ALGEBRAIC TOPOLOGY DIARY

JENS HARLANDER, MATH 512 SPRING 2021

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Syllabus and Intro. In algebraic topology we assign to topological spaces algebraic objects, like numbers, polynomials, groups, vector spaces etc. These algebraic objects provide homes for obstructions on things you might want to do with your space. Here is a very typical example. A graph is a set of vertices V together with a set of edges $E \subseteq V \times V$. An edge $e = (a, b)$ connects the vertex a to the vertex b. You might wonder when is a graph planar: that is when, can you draw the graph in the plane so that edges do not cross. Given a graph Γ drawn in the plane define

$$\chi(\Gamma) = |V| - |E| + |R|$$

where R is the set of regions and $|V|$ is the number of vertices etc. The number $\chi(\Gamma)$ is the algebraic object that obstruct (or constricts) graphs from being planar.

Let $K_{3,3}$ be the graph with 6 vertices $\pm 1, \pm 2, \pm 3$ and 9 edges $(+i, -j)$, $i, j = 1, 2, 3$. Try to draw it in the plane...

Suppose you could draw it on the plane. Note that the shortest cycle in $K_{3,3}$ has length 4. Thus if R is a region, then $\ell(R) \geq 4$ (number of edges in the boundary of the region). We have

$$\chi(K_{3,3}) = 6 - 9 + |R| = 2$$

so $|R| = 5$. We have 5 regions. Now note that

$$18 = 2|E| = \ell(R_1) + \ell(R_2) + \ell(R_3) + \ell(R_4) + \ell(R_5) \geq 5 \cdot 4 = 20.$$

A contradiction to the assumption that $K_{3,3}$ is planar. □

Vocabulary

Spaces, continuous maps and cell complexes

A *space* is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a collection of subsets satisfying three axioms: (1) X and $\emptyset$ are in $\mathcal{T}$; (2) A union of sets from $\mathcal{T}$ is in $\mathcal{T}$; (3) a finite intersection of sets from $\mathcal{T}$ is in $\mathcal{T}$. The sets in $\mathcal{T}$ are called *open* sets of X.

A map $f\colon X \to Y$ between spaces is continuous if the preimage of every open set in Y is an open set in X.

Consider a ball B^n in $\mathbb{R}^n$ and assume we have a map $S^{n-1} = \partial B^n \to X$. Then we can attach the ball to X to obtain a space $X \cup_f B^n$ defined as follows. Consider the disjoint union $X \cup B^n$ with the following partition: $\{x\}$ (singleton) if $x \in X - f(S^{n-1})$, $\{y\}$ (singleton) in y is in the interior of B^n, $\{y, f(y)\}$ if $y \in S^n$. Then $X \cup_f B^n$ is the quotient space arising from this partition.

A *cell complex* is a space that is inductively obtained by attaching balls of increasing dimension. Start with 0-balls (vertices), attach a bunch of 1-balls (edges), attach a bunch of 2-balls (triangles if you want), attach a bunch of 3-balls, and so on. The spaces we will deal with in this course are cell complexes, which are a generalization of graphs to higher dimensions.

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Example (standard 2-complex): A presentation consists of letters and words in these letters, like $P = \langle a, b, c \mid abc^2ab^{-1}a^{-1}c^{-2}b^{-1}, bca^2bc^{-1}b^{-1}a^{-2}c^{-1}\rangle$. This can be used to build a 2-complex. Start with a single vertex, attach edges a, b, c, and now attach 2 discs along the given words. This complex is called *the Jens complex*. Please make yourself a good complex.

Deformation retraction.

Let $A \subseteq X$. We say A is a *deformation retraction of* X if X can be continuously deformed into A. So X and A have the same shape. This means there is a map $F\colon X \times [0,1] \to X$ so that $F(x,0) = x$ and $F(a,t) = a$, for all t, and $F(x,1) \in A$. You should visualize this as follows: Every point in X flows into A over time, and once in A stops moving. Points in A never move. The flow line for a point x is the path $F(x,t)$.

Homotopy and homotopic maps

A homotopy is a map $F\colon X \times [0,1] \to Y$. Define $f_t\colon X \to Y$ by $f_t(x) = F(x,t)$. A homotopy should be thought of as deforming the map f_0 to the map f_1. Two maps $a, b\colon X \to Y$ are homotopic if there is a homotopy $F\colon X \times [0,1] \to Y$ so that $a = f_0$ and $b = f_1$.

Homotopy Equivalence

We already saw what it means to deform a space into a subspace (deformation retraction). How do we deform spaces into each other if they do not share an ambient space? Recall that topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic (considered the same) if if there are continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ so that $f \circ g = 1_Y$ and $g \circ f = 1_X$ (this is the same as saying there exists a bijection $f\colon X \to Y$ so that $f(\mathcal{T}_X) = \mathcal{T}_Y$). We say spaces X, and Y are homotopically equivalent (the same up to deformation) if there are continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ so that $f \circ g \cong 1_Y$ and $g \circ f \cong 1_X$ (here $\cong$ means homotopic). The maps f and g are called homotopy equivalences. Note that they are inverse to each only up to homotopy (or deformation).

Contractible space

A space X is called *contractible* if $X \to y$ (y a single point) is a homotopy equivalence. Example: Bing's house with 2-rooms is, contractible. And so is the Dunce hat.

Operations on spaces: Products $X \times Y$, quotients X/A and joints $X * Y$.

Changing a space without changing its homotopy type

1) If $A \subseteq X$ is a contractible subcomplex, then the quotient map $X \to X/A$ is a homotopy equivalence.

2) Changing attaching maps of cells up to homotopy does not change the homotopy type of the cell complex. That is why the dunce hat is homotopically equivalent to the disc.

Example: If $A = \mathbb{R}^2$ and $f\colon S^1 \to \mathbb{R}^2$ is any map, then $X = A \cup_f B^2$ is homotopy equivalent to a 2-sphere.

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We are starting Chapter 1 Fundamental Group in Hatcher. First we review some basic vocabulary fro group theory.

Group: A set G with an associative multiplication that contains an identity element denoted by 1 (or sometimes e) and inverses.

Free group $F(\mathbf{x})$: Let $\mathbf{x}$ be a set. The elements of $F(\mathbf{x})$ are words in $\mathbf{x}^{\pm\mathbf{1}}$. Multiplication is concatenation. We do allow cancellation of pairs $x^\epsilon x^{-\epsilon}$, where $\epsilon = \pm 1$. Lets do an example. If $\mathbf{x} = \{a, b, c\}$, then $g = aba^{-1}ccbb^{-1}ab^{-1}$ would be an element of $F(a, b, c)$. Note that we can cancel bb^{-1} and obtain $g = aba^{-1}ccab^{-1}$. If $h = babcabc$ then $g \cdot h = (aba^{-1}ccab^{-1}) \cdot (babcabc) = aba^{-1}ccab^{-1}babcabc = aba^{-1}ccaabcabc$.

Normal subgroup: Let G be a group and H be a subgroup. We say H is normal in G if $gHg^{-1} = H$ for all $g \in G$. In that case $G/H = \{gH \mid g \in G\}$ is a group, called the quotient group. Multiplication is defined by $g_1H \cdot g_2H = g_1g_2H$.

Generation and closure: Let $\mathbf{g}$ be a subset of G. Then $\langle \mathbf{g} \rangle$ is the set of words in $\mathbf{g}^{\pm\mathbf{1}}$. Note that $\langle \mathbf{g} \rangle$ is a subgroup of G, called the subgroup *generated* by $\mathbf{g}$. Let $\hat{\mathbf{g}} = \{aga^{-1} \mid g \in$

$\mathbf{g}, \mathbf{a} \in \mathbf{G}\}$, the set of all conjugates of elements in $\mathbf{g}$. Define $\langle\langle \mathbf{g} \rangle\rangle = \langle \hat{\mathbf{g}} \rangle$. The group $\langle\langle \mathbf{g} \rangle\rangle$ is the normal subgroup of G normally generated by $\mathbf{g}$.

Presentation: Let $\mathbf{x}$ be a set and $\mathbf{r}$ be a set of words in $\mathbf{x}^{\pm \mathbf{1}}$. Then

$$P = \langle \mathbf{x} \mid \mathbf{r} \rangle$$

is called a presentation.

2-complex defined by a presentation: Construct a bouquet of circles in one-to-one correspondence with the elements of $\mathbf{x}$. Attach discs to that bouquet in one-to-one correspondence to the elements of $\mathbf{r}$, using the words $r \in \mathbf{r}$ as attaching maps. Denote the 2-complex obtained by $K(P)$.

Group defined by a presentation: Let $G(P) = F(\mathbf{x})/\langle\langle \mathbf{r} \rangle\rangle$ be the quotient group. Note that group elements in $G(P)$ are words but every $r \in \mathbf{r}$ is now the trivial element in $G(P)$. We have imposed relations among the generators of $G(P)$. Every group arises as a $G(P)$ for some presentation because every group is a quotient of a free group.

Combinatorial group theory: The part of group theory that studies groups given by a presentation. Given P, what can you tell me about $G(P)$: is it trivial, finite, abelian, does it have free subgroups, etc. These are difficult questions that can not be answered by a computer, even if both $\mathbf{x}$ and $\mathbf{r}$ are finite sets (in which case P is called a finite presentation). One can prove that there does not exist a computer program that takes a finite presentation as an input and decides if $G(P)$ is trivial or not.

How to work with presentations: Given a presentation $P = \langle \mathbf{x} \mid \mathbf{r} \rangle$ the elements of $G(P)$ are words in $\mathbf{x}$. Elements in $\mathbf{r}$ give rewriting laws for the words.

Example: Let $P = \langle a, b \mid aba^{-1}b^{-1} \rangle$. Note $aba^{-1}b^{-1} = 1$ in $G(P)$, we can rewrite this to obtain the law $ab = ba$ in $G(P)$. So the element $ababaab^{-1}$ can be rewritten to a^4b^2. Every element in $G(P)$ can be written as $a^m b^n$ and that sets up a group isomorphism $G(P) \to \mathbb{Z} \times \mathbb{Z}$.

Example: Let $P = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$. Lets derive some laws: $aa = 1$, so $a^{-1} = b$. Also $b^{-1} = b$. Thus elements in $G(P)$ can be represented by positive words. Also $a^5 = aaaaa = (aa)(aa)a = a$, etc. Also $ababab = 1$, so $aba = bab$. We can now list the elements of G by listing positive words according to length

1. length 0: $\emptyset$, the empty word which is the identity in $G(P)$;
2. length 1: a, b;
3. length 2: ab, ba;
4. length 3: aba, $bab = aba$, so there is only one element of length 3;
5. length 4: $abab = (aba)b = (bab)b = babb = ba$, is actually already listed under length 2. All length 4 elements are already listed;

(6) length 5: already listed, etc.

Thus $\emptyset$, a, b, ab, ba, aba represent all the elements in $G(P)$. Are there redundancies on the list? Can you write down a group table?

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Free product of groups. Let A and B be groups, then the free product $A * B$ is defined as follows. Elements are words $x_1 * x_2 * \cdots * x_n$ where the x_i come alternately from A and B, like $a_1 * b_1 * a_2$, or $b_1 * a_1 * b_2 * a_2 * b_3$. Multiplication is concatenation, wherby elements from the same group can be combined. For example $(a_1 * b_1 * a_2) \cdot (a_3 * b_2 * a_4 * b_3) = a_1 * b_1 * a_2 * a_3 * b_2 * a_4 * b_3 = a_1 * b_1 * (a_2 \cdot a_3) * b_2 * a_4 * b_3$.

Free product with amalgamation. Now suppose that C is a third group and we have homomorphisms $f_A\colon C \to A$ and $f_B\colon C \to B$. Then we define

$$A *_C B = A * B / \langle\langle f_A(c) f_B^{-1}(c),\ c \in C \rangle\rangle.$$

This is a free product with the imposed law $f_A(c) = f_B(c)$ if $c \in C$. Thus if $b_1 * a_1 * b_2 * a_2 * b_3$ is in $A * B$ and $b_2 = f_B(c) \in B$ for some c, you are allowed to switch it to to $a_3 = f_A(c)$. So

$$b_1 * a_1 * b_2 * a_2 * b_3 = b_1 * a_1 * f_B(c) * a_2 * b_3 = b_1 * a_1 * f_A(c) * a_2 * b_3 = b_1 * a_1 * a_3 * a_2 * b_3 = b_1 * (a_1 a_3 a_2) * b_3.$$

If both f_A and f_B are injective (that is C is a subgroup of both A and B), we call $A *_C B$ the amalgamated product of A and B over C. The idea is to glue the two groups together along a common subgroup. It is the group theoretic analogue of gluing two spaces together along a common subspace.

Example: Let $A = GL(2, \mathbb{R})$ and $B = D_3$, the dihedral group. Let $C = \mathbb{Z}_2$. Both A and B have subgroups isomorphic to C, namely the subgroups generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and the transposition (12). Thus we can form the amalgamated product $GL(2, \mathbb{R}) *_{\mathbb{Z}_2} D_3$.

We now properly start with Chapter 1 in Hatcher. Read “the idea of the fundamental group”.

Loops (closed paths). Let X be a space. A loop in X is a continuous map $f\colon [0,1] \to X$ such that $f(0) = f(1)$. If $f(0) = f(1) = x_0$ we say the loop f is based at x_0. We denote by $[f, x_0]$ the set of all loops based at x_0 that are homotopic to f under a base point preserving homotopy $F\colon [0,1] \times [0,1] \to X$, $F(0,t) = F(1,t) = x_0$ for all $t \in [0,1]$.

Fundamental group. Let X be a space and $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of all $[f, x_0]$. We can define a multiplication in the following way: $[f] \cdot [g] = [f \cdot g]$, where $f \cdot g$ means run the loop f followed by the loop g. Said precisely $f \cdot g\colon [0,1] \to X$ where $f \cdot g(s) = f(2s)$ if $s \in [0, 1/2]$ and $f \cdot g(s) = g(2s - 1)$ if $s \in [1/2, 1]$. One can now check the following things

(1) The multiplication is well defined;
(2) If e is the constant path $e(s) = x_0$, then $[e] \cdot [f] = [f] \cdot [e] = [f]$;
(3) If f is a loop define $\bar{f}(s) = f(1-s)$. Then $[f] \cdot [\bar{f}] = [\bar{f}] \cdot [f] = [e]$;
(4) $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$.

Thus $\pi_1(X, x_0)$ is a group called the fundamental group of X based at x_0. A couple of lectures down the road we will prove the van Kampen Theorem:

Theorem 0.1. *(van Kampen) Suppose $X = X_1 \cup_Y X_2$ ($X_1 \cap X_2 = Y$), everybody is connected and $x_0 \in Y$, then*

$$\pi_1(X, x_0) = \pi_1(X_1, x_0) *_{\pi_1(Y, x_0)} \pi_1(X_2, x_0).$$

That's why we talked about all this group theory at the beginning of the lecture. Van Kampen's theorem is a device for computing fundamental groups, especially in the context of cell complexes. The inductive procedure for building a cell complex inductively also builds its fundamental group.

We end this lecture with the following observation. Looking at $[f] \in \pi_1(X, x_0)$ you should pay attention to the loop you see in X and not the way an ant might traverse it. Let $\phi\colon [0,1] \to [0,1]$ be a continuous map such that $\phi(0) = 0$ and $\phi(1) = 1$. Given a loop $f\colon [0,1] \to X$ we say $f \circ \phi\colon [0,1] \to X$ is a reparametrization of f. Note that the images of f and $f \circ \phi$ are the same (only speed and direction of the crawling ant changes).

Lemma 0.2. $[f] = [f \circ \phi]$.

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Lemma 0.3. $[f] = [f \circ \phi]$.

Proof. Note that ϕ is homotopic to the identity via a straight line homotopy $H(s,t) = (1-t)s + t\phi(s)$. Now define $F(s,t) = H(f(s), t)$. Then $F(s,0) = H(f(s), 0) = f(s)$ and $F(s,1) = H(f(s), 1) = \phi(f(s))$. □

Theorem 0.4. *$\pi_1(X, x_0)$ is a group.*

Proof.

(1) The multiplication is well defined. That is if $[f] = [f']$ and $[g] = [g']$ then $[f \cdot g] = [f' \cdot g']$. To see this let A be a base point fixing homotopy from f to f' and B be a base point fixing homotopy from g to g'. Then $A \cdot B$ is a base point fixing homotopy from $f \cdot g$ to $f' \cdot g'$. Here $A \cdot B(s,t) = A(2s, t)$ on $[0, 1/2]$ and $A \cdot B(s,t) = A(2s-1, t)$ on $[1/2, 1]$.

(2) The multiplication is associative because $f \cdot (g \cdot h)$ is a re-parametrization of $(f \cdot g) \cdot h$. So $[f \cdot (g \cdot h)] = [(f \cdot g) \cdot h]$ by the Lemma.

(3) $[e]$, where $e(s) = x_0$, is the identity because $f \cdot e$ and $e \cdot f$ are re-parametrizations of f.

(4) We have left to show that $[f \cdot \bar{f}] = [\bar{f} \cdot f] = [e]$ where $\bar{f}(s) = f(1-s)$. This can not be done using reparametrizations. For $t \in [0,1]$ define $f_t\colon [0,1] \to X$ by $f_t(s) = f(s)$ for $s \in [0, 1-t]$ and $f_t(s) = f(1-t)$ for $s \in [1-t, 1]$. Visualize this as follows: the ant crawls

along the path f until its clock shows the time t, then the ant stops and waits until the time is up. Note that $f_0 = e$ and $f_1 = f$. Now define $F(s,t) = f_t \cdot \bar{f}_t(s)$. □

Let $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle. We want to compute $\pi_1(S^1)$ (basepoint is assumed to be $(1,0)$). Loops we like are $\omega_n \colon [0,1] \to S^1$, defined by $\omega_n(s) = (\cos(2\pi s), \sin(2\pi s))$. Draw graphs of ω_n in $[0,1] \times S^1$ to get a feeling. It is believable that every loop in S^1 can be straightened out to look like one of the ω_n.

Theorem 0.5. *The map $\Phi \colon \mathbb{Z} \to \pi_1(S^1)$, defined by $\Phi(n) = [\omega_n]$ is a group isomorphism.*

Proof. We have a map $p \colon \mathbb{R} \to S^1$ defined by $p(s) = (\cos(2\pi s), \sin(2\pi s))$. (This map is a local homeomorphism with discrete fiber, a so called *covering projection*). Define $\tilde{\omega}_n \colon [0,1] \to \mathbb{R}$ by $\tilde{\omega}_n(s) = s$ and note that $p \circ \tilde{\omega}_n = \omega_n$. $\tilde{\omega}_n$ is called a lift of ω_n.

(1) Φ is a group homomorphism. We have to check that $\Phi(m+n) = \Phi(m)\Phi(n)$, that is $[\omega_{m+n}] = [\omega_m \cdot \omega_n]$. Let T_m be the translation $T_m(x) = x + m$. Now $\tilde{\omega}_{m+n}$ is homotopic (via a straight line homotopy) to $\tilde{\omega}_m \cdot (T_m \circ \tilde{\omega}_n)$. Thus $\omega_{m+n} = p \circ \tilde{\omega}_{m+n}$ is homotopic to $\omega_m \cdot \omega_n = p \circ (\tilde{\omega}_m \cdot (T_m \circ \tilde{\omega}_n))$.

In order to see that Φ is an isomorphism we need unique path and homotopy lifting.

(a) Every path $f \colon [0,1] \to S^1$ starting at $(1,0)$ has a unique lift $\tilde{f} \colon [0,1] \to \mathbb{R}$ $(p \circ \tilde{f} = f)$ starting at 0.

(b) Every homotopy of paths $F \colon [0,1] \times [0,1] \to S^1$ starting at $(0,0)$ has a unique lift $\tilde{F} \colon [0,1] \times [0,1] \to \mathbb{R}$ $(p \circ \tilde{F} = F)$ starting at 0.

(2) Φ is injective. Suppose $\Phi(n) = [\omega_n] = [\omega_0]$. Let F be the homotopy from ω_n to ω_0 and let $\tilde{F}$ be its unique lift starting at 0 ($\tilde{F}(0,t) = 0$ for all t). From (a) it follows that $\tilde{F}(s,0) = \tilde{\omega}_n(s)$ and $F(s,1) = \tilde{\omega}_0(s) = 0$. Let $\tilde{\gamma}(t) = \tilde{F}(1,t)$, which is a path stating at n and ending at 0. If $n \neq 0$ then by the intermediate value theorem there exists a t_0 so that $\gamma(t_0) = 1/2$. Now $(\cos\pi, \sin\pi) = \pi p(1/2) = p(\tilde{\gamma}(t_0)) = p(\tilde{F}(1,t_0)) = F(1,t_0) = (1,0)$, a contradiction. So $n = 0$.

(3) Φ is surjective. Suppose $[f] \in \pi_1(S^1)$. Let $\tilde{f}$ be its unique lift starting at 0 and let $\tilde{f}(1) = n \in \mathbb{Z}$. Now $\tilde{f}$ and $\tilde{\omega}_n$ are homotopic by a straight line homotopy $\tilde{F}$ that fixes 0 and n. Then $F = p \circ \tilde{F}$ is a base point fixing homotopy between f and ω_n, so $\Phi(n) = [f]$.

This completes the proof. □