MATH 512: Advanced Topology
Spring 2021 | Homework #1 Solutions

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1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution: Let $I = [-1, 1]$ be an interval on $\mathbb{R}^2$. Let the origin $(0, 0)$ be the point deleted from the torus. Consider the map $f : I^2 - \{0\} \to \mathbb{S}$

$$f(x) = \frac{x}{\|x\|}.$$

Notice that f is a deformation retraction onto $\mathbb{S}$.

Consider $g = f|_{\partial I^2}$, which sends all points on ∂I^2 onto $\mathbb{S}$. With this definition, $g^{-1} \circ f$ is a deformation retraction sending all points $I^2 - \{0\}$ to ∂I^2.

Now, define the homotopy $H : \mathbb{T}^2 - \{*\} \times I \to \partial I^2$ by

$$H(x, t) = (1 - t)x + t(g^{-1} \circ f).$$

This is a deformation retraction of the torus with one point deleted onto ∂I^2, which is a graph of two circles intersecting at a point. ■

2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto $\mathbb{S}^{n-1}$.

Solution: Consider the family of functions $f_t : \mathbb{R}^n - \{0\} \to \mathbb{S}^{n-1}$ for all $t \in \mathbb{R}$.

$$f_t(x) = x + \left(\frac{1}{\|x\|} - 1\right) tx.$$

Notice that $f_0(x) = x$, so that $f_0 = \mathbb{1}$. We compute

$$f_t(x) \cdot f_t(x) = \|x\|^2 \left(1 + \left(\frac{1}{\|x\|} - 1\right) t\right)^2.$$

Hence $\|f_1(x)\|^2 = 1$, showing that $f_1(x) \in \mathbb{S}^{n-1}$ for any $x \in \mathbb{R}^n - \{0\}$.

Finally, if $x \in \mathbb{S}^{n-1}$, then $\|x\| = 1$ so f_t reduces to the identity map. ■

3. (a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

Solution: Suppose that $f_1 : X \to Y$ and $f_2 : Y \to Z$ are homotopy equivalences with homotopy inverses g_1, g_2, respectively. We want to show that $f_2 f_1 : X \to Z$ is a homotopy equivalence. Note that $g_1 g_2 : Z \to X$ for which

$$(f_2 f_1) \circ (g_1 g_2) = f_2 \circ (f_1 g_1) \circ g_2 \simeq f_2 \circ \mathbb{1}_Y \circ g_2 = f_2 g_2 \simeq \mathbb{1}_Z,$$

and

$$(g_1 g_2) \circ (f_2 f_1) = g_1 \circ (g_2 f_2) \circ f_1 \simeq g_1 \circ \mathbb{1}_Z \circ f_1 = g_1 f_1 \simeq \mathbb{1}_X.$$

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

Solution:

This relation is *reflexive* because $f : X \to Y$ is homotopic to itself by the identity homotopy: $F(x,t) = x$.

The relation is *symmetric* because if $f, g : X \to Y$ are homotopic and $f \simeq g$ by a homotopy φ_t, then $g \simeq f$ by the inverse homotopy φ_t^{-1}, defined by $\varphi_t^{-1}(x) = \varphi_{(1-t)}(x)$.

Finally, the relation is *transitive*. To show this, suppose that $F : X \times I \to Y$ is a homotopy connecting f to g and that $G : X \times I \to Y$ is a homotopy connecting g to h, where f, g, and h map X to Y. Now, define the map $H : X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ G(x, 2t-1) & \text{if } \frac{1}{2} < t \le 1 \end{cases}.$$

This map is continuous by construction. The only interesting point is $t = 1/2$ where it is continuous since $F(x,1) = g(x) = G(x,0)$. It follows that $f \simeq h$.

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Solution: Suppose that $f : X \to Y$ is a homotopy equivalence with the homotopy inverse $g : Y \to X$ and that $h \simeq f$ for some $h : X \to Y$. Then, there exists a family $\{f_t : X \to Y\}_{t \in I}$ connecting h to f.

Now, consider the family $\{g \circ f_t : X \to X\}_{t \in I}$. This family connects $g \circ h$ to $g \circ f \simeq \mathbb{1}_X$. By part (b), $h \simeq f$ if and only if $f \simeq h$ so that $h \circ g \simeq f \circ g \simeq \mathbb{1}_Y$. We have therefore shown that $h : X \to Y$ is also a homotopy equivalence.

4. A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}$, $f_1(X) \subseteq A$, and $f_t(A) \subseteq A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Solution: Let $i : A \hookrightarrow X$ be the inclusion and let $f : X \to A$ be the range restriction of $f_1 : X \to X$. We want to show that

$$if \simeq \mathbb{1}_X \quad \text{and} \quad fi \simeq \mathbb{1}_A.$$

Let $\overline{f}_t = f_t|_A$, range restricted to A. This is a homotopy between $\mathbb{1}_A$ and fi.

Since $if = f_1$ and $f_0 = \mathbb{1}$, f_t itself is a homotopy between $\mathbb{1}_X$ and if.

5. Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of $x \in X$ there exists a neighborhood $V \subseteq U$ of x such that the inclusion $V \hookrightarrow U$ is nullhomotopic.

Solution: Since X retracts to a point $x \in X$, there is a homotopy $f : X \times I \to X$ with $f_0 = \mathbb{1}_X$ and $f(x,t) = x$ for all t. By continuity, $f^{-1}(U)$ is an open set for all open sets $U \subseteq X$. In other words, for each t there is a neighborhood $V_t \ni x$ and an open interval $I_t \ni t$ such that $f(V_t, I_t) \subseteq U$.

I is compact so there is a finite set of intervals $I_1, \ldots, I_n$, covering I, with corresponding sets $V_1, \ldots, V_n$ so that $f(V_i \times I_i) \subseteq U$ for $i = 1, \ldots, n$.

Now, set $V = U \cap V_1 \cap \cdots \cap V_n$ and notice that $f(V \times I) \subseteq U$ and $f|_{V \times I}$ is a homotopy from $V \hookrightarrow U$ to a constant map. ■

6. (a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point.

Solution: The existence of a deformation retract from X to any point $(r,0)$ is obvious: Compose the map that shrinks the interval $[0,1] \times \{0\}$ to $(r,0)$ with the map that shrinks the interval $\{r\} \times [0, 1-r]$ to $(r,0)$.

Now, consider an arbitrary point (r,s) with $s > 0$ and hence $r \in \mathbb{Q}$. Suppose X deformation retracts to (r,s). Choose a neighborhood U of (r,s) small enough to be disjoint from the "base," i.e., containing no points for the form $(\overline{r}, 0)$. Let $V \subseteq U$ as in Exercise 5, so $f : V \times I \to U$ is a homotopy from the inclusion map $V \hookrightarrow U$ to the constant map sending all of V to (r,s).

By Exercise 5, if X deformation retracts to (r,s) then $V \hookrightarrow U$ is nullhomotopic. However, this requires V to be path-connected since the map $t \mapsto f_t(x)$ yields a path connecting x to $x_0 = f_1(x)$. On the other hand, the neighborhood V we just picked is clearly not path-connected.

(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure (in the book). Show that Y is contractible but does not deformation retract onto any point.

(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show that there is a deformation retraction in the weak sense of Y onto Z, but no true deformation retraction.