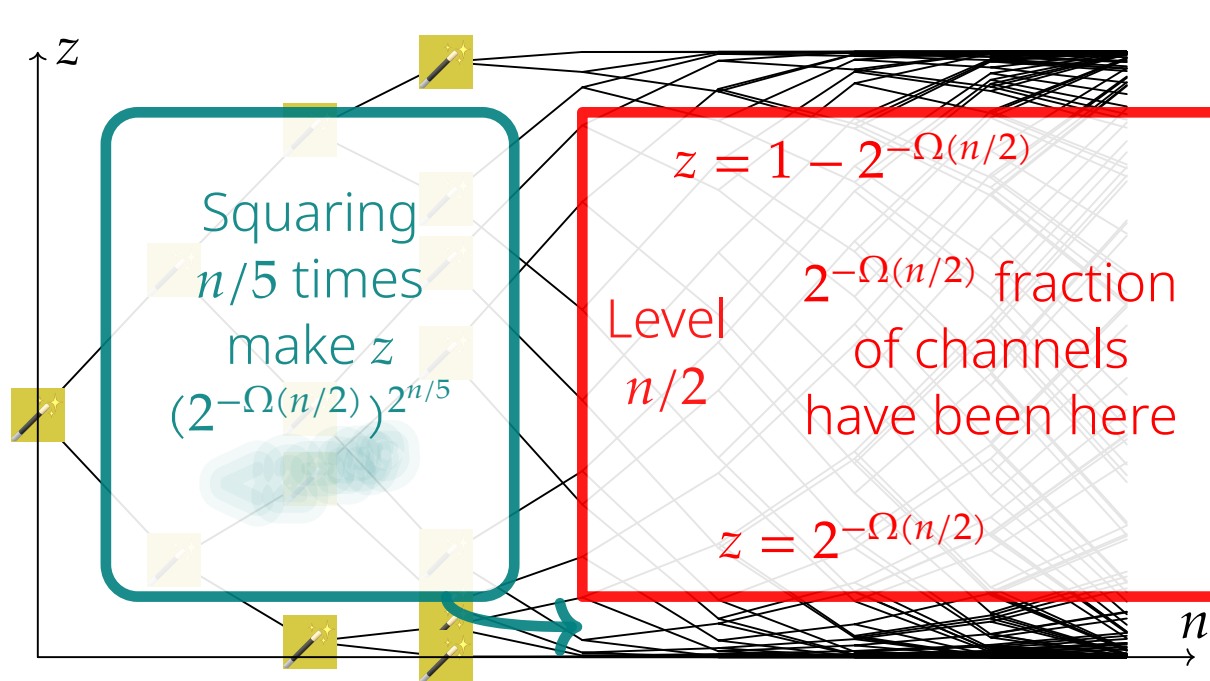

z
z^2
n

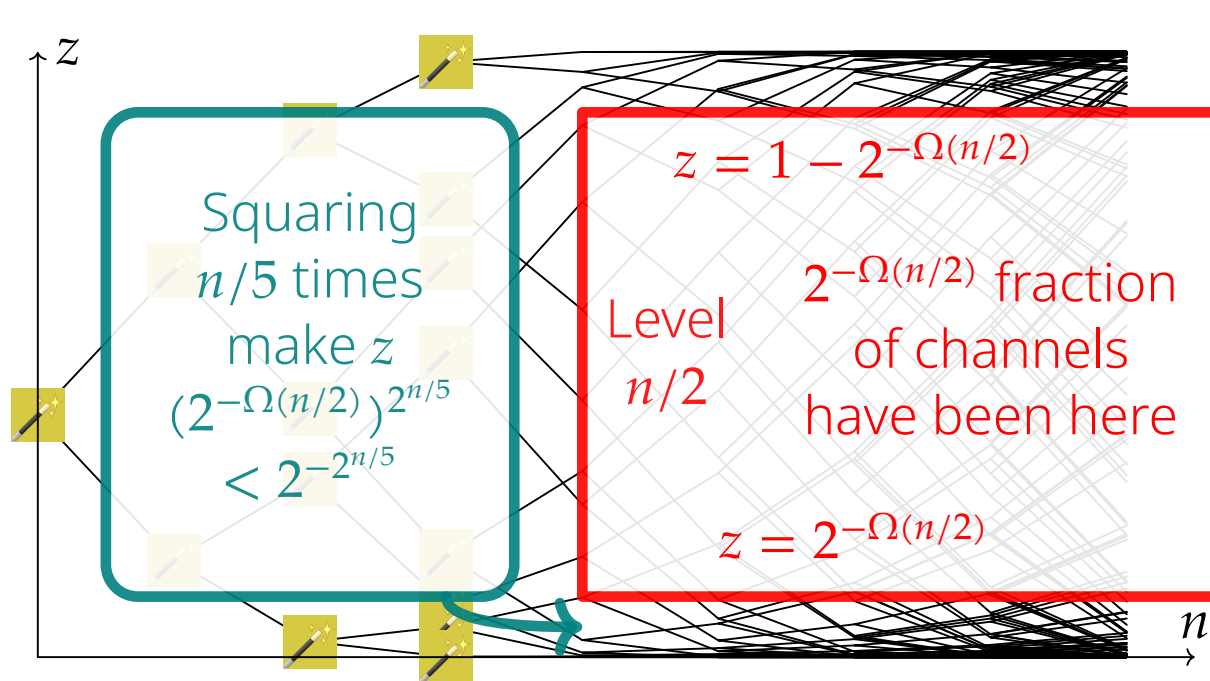

z
Level
n/2
n


z
$z = 1 - 2^{-\Omega(n/2)}$
Level
$n/2$
$z = 2^{-\Omega(n/2)}$
n


z
n
$z = 1 - 2^{-\Omega(n/2)}$
Level $n/2$
$2^{-\Omega(n/2)}$ fraction of channels have been here
$z = 2^{-\Omega(n/2)}$


z
n
These channels, most of them (Chernoff) undergo $z \mapsto z^2$ $n/5$ times
$z = 1 - 2^{-\Omega(n/2)}$
Level $n/2$
$2^{-\Omega(n/2)}$ fraction of channels have been here
$z = 2^{-\Omega(n/2)}$

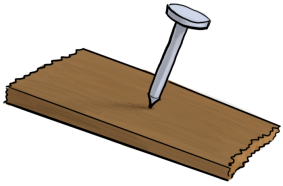

z
Squaring $n/5$ times make z $(2^{-\Omega(n/2)})^{2^{n/5}}$
$z = 1 - 2^{-\Omega(n/2)}$
Level $n/2$
$2^{-\Omega(n/2)}$ fraction of channels have been here
$z = 2^{-\Omega(n/2)}$
n


z
Squaring $n/5$ times make z $(2^{-\Omega(n/2)})^{2^{n/5}} < 2^{-2^{n/5}}$
$z = 1 - 2^{-\Omega(n/2)}$
Level $n/2$
$2^{-\Omega(n/2)}$ fraction of channels have been here
$z = 2^{-\Omega(n/2)}$
n

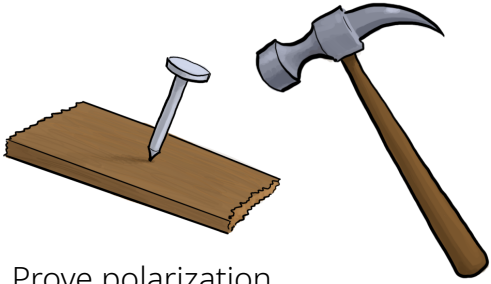
If k of 2^n are less than p,
polar code has rate $k/2^n$
& block error prob $< kp$

$$k/2^n = 60\% - 2^{-\Omega(n)}$$

$$p = 2^{-\Omega(n)} \qquad p = 2^{-2^{\Omega(n)}}$$

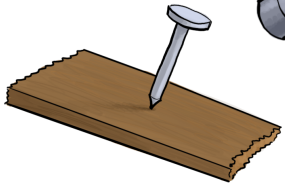


Prove polarization



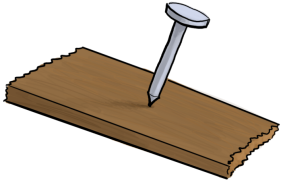
Prove polarization

Jensen’s

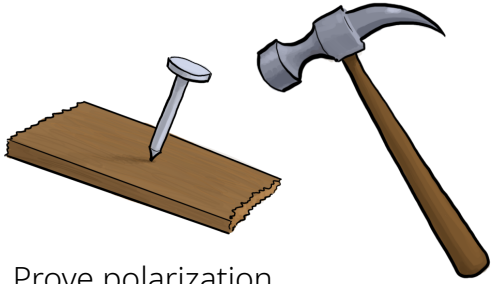


Prove polarization

Jensen's

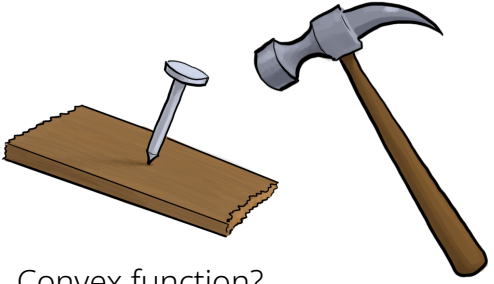


Convex function?



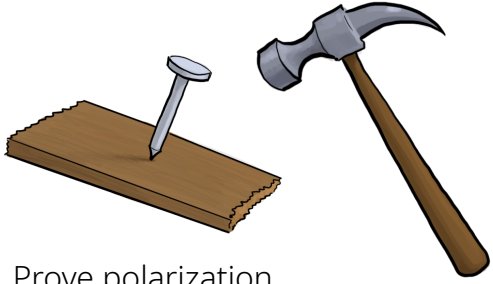
Prove polarization

Jensen's



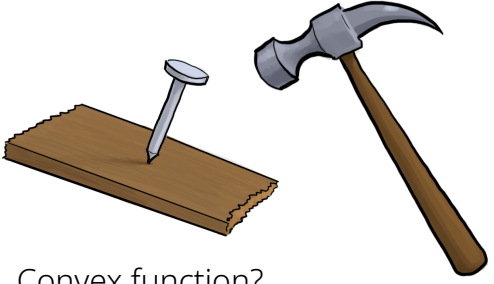
Convex function?

$$(z(1-z))^{0.663}$$



Prove polarization

Jensen's



Convex function?

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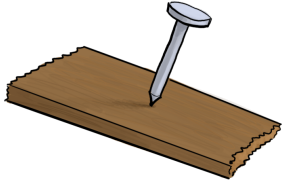
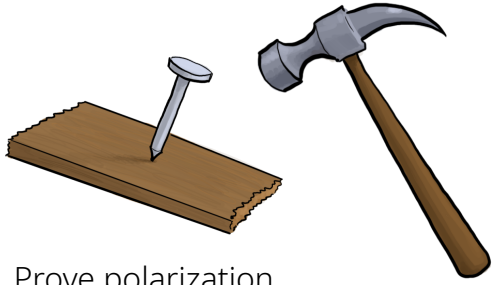
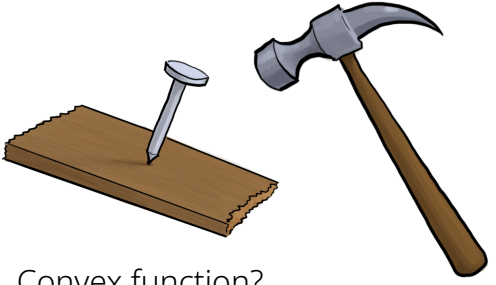


p too bad



Prove polarization

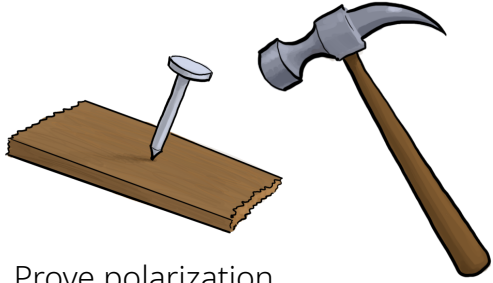


Convex function?

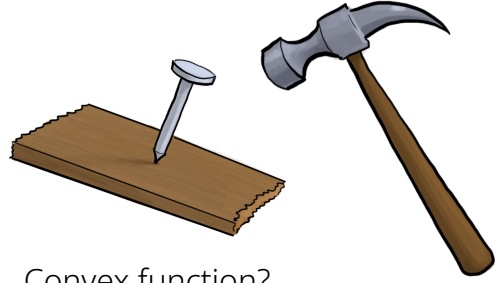
$$(z(1-z))^{0.663}$$

p too bad

Chernoff p



Prove polarization

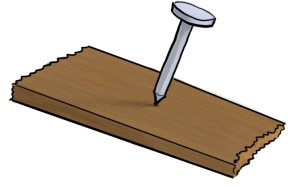


Convex function?

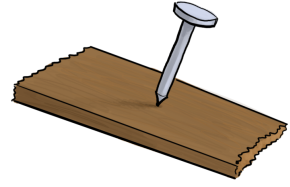
$(z(1-z))^{0.663}$

p too bad

Chernoff p



Improvements



Generalizations


ChatGPT


successive
cancellation
channel
polarization
ChatGPT

better gap to capacity

better error probability

better gap to capacity

better error probability

large kernel/RS/AG

BMS/large alphabet

better gap to capacity

better error probability

large kernel/RS/AG

BMS/large alphabet

classical–quantum

insdel channel

better gap to capacity

better error probability

large kernel/RS/AG

BMS/large alphabet

classical–quantum

insdel channel

coded computation

PAC & list decoding

dummy

dummy