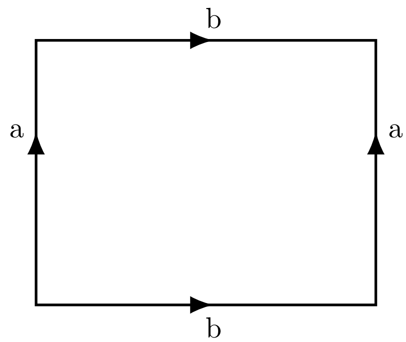
Chapter 0

0.1 Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

The torus $S^1 \times S^1$ can be obtained by identifying opposite sides of a square.



The longitude and meridian circles are precisely the edge a and b.

Let $I = [-1, 1]$ be an interval on $\mathbb{R}^2$. Then I^2 is a square. Without loss of generality, let the origin $(0, 0)$ be the point deleted from the torus. Consider the map f on $I^2 \backslash \{0\}$ defined by $f(x) = \dfrac{x}{|x|}$. f is a retraction onto S^1. Let $g = f|_{\partial I^2}$, then g sends all points on the boundary of the square I^2 onto S^1. Then $g^{-1} \circ f$ is a retraction sending all points on $I^2 \backslash \{0\}$ to ∂I^2.

Define the homotopy H by $H(x, t) = (1 - t)x + t(g^{-1} \circ f)$, then H is the desired deformation retraction of the torus with one point deleted onto ∂I^2, which is a graph of two circles intersecting in a point.

0.2 Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

Let $X = \mathbb{R}^n \backslash \{0\}$. And $i : S^{n-1} \hookrightarrow X$ an inclusion map. Note that $r : X \to S^{n-1}$ defined by $r = \dfrac{x}{|x|}$ is a retraction. Now, define $H : X \times I \to X$ by $H(x, t) = (1 - t)x + t\dfrac{x}{|x|}$ for $0 \leq t \leq 1$, we see that H is a homotopy between the identity map of X and the retraction of X onto S^{n-1}. And it is clear that $i \cdot r = 1_X$ and $r \cdot i = 1_{S^{n-1}}$. Hence H is the deformation retraction of X onto S^{n-1}.

0.3 (a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalent $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.
(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.
(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Proof
(a) Suppose $f : X \to Y$ and $g : Y \to Z$ are homotopy equivalence and f' and g' their inverses.

Want to show that $(g \circ f) \circ (f' \circ g') = 1_Z$ and $(f' \circ g') \circ (g \circ f) = 1_X$.

Lemma: Show that if $f \simeq h$, then $f \circ g \simeq h \circ g$ and $g \circ f \simeq g \circ h$.

Proof of Lemma: Let's prove $f \circ g \simeq h \circ g$. The other one is similar.

Suppose: $f, h : X \to Y$ are homotopic, and $g : Z \to X$. Let $H : X \times I \to Y$ be the homotopy from f to h. Define $F : Z \times I \to Y$ by $F(z,t) = H(g(z),t)$, then $F(z,0) = f \circ g$ and $F(z,1) = h \circ g$. Hence $f \circ g \simeq h \circ g$ as desired. $\square$

Now, by Lemma we have $(g \circ f) \circ (f' \circ g') = g \circ (f \circ f') \circ g' \simeq g \circ 1_Y \circ g' = g \circ g' \simeq 1_Z$. Similarly, $(f' \circ g') \circ (g \circ f) = f' \circ (g' \circ g) \circ f \simeq f' \circ 1_Y f = f' \circ f \simeq 1_X$.

Hence $g \circ f : X \to Z$ is a homotopy equivalence, and the homotopy equivalence is transitive. X is reflexive because X is clearly homotopic equivalent to itself. On the other hand, symmetry is also obvious. Therefore, homotopy equivalence is a equivalence relation.

(b) First, it is clear that $f \simeq f$. Suppose $f \simeq g$, want to show that $g \simeq f$. Let F be a homotopy between f and g, then $G(x,t) = F(x, 1-t)$ is the homotopy from g to f. Now, suppose $f \simeq g$ and $g \simeq h$. Let F be a homotopy between f and g, and H a homotopy between g and h.

Define $K : X \times I \to Y$ by
$$K(x,t) = \begin{cases} F(x,2t) & \text{for } t \in [0, \frac{1}{2}] \\ H(x, 2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$
Then K is well defined since if $t = 1/2$, we have $F(x,2t) = g(x) = H(x, 2t-1)$. Since K is continuous on the two closed subsets $X \times [0,1/2]$ and $X \times [1/2,1]$ of $X \times I$, it is continuous on all of $X \times I$, by the pasting lemma (Munkres's Topology page 108). Thus, K is a homotopy between f and h. Hence the relation of homotopy among maps $X \to Y$ is an equivalence relation.

(c) Let $f : X \to Y$ be a homotopy equivalence and g be its inverse. Let $h \simeq f$. Since $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$, by Lemma in (a) we have $h \circ g \simeq f \circ g \simeq 1_Y$ and $g \circ h \simeq g \circ f \simeq 1_X$. Hence h is a homotopy equivalence.

0.5 Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exits a neighborhood $V \subset U$ of x such that the inclusion $V \hookrightarrow U$ is nulhomotopic.

Proof

If X deformation retracts to a point $x_0 \in X$, by definition there exists a homotopy such that $H : X \times I \to X$ with

$$H(x,0) = 1_X \qquad H(x,1) = c_{x_0} \qquad H(x_0,t) = \{x_0\}$$

Since H is continuous. For an open set U containing x_0, $H^{-1}(U)$ is open in $X \times I$ in product topology and $H^{-1}(U)$ is the union of open sets of the form $W_X \times W_I$ containing $x \times I$. Since I is compact, by Tube Lemma $W_X \times W_I$ contains a tube $V \times I$ about $x \times I$ where V is a neighborhood of x. So the restriction of H on $V \times I$ is a map from $V \times I$ to U.

Let $i : V \hookrightarrow U$ be an inclusion. Then $i \simeq c_{x_0}$ since $H(x, 0) = i$ and $H(x, 1) = c_{x_0}$ for $x \in U$. Therefore, this V is the desired neighborhood of x such that the inclusion $V \hookrightarrow U$ is nulhomotopic.

0.6 (a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0, 1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1 - r]$ for r a rational number in $[0, 1]$. Show that X deformation retracts to any point in the segment $[0, 1] \times \{0\}$, but not to any other point.
(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.
(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show there is a deformation retraction in the weak sense of Y onto Z, but no true deformation retraction.

Proof

(a) Let $H(r, y, t) : X \times Y \times I \to X$ be defined by $H(r, y, t) = (r, (1 - t)y)$ for $y = 1 - r$ and $0 \leq t \leq 1$. H is a homotopy between the identity map to the retraction that is a projection onto its image on the horizontal segment $[0, 1] \times \{0\}$. Let $F(r, 0, s) : X \times I \to X$ be defined by $F(r, 0, s) = ((1 - s)r, sp)$, where p is any point on $[0, 1] \times \{0\}$, then F is a homotopy between the identity map on the line segment $[0, 1] \times \{0\}$ to the a point p. Both H and F are continuous, hence the composition of $F \circ H$ is a homotopy of the deformation retract to any point in $[0, 1] \times \{0\}$.

Now suppose that X deformation retracts to a point, say x, not on the $[0, 1] \times \{0\}$. Let U be a neighborhood of x, by exercise 0.5 there exists a neighborhood $V \subset U$ of x such that the inclusion $V \hookrightarrow U$ is nullhomotopic. However, the image of X in V consists of discrete vertical line segment that cannot be contractible to a point, hence the inclusion $V \hookrightarrow U$ is not nullhomotopic, a contraction. Therefore, X does not deformation retracts to any other point.

(b) I will do this by drawing a graph.

For each pair of adjacent of X's as below, denoted by X_1 and X_2. From (a) we know that each comb X can be deformation retracts onto a point p on $[0, 1] \times \{0\}$. But if there are more than one copies of X as in the figure, then the composition $F \circ H$ constructed in (a) will not be the deformation retraction any longer since the map is not continuous on the connecting part, the segment $[0, 1] \times \{0\}$, of the pair. Instead, we construct another homotopy as follows:

First, divide each comb space X_1 into three parts as below, keep the top part (1) of the fixed, while deformation retracts the third part to a point on the vertex p. This way X_2 will be contracted to the zigzag line. Finally, we proceed as part (a) to contract all X_1 to the zigzag line, since the zigzag line is homeomorphic to the real line $\mathbb{R}$. Hence Y is contractible.

However, Y does not deformation retracts to any point. The reason is, by (a) we know that Y does not deformation retracts to any point other than the zigzag line. By defintion of deformation retract, there is a subspace $A \subset X$ such that there is a family of maps with $f_t : X \to X$, $t \in I$, $f_0 = 1_X$, $f_1(X) = A$ and $f_t(X) = A$ for all t. But there's no such $A \subset X$ exists since no any point on this line stays fixed in the process during the contraction of Y to a point on the zigzag line. Hence Y does not deformation retracts onto any point.

(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Let $A = Z$. Then the contraction constructed in (b) is actually the deformation retraction in the weak sense of Y onto Z. Now suppose there is a true deformation retraction of Y onto Z, but Z is homeomorphic to $\mathbb{R}$ so can be contracted to a point. The composition gives a deformation retracts of Y to a point which is a contradiction by (b). This proves (c).

0.9 Show that a retract of a contractible space is contractible.

Proof

Let X be a contractible space, then $H(x, 0) = 1_X$, for $x \in X$; $H(x, 1) = c_{x_0}$, and $1_X \simeq c_{x_0}$. Assume that X retracts to A, then $r : X \to A$ is a retract such that $r(X) = A$, $r|_A = 1$.

Define $G = r \circ H$, then $G(a, t) = r(H(a, t))$.

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "r"'] & X \arrow[d, "r"] \\ A \times I \arrow[r, "G"] & A \end{tikzcd}$$

Since

$$G(a, 0) = r(H(a, 0)) = r(A) = A$$

so $G_0 = G(A, 0) = 1_A$.

Also

$$G(a, 1) = r(H(a, 1)) = r(x_0) = x_0$$

so $G_1 = G(A, 1) = c_{x_0}$.

Since the composition of continuous maps is continuous. Hence, G is the desired homotopy. i.e. the retract of X is contractible.

0.10 Show that a space X is contractible iff every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \to X$ is nullhomotopic.

Proof

(1) Show that X is contractible iff every map $f : X \to Y$.

" $\Rightarrow$ " If X is contractible, there exits a homotopy $H : X \times I \to X$ with $H(x, 0) = 1_X$; $H(x, 1) = c_{x_0}$. i.e. $1_X \simeq c_{x_0}$.

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[dr, "G"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Define $G = f \circ H$, then we have $G(x, 0) = f \circ 1_X = f$, $G(x, 1) = f \circ c_{x_0} = c_{f(x_0)}$. i.e. $f \simeq c_{f(x_0)}$.

" $\Leftarrow$ " If $f : X \to Y$ is nullhomotopic for any Y. Take $Y = X$, then $f \simeq c_{x_0}$. There exists a homotopy $H : X \times I \to X$ such that $H(x, 0) = f$ and $H(x, 1) = c_{x_0}$. Since f is any map, in particular let $f = 1_X$, then $1_X \simeq c_{x_0}$, hence X is contractible.

(2) Show X is contractible iff every map $f : Y \to X$ is nullhomotopic.

" $\Rightarrow$ " If X is contractible, there exists a homotopy $H : X \times I \to X$ such that $H(x, 0) = 1_X$, $H(x, 1) = c_{x_0}$. Define $G = H \circ f$, then $G(y, 0) = H(f(y), 0) = id_{f(y)}$, and $G(y, 1) = H(f(y), 1) = c_{f(y)} = c_{x_0}$. Hence, $f \simeq c_{x_0}$. i.e $f : Y \to X$ is nullhomotopic.

$$\begin{tikzcd} X \times I \arrow[r, "H"] & X \\ Y \times I \arrow[u, "f"] \arrow[ur, "G"] & \end{tikzcd}$$

" $\Leftarrow$ " If $f : Y \to X$ is nullhomotopic for all Y. Take $Y = X$, then there exists a homotopy such that $H(x, 0) = f$ and $H(x, 1) = c_{x_0}$. This is true for every map f, let $f = 1_X$. Then $1_X \simeq c_{x_0}$, i.e. X is contractible.

0.11 Show that $f : X \to Y$ is a homotopiy equivalence if there exists maps $g, h : Y \to X$ such that $fg \simeq 1$ and $hf \simeq 1$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalence.

Proof

Suppose fg and hf are homotopic equivalence. By definition there exists a map $\alpha : Y \to X$ such that $fg\alpha \simeq 1_Y$ and $\alpha fg \simeq 1_Y$. Also, there exists a map $\beta : X \to Y$ such that $hf\beta \simeq 1_X$ and $\beta hf \simeq 1_X$.

Now, we have $(\beta h)f \simeq 1_X$. It remains to prove that $f(\beta h) \simeq 1_X$. Since

$$f(\beta h) \simeq f(\beta h) \cdot 1_Y \simeq f\beta h \cdot fg\alpha \simeq f(\beta hf)g\alpha \simeq f \cdot 1_X \cdot ga \simeq fg\alpha \simeq 1_Y$$

Hence, we prove that (βh) is the homotopy inverse of f. Therefore, f is a homotopy equivalence.

0.12 Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f retricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Prove also that the corresponding statement with components instead of path-components. Dedece from this that if the components and path-components of a space coincide, then the same is true for any homotopy equivalenct space.

Proof

Suppose $f : X \to Y$ is a homotopy equivalence. There exists continuous map g such that $fg \simeq 1_Y$ and $gf \simeq 1_X$. Let H be the homotopy sending gf to 1_X, so

$$H(x, 0) = gf \qquad H(x, 1) = 1_X$$

For each $x \in X$, the homotopy H is a path between $x \in X$ and $gf(x) \in X$. So x and $gf(x)$ are in the same path component. Since the continuous image of a path connected space is path connected. Hence, f induces a map on path components.

Now, since the map induced by gf on path components is an isomorphism, the map induced by f is injective. Similarly, the map induced by fg on path components is an isomorphism, so the map induced by f is surjective. Therefore, the map induced by f is a bijection.

Given the above argument, let A be a path component of X and B the corresponding path component of Y. The homotopy H restricted on A is a homotopy equivalence.

Since the continuous image of connected space is connected. f induces a map on components, hence the same argument applies. Therefore, if the components and path-components of a space coincide, then the same is true for any homotopy equivalent space.

0.14 Given positive integers v, e and f satisfying $v - e + f = 2$, construct a cell structure on S^2 having v 0-cell, e 1-cell, and f 2-cell.

Suppose $v = 1$. Then $f = 1 + e$. Attach the two ends of all 1-cells to this only 0-cell, like a bouquet of circles. Then fill in each circle with 2-cells. Then Attach the last 2-cell to the boundary of the bouquet of circles.

Suppose $v = 2$. Then $e = f$. Use all 1-cells to connect the two 0-cells, making it as lens shape. Then fill in 2-cells in each of the lens space.

Suppose $v > 2$. Then $v = f + e - 2$. Take two 0-cells and f of 1-cells and do as in previous case to a lens shape. Use f-1 of 2-cells to fill in the lens. Then take the last 2-cell to attach the boundary of the outside of the lens.

0.16 Show that S^∞ is contractible.

Proof

Consider a shifting map $f : S^\infty \to S^\infty$ defined by $f(x_1, x_2 \cdots) = (0, x_1, x_2, \cdots)$. Let $H : S^\infty \times I \to S^\infty$ be define by

$$H(x_1, x_2, \cdots, t) = \frac{(\cos(\frac{\pi}{2}t)x_1, \sin(\frac{\pi}{2}t)x_1 + \cos(\frac{\pi}{2}t)x_2, \sin(\frac{\pi}{2}t)x_2 + \cos(\frac{\pi}{2}t)x_3, \cdots)}{\sqrt{[(\cos(\frac{\pi}{2}t)x_1]^2 + [\sin(\frac{\pi}{2}t)x_1 + \cos(\frac{\pi}{2}t)x_2]^2 + [\sin(\frac{\pi}{2}t)x_2 + \cos(\frac{\pi}{2}t)x_3)]^2 + \cdots}}$$

$0 \leq t \leq 1$

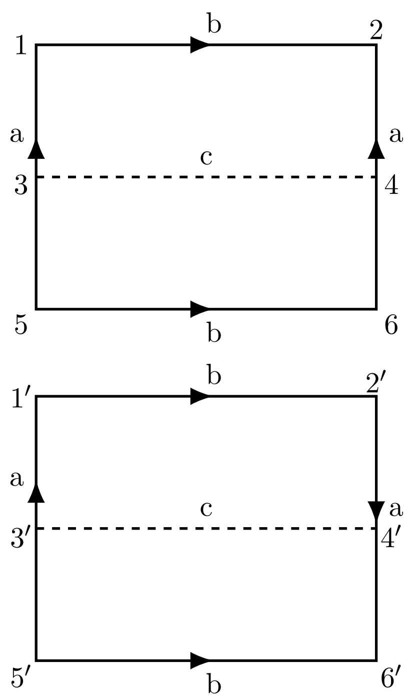
H is well defined since the domain is on S^∞ so is never zero. H is clearly continuous and $H_0 = 1_{S^\infty}$, $H_1 = f$, hence H is a homotopy between the identity map 1_{S^∞} and the shifting map f.

Now, define a map $F(x, t) : S^\infty \times I \to S^\infty$ by $F(x, t) = \cos(\frac{\pi}{2}t)f + \sin(\frac{\pi}{2}t)e_1$ where $e_1 = (1, 0, 0, \cdots)$. Then F is homotopy between f and the constant map c_{e_1}. Hence, the composition $F \circ H$ is a homotopy between f and c_{e_1}. Therefore, S^∞ is contractible.

(Or see Example 1B.3)

0.17 Construct a 2-dimensional cell complex that contains both an annulus $S^1 \times I$ and a Mobius band as deformation retracts.

Construct a 2-dimensional cell complex as follows:

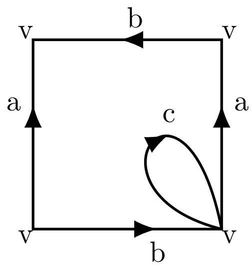


For 6 0-cells marked $1, 2, \cdots, 6$ above, connect 1 and 2, 1 and 3, 3 and 5, 2 and 4, 4 and 6, 3 and 4, 5 and 6 with 1-cells. Then attach 2-cells on top and bottom of the square. Now, identify the two 1-cells: 3 and 4, $3'$ and $4'$. The quotient space of the top is $S^1 \times I$ while the quotient space of bottom is Mobius band. Note that in both spaces, c and c' are deformation retracts of whole space. i.e. $S^1 \times I$ and M can be retracted to that 1-cell.

0.20 Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

Proof

Note that the self-intersecting Klein bottle can be given a CW structure with one 0-cell, three 1-cells and one 2-cell as follows:



Note that Klein bottle self-intersects on a small closed disk. By Contracting the the disk to a point, we see that it is now a 2-sphere with two distinct points identified. In example 0.8 in Hatcher's, this is homotopy equivalent to $S^2 \vee S^1$. But the boundary circle of the closed disk forms another S^1 with one point identified with those two distinct points on S^2. Hence the Klein bottle intersecting itself in a circle is homotopy equivalent to $S^2 \vee S^1 \vee S^1$.

0.23 Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

Proof

Let X, Y be the two contractible subcomplexes, $A = X \cap Y$, then A is contractible by assumption. Since (X, A) is a CW pair and A is contractible. By Proposition 0.16, (X, A) has the homotopy extension property, then by Proposition 0.17, $X \to X/A$ is a homotopy equivalence. i.e. $X \simeq X/A$. Similarly, $Y \simeq Y/A$. Now, since $(X \cup Y, A)$ is also a CW pair. By Proposition 0.16 and 0.17 again, $X \cup Y \simeq (X \cup Y)/A$.

But $(X \cup Y)/A$ is homeomorphic to $X/A \vee Y/A$. To see this, define a map $f : X/A \vee Y/A \to (X \cup Y)/A$ by $f(x) = x$ for $x \in X/A$ and $f(y) = y$ for $y \in Y/A$ and maps $a \in A$ to a point. Clearly f is well-defined and continuous. And $g : X \cup Y \to X/A \vee Y/A$ sends $x \in X$ to its image in X/A and $y \in Y$ to its image in Y/A. So g is well-defined and continuous. Hence we have

$$X \cup Y \simeq (X \cup Y)/A \cong X/A \vee Y/A$$

Finally, since X/A and Y/A are contractible. Let $H_1(x,t) : X/A \times I \to X/A$ and $H_2(y,t) : Y/A \times I \to Y/A$ be the homotopies respectively that send the identity maps to constant maps. i.e. $1_{X/A} \simeq c_{x_0}$ and $1_{Y/A} \simeq c_{y_0}$. Now, define a map $F : (X/A \vee Y/A) \times I \to X/A \vee Y/A$ by $F(z,t) = H_1(z,t)$ for $z \in X/A$ and $F(z,t) = H_2(z,t)$ for $z \in Y/A$, then F is the desired homotopy between the identity map on $X/A \vee Y/A$ and a constant map. Therefore $X \cup Y$ is contractible.

Chapter 1

1.1.1 Show that composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.

Proof

Note that path-homotopy classes satisfy properties of associativity, existence of right and left identities and inverse. Let e denote the identity and $\bar{f_i}$ and $\bar{g_i}$ denote the inverse of f_i and g_i respectively for $i = 0, 1$.

Lemma If $g_0 \simeq q_1$ then $\bar{g_0} \simeq \bar{g_1}$.

Proof

Since the inverse of g_i is defined by $\bar{g_i}(s) = g_i(1-s)$. Let $F : I \times I \to X$ be a path homotopy between g_0 and g_1 such that $F_0(s) = g_0(s)$ and $F_1(s) = g_1(s)$. Now, define $\bar{F} : I \times I \to X$ by $\bar{F}(t,s) = F(t, 1-s)$. Then $\bar{F_0}(s) = F_0(1-s) = g_0(1-s) = \bar{g_0}(s)$ and $\bar{F_1}(s) = F_1(1-s) = g_1(1-s) = \bar{g_1}(s)$. Hence $\bar{F}$ is the homotopy between $\bar{g_0}$ and $\bar{g_1}$. $\square$

By the properties of path-homotopy classes and above Lemma we have

$$f_0 \simeq f_0 \cdot e \simeq f_0 \cdot (g_0 \cdot \bar{g_0}) \simeq (f_0 \cdot g_0) \cdot \bar{g_0} \simeq (f_1 \cdot g_1) \cdot \bar{g_1} = f_1 \cdot (g_1 \cdot \bar{g_1}) \simeq f_1 \cdot e \simeq f_1$$

as desired.

1.1.3. For a path-connected space X, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphism β_h depend only on the endpoints of the path h.

Proof

Let $x_0, x_1 \in X$. Let $[f] \in \pi_1(X, x_1)$ be a loop and h_1, h_2 be two paths from x_0 to x_1. Want to prove that if $\pi_1(X)$ is abelian, then $\beta_{h_1} = \beta_{h_2}$. By definition, $\beta_h[f] = [h \cdot f \cdot \bar{h}]$. We have

$$\begin{aligned}
\beta_{h_1}[f] &= [h_1 \cdot f \cdot \bar{h_1}] \\
&= [h_1 \cdot f \cdot \bar{h_2} \cdot h_2 \cdot \bar{h_1}] \\
&= [h_1 \cdot f \cdot h_2] \cdot [h_2 \cdot \bar{h_1}] \\
&= [h_2 \cdot \bar{h_1}] \cdot [h_1 \cdot f \cdot h_2] \\
&= [h_2 \cdot \bar{h_1} \cdot h_1 \cdot f \cdot h_2] \\
&= [h_2 \cdot f \cdot h_2] \\
&= \beta_{h_2}[f]
\end{aligned}$$

Conversely, let $[f], [g] \in \pi_1(X, x)$ and let e_x denote the constant map at x. And note that $\beta_{e_x} = \beta_g$ by assumption that all basepoint-change homomorphism β depend only on the endpoints of the path, hence we have

$$[f][g] = [f \cdot g] = [e_x \cdot f \cdot g \cdot \bar{e_x}] = \beta_{e_x}[fg] = \beta_g[fg] = [g \cdot f \cdot g \cdot \bar{g}] = [g \cdot f] = [g][f]$$

Therefore, $\pi_1(X)$ is abelian.

1.1.5 Show that for a space X, the following three conditions are equivalent:
(a) Every $S^1 \to X$ is homotopic to a constant map, with image a point.
(b) Every $S^1 \to X$ extends to a map $D^2 \to X$.
(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic.

Proof

(a) $\Rightarrow$ (b)

Let $f : S^1 \to X$ be a map. Let $F : S^1 \times I \to X$ be a homotopy between f and a constant map c_{x_0} for some $x_0 \in S^1$. i.e. $F_0 = f$ and $F_1 = c_{x_0}$. Since (D^2, S^1) is a CW pair, by Proposition 0.16 (D^2, S^1) has a homotopy extension property $F : S^1 \times I \to X$ can be extended to $\bar{F} : D^2 \times I \to X$ such that $\bar{F}|_{S^1 \times I} = F$. The restriction of $\bar{F}$ on $D^2 \times \{0\}$ is the desired map $D^2 \to X$.

$$\begin{tikzcd} S^1 \times I \arrow[r, "F"] \arrow[d, "\text{H.E.P.}"'] & X \\ D^2 \times I \arrow[ur, "\bar{F}"'] & \end{tikzcd}$$

(b) $\Rightarrow$ (c)

For $[f] \in \pi_1(X, x_0)$. $[f]$ is a set of homotopy classes of loops $f : I \to X$ based at $x_0 \in X$. By definition, we want to show that f is homotopic to the constant loop, hence $\pi_1(X, x_0) = 0$. Since the loop is a path with endpoints identifies, i.e. $f(0) = f(1) = x_0$. So f can be regarded as a map $S^1 \to X$ with the based point of S^1 mapped to x_0. Let $i : S^1 \hookrightarrow D^2$ be an inclusion. By assumption, f can be extended to a map $\bar{f} : D^2 \to X$. Since D^2 is contractible, it deformation retracts onto a point s_0 on S^1. So there is a homotopy $\bar{F} : D^2 \times I \to X$ from identity map of D^2 rel x_0 to a constant map c_{x_0}. Hence, $F = \bar{F} \circ (i \times Id) : S^1 \times I \to X$ is the desired homotopy between $F(x, 0) = f$ and a constant map $F(x, 1) = c_{x_0}$. Hence $\pi_1(X, x_0) = 0$.

$$\begin{tikzcd} S^1 \times I \arrow[r] \arrow[d, "i \times Id"'] & X \\ D^2 \times I \arrow[ur, "\bar{F}"'] & \end{tikzcd}$$

(c) $\Rightarrow$ (a)

Let $[f] \in \pi_1(X, x_0)$ be the set of homotopy classes of loops, then f can be seen as a map $S^1 \to X$. Suppose $\pi_1(X, x_0) = 0$, then f is homotopic to a constant map with image a point.

Finally, we deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic as follows: suppose X is simply-connected, then X is path-connected, hence has trivial fundamental groups. Then every loop $f : S^1 \to X$ is homotopic to a constant loop. Since X is path-connected, all maps $S^1 \to X$ are homotopic.

Conversely, suppose all maps $S^1 \to X$ are homotopic, this implies that all loop are homotopic. Constant map loop is one of them, hence all loops are homotopic to a constant loop. i.e. the fundamental group $\pi_1(X, x_0)$ is a trivial group.

1.1.6. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

Proof

If X is path-connected. For $[f] \in [S^1, X]$, $f : S^1 \to X$ is a map. Let h be a path from $f(s_0)$ to the based point $x_0 \in X$. Then $h : I \to X$ is a homotopy on s_0. Since (S^1, s_0) is a CW pair. By Homotopy Extension Property, h can be extended to $\bar{H} : S^1 \times I \to X$ with $\bar{H}|_{s_0 \times I} = h$. And $\bar{H}(s, 0) = f$ and $\bar{H}(s, 1)$ is some map g such that $g(s_0) = x_0$. Thus, $\Phi(\bar{H}|_{S^1 \times \{1\}}) = [f]$. Hence Φ is onto.

$$\begin{tikzcd}
S^1 \times \{0\} \cup s_0 \times I \arrow[r] \arrow[d, "i"'] & X \\
S^1 \times I \arrow[ur, "\bar{H}"'] &
\end{tikzcd}$$

$\Leftarrow$

Prove that Φ maps the conjugacy classes to homotopy classes.

If $[f]$, $[g] \in \pi_1(X, x_0)$ are conjugate, $f = h \circ g \circ \bar{h}$ for some $[h] \in \pi_1(X, x_0)$ want to prove that g and $h \circ g \circ \bar{h}$ are homotopic.

Fix t, we define

$$H(t,s)= \begin{cases} h(3t+s) & \text{if } \le t \le \dfrac{1-s}{3} \\ g\left(\dfrac{3}{1+2s}(t - \dfrac{1-s}{3})\right) & \text{if } \dfrac{1-s}{3} \le t \le \dfrac{2+s}{3} \\ \bar{h}\left(3(t - \dfrac{2+s}{3})\right) & \text{if } \dfrac{2+s}{3} \le t \end{cases}$$

This is the desired homotopy. Since $H(0, s) = H(1, s)$ for all s, but $H(0, s)$ are moving along h. And $H(t, 1) = g$. Hence g and $h \circ g \circ \bar{h}$ are homotopic.

$\Rightarrow$

Suppose $\Phi([f]) = \Phi([g])$, prove that $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

Let H be the homotopy between $[f]$ and $[g]$ as in (1), $f, g \in \pi_1(X, x_0)$, ignoring basepoints. Let h be $H(0, s)$, a path moving along the starting points of the loops for all s. Then h is a loop. Now, prove f is homotopic to $h^{-1}gh$.

Define a homotopy K as follows:

$$K(t,s)=\begin{cases} h(t) & \text{if } 0 \le t \le \dfrac{s}{3} \\ H\left(\dfrac{3}{3-2s}(t-\dfrac{s}{3}), s\right) & \text{if } \dfrac{s}{3} \le t \le \dfrac{3-s}{3} \\ h^{-1}(t) & \text{if } \dfrac{3-s}{3} \le t \end{cases}$$

So we have $K(t,0) = f(t)$ and

$$K(t,1)=\begin{cases} h(t) & \text{for } 0 \le t \le \dfrac{s}{3} \\ g(3(t-\dfrac{1}{3})) & \text{for } \dfrac{s}{3} \le t \le \dfrac{2}{3}s \\ h^{-1}(t) & \text{for } \dfrac{2}{3}s \le t \end{cases}$$

Hence $[f]$ and $[g]$ are conjugate. Therefore Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$ when X is path-connected.

1.1.8 Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?

Proof

Borsuk-Ulam Theorem For every continuous map $f : S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

Let a torus $S^1 \times S^1$ be embedded in $\mathbb{R}^3$ with the wedge point on the origin $(0, 0, 0)$ as in figure below. Let $f : S^1 \times S^1 \to \mathbb{R}^2$ be a projection onto the $x-y$ plane. We see that $f(x) = -f(-x)$. We claim that there does not exist a pair of antipodal points $x \in S^1 \times S^2$ with $f(x) = f(-x)$. Suppose that there is a pair of antipodal point x and $-x$ in $S^1 \times S^1$ with the above property. Then $-f(x) = f(x)$ which implies that $f(x) = 0$. But $f(x)$ is never 0. Therefore, the Borsuk-Ulam Theorem does not hold for the torus.

1.1.12 Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

Proof

Let $f : \pi_1(S^1) \to \pi_1(S^1)$ be a homomorphism. Since $\pi_1(S) \cong \mathbb{Z}$, the image of the homomorphism $f : \mathbb{Z} \to \mathbb{Z}$ is completely determined by where the generator 1 goes. Suppose $f : 1 \mapsto k$ for some $k \in \mathbb{Z}$. Let $\omega : I \to S^1$ be a loop defined by $\omega(t) = e^{2\pi it}$. Since f sends loops to loops, this corresponds to a map $[\omega_1] \mapsto [\omega_k]$ where $[\omega_1]$ is a loop class with basepoint at 1 and $[\omega_k]$ is a loop class with k-times faster. So there is a degree map $\varphi : S^1 \to S^1$ defined by $\varphi(e^{2\pi it}) = e^{2\pi ikt}$ for $t \in [0, 1]$ and φ has an induced map defined by $\varphi_*([\omega]) = [\varphi \circ \omega]$ for a loop $\omega : I \to S^1$ with $\omega(t) = e^{2\pi it}$. Now, it remains to show that $\varphi_* \equiv f$. This is clear since $\varphi_*([\omega]) = [\varphi \cdot \omega] = [\varphi(e^{2\pi it})] = [e^{2\pi ikt}] = [\omega_k] = f([\omega_1])$.

1.1.13 Given a space X and a path-connected subspace A containing the base point x_0, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in X with endpoint in A is homotopic to a path in A.

Proof

If every path in X with endpoints in A is homotopic to a path in A, then in particular every loop in X with based point x_0 in A is homotopic to a loop in A. This implies that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective.

Conversely, let α be a path in X with endpoints x_1 and x_2 in A. Since A is path-connected, there are paths β_1 and β_2 connecting the base point x_0 with x_1 and x_2 respectively. So the composition $\beta_1 \cdot \alpha \cdot \bar{\beta}_2$ is a loop in X with basepoint x_0. Since the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective. There is a corresponding loop γ with basepoint x_0 lying entirely in A and is homotopic to the loop $\beta_1 \cdot \alpha \cdot \bar{\beta}_2$. i.e. $\gamma \simeq \beta_1 \cdot \alpha \cdot \bar{\beta}_2$. Equivalently, we have $\bar{\beta}_1 \cdot \gamma \cdot \beta_2 \simeq \alpha$. Since the paths γ, $\bar{\beta}_1$ and β_2 all lie in A, hence $\beta_2 \cdot \gamma \cdot \bar{\beta}_1$ lies in A. This proves that α is homotopic to a path in A.

1.1.16 Show that there are no retractions $r : X \to A$ in the following cases:
(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
(c) $X = S^1 \times D^2$ with A the circle shown in the figure.
(d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.
(e) X a disk with two points on its boundary identified and A its boundary $S^1 \times S^1$.
(f) X the Mobius band and A its boundary circle.

Proof

Lemma If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $A \hookrightarrow X$ is injective.

Proof of Lemma

Let $r : X \to A$ be a retraction, then the composition map $r \circ j = 1_A$. It follows that $r_* \circ j_*$ is the identity map of $\pi_1(A, a)$, so j_* must be injective. $\square$

(a) Suppose $r : \mathbb{R}^3 \to A$ is a retraction. By Lemma the homomorphism induced by inclusion $j : A \hookrightarrow \mathbb{R}^3$ is injective. But the $\pi_1(A) \cong \pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(\mathbb{R}^3) = 0$, a contradiction.

(b) Suppose $r : S^1 \times D^2 \to S^1 \times S^1$ is a retraction. The homomorphism induced by inclusion $j : S^1 \times S^1 \hookrightarrow S^1 \times D^2$ is injective. Proposition 1.12 says that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ if X and Y are path-connected. So we have $j_* : \pi_1(S^1) \times \pi_1(S^1) \to \pi_1(S^1) \times \pi_1(D^2)$. Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2) = 0$, this implies that $j_* : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is injective, a contradiction.

(c) If A is a retract of X, then the induced homomorphism on the inclusion $j : A \hookrightarrow X$ is injective. But A is nullhomotopic (the linked loops can be resolved in X during the homotopy), the induced map $\pi_1(A) \to \pi_1(X)$ is trivial, not injective. So this is a contradiction.

(d) Suppose $r : D^2 \vee D^2 \to S^1 \vee S^1$ is a retraction. The homomorphism induced by inclusion $j : S^1 \times S^1 \hookrightarrow D^2 \times D^2$ is injective. Since $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$ and $\pi_1(D^2 \vee D^2) = 0$ by exercise 0.23 saying that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible. So we have an injective induced homomorphism $j_* : \pi_1(S^1) \times \pi_1(S^1) \to \pi_1(D^2) \times \pi_1(D^2)$ which is $j_* : \mathbb{Z} * \mathbb{Z} \to 0$, a contradiction.

(e) First note that the space X is homotopic to a circle. This is because if we identify two points on the boundary of a disk, say a and b. Since the disk is contractible, it deformation retracts to the arc connecting a and b. Now Suppose $r : X \to S^1 \vee S^1$ is a retraction. The homomorphism induced by inclusion $j : S^1 \vee S^1 \hookrightarrow X$ is injective. However, $\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}$ and $\pi_1(X) = \mathbb{Z}$, a contradiction.

(f) Since the Mobius band deformation retracts to its center circle. It's fundamental group is thus isomorphic to $\mathbb{Z}$. But it's boundary circle is a circle that winds around twice. Suppose $f : M \to A$ is a retraction. The homomorphism induced by inclusion $j : A \to M$ is injective. But this implies that $j_* : \mathbb{Z} \to 2\mathbb{Z}$ is injective, a contradiction.

1.1.17 Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$.

Proof

For the wedge of two circles $S^1 \vee S^1$, denote them by A and B respectively. Without loss of generality, we retract the circle B to the circle A. i.e. Want to construct a retraction $r : S^1 \vee A \to B$ onto B such that $r(A \vee B) = A$ and $r|_A = 1_A$.

In the Theorem 1.7, we prove $\pi_1(S^1) \cong \mathbb{Z}$ by sending an integer n to the homotopy class of loop $\omega_n(S) = (\cos 2\pi ns, \sin 2\pi ns)$ based at $(1, 0)$. Now, consider the circle A and B on the complex plane. For each n, define a map

$$f_n(z) = \begin{cases} z^n & \text{for } z \in \{1\} \vee B \\ z & \text{for } z \in A \vee \{1\} \end{cases}$$

f_n is indeed a retraction since $f_n(A \vee B) = A$ and $f_n(z) = z^n$. This maps will wind the circle B around A n times. For $m, n \in \mathbb{Z}$, $m \neq n$ if and only $f(m)$ and $f(n)$ are not homotopic.
Therefore, we have constructed infinitely many nonhomotopic retractions f_n for $n \in \mathbb{Z}$ from $S^1 \vee S^1$ onto S^1.

1.2.3 Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

Proof

We prove by induction on the set of finite points $\{x_1, \cdots, x_m\}$ in $\mathbb{R}^n$.

If $m = 1$, $\mathbb{R}^n \backslash \{x_1\} \cong S^{n-1}$ is simply connected.
If $m = 2$, $\mathbb{R}^n \backslash \{x_1, x_2\}$ is homeomorphic to the wedge of two spheres $S^{n-1} \vee S^{n-1}$, hence is simply connected.

Suppose it is true for k points. Let A be the neighborhood of x_{k+1} and A' be a closed ball in A containing x_1. Let $B = \mathbb{R}^n - A'$, then B is a neighborhood of k points in $\mathbb{R}^n$. Since A, B are simply connected, $A \cap B$ is path connected and $\mathbb{R}^n = A \cup B$. By Van Kampen, $R^n - \{x_1, \cdots, x_{k+1}\}$ is simply connected.

By induction, $\mathbb{R}^n - \{x_1, \cdots, x_m\}$ is simply connected.

1.2.4 Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

Proof

Since $\mathbb{R}^3 - \{0\}$ deformation retracts to S^2 by Exercise 0.2. By restriction of the deformation retraction, $\mathbb{R}^3 - X$ deformation retracts onto $S^2 - (X \cap S^2)$, the latter is precisely $S^2 - \{2n \text{ points}\}$. By Proposition 1.17, $\pi_1(\mathbb{R}^3 - X) \cong \pi_1(S^2 - \{2n \text{ points}\})$.

Note that $S^n - \{x_0\} \cong \mathbb{R}^n$, so $S^2 - \{2n \text{ points}\} \cong \mathbb{R}^2 - \{2n - 1 \text{ points}\}$. Note that $S^2 - \{2n \text{ points}\}$ deformation retracts onto, hence is homotopic equivalent to the wedge of $2n - 1$ circles. Hence

$$\pi_1(\mathbb{R}^3 - X) \cong \pi_1(S^2 - \{2n \text{ points}\}) \cong \pi_1(S^1 \vee S^1 \vee \cdots \vee S^1) \cong \mathbb{Z} \times \mathbb{Z} \cdots \times \mathbb{Z}$$

free group of $2n - 1$ generators.

1.2.5 Let $X \subset \mathbb{R}^2$ be a connected graph that is the union of a finite number of straight line segments. Show that $\pi_1(X)$ is free with a basis consisting of loops formed by the boundaries of the bounded complementary regions of X, joined to a basepoint by suitable chosen paths in X.

Proof

We will proceed the proof by inductively looking at the graph X starting from a point, say $x_0 \in X$. In each step we add a line segment to x_0 and denote these added line segments, part of the graph, by Y. If we see a line segment L attached to Y with only one of its two vertices, the fundamental group is unchanged since the line segment is contractible. If L has both vertices attached to Y. Let h_1 and h_2 be the two line segments in Y connecting to L. Then this is a closed curve polygonal curve since we assume X is a connected graph. Now we want to use Van Kampan's Theorem to prove that this is precisely "joining" a cyclic group to the fundamental group of Y. Note that we do this under the assumption of Jordan Curve Theorem for polygonal simple closed curves.

Let $A = Y$ and B the new line segment L together with h_1 and h_2, i.e. $B = L \cup \{h_1, h_2\}$. Then $A \cup B = Y \cup L$ and $A \cap B = h_1 \cup h_2$ which is contractible hence simply connected. By Van Kampen, $\pi_1(Y \cup L) = \pi_1(A \cup B) = \pi_1(A) * \pi_1(B) = \pi_1(A) * \mathbb{Z}$. Hence we "joined" one cyclic group into the fundamental group of Y. Inductively we see that $\pi_1(X)$ is free with a basis consisting of loops.

1.2.6 Suppose a space Y is obtained from a path-connected subspace X by attaching n-cells for a fixed $n \geq 3$. Show that the inclusion $X \hookrightarrow Y$ induces an isomorphism on π_1. Apply this to show that the complement of a discrete subspace of $\mathbb{R}^n$ is

simply-connected if $n \geq 3$.

Proof

We proceed the proof by the construction in the proof of Proposition 1.26. Suppose we attach n-cells e^n_α to a path-connected space X via maps $\varphi_\alpha : S^{n-1} \to X$, producing a space Y. Let s_0 be a basepoint of S^{n-1}, then φ_α determines a loop at $\varphi_\alpha(s_0)$. For different α's the basepoints $\varphi_\alpha(s_0)$ of these loops φ_α may not all coincide. So we choose a basepoint $x_0 \in X$ and a path γ_α in X from x_0 to $\varphi_\alpha(s_0)$ for each α. Then $\gamma_\alpha\varphi_\alpha\bar{\gamma}_\alpha$ is a loop at x_0.

Now, let us expand Y to a larger space Z, as in the proof of Proposition 1.26, that deformation retracts onto Y. The space Z is obtained from Y by attaching rectangular strips $S_\alpha = I \times I$, with the lower edge $I \times \{0\}$ attached along γ_α, the right edge $\{1\} \times I$ attached along an arc in e^n_α, and all the left edges $\{0\} \times I$ of the different strips identified together.

In each cell e^n_α choose a point y_α not in the arc along which S_α is attached. Let $A = Z - \bigcup_\alpha \{y_\alpha\}$ and let $B = Z - X$. Then A deformation retracts onto X, and B is contractible. Since $\pi_1(B) = 0$, by Van Kampen Theorem $\pi_1(Z)$ is isomorphic to the quotient of $\pi_1(A)$ by the normal subgroup generated by the image of the map $\pi_1(A \cap B) \to \pi_1(A)$.

It remains to show that $\pi_1(A \cap B)$ is trivial. Note that $\pi_1(A \cap B)$ is generated by the loops $\gamma_\alpha\varphi_\alpha\bar{\gamma}_\alpha$. This can be proved by another Van Kampen Theorem by letting $A_\alpha = A \cap B - \cup_{\beta \neq \alpha} e^n_\beta$. Since A_α deformation retracts onto a S^{n-1} on $e^n_\alpha - \{y_\alpha\}$, we have $\pi_1(A_\alpha) \cong \pi_1(S^{n-1}) = 0$ since $n \geq 3$. Hence $\pi_1(A \cap B) = 0$ and $\pi_1(Z) \cong \pi_1(X) \cong \pi_1(A)$.

Finally, the complement of a discrete subspace of $\mathbb{R}^n$ for $n \geq 3$ deformation retracts to a wedge sum of S^{n-1} spheres. But the fundamental group of S^n is 0 for $n \geq 2$. Hence it has fundamental group 0, therefore, is simply connected. (Or, we should use above conclusion: construct a space Y obtained from $\mathbb{R}^n$ by attaching n-cells for $n \geq 3$. Then every loop in Y is nulhomotopic. Hence, $\pi_1(X) \cong \pi_1(Y) = 0$).

1.2.7 Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

Refer Example 0.8 on page 11. Let 0-cell be the image of the quotient of N and S. Let arc B connecting the north pole N and the south pole S be the 1-cell, and the rest of S^2 2-cell. That is, we attach the 2-cell to 1-cell by the map

$$\varphi : S^1 = \partial D^2 \to B$$

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr] & \\
\pi_1(U \cap V) \arrow[ur, "\varphi_1"] \arrow[rr] \arrow[dr, "\varphi_2"'] & & \pi_1(X) \\
 & \pi_1(V) \arrow[ur] &
\end{tikzcd}$$

Let U be the arc B together with its neighborhood on S^2, and V be $S^2 \backslash B$ or the interior of D^2. Then $\pi_1(U) = \mathbb{Z}$ and let α be the generator. It is clear that V is simply connected so $\pi_1(V) = 0$ and $U \cap V$ is path connected. Let $\gamma \in \pi_1(U \cap V)$ be a generator, then $\varphi_1(\gamma) = \alpha\alpha^{-1} = 0$. By Van Kampen,

$$\pi_1(X) = \frac{\pi_1(U) * \pi_1(V)}{N} = \frac{\mathbb{Z} * 0}{0} = \mathbb{Z}$$

1.2.8 Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

Let X be the surface, the identification of two tori $S^1 \times S^1$ as described in the exercise. And we know the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$. Let's assume the two tori T_1, T_2 are identified by "stacking" one on the other. i.e. If a, b and c, d are the generators of the fundamental groups respectively. And a and c are their longitudes. The way we stack the two tori will make a and c identified. To use the Van Kampen's Theorem, let A be the top torus T_1 together with a strip of open neighborhood of a on itself and on the bottom torus T_2. Similarly, let B the bottom torus T_2 together with a strip of open neighborhood of c on itself and the top one T_1. Then A and B are open subset of X and $A \cap B$ is open and path connected. Since A and B deformation retracts to T_1 and T_2 respectively, so $\pi_1(A) = \pi_1(B) = \mathbb{Z} \times \mathbb{Z}$. Since $A \cap B$ deformation retracts to a circle, we have $\pi_1(A \cap B) \cong \mathbb{Z}$, the generator has its image a, c in A, B respectively.

$$\begin{tikzcd}
 & \pi_1(A) \arrow[dr] & \\
\pi_1(A \cap B) \arrow[ur, "\varphi_1"] \arrow[rr] \arrow[dr, "\varphi_2"'] & & \pi_1(X) \\
 & \pi_1(B) \arrow[ur] &
\end{tikzcd}$$

By Van Kampen, $\pi_1(X)$ is isomorphic to the quotient of $\pi_1(A) * \pi_1(B)$ by the normal subgroup generated by $\langle ac^{-1} \rangle$.

$$\pi_1(X) \cong \frac{(\mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z})}{\langle ac^{-1} \rangle} \cong (\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}$$

1.2.9 In the surface M_g of genus g, let C be a circle that separates M_g into two compact subspaces M'_h and M'_k obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M'_h does not retract onto its boundary circle C, and hence M_g does not retract onto C. But show that M_g does retract onto the nonseparating circle C' in the figure.

Proof

Suppose M'_h retracts to C. Let $r : M'_h \to C$ be a retraction and $i : C \hookrightarrow M'_h$ an inclusion. Then $r \circ i = 1_C$ hence $r_* \circ i_* = 1_C*$. This implies that r_* is surjective.

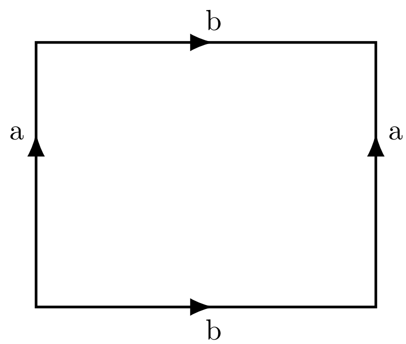
$$\pi_1(C) \xrightarrow{i_*} \pi_1(M'_h) \xrightarrow{r_*} \pi_1(C)$$

Take the abelianization on the maps,

$$\pi_1(C) \xrightarrow{i_*^{ab}} ab(\pi_1(M'_h)) \xrightarrow{r_*} \pi_1(C)$$

Abelianization is a functor that preserves the injectivity and surjectivity. Since the generator of C deformation to the boundary of the $2h$-gon, the map i_*^{ab} sends a generator of $\pi_1(C) \cong \mathbb{Z}$ to the commutator $[a_1, b_1][a_2, b_2] \cdots [a_h, b_h]$. After abelianization, the commutators vanish, so the image of i_*^{ab} becomes trivial, which is a contradiction to the injectivity of the map i_*^{ab}. Therefore M'_h does not retracts to C hence the M_g does not retract to C.

But M_g does retracts onto the nonseparating circle C' as in the figure in the text. We know that a torus can retracts to its meridian, say a, as follows:



M_g of genus g can therefore retracts to a if we extend the retraction to the entire polygon and respect their orientation. This can be seen from the figure below: we can retract M_g onto C' by collapsing all other $2g - 1$ circles to a.

1.2.10 Consider two arcs α and β embedded in $D^2 \times I$ as shown in the figure. The loop γ is obviously nullhomotopic in $D^2 \times I$, but that there is no null-homotopy of γ in the complement of $\alpha \cup \beta$.

Proof

Let X denote the complement of α and β in $D^2 \times I$. Since the two arcs α, β can deformation retract to two parallel lines. We see that X is homeomorphic to a disk minus two distinct points, i.e. $X \cong D^2 - \{a, b\}$. The loop γ is just the boundary of the disk. And X deformation retracts to wedge of two circles. Hence the fundamental group of X is free on two generators, i.e. $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$. Therefore there is no nullhomotopy of γ in X.

1.2.11 The mapping torus T_f of a map $f : X \to Y$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same with when $X = S^1 \times S^1$.

Proof

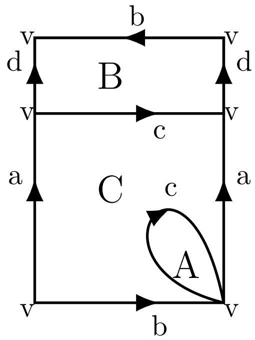
Let $x_0 \in X$ be a basepoint. Suppose $X = S^1 \vee S^1$, then $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$. Let $[a], [b]$ be the two generators, $[a], [b]$ are loops. Under the identification of $(x, 0) \sim (f(x), 1)$ with basepoint preserved, their images $f_*([a])$, $f_*([b])$ thus are also loops in $\pi_1(X, f(x_0))$ based at $f(x_0)$. Instead of seeing T_f as $(S^1 \vee S^1) \times I / \sim$, we can regard T_f as $(S^1 \vee S^1) \vee S^1$ then attach 2 cells on it. This is precisely stacking one torus on the other. Hence $\pi_1(T_f) \cong \langle a, b, c \mid a^{-1} f_*([a]) = b^{-1} f_*([b]) = e \rangle$.

When $X = S^1 \times S^1$. Again we regards T_f as $(S^1 \times S^1) \vee S^1$. So c commutes the generators a, b. Hence we have $a^{-1} f_*[a] = b^{-1} f_*[b] = [a, c] = [a, b] = e$. Therefore, the fundamental group of X is $\pi_1(T_f) = \langle a, b, c \mid a^{-1} f_*([a]) = b^{-1} f_*([b]) = [a, c] = [a, b] = e \rangle$.

1.2.12 The Klein bottle is usually pictured as a subspace of $\mathbb{R}^3$ like the subspace $X \subset \mathbb{R}^3$ shwon in the figure in the text. If one wanted a model that could actually function as a bottle, one would delete the open disk bounded by the circle of self-intersection of X, producing a subspace $Y \subset X$. Show that $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$ and that $\pi_1(Y)$ has the presentation $\langle a, b, c \mid aba^{-1}b^{-1}cb^{\epsilon}c^{-1} \rangle$ for $\epsilon = \pm 1$. Show also that $\pi_1(Y)$ is isomorphic to $\pi_1(\mathbb{R}^3 - Z)$ for Z the graph shown in the figure.

Proof

The space X can be constructed as one 0-cell v, four 1-cells a, b, c, d and three 2-cells A, B, C as below:



A is attached to c, B is attached to $b^{-1}d^{-1}c^{-1}d$ and C is attached to $aca^{-1}c^{-1}b^{-1}$.

So we have the fundamental group

$$\pi_1(X) \cong \langle a, b, c, d \rangle / \langle c, b^{-1}d^{-1}c^{-1}d, aca^{-1}c^{-1}b^{-1} \rangle$$

Now, since $c = 1$, $b^{-1}d^{-1}c^{-1}d = 1$ and $aca^{-1}c^{-1}b^{-1} = 1$. Hence, $c^{-1} = 1$ and $b^{-1}d^{-1}c^{-1}d = b^{-1}d^{-1}d = b^{-1} = 1$.

On the other hand, $acd^{-1}c^{-1}b^{-1} = aa^{-1}b^{-1} = b^{-1} = 1$ implies that $b = 1$. Hence

$$\pi_1(X) \cong \langle a, b, c, d \rangle / \langle c, b \rangle \cong \langle a, d \rangle \cong \mathbb{Z} \times \mathbb{Z}$$

The space Y is actually X with A deleted.

1.2.16 Show that the fundamental group of the surface of infinite genus shown below is free on an infinite number of generators.

Proof

When computing the fundamental group of the surface of genus 2, we let U, V be the subspace that is a torus with a disk removed. Then we apply Van Kampen's Theorem.

Now we want to use Van Kampen to add one torus at a time. Let U be a surface of genus g with boundary as below and V be a torus with two disks removed. Then $U \cap V$ is path connected. And $\pi_1(U)$ is free on $2g$ generators since U deformation retracts to the union of $2g$ circles $a_1, b_1, \cdots, a_g, b_g$. By adding V and applying Van Kampen, we are actually "adding" two generators a_{g+1}, b_{g+1} to the generators of $\pi_1(U)$. Inductively, we see that the fundamental group of the surface of infinite genus is free on infinite number of generators.

1.2.19 Show that the subspace of $\mathbb{R}^3$ that is the union of the spheres of radius $1/n$ and center $(1/n, 0, 0)$ for $n = 1, 2, \cdots$ is simply-connected.

Proof

Let X denote the union of these spheres. All spheres are wedged at the origin $\{0\}$, let's call x_0. Note that every loop on X can be broken down to loops in each individual sphere. Any this loops must pass x_0 and can be contracted to x_0. Hence every loop on X is contractible, to the point p. Therefore $\pi_1(X, x_0) = 0$.

Or, seeing as a CW complex, X has 1-skeleton $X^{(1)}$ which can be contracted to the origin x_0. By Proposition 1.26 the inclusion $i : X^{(1)} \hookrightarrow X$ induces a surjection on $i_* : \pi_1(X^{(1)}) \to \pi_1(X)$. i.e. $i_* : 0 \to \pi_1(X)$. Hence $\pi_1(X) = 0$.

1.2.20 Let X be the subspace of $\mathbb{R}^2$ that is the union of the circles C_n of radius n and center $(n, 0)$ for $n = 1, 2, \cdots$. Show that $\pi_1(X)$ is the free group $*_n\pi_1(C_n)$, the same as for the infinite wedge sum $\bigvee_\infty S^1$. Show that X and $\bigvee_\infty S^1$ are in fact homotopy equivalent, but not homeomorphic.

Proof

Let U_n be an open annulus neighborhood of C_n for each circle as in the figure. We see that U_n deformation retracts to C_n hence the fundamental group of U_n is $\mathbb{Z}$. The union of U_n deformation retracts to the union of C_n, hence the induced homomorphism on π_1 is an isomorphism. Also it is clear that the intersection of any two U_n is contractible since it is just a strip of open set close to the origin. So $\pi_1(\bigcap_n U_n) = 0$. By Van Kampen's Theorem, $\pi_1(\bigcup_n C_n) \cong \pi_1(\bigcup U_n) \cong *_n\pi_1(C_n) \cong *_n\mathbb{Z}$ which is the same as for the infinite wedge sum $\bigvee_\infty S^1$.

Show that the infinite wedge sum of circles and X are homotopy equivalent. By the discussion of cell complexes construction in Chapter 0, we could form the wedge sum of arbitrary collection of spaces X_α by starting with the disjoint union $\coprod_\alpha X_\alpha$ and identifying points $x_\alpha \in X_\alpha$ to a single point. Since each circle S^1 has a cell structure, with one 0-cell and one 1-cell attaching to the 0-cell on its both vertices. So $\vee_\alpha X_\alpha$ is a cell complex since it is obtained from the cell complex $\coprod_\alpha X_\alpha$ by collapsing a subcomplex to a point. We see that both X and the infinite wedge sum of circles can be constructed in the same manner. Hence they are homotopy equivalent.

However, they are not homeomorphism since X is first countable but the infinite wedge sum of circles are not. First, note that X, as a subspace of R^2, is first countable. We assume the infinite wedge sum of circles is first countable, so it has countable basis for each point. Let $\{B_1, B_2, \cdots\}$ be a basis for the common point, say a. We see the neighborhood of a has the form of $\bigvee_i U_i$ for each $U_i \in S^i$ open. So we can choose an open neighborhood of a of this form such that U_j is small, for some j, enough so that no basis element B_i can be contained in it. Therefore, X and infinite wedge sum of circles are not homeomorphic.

1.3.1 For a covering space $p : \tilde{X} \to X$ and a subspace $A \subset X$, let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p : \tilde{A} \to A$ is a covering space.

Proof

Since $p : \tilde{X} \to X$ is a covering space, there exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U_α. Take $V_\alpha = U_\alpha \cap A$, then V_α is open in subspace topology in A.

$$\begin{aligned} p^{-1}(V_\alpha) &= p^{-1}(U_\alpha \cap A) \\ &= p^{-1}(U_\alpha) \cap p^{-1}(A) \\ &= p^{-1}(U_\alpha) \cap \tilde{A} \end{aligned}$$

Since p is continuous, $p^{-1}(V_\alpha)$ is open in X. And $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$. Let $p^{-1}(U_\alpha) = \bigcup O_i$ where $O_i \cap O_j = \phi$ for $i \neq j$. Then we have

$$\begin{aligned} p^{-1}(V_\alpha) &= p^{-1}(U_\alpha) \cap \tilde{A} \\ &= (\bigcup O_i) \cap \tilde{A} \\ &= \bigcup (O_i \cap \tilde{A}) \end{aligned}$$

is also a disjoint union of open sets in $\tilde{X}$. (Note that $O_i \cap \tilde{A}$ is open in subspace topology).

Finally, since each O_i is mapped homeomorphically onto U_α, $O_i \cap \tilde{A}$ is mapped homeomorphically onto $U_\alpha \cap A$. Hence, $p|_{\tilde{A}}$ is a covering space.

1.3.2 Show that if $p_1 : \tilde{X}_1 \to X_1$ and $p_2 : \tilde{X}_2 \to X_2$ are covering spaces, so is their product $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$.

Proof

If $p_1 : \tilde{X}_1 \to X_1$ and $p_2 : \tilde{X}_2 \to X_2$ are covering spaces. There is an open cover $\{U_\alpha\}$ of X_1 such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}_1$. Also there is an open cover $\{V_\beta\}$ of X_2 such that for each β, $p^{-1}(V_\beta)$ is a disjoint union of open sets in $\tilde{X}_2$.

Consider the open sets $U_\alpha \times V_\beta$ in $X_1 \times X_2$. Its preimage $(p_1 \times p_2)^{-1}(U_\alpha \times V_\beta) = p^{-1}(U_\alpha) \times p^{-1}(V_\beta)$ is product of disjoint union of open sets in X_1 and X_2. i.e.

$$(p_1 \times p_2)^{-1}(U_\alpha \times V_\beta) = p^{-1}(U_\alpha) \times p^{-1}(V_\beta) = \bigcup A_i \times \bigcup B_j = \bigcup (A_i \times B_j)$$

which is a disjoint union of open sets of $\tilde{X}_1 \times \tilde{X}_2$.

Since each A_i is mapped homeomorphically by p_1 onto U_α, and each B_j is mapped homeomorphically by p_2 onto V_β. Hence each $A_i \times B_j$ is mapped homeomorphically by $p_1 \times p_2$ onto $U_\alpha \times V_\beta$. Therefore, $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$ is a covering space.

1.3.3 Let $p : \tilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $\tilde{X}$ is compact Housdorff iff X is compact Hausdorff.

Proof

(1) If $\tilde{X}$ is compact. Since p is continuous and X is the continuous image of a compact set hence is compact.

(2) If $\tilde{X}$ is Hausdorff, show that X is Hausdorff.

For $x, y \in X$. Suppose there is an evenly covered open set U containing both x, y. Then $p^{-1}(U)$ contains $p^{-1}(x)$ and $p^{-1}(y)$. Take one of the preimage of this open set, say, $p^{-1}(U)$. Since $\tilde{X}$ is Hausdorff, there exist open sets U_x and U_y in $p^{-1}(U)$ containing $p^{-1}(x)$ and $p^{-1}(y)$ and $U_x \cap U_y = \phi$. i.e. $p^{-1}(x) \in U_x$ and $p^{-1}(y) \in U_y$. So we have $x \in p(U_x)$ and $y \in P(U_y)$ and $p(U_x) \cap p(U_y) = \phi$.

Now, suppose there is no such open set exists. Let V_x be an open set containing x but not y and V_y be an open set containing y but not x. Then $p^{-1}(V_x) = \bigcup \tilde{U}_{\tilde{x}}$ and $p^{-1}(V_y) = \bigcup \tilde{V}_{\tilde{y}}$. Fix $\tilde{x} \in p^{-1}(V_x) \subset U_\alpha$ for some α. Since $\tilde{X}$ is Hausdorff. Pick an open set $\tilde{V}_x \subset p^{-1}(V_x)$ containing $\tilde{x}$ and $\tilde{V}_{\tilde{y}} \subset p^{-1}(V_y)$ containing $\tilde{y}$ for each $\tilde{y} \in p^{-1}(y)$ such that $\tilde{V}_{\tilde{x}} \cap \tilde{V}_{\tilde{y}} = \phi$. Then $((\cap \tilde{V}_x) \cap \tilde{V}_y = \phi$ for all y. Since $p((\cap \tilde{V}_{\tilde{x}}) \cap \tilde{V}_{\tilde{y}}) = p(\cap \tilde{V}_{\tilde{x}}) \cap p(\tilde{V}_{\tilde{y}})$ and $x \in p(\cap \tilde{V}_{\tilde{x}})$ and $y \in p(\tilde{V}_{\tilde{y}})$.

Claim that $p(\cap \tilde{V}_{\tilde{x}}) \cap p(\tilde{V}_{\tilde{y}}) = \phi$.

Suppose $p(\cap \tilde{V}_{\tilde{x}}) \cap p(\tilde{V}_{\tilde{y}}) = \{z\}$. Then $p^{-1}(z) \subset \cap \tilde{V}_{\tilde{x}}$ and $p^{-1}(z) \subset \cap \tilde{V}_{\tilde{y}}$ for all y. This is a contradiction to the fact that $(\cap \tilde{V}_{\tilde{x}}) \cap \tilde{V}_{\tilde{y}} = \phi$. Hence X is Hausdorff.

(3) If X is Hausdorff, prove that $\tilde{X}$ is Hausdorff.

For $x, y \in \tilde{X}$, if $p(x) \neq p(y)$. Since X is Hausdorff, there is an open set U_x containing $p(x)$ and open set U_y containing $p(y)$ such that $U_x \cap U_y = \phi$. So their preimages $p^{-1}(U_x)$ and $p^{-1}(U_y)$ are also disjoint. If not, suppose $p^{-1}(U_x) \cap p^{-1}(U_y) = \{z\}$. Then $p(z) \in U_x$ and $p(z) \in U_y$ which is a contradiction.

Now, suppose $p(x) = p(y)$ in X, take the open set U_x containing $p(x)$. Then $x \in P^{-1}(U_\alpha)$ and $y \in p^{-1}(U_\alpha)$. Hence x and y are either the same point or on different sheet. Therefore, $\tilde{X}$ is Hausdorff.

(4) If X is compact, prove $\tilde{X}$ is compact.

1.3.4 Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter.

Let the diameter be outside the sphere S^2 as Example 0.3 on page 11. The covering space is an infinite sequence of S^2 with interval connecting their north and south poles.

Prove that the covering space is simply-connected. We construct S^2 as follows: Let N/S (north and south poles) be 0-cells and two 1-cells connecting them, and two 2-cells attaching to the 1-cells. Now, one 1-cells connecting N/S on each sphere together with all intervals connecting these S^2 is homeomorphic to $\mathbb{R}$ which is contractible. So it is homotopic to a point.

This makes the covering space a wedge of S^2s. Prove that it is simply connected.

Let $U = S_1^2 \cup W_2 \cup W_3 \cup \cdots$ where W_i denotes the sphere with one point deleted and $V = W_1 \cup S_2^2 \cup S_3^2 \cup \cdots$. Then $U \cap V = W_1 \cup W_2 \cup \cdots$. Since $U \cap V$ deformation retracts onto a point, say, p. Also $U \cap V$ is path connected. Also, since U deformation retracts onto S^2, so $\pi_1(U) = 0$ and V deformation retracts onto $S^2 \vee S^2 \vee \cdots$, so $\pi_1(V) = 0$. By Van Kampen, $\pi_1(X) = 0$. i.e. the covering space is simply connected.

1.3.8 Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ then $\tilde{X} \simeq \tilde{Y}$.

Proof

Let $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ be covering maps. Since $X \simeq Y$, there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq 1_Y$ and $gf \simeq 1_X$. Let $x_0 \in X$ be a base point so $y_0 = f(x_0)$ is its image in Y. And $\tilde{x_0} = p^{-1}(x_0)$ and $\tilde{y_0} = p^{-1}(y_0)$ are their lifts in $\tilde{X}$ and $\tilde{Y}$ respectively.

Since $gf \simeq 1_X$, there is a homotopy $H : X \times I \to X$ such that $H_0 = gf$ and $H_1 = 1_X$. That is, there is a path γ connecting x_0 and $gf(x_0)$. Note that fp is a map from $\tilde{X}$ to Y. Since $\tilde{x_0} \xrightarrow{p} x_0 \xrightarrow{f} y_0 = q(\tilde{y_0})$, then $f_* g_*(\pi_1(\tilde{X}, \tilde{x_0})) \subseteq q_*(\pi_1(\tilde{Y}, \tilde{y_0}))$. By Lifting Criterion, fp lifts to $\tilde{f} : \tilde{X} \to \tilde{Y}$.

On the other hand, gq is a map from $\tilde{Y}$ to X and since $\tilde{y_0} \xrightarrow{q} y_0 \xrightarrow{g} gf(x_0) = p(\tilde{\gamma}(1))$. i.e. $\tilde{y_0}$ in $\tilde{Y}$ that gets mapped into a point $gf(x_0)$ in X is in the image of p. Hence $g_* q_*(\pi_1(\tilde{Y}, \tilde{y_0})) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x_0}))$. By Lifting Criterion, gq lifts to $\tilde{g} : \tilde{Y} \to \tilde{X}$.

Now,

$$\tilde{X} \times I \xrightarrow{p \times 1} X \times I \xrightarrow{H} X$$

So $f \circ H \circ (p \times 1) : \tilde{X} \times I] \to Y$ is a homotopy when restricted on $\tilde{X} \times \{0\}$. Since gf lifts to $\tilde{g}\tilde{f}$, by Homotopy Lifting Property, there is a unique homotopy

$$\tilde{H} : \tilde{X} \times I \to \tilde{Y}$$

Since $H : X \times I \to X$ is a homotopy from gf to 1_X and $\tilde{H}_0(x) = \tilde{g}\tilde{f}$. Then $\tilde{g}\tilde{f} \simeq \tilde{H}_1(x)$. And since $H(x_0)$ lifts to $\tilde{H}(x_0)$ and H is a path from x_0 to $gf(x_0)$ in X. So $\tilde{H}$ is a path from $x_0 = \tilde{\gamma}(0)$ to $\tilde{\gamma}(1)$. Hence $\tilde{H}_1(\tilde{x_0}) = \tilde{x_0}$.

Also $H_1 = 1_X$ lifts to $\tilde{H}_1$, by the uniqueness of the Lifting Property, we have $\tilde{H}_1 = 1_{\tilde{X}}$. Hence $\tilde{H}_0 = \tilde{g}\tilde{f} \simeq 1_{\tilde{x}} = \tilde{H}_1$.

Finally, prove that $\tilde{f}\tilde{g} \simeq 1_{\tilde{Y}}$. Since $fg \simeq 1_Y$, there is a homotopy $H' : Y \times I \to Y$ such that $H'_0 = fg$, $H'_1 = 1_Y$. That is, there is a path connecting y_0 and $fg(y_0)$. By the Lifting Criterion, we have $\tilde{f}$ and $\tilde{g}$ as before. Hence $\tilde{f}$ is a homotopy inverse of $\tilde{g}$ and $\tilde{X} \simeq \tilde{Y}$.

1.3.9 Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic.

Proof

Let $p : \mathbb{R} \to S^1$ be the covering space. Since $\pi_1(X)$ is finite and $\pi_1(S^1) = \mathbb{Z}$. But there is no group homomorphism mapping from a finite group to $\mathbb{Z}$. Hence f is zero.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

So $0 = f_*(\pi(X)) \subset p_X(\pi_1(\mathbb{R}))$. By Lifting Criterion, f lifts to $\tilde{f} : X \to \mathbb{R}$. Since $\mathbb{R}$ is contractible. By problem 0.10 in Chapter 0, $\tilde{f}$ is nullhomotopic. By 0.10 again, X is contractible, hence f is nullhomotopic.

1.3.12 Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2 and $(ab)^4$, and prove that this covering space is indeed the correct one.

Consider a graph of 8-gon as below:

Choose one of the vertex as the basepoint x_0. Clearly this is a covering space of $S^1 \times S^1$, call it $\tilde{X}$. Let N be the normal subgroup generated by a^2, b^2 and $(ab)^4$. The loops adjacent to x_0 correspond to a^2 and b^2. And the loops tracing outer edges of the graph corresponds to $(ab)^4 = abababab$. So $\pi_1(\tilde{X}, \tilde{x}_0)$ contains N. To show they are equal. It is enough to show that $\pi_1(\tilde{X}, \tilde{x}_0)$ is a normal subgroup. The covering space $\tilde{X}$ is like the normal covering space of (8) of $S^1 \vee S^1$ on page 58 except here $\tilde{X}$ is 8-gon, so this is a normal covering space because it has maximal symmetry as the discussion on the top of page 71 in Hatcher's. By Proposition 1.39(a), $p_*\pi_1(\tilde{X}, \tilde{x}_0)$ is a normal subgroup of $\pi_1(X, x_0)$. But the induced map p_* is injective by Proposition 1.31, so $\pi_1(\tilde{X}, \tilde{x}_0)$ is a normal subgroup hence $N = \pi_1(\tilde{X}, \tilde{x}_0)$.

1.3.14 Find all the connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

Let x_0 be a basepoint in $\mathbb{R}P^2$. Note that $p : S^2 \to \mathbb{R}P^2$ is a covering space of $\mathbb{R}P^2$ with two sheets:

$$2 = |p^{-1}(x_0)| = |\pi_1(\mathbb{R}P^2)/p_*\pi_1(S^2)| = |\pi_1(\mathbb{R}P^2)|$$

since $\pi_1(S^2) = 0$. Hence $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$. So $\pi_1(\mathbb{R}P^2)$ has two subgroups: trivial group and itself $\mathbb{Z}/2\mathbb{Z}$. So by Proposition 1.36, there are two covering spaces of $\mathbb{R}P^2$, up to isomorphism: (1) the identity $\mathbb{R}P^2 \to \mathbb{R}P^2$ and (2) $S^2 \to \mathbb{R}P^2$ which is the quotient by the antipodal action. In covering space (1) the preimage of the basepoint is the basepoint itself and in (2) the preimage of the basepoint is the two poles of the spheres.

Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$. The covering space of X restricts to the covering space of each summand $\mathbb{R}P^2$ of X. Let a and b denote each $\mathbb{R}P^2$ of the wedge sum X. Then any such covering space is the union of the copies of $\mathbb{R}P^2$ and S^2.

Let a and b intersect at the basepoint, say x_0. By definition of the covering space, for an open set U containing x_0 in X, $p^{-1}(U)$ must be a disjoint union of open sets in the covering space $\tilde{X}$, each of which is mapped by p homeomorphically onto U. So there are only couple ways for the covering spaces: (1) the two basepoints of two copies of $\mathbb{R}P^2$ can be identified, (2) let one of the poles be the basepoint of the sphere S^2, then the basepoints of two spheres can be identified, or (3) the basepoint of $\mathbb{R}P^2$ and S^2 can be identified. In all cases, the intersection of two spaces must bear different labels. Hence we have the following covering spaces:

The string of spheres with two $\mathbb{R}P^2$s at both ends. (if there is an odd number of spheres, there are two nonisomorphic ways of labeling the components.)

Or the infinite string of spheres with one copy $\mathbb{R}P^2$ at the end. Or we can have a chain of spheres that does not terminate (this is the universal cover.)

Finally, the loop of spheres is also a covering space.

1.3.15 Let $p : \tilde{X} \to X$ be a simply-connected covering space of X and let $A \subset X$ be a path-connected, locally path-connected subspace, with $\tilde{A} \subset \tilde{X}$ a path-component of $p^{-1}(A)$. Show that $p : \tilde{A} \to A$ is the covering space corresponding to the kernel of the map $\pi_1(A) \to \pi_1(X)$.

Proof

By assumption $A \subset X$ and $\pi_1(\tilde{X}) = 0$. Since $p : \tilde{A} \to A$ is a covering space. Let $x_0 \in A$ be a basepoint and $\tilde{x_0} \in \tilde{A}$ be its lift. Theorem 1.38 says that the path-connected covering spaces corresponds to a subgroup $p_*(\pi_1(\tilde{A}, \tilde{x_0}))$ of $\pi_1(A, x_0)$. Prove that the covering space $p : \tilde{A} \to A$ corresponds to the subgroup which is the kernel of $i_* : \pi_1(A) \to \pi_1(X)$, i.e. want to prove that $p_*(\pi_1(\tilde{A}, \tilde{x_0}) = \ker i_*$.

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d, "p"'] & \tilde{X} \arrow[d, "p"] \\ A \arrow[r, "i"] & X \end{tikzcd}$$

($\subseteq$)

For a loop $[\gamma]$ in $p_*(\pi_1(\tilde{A}, \tilde{x_0}))$, $[\gamma]$ lifts to a loop γ starting at $\tilde{x_0}$. Since $\tilde{A} \subset \tilde{X}$, γ is a loop in $\tilde{X}$.

But $\tilde{X}$ is simply-connected, so $\pi_1(\tilde{X}) = 0$. Hence $p_*(\pi_1(\tilde{X}, \tilde{x_0})) = 0$ is a trivial subgroup of $\pi_1(X)$. i.e. $\tilde{X}$ corresponds to trivial subgroup of $\pi_1(X, x_0)$. Hence γ is homotopic to a constant loop in X. i.e. $[\gamma] \simeq c_{x_0}$. Therefore, $[\gamma] \in \ker i_* : \pi_1(A) \to \pi_1(X)$.

($\supseteq$)

For $[\gamma] \in \ker i_*$, $[\gamma] \in \pi_1(A)$ and $i_*([\gamma]) = 0$. That is, $i_*[\gamma] \simeq c_{x_0}$ in X. But $c_{x_0} \in p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, there is a loop $[\tilde{\gamma}]$ in $\tilde{X}$ based at $\tilde{x_0}$. Since $\pi_1(\tilde{X}) = 0$, then $[\tilde{\gamma}] \simeq C_{\tilde{x_0}}$ in $\tilde{X}$.

We claim that $[\tilde{\gamma}]_{\tilde{x_0}} \in \pi_1(A)$. Since $\tilde{x_0}$ is the lift of x_0 and $\tilde{A}$ is mapped homeomorphically onto A containing x_0. So $\tilde{x_0}$ must be in $\tilde{A}$. Hence $[\gamma] \in p_*(\pi_1(\tilde{A}, \tilde{x_0}))$.

Therefore, $p_*(\pi_1(\tilde{A}, \tilde{x_0}) = \ker i_*$. i.e. the covering space $p : \tilde{A} \to A$ corresponds to the subgroup which is the kernel of $p_* : \pi_1(A) \to \pi_1(X)$

1.3.16 Given maps $X \to Y \to Z$ such that both $Y \to Z$ and the composition $X \to Z$ are covering spaces, show that $X \to Y$ is a covering space if Z is locally path-connected and show that this covering space is normal if $X \to Z$ is a normal covering space.

Proof

Consider the following diagram:

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ X \arrow[ur, "f"] \arrow[r, "q"'] & Z \end{tikzcd}$$

Since $p : Y \to Z$ is a covering space. For each $y \in Y$, choose an open set U containing y such that the restriction $p|_U$ maps homeomorphically onto $p(U)$. Without loss of generality we assume that the collection $\{p(U)\}$ is a cover of Z (Hatcher says that the covering map p need not be surjective. In this case, we simply add the complement of $\{p(U)\}$ into the set of cover so that the collection cover Z). Now, since $pf : X \to Z$ is also a covering space, $(pf)^{-1} = f^{-1}p^{-1}(p(V)) = f^{-1}(U)$ is a disjoint union of open sets in X. Since $pf|_{f^{-1}(U)}$ is a homeomorphism onto $p(U)$ and $p|_U$ is a homeomorphism onto $P(U)$, hence $f|_{f^{-1}(U)}$ must a homeomorphism onto U. Therefore $f : X \to Y$ is a covering space.

If $q : X \to Z$ is a normal covering space. By Proposition 1.39, $q_*\pi_1(X) = \pi_1(X)$ (since the covering maps q_* is injective) is a normal subgroup of $\pi_1(Z)$. Also, $p_*\pi_1(Y) = \pi_1(Y)$ is a subgroup of $\pi_1(Z)$. But $f_*\pi_1(X) = \pi_1(X)$ is a subgroup of $\pi_1(Y)$. i.e. $\pi_1(X) \leq \pi_1(Y) \leq \pi_1(Z)$, and $\pi_1(X) \trianglelefteq \pi_1(Z)$. Hence $\pi_1(X)$ must be a normal subgroup of $\pi_1(Y)$, therefore $f : X \to Y$ is a normal covering space.

1.3.17 Given a group G and a normal subgroup N, show that there exists a normal covering space $\tilde{X} \to X$ with $\pi_1(X) \approx G$, $\pi_1(\tilde{X}) \approx N$, and deck transformation group $G(\tilde{X}) \approx G/N$.

Proof

By Corollary 1.28, for every group G there is a 2-dimensional cell complex X such that

$$\pi_1(X) \cong G$$

Since N is a normal subgroup of G, by Corollary 1.36 there is a covering space $p : \tilde{X} \to X$ such that $p_*(\pi_1(\tilde{X})) = N$. Now, since N is a normal subgroup. By Proposition 1.39(a), $p : \tilde{X} \to X$ is a normal covering space. By the same proposition (b) we have

$$G(\tilde{X}) \cong G/N$$

Finally, since p_* is injective hence

$$p_*(\pi_1(\tilde{X})) = N \cong \pi_1(\tilde{X})$$

1.3.18 For a path-connected, locally path-connected, and semilocally simply-connected space X, call a path-connected covering space $\tilde{X} \to X$ abelian if it is normal and

has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X, and that such "universal" abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.

Proof

Let $G = \pi_1(X)$, and N the commutator subgroup. Then by Corollary 1.36 there is a covering space $p : \tilde{X} \to X$ such that $N = p_*(\pi_1(\tilde{X}))$. Since N is a normal subgroup, by Proposition 1.39 $p : \tilde{X} \to X$ is a normal covering space and $G(\tilde{X}) = G/N$ which is abelian. Hence $\tilde{X}$ is an abelian covering.

Now, prove that $\tilde{X}$ is covering space of every other abelian covering space of X.

Let $q : Y \to X$ be another abelian covering space. Since it is normal, by Proposition 1.39

$$G(Y) \cong \pi_1(X)/q_*\pi_1(Y)$$

since q_* is injective.

But $G(Y)$ is abelian, so $q_*\pi_1(Y)$ must contain $N = p_*\pi_1(\tilde{X})$. By Lifting Criterion Proposition 1.33, the map $p : \tilde{X} \to X$ lifts to $\tilde{p} : \tilde{X} \to Y$. i.e.

$$\begin{tikzcd} & Y \arrow[d, "q"] \\ \tilde{X} \arrow[ur, "\tilde{p}"] \arrow[r, "p"'] & X \end{tikzcd}$$

with $p = q\tilde{p}$. And $\tilde{p}$ is a covering map since both p and q are covering maps. Hence $\tilde{X}$ is the universal abelian covering space in the sense that it is a covering space of every other abelian covering space of X. Now, show that such a universal covering space is unique. Suppose $q : Y \to X$ is also a universal abelian covering space. We have a covering map $\tilde{q} : Y \to \tilde{X}$ with $q = p\tilde{q}$. Since $\tilde{X}$ is a universal abelian covering space, we have a covering map $\tilde{p} : \tilde{X} \to Y$ with $p = q\tilde{p}$. i.e.

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\tilde{p}"'] \arrow[d, "p"] \\ Y \arrow[r, "q"'] & X \end{tikzcd} \qquad \begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{q}"] \arrow[r, "q"'] & X \end{tikzcd}$$

Hence $p = p\tilde{q}\tilde{p}$. By Proposition 1.34, since both $1_{\tilde{X}}$ and $\tilde{q}\tilde{p}$ are lifts of p and they carry the basepoint to the basepoint, so the two lifts must agree on $\tilde{X}$. Similar argument shows that $\tilde{q}\tilde{p} = 1_Y$. Hence $\tilde{p}$ and $\tilde{q}$ are inverse isomorphisms. Therefore, the universal abelian covering space is unique up to isomorphism.

When $X = S^1 \vee S^1$, consider the integer grid lines in $\mathbb{R}^2$ as below:

This is clearly a covering space of X, called $\tilde{X}$. Recall that for the covering space $\mathbb{R} \to S^1$, the deck transformation are the vertical translation taking the helix onto itself so it has deck transformation group $\mathbb{Z}$. So when $X = S^1 \vee S^1$, the abelian deck transformation group is $G(\tilde{X})^{ab} = \mathbb{Z} \times \mathbb{Z}$. To show this is the universal abelian covering space, it is enough to show that $p_*\pi_1(\tilde{X})$ is the commutator subgroup N. Indeed, for $a, b \in \pi_1(X)$, $aba^{-1}b^{-1}$ corresponds a loop in $\tilde{X}$. It represents a homotopy class whose image in $\pi_1(X)$ is $[aba^{-1}b^{-1}]$. So $p_*\pi_1(\tilde{X})$ contains the commutator subgroup N. But since

$$\mathbb{Z} \times \mathbb{Z} = G(\tilde{X})^{ab} = \pi_1(X)/p_*\pi_1(\tilde{X}) \cong \mathbb{Z} * \mathbb{Z}/p_*\pi_1(\tilde{X})$$

so $p_*\pi_1(\tilde{X})$ must be equal to the commutator subgroup N. Therefore, $\tilde{X}$ is an abelian universal covering space.

When $X = S^1 \vee S^1 \vee S^1$, then we consider the integer grid on $\mathbb{R}^3$. Follow similar argument we can show that this is an ableian universal covering space of X.

1.3.20 Construct nonnormal covering spaces of the Klein bottle by a Klein bottle and by a torus.

(1) Find a nonnormal covering space of Klein bottle by a Klein bottle.

The Klein bottle, K, has the representation $G = \langle a, b | abab^{-1} = 1 \rangle$. By Proposition 1.39, finding a nonnormal covering space is equivalent to finding a nonnormal subgroup of G. We see that $a^3ba^3b^{-1} = a^2(abab^{-1})ba^2b^{-1} = a^2ba^2b^{-1} = a(abab^{-1})bab^{-1} = abab^{-1} = 1$. So this induced a homomorphism $\varphi : G \to G$ sending $a \mapsto a^3$ and $b \mapsto b$. So the subgroup $H_1 = \varphi(G) = \langle a^3, b \rangle \leq G$ corresponds a covering space $p : K \to K$ as below:

This is a nonnormal covering since conjugating the generator b by a is not in H_1: $aba^{-1} = a(aba)a^{-1} = a^2b \notin H_1$. And from above graph we see that this covering space is indeed a Klein bottle, and it has degree 3 as a covering space of itself.

(2) Find a nonnormal covering space of Klein bottle by a torus.

The relation $abab^{-1} = 1$ is equivalent to $bab^{-1} = a^{-1}$. This implies that conjugation by b sends a to a^{-1}. Consider the subgroup $\langle a, b^2 \rangle$ of G, it has index 2, so it is a normal subgroup of G. So of course this is not the subgroup we are looking for. But if we look at its subgroups, we see that $H_2 = \langle a^3, ab^2 \rangle$ is not a normal subgroup of G since $bab^2b^{-1} = bab^{-1} = a^{-1} \notin H_2$. So H_2 is not a normal subgroup of G. Since H_2 is a subgroup of $\langle a, b^2 \rangle$ (of index 3 ?) and $\langle a, b^2 \rangle$ is subgroup of G of index 2, hence H_2 is a subgroup of G (of index 6 ?). So it corresponds a covering space of the Klein bottle K by a torus by Proposition 1.36.

1.3.25 Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation $\varphi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 - \{0\}$. Show this action is a covering space action and compute $\pi_1(X/\mathbb{Z})$. Show the orbit space $X/\mathbb{Z}$ is non-Hausdorff, and describe how

it is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, coming from the complementary components of the x-axis and y-axis.

Proof

First note that φ is a hyperbola $xy = c$ where c is a constant in $\mathbb{R}$. And it is invariant under the action induced by φ.

The action induced by φ is a $\mathbb{Z}$-action $n\varphi(x, y) = (2^n x; y/2^n)$. To show that this action is a covering space action, want to show that for each (x, y), there exists neighborhood U containing (x, y) such that if $g_1 \neq g_2$ for $g_1, g_2 \in \mathbb{Z}$, then $g_1(U) \cap g_2(U) = \phi$.

choose a closed ball B containing (x, y) with radius $r = sup_{(a,b)\in B} \sqrt{(a-x)^2 + (b-y)^2}$. Let $B_n := n \cdot B$, then B_n is an ellipse centered at $(2^n x, y/2^n)$ with axes of length $a_n = 2^n r_n$ and $b_n = 2^n r_n$, where $r_n = sup_{(a,b)\in B} \sqrt{2^{2n}(a-x)^2 + 2^{2n}(b-y)^2}$. Now, want to show that there exists $r > 0$ such that $B_n \cap B_m = \phi$. The distance d between the centers of the two balls B_n and B_m is given by

$$d_{m,n} = \sqrt{(2^n - 2^m)^2 x^2 + (2^{-n} + 2^{-m})^2 y^2}$$

If we choose (a, b) on the boundary of B giving r_n, this give $d_{n,m} > max\{a_n, a_m\}$, and $d_{n,m} > max\{b_n, b_m\}$ for all n, m. And it reduced to two inequalities to solve for two unknown variables (a, b). A choice of a, b determines r. Hence an open set U exists. Therefore, φ induces a covering space action.

Since $X = \mathbb{R}^2 - \{0\} \simeq S^1$, so $\pi_1(X) \cong \mathbb{Z}$. Since the $\mathbb{Z}$-action is a covering space action and X is path-connected and locally path-connected, by Proposition 1.40, the quotient map $p : X \to X/\mathbb{Z}$ is a normal covering space. And $\mathbb{Z} = \pi_1(X/\mathbb{Z})/p_*\pi_1(X) = \pi_1(X/\mathbb{Z})/\pi_1(X)$ because p_* is injective. But $\pi_1(X) \cong \mathbb{Z}$, so $\pi_1(X/Z) = \mathbb{Z}^2$.

Now, show that X is not Hausdorff. We know that the hyperbola $xy = c$ is invariant under the $\mathbb{Z}$-action of φ. And each hyperbola correspond to a different point in the quotient $X/\mathbb{Z}$. We know that on $\mathbb{R}^2$ any two hyperbola in the family $xy = c$ intersects when $x, y \to \infty$. At the tails of any two hyperbola, we can't find two neighborhoods U, V of two distinct points (a_1, b_1) and (a_2, b_2) on different hyperbola such that $U \cap V = \phi$. Hence X is not Hausdorff.

Finally, we need to describe how $X/\mathbb{Z}$ is the union of four subspaces homeomorphism to $S^1 \times \mathbb{R}$. Consider the four quadrants of $\mathbb{R}^2 - \{0\}$ and define $\mathbb{R}$ in $S^1 \times \mathbb{R}$ to be the distance from the origin in each quadrant. The orbits of the action by $\mathbb{Z}$ are all contained in the hyperbolas $xy = c$. It is easy to see that the action of $\mathbb{Z}$ on a single hyperbola is equivalent to the action of $\mathbb{Z}$ on $\mathbb{R}$ given by translation. Hence, the orbit space of this action on a single hyperbola is the circle S^1 (as this is the orbit space of $\mathbb{Z}$ acting on $\mathbb{R}$ by translation). Thus we get a copy of S^1 for each $c > 0$, giving us that the orbit space of $\mathbb{Z}$ acting on a single quadrant is $S^1 \times (0, \infty) \cong S^1 \times \mathbb{R}$. This is true for all four quadrants, so $X/\mathbb{Z}$ is the union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$.

1.3.29 Let Y be a path-connected, locally path-connected, and simply-connected, and let G_1 and G_2 be subgroups of Homeo(Y) defining covering space actions on Y. Show that the orbit spaces Y/G_1 and Y/G_2 are homeomorphic iff G_1 and G_2 are

conjugate subgroups of Homeo(Y).

Proof

Let $G = \text{Homeo}(Y)$. Since G defines covering space action, it satisfied the condition:

$(*)$ – Each $y \in Y$ has a neighborhood U such that all the images $g(U)$ for varying $g \in G$ are disjoint. In other words, $g_1(U) \cap g_2(U) \neq \phi$ implies $g_1 = g_2$.

Let $y_0 \in Y$ be a basepoint. Note that the subgroups G_1 and G_2 in G determine a composition of covering spaces $Y \xrightarrow{p} Y/G_1 \to Y/G$ and $Y \xrightarrow{p} Y/G_2 \to Y/G$. Let $f : Y/G_1 \to Y/G_2$, consider the following diagram.

$$\begin{tikzcd}
& Y/G_1 \arrow[dl, "f"'] \arrow[d, "p_1"] \\
Y/G_2 \arrow[r, "p_2"'] & Y/G
\end{tikzcd}$$

Since G satisfies the condition $(*)$ above, by theorem 1.40 we have

$$G_1 \cong \pi_1(Y/G_1)/p_*\pi_1(Y) = \pi_1(Y/G_1)$$

since $p_*\pi_1(Y) = 0$ because Y is simply connected. Similarly,

$$G_2 \cong \pi_1(Y/G_2)$$

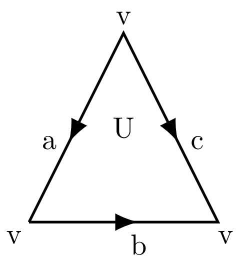
By assumption, G_1 and G_2 are conjugate subgroups of G, so $\pi_1(Y/G_1)$ and $\pi_1(Y/G_2)$ are conjugate, hence $p_{1*}\pi_1(Y/G_1)$ and $p_{2*}\pi_1(Y/G_2)$ are conjugate subgroups of $\pi_1(Y/G)$. Let $y_0 \in Y$ be a basepoint, then $p_{1*}\pi_1(Y/G_1, y_1)$ and $p_{2*}\pi_1(Y/G_2, y_2)$ are conjugate subgroups of $\pi_1(Y/G, y_0)$ where $y_1 = p_1^{-1}(y_0)$ and $y_2 = p_2^{-1}(y_0)$. By Proposition 1.38 the Correspondence Theorem, the map $f : Y/G_1 \to Y/G_2$ is an isomorphism sending $y_1 = p_1^{-1}(y_0)$ to $y_2 = p_2^{-1}(y_0)$. Since G is a group of homeomorphism on Y, so f is in fact a homeomorphism.

The other direction is precisely the same idea: suppose $f : Y/G_1 \to Y/G_2$ is a homeomorphism, it is an isomorphism. Then $H_1 = p_{1*}\pi_1(Y/G_1, y_1)$ and $H_2 = p_{2*}\pi_1(Y/G_2, y_2)$ are conjugate subgroups of $\pi_1(Y/G, y_0)$ where $y_1 = p_1^{-1}(y_0)$ and $y_2 = p_2^{-1}(y_0)$. This implies that G_1 and G_2 are conjugate subgroups of G.

Chapter 2

2.1.4 Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

Let X be the triangular parachute obtained from Δ^2.



We have the chain complex for X as follows:

$$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where $C_2 = \langle U \rangle$, $C_1 = \langle a, b, c \rangle$, $C_0 = \langle v \rangle$.

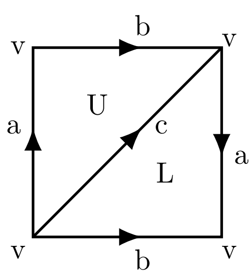
Since $\partial_2(U) = a + b - c$ and $\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0$. The homology groups are

$$H_0(X) = \frac{\ker \partial_0}{\mathrm{Im}\partial_1} = \frac{\langle v \rangle}{0} = \mathbb{Z}$$
$$H_1(X) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \frac{\langle a, b, c \rangle}{\langle a + b - c \rangle} = \frac{\langle a, b, a + b - c \rangle}{\langle a + b - c \rangle} = \langle a, b \rangle = \mathbb{Z}^2$$
$$H_2(X) = \frac{\ker \partial_2}{\mathrm{Im}\partial_3} = \frac{0}{0} = 0$$

Hence, the homology groups are $H_n(X) = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}^2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$

2.1.5 Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of theis section.

Let X be the Klein bottle. We have the chain complex for X as follows:



We have the chain complex,

$$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where $C_2 = \langle U, L\rangle$, $C_1 = \langle a, b, c\rangle$, $C_0 = \langle v\rangle$.

Since $\partial_2(U) = a + b - c$, $\partial_2(L) = a - b + c$, $\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0$. The homology groups are

$$H_0(X) = \frac{\ker \partial_0}{\mathrm{Im}\partial_1} = \frac{\langle v\rangle}{0} = \mathbb{Z}$$

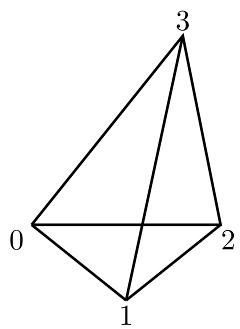
$$H_1(X) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \frac{\langle a, b, c\rangle}{\langle a + b - c, a - b + c\rangle} = \frac{\langle a, b, a + b - c\rangle}{\langle a + b - c, 2a\rangle} = \frac{\langle a, b\rangle}{\langle 2a\rangle} = \mathbb{Z}_2 \oplus \mathbb{Z}$$

where $2a = (a + b - c) + (a - b + c)$.

$$H_2(X) = \frac{\ker \partial_2}{\mathrm{Im}\partial_3} = \frac{0}{0} = 0$$

Hence, the homology groups are $H_n(X) = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z} & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$

2.1.7 Find a way of identifying pairs of faces of Δ^3 to produce a Δ-complex structure on S^3 having a single 3-simplex, and compute the simplicial homology groups of this Δ-complex.



Identifying (012) with (312), (013) with (023), we obtain the following simplexes:

two 0-simplices: (0) = (3), (1) = (2)
three 1-simplices: (03), (12), (01) = (02) = (31) = (32)
two 2-simplices: (012) = (312), (013) = (023)
one 3-simplex: (0123)

So we have the chain complex :

$$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where $C_3 = \langle(0123)\rangle$, $C_2 = \langle(012), (013)\rangle$, $C_1 = \langle(03), (12), (01)\rangle$, $C_0 = \langle(0), (1)\rangle$.

Since

$\partial_3(0123) = (123) - (023) + (013) - (012) = 0$
$\partial_2(012) = (12) - (02) + (01) = (12)$, $\partial_2(013) = (13) - (03) + (01) = (03)$
$\partial_1(03) = (3) - (1) = 0$, $\partial_1(12) = (2) - (1) = 0$, $\partial_1(01) = (1) - (0)$

$\partial_0(0) = 0$, $\partial_0(1) = (0)$.

Hence, the homology groups are

$$H_0(X) = \frac{\ker \partial_0}{\mathrm{Im}\partial_1} = \frac{\mathbb{Z}^2}{\langle (1) - (0) \rangle} = \mathbb{Z}$$
$$H_1(X) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \frac{\langle (03), (12) \rangle}{\langle (12), (03) \rangle} = 0$$
$$H_2(X) = \frac{\ker \partial_2}{\mathrm{Im}\partial_3} = \frac{0}{0} = 0$$
$$H_3(X) = \frac{\ker \partial_3}{\mathrm{Im}\partial_4} = \frac{\langle (0123) \rangle}{0} = \mathbb{Z}$$

Hence, the homology groups are $H_n(X) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 3 \\ 0 & \text{otherwise} \end{cases}$

2.1.8 Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \cdots, T_n$ by the following two steps. First arrange the tetrahedra in a cycle pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions $0, 1, 2, 3$ are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$ respectively.

Proof

Denote the vertices $1, 2, 3, 4$ for each T_i of the tetrahedra as in the figure. If we identify the bottom face of T_i with the top face of T_{i+1} for each i, this implies that we identify the right-hand side face (123) of T_i with the left-hand side face (023) of T_{i+1}, and identify the bottom (012) of T_i with (013) of T_{i+1}. Hence we have the following identification:

$(123)_i = (023)_{i+1}$
$(012)_i = (013)_{i+1}$

This implies that $(0)_i = (1)_{i+1}$ and $(2)_i = (3)_{i+1}$ for all i.

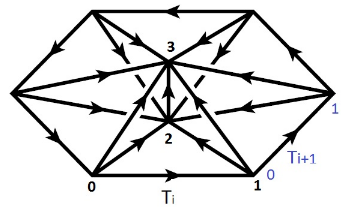
We summarize the simplices as follows:

two 0-simplices: $(0) = (1)$, $(2) = (3)$
$n+2$ 1-simplices: (01), (23), $(02)_i = (03)_{i+1}$ and $(02)_{i+1} = (12)_i = (13)_{i+1}$ for $i = 1, \cdots, n$
$2n$ 2-simplices: $(023)_i = (123)_{i-1}$, $(012)_i = (013)_{i+1}$ for $i = 1, \cdots, n$
n 3-simplices: $(0123) = T_i$, for $i = 1, \cdots, n$

So we have the chain complex :

$$0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where C_i are i-simplices as above.



Since

$\partial_3(0123) = (123)_i - (023)_i + (013)_i - (012)_i = (023)_{i+1} - (023)_i + (012)_{i-1} - (012)_i$
$\partial_2(012) = (12)_i - (02)_i + (01)_i = (02)_{i+1} - (02)_i + (01)_i$, $\partial_2(023) = (23) - (03)_i + (02)_i$
$\partial_1(01) = (1) - (0) = 0$, $\partial_1(23) = (3) - (2) = 0$, $\partial_1(02) = (2) - (0) = 0$
$\partial_0(0) = 0$, $\partial_0(2) = (0)$.

Hence, the homology groups are

(1) Compute $H_0(X)$

$$H_0(X) = \frac{\ker \partial_0}{\mathrm{Im}\partial_1} = \frac{\langle (0), (2)\rangle}{\langle (2) - (0)\rangle} = \frac{\langle (0), (2) - (0)\rangle}{\langle (2) - (0)\rangle} = \mathbb{Z}$$

(2) Compute $H_1(X)$

$\ker \partial_1 = \langle (01), (23), (02)_i - (02)_{i+1}\rangle = \langle (01), (23), (02)_1 - (02)_2, (02)_2 - (02)_3, \cdots, (02)_{n-1} - (02)_n, (02)_n - (02)_1\rangle$
$\mathrm{Im}\ \partial_2 = \langle (23) - (02)_{i-1} + (02)_i, (02)_{i+1} - (02)_i + (01)\rangle = \langle (23) - (02)_1 + (02)_2, (23) - (02)_2 + (02)_3, \cdots, (23) - (02)_n + (02)_1, (02)_2 - (02)_1 + (01), (02)_3 - (02)_2 + (01), \cdots, (02)_1 - (02)_n + (01)\rangle$

Now, since

(a) $\displaystyle\sum_{i=1}^{n} (23) - (02)_{i-1} + (02)_i = n(23) = 0$
(b) $(23) - (02)_{i-1} + (02)_i = 0$ so $(23) = (02)_{i-1} - (02)_i$
(c) $(02)_{i+1} - (02)_i + (01) = 0$ implies $(01) = (02)_i - (02)_{i+1} = (23)$

Hence,

$$H_1(X) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \frac{\langle (01), (23), (02)_i - (02)_{i+1}\rangle}{\langle (23) - (02)_{i-1} + (02)_i, (02)_{i+1} - (02)_i + (01)\rangle} = \frac{\langle (23)\rangle}{\langle n(23)\rangle} = \mathbb{Z}_n$$

(3) Compute $H_2(X)$

$\ker \partial_2((023)_i) = (23) - (02)_{i-1} + (02)_i$
$\ker \partial_2((012)_i) = (02)_{i+1} - (02)_i + (01)$

So,

$$\begin{aligned}
\partial_2(\sum_{i=1}^{n} a_i(023)_i + b_i(012)_i) &= \sum_{i=1}^{n} a_i\partial_2(023)_i + \sum_{i=1}^{n} b_i\partial_2(012)_i \\
&= \sum_{i=1}^{n} a_i\left((23) - (02)_{i-1} + (02)_i\right) + \sum_{i=1}^{n} b_i\left((02)_{i+1} - (02)_i + (01)\right)
\end{aligned}$$

The cycle must be
$(023)_{i+1} - (023)_i + (012)_{i-1} - (012)_i = ((23) - (02)_i + (02)_{i+1}) - ((23) - (02)_{i-1} + (02)_i) + ((02)_i - (02)_{i-1} + (02)) - ((02)_i - (02)_{i-1} + (02)) = 0$ for $1 \leq i \leq n$.

And we see that this cycle is exactly Im ∂_3 since
$\partial_3(0123) = (123)_i - (023)_i + (013)_i - (012)_i = (023)_{i+1} - (023)_i + (012)_{i-1} - (012)_i$.

Hence $H_2(X) = \dfrac{\ker \partial_2}{\mathrm{Im}\partial_3} = \dfrac{0}{0} = 0$.

(4) Compute $H_3(X)$

Since $\partial_3\left(\sum_{i=1}^{n} T_i\right) = \sum_{i=1}^{n} a_i\partial T_i = \sum_{i=1}^{n}((023)_{i+1} - (023)_i + (012)_{i-1} - (012)_i)$. So a_i must be equal to a_j for $i \neq j$ to get zero. Hence, $\ker \partial_3 = \sum_{i=1}^{n} T_i \cong \mathbb{Z}$.

Hence $H_3(X) = \dfrac{\ker \partial_3}{\mathrm{Im}\partial_4} = \sum_{i=1}^{n} T_i \cong \mathbb{Z}$.

Therefore, the homology groups are $H_n(X) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 3 \\ \mathbb{Z}_n & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}$

2.1.9 Compute the homology groups of the Δ-complex X obtained from Δ^n by identifying all faces of the same dimension. Thus X has a single K-simplex for each $k \leq n$.

For $n = 0$, X has only one 0-simplex. So $H_i(X) \cong \mathbb{Z}$ for $i = 0$ and $H_i(X) = 0$ otherwise.

For $k \leq n$, X has single k-simplex for each k. By the definition of boundary map, we have

$$\partial(\sigma_n) = \sum_{i}^{n+1}(-1)^i\sigma_{n-1} = \begin{cases} 0 & \text{for } n \text{ odd} \\ \sigma_{n-1} & \text{for } n \text{ even} \end{cases}$$

So now, when $k < n$, there are two cases. First, when k is odd, $\ker \partial_k = \mathbb{Z}$ and im $\partial_k = 0$. When k is even, $\ker \partial_k = 0$ and im $\partial_k = \mathbb{Z}$. Hence we have

$$H_k(X) = \frac{\ker \partial_k}{\mathrm{im}\ \partial_{k+1}} = \begin{cases} \frac{\mathbb{Z}}{\mathbb{Z}} = 0 & \text{for } k \text{ odd} \\ \frac{0}{0} = 0 & \text{for } k \text{ even} \end{cases}$$

Finally, when $n = k$. Since then im $\partial_{n+1} = 0$, thus,

$$H_n(X) = \frac{\ker \partial_n}{\text{im } \partial_{n+1}} = \begin{cases} \frac{\mathbb{Z}}{0} = \mathbb{Z} & \text{for } n \text{ odd} \\ \frac{0}{0} = 0 & \text{for } n \text{ even} \end{cases}$$

In summary, we have

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{for } n = 0, \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

2.1.11 Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Proof

Let $i : A \to X$ be an inclusion, $r : X \to A$ be a retract. By definition, $r \circ i = 1_A$. So $r_* \circ i_* = 1_*$. Hence, the induced map $i_* : H_n(A) \to H_n(X)$ is injective.

2.1.15 For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphism on all homology groups iff $H_n(X, A) =$ for all n.

Proof

Suppose $C = 0$, we have

$$A \xrightarrow{f} B \xrightarrow{g} 0 \xrightarrow{h} D \xrightarrow{i} E$$

Since the sequence is exact, Im $f = \ker g = B$, so f is surjective. Also, Im $h = 0 = \ker i$, so i is injective. Hence, the map $f : A \to B$ is surjective and $i : D \to E$ is injective.

On the other hand, if $f : A \to B$ is surjective and $i : D \to E$ is injective. Then we have Im $h = \ker i = 0$. So $C = \text{Im} g \cong B/\text{Im} f = B/B = 0$ since f is surjective. Hence, $C = 0$.

2.1.16 (a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.
(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof

(a) For any pair (X, A), we have the long exact sequence

$$\cdots \to H_1(X, A) \to H_0(A) \xrightarrow{f} H_0(X) \to H_0(X, A) \to 0$$

Proposition 2.7 says that for any space, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X. If $H_0(X, A) = 0$, the map $f : H_0(A) \to H_0(X)$ is surjective. The number of path components in A is equal to the number of path components in X. But A is a subspace of X, hence A meets each path component of X.

Conversely, if A meets each path component of X, there is a one-to-one correspondence between the path components of A and path components of X. Hence f is surjective. So $H_0(X, A) = 0$.

(b) The long exact sequence for the pair (X,A) is

$$\cdots \to H_1(A) \xrightarrow{g} H_1(X) \to H_1(X,A) \to H_0(A) \xrightarrow{f} H_0(X) \to H_0(X,A) \to 0$$

If $H_1(X,A) = 0$, the map g is surjective and f is injective which implies each path component of X contains at most one path component of A. Conversely, if g is surjective and each path-component of X contains at most one path-component of A. The latter implies that f is injective. Hence $H_1(X,A) = 0$.

2.1.17 (a) Compute the homology groups $H_n(X,A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.
(b) Compute the groups $H_n(X,A)$ and $H_n(X,B)$ where X is a closed orientable surface of genus two and A and B are the circles shown.

Proof

(a) **Case I :** $X = S^2$

For the pair (S^2, A) where A is a finite set of points, we have a long exact sequence :

$$\cdots \to H_2(A) \to H_2(S^2) \to H_2(S^2,A) \to H_1(A) \to H_1(S^2) \to H_1(S^2,A) \to H_0(A) \to H_0(S^2)$$
$$\to H_0(S^2,A) \to 0$$

(1) $H_2(S^2,A) = \mathbb{Z}$

Note that $H_n(A) = 0$ for $n > 0$ if A is a finite set of points. Also $H_n(S^2) = \mathbb{Z}$ for $n = 0, 2$ and $H_n(S^2) = 0$ otherwise. Hence from the long exact sequence above we got an isomorphism $H_2(S^2) \cong H_2(S^2,A) = \mathbb{Z}$.

(2) $H_0(S^2,A) = 0$

Suppose A has n points. Since $H_0(A)$ is a direct sum of $\mathbb{Z}$, one for each path component in A, so $H_0(A) = \mathbb{Z}^n$ and $H_0(S^2) = \mathbb{Z}$. So the map $f : H_0(A) \to H_0(S^2)$ is surjective by sending one generator, corresponding to one point in A, to the generator of the path component in S^2. Hence $H_0(S^2,A) = 0$ by exactness.

(3) $H_1(S^2,A) = \mathbb{Z}^{n-1}$

Since $H_1(S^2) = 0$ and $H_0(S^2,A) = 0$, we have a short exact sequence

$$0 \to H_1(S^2,A) \xrightarrow{g} H_0(A) \xrightarrow{f} H_0(S^2) \to 0$$

Also since $H_0(A) = \mathbb{Z}^n$ and $H_0(S^2) = \mathbb{Z}$, this sequence is actually

$$0 \to H_1(S^2,A) \xrightarrow{g} \mathbb{Z}^n \xrightarrow{f} \mathbb{Z} \to 0$$

So $H_1(S^2,A) = \operatorname{im} g = \ker f$. But f is surjective, so $\mathbb{Z}^n / \ker f = \mathbb{Z}$, so $\ker f = \mathbb{Z}^{n-1}$. Therefore, $H_1(S^2,A) \cong \mathbb{Z}^{n-1}$.

Another way of seeing this is: the map $f : H_0(A) \to H_0(S^2)$ surjective by sending the generator of each path components, i.e. each point x_i, to the generator of the component in S^2. So $\ker f$ is free on the $\{[x_i] - [x_1], i = 1, \cdots, n\}$. So $\ker f = \mathbb{Z}^{n-1}$. But $\ker f = \operatorname{im}\, g$ by exactness and

g is injective. Thus, $H_1(S^2, A) \cong \mathbb{Z}^{n-1}$.

Or, using the generalized version of Example 0.8 we could see that $X/A \cong S^2 \vee_{n-1} S^1$, the wedge sum of the sphere S^2 and $n-1$ circles. Then by Proposition 2.22 and Corollary 2.25, we have $H_n(X, A) \cong H_n(X/A) = H_n(S^2) \oplus H_n(S^1) \oplus \cdots \oplus H_n(S^1)$, for $n-1$ terms of $H_n(S^1)$. Thus we compute $H_n(S^2, A)$ easily.

Since $H_n(S^2, A) = \begin{cases} \mathbb{Z}^{n-1} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases}$

Case II : $X = S^1 \times S^1$

For the pair $(S^1 \times S^1, A)$, the long exact sequence is as below

$$\cdots \to H_2(A) \to H_2(S^1 \times S^1) \to H_2(S^1 \times S^1, A) \to H_1(A) \to H_1(S^1 \times S^1) \to H_1(S^1 \times S^1, A)$$

$$\to H_0(A) \to H_0(S^1 \times S^1) \to H_0(S^1 \times S^1, A) \to 0$$

(1) $H_2(S^1 \times S^1, A) = \mathbb{Z}$

Since $H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} \times \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$

Since $H_1(A) = H_2(A) = 0$. By exactness of the sequence, $H_2(S^1 \times S^1, A) = \mathbb{Z}$.

(2) $H_0(S^1 \times S^1, A) = 0$

This is because of $H_0(S^1 \times S^1) = \mathbb{Z}$ and by the same reason as in the case when $X = S^2$.

(3) $H_1(S^1 \times S^1, A) = \mathbb{Z}^{n+1}$

Since $H_1(A) = 0$ and $H_1(S^1 \times S^1) = Z^2$ we have an exact sequence

$$0 \to H_1(S^1 \times S^1) \xrightarrow{h} H_1(S^1 \times S^1, A) \xrightarrow{g} H_0(A) \xrightarrow{f} H_0(S^1 \times S^1) \to 0$$

This is actually

$$0 \to \mathbb{Z}^2 \xrightarrow{h} H_1(S^1 \times S^1, A) \xrightarrow{g} \mathbb{Z}^n \xrightarrow{f} Z \to 0$$

Since A a finite set of points. Let $A = (x_1, \cdots, x_n)$. Recall that the kernel $\ker(f)$ is free on the generators $\{[x_i] - [x_1], i = 1, \cdots, n\}$ which is $\mathrm{im}(g)$ by exactness. So $\mathrm{im}(g) \cong Z^{n-1}$. In addition, h is injective and by exactness $\mathrm{im}(h) = \ker(g) \cong \mathbb{Z}^2$. The sum of $\ker(g)$ and $\mathrm{im}(g)$ must be equal to the dimension of $H_1(S^1 \times S^1, A)$, hence $H_1(S^1 \times S^1, A) = \mathbb{Z}^{n+1}$.

(b) Let X be a closed orientable surface of genus two and A, B the circles shown in the figure in 2.1.17(b) in Hatcher's.

Proposition 2.22 says $H_n(X, A) \cong \widetilde{H}_n(X/A)$ for all n. Since X/A is two tori with one point connected. By Cor 2.25, we have

$$H_n(X/A) = H_n(S^1 \times S^1) \oplus H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}^4 & \text{for } n = 1 \\ \mathbb{Z}^2 & \text{for } n = 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Similarly, $H_n(X, B) \cong \widetilde{H}_n(X/B)$ for all n. But X/B is homeomorphic to a torus with two points identified. So applying (a) we get

$$H_n(X, B) = \begin{cases} \mathbb{Z}^3 & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

2.1.18 Show that for the subspace $\mathbb{Q} \subset \mathbb{R}$, the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian and find a basis.

Proof

For the pair $(\mathbb{R}, \mathbb{Q})$, we have a long exact sequence

$$\cdots \to H_1(\mathbb{Q}) \to H_1(\mathbb{R}) \to H_1(\mathbb{R}, \mathbb{Q}) \to H_0(\mathbb{Q}) \xrightarrow{f} H_0(\mathbb{R}) \to H_0(\mathbb{R}, \mathbb{Q}) \to 0$$

Note that $\mathbb{R}$ is contractible, so $H_1(\mathbb{R}) = 0$. Also $H_0(\mathbb{R}) = \mathbb{Z}$ since it is path connected and $H_0(\mathbb{Q})$ is free abelian on generators rationals and f maps one generator (corresponding to one rational) to the (only one) generator of the path component in $\mathbb{R}$, hence f is surjective and $H_0(\mathbb{R}, \mathbb{Q}) = 0$. Hence we have a short exact sequence

$$0 \to H_1(\mathbb{R}, \mathbb{Q}) \xrightarrow{g} H_0(\mathbb{Q}) \xrightarrow{f} H_0(\mathbb{R}) \to 0$$

So g is injective and $H_1(\mathbb{R}, \mathbb{Q}) \cong \operatorname{im}(g) = \ker(f)$. Since $H_0(\mathbb{R})$ and $H_0(\mathbb{Q})$ are free abelian, by exactness $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian. To find a basis, recall that $H_0(\mathbb{Q}) \cong \tilde{H}_0(\mathbb{Q}) \oplus \mathbb{Z}$, i.e. we have SES

$$0 \to \tilde{H}_0(\mathbb{Q}) \to H_0(\mathbb{Q}) \xrightarrow{\varphi} \mathbb{Z} \to 0$$

where $\varphi : H_0(X) \to \mathbb{Z}$ is a map induced by the augmented map $\epsilon : C_0(X) \to \mathbb{Z}$ defined by $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$. So, $\ker f$ is actually $\ker \varphi = \tilde{H}_0(\mathbb{Q})$. Since $\sigma_q : \Delta^0 \to \mathbb{Q}$ is 0-simplex, $\ker \varphi$ consists of finite integer linear combination of 0-simplices such that the sum of the coefficient is zero. As in the proof of Proposition 2.7, the basis for $\ker \varphi$ is the set $\{\sigma_q - \sigma_0 \mid q \in \mathbb{Q}\}$. And this is also a basis for $H_1(\mathbb{R}, \mathbb{Q})$.

2.1.20 Show that $\tilde{H}_n(X) \approx \tilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of n cones CX with their bases identified.

Proof

We can see the suspension SX as the union of two cones $C_N \cup C_S$. Look at the long exact sequence for the pair (SX, C_N),

$$\cdots \to H_i(C_N) \to H_i(SX) \xrightarrow{f} H_i(SX, C_N) \to H_{i-1}(C_N) \to \cdots$$

The cone C_N is contractible, $H_i(C_N) = 0$ for all i. So f is an isomorphism. Let p be the point on the tip of th cone, by Excision Theorem the inclusion $(SX - p, C_N - p) \hookrightarrow (SX, C_N)$ induces isomorphisms $H_i(SX - p, C_N - p) \cong H_i(SX, C_N)$ for all i. Now look at the long exact sequence for the pair $(SX - p, C_N - p)$ is

$$\cdots \to H_i(SX - p) \to H_i(SX - p, C_N - p) \xrightarrow{g} H_{i-1}(C_N - p) \to H_{i-1}(SX - p) \to \cdots$$

$SX - p$ is contractible, $H_i(SX - p) = 0$. And $C_N - p$ is homeomorphic to X. So

$$H_i(SX - p, C_N - p) \cong H_{i-1}(X)$$

Hence

$$\widetilde{H}_i(SX) \cong \widetilde{H}_{i-1}(X)$$

for all i.

In general, let Z_m be the union of m cones CX at X, Z_{m-1} the union of $m-1$ cones. Let A be the set of the $m-1$ cone points.

2.1.22 Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres:
(a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.
(b) $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.
(c) If X has k n-cells, then $H_n(X)$ is generated by at most k elements.

Proof

(a) When $n = 0$ this is clear. Suppose X has dimension k (then $X \cong X^k$) and $H_i(X^{k-1}) = 0$ by induction hypothesis. Since $H_i(X^k, X^{k-1}) \cong \widetilde{H}_i(X^k/X^{k-1}) \cong \widetilde{H}_i(\vee S^k) = 0$ if $i \neq k$. Then LES gives

$$\cdots \to 0 = H_i(X^{k-1}) \to H_i(X^k) \to H_i(X^k, X^{k-1}) = 0 \to \cdots$$

Hence by induction $H_i(X^k) = H_i(X)$ must be 0.

To show that $H_n(X)$ is free, look at the long exact sequence for the pair (X^n, X^{n-1}) when $i = n$.

$$\cdots \to 0 = H_n(X^{n-1}) \to H_n(X^n) \xrightarrow{f} H_n(X^n, X^{n-1}) \to \cdots$$

$H_n(X^{n-1}) = 0$ so f is injective and $H_n(X^n, X^{n-1}) \cong \widetilde{H}_n(X^n/X^{n-1}) \cong \widetilde{H}_n(\vee S^n) = \bigoplus \mathbb{Z}$ which is free abelian on n-cells. Thus $H_n(X) = H_n(X^n)$ is a subgroup of a free abelian group, hence is free.

(b) Consider the long exact sequence

$$\cdots \to H_n(X^{n-2}) \to H_n(X^n) \xrightarrow{f} H_n(X^n, X^{n-2}) \to H_{n-1}(X^{n-2}) \to \cdots$$

Since there are no cells of dimension $n+1$, we have $H_n(X) \cong H_n(X^{n+1}) \cong H_n(X^n)$. From (a) we know that $H_n(X^{n-2}) = H_{n-1}(X^{n-2}) = 0$. Also $H_n(X^n, X^{n-1}) \cong H_n(X^n, X^{n-2})$ because there are no cells of dimension $n-1$. Hence f is an isomorphism.

$$0 \to H_n(X) \xrightarrow{f} H_n(X^n, X^{n-2}) \cong \bigoplus \mathbb{Z} \to 0$$

Therefore $H_n(X)$ is free abelian with basis in bijective correspondence with n-cells.

(c) Suppose X has k n-cells, from (a) we proved that $H_n(X^n)$ is free abelian as a subgroup of $\bigoplus \mathbb{Z}$ which is free on n-cells. Thus $H_n(X^n)$ is free on a number of generators no larger than the number of n-cells, which is k. Show that this is also true for $H_n(X^{n+1})$. Consider the LES for the pair (X^{n+1}, X^n),

$$\cdots \to H_{n+1}(X^{n+1}, X^n) \to H_n(X^n) \xrightarrow{f} H_n(X^{n+1}) \to H_n(X^{n+1}, X^n) \to \cdots$$

Since $H_n(X^{n+1}, X^n) \cong H_n(\vee S^{n+1}) = 0$, f is surjective.

$$\cdots \to H_n(X^n) \xrightarrow{f} H_n(X^{n+1}) \to 0$$

So $H_n(X^{n+1})$ has no more generators than $H_n(X^n)$ does. By induction, $H_n(X)$ is generated by at most k elements.

2.1.27 Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences.
(a) Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.
(b) For the case of the inclusion $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs - there is no $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs.

Proof

(a) Use LES for the pairs (X, A) and (Y, B) and five lemma, we have

$$\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(X) \arrow[d, "f_*"] \\
H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(Y)
\end{tikzcd}$$

Since $f : X \to Y$ and $f : A \to B$ are homotopy equivalences, they induces isomorphisms on homology. Then by five lemma, $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism.

(b) Suppose $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$ and $g : (D^n, D^n - \{0\}) \hookrightarrow (D^n, S^{n-1})$ is its homotopy inverse such that $gf \simeq 1$ and $fg \simeq 1$. Let $H : D^n \times I \to D^n$ be the homotopy with $H(x, 0) = 1$ and $H(x, 1) = fg$. $f : D^n \to D^n$ is an isomorphism, so $H(x, 1) = g$.

Now, look at the restriction $H(x, 1) = g$ on $D^n - \{0\}$. By assumption, the restriction $g : D^n - \{0\} \to S^{n-1}$ is a homotopy equivalence, $g|_{D^n-\{0\}}$ lands in S^{n-1}. Homotopy H is a continuous map, so is its restriction. Since S^{n-1} is closed hence $g^{-1}(S^{n-1})$ is also closed. So $\{0\}$ is contained in the preimage $g^{-1}(S^{n-1})$ or $g(0) \in S^{n-1}$. Let $h : D^n \to S^{n-1}$ be this map.

Consider the composition $D^n - \{0\} \cong S^{n-1} \hookrightarrow D^n \xrightarrow{h} S^{n-1}$. This is actually g which is homotopic to the identity hence induces an identity on homology.

$$H_n(S^{n-1}) \xrightarrow{i_*} H_n(D^n) \xrightarrow{h_*} H_n(S^{n-1})$$

But $H_n(D^n) = 0$ and $H_n(S^{n-1}) = \mathbb{Z}$. Contradiction. Hence g does not induce an isomorphism on homology. There is no such g exists!

2.1.29 Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Proof

Note that $H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z}^2 & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$

and

$H_n(S^1 \vee S^1 \vee S^2) = H_n(S^1) \oplus H_n(S^1) \oplus H_n(S^2) = \begin{cases} \mathbb{Z}^2 & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$

The universal covering for $S^1 \times S^1$ is $\mathbb{R}^2$ which is contractible hence has homology group of 0. On the other hand, the universal covering X for $S^1 \vee S^1 \vee S^2$ is a infinite TV antenna with S^2 glued on at each vertex. Want to prove that $H_2(X) \neq 0$. Since $X^{(1)}$ is contractible, by Proposition 0.17, $X \simeq X/X^{(1)} \cong \vee S^2$, X is homotopy equivalence to the wedge sum of 2-spheres. Hence $H_2(X) \cong \oplus H_2(S^2) \neq 0$.

2.1.31 Using the notation of the five-lemma, give an example where the maps α, β, δ and ϵ are zero but γ is nonzero. This can be done with short exact sequences in which all the groups are either $\mathbb{Z}$ or 0.

Consider the diagram below :

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d, "\beta"] & \mathbb{Z} \arrow[r] \arrow[d, "\gamma"] & \mathbb{Z} \arrow[r] \arrow[d, "\delta"] & 0 \arrow[d, "\epsilon"] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

2.2.2 Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

Proof

Given a map $f : S^{2n} \to S^{2n}$ and a point $x \in S^{2n}$. Assume $f(x) \neq x$, that is, there is no fixed point, by the basic properties of degree (f) and (g), f is homotopic to the antipodal map. So x is sent to its antipodal. i.e. deg $(f) = (-1)^{2n+1} = -1$ and $f(x) = -x$.

Let $g : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ be a map. Pre-composing with the covering map $q : S^{2n} \to \mathbb{R}P^{2n}$ we obtain the map $g \circ q : S^{2n} \to \mathbb{R}P^{2n}$. Now, $g \circ q$ lifts to $h : S^{2n} \to S^{2n}$ by lifting criterion since $\pi_1(S^{2n}) = 0$.

$$\begin{tikzcd}
 & S^{2n} \arrow[d, "q"] \\
S^{2n} \arrow[ur, "h"] \arrow[r, "g \circ q"'] & \mathbb{R}P^{2n}
\end{tikzcd}$$

The diagram commutes. So for $x \in S^{2n}$, $gq(x) = g([x])$, but $gq(x) = qh(x) = q(h(x)) = q(\pm x) = [x]$, hence $g([x]) = [x]$. Therefore, g has a fixed point.

Or, suppose g has no fixed point, it must be $g(x) \neq x$ and $g(x) \neq -x$. But $g(x) \neq x$ means that $g \simeq$ **-1**, thus $\deg(g) = (-1)^{2n+1} = -1$. However, $g(x) \neq -x$ means $g \simeq \mathrm{Id}$, hence $\deg(g) = 1$, a contradiction.

To construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixing points from linear transformation $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors, let A be a $2n \times 2n$ matrix with

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on the main diagonal blocks. This transformation sends $(x_1, x_2, \cdots, x_{2n-1}, x_{2n})$ to $(x_2, -x_1, \cdots, x_{2n}, -x_{2n-1})$, it has no fixed points. And the characteristic polynomial is $x^{2n} + x^{2n-1} + \cdots + x^2 + x + 1$ which is irreducible over $\mathbb{R}$, hence has no eigenvalue hence has no eigenvectors.

2.2.4 Construct a surjective map $S^n \to S^n$ of degree zero, for each $n > 1$.

First, look at the map $f : S^1 \to D^1$, defined by $f(x_1, x_2) = x_1$, so f is surjective. In general, for $S^n \in \mathbb{R}^{n+1}$, let $f : S^n \to D^n$ be the projection defined by $(x_1, x_2, ..., x_{n+1}) \to (x_1, x_2, ...x_n)$, then the restriction of f to S^{n+1} maps it onto a disk. Let $q : D^n \to D^n/S^{n-1} \cong S^n$ be the quotient map, then $q \circ f : S^n \to D^n \to D^n/S^{n-1} \cong S^n$ is the composition of the attaching map of e^n with the quotient map q collapsing $D^n - S^{n-1}$ to a point (refer the definition of **Cellular Boundary Formula** on page 140). It is clearly surjective. Since the induced map on $H_n(S^n)$ factors through $H_n(D^n) = 0$, so has degree 0.

2.2.9 Compute the homology groups of the following 2-complexes:
(a) The quotient of S^2 obtained by identifying north and south poles to a point.
(b) $S^1 \times (S^1 \times S^1)$.
(c) The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.
(d) The quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

Proof

(a) By Example 0.8 in Hatcher, we know that the quotient of S^2 obtained by identifying two points is homotopic equivalent to the wedge of a 2-sphere and a circle. i.e. $X \simeq S^2 \vee S^1$. So they induce an isomorphism on homology groups. Then by Corollary 2.25 we have $\tilde{H}_n(X) \cong \tilde{H}_n(S^2 \vee S^1) \cong H_n(S^2) \oplus H_n(S^1)$. Hence

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

(b)

(c) The space, called X, obtained from D^2 is as figure below:

So there is one 0-cell v, three 1-cells a, b, c and one 2-cell A. So we have the cellular chain complex.

$$0 \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

with $C_0 \cong \mathbb{Z} = \langle v \rangle$, $C_1 \cong \mathbb{Z}^3 = \langle a, b, c \rangle$ and $C_2 \cong \mathbb{Z} = \langle A \rangle$. i.e.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

where $d_1(a) = d_1(b) = d_1(c) = v - v = 0$. And A is attached via the word $aba^{-1}b^{-1}ca^{-1}c^{-1}$. Abelianization gives $-a$ hence $d_2(A) = -a$. So $H_0(X) = \mathbb{Z}$ since X is path-connected. $H_1(X) = \dfrac{\ker d_1}{\text{im } d_2} = \dfrac{\langle a, b, c \rangle}{\langle -a \rangle} = \langle b, c \rangle = \mathbb{Z}^2$. Hence $H_2(X) = \dfrac{\ker d_2}{\text{im } d_3} = \dfrac{0}{0} = 0$. Hence

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}^2 & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

(d) Let X be the space as described. Then X has the CW structure of one 0-cell v, two 1-cells a, b and one 2-cell A. The cellular chain complex has the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

where $d_1(a) = d_1(b) = v - v = 0$. Hence $H_0(X) = \mathbb{Z}$ since X is path-connected. Since A is attached to 1-skeleton $S^1 \vee S^1$ via the word $a^n b^m a^{-n} b^{-m}$. Abelianization gives us 0. So $d_2 = 0$. So $H_1(X) = \dfrac{\ker d_1}{\text{im } d_2} = \dfrac{\langle a, b \rangle}{0} \cong \mathbb{Z}^2$. And $H_2(X) = \dfrac{\ker d_2}{\text{im } d_3} = \dfrac{\langle A \rangle}{0} = \mathbb{Z}$. $H_0(X) = \mathbb{Z}$ since X is path-connected. In summary, we have

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2, \\ \mathbb{Z}^2 & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

2.2.12 Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nulhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is nulhomotopic.

Proof

Consider the LES for the pair $(S^1 \times S^1, S^1 \vee S^1)$,

$$\cdots \to H_2(S^1 \vee S^1) \to H_2(S^1 \times S^1) \xrightarrow{f} H_2(S^1 \times S^1, S^1 \vee S^1) \to H_1(S^1 \vee S^1) \xrightarrow{g} H_1(S^1 \times S^1) \to 0$$

Since $H_2(S^1 \vee S^1) = 0$, f is injective. Also $H_2(S^1 \times S^1, S^1 \vee S^1) \cong H_2(S^1 \times S^1 / S^1 \vee S^1) \cong H_2(S^2) = \mathbb{Z}$ (because $S^m \wedge S^n = S^{m+n}$ by page 10) and $H_1(S^1 \vee S^1) = H_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$. So the long exact sequence is actually

$$\cdots \to 0 \to H_2(S^1 \times S^1) \xrightarrow{f} \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z} \to 0$$

Hence f is an isomorphism.

On the other hand, Let $f : S^2 \to S^1 \times S^1$ be a map. Let $q : \mathbb{R}^1 \times \mathbb{R}^1 \to S^1 \times S^1$ be a covering map. f lifts to $\tilde{f} : R^2 \to \mathbb{R}^1 \times \mathbb{R}^1$ by lifting criterion since $\pi_1(S^2) = 0$. Now, since $\mathbb{R}^1 \times \mathbb{R}^1$

is contractible, $\tilde{f}$ is nulhomotopic (by exercise 10 in chapter 0). Hence f is also nulhomotopic because of $f = q \circ \tilde{f}$.

$$\begin{tikzcd} & \mathbb{R}^1 \times \mathbb{R}^1 \arrow[d, "q"] \\ S^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & S^1 \times S^1 \end{tikzcd}$$

2.2.13 Let X be the 2-complex obtained from S^1 with its usual cell structure by attaching two 2-cells by maps of degree 2 and 3, respectively.
(a) Compute the homology groups of all the subcomplexes $A \subset X$ and the corresponding quotient complexes X/A.
(b) Show that $X \approx S^2$ and that the only subcomplex $A \subset X$ for which the quotient map $X \to X/A$ is a homotopy equivalence is the trivial subcomplex, the 0-cell.

Proof

(a)
(1) Compute $H_n(A)$
Among all the subcomplexes $A \in X$, two easy ones are 0-cell and S^1. We have computed the homology groups for these two spaces.

When $A = S^1 \cup_2 e^2$ (a circle with the boundary of a two cell e^2 wrapping around twice). This is actually $R\mathbb{P}^2$ or the Moose space $M(\mathbb{Z}/2, 1)$ which has homology groups $H_0(A) = \mathbb{Z}$, $H_1(A) = \mathbb{Z}_2$ and zero otherwise.

Or, we could use cellular chain

$$0 \to \mathbb{Z} \xrightarrow{d_2=2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$$

with 0-cell $\langle v \rangle \cong \mathbb{Z}$, 1-cell $\langle x \rangle \cong \mathbb{Z}$ and 2-cell $\langle a, b \rangle \cong \mathbb{Z}^2$. Then we compute $H_1(A) = \ker d_1/\text{im } d_2 = \langle x \rangle/\langle 2x \rangle = \mathbb{Z}_2$.

When $A = S^1 \cup_3 e^2$ which is a circle with the boundary of e^2 attaching to it by wrapping around 3 times, then $A = M(\mathbb{Z}/3, 1)$. The homology groups for this space is $H_0(A) = \mathbb{Z}$, $H_1(A) = \mathbb{Z}_3$ and zero otherwise.

Or, we could use cellular chain

$$0 \to \mathbb{Z} \xrightarrow{d_2=3} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$$

with 0-cell $\langle v \rangle \cong \mathbb{Z}$, 1-cell $\langle x \rangle \cong \mathbb{Z}$ and 2-cell $\langle a, b \rangle \cong \mathbb{Z}^2$. Then we compute $H_1(A) = \ker d_1/\text{im } d_2 = \langle x \rangle/\langle 3x \rangle = \mathbb{Z}_3$.

Now, we compute the space $A = X$. Since X has one 0-cell, one 1-cell, two 2-cells. The cellular chain complex for X is

$$0 \to \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$$

Note that $d_1 = 0$. $d_2 : \langle x, y \rangle \mapsto 2x + 3y$. So $H_1(X) = \dfrac{ker(d_1)}{im(d_2)} = \dfrac{\mathbb{Z}}{\langle 2x + 3y \rangle} = \dfrac{\mathbb{Z}}{\mathbb{Z}} = 0$

NO! The above doesn't make sense, also notation was wrong. It should be: $d_2(a) = \langle 2x \rangle$ and $d_2(b) = \langle 3x \rangle$. Hence, $H_1(X) = \dfrac{\langle a \rangle}{\langle 2a - 3a \rangle} = \dfrac{\langle a \rangle}{\langle a \rangle} = 0$.
Also, $H_2(X) = \dfrac{ker(d_2)}{im(d_3)} = \dfrac{\langle (3,-2) \rangle}{0} = \dfrac{\mathbb{Z}}{0} = \mathbb{Z}$ and $H_0(X) = \mathbb{Z}$ since X is path connected.

(2) Compute $H_n(X/A)$

When A is 0-cell, $H_n(X/A) \cong H_n(X) = \mathbb{Z}$ for $n = 0, 2$ and is zero otherwise.

When $A = S^1$, $X/A \cong S^2 \vee S^2$, so

$$H_n(X/A) = H_n(S^2 \vee S^2) = H_n(S^2) \oplus H_n(S^2) = \begin{cases} \mathbb{Z}^2 & \text{for } n = 2 \\ \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

When $A = S^1 \cup_2 e^2$, or $S^1 \cup_2 e^3$, $X/A \cong S^2$, so

$$H_n(X/A) = H_n(S^2) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

When $A = X$, $X/A \cong$ a point, $H_n(A) = \mathbb{Z}$ for $n = 0$ and $= 0$ otherwise.

(b)
$X = S^1 \cup_2 e^2 \cup_3 e^2$, show $X \cong S^2$.

Since $S^1 \cup_2 e^2 = \mathbb{R}P^2$, the degree 3 map $\cdot 3 : S^1 \to S^1 \cup_2 e^2$ and the degree 1 map $\cdot 1 : S^1 \to S^1 \cup_2 e^2$ are homotopic. (Note that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$, it is zero when the loop in on the e^2, and is 1 when the loop is on S^1.)

Hence $S^1 \cup_2 e^2 \cup_3 e^2 \simeq S^1 \cup_2 e^2 \cup_1 e^2 \simeq (S^1 \cup_2 e^2 \cup_1 e^2)/(S^1 \cup_1 e^2) \simeq S^2$, the last homotopy came from that $S^1 \cup_1 e^2 \simeq D$ which is contractible and by Proposition 0.17.

2.2.15 Show that if X is a CW complex then $H_n(X^n)$ is free by identifying it with the kernel of the cellular boundary map $H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$.

Proof

Look at the diagram in Hatcher page 139. $d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ is the cellular boundary map. Since the map $j_n : H_n(X^n) \to H_n(X^n, X^{n-1})$ is injective, $H_n(X^n) = Im(j_n) = ker(\partial_n)$. The diagram commutes, i.e., $d_n = j_{n-1}\partial_n$, so $ker(\partial_n) = ker(d_n)$. Hence $H_n(X^n) = ker(d_n)$ is free (because $H_n(X^n, X^{n-1})$ is free).

2.2.20 For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

Proof

For a finite CW complex X and Y, $\chi(X) = \Sigma_n (-1)^n \text{rank } H_n(X)$, $\chi(Y) = \Sigma_n (-1)^n$ rank $H_n(Y)$. Want to prove $\chi(X \times Y) = \chi(X)\chi(Y)$

Note that the product $X \times Y$ has the structure of a cell complex with cells the products $e^i_\alpha \times e^j_\beta$ where e^i_α ranges over the cells of X and e^j_β ranges over the cells of Y. Let a_i be the number of i-cells of X and b_j be the number of j-cells of Y. Then the the number of n-cell in $X \times Y$ is $a_i b_j$ with $i + j = n$. Hence,

$$\chi(X)\chi(Y) = \sum_i (-1)^i a_i \sum_j (-1)^j b_j = \sum_n \sum_{i+j=n} (-1)^{i+j} a_i b_j = \chi(X \times Y).$$

2.2.21 If a fintie CW complex X is the union of subcomplexes A and B, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

Proof

If a finite CW complex X is the union of subcomplexes A and B, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$. Let a_n be the number of n-cells of A, b_n be the number of n-cells of B and c_n be the number of n-cells of $A \cap B$. So the number of n-cells in $A \cup B$ is $a_n + b_n - c_n$ and

$$\chi(X) = \sum_n (-1)^n (a_n + b_n - c_n) = \sum_n (-1)^n a_n + \sum_n (-1)^n b_n - \sum_n (-1)^n c_n = \chi(A) + \chi(B) - \chi(A \cap B)$$

2.2.22 For X a finite CW complex and $p : \tilde{X} \to X$ an n-sheeted covering space, show that $\chi(\tilde{X}) = n\chi(X)$.

Proof

If X is a finite CW complex X and $p : \widetilde{X} \to X$ an n-sheeted covering space. For each n-cell e^n_α (an n-cell is an open disk hence open), by definition of the covering space, $p^{-1}(e^n_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is homeomorphically mapped onto e^n_α. These open sets must be homeomorphic to n-cells e^n_β in $\widetilde{X}$.

Hence by 2.2.21, the number of i-cell in $\widetilde{X}$ is $n \cdot a_i$, where a_i is the number of i-cells in X. And $\chi(\widetilde{X}) = \sum_i (-1)^i n a_i = n \sum_i (-1)^i a_i = n\chi(X)$

2.2.28 (a) Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Mobius band via a homeomorphism from the boundary circle of the Mobius band to the circle $S^1 \times \{x_0\}$ in the torus.
(b) Do the same for the space obtained by attaching a Mobius band to $\mathbb{R}P^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{R}P^1 \subset \mathbb{R}P^2$.

Proof

(a) Let A be the Torus with a neighborhood of the attached Mobius band, B be the Mobius band with a neighborhood of the attached torus. So $X = A \cup B$. Look at Mayer-Vietoris Sequence for the pair (A, B).

$$0 \to H_2(A \cap B) \to H_2(A) \oplus H_2(B) \xrightarrow{f} H_2(X) \to H_1(A \cap B) \xrightarrow{g} H_1(A) \oplus H_1(B) \xrightarrow{h} H_1(X)$$
$$\to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0$$

Note that Mobius band is homotopic equivalent to S^1, and $A \cap B$ retracts to a circle hence is homotopic equivalent to S^1, so $H_n(A \cap B) = H_n(B) = \mathbb{Z}$ for $n = 0, 1$ and 0 otherwise. Also,

$$H_n(A) = H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} \times \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Now, $g : \mathbb{Z} \to (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z}$ is injective sending 1 to $((1, 0), 2)$ since the cirle in $A \cap B$ was mapped onto the circle $S^1 \times \{x_0\}$ in $S^1 \times S^1$ and onto the boundary circle of Mobius band by wrapping around twice. The long exact sequence breaks up into short exact sequence.

$$0 \to \mathbb{Z} \xrightarrow{g} (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} \xrightarrow{h} H_1(X) \to 0$$

Hence $H_1(X) = \dfrac{\mathbb{Z}^3}{\ker h} = \dfrac{\mathbb{Z}^3}{\text{im } g} = \dfrac{\mathbb{Z}^3}{\mathbb{Z} \oplus 2\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}_2$.

On the other hand, f is an isomorphism and $H_2(X) = \mathbb{Z}$ by the short exact sequence:

$$0 \to \mathbb{Z} \oplus 0 \xrightarrow{f} H_2(X) \to 0$$

Finally $H_0(X) = \mathbb{Z}$ since X is path connected. Therefore, we have

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 3 \end{cases}$$

(b) Let A be the $\mathbb{R}P^2$ with a neighborhood of the Mobius band, and B be the Mobius band with a neighborhood on $\mathbb{R}P^2$. So $X = A \cup B$. Look at the Mayer-Vietoris Sequence for the pair (A, B):

$$0 \to H_2(A \cap B) \to H_2(A) \oplus H_2(B) \xrightarrow{f} H_2(X) \to H_1(A \cap B) \xrightarrow{g} H_1(A) \oplus H_1(B) \to H_1(X)$$
$$\to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0$$

As in (a), Mobius band and $A \cap B$ are both homotopic equivalent to S^1, so $H_n(B) = H_n(A \cap B) = H_n(S^1)$. Remember that $H_n(\mathbb{R}P^2) = \mathbb{Z}$ for $n = 0$, $\mathbb{Z}_2$ for $n = 1$ and 0 otherwise. So we obtain

$$0 \to 0 \to 0 \oplus 0 \xrightarrow{f} H_2(X) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_2 \oplus \mathbb{Z} \xrightarrow{h} H_1(X) \to 0$$

where $g : 1 \mapsto (1, 2)$ is injective and $\ker g = 0 = \text{im}\partial$. So f is an isomorphism, hence $H_2(X) = 0$.

Now, compute $H_1(X)$. Since h is surjective, we have

$$H_1(X) \cong \frac{\mathbb{Z}_2 \oplus \mathbb{Z}}{ker(h)} \cong \frac{\mathbb{Z}_2 \oplus \mathbb{Z}}{im(g)} \cong \frac{\langle a, b \mid 2a \rangle}{\langle a + 2b \rangle} \cong \langle a, b \mid 4b \rangle \ \cong \mathbb{Z}_4.$$

Or, it should be that

$$H_1(X) \cong \frac{\mathbb{Z}_2 \oplus \mathbb{Z}}{ker(h)} \cong \frac{\mathbb{Z}_2 \oplus \mathbb{Z}}{im(g)} \cong \frac{\mathbb{Z}_2 \oplus \mathbb{Z}}{\mathbb{Z} \oplus 2\mathbb{Z}} \cong \mathbb{Z}_4(?)$$

Finally, $H_0(X) = \mathbb{Z}$ since X is path connected. Therefore, $H_n(X) = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}_4 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$

2.2.29 The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies R, glued together by the identity map between their boundary surfaces M_g, form a closed 3-manifold X. Compute the homology groups of X via the Mayer-Vietoris sequence for this decomposition of X into two copies of R. Also compute the relative groups $H_i(R, M_g)$.

(1) Compute the homology groups of X -

Let A and B be two copies of compact regions R, U and V be the neighborhood of A and B respectively. So $X = A \cup B$, A is the deformation retract of U and B is the deformation retract of V.

Note that $A \cap B$ deformation retracts to M_g. Also R deformation retracts to the wedge of g circles. i.e. $R \cong \vee_g S^1$ Hence, the Mayer-Vietoris sequence then gives us :

$$0 \to H_3(X) \to H_2(M_g) \to H_2(R) \oplus H_2(R) \to H_2(X) \to H_1(M_g) \to H_1(R) \oplus H_1(R) \to H_1(X)$$

$$\to H_0(M_g) \xrightarrow{f} H_0(R) \oplus H_0(R) \xrightarrow{g} H_0(X) \to 0$$

Since

$$H_n(R) = H_n(S^1 \vee S^1 \vee \cdots \vee S^1) = \bigoplus_g H_n(S^1) = \begin{cases} \mathbb{Z}^g & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{for } n \geq 2 \end{cases}$$

and

$$H_n(M_g) = \begin{cases} \mathbb{Z}^{2g} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

So $H_2(M_g) = \mathbb{Z}$, $H_2(R) = 0$, and we have the exact sequence

$$0 \to H_3(X) \xrightarrow{\cong} \mathbb{Z} \to 0 \to H_2(X) \xrightarrow{\phi} \mathbb{Z}^{2g} \xrightarrow{\psi} \mathbb{Z}^g \oplus \mathbb{Z}^g \xrightarrow{\eta} H_1(X) \to \cdots$$

Hence $H_3(X) = \mathbb{Z}$.

To compute $H_2(X)$, note that the map $\psi : M_g \to R$ sends the generators $\langle a_1, b_1, \cdots, a_g, b_g \rangle$ of M_g to $\langle b_1, \cdots, b_g \rangle$ since $a_1, \cdots, a_g$ retract to a point through the compact solid R. Since ϕ is injective, $H_2(X) \cong \mathrm{im}\phi = \ker \psi = \langle a_1, \cdots a_g \rangle \cong \mathbb{Z}^g$.

Now, compute $H_0(X)$. From LES we see that $g : H_0(R) \oplus H_0(R) \to H_0(X)$ is surjective, so $H_0(X) \cong \mathbb{Z}$.

To computer $H_1(X)$, note that im $\psi = \langle b_1, \cdots, b_g \rangle = \mathbb{Z}^g = \ker \eta$. Since $f : H_0(M_g) \to H_0(R) \oplus H_0(R)$ is injective, so $H_1(X) \to H_0(M_g)$ is a zero map, hence η is surjective. Thus $H_1(X) = \dfrac{\mathbb{Z}^g \oplus Z^g}{\ker \eta} = \dfrac{\mathbb{Z}^g \oplus \mathbb{Z}^g}{\mathbb{Z}^g} = \mathbb{Z}^g$. Hence,

$$H_n(X) = \begin{cases} \mathbb{Z}^g & \text{for } n = 1, 2 \\ \mathbb{Z} & \text{for } n = 0, 3 \\ 0 & \text{for } n \geq 4 \end{cases}$$

(2) Compute $H_i(R, M_g)$

Consider the LES for the pair (R, M_g)...

$0 \to H_3(R, M_g) \to H_2(M_g) \to H_2(R) \to H_2(R, M_g) \to H_1(M_g) \to H_1(R) \to H_1(R, M_g) \to H_0(M_g) \to H_0(R) \to$

Using the homology groups for R and M_g mentioned in (a), we have the following sequence...

$$0 \to H_3(R, M_g) \xrightarrow{\cong} \mathbb{Z} \to 0 \to H_2(R, M_g) \xrightarrow{\phi} \mathbb{Z}^{2g} \xrightarrow{\psi} \mathbb{Z}^g \xrightarrow{\eta} H_1(R, M_g) \to \cdots$$

So $H_3(R, M_g) = \mathbb{Z}$.

To compute $H_2(R, M_g)$, note that the map ψ sends the generators $\langle a_1, b_1, \cdots a_g, b_g \rangle$ of M_g to the generator $\langle b_1, \cdots\ b_g \rangle$ of R by the same reasoning in part (a). Hence $H_2(R, M_g) = im(\phi) = ker(\psi) = \langle a_1, \cdots, a_g \rangle = \mathbb{Z}^g$.

Now, compute $H_0(R, M_g)$. Since $H_0(M_g) \to H_0(R)$ is an isomorphism. Hence $H_0(R, M_g) = 0$. So we have a SES

$$0 \to H_0(M_g) \xrightarrow{\cong} H_0(R) \to H_0(R, M_g) \to 0$$

which implies that η is surjective. So $H_1(R, M_g) = \dfrac{\mathbb{Z}^g}{ker(\eta)} = \dfrac{\mathbb{Z}^g}{im(\psi)} = \dfrac{\mathbb{Z}^g}{Z^g} = 0$.

Hence we have $H_n(R, M_g) = \begin{cases} \mathbb{Z}^g & \text{for } n = 2 \\ \mathbb{Z} & \text{for } n = 3 \\ 0 & \text{otherwise} \end{cases}$

2.2.32 For SX the suspension of X, show by a Mayer-Vietoris sequence that there are isomorphisms $\tilde{H}_n(SX) \approx \tilde{H}_{n-1}(X)$ for all n.

Proof

Recall that SX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. Let a, b be these two points. Let $A = SX - \{a\}$ and $B = SX - \{b\}$. Then $SX = A \cup B$ and $A \cap B \simeq X$. The Mayer-Vietoris sequence gives:

$\cdots \to H_{n+1}(SX) \to H_n(X) \to H_n(A) \oplus H_n(B) \to H_n(SX) \to H_{n-1}(X) \to H_{n-1}(A) \oplus H_{n-1}(B)$
$\to H_{n-1}(X) \to \cdots$

Since $A \simeq CX$ and $B \simeq CX$ and CX is contractible. So $\tilde{H}_i(CX) = 0$ for all i. Hence $\tilde{H}_i(A) \oplus \tilde{H}_i(B) = 0$, the sequence becomes

$$\cdots \to 0 \to H_{n+1}(SX) \xrightarrow{\cong} H_n(X) \to 0 \to H_n(SX) \xrightarrow{\cong} H_{n-1}(X) \to 0 \to H_{n-1}(X) \to \cdots$$

This is true for $n \neq 0$. Therefore $\tilde{H}_n(SX) \approx \tilde{H}_{n-1}(X)$ for all n.

Chapter 3

3.1.2 Show that the maps $G \xrightarrow{n} G$ and $H \xrightarrow{n} H$ multiplying each element by the integer n induce multiplication by n in Ext(H, G).

Proof

Let $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$ be a free resolution of an abelian group H. Taking $\text{Hom}(-, G)$, we obtain $0 \leftarrow \text{Hom}(F_1, G) \xleftarrow{f_1^*} \text{Hom}(F_0, G) \xleftarrow{f_0^*} \text{Hom}(H, G) \leftarrow 0$.

Consider the map $\cdot n : G \to G$ and $\phi \in \text{Hom}(F_1, G)$, the composition $\cdot n \circ \phi : F_1 \xrightarrow{\phi} G \xrightarrow{\cdot n} G$ is the induced multiplication by n in $\text{Hom}(F_1, G)$.

$$\begin{tikzcd}
0 & \text{Hom}(F_1,G) \arrow[l] \arrow[d, "n^*"] & \text{Hom}(F_0,G) \arrow[l, "f_1^*"'] \arrow[d, "n^*"] & \text{Hom}(H,G) \arrow[l, "f_0^*"'] \arrow[d, "n^*"] & 0 \arrow[l] \\
0 & \text{Hom}(F_1,G) \arrow[l] & \text{Hom}(F_0,G) \arrow[l, "f_1^*"'] & \text{Hom}(H,G) \arrow[l, "f_0^*"'] & 0 \arrow[l]
\end{tikzcd}$$

And by definition,

$$\text{Ext}(H, G) = H^1(F, G) = \frac{\ker f_2^*}{\text{im} f_1^*} = \frac{\text{Hom}(F_1, G)}{\text{im} f_1^*}$$

Thus, multiplying by n on G induces multiplication by n on in $\text{Hom}(F_1, G)$, hence induces multiplication by n on $\text{Ext}(H, G)$.

Now consider the map $\cdot n : H \to H$, the multiplication by n on abelian groups commutes with group homomorphism hence gives the commutative digram below.

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r] \arrow[d, "\cdot n"] & F_0 \arrow[r] \arrow[d, "\cdot n"] & H \arrow[r] \arrow[d, "\cdot n"] & 0 \\
0 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & H \arrow[r] & 0
\end{tikzcd}$$

Apply $\text{Hom}(-, G)$ we obtain

$$\begin{tikzcd}
0 & \text{Hom}(F_1,G) \arrow[l] & \text{Hom}(F_0,G) \arrow[l, "f_1^*"'] & \text{Hom}(H,G) \arrow[l, "f_0^*"'] & 0 \arrow[l] \\
0 & \text{Hom}(F_1,G) \arrow[l] \arrow[u, "n^*"'] & \text{Hom}(F_0,G) \arrow[l, "f_1^*"'] \arrow[u, "n^*"'] & \text{Hom}(H,G) \arrow[l, "f_0^*"'] \arrow[u, "n^*"'] & 0 \arrow[l]
\end{tikzcd}$$

The diagram above commutes. For $\phi \in \text{Hom}(H, G)$, $(f_0^* n)(\phi) = f_0^*(n(\phi)) = n f_0^*(\phi) = (n f_0^*)(\phi)$ (No! it should be $n^*(\phi(x)) = \phi(nx) = n\phi(x)$, refer D&F page 392 to see how $\text{Hom}(-, G)$ works.). So multiplication by n on the abelian group H induces a multiplication by n on every group in the top row of the diagram, in particular, a multiplication by n on $\text{Hom}(F_1, G)$ hence induces a multiplication by n on $\text{Ext}(H, G)$.

3.1.3 Regarding $\mathbb{Z}_2$ as a module over the ring $\mathbb{Z}_4$, construct a resolution of $\mathbb{Z}_2$ by free modules over $\mathbb{Z}_4$ and use this to show that $\text{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all n.

Consider a free resolution of $\mathbb{Z}_2$ by a free module $\mathbb{Z}_4$ over $\mathbb{Z}_4$:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 1} \mathbb{Z}_2 \to 0$$

Apply the functor $\mathrm{Hom}(-, \mathbb{Z}_2)$, we obtain

$$\cdots \xleftarrow{\cdot 2^*} \mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \xleftarrow{\cdot 2^*} \mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \xleftarrow{\cdot 2^*} \mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \xleftarrow{\cdot 2^*} \mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) \longleftarrow 0$$

But $\mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \cong \mathbb{Z}_2$, so this sequennce is

$$\cdots \xrightarrow{\cdot 2^*} \mathbb{Z}_2 \xrightarrow{\cdot 2^*} \mathbb{Z}_2 \xrightarrow{\cdot 2^*} \mathbb{Z}_2 \xrightarrow{\cdot 2^*} \mathbb{Z}_2 \to 0$$

Now, since $\cdot 2^*$ is just the zero map. So $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) = \dfrac{\ker f^*_{n+1}}{\mathrm{Im}\ f^*_n} = \dfrac{\mathbb{Z}_2}{0} = \mathbb{Z}_2$ which is non-zero for all n.

3.1.4 What happens if one defines homology groups $h_n(X; G)$ as the homology groups of the chain complex $\cdots \to \mathbf{Hom}(G, C_n(X)) \to \mathbf{Hom}(G, C_{n-1}(X)) \to \cdots$? More specifically, what are the groups $h_n(X; G)$ when $G = \mathbb{Z}, \mathbb{Z}_m$, and $\mathbb{Q}$?

When $G = \mathbb{Z}$, then $\mathrm{Hom}(\mathbb{Z}, C_n(X)) = C_n(X)$, the chain complex is:

$$\cdots \to C_3 \to C_2 \to C_1 \to C_0 \to 0$$

So the homology group $h_n(X, G)$ is just the ordinary homology groups studied in chapter 2.

If $G = \mathbb{Z}_m$, $\mathrm{Hom}(\mathbb{Z}_m, C_n(X)) = 0$ since free abelian groups have no element of finite order so no homomorphism sends from a finite or torsion group to a free abelian group. Hence the chain complex is all zero. Same for the case $G = \mathbb{Q}$.

3.1.6 (a) Directly from the definitions, compute the simplicial cohomology groups of $S^1 \times S^1$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients, using the Δ-complex structure given in 2.1. (b) Do the same for $\mathbb{R}P^2$ and the Klein bottle.

(a) Compute $H^n(S^1 \times S^1; \mathbb{Z})$

Remember in chapter 2 we had a chain complex

$$0 \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

where $C_0 = \mathbb{Z}\langle v\rangle$, $C_1 = \mathbb{Z}\langle a, b, c\rangle$ and $C_2 = \mathbb{Z}\langle U, L\rangle$. Apply $\mathrm{Hom}(-, Z)$, we obtain the cochain complex

$$0 \leftarrow C^2 \xleftarrow{\delta_2} C^1 \xleftarrow{\delta_1} C^0 \xleftarrow{\delta_0} 0$$

Now, find a basis for each cochain: $C^0 = \mathbb{Z}\langle \nu\rangle$ where $\nu(v) = 1$. $C^1 = \mathbb{Z}^3\langle \alpha, \beta, \gamma\rangle$ where $\alpha(a) = 1$, $\alpha(b) = \alpha(c) = 0$; $\beta(b) = 1$, $\beta(a) = \beta(c) = 0$ and $\gamma(c) = 1$, $\gamma(a) = \gamma(b) = 0$. Also, $C^2 = \mathbb{Z}^2\langle \lambda, \mu\rangle$ where $\lambda(U) = 1$, $\lambda(L) = 0$; $\mu(L) = 1$, $\mu(U) = 0$.

Compute $H^0(X)$ -
$\delta_1\nu(a) = \nu d_1(a) = \nu(v - v) = 0$, similarly $\delta_1\nu(b) = \delta_1\nu(c) = 0$. So $\delta_1 = 0$. Hence $H^0(X) = \dfrac{\ker \delta_1}{\mathrm{im} d_0} = \dfrac{C^0}{0} = \dfrac{\mathbb{Z}}{0} = \mathbb{Z}$

Compute $H^1(X)$ -
Since $\delta_1 = 0$, so $\mathrm{Im}\delta_1 = 0$. Now, to compute δ_2, we have
$\delta_2\alpha(U) = \alpha d_2(U) = \alpha(a+b-c) = \alpha(a)+\alpha(b)-\alpha(c) = 1 = \delta_2\alpha(L)$
$\delta_2\beta(U) = \beta d_2(U) = \beta(a+b-c) = \beta(a)+\beta(b)-\beta(c) = 1 = \delta_2\beta(L)$
$\delta_2\gamma(U) = \gamma d_2(U) = \gamma(a+b-c) = \gamma(a)+\gamma(b)-\gamma(c) = -1 = \delta_2\gamma(L)$

We could pick $\alpha-\beta$ and $\alpha+\gamma$ as a basis for $\ker\delta_2$ since $\delta_2(\alpha-\beta)(U) = \delta_2(\alpha+\gamma)(U) = \delta_2(\alpha-\beta)(L) = \delta_2(\alpha+\gamma)(L) = 0$. So $\ker\delta_2 = \langle\alpha-\beta, \alpha+\gamma\rangle$. Hence $H^1(X) = \dfrac{\ker\delta_2}{\mathrm{im}\delta_1} = \dfrac{\mathbb{Z}^2}{0} = \mathbb{Z}^2$.

Compute $H^2(X)$ -
Note that $\ker\delta_3 = \langle\lambda,\mu\rangle$. To compute $\mathrm{Im}\delta_2$, write it in terms of λ and μ.
$\delta_2\alpha(U) = 1 = (\lambda+\mu)(U) = \delta_2\alpha(L)$
$\delta_2\beta(U) = 1 = (\lambda+\mu)(U) = \delta_2\beta(L)$
$\delta_2\gamma(U) = -1 = -(\lambda+\mu)(U) = \delta_2\gamma(L)$
So $\mathrm{Im}\delta_2 = \langle\lambda+\mu\rangle$. Hence $H^2(X) = \dfrac{\ker\delta_3}{\mathrm{Im}\delta_2} = \dfrac{\langle\lambda,\mu\rangle}{\langle\lambda+\mu\rangle} = \dfrac{\langle\lambda,\lambda+\mu\rangle}{\langle\lambda+\mu\rangle} = \langle\lambda\rangle = \mathbb{Z}$
Changing coefficient from $\mathbb{Z}$ to $\mathbb{Z}_2$ will not affect the computation.
Hence we have $H^n(S^1\times S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0,2 \\ \mathbb{Z}^2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 3 \end{cases}$, and
$H^n(S^1\times S^1;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } n = 0,2 \\ \mathbb{Z}_2^2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 3 \end{cases}$
(b) Compute $H^n(\mathbb{R}P^2;\mathbb{Z})$

First, we have a chain complex

$$0 \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

where $C_0 = \langle v,w\rangle$, $C_1 = \langle a,b,c\rangle$ and $C_2 = \mathbb{Z}\langle U,L\rangle$. Apply $\mathrm{Hom}(-,Z)$, we obtain the cochain complex

$$0 \to C^2 \xrightarrow{\delta_2} C^1 \xrightarrow{\delta_1} C^0 \xrightarrow{\delta_0} 0$$

Now, find a basis for each cochain: $C^0 = \langle\nu,\eta\rangle$ where $\nu(v) = 1$, $\nu(w) = 0$; $\eta(w) = 1$, $\eta(v) = 0$. $C^1 = \langle\alpha,\beta,\gamma\rangle$ where $\alpha(a) = 1$, $\alpha(b) = \alpha(c) = 0$; $\beta(b) = 1$, $\beta(a) = \beta(c) = 0$ and $\gamma(c) = 1$, $\gamma(a) = \gamma(b) = 0$. Also, $C^2 = \langle\lambda,\mu\rangle$ where $\lambda(U) = 1$, $\lambda(L) = 0$; $\mu(L) = 1$, $\mu(U) = 0$.

Compute $H^0(X)$ -
$\delta_1\nu(a) = \nu d_1(a) = \nu(w-v) = \nu(w)-\nu(v) = -1$
$\delta_1\nu(b) = \nu d_1(b) = \nu(w-v) = \nu(w)-\nu(v) = -1$
$\delta_1\nu(c) = \nu d_1(c) = \nu(v-v) = \nu(v)-\nu(v) = 0$
$\delta_1\eta(a) = \eta d_1(a) = \eta(w-v) = \eta(w)-\eta(v) = 1$
$\delta_1\eta(b) = \eta d_1(b) = \eta(w-v) = \eta(w)-\eta(v) = 1$
$\delta_1\eta(c) = \eta d_1(c) = \eta(v-v) = \eta(v)-\eta(v) = 0$

So $\ker\delta_1 = \langle\nu+\eta\rangle$. Note that $\mathrm{Im}\delta_0 = 0$. Hence $H^0(X) = \dfrac{\ker\delta_1}{\mathrm{Im}\delta_0} = \dfrac{\langle\nu+\eta\rangle}{0} = \mathbb{Z}$.

Compute $H^1(X)$ -
Write $\mathrm{Im}\delta_1$ in terms of α, β and γ. From above, we see that $\delta_1\nu = -\alpha-\beta$ and $\delta_1\eta = \alpha+\beta$,

so $\text{Im}\delta_1 = \langle \alpha + \beta \rangle$. Now, compute $\ker \delta_2$, since

$$\begin{aligned}
&\delta_2\alpha(U) = \alpha d_2(U) = \alpha(-a+b+c) = -\alpha(a) + \alpha(b) + \alpha(c) = -1\\
&\delta_2\beta(U) = \beta d_2(U) = \beta(-a+b+c) = -\beta(a) + \beta(b) + \beta(c) = 1\\
&\delta_2\gamma(U) = \gamma d_2(U) = \gamma(-a+b+c) = \gamma(a) + \gamma(b) - \gamma(c) = 1\\
&\delta_2\alpha(L) = \alpha d_2(L) = \alpha(a-b+c) = \alpha(a) - \alpha(b) + \alpha(c) = 1\\
&\delta_2\beta(L) = \beta d_2(L) = \beta(a-b+c) = \beta(a) - \beta(b) + \beta(c) = -1\\
&\delta_2\gamma(L) = \gamma d_2(L) = \gamma(a-b+c) = \gamma(a) - \gamma(b) - \gamma(c) = 1
\end{aligned}$$

So $\alpha+\beta$ is a basis for $\ker \delta_2$ since $\delta_2(\alpha+\beta)(U) = \delta_2(\alpha+\gamma)(L) = 0$. So $\ker \delta_2 = \langle \alpha+\beta \rangle$. Hence $H^1(X) = \dfrac{\ker \delta_2}{\text{im}\delta_1} = \dfrac{\langle \alpha+\beta \rangle}{\langle \alpha+\beta \rangle} = \dfrac{\mathbb{Z}}{\mathbb{Z}} = 0$.

Compute $H^2(X)$ -
Note that $\ker \delta_3 = \langle \lambda, \mu \rangle$. To compute $\text{Im}\delta_2$, write it in terms of λ and μ.
$\delta_2\alpha(U) = -1 = (-\lambda + \mu)(U)$
$\delta_2\beta(U) = 1 = (\lambda + \mu)(U)$
$\delta_2\gamma(U) = 1 = (\lambda + \mu)(U)$
$\delta_2\alpha(L) = 1 = (\lambda + \mu)(L)$
$\delta_2\beta(L) = -1 = (\lambda - \mu)(L)$
$\delta_2\gamma(L) = 1 = (\lambda + \mu)(L)$
So $\text{Im}\delta_2 = \langle -\lambda + \mu, \lambda + \mu \rangle$.
So $H^2(X) = \dfrac{\ker \delta_3}{\text{Im}\delta_2} = \dfrac{\langle \lambda, \mu \rangle}{\langle -\lambda + \mu, \lambda + \mu \rangle} = \dfrac{\langle \lambda + \mu, \mu \rangle}{\langle \lambda + \mu, 2\mu \rangle} = \langle \lambda \rangle = \dfrac{\mu}{2\mu} = \mathbb{Z}_2$

Hence we have $H^n(\mathbb{R}P^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}_2 & \text{for } n = 2 \\ 0 & \text{for } n = 1, n \geq 3 \end{cases}$

Now, compute $H^n(\mathbb{R}P^2; \mathbb{Z}_2)$ -

The bases for cochain C^i are the same as in the case when $G = \mathbb{Z}$.

Compute $H^0(X)$ -
Note that in $\mathbb{Z}_2$, we have $\delta_1\nu(a) = \delta_1\nu(b) = \delta_1\eta(a) = \delta_1\eta(b) = 1$ and $\delta_1\nu(c) = \delta_1\eta(c) = 0$.

So $\ker \delta_1 = \langle \nu - \eta \rangle$. Since $\text{Im}\delta_0 = 0$, we have $H^0(X) = \dfrac{\ker \delta_1}{\text{Im}\delta_0} = \dfrac{\langle \nu - \eta \rangle}{0} = \mathbb{Z}_2$.

Compute $H^1(X)$ -
Write $\text{Im}\delta_1$ in terms of α, β and γ. We see that $\delta_1\nu = \delta_1\eta = \alpha + \beta$, so $\text{Im}\delta_1 = \langle \alpha + \beta \rangle$. Now, compute $\ker \delta_2$, since

$\delta_2\alpha(U) = \delta_2\beta(U) = \delta_2\gamma(U) = \delta_2\alpha(L) = \delta_2\beta(L) = \delta_2\gamma(L) = 1$ in $\mathbb{Z}_2$.

So the basis for $\ker \delta_2$ would be $\langle \alpha + \beta, \alpha + \gamma \rangle$ since $\delta_2(\alpha + \beta)(U) = \delta_2(\alpha + \gamma)(L) = 2 = 0$ in $\mathbb{Z}_2$. Hence $H^1(X) = \dfrac{\ker \delta_2}{\text{im}\delta_1} = \dfrac{\langle \alpha + \beta, \alpha + \gamma \rangle}{\langle \alpha + \beta \rangle} = \langle \alpha + \gamma \rangle = \mathbb{Z}_2$.

Compute $H^2(X)$ -
Note that $\ker \delta_3 = \langle \lambda, \mu \rangle$. On the other hand,

$\delta_2\alpha(U) = \delta_2\beta(U) = \delta_2\gamma(U) = (\lambda+\mu)(U) = 1$
$\delta_2\alpha(L) = \delta_2\beta(L) = \delta_2\gamma(L) = (\lambda+\mu)(L) = 1$

So $\mathrm{Im}\delta_2 = \langle \lambda + \mu \rangle$.
So $H^2(X) = \dfrac{\ker \delta_3}{\mathrm{Im}\delta_2} = \dfrac{\langle \lambda, \mu \rangle}{\langle \lambda + \mu \rangle} = \dfrac{\langle \lambda + \mu, \mu \rangle}{\langle \lambda + \mu \rangle} = \langle \mu \rangle = \mathbb{Z}_2$
Hence we have $H^n(\mathbb{R}P^2; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } n = 0, 1, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$

(c) Compute $H^n(K, \mathbb{Z})$
Again we had a chain complex

$$0 \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

where $C_0 = \langle v \rangle$, $C_1 = \langle a, b, c \rangle$ and $C_2 = \langle U, L \rangle$. Apply $\mathrm{Hom}(-, Z)$, we obtain the cochain complex

$$0 \to C^2 \xrightarrow{\delta_2} C^1 \xrightarrow{\delta_1} C^0 \xrightarrow{\delta_0} 0$$

Now, find a basis for each cochain: $C^0 = \langle \nu \rangle$ where $\nu(v) = 1$. $C^1 = \langle \alpha, \beta, \gamma \rangle$ where $\alpha(a) = 1$, $\alpha(b) = \alpha(c) = 0$; $\beta(b) = 1$, $\beta(a) = \beta(c) = 0$ and $\gamma(c) = 1$, $\gamma(a) = \gamma(b) = 0$. Also, $C^2 = \langle \lambda, \mu \rangle$ where $\lambda(U) = 1$, $\lambda(L) = 0$; $\mu(L) = 1$, $\mu(U) = 0$.

Compute $H^0(X)$ -

$\delta_1\nu(a) = \nu d_1(a) = \nu(v - v) = 0$, similarly $\delta_1\nu(b) = \delta_1\nu(c) = 0$. So $\delta_1 = 0$. Hence $H^0(X) = \dfrac{\ker \delta_1}{\mathrm{im} d_0} = \dfrac{C^0}{0} = \dfrac{\mathbb{Z}}{0} = \mathbb{Z}$

Compute $H^1(X)$ -

Since $\delta_1 = 0$, so $\mathrm{Im}\delta_1 = 0$. Now, to compute δ_2, we have
$\delta_2\alpha(U) = \alpha d_2(U) = \alpha(a + b - c) = \alpha(a) + \alpha(b) - \alpha(c) = 1$
$\delta_2\beta(U) = \beta d_2(U) = \beta(a + b - c) = \beta(a) + \beta(b) - \beta(c) = 1$
$\delta_2\gamma(U) = \gamma d_2(U) = \gamma(a + b - c) = \gamma(a) + \gamma(b) - \gamma(c) = -1$
$\delta_2\alpha(L) = \alpha d_2(L) = \alpha(a - b + c) = \alpha(a) - \alpha(b) + \alpha(c) = 1$
$\delta_2\beta(L) = \beta d_2(L) = \beta(a - b + c) = \beta(a) - \beta(b) + \beta(c) = -1$
$\delta_2\gamma(L) = \gamma d_2(L) = \gamma(a - b + c) = \gamma(a) - \gamma(b) + \gamma(c) = 1$

So $\ker \delta_2 = \langle \beta + \gamma \rangle$ since $\delta_2(\beta + \gamma)(U) = \delta_2(\beta + \gamma)(L) = 0$.

Hence $H^1(X) = \dfrac{\ker \delta_2}{\mathrm{im}\delta_1} = \dfrac{\langle \beta + \gamma \rangle}{0} = \mathbb{Z}$.

Compute $H^2(X)$ -

Note that $\ker \delta_3 = \langle \lambda, \mu \rangle$. To compute $\mathrm{Im}\delta_2$, write it in terms of λ and μ.

$\delta_2\alpha(U) = 1 = (\lambda + \mu)(U)$
$\delta_2\beta(U) = 1 = (\lambda + \mu)(U)$
$\delta_2\gamma(U) = -1 = (-\lambda - \mu)(U)$
$\delta_2\alpha(L) = 1 = (\lambda + \mu)(L)$

$\delta_2\beta(L) = -1 = (-\lambda + \mu)(L)$
$\delta_2\gamma(L) = 1 = (\lambda + \mu)(L)$

So $\text{Im}\delta_2 = \langle -\lambda + \mu, \lambda + \mu \rangle$.

And $H^2(X) = \dfrac{\ker \delta_3}{\text{Im}\delta_2} = \dfrac{\langle \lambda, \mu \rangle}{\langle -\lambda + \mu, \lambda + \mu \rangle} = \dfrac{\langle \lambda + \mu, \mu \rangle}{\langle \lambda + \mu, 2\mu \rangle} = \langle \lambda \rangle = \dfrac{\mu}{2\mu} = \mathbb{Z}_2$

Hence we have $H^n(K;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ \mathbb{Z}_2 & \text{for } n = 2 \\ 0 & \text{for } n \geq 3 \end{cases}$

Compute $H^n(K;\mathbb{Z}_2)$

The bases for cochain C^i are the same as in the case when $G = \mathbb{Z}$.

Compute $H^0(X)$ -

$\delta_1\nu(a) = \delta_1\nu(b) = \delta_1\nu(c) = 0$, so $\delta_1 = 0$.

So $\ker \delta_1 = \mathbb{Z}$. Note that $\text{Im}\delta_0 = 0$. Hence $H^0(X) = \dfrac{\ker \delta_1}{\text{Im}\delta_0} = \dfrac{\mathbb{Z}}{0} = \mathbb{Z}$.

Compute $H^1(X)$ -

From above we had $\delta_1 = 0$ so $\text{Im}\delta_1 = 0$. Now, compute $\ker \delta_2$, since

$\delta_2\alpha(U) = \delta_2\beta(U) = \delta_2\gamma(U) = \delta_2\alpha(L) = \delta_2\beta(L) = \delta_2\gamma(L) = 1$ in $\mathbb{Z}_2$.

So the basis for $\ker \delta_2$ would be $\langle \alpha + \beta, \alpha + \gamma \rangle$ since $\delta_2(\alpha + \beta)(U) = \delta_2(\alpha + \gamma)(L) = 2 = 0$ in $\mathbb{Z}_2$. Hence $H^1(X) = \dfrac{\ker \delta_2}{\text{im}\delta_1} = \dfrac{\langle \alpha + \beta, \alpha + \gamma \rangle}{0} = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Compute $H^2(X)$ -
Note that $\ker \delta_3 = \langle \lambda, \mu \rangle$. On the other hand, we have
$\delta_2\alpha(U) = \delta_2\beta(U) = \delta_2\gamma(U) = (\lambda + \mu)(U) = 1$
$\delta_2\alpha(L) = \delta_2\beta(L) = \delta_2\gamma(L) = (\lambda + \mu)(L) = 1$

So $\text{Im}\delta_2 = \langle \lambda + \mu \rangle$.
So $H^2(X) = \dfrac{\ker \delta_3}{\text{Im}\delta_2} = \dfrac{\langle \lambda, \mu \rangle}{\langle \lambda + \mu \rangle} = \dfrac{\langle \lambda + \mu, \mu \rangle}{\langle \lambda + \mu \rangle} = \langle \mu \rangle = \mathbb{Z}_2$

Hence we have $H^n(K;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } n = 0, 2 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 3 \end{cases}$

3.1.8 Many basic homology arguments work just as well for cohomology even though maps go in the opposite direction. Verify this in the following cases:
(a) Compute $H^i(S^n;G)$ by induction on n in two ways; using the long exact sequence of a pair, and using the Mayer-Vietoris sequence.
(b) Show that if A is a closed subspace of X that is a deformation retract of some neighborhood, then the quotient map $X \to X/A$ induces isomorphism $H^n(X, A;G) \approx$

$\tilde{H}^n(X/A; G)$ **for all** n**.**
(c) Show that if A **is a retract of** X **then** $H^n(X;G) \approx H^n(A;G) \oplus H^n(X,A;G)$**.**

Proof

(a) When $n = 0$, S^0 is a two-point set. As in homology groups since $H^0(S^0) = G \oplus G$ and $H^0(S^0) = \widetilde{H}^0(S^0) \oplus G$, so $\widetilde{H}^0(S^0) = G$.

When $n > 0$, consider the LES for the pair (D^n, S^{n-1}).

$$\cdots \to H^{i-1}(D^n) \to H^{i-1}(S^{n-1}) \to H^i(D^n, S^{n-1}) \to H^i(D^n) \to H^i(S^{n-1}) \to \cdots$$

Since $\widetilde{H}^i(D^n) = 0$ because D^n is contractible. Also $H^i(D^n, S^{n-1}) \cong H^i(D^n/S^{n-1}) \cong H^i(S^n)$. So the sequence becomes...

$$0 \to H^{i-1}(S^{n-1}) \xrightarrow{\cong} H^i(S^n) \to 0$$

Hence the maps $f : H^{i-1}(S^{n-1}) \to H^i(S^n)$ are isomorphism for all $i > 0$. By induction,

$$H^i(S^n) = \begin{cases} G & \text{for } i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Now, using Mayer-Vietoris sequence.

For $n = 1$, $H^0(S^1) \cong \operatorname{Hom}(H_0(S^1), G) \cong \operatorname{Hom}(G, G) \cong G$ by Universal Coefficient Theorem in which no Ext term.

For $n > 1$, let A, B be upper and lower semi-spheres D_N, D_S of S^n. So $D_N = D_S = D^{n-1}$. Let U, V be the neighborhood of A, B respectively. So A, B are deformation retracts of U, V with $U \cap V$ deformation retracting to $A \cap B \cong S^{n-1}$. We have

$$\cdots \to H^{i-1}(S^n) \to H^{i-1}(A) \oplus H^{i-1}(B) \to H^{i-1}(S^{n-1}) \to H^i(S^n) \to H^i(A) \oplus H^i(B) \to \cdots$$

and it becomes...

$$0 \to H^{i-1}(S^{n-1}) \xrightarrow{\cong} H^i(S^n) \to 0$$

By induction, we have the same result.

(b)
If A is a closed subspace of X that is a deformation retract of some neighborhood, prove that the quotient map $X \to X/A$ induces isomorphisms $H^n(X, A; G) \cong \widetilde{H}^n(X/A; G)$ for all n. The proof is actually similar to the proof of Proposition 2.22 except the arrows go in the opposite direction.

3.1.9 Show that if $f : S^n \to S^n$ **has degree** d **then** $f^* : H^n(S^n; G) \to H^n(S^n; G)$ **is multiplication by** d**.**

Proof

If the map $f : S^n \to S^n$ has degree d, then $f_* : \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$, $\alpha \mapsto d\alpha$ for an integer d. i.e. $f_* = \cdot d$

Use the Universal Coefficient Theorem and consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(S^n);G) \arrow[r] & H^n(S^n,G) \arrow[r] & \operatorname{Hom}(H_n(S^n),G) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(S^n);G) \arrow[r] \arrow[u] & H^n(S^n,G) \arrow[r] \arrow[u] & \operatorname{Hom}(H_n(S^n),G) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Since $\operatorname{Ext}(H_{n-1};G) = 0$ so we have the diagram.

$$\begin{tikzcd}
0 \arrow[r] & H^n(S^n,G) \arrow[r, "g"] & \operatorname{Hom}(\mathbb{Z},G) \arrow[r] & 0 \\
0 \arrow[r] & H^n(S^n,G) \arrow[r, "g"] \arrow[u, "f^*"] & \operatorname{Hom}(\mathbb{Z},G) \arrow[r] \arrow[u, "d^*"] & 0
\end{tikzcd}$$

Since two horizontal maps g are isomorphisms. Want to show that the right vertical map is a multiplication by d, then so is f^*. To see this, for $\varphi \in \operatorname{Hom}(\mathbb{Z};G)$ in the right lower corner, the dual homomorphism $d^* : \operatorname{Hom}\ (\mathbb{Z},G) \to \operatorname{Hom}(\mathbb{Z},G)$ defined by $d^*\varphi = \varphi d$,

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, "d"'] \arrow[dr, "\varphi\cdot d"] & \\
\mathbb{Z} \arrow[r, "\varphi"'] & G
\end{tikzcd}$$

(I think this is wrong, this is actually like 3.1.2 where says that multiplication by $n : H \to H$ on the abelian group H will result in multiplication by n on $\operatorname{Hom}(H,G)$. Here H is $H_n(S^n)$. But the following is correct though):

hence $d^*\varphi(1) = (\varphi \circ d)(1) = \varphi(d \cdot 1) = \varphi(d) = d\varphi(1)$ (refer 3.1.2 and D&D page 392). So the right hand side vertical map d^* is a multiplication by d. We could check further that for $x \in H^n(S^n,G)$, since $f^* = g^{-1} \circ d \circ g$, so $f^*(x) = (g^{-1} \circ d \circ g)(x) = (g^{-1}dg(x)) = dg^{-1}g(x) = dx$.

Hence $f^* : H^n(S^n) \to H^n(S^n)$ is a degree d map.

3.1.11 Let X be a Moore space $M(\mathbb{Z}_m,n)$ obtained from S^n by attaching a cell e^{n+1} by a map of degree m.
(a) Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\widetilde{H}_i(-;\mathbb{Z})$ for all i, but not on $H^{n+1}(-;\mathbb{Z})$. Deduce that the splitting in the universal coefficient theorem for cohomology cannot be natural.
(b) Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on $\tilde{H}^i(-;\mathbb{Z})$ for all i, but not on $H_n(-;\mathbb{Z})$.

Proof

(a) Let X denote the Moore space $M(\mathbb{Z}_m,n)$, in chapter 2 we know that $\widetilde{H}_i(X) = \mathbb{Z}_m$ for $i = n$. Also, $H_i(S^{n+1}) = \mathbb{Z}$ for $i = n+1$ and 0 otherwise. So the quotient map $q : X \to X/S^n = S^{n+1}$ induces a trivial map $q_* : \widetilde{H}_i(X;\mathbb{Z}) \to \widetilde{H}_i(S^{n+1};\mathbb{Z})$ for all i.

To prove that it does not induce a trivial map on $H^{n+1}(-;\mathbb{Z})$, Consider the LES for the pair (X,S^n).

$$\cdots \to H^n(X) \to H^n(S^n) \to H^{n+1}(X,S^n) \to H^{n+1}(X) \to H^{n+1}(S^n) \to \cdots$$

First, we need to compute $H^n(X)$ and $H^{n+1}(X)$. By Universal Coefficient Theorem, we have

$$0 \to \text{Ext}(H_{n-1}(X);\mathbb{Z}) \to H^n(X;\mathbb{Z}) \to \text{Hom}(H_n(X);\mathbb{Z}) \to 0$$

Since $\text{Hom}(H_n(X);\mathbb{Z}) = 0$ and $\text{Ext}(H_{n-1}(X);\mathbb{Z}) = \text{Ext}(0;\mathbb{Z}) = 0$. So $H^n(X;\mathbb{Z}) = 0$.

On the other hand,

$$0 \to \text{Ext}(H_n(X);\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}) \to \text{Hom}(H_{n+1}(X);\mathbb{Z}) \to 0$$

Since $\text{Hom}(H_{n+1}(X);\mathbb{Z}) = 0$ and $\text{Ext}(H_n(X);\mathbb{Z}) = \text{Ext}(\mathbb{Z}_m;\mathbb{Z}) = \mathbb{Z}_m$. So $H^{n+1}(X;\mathbb{Z}) = \mathbb{Z}_m$.

Also, $H^{n+1}(X, S^n) \cong H^{n+1}(X/S^n) \cong H^{n+1}(S^{n+1}) \cong \mathbb{Z}$. Hence the LES above becomes

$$\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_m \to 0 \to \cdots$$

Thus the induced map $q^* : H^{n+1}(X, S^n;\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z})$ is surjective and clearly non-zero.

Finally, show that the splitting in the UCT for cohomology cannot be natural. Consider the diagram,

$$\begin{tikzcd}
H^{n+1}(X,\mathbb{Z}) \arrow[r, phantom, "\cong"] & \text{Ext}(H_n(X);\mathbb{Z}) \arrow[r, phantom, "\bigoplus"] & \text{Hom}(H_{n+1}(X;\mathbb{Z})) \\
H^{n+1}(S^{n+1},\mathbb{Z}) \arrow[u, "q^*"] \arrow[r, phantom, "\cong"] & \text{Ext}(H_n(S^{n+1});\mathbb{Z}) \arrow[u] \arrow[r, phantom, "\bigoplus"] & \text{Hom}(H_{n+1}(S^{n+1};\mathbb{Z})) \arrow[u]
\end{tikzcd}$$

With what we have calculated, this diagram is actually

$$\begin{tikzcd}
\mathbb{Z}_m \arrow[r, phantom, "\cong"] & \mathbb{Z}_m \arrow[r, phantom, "\bigoplus"] & 0 \\
\mathbb{Z} \arrow[u, "q^*"] \arrow[r, phantom, "\cong"] & 0 \arrow[u] \arrow[r, phantom, "\bigoplus"] & \mathbb{Z} \arrow[u]
\end{tikzcd}$$

which apparently does not commute. Hence the splitting in the universal coefficient theorem for cohomology cannot be natural.

(b) Show that the inclusion $i : S^n \to X$ induces the trivial map on $\widetilde{H}^i(-;\mathbb{Z})$ for all i.

Recall that 3.1.8(a) $\widetilde{H}^i(S^n;\mathbb{Z}) = \mathbb{Z}$ when $i = n$. From the LES in (a) we see that the induced map $H^i(X;\mathbb{Z}) \to H^i(S^n;\mathbb{Z})$ are indeed trivial when $i = n, n+1$, and the map are zero at all other i.

Now show that the inclusion induces a non-trivial map on $H_n(-;\mathbb{Z})$. Consider the LES for the pair (X, S^n),

$$\cdots \to H_{n+1}(S^n) \to H_{n+1}(X) \to H_{n+1}(X, S^n) \to H_n(S^n) \to H_n(X) \to H_n(X, S^n) \cdots$$

From information in (a), we see that the LES becomes

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z}_m \longrightarrow 0$$

in which the map $f : \mathbb{Z} \to \mathbb{Z}_m$ is non-zero.

3.2.1 Assuming as known that cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to a wedge sum of g tori.

First, $H^i(M_g)= \begin{cases} \mathbb{Z} & \text{for } i=0,2 \\ \mathbb{Z}^{2g} & \text{for } i=1 \\ 0 & \text{for } i \geq 3 \end{cases}$

by UCT.

Look at the LES for the pair (M_g, A) where A is the subspace of M_g that is mod out (see the graph in Hatcher's p228), we have

$$\cdots \to H^1(M_g, A) \to H^1(M_g) \to H^1(A) \to H^2(M_g, A) \to H^2(M_g) \to \cdots$$

Note that A is homeomorphic to S^2 with $g-1$ disks removed, hence is homeomorphic to the wedge of $g-1$ circles. So $\widetilde{H}^i(A) \cong \widetilde{H}^i(\vee_{g-1} S^1) = \mathbb{Z}^{g-1}$ for $i=1$ and 0 otherwise. Also $\widetilde{H}^i(M_g, A) \cong \widetilde{H}^i(M_g/A) \cong \widetilde{H}^i(\vee_g T) \cong \bigoplus_g H^i(T)$. We have computed the cohomology groups for torus in problem 3.1.6(1), so we have

$\widetilde{H}^i(M_g, A)= \begin{cases} \mathbb{Z}^g & \text{for } i=2 \\ \mathbb{Z}^{2g} & \text{for } i=1 \\ 0 & \text{for } i \geq 3 \end{cases}$

Hence the above sequence is actually ...

$$\cdots \to \mathbb{Z}^{2g} \xrightarrow{f} \mathbb{Z}^{2g} \to \mathbb{Z}^{g-1} \xrightarrow{g} \mathbb{Z}^g \xrightarrow{h} \mathbb{Z} \to \cdots$$

Since g is injective and h is surjective, so this sequence breaks into two short exact sequences.

$$0 \to \mathbb{Z}^{2g} \xrightarrow{f} \mathbb{Z}^{2g} \to 0$$

and

$$0 \to \mathbb{Z}^{g-1} \xrightarrow{g} \mathbb{Z}^g \xrightarrow{h} \mathbb{Z} \to 0$$

So f is an isomorphism. Hence we have $H^1(M_g, A) \cong H^1(M_g)$.

Now, want to compute the cup product structure on $H^*(M_g)$ by using $H^*(M_g/A) = H^*(\vee_g T)$. By Example 3.11, for the cohomology classes in i-th Torus, α_{i_1}, α_{i_2} in $H^1(T_i)$, we have $\alpha_{i_1} \smile \alpha_{i_2} = \beta_i$ which is a generator in $H^2(T_i)$. But for $i \neq j$, then $\alpha_i \smile \alpha_j = 0$

I will come back to this problem later.

3.2.2 Using the cup product $H^k(X, A; R) \times H^l(X, B; R) \to H^{k+l}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. This applies in particular if X is a suspension. Generalize to the situation that X is the union of n contractible open subsets, to show that all n-fold cup products of positive-dimensional classes are zero.

Proof

$\widetilde{H}^i(A) = \widetilde{H}^i(B) = 0$ since A, B are contractible. By LES for the pair (X, A), we have

$$\cdots \to H^{i-1}(A) \to H^i(X,A) \xrightarrow{\cong} H^i(X) \to H^i(A) \to \cdots$$

Similarly, we also have $H^i(X,B) \cong H^i(X)$.

Consider the diagram

$$\begin{tikzcd}
\widetilde{H}^k(X;R) \times \widetilde{H}^l(X;R) \arrow[r, "f"] & \widetilde{H}^{k+l}(X;R) \\
\widetilde{H}^k(X,A;R) \times \widetilde{H}^l(X,B;R) \arrow[u, "\cong"] \arrow[r, "g"] & \widetilde{H}^{k+l}(X, A\cup B;R) \arrow[u]
\end{tikzcd}$$

The bottom line is $(*)$ on page 209. The left vertical isomorphism came from the above discussion. Also, $\widetilde{H}^{k+l}(X, A\cup B;R) = \widetilde{H}^{k+l}(X/A\cup B;R) = 0$, hence $\widetilde{H}^{k+l}(X,R) = 0$. Thus for cohomology classes $\phi \in \widetilde{H}^k(X,A;R)$ and $\psi \in \widetilde{H}^k(X,A;R)$, the cup product $\phi \smile \psi = 0$ since g is a zero map. By the naturality of the cup product, i.e. the commutativity of the diagram, $H^*(X;R) = 0$ for all positive dimensional classes.

In general, let $X = A_1 \cup \cdots \cup A_n$ and A_i contractible. Consider the diagram,

$$\begin{tikzcd}
\widetilde{H}^{k_1}(X;R) \times \cdots \times \widetilde{H}^{k_n}(X;R) \arrow[r, "\smile"] & \widetilde{H}^{\sum k_i}(X;R) \\
\widetilde{H}^{k_1}(X,A_1;R) \times \cdots \times \widetilde{H}^{k_n}(X,A_n;R) \arrow[u, "\cong"] \arrow[r, "\smile g"] & \widetilde{H}^{\sum k_i}(X, A_1\cup\cdots\cup A_n;R) \arrow[u]
\end{tikzcd}$$

Since for $\phi_i \in \widetilde{H}^{k_i}(X,A_1;R)$, $g : \phi_1 \smile \cdots \smile \phi_n \mapsto 0 \in \widetilde{H}^{\sum k_i}(X, A_1\cup\cdots\cup A_n;R)$, then by the commutativity of the diagram, $\widetilde{H}^{\sum k_i}(X;R) = 0$ for all i. Hence all n-fold cup products of positive dimensional classes are zero.

3.2.3 (a) Using the cup product structure, show there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m;\mathbb{Z}_2) \to H^1(\mathbb{R}P^n;\mathbb{Z}_2)$ if $n > m$. What is the corresponding result for maps $\mathbb{C}P^n \to \mathbb{C}P^m$?
(b) Prove the Bursuk-Ulam theorem by the following argument. Suppose on the contrary that $f : S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all x. Then define $g : S^n \to S^{n-1}$ by $g(x) = (f(x) - f(-x))/|f(x) - f(-x)|$, so $g(-x) = -g(x)$ and g induces a map $\mathbb{R}P^n \to \mathbb{R}P^{n-1}$. Show that part (a) applies to this map.

Proof

(a) Suppose $f : \mathbb{R}P^n \to \mathbb{R}P^m$ induces a nontrivial map on $H^1(-;\mathbb{Z}_2)$. We have $f^* : H^1(\mathbb{R}P^m;\mathbb{Z}_2) \to H^1(\mathbb{R}P^n;\mathbb{Z}_2)$ for $n > m$. By Theorem 3.12, this implies $f^* : \mathbb{Z}_2[\alpha]/(\alpha^{m+1}) \to \mathbb{Z}_2[\beta]/(\beta^{n+1})$. So this implies that $f^*(\alpha) = \beta$ is non-trivial. Since $\alpha^{m+1} = 0$ in $\widetilde{H}(\mathbb{R}P^m;\mathbb{Z}_2)$ but $0 = f^*(\alpha^{m+1}) = \beta^{m+1} \neq 0$ since $n > m$, a contradiction. Hence there's no such an map f exists.

Alternatively, assume there is an induced nontrivial map $f^* : H^1(\mathbb{RP}^m; \mathbb{Z}_2) \to H^1(\mathbb{RP}^n; \mathbb{Z}_2)$. Consider the following diagram

$$\begin{tikzcd} H^1(\mathbb{RP}^m; \mathbb{Z}_2) \times H^m(\mathbb{RP}^m; \mathbb{Z}_2) \arrow[r, "\smile"] \arrow[d, "f^*"] & H^{m+1}(\mathbb{RP}^m; \mathbb{Z}_2) \arrow[d, "f^*"] \\ H^1(\mathbb{RP}^n; \mathbb{Z}_2) \times H^m(\mathbb{RP}^n; \mathbb{Z}_2) \arrow[r, "\smile"] & H^{m+1}(\mathbb{RP}^n; \mathbb{Z}_2) \end{tikzcd}$$

Suppose $n > m$. On the top row, for the generator $\alpha \in H^1(\mathbb{RP}^m; \mathbb{Z}_2)$, cup product gives $\alpha^m \in H^m(\mathbb{RP}^m; \mathbb{Z}_2)$, hence $\alpha^{m+1} = 0 \in H^{m+1}(\mathbb{RP}^m; \mathbb{Z}_2)$. However, on the bottom row, $\beta \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$, cup product gives $\beta^m \in H^m(\mathbb{RP}^n; \mathbb{Z}_2)$, hence $\beta^{m+1} \neq 0 \in H^{m+1}(\mathbb{RP}^n; \mathbb{Z}_2)$ since $|\beta| = n + 1 > m + 1$. A contradiction! Hence there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}_2) \to H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ if $n > m$.

So what is the corresponding result for maps $\mathbb{C}P^n \to \mathbb{C}P^m$?
Similarly, suppose there is an induces a map $f^* : H^2(\mathbb{C}P^m; \mathbb{Z}) \to H^2(\mathbb{C}P^n; \mathbb{Z})$ for $n > m$. By Theorem 3.12, this implies $f^* : \mathbb{Z}[\alpha]/(\alpha^{m+1}) \to \mathbb{Z}[\beta]/(\beta^{n+1})$. Suppose $f^*(\alpha) = \beta$ is non-trivial, then $0 = f^*(\alpha^{m+1}) = \beta^{m+1} \neq 0$ since $n > m$, a contradiction. Hence, no such a map exists.

(b) Suppose on the contrary that $f(x) \neq -f(x)$ for all x. Define $g : S^n \to S^{n-1}$ by $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$, so $g(-x) = -g(x)$ and g induces a map $\mathbb{R}P^n \to \mathbb{R}P^{n-1}$ as follows

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d] & S^{n-1} \arrow[d] \\ \mathbb{R}P^n \arrow[r, "g"] & \mathbb{R}P^{n-1} \end{tikzcd}$$

Let $[f] \in \mathbb{R}P^n$ be a loop. It lifts to a path in its universal cover S^n from a point x to its antipodal $-x$. The image of that path under g is a path from $g(x)$ to $g(-x) = -g(x)$. The image of the path under the covering map is a loop that is not nullhomotopic. So we have that $g_* : \pi_1(\mathbb{R}P^n) \to \pi_1(\mathbb{R}P^{n-1})$ is nontrivial.

If $n = 2$, this is a contradiction since there is no nontrivial homomorphism $\mathbb{Z}_2 \to \mathbb{Z}$. If $n = 1$, then $\mathbb{R}P^0$ is a point, and we have a contradiction because there is no nontrivial homomorphism $\mathbb{Z} \to 0$. So we assume that $n > 2$. In particular, the nontrivial homomorphism g_* is in fact an isomorphism.

Note that since H_1 is the abelianization of π_1. Since $\pi_1(X)$ is abelian, we have

$$\begin{tikzcd} \pi_1(\mathbb{R}P^n) \arrow[r, "g_*"] \arrow[d, "\cong"] & \pi_1(\mathbb{R}P^{n-1}) \arrow[d, "\cong"] \\ H_1(\mathbb{R}P^n) \arrow[r, "g_*"] & H_1(\mathbb{R}P^{n-1}) \end{tikzcd}$$

The top map is an isomorphism, so the bottom map is also an isomorphism. Now, by UCT, its dual $g^* : H^1(\mathbb{R}P^n) \to H^1(\mathbb{R}P^{n-1})$ is also an isomorphism, taking the generator to the generator. This is a contradiction to (a).

3.2.7 Use cup products to show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

Recall that $\widetilde{H}^i(\mathbb{R}P^3;\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2 & \text{for } i\le 3\\ 0 & \text{else}\end{cases}$

$\widetilde{H}^i(\mathbb{R}P^2\vee S^3;\mathbb{Z}_2)=\widetilde{H}^i(\mathbb{R}P^2;\mathbb{Z}_2)\oplus\widetilde{H}^i(S^3;\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2 & \text{for } i\le 3\\ 0 & \text{else}\end{cases}$

Prove that $\mathbb{R}P^3\not\simeq\mathbb{R}P^2\vee S^3$ by cup product.

Suppose $f:\mathbb{R}P^2\vee S^3\to\mathbb{R}P^3$ induces an isomorphism on H^*. Considering the following diagram,

$$\begin{array}{ccc} H^1(\mathbb{R}P^3;\mathbb{Z}_2)\times H^2(\mathbb{R}P^3;\mathbb{Z}_2) & \xrightarrow{\smile} & H^3(\mathbb{R}P^3;\mathbb{Z}_2)\\ \Big\downarrow f^* & & \Big\downarrow f^*\\ H^1(\mathbb{R}P^2\vee S^3;\mathbb{Z}_2)\times H^2(\mathbb{R}P^2\vee S^3;\mathbb{Z}_2) & \xrightarrow{\smile} & H^3(\mathbb{R}P^2\vee S^3;\mathbb{Z}_2)\end{array}$$

In the top row, for the cohomology class $\alpha\in H^1(\mathbb{R}P^3;\mathbb{Z}_2)$, we have $\alpha^2\in H^2(\mathbb{R}P^3;\mathbb{Z}_2)$ hence $\alpha\smile\alpha^2=\alpha^3\neq 0$ in $H^3(\mathbb{R}P^3;\mathbb{Z}_2)$. In the bottom row, note that $H^i(\mathbb{R}P^2\vee S^3;\mathbb{Z}_2)=H^i(\mathbb{R}P^2;\mathbb{Z}_2)\bigoplus H^i(S^3;\mathbb{Z}_2)$ for $i=1,2$. Since f^* is an isomorphism, let cohomology classes $a=f^*(\alpha)\in H^1(\mathbb{R}P^2;\mathbb{Z}_2)$ and $b=f^*(\alpha^2)\in H^2(\mathbb{R}P^2;\mathbb{Z}_2)$ be the images, we have the cup product $(a,0)\smile(b,0)=(ab,0)$ in $H^3(\mathbb{R}P^2\vee S^3;\mathbb{Z}_2)$. But $ab=0$ in $H^3(\mathbb{R}P^2;\mathbb{Z}_2)$.

By the commutativity of the diagram, the right vertical map which is an isomorphism $f^*:H^3(\mathbb{R}P^2\vee S^3;\mathbb{Z}_2)\to H^3(\mathbb{R}P^3;\mathbb{Z}_2)$ maps $\alpha\neq 0$ to $(ab,0)=(0,0)$, a contradiction!

3.2.8 Let X be $\mathbb{C}P^2$ with a cell e^3 attached by a map $S^2\to\mathbb{C}P^1\subset\mathbb{C}P^2$ of degree p, and let $Y=M(\mathbb{Z}_p,2)\vee S^4$. Thus X and Y have the same 3-skeleton but differ in the way their 4-cells are attached. Show that X and Y have isomorphic cohomology rings with $\mathbb{Z}$ coefficeints but not with $\mathbb{Z}_p$ coefficients.

Proof

First, show that X and Y have isomorphic cohomology rings with $\mathbb{Z}$ coefficient.
Since X is a $\mathbb{C}P^2=e^0\cup e^2\cup e^4$ with e^3 attached by a map $S^2\to\mathbb{C}P^1\subset\mathbb{C}P^2$ of degree p. So X has one 0-cell, one 2-cell, one 3-cell and one 4-cell. The cellular chain complex for X is

$$0\longrightarrow\mathbb{Z}\xrightarrow{d_4}\mathbb{Z}\xrightarrow{d_3=p}\mathbb{Z}\xrightarrow{d_2}0\xrightarrow{d_1}\mathbb{Z}\xrightarrow{d_0}0$$

where $d_1=d_3=d_4=0$. So we have the cochain complex...

$$0\longleftarrow\mathbb{Z}\xleftarrow{\delta_4}\mathbb{Z}\xleftarrow{\delta_3=p}\mathbb{Z}\xleftarrow{\delta_2}0\xleftarrow{\delta_1}\mathbb{Z}\xleftarrow{\delta_0}0$$

Hence $\widetilde{H}^i(X,\mathbb{Z})=\begin{cases}\mathbb{Z}_p & \text{for } i=3\\ \mathbb{Z} & \text{for } i=4\end{cases}$

Alternative, we could compute homology groups which $\widetilde{H}_2(X;\mathbb{Z})=\mathbb{Z}_p$ and $\widetilde{H}_4(X;\mathbb{Z})=\mathbb{Z}$. Then use UCT.

On the other hand, recall that $M(\mathbb{Z}_p,2)$ is space obtained from S^2 with a cell e^3 attached by a map $S^2\to S^2$ of degree p. The homology groups are $\widetilde{H}_i(M(\mathbb{Z}_p,2),\mathbb{Z})=\mathbb{Z}_p$ for $i=2$. Hence we have $\widetilde{H}^i(M(\mathbb{Z}_p,2),\mathbb{Z})=\mathbb{Z}_p$ for $i=3$ by UCT. Hence, if $Y=M(\mathbb{Z}_p,2)\vee S^4$, then

$$\widetilde{H}^i(Y,\mathbb{Z}) = \widetilde{H}^i(M(\mathbb{Z}_p,2),\mathbb{Z}) \bigoplus \widetilde{H}^i(S^4,\mathbb{Z}) = \begin{cases} \mathbb{Z}_p & \text{for } i=3 \\ \mathbb{Z} & \text{for } i=4 \end{cases}$$

Therefore, $\widetilde{H}^i(X,\mathbb{Z}) = \widetilde{H}^i(Y,\mathbb{Z})$. But the cup product structure is 0 since there's no room for any cup product.

Now, show that X and Y do not have isomorphic cohomology rings with $\mathbb{Z}_p$ coefficient.

In $\mathbb{Z}_p$, the cellular cochain complex for X is

$$0 \longleftarrow \mathbb{Z}_p \xleftarrow{\delta_4} \mathbb{Z}_p \xleftarrow{\delta_3=p} \mathbb{Z}_p \xleftarrow{\delta_2} 0 \xleftarrow{\delta_1} \mathbb{Z}_p \xleftarrow{\delta_0} 0$$

Hence, $\widetilde{H}^i(X,\mathbb{Z}_p) = \mathbb{Z}_p$ for $i = 2,3,4$.

Or alternatively, since we already know that $\widetilde{H}_2(X;\mathbb{Z}) = \mathbb{Z}_p$ and $\widetilde{H}_4(X;\mathbb{Z}) = \mathbb{Z}$. By UCT with coefficients $\mathbb{Z}_p$ we get the same result.

Similarly, since homology groups for space Y are $\widetilde{H}_2(Y;\mathbb{Z}) = \mathbb{Z}_p$ and $\widetilde{H}_4(Y;\mathbb{Z}) = \mathbb{Z}$. By UCT we have $\widetilde{H}^i(Y,\mathbb{Z}_p) = \mathbb{Z}_p$ for $i = 2,3,4$.

Now, let's see the cup product structure of in $\mathbb{Z}_2$ coefficient. First, the cup product for Y is

$$H^2(Y,\mathbb{Z}_p) \times H^2(Y,\mathbb{Z}_p) \longrightarrow H^4(Y,\mathbb{Z}_p)$$

Since $Y = M(\mathbb{Z}_p;2) \vee S^4$. The above becomes

$$H^2(M(\mathbb{Z}_p,2);\mathbb{Z}_p)\oplus H^2(S^4;\mathbb{Z}_p)\times H^2(M(\mathbb{Z}_p,2);\mathbb{Z}_p)\oplus H^2(S^4;\mathbb{Z}_p) \longrightarrow H^4(M(\mathbb{Z}_p,2);\mathbb{Z}_p)\oplus H^4(S^4;\mathbb{Z}_p)$$

But $H^2(S^4;\mathbb{Z}_p) = 0$. And note that $H^i(M(\mathbb{Z}_p,2);\mathbb{Z}_p) = \mathbb{Z}_p$ for $i = 2$ zero otherwise by UCT. For $\alpha \in H^2(M(\mathbb{Z}_p,2);\mathbb{Z}_p)$, $\alpha \smile \alpha = \alpha^2 = 0$ in $H^4(M(\mathbb{Z}_p,2);\mathbb{Z}_p)$. Hence the cup product is zero.

To see the cup product for X, since

$$H^2(X;\mathbb{Z}_2) \times H^2(X;\mathbb{Z}_2) \xrightarrow{\smile} H^4(X;\mathbb{Z}_2)$$

For $\beta \in H^2(X,\mathbb{Z}_p)$, the cup product $\beta \smile \beta = \beta^2 \neq 0$ in $H^4(\mathbb{C}P^2,\mathbb{Z}_p)$. i.e. the cup product is non-zero in this case.

Hence X and Y do not have isomorphic cohomology rings with $\mathbb{Z}_p$ coefficient since they have different cup product structure.

3.2.9 Show that if $H_n(X;\mathbb{Z})$ is finitely generated and free for each n, then $H^*(X;\mathbb{Z}_p)$ and $H^*(X;\mathbb{Z}) \otimes \mathbb{Z}_p$ are isomorphic as rings, so in particular the ring structure with $\mathbb{Z}$ coefficients determines the ring structure with $\mathbb{Z}_p$ coefficients.

Proof

If $H_n(X;\mathbb{Z})$ is finitely generated and free for each n, this implies that $H_n(X;\mathbb{Z}) \cong \mathbb{Z}^r$ for some integer r. By UCT, $H^n(X;\mathbb{Z}_p) \cong \mathrm{Hom}(H_n(X);\mathbb{Z}_p) \cong \mathrm{Hom}(\mathbb{Z}^r;\mathbb{Z}_p) \cong (\mathbb{Z}_p)^r$ for all n since Ext term is zero in each dimension. On the other hand, in coefficient $\mathbb{Z}$, $H^n(X;\mathbb{Z}) \cong \mathrm{Hom}(H_n(X);\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}^r;\mathbb{Z}) \cong \mathbb{Z}^r$ by UCT again. So $H^n(X;\mathbb{Z}) \otimes \mathbb{Z}_p \cong \mathbb{Z}^r \otimes \mathbb{Z}_p \cong (\mathbb{Z}_p)^r$. Hence $H^*(X;\mathbb{Z}_p)$ and $H^*(X;\mathbb{Z}) \otimes \mathbb{Z}_p$ have the same cohomology groups hence are isomorphic as rings (No! this is not correct. 3.2.7 is an example.) This says that in particular the ring structure with $\mathbb{Z}$-coefficients determines the ring structure with $\mathbb{Z}_p$-coefficients.

(Not finished yet...)

3.2.11 Using cup products, show that every map $S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism $H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$, assuming $k > 0$ and $l > 0$.

Proof

Suppose $f : S^{k+l} \to S^k \times S^l$ induces the non-trivial homomorphism $H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$. Since all non zero homomorphisms for S^k, S^l and S^{k+l} are free, we could just look at its dual map $f^* : H^{k+l}(S^k \times S^l) \to H^{k+l}(S^{k+l})$ since all Ext terms are zero in UCT. Let $p_1 : S^k \times S^l \to S^k$ and $p_2 : S^k \times S^l \to S^l$ be the projections.

$$\begin{tikzcd}
H^k(S^{k+l};\mathbb{Z}) \times H^l(S^{k+l};\mathbb{Z}) \arrow[rr, "\smile"] & & H^{k+l}(S^{k+l};\mathbb{Z}) \\
H^k(S^k \times S^l;\mathbb{Z}) \times H^l(S^k \times S^l;\mathbb{Z}) \arrow[u, "f^* \times f^*"] \arrow[rr, "\smile"] & & H^{k+l}(S^k \times S^l;\mathbb{Z}) \arrow[u, "f^*"] \\
H^k(S^k;\mathbb{Z}) \times H^l(S^l;\mathbb{Z}) \arrow[u, "p_1^* \times p_2^*"] \arrow[urr, "\times"] & & H^k(S^k;\mathbb{Z}) \otimes H^l(S^l;\mathbb{Z}) \arrow[u, "\times"]
\end{tikzcd}$$

Let $\alpha \in H^k(S^k;\mathbb{Z})$ and $\beta \in H^l(S^l;\mathbb{Z})$ be generators. Then $p_1^*(\alpha) \smile p_2^*(\beta) = \alpha \times \beta \in H^{k+l}(S^k \times S^l;\mathbb{Z})$ is a generator by the definition of **cross product** on page 210. It is nonzero since $\alpha \otimes \beta \neq 0$. Now, if f^* is nonzero, $f^*(p_1^*(\alpha) \cup p_2^*(\beta)) = f^*(p_1^*(\alpha)) \cup f^*(p_2^*(\beta)) \in H^{k+l}(S^{k+l}) \cong \mathbb{Z}$ must be nonzero. By the naturality of the diagram, however, we got zero since the two cohomology groups $H^k(S^{k+l};\mathbb{Z})$ and $H^l(S^{k+l};\mathbb{Z})$ on the top row are zeros, a contradiction. Thus, f^* must be a trivial map.
(Refer Maxim's notes page 69).

Finally, by UCT we have

$$H^{k+1}(S^k \times S^l;\mathbb{Z}) \cong \mathrm{Hom}(H_{k+l}(S^k \times S^l),\mathbb{Z})$$

$$H^{k+1}(S^{k+1},\mathbb{Z}) \cong \mathrm{Hom}(H_{k+l}(S^{k+l});\mathbb{Z})$$

since all Ext terms are zero. So if $f^* : H^{k+l}(S^k \times S^l) \to H^{k+l}(S^k \times S^l)$ is a trivial map, then the induced map $f_* : H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$ is also a trivial map.

3.2.13 Describe $H^*(\mathbb{C}P^\infty/\mathbb{C}P^1;\mathbb{Z})$ as a ring with finitely many multiplicative generators. How does this ring compare with $H^*(S^6 \times \mathbb{H}P^\infty;\mathbb{Z})$?

This is similar to Example 3.24. Since $\mathbb{C}P^\infty = e^0 \cup e^2 \cup e^4 \cdots$. Let X be the quotient space $\mathbb{C}P^\infty/\mathbb{C}P^1$, so $X = e^4 \cup e^6 \cup \cdots$. The quotient map $\mathbb{C}P^\infty \to X$ induces an injection $H^*(X;\mathbb{Z}) \to H^*(\mathbb{C}P^\infty;\mathbb{Z})$ embedding $H^*(X;\mathbb{Z})$ in $\mathbb{Z}[\alpha]$ as a subring generated by $1, \alpha^2, \alpha^3, \cdots$. (Note that $\alpha \in H^2(X;\mathbb{Z})$ as $|\alpha| = 2$). If we view this subring as a module over $\mathbb{Z}[\alpha^2]$, it is free with basis $\{1, \alpha^2, \alpha^3, \cdots\}$. i.e. $H^*(X;\mathbb{Z}) \cong \mathbb{Z}[\alpha^2, \alpha^3]$.

On the other hand, by Kunneth formula (Theorem 3.16) we have

$$H^*(S^6 \times \mathbb{H}P^\infty;\mathbb{Z}) \cong H^*(S^6;\mathbb{Z}) \otimes H^*(\mathbb{H}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2) \otimes \mathbb{Z}[\beta] \cong \mathbb{Z}[\alpha,\beta]/(\alpha^2)$$

with $|\alpha| = 6$ and $|\beta| = 4$.

So the two rings are not isomorphic, hence $\mathbb{CP}^\infty/\mathbb{CP}^1$ and $S^6 \times \mathbb{HP}^\infty$ are not homotopy equivalent.

3.2.15 For a fixed coefficient field F, define the Poincare series of a space X to be the formal power series $p(t) = \Sigma_i a_i t^i$ where a_i is the dimension of $H^i(X;F)$ as a vector space over F, assuming this dimension is finite for all i. Show that $p(X \times Y) = p(X)p(Y)$. Compute the Poincare seres for S^n, $\mathbb{R}P^n$, $\mathbb{R}P^\infty$, $\mathbb{C}P^n$, $\mathbb{C}P^\infty$, and the spaces in preceding three exercises.

For a fixed coefficient field F, the Poincare series of a space X is the formal power series $p(t) = \Sigma_i a_i t^i$ where a_i is the dimension of $H^i(X;F)$ as a vector space over F.

(1) S^n

By UCT, $H^i(S^n;F) = F$ for $i = 0, n$ and 0 otherwise. And F is one-dimension over F. Hence $p(t) = 1 + t^n$.

(2) $\mathbb{R}P^n$

Note that $H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \text{and } i = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } i \text{ odd}, 0 < i < n \\ 0 & \text{otherwise} \end{cases}$

By UCT, we have $H^i(\mathbb{R}P^n;F) = \begin{cases} F & \text{for } i = 0, \text{and } i = n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$

Hence $p(t) = \begin{cases} 1 + t^n & \text{for } n \text{ odd} \\ 1 & \text{otherwise} \end{cases}$

(3) $\mathbb{R}P^\infty$

Same as (2)

(4) $\mathbb{C}P^n$

Since $H^i(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, \cdots \\ 0 & \text{otherwise} \end{cases}$

Hence $p(t) = 1 + t^2 + t^4 + \cdots + t^{2n}$.

(5) $\mathbb{C}P^\infty$

Similar to above, $p(t) = 1 + t^2 + t^4 + \cdots + t^{2n} + t^{2n+2} + \cdots$.

3.2.16 Show that if X and Y are finite CW complexes such that $H^*(X;\mathbb{Z})$ and $H^*(Y;\mathbb{Z})$ contain no elements of order a power of a given prime p, then the same is true for $X \times Y$.

Proof

If X and Y are finite CW complexes, $H^*(X;\mathbb{Z})$ and $H^*(Y;\mathbb{Z})$ contains no elements of order a power of a given p. This implies that $H^*(X;\mathbb{Z})$ and $H^*(Y;\mathbb{Z})$ are torsion free hence is free.

i.e. $H^*(X;\mathbb{Z}) \cong \mathbb{Z}^m$ for some integer $m > 0$. So Ext terms are all zero in UCT. Hence we have $H^*(X;\mathbb{Z}) \cong \mathrm{Hom}(H_i(X;\mathbb{Z});\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}^m;\mathbb{Z}) \cong \mathbb{Z}^m$. Similarly, $H^*(Y;\mathbb{Z}) \cong \mathbb{Z}^n$ for some integer $n > 0$.

By Kunneth formula, $H^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(Y;\mathbb{Z}) \cong H^*(X \times Y;\mathbb{Z})$ which is $\mathbb{Z}^m \otimes \mathbb{Z}^n \cong \mathbb{Z}^{mn}$. (Dummit & Foote Corollary 19 in Section 10.4). Therefore, $H^*(X \times Y;\mathbb{Z})$ is free, so contains no elements of order a power of p.

3.3.1 Show that there exist nonorientable 1-dimensional manifolds if the Hausdorff condition is dropped from the definition of a manifold.

Proof

Let M be a line with two origins. Then M is not Hausdorff. And it is not orientable at the doubled origin.

3.3.2 Show that deleting a point from a manifold of dimension greater than 1 does not affect orientability of the manifold.

Let M be a manifold of dimension greater than 1. Suppose M is orientable, then $M - \{x\}$ is also orientable (why?). On the other hand, if M is not orientable, we claim that $M - \{x\}$ is also not orientable. Suppose $M - \{x\}$ is orientable, by Proposition 3.25, $\widetilde{M} - p^{-1}(x)$ still has two path components because M has dimension greater than 1. This implies that $\widetilde{M}$ must have two components. By this proposition again, M is orientable, a contradiction. Hence, $M - \{x\}$ ia orientable. Therefore, deleting a point from a manifold of dimension greater than 1 does not affect orientability of the manifold.

3.3.3 Show that every covering space of an orientable manifold is an orientable manifold.

Proof

Let M be an orientable manifold. Want to prove that the covering space $p : \widetilde{M} \to M$ is orientable. For any $\tilde{x} \in \widetilde{M}$, its image $x = p(\tilde{x})$ is in M. Since M is orientable, let μ_x be an orientation. By definition, there is an open ball B_x such that all μ_y at $y \in B_x$ are the image of one generator μ_B in $H_n(M|B_x) \cong H_n(\mathbb{R}^n|B_x)$ under the natural maps $H_n(M|B_x) \to H_n(M|y)$. Note that we can choose B_x so that it is evenly covered since $p : \widetilde{M} \to M$ is a covering map. Now, let $\tilde{B}_{\tilde{x}} \in \widetilde{M}$ be the sheet containing $\tilde{x} \in \widetilde{M}$ such that $\tilde{B}_{\tilde{x}}$ is mapped homeomorphically onto B_x. Hence we have $H_n(\tilde{B}_{\tilde{x}}, \tilde{B}_{\tilde{x}} - \tilde{x}) \cong H_n(B_x, B_x - x)$.

Now, since by excision we also have $H_n(\widetilde{M}, \widetilde{M} - \tilde{x}) \cong H_n(\tilde{B}_{\tilde{x}}, \tilde{B}_{\tilde{x}} - \tilde{x})$ and $H_n(M, M - x) \cong H_n(B_x, B_x - x)$. Therefore, we have the following isomorphisms.

$$H_n(\widetilde{M}, \widetilde{M} - \tilde{x}) \cong H_n(\tilde{B}_{\tilde{x}}, \tilde{B}_{\tilde{x}} - \tilde{x}) \cong H_n(B_x, B_x - x) \cong H_n(M, M - x)$$

Hence the orientation $\mu_{\tilde{x}}$ at $\tilde{x} \in \widetilde{M}$ can be obtained by these isomorphisms. To prove the local consistency, consider the diagram,

$$
\begin{tikzcd}
H_n(\widetilde{M}|\tilde{B}_{\tilde{x}}) \arrow[r, "\varphi"] \arrow[d, "\cong"'] & H_n(\widetilde{M}|\tilde{y}) \ni \mu_{\tilde{y}} \arrow[d, "p"] \\
H_n(M|B_x) \arrow[r, "\psi"'] & H_n(M|y) \ni \mu_y
\end{tikzcd}
$$

The left vertical isomorphism is coming from the facts that $H_n(M|B_x) \cong H_n(\mathbb{R}^n|B_x) \cong H_n(\widetilde{M}|\tilde{B}_x)$ for some n, and that $\tilde{B}$ is homeomorphic to B via the covering map p. By the commutativity of the diagram, the local orientations $\mu_{\tilde{y}}$ at point $\tilde{y} \in \tilde{B}$ are the images of the chosen generator $\mu_{\tilde{B}}$ in $H_n(\widetilde{M}|\tilde{B})$ since the bottom map ψ is surjective due to the orientability of M. Therefore, we prove that the covering space $\widetilde{M}$ of M is orientable.

3.3.4 Given a covering space action of a group G on an orientable manifold M by orientation-preserving homeomorphisms, show that M/G is also orientable.

Proof

For $x \in M$, let B the a neighborhood of x homeomorphic to a disk and satisfying the covering space action: $gB \cap B = \phi$ for all $g \in G$. Let μ_x be the orientation at x satisfying the local consistency.

As in previous exercise, $H_n(M, M - x) \cong H_n(B, B - x)$ by excision. Also, by assumption, the action is a homeomorphism. So $B \approx B/G$. This gives an isomorphism $H_n(B, B - x) \cong H_n(B/G, B/G - [x])$. Hence, the orientation $\mu_{[x]}$ can be chosen to be the image of the following isomorphisms.

$$H_n(M, M - x) \cong H_n(B, B - x) \cong H_n(B/G, B/G - [x]) \cong H_n(M/G, M/G - [x])$$

Note that the last isomorphism is from excision.

Since the action preserves the orientation, the map $H_n(M, M - x) \to H_n(M, M - gx)$ takes μ_x to μ_{gx}. So $\mu_{[x]}$ is independent of the choice of x. Now, consider the diagram

$$
\begin{tikzcd}
H_n(M/G|B/G) \arrow[r, "\varphi"] \arrow[d, "\cong"'] & H_n(M/G|[y]) \ni \mu_{[y]} \arrow[d, "q"] \\
H_n(M|B) \arrow[r, "\psi"'] & H_n(M|y) \ni \mu_y
\end{tikzcd}
$$

The left vertical isomorphism is coming from the facts that $H_n(M|B_x) \cong H_n(\mathbb{R}^n|B) \cong H_n(M/G|B/G)$ for some n, and that $B \approx B/G$. By the commutativity of the diagram, the local orientations $\mu_{[y]}$ at $[y] \in B/G$ are the images of the chosen generator μ_B in $H_n(M/G|B/G)$ since the bottom map ψ is surjective due to the orientability of M. Therefore, we proved that M/G is orientable.

3.3.5 Show that $M \times N$ is orientable iff M and N are both orientable.

Proof

In Section 3B, the relative version of cross product homomorphism is

$$\alpha : H_i(M, M-x) \otimes H_j(N, N-y) \xrightarrow{\times} H_{i+j}(M \times N, M \times N - \{(x,y)\})$$

Let $x \mapsto \mu_x$ and $y \mapsto \mu_y$ be functions that assign to $x \in M$ and $y \in N$ local orientations respectively. Let $\mu_x \times \mu_y$ denote the function which assigns to each $(x, y) \in M \times N$ the homology class

$$\alpha(\mu_x \otimes \mu_y) \in H_{m+n}(M \times N, M \times N - \{(x,y)\})$$

It is easy to see that $\alpha(\mu_x \otimes \mu_y)$ is a generator of the relative homology group. Thus, $\mu_x \times \mu_y$ is an orientation for $M \times N$.

3.3.6 Given two disjoint connected n-manifolds M_1 and M_2, a connected n-manifold $M_1 \# M_2$, their connected sum, can be constructed by deleting the interior of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. (Assume that each B_i embeds nicely in a larger ball in M_i.)
(a) Show that if M_1 and M_2 are closed then there are isomorphisms

$$H_i(M_1 \# M_2; \mathbb{Z}) \cong M_i(M_1; \mathbb{Z}) \oplus H_i(M_2; \mathbb{Z}), \quad \textbf{for } 0 < i < n$$

with one exception: If both M_1 and M_2 are non-orientable, then $H_{n-1}(M_1 \# M_2; \mathbb{Z})$ is obtained from $H_{n-1}(M_1; \mathbb{Z}) \oplus H_{n-1}(M_2; \mathbb{Z})$ by replacing one of the two $\mathbb{Z}_2$-summands by a $\mathbb{Z}$-summand.
(b) Show that $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$ if M_1 and M_2 are closed.

Proof

(a) Let M denote $M_1 \# M_2$. Since M is constructed by deleting the interior of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. The boundary sphere is homeomorphic to S^{n-1}. Since $(M_i - B_i)/\partial B_i \cong (M_i - B_i)/S^{n-1} \cong M_i$ for $i = 1, 2$.

Consider the LES of the pair (M, S^{n-1})

$$\cdots \longrightarrow H_i(S^{n-1}) \longrightarrow H_i(M) \xrightarrow{f} H_i(M, S^{n-1}) \xrightarrow{\alpha} H_{i-1}(S^{n-1}) \longrightarrow \cdots$$

Since $H_i(M, S^{n-1}) \cong H_i(M/S^{n-1}) \cong H_i(M_1 \vee M_2)$. This sequence becomes

$$\cdots \longrightarrow H_i(S^{n-1}) \longrightarrow H_i(M) \xrightarrow{f} H_i(M_1) \oplus H_i(M_2) \xrightarrow{\alpha} H_{i-1}(S^{n-1}) \longrightarrow \cdots$$

So for $0 < i < n-1$, $H_i(S^{n-1}) = H_{i-1}(S^{n-1}) = 0$, hence $H_i(M) \cong H_i(M_1) \oplus H_i(M_2)$.

Now, prove that M is orientable if and only if both M_1 and M_2 are orientable. Want to use Proposition 3.26 that $H_n(M_i; \mathbb{Z}) = \mathbb{Z}$ if M_i is orientable and 0 if M_i is nonorientable. Same for M.

Suppose M_1 and M_2 are orientable, from the LES above we have

$$0 \longrightarrow H_n(M) \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \longrightarrow H_{n-1}(M) \longrightarrow \cdots$$

f is injective, so $H_n(M)$ must be either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. Since M is closed and connected hence $H_n(M) = \mathbb{Z}$. i.e. M is orientable. (why?)

On the other hand, if M is orientable. By construction, M is the connected sum $M_1 \# M_2$, so if M_1, say, is non orientable, clearly M can not be orientable. Hence M is orientable if and only if M_1 and M_2 are orientable.

So if both M_1 and M_2 are orientable, from the LES above, we have

$$0 \longrightarrow H_n(M) \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \xrightarrow{h} H_{n-1}(M) \longrightarrow H_{n-1}(M_1) \oplus H_{n-1}(M_2) \longrightarrow 0$$

So g is surjective hence h is a zero map (or, f is injective, Im $f = \ker g = \mathbb{Z}$. But Im $g + \ker g = \mathbb{Z} \oplus \mathbb{Z}$, which implies that Im$g = \ker h = \mathbb{Z}$. Thus, h is a zero map), hence $H_{n-1}(M) \cong H_{n-1}(M_1) \oplus H_{n-1}(M_2)$. (Note that this is still true if only one of M_1 and M_2 is orientable.) Hence we proved that $H_i(M) \cong H_i(M_1) \oplus H_i(M_2)$ for $0 < i < n$.

Now, if neither M_1 nor M_2 is orientable. Then $H_n(M_1) = H_n(M_2) = 0$. From LES above we have

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} H_{n-1}(M) \longrightarrow H_{n-1}(M_1) \oplus H_{n-1}(M_2) \longrightarrow 0$$

By Proposition 3.28 the torsion subgroup of $H_{n-1}(M_1) \oplus H_{n-1}(M_2)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and the torsion subgroup of $H_{n-1}(M)$ is $\mathbb{Z}_2$. So the map f takes the generator of $\mathbb{Z}$ to the generator of the free abelian summand in $H_{n-1}(M)$. It follows that $H_{n-1}(M; \mathbb{Z})$ is obtained from $H_{n-1}(M_1; \mathbb{Z}) \oplus H_{n-1}(M_2; \mathbb{Z})$ by replacing one of the two $\mathbb{Z}_2$-summands by a $\mathbb{Z}$-summand.

(b) We proved in (a) that $H_i(M) \cong H_i(M_1) \oplus H_i(M_2)$ for $0 < i < n$. By definition $\chi(M) = \sum_i (-1)^i \text{rank } H_i(M) = \sum_i (-1)^i b_i$. So $b_i = b_{i,1} + b_{i,2}$ for $0 < i < n$. Thus, $\chi(M) - \chi(M_1) - \chi(M_2) = \sum_i (-1)^i (b_i - b_{i,1} - b_{i,2}) - 1$ where 1 is for H_0. So if both M_1 and M_2 are orientable, all terms $i < n$ are cancelled. For $i = n$, $H_n(M) = H_n(M_1) = H_n(M_2) = \mathbb{Z}$, so $b_n - b_{n,1} - b_{n,2} = 1 - 1 - 1 = -1$. Thus, $\chi(M) - \chi(M_1) - \chi(M_2) = (-1)^n(-1) - 1 = -((-1)^n + 1) = -\chi(S^n)$. If one of them is orientable (say only M_1 is orientable) , we still get $\chi(M) - \chi(M_1) - \chi(M_2) = (-1)^n(0 - 1 - 0) - 1 = -(-1)^n - 1 = -\chi(S^n)$. If both of them are nonorientable, $b_{n-1} - b_{n-1,1} - b_{n-1,2} = 1$ (why?). So we have $\chi(M) - \chi(M_1) - \chi(M_2) = (-1)^{n-1}(1) - 1 = \chi(S^n)$. In all cases we get $\chi(M) = \chi(M_1) + \chi(M_2) - \chi(S^n)$.

3.3.7 For a map $f : M \to N$ between connected close orientable n-manifolds with fundamental class $[M]$ and $[N]$, the degree of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifolds M there is a degree 1 map $M \to S^n$.

Proof

Let B be a subset of M homeomorphic to an open disk. Define a map $\varphi : M \to S^n$ as follows

$$\begin{tikzcd} H_n(M) \arrow[r, "\cong"] \arrow[d, "\varphi_*"] & H_n(M, M - B) \arrow[d, "\cong"] \\ H_n(S^n) \arrow[r, "\cong"] & H_n(M/M - B) \end{tikzcd}$$

Since M is closed orientable, by Theorem 3.26(a) we have $H_n(M) \cong H_n(M, M-x)$. But $H_n(M|B) \to H_n(M|x)$ is a natural map.

$$H_n(M) \longrightarrow H_n(M, M-B) \longrightarrow H_n(M, M-x)$$

So $H_n(M) \to H_n(M|B)$ is an isomorphism. The right-hand vertical isomorphism in the diagram comes from Theorem 2.22. And the bottom horizontal isomorphism is induced by the homeomorphism $S^n \to M/(M-B)$. The left-hand vertical map can be chosen to make the diagram commute.

Hence the fundamental class $[M] \in H_n(M)$ can be taken to the fundamental class $\pm[S^n] \in H_n(S^n)$. If it is negative, we can replace μ_M with the other orientation.

Correction: since the composition of the maps

$$H_n(M, M-B) \xrightarrow{\text{surjective}} H_n(M, M-x) \xrightarrow{\cong} H_n(M)$$

is surjective, we have

$$\begin{tikzcd} H_n(M, M-B) \arrow[r, "\text{surjective}"] \arrow[d, "\cong"] & H_n(M) \arrow[dl, "\varphi^*"] \\ H_n(S^n) & \end{tikzcd}$$

The conclusion applies.

More correction on April 20, 2015 I think the solution might not be correct. Note that $H_n(M, M-x; \mathbb{Z}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}) \cong \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}) = \tilde{H}_{n-1}(S^{n-1}; \mathbb{Z}) \cong \mathbb{Z}$ as on page 231. Since M is closed and orientable, we have $H_n(M) = H_n(M, M-x) \cong \mathbb{Z}$. So we have the following diagram:

$$\begin{tikzcd} H_n(M) \arrow[r, "\cong\ \text{Thm 3.26(a)}"] \arrow[d] & H_n(M, M-x) \arrow[d, "\cong"] \\ H_n(S^n) \arrow[r, "\cong"] & H_{n-1}(S^{n-1}) \cong \mathbb{Z} \end{tikzcd}$$

Now, let $[M] \in H_n(M)$ be a fundamental class for M, by definition, its image in $H_n(M, M-x) = \mathbb{Z}$ is a generator. Since The bottom is an isomorphism by LES of the pair (D^n, S^{n-1}) because $H_i(D^n) = 0$ for all i. So the left hand vertical takes $[M]$ to $\pm[S^n]$. If it is negative, then replace $[M]$ with the other orientation.

3.3.9 Show that a p-sheeted covering space projection $M \to N$ has degree $\pm p$, when M and N are connected closed orientable manifolds.

Proof

Let $\mu_M \in H_n(M)$ be a fundamental class for M. For a point $x \in N$, choose an open set B containing x such that $p^{-1}(B) = \sqcup_{i=1}^{p} U_i$ where $U_i \in M$ and each U_i is mapped homeomorphically onto B. Consider the following diagram,

$$\begin{tikzcd} H_n(M) \arrow[r, "\cong"] \arrow[d, "p_*"] & H_n(M, M - \sqcup_{i=1}^p U_i) \arrow[r, "\cong"] \arrow[d] & H_n(\vee S^n) \cong \bigoplus \mathbb{Z} \arrow[d, "p_*"] \\ H_n(N) \arrow[r, "\cong"] & H_n(N, N - B) \arrow[r, "\cong"] & H_n(S^n) \cong \mathbb{Z} \end{tikzcd}$$

The top left and bottom left isomorphisms are by the same reason as in 3.3.7. The bottom right isomorphism is by the isomorphism $H_n(N, N-B) \cong H_n(\overline{B}, \overline{B} - B) \cong H_n(S^n)$. Hence $[M]$ was sent via the diagram to $p[S^n]$. Therefore, p-sheeted covering space projection $M \to N$ has degree $\pm p$.

Or, let $x_1, \cdots, x_p$ be the pre-images of x under f. Since U_i maps homeomorphically onto B, we have $\deg_{x_i}(f) = \pm 1$ for each i. Each $x_i \in M$ has a neighborhood which maps homeomorphically to a neighborhood in N. So the local degree for preimage $x_i \in U_i$ is a locally constant map $M \to \{\pm 1\}$ (refer Standford's solution and the proof of Proposition 2.29 on page 135). **But M is connected, local degree has to be constant on** M, so we have $\deg(f) = \sum_{i=1}^p \deg_{x_i} f = \pm p$. Therefore, p-sheeted covering space projection $M \to N$ has degree $\pm p$.

Correction: The map $H_n(M, M - B) \to H_n(M)$ should be surjective as mentioned in previous exercise.

3.3.10 Show that for a degree 1 map $f : M \to N$ of connected closed orientable manifolds, the induced map $f_* : \pi_1 M \to \pi_1 N$ is surjective, hence also $f_* : H_1(M) \to H_1(N)$.

Proof

Consider the diagram

$$\begin{tikzcd} & \tilde{N} \arrow[d, "p"] \\ M \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & N \end{tikzcd}$$

i.e. Lift f to the covering space $\tilde{N} \to N$ corresponding to the subgroup $\text{Im} f_* \subset \pi_1 N$, by Theorem 1.36, such that $p_*(\pi_1(\tilde{N})) = \text{Im} f_*$. Since $f_*(\pi_1(M)) \subset p_*(\pi_1(\tilde{N})) = \text{Im } f_*$. By Lifting Criterion $\tilde{f} : M \to \tilde{N}$ exists. Now, since $f : M \to N$ is a degree 1 map, the fundamental class $[M]$ for M was sent to the fundamental class $[N]$ for N. On the other hand, suppose the covering space has degree p. Since $[M]$ was sent to $k[M] \in H_n(\tilde{N})$ for some $k \in \mathbb{Z}$, and $[\tilde{N}]$ was sent to $p[N]$. Hence $[M]$ was sent to $kp[N] \in H_n(N)$. (Or, recall that if $f = p\tilde{f}$, then $\deg(f) = \deg(p) \cdot \deg(\tilde{f})$ and by 3.3.9 $|\deg(p)| =$ the number of sheets.) Since f is a degree 1 map. Therefore, $kp = \pm 1$. i.e. k and p must be ± 1. The map $p : \tilde{N} \to N$ has degree ± 1, hence $\tilde{N} \cong N$. Since $\text{Im} f_* = p_*(\pi_1(\tilde{N})) = \pi_1(\tilde{N}) = \pi_1 N$ since p_* is injective by Theorem 1.31. Hence $f_* : \pi_1 M \to \pi_1 N$ is surjective. The surjectivity of π_1 implies the surjectivity of H_1 since H_1 is just the abelianization of π_1.

If the covering $\tilde{N} \to N$ has infinite degree. Then the cover $\tilde{N}$ is not compact, so $H_n(\tilde{N}) = 0$ (by Proposition 3.29). Hence $f_* : H_n(M) \to H_n(N)$ factors through 0, a contradiction.

3.3.11 If M_g denotes the closed orientable surface of genus g, show that degree 1 maps $M_g \to M_h$ exist iff $g \geq h$.

Proof

Since $\pi_1(M_g)$ has $2g$ generators and $\pi_1(M_h)$ has $2h$ generators. Suppose $g < h$, by previous problem $\pi_1(M_g) \leq \pi_1(M_h)$, a contradiction since the map $\pi_1(M_g) \to \pi_i(M_h)$ is not surjective. So no degree 1 map $M_g \to M_h$ exists.

To prove the converse, we can write $M_g = M_h \# M_{g-h}$ if $g > h$. Define a degree-1 map $f : M_g \to M_h$ by collapsing $M_{g-h} - D^2$ together with ∂D^2 to a point, and identity if $g = h$. Let $x \in M_h$ and $U \subset M_h$ an open set containing x. To see that this is a degree-1 map, consider the following diagram:

$$\begin{tikzcd}
H_2(M_g) \arrow[r, "\cong"] \arrow[d, "f_*"] & H_2(M_g, M_g - x) \arrow[r, "\text{excision}"] \arrow[d, "f_*"] & H_2(U, U-x) \arrow[d, "(f|_U)_*"] \\
H_2(M_h) \arrow[r, "\cong"] & H_2(M_h, M_h - x) \arrow[r, "\text{excision}"] & H_2(U, U-x)
\end{tikzcd}$$

The map $(f_U)_*$ is an identity on U, so f_* is an isomorphism. Hence $f : M_g \to M_h$ has degree 1.

Or, we could reproduce a map $M_{h+1} \to M_h$ by replacing one torus with a point. i.e. Looking at the polygonal representation, this is simply mapping the edges of the $(h+1)^t h$ torus to a point, and by extending that map to a homeomorphism on the interior of the disk. The fact that the homeomorphism on the interior of the disk guarantee that the map has degree 1.

3.3.16 Show that $(\alpha \frown \varphi) \frown \psi = \alpha \frown (\varphi \frown \psi)$ for all $\alpha \in C_k(X;R)$, $\varphi \in C^l(X;R)$, and $\psi \in C^m(X;R)$. Deduce that cap product makes $H_*(X;R)$ a right $H^*(X;R)$-module.

Proof

With loss of generality, assume that $l \leq m \leq k$. By definition,

$$\alpha \frown \varphi = \varphi(\alpha|_{[v_0,\cdots,v_l]})\alpha|_{[v_l,\cdots,v_k]}$$

So

$$\begin{aligned}
(\alpha \frown \varphi) \frown \psi &= \left(\varphi(\alpha|_{[v_0,\cdots,v_l]})\alpha|_{[v_l,\cdots,v_k]}\right) \frown \psi \\
&= (\varphi(\alpha|_{[v_0,\cdots,v_l]})\psi(\alpha|_{[v_l,\cdots,v_{l+m}]})\alpha|_{[v_{l+m},\cdots,v_k]}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\alpha \frown (\varphi \smile \psi) &= (\varphi \smile \psi)\left(\alpha|_{[v_0,\cdots,v_{l+m}]}\right)\alpha|_[v_{l+m},\cdots,v_k] \\
&= (\varphi(\alpha|_{[v_0,\cdots,v_l]})\psi(\alpha|_{[v_l,\cdots,v_{l+m}]})\alpha|_{[v_{l+m},\cdots,v_k]}
\end{aligned}$$

Hence, $(\alpha \frown \varphi) \frown \psi = \alpha \frown (\varphi \frown \psi)$.

To see that $H_*(X;R)$ is a right $H^*(X;R)$-module. Note that the action is defined by

$$H_*(X;R) \times H^*(X;R) \to H_*(X;R)$$

sending $\alpha \times \varphi \mapsto \alpha \frown \varphi$. For $\alpha, \beta \in H_*(X;R)$, $\varphi, \psi \in H^*(X;R)$, we need

(1) $(\alpha + \beta) \frown \varphi = (\alpha \frown \varphi) + (\beta \frown \varphi)$
(2) $\alpha \frown (\varphi + \psi) = (\alpha \frown \varphi) + (\alpha \frown \psi)$
(3) $\alpha \frown (\varphi \smile \psi) = (\alpha \frown \varphi) \frown \psi)$
(4) $1 \cdot \alpha = \alpha$

We have just done (3), the rests are clear. Hence $H_*(X; R)$ is a right $H^*(X; R)$-module.

3.3.17 Show that a direct limit of exact sequences is exact. More generally, show that homology commutes with direct limits: If $[C_\alpha, f_{\alpha\beta}]$ is a directed system of chain complexes, with the maps $f_{\alpha\beta} : C_\alpha \to C_\beta$ chain maps, then $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$.

Proof

Suppose $A_\alpha \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} C_\alpha$ is exact. Let $A = \varinjlim A_\alpha$, $B = \varinjlim B_\alpha$ and $C = \varinjlim C_\alpha$. Show that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$.

(1) $\mathrm{im} f \subseteq \ker g$

Consider the diagram

$$\begin{tikzcd}
A_\alpha \arrow[r, "f_\alpha"] \arrow[d, "i"] & B_\alpha \arrow[r, "g_\alpha"] \arrow[d, "j"] & C_\alpha \arrow[d, "k"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

For $a \in A$, $i(a_\alpha) = a$, so $gf(a) = gf(i(a_\alpha)) = gjf_\alpha(a_\alpha) = kg_\alpha f_\alpha(a_\alpha) = k(0) = 0$ by the commutativity of the diagram and the exactness the top row. Hence, $\mathrm{im} f \subset \ker g$.

(2) $\ker g \subseteq \mathrm{im}\ f$

$$\begin{tikzcd}
A_\alpha \arrow[rr, "f_\alpha"] \arrow[dr, "i^A_{\alpha\beta}"] \arrow[dd, "i^A_\alpha"'] & & B_\alpha \arrow[rr, "g_\alpha"] \arrow[dr, "i^B_{\alpha\beta}"] \arrow[dd, "i^B_\alpha" near start] & & C_\alpha \arrow[dr, "i^C_{\alpha\beta}"] \arrow[dd, "i^C_\alpha" near start] & \\
& A_\beta \arrow[rr, "f_\beta" near start] \arrow[dl, "i^A_\beta"] & & B_\beta \arrow[rr, "g_\beta" near start] \arrow[dl, "i^B_\beta"] & & C_\beta \arrow[dl, "i^C_\beta"] \\
A \arrow[rr, "f"'] & & B \arrow[rr, "g"'] & & C &
\end{tikzcd}$$

For $b \in \ker g$, $g(b) = 0$. There is $b' \in B_\alpha$ such that $i^B_\alpha(b') = b$ for some α. By the commutativity of the diagram, $i^C_\alpha g_\alpha(b') = gi^B_\alpha(b') = g(b) = 0$. But $0 = i^C_\alpha g_\alpha(b') = i^C_\beta i^C_{\alpha\beta} g_\alpha(b')$, hence $i^C_{\alpha\beta} g_\alpha(b') = 0$ since i^C_β is injective.

Now, $i^C_{\alpha\beta} g_\alpha(b') = g_\beta i^B_{\alpha\beta}(b') = 0$ by the commutativity of the diagram. Denote $b'' = i^B_{\alpha\beta}(b')$, then $b' \subset \ker g_\beta = \mathrm{im}\ f_\beta$. So there exists $a'' \in A_\beta$ such that $f_\beta(a'') = b''$. Denote $a = i^A_\beta(a'')$, then $b = i^B_\alpha(b') = i^B_\beta i^B_{\alpha\beta}(b') = i^B_\beta(b'') = i^B_\beta f_\beta(a'') = fi^A_\beta(a'') = f(a)$. Hence, $b \in \mathrm{im} f$. Therefore, $\ker g \subseteq \mathrm{im}\ f$.

In general, show that $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$.

Note that we have the following exact sequences:
(1) $0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n \longrightarrow 0$ (defining H_n).
(2) $0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow 0$ (defining Z_n).
(3) $0 \longrightarrow Z_{n+1} \longrightarrow C_{n+1} \xrightarrow{j} B_n \longrightarrow 0$. (defining B_n).

By the first part, we have
(1)′ $0 \longrightarrow \varinjlim B_n \longrightarrow \varinjlim Z_n \longrightarrow \varinjlim H_n \longrightarrow 0$
(2)′ $0 \longrightarrow \varinjlim Z_n \xrightarrow{i_*} \varinjlim C_n \xrightarrow{\partial_*} \varinjlim C_{n-1} \longrightarrow 0$
(3)′ $0 \longrightarrow \varinjlim Z_{n+1} \longrightarrow \varinjlim C_{n+1} \xrightarrow{j_*} \varinjlim B_n \longrightarrow 0$

Now, since $H_n = \dfrac{Z_n}{B_n}$, by (1)′, $\varinjlim H_n = \dfrac{\varinjlim Z_n}{\varinjlim B_n}$. On the other hand, by (2)′ and (3)′, we have $\varinjlim Z_n = \ker \partial_*$ and $\varinjlim B_n = \mathrm{im}\partial_*$. Hence, $H_n(\varinjlim C_n) = \dfrac{\ker \partial_*}{\mathrm{im}\partial_*} = \dfrac{\varinjlim Z_n}{\varinjlim B_n}$.

Therefore, $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$.

3.3.20 Show that $H_c^0(X;G) = 0$ if X is path-connected and noncompact.

Proof

By the discussion following Proposition 3.33, the compact subsets $K \subset X$ form a directed set under inclusion since the union of two compact sets is compact. To each compact $K \subset X$ we associate the group $H^i(X, X-K; G)$, with a fixed i and coefficient group G, and to each inclusion $K \subset L$ of compact sets we associate the natural homomorphism $H^i(X, X-K; G) \to H^i(X, X-L; G)$. The resulting limit group $\varinjlim H^i(X, X-K; G)$ is then equal to $H_c^i(X;G)$. i.e. $\varinjlim H^i(X, X-K; G) = H_c^i(X;G)$. Note that $X - K \not\psi$ since K is compact and X is noncompact.

Since X is connected, so for any two points v_0, v_1, there is a path connecting them. Let $\sigma : [v_1, v_1] \to X$ be 1-simplex (that path). If $\varphi \in C^0(X, X-K)$, then $\varphi(v_0) = \varphi(v_1) = 0$. If φ is a cocycle, the $\delta\varphi = 0$. So $0 = \delta\varphi(\sigma) = \varphi(d\sigma) = \varphi(v_1) - \varphi(v_0) = 0$. So φ takes the same value on all points in X. But φ vanishes on $X - K$ hence $\varphi \equiv 0$.

3.3.24 Let M be a closed connected 3-manifold, and write $H_1(M;\mathbb{Z})$ as $\mathbb{Z}^r \oplus F$, the direct sum of a free abelian group of rank r and a finite group F. Show that $H_2(M;\mathbb{Z})$ is $\mathbb{Z}^r$ if M is orientable and $\mathbb{Z}^{r-1} \oplus \mathbb{Z}_2$ if M is nonorientable. In particular, $r \geq 1$ when M is nonorientable. Using Exercise 6, construct examples showing there are no other restrictions on the homology groups of closed 3-manifolds. [In the nonorientable case consider the manifold N obtained from $S^2 \times I$ by identifying $S^2 \times \{0\}$ with $S^2 \times \{1\}$ via a reflection of S^2.]

Proof

Since rank $H_1(M;\mathbb{Z}) = r$. By Proposition 3.37, a closed manifold of odd dimension has Euler characteristic zero. Let rank $H_2(M;\mathbb{Z}) = s$, we have $\sum_{i=0}^{3}(-1)^i H_i(M;\mathbb{Z}) = 1 - r + s - 1 = 0$, if M is orientable because $H_3(M;\mathbb{Z}) = \mathbb{Z}$ and $1 - r + s = 0$ if M is nonorientable because in this case $H_3(M;\mathbb{Z}) = 0$. And $H_0(M;\mathbb{Z}) = \mathbb{Z}$ in both cases. Thus, we have $s = r$ if M is orientable and $s = r - 1$ if M is nonorientable.

$H^2(M;\mathbb{Z}) = H_1(M;\mathbb{Z}) = \mathbb{Z}^r \oplus F$ by Poincare Duality. Then by UCT, we have

$$H^2(M;\mathbb{Z}) = \mathbb{Z}^r \oplus F = \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_1(M);\mathbb{Z}).$$

Since Ext $(H_1(M;\mathbb{Z}),\mathbb{Z})$ = Ext $(\mathbb{Z}^r \oplus F,\mathbb{Z}) = F$ and $H_2(M;\mathbb{Z})$ has no torsion. Thus, $H_2(M,\mathbb{Z}) = \mathbb{Z}^r$ if M is orientable and $H_2(M;\mathbb{Z}) = \mathbb{Z}^s \oplus \mathbb{Z}_2 = \mathbb{Z}^{r-1} \oplus \mathbb{Z}_2$ if M is nonorientable by Proposition 3.28.

3.3.25 Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M;\mathbb{Z})$ torsionfree, then $H_k(M;\mathbb{Z})$ is also torsionfree.

Proof

By the UCT, we have

$$H^k(M;\mathbb{Z}) = \mathrm{Hom}(H_k(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_{k-1}(M);\mathbb{Z})$$

$H_{k-1}(M;\mathbb{Z})$ is torsion free, so the Ext term, the torsion part, of $H^k(M;\mathbb{Z})$ is zero. Thus, $H^k(M;\mathbb{Z})$ is torsion free. But $H^k(M;\mathbb{Z}) = H_k(M;\mathbb{Z})$ by Poincare Duality, $H_k(M;\mathbb{Z})$ is also torsion free.

3.3.32 Show that a compact manifold does not retract onto its boundary.

Proof

Suppose there is a retract $r : M \to \partial M$, let $i : \partial M \to M$ be the inclusion. We have $r \circ i = id$, so $r_* \circ i_* = id_*$. i.e.

$$0 \longrightarrow H_{n-1}(\partial M) \xrightarrow{i_*} H_{n-1}(M) \xrightarrow{r_*} H_{n-1}(\partial M) \longrightarrow 0$$

So i_* is injective.

To prove this is a contradiction, look at the LES for the pair $(M,\partial M)$ in coefficent $\mathbb{Z}_2$,

$$0 \longrightarrow H_n(M;\mathbb{Z}_2) \longrightarrow H_n(M,\partial M;\mathbb{Z}_2) \xrightarrow{\alpha} H_{n-1}(\partial M;\mathbb{Z}_2) \xrightarrow{i_*} H_{n-1}(M;\mathbb{Z}_2) \longrightarrow \cdots$$

$H_n(M,\partial M;\mathbb{Z}_2) = H^0(M;\mathbb{Z}_2) = \mathbb{Z}_2$ by Lefschetz duality (when $A = \phi$ in Theorem 3.43), and $H_{n-1}(\partial M;\mathbb{Z}_2) = \mathbb{Z}_2$ by Proposition 3.26 since ∂M is closed and have dimension $n-1$.

If i_* is injective, α is a zero map. However, by the result of Exercise 3.3.31 α sends a fundamental class for $(M,\partial M)$ to a fundamental class for ∂M. This is a contradiction. Thus, i_* cannot be injective hence M does not retract onto ∂M.

Or simply look at the the diagram

$$\begin{tikzcd}
0 \arrow[r] & H_n(M;\mathbb{Z}_2) \arrow[r, "\cong"] \arrow[d, "\cong"] & H_n(M,\partial M;\mathbb{Z}_2) \arrow[r, "\alpha=0"] \arrow[d, "\cong"] & H_{n-1}(\partial M;\mathbb{Z}_2) \arrow[r, "i_*"] & H_{n-1}(M;\mathbb{Z}_2) \\
0 \arrow[r] & H^0(M,\partial M;\mathbb{Z}_2) \arrow[r, "\cong"] & H^0(M;\mathbb{Z}_2) \arrow[r, "i^*"] & H^0(\partial M;\mathbb{Z}_2) &
\end{tikzcd}$$

Two vertical isomorphisms are by Lefschetz duality, so the bottom map is an isomorphism. This implies that i^* is an surjective, but $H^0(\partial M; \mathbb{Z}_2) \neq 0$, a contradiction!

Remark: if the problem changes to be "show that an orientable manifold does not retract onto its boundary". Then i^* cannot be injective since $H_{n-1}(\partial M; \mathbb{Z}_2) = \mathbb{Z}_2$ but the torsion part of $H_{n-1}(M)$ is zero.

3.3.33 Show that if M is a compact contractible n-manifold then ∂M is a homology $(n-1)$-sphere, that is, $H_i(\partial M; \mathbb{Z}) \cong H_i(S^{n-1}; \mathbb{Z})$ for all i.

Proof

Consider the LES for the pair $(M, \partial M)$,

$$\cdots \longrightarrow H_i(\partial M) \longrightarrow H_i(M) \longrightarrow H_i(M, \partial M) \longrightarrow H_{i-1}(\partial M) \longrightarrow \cdots$$

M is contractible, so $H_i(M) = 0$ for all $i > 0$ and $\mathbb{Z}$ for $i = 0$, hence $H^i(M) = 0$ for $i > 0$ and $\mathbb{Z}$ for $i = 0$ by the UCT. Thus, from above sequence we have $H_i(M, \partial M) \cong H_i(\partial M)$ for all $0 < i < n-1$. By Lefschetz Duality, $H_i(M, \partial M) \cong H^{n-i}(M) = 0$. So $H_i(M) = 0$ for $0 < i < n-1$. When $i = 0$, look at the LES,

$$0 \longrightarrow H_0(\partial M) \longrightarrow H_0(M) \longrightarrow H_0(M, \partial M) \longrightarrow 0$$

$H_0(M, \partial M) = H^n(M) = 0$ by Lefschetz duality. This implies that $H_0(\partial M) \cong H_0(M) = \mathbb{Z}$. When $i = n$, we have the LES

$$0 \longrightarrow H_n(\partial M) \longrightarrow H_n(M) \longrightarrow H_n(M, \partial M) \longrightarrow H_{n-1}(\partial M) \longrightarrow 0$$

We see that $H_n(\partial M) = 0$ because $H_n(M) = 0$. Finally, when $i = n-1$, we have

$$0 \longrightarrow H_n(M, \partial M) \longrightarrow H_{n-1}(\partial M) \longrightarrow H_{n-1}(M) \longrightarrow H_{n-1}(M, \partial M) \longrightarrow 0$$

Since $H_{n-1}(M) = 0$ and $H_n(M, \partial M) = H^0(M) = \mathbb{Z}$, we have $H_{n-1}(\partial M) = \mathbb{Z}$. We summarize that

$$H_i(\partial M) = \begin{cases} \mathbb{Z} & \text{for } i = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, ∂M is a homology $(n-1)$-sphere, i.e. $H_i(\partial M; \mathbb{Z}) \cong H_i(S^{n-1}; \mathbb{Z})$ for all i.

3.B.3 Show that the splitting in the topological Kunneth formula cannot be natural by considering the map $f \times 1 : M(\mathbb{Z}_m, n) \times M(\mathbb{Z}_m, n) \to S^{n+1} \times M(\mathbb{Z}_m, n)$ where f collapses the n-skeleton of $M(\mathbb{Z}_m, n) = S^n \cup e^{n+1}$ to a point.

Proof

Let $X = M(\mathbb{Z}_m, n)$. Consider the diagram,

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus H_i(X) \otimes H_{2n+1-i}(X) \arrow[r] \arrow[d, "f_*"] & H_{2n+1}(X \times X) \arrow[r] \arrow[d, "f_*"] & \bigoplus H_i(X) * H_{2n-i}(X) \arrow[r] \arrow[d, "f_*"] & 0 \\
0 \arrow[r] & \bigoplus H_i(S^{n+1}) \otimes H_{2n+1-i}(X) \arrow[r] & H_{2n+1}(S^{n+1} \times X) \arrow[r] & \bigoplus H_i(S^{n+1}) * H_{2n-i}(X) \arrow[r] & 0
\end{tikzcd}$$

Since $H_n(X) = \mathbb{Z}_m$ and $H_i(X) = 0$ for all $i \neq n$. So this diagram becomes

$$\begin{tikzcd}
0 \arrow[r] & H_n(X) \otimes H_{n+1}(X) \oplus H_{n+1}(X) \otimes H_n(X) \arrow[r] \arrow[d, "f_*"] & H_{2n+1}(X \times X) \arrow[r] \arrow[d, "f_*"] & \bigoplus H_n(X) * H_n(X) \arrow[r] \arrow[d, "f_*"] & 0 \\
0 \arrow[r] & \bigoplus H_{n+1}(S^{n+1}) \otimes H_n(X) \arrow[r] & H_{2n+1}(S^{n+1} \times X) \arrow[r] & \bigoplus H_n(S^{n+1}) * H_n(X) \arrow[r] & 0
\end{tikzcd}$$

Now, since $\mathbb{Z}_m * \mathbb{Z}_m = \mathbb{Z}_m$, $0 * \mathbb{Z}_m = 0$ and $\mathbb{Z}_m \otimes \mathbb{Z}_m = \mathbb{Z}_m$.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d, "f_*"] & H_{2n+1}(X \times X) \arrow[r] \arrow[d, "f_*"] & \mathbb{Z}_m \arrow[r] \arrow[d, "f_*"] & 0 \\
0 \arrow[r] & \mathbb{Z}_m \arrow[r] & H_{2n+1}(S^{n+1} \times X) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Show that f_* is an isomorphism by looking at the cellular chain complex for $X \times X$ and $S^{n+1} \times X$. Since $X = M(\mathbb{Z}_m, n)$ has cells e^0, e^n, e^{n+1}. And $X \times X$ has cells $e^0, e^n, e^{n+1}, e^{2n}, e^{2n+1}, e^{2n+2}$. Also $S^{n+1} = \langle e^0, e^{n+1} \rangle$. So $S^{n+1} \times X$ has cells $e^0, e^n, e^{n+1}, e^{2n+1}, e^{2n+2}$.

Hence, we consider the dimensions of $2n+2$, $2n+1$ and $2n$ for $X \times X$ and $S^{2n+1} \times X$,

$$\begin{tikzcd}
\cdots \arrow[r] & \langle e^{n+1} \times e^{n+1} \rangle \arrow[r, "{(m,\pm m)}"] \arrow[d, "\cong f"] & \langle e^n \times e^{n+1}, e^{n+1} \times e^n \rangle \arrow[r, "{(m,\mp m)}"] \arrow[d, "{\pm(0,1)}"] & \langle e^n \times e^n \rangle \arrow[r] & \cdots \\
\cdots \arrow[r] & \langle e^{n+1} \times e^{n+1} \rangle \arrow[r, "m"] & \langle e^{n+1} \times e^n \rangle \arrow[r] & \cdots &
\end{tikzcd}$$

which is,

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z} \arrow[r, "{(m,\pm m)}"] \arrow[d, "\cong f"] & \mathbb{Z} \times \mathbb{Z} \arrow[r, "{(m,\mp m)}"] \arrow[d, "{\pm(0,1)}"] & \mathbb{Z} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathbb{Z} \arrow[r, "m"] & \mathbb{Z} \arrow[r] & \cdots &
\end{tikzcd}$$

Hence, f_* is an isomorphism.

Finally, show that the splitting is not natural. From the diagram above we have,

$$\begin{tikzcd}
H_{2n+1}(X \times X) \arrow[d, "\cong f_*"] & \cong & \oplus H_i(X) \otimes H_{2n+1-i}(X) \arrow[d] & \bigoplus & \oplus H_i(X) * H_{2n-i}(X) \arrow[d] \\
H_{2n+1}(S^{n+1} \times X) & \cong & \oplus H_i(S^{n+1}) \otimes H_{2n+1-i}(X) & \bigoplus & \oplus H_i(S^{n+1}) * H_{2n-i}(X)
\end{tikzcd}$$

which is

$$\begin{tikzcd}
H_{2n+1}(X \times X) \arrow[d, "\cong f_*"] & \cong & 0 \arrow[d] & \bigoplus & \mathbb{Z}_m \arrow[d] \\
H_{2n+1}(S^{n+1} \times X) & \cong & \mathbb{Z}_m & \bigoplus & 0
\end{tikzcd}$$

This diagram is not commutative, hence the splitting can not be natural.

Chapter 4

4.1.2 Show that if $\varphi : X \to Y$ is a homotopy equivalence, then the induced homomorphisms $\varphi_* : \pi_n(X, x_0) \to \pi_1(Y, \varphi(x_0))$ are isomorphisms for all n. [The case $n = 1$ is Proposition 1.18]

Proof

The proof is similar to Proposition 1.18, the case for $n = 1$. For $n \geq 2$, as the statement on page 342, since π_n is a functor. A map $\varphi : (X, x_0) \to (Y, y_0)$ induces $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ defined by $\varphi_*([f]) = [\varphi f]$. If $\varphi_t : (X, x_0) \to (Y, y_0)$ is a homotopy then $\varphi_{0_*} = \varphi_{1_*}$.

Now, suppose $\varphi : X \to Y$ is a homotopy equivalence. Let $\psi : Y \to X$ be a homotopy inverse for φ, so that $\varphi\psi \simeq 1$ and $\psi\varphi \simeq 1$. Consider the following compositions

$$\pi_n(X, x_0) \xrightarrow{\varphi_*} \pi_n(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_n(X, \psi\varphi(x_0))$$

$$\pi_n(Y, y_0) \xrightarrow{\psi_*} \pi_n(X, \psi(y_0)) \xrightarrow{\varphi_*} \pi_n(Y, \varphi\psi(y_0))$$

In each composition, the first map is an injection and the second is an surjection. Take $y_0 = \varphi(x_0)$ in the second composition, we see that ψ_* is an isomorphism. Thus, so is φ_*. Therefore, we proved that $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, \varphi(x_0))$ are isomorphisms for all n.

(Reference: A Concise Course in Algebraic Topology, J. P. May)

4.1.3 For an H-space (X, x_0) with multiplication $\mu : X \times X \to X$, show that the group operation in $\pi_n(X, x_0)$ can also be defined by the rule $(f + g)(x) = \mu(f(x), g(x))$.

Proof

4.1.11 Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nulhomotopic, a condition sometimes expressed by saying X_i is contractible in X_{i+1}. An example is S^∞, or more generally the infinite suspension $S^\infty X$ of any CW complex X, the union of the iterated suspensions $S^n X$.

Proof

By Corollary 4.12 the inclusion $X_i \hookrightarrow X$ induces a surjection $\pi_n(X_i) \to \pi_n(X)$. For $[f] \in \pi_n(X)$, $[f]$ must be in $\pi_n(X_i)$ for some i. Since X_i is contractible in X_{i+1}, its image in $\pi_n(X_{i+1})$ is 0. Since this CW complex is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$. Hence $[f] = 0$ in $\pi_n(X)$.

Or, let $f : S^n \to X$ be a map. By Cellular Approximation Theorem, we can assume that f is cellular. Since the image of f is compact, it intersects finitely many n-cells of X, so its image lies in X_k for some k. Since $X_i \hookrightarrow X_{i+1}$ is nullhomotopic, f is nullhomotopic, hence $\pi_n(X) = 0$ (by definition of n-connectedness on page 346). Now, let $\varphi : X \to \{*\}$ be a map, φ induces a isomorphisms on π_n for all n. By Whitehead, φ is a homotopy equivalence, which implies that X is contractible.

As the bottom paragraph in Hatcher's on page 7, if we give S^n a cell structure in which each of the subspheres S^k is a subcomplex, by regarding each S^k as being obtained inductively from the equatorial S^{k-1} by attaching two k-cells, the components of $S^k - S^{k-1}$. Hence the infinite-dimensional sphere $S^\infty = \cup_n S^n$ is a CW complex and we have an increasing sequence of subcomplexes $S^0 \subset S^1 \subset \cdots$ such that S^i is contractible in S^{i+1}. By above argument we proved that S^∞ is contractible.

4.1.12 Show that an n-connected, n-dimensional CW complex is contractible.

Proof

In 4.6 Compression Lemma, let $A = B = x_0$ be a point in X, and let $Y = X$. Then the identity map $f : (X, x_0) \to (X, x_0)$ is homotopic to a map $X \to x_0$. i.e. the identity map is homotopic to a constant map c_{x_0}. Hence f is nulhomotopic. Therefore, X is contractible.

Or, is this just part of 4.1.11? Let $f : X \to \{*\}$ be a map. Since X is n-connected, we have $\pi_n(X) = 0$ and X has no cells of dimension bigger than n. So f induces isomorphisms $f^* : \pi_n(X) \to \pi_n(*)$ for all n. By Whitehead, f is a homotopy equivalence.

4.1.14 Use cellular approximation to show that the n-skeletons of homotopy equivalent CW complexes without cells of dimension $n+1$ are also homotopy equivalence.

Proof

Let X, Y be CW complexes, and $f : X \to Y$ be a homotopy equivalence and $g : Y \to X$ the homotopy inverse. By Theorem 4.8 Cellular Approximation Theorem, we can assume that f is a cellular map. Since f is a homotopy equivalence, there is a homotopy $H : X \times I \to X$ such that $gf \simeq 1_X$.

Since $H_0 = gf$ and $H_1 = 1_X$ are cellular. By Cellular Approximation Theorem, H can be deformed to a cellular map that is stationary on $X \times \{0, 1\}$. Now, consider the restriction $G = H|_{X^n \times I}$. Since H is a cellular map, the restriction G maps X^n to $X^{n+1} = X^n$ because there is no $(n+1)$-cell. Restricting everything to n-skeletons we get a homotopy equivalence $f|_{X^n} : X^n \to Y^n$ and $g|_{Y^n} : Y^n \to X^n$ such that $g|_{Y^n} \circ f|_{X^n} \simeq 1_{X^n}$.

Similarly, we can apply the same proof to the homotopy $H : Y \times I \to Y$ to obtain $f|_{X^n} \circ g|_{Y^n} \simeq 1_{Y^n}$. Therefore, n-skeletons of homotopy equivalent CW complexes without cells of dimension $n+1$ are also homotopy equivalence.

4.1.16 Show that a map $f : X \to Y$ between connected CW complexes factors as a composition $X \to Z_n \to Y$ where the first map induces isomorphisms on π_i for $i \leq n$ and the second map induces isomorphisms on π_i for $i \geq n+1$.

Proof

As in Example 4.17, we can construct Z_n such that $\pi_i(Z_n) \cong \pi_(X)$ for $i \leq n$ and $\pi_i(Z_n) = 0$ for $i > n$. Not finish yet...

4.1.17 Show that if X and Y are CW complexes with X m-connected and Y n-connected, then $(X \times X, X \vee Y)$ is $(m+n+1)$-connected, as in the smash product $X \wedge Y$.

Proof

In Corollary 4.16, let A be a point x_0, then (X, x_0) is homotopic to a CW complex X' such that all cells of X' have dimension greater than m except 0-cell. i.e. X' has only one 0-cell and no k-cell for $k < m+1$. Similarly, by this Corollary, Y is homotopic to a CW complex Y' such that all cells of Y' have dimension greater than n, except 0-cell. i.e. Y' has only one 0-cell and no k-cell for $k < n+1$.

Hence, $X \times Y \simeq X' \times Y'$, where $X' \times Y'$ has no k-cell for $k < m+n+2$ except 0-cell. Since the first cell of $X \times Y$ that is not in $X \vee Y$ is in dimension $m+n+2$, we have

$$(X \times Y)^{m+n+1} = (X \vee Y)^{m+n+1}$$

Hence the top horizontal map of the following diagram is an isomorphism.

$$\begin{tikzcd}
\pi_k((X \vee Y)^{m+n+1}) \arrow[r, "\cong"] \arrow[d] & \pi_k((X \times Y)^{m+n+1}) \arrow[d] & & & \\
\pi_k(X \vee Y) \arrow[r, "f_*"] & \pi_k(X \times Y) \arrow[r] & \pi_k(X \times Y, X \vee Y) \arrow[r] & \pi_{k-1}(X \vee Y) \arrow[r, "g_*"] & \pi_{k-1}(X \times Y)
\end{tikzcd}$$

The right-hand and left-hand vertical map are surjective by Corollary 4.12. Hence the bottom horizontal map f_* is surjective and g_* in injective for $k \leq m+n+1$. Therefore, $\pi_k(X \times Y, X \vee Y) = 0$ for $k \leq m+n+1$. i.e. $(X \times Y, X \vee Y)$ is $(m+n+1)$-connected.

4.1.19 Consider the equivalence relation $\simeq_\omega$ generated by weak homotopy equivalence: $X \simeq_\omega Y$ if there are spaces $X = X_1, X_2, \cdots, X_n = Y$ with weak homotopy equivalences $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$ for each i. Show that $X \simeq_\omega Y$ iff X and Y have a common CW approximation.

Proof

Suppose X and Y have a common CW approximation Z, let $X_2 = Z$, then we are done.

On the other hand, if $X \simeq_\omega Y$. Consider the following diagram,

Case I : If $X_{i-1} \xrightarrow{f} X_i \xrightarrow{g} X_{i+1}$

$$\begin{tikzcd}
X_{i-1} \arrow[r, "f"] & X_i \arrow[r, "g"] & X_{i+1} \arrow[r] & {} \\
Z \arrow[u, "\alpha"'] & & &
\end{tikzcd}$$

Suppose Z is a CW-approximation to X_{i-1}, then $f \circ \alpha : Z \to X_i$ is a CW approximation to X_i.

Case II : If $X_{i-1} \xrightarrow{f} X_i \xleftarrow{g} X_{i+1}$

$$
\begin{tikzcd}
\arrow[r] & X_{i-1} \arrow[r, "f"] & X_i & X_{i+1} \arrow[l, "g"'] \arrow[r] & {} \\
\arrow[r] & Z_{i-1} \arrow[u, "\alpha"] \arrow[r, "f'"] & Z_i \arrow[u, "\beta"] & &
\end{tikzcd}
$$

Let $\alpha : Z_{i-1} \to X_{i-1}$ and $\beta : Z_i \to X_i$ be CW approximations to X_{i-1} and X_i respectively. So f induces a map f' such that the diagram commutes. Since α, β and f are all weak homotopy equivalences, hence induces isomorphism on π_n. Hence f' also induces isomorphism on π_n for all $n \geq 0$. By Whitehead's Theorem, f' is a homotopy equivalence. Let $g' : Z_i \to Z_{i-1}$ be the homotopy inverse. Hence, $\alpha \circ g' : Z_i \to X_{i-1}$ is a CW approximation to X_{i-1}. Hence, Z_i is the common CW approximation to X_{i-1} and X_i.

Inductively, we will find a common CW approximation between X and Y.

4.2.1 Use homotopy groups to show there is no retraction $\mathbb{R}P^n \to \mathbb{R}P^k$ if $n > k > 0$.

Proof

The map $i : \mathbb{R}P^k \to \mathbb{R}P^n$ in an inclusion. Suppose there is a retraction $r : \mathbb{R}P^n \to \mathbb{R}P^k$, then $r \circ i = 1_{\mathbb{R}P^k}$. However,

$$r \circ i : \mathbb{R}P^k \to \mathbb{R}P^n \to \mathbb{R}P^k$$

induces

$$r_* \circ i_* : \pi_k(\mathbb{R}P^k) \to \pi_k(\mathbb{R}P^n) \to \pi_k(\mathbb{R}P^k)$$

on homotopy group π_k.

By Proposition 4.1: A covering space $p : \tilde{X} \to X$ induces isomorphisms $p_* : \pi_n(\tilde{X}) \to \pi_n(X)$ for all $n \geq 2$, we have $\pi_k(\mathbb{R}P^k) = \pi_k(S^k) = \mathbb{Z}$ and $\pi_k(\mathbb{R}P^n) = 0$. Hence we get a contradiction since there's no such a sequence of maps exist.

4.2.8 Show the suspension of an acyclic CW complex is contractible.

Proof

Let X be an acyclic CW complex, then $\tilde{H}_i(X) = 0$ for all i. Note that $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$, so $H_0(X) \cong \mathbb{Z}$, hence X is path connected. Thus SX is simply connected, $\pi_1(SX) = 0$. By Hurewicz Theorem, $H_1(SX) = 0$ and $H_2(SX) = \pi_2(SX)$. By Exercise 2.2.32, $H_2(SX) = H_1(X) = 0$, so $\pi_2(SX) = 0$. Applying Hurewicz inductively, we see that $\pi_i(SX) = 0$ for all i. Hence SX is contractible.

Or: Let $x_0 \in SX$ and $f : SX \to x_0$ be a map. Since SX and x_0 are both simply connected and the induced map $f_* : H_n(SX) \to H_n(x_0)$ is an isomorphism for all n since $H_n(SX) = 0)$ for all n by the same procedure as above (Exercise 2.2.32). Then by Corollary 4.33 (alternative version of Whitehead), $SX \simeq \{*\}$, i.e. SX is contractible.

4.2.9 Show that a map between simply-connected CW complexes is a homotopy equivalence if its mapping cone is contractible. Use the proceeding exercise to give an example where this fails in the nonsimply-connected case.

Proof

Let $f : X \to Y$ for X, Y simply-connected CW complexes. Since $C_f \simeq M_f/X \simeq Y/X$ and by assumption the mapping cone is contractible, we have $H_i(C_f) = H_i(M_f/X) = H_i(Y/X) = 0$. The LES of the pair (M_f, X) gives

$$H_{i+1}(M_f, X) \longrightarrow H_i(X) \xrightarrow{\cong} H_i(M_f) \longrightarrow H_i(M_f, X) \longrightarrow$$

which is

$$0 \longrightarrow H_i(X) \xrightarrow{\cong} H_i(M_f) \longrightarrow 0$$

Hence $H_i(X) \cong H_i(M_f) \cong H_i(Y)$ for all $i > 0$. Now, for $i = 0$, since X, Y are simply connected, hence path connected (by definition of n-connectedness: 1-connected implies 0-connected, page 346). This is because So $H_0(X) = H_0(Y) = \mathbb{Z}$. Therefore, $H_i(X) \cong H_i(M_f) \cong H_i(Y)$ for all $i \geq 0$. By Corollary 4.33 (a version of Whitehead), $X \simeq Y$.

As in Example 2.38 in the text, let X be obtained from $S^1 \vee S^1$ by attaching two 2-cells by the words a^5b^{-3} and $b^3(ab)^{-2}$. Then X is acyclic and $\tilde{H}_i(X) = 0$ for all i. And X is not contractible and $\pi_1(X) \neq 0$ hence X is not simply-connected. By preceding exercise, SX is contractible, so its mapping cone is contractible (why?). But the map $f : X \to \{*\}$ is not a homotopy equivalence.

4.2.12 Show that a map $f : X \to Y$ of connected CW complexes is a homotopy equivalence if it induces an isomorphism on π_1 and if a lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ to the universal covers induces an isomorphism on homology.

Proof

$\tilde{X}$ and $\tilde{Y}$ are simply-connected because they are universal covers. Since $H_i(\tilde{X}) \cong H_i(\tilde{Y})$ for all i. So by Corollary 4.33 $\tilde{X} \simeq \tilde{Y}$, hence $\pi_i(\tilde{X}) = \pi_i(\tilde{Y})$ for all i.

But Proposition 4.1 says that $\pi_i(\tilde{X}) = \pi_i(X)$ and $\pi_i(\tilde{Y}) = \pi_i(Y)$ for $i \geq 2$. And by assumption $\pi_1(X) = \pi_1(Y)$. Also $\pi_0(X) \cong \pi_0(Y)$. So we have $\pi_i(X) = \pi_i(Y)$ for all i. Since the map $f : X \to Y$ of connected CW complexes induces isomorphisms on all homotopy groups, $f_* : \pi_i(X) \to \pi_i(Y)$, so by Whitehead's Theorem f is a homotopy equivalence, or $X \simeq Y$.

4.2.13 Show that a map between connected n-dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on π_i for $i \leq n$.

Proof

Let $f : X \to Y$ be a map and X, Y are n-dimensional CW complexes. Then $\pi_i(X) = \pi_i(Y)$ for $i \leq n$. Let $\tilde{X}$ and $\tilde{Y}$ be the universal covering of X and Y respectively. So $\pi_1(\tilde{X}) = \pi_1(\tilde{Y}) = 0$. Since $\pi_i(\tilde{X}) = \pi_i(X)$ and $\pi_i(\tilde{Y}) = \pi_i(Y)$ for $2 \leq i \leq n$ by Proposition 4.1. So $\pi_i(\tilde{X}) = \pi_i(\tilde{Y})$ for $i \leq n$.

Without loss of generality, replacing $\tilde{Y}$ by the mapping cylinder $M_{\tilde{f}}$ we may take $\tilde{f}$ to be an inclusion $\tilde{X} \hookrightarrow \tilde{Y}$. Then LES of the pair $(M_{\tilde{f}}, \tilde{X})$ is as follows:

$$\longrightarrow \pi_{i+1}(M_{\tilde f}, \tilde X) \longrightarrow \pi_i(\tilde X) \xrightarrow{=} \pi_i(M_{\tilde f}) \longrightarrow \pi_i(M_{\tilde f}, \tilde X) \longrightarrow$$

Since $\pi_i(\tilde Y) = \pi_i(\tilde X)$, we have $\pi_i(M_{\tilde f}, \tilde X) = 0$ for $i \le n$, i.e. $(M_{\tilde f}, \tilde X)$ is n-connected. By Hurewicz Theorem, $H_i(\tilde Y, \tilde X) = 0$ for $i \le n$.

Now, consider the diagram below:

$$\begin{tikzcd}
{} \arrow[r] & \pi_{i+1}(\tilde Y, \tilde X) \arrow[r] \arrow[d, "\cong h"] & \pi_i(\tilde X) \arrow[r, "\cong"] \arrow[d] & \pi_i(\tilde Y) \arrow[r] \arrow[d] & \pi_i(\tilde Y, \tilde X) \arrow[r] \arrow[d, "\cong_h"] & {} \\
{} \arrow[r] & 0 = H_{i+1}(\tilde Y, \tilde X) \arrow[r] & H_i(\tilde X) \arrow[r, "\cong \tilde f_*"] & H_i(\tilde Y) \arrow[r] & H_i(\tilde Y, \tilde X) = 0 \arrow[r] & {}
\end{tikzcd}$$

For $i < n$, the two-end vertical maps h are isomorphism by Hurewicz Theorem. Since $H_i(\tilde Y, \tilde X) = 0$ for $i \le n$, we have $H_i(\tilde X) \cong H_i(\tilde Y)$ for $i \le n$. Hence, $H_i(\tilde X) \cong H_i(\tilde Y)$ for all n since $\tilde X$ and $\tilde Y$ have no cells of dimension greater than n. By Corollary 4.33 (Whitehead's Theorem), $\tilde f$ is a homotopy equivalence, hence it is an isomorphism on homotopy groups. As mentioned earlier, $\pi_i(\tilde X) = \pi_i(X)$ and $\pi_i(\tilde Y) = \pi_i(Y)$ for all i. So $f : X \to Y$ is an isomorphism on homotopy groups. Then By Whitehead's Theorem, f is a homotopy equivalence.

4.2.14 If an n-dimensional CW complex X contains a subcomplex Y homotopy equivalent to S^n, show that the map $\pi_n(Y) \to \pi_n(X)$ induced by inclusion is injective.

Proof

Since X has no $n+1$ cells, $H_n(X)$ is a subgroup of the free abelian group on the cells on X. Of course the same is true for Y. Since Y is homotopic equivalent to S^n, take a generator for $H_n(Y)$. It is a linear combination of n-cells of Y, so it is a linear combination of n-cells of X. Since it is the image of a cycle, it is a cycle. So it generates a copy of $\mathbb{Z}$ as a subgroup of $H_n(X)$. So $H_n(Y) \to H_n(X)$ is an injection. Consider the diagram

$$\begin{tikzcd}
 & \pi_n(Y) \arrow[r] \arrow[d, "\cong"', "h"] & \pi_n(X) \arrow[d] \\
0 \arrow[r] & H_n(Y) \arrow[r, "\text{injective}"] & H_n(X)
\end{tikzcd}$$

So the bottom map is injective. Now, Since $Y \simeq S^n$ and S^n is $(n-1)$-connected, by Hurewicz Theorem, $\pi_n(Y) \approx H_n(Y)$ for $i \le n$ and the left vertical maps is an isomorphism. The diagram commutes, so the top map must be also injective.

4.2.15 Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3.

Proof

Let X be a closed simply-connected 3-manifold, X is homotopy equivalent to a CW complex (so later we could use Whitehead). Since $\pi_1(X) = 0$, by Hurewicz $H_1(X) = 0$ and $\pi_2(X) = H_2(X)$. By Poincare Duality, $H^2(X) = 0$, then by UCT $H_2(X) = 0$. Also, $H_3(X) = H^0(X) = \mathbb{Z}$ (or since X is simply-connected, it is orientable). Let $f : S^3 \to X$ be a map sending generator to generator of $\pi_3(X) = \mathbb{Z}$ (since by Hurewicz $H_3(X) = \mathbb{Z}$). Now, since the induced map $f_* : H_n(X) \to H_n(S^3)$ is an isomorphism, by Corollary 4.33 (Whitehead) $X \simeq S^3$.

4.2.31 For a fiber bundle $F \to E \to B$ such that the inclusion $F \hookrightarrow E$ is homotopic to a constant map, show that the long exact sequence of homotopy groups breaks up into split short exact sequence giving isomorphisms $\pi_n \approx \pi_n(E) \oplus \pi_{n-1}(F)$. In particular, for the Hopf bundles $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$ yields isomorphisms $\pi_n(S^4) \approx \pi_n(S^7) \oplus \pi_{n-1}(S^3)$ and $\pi_n(S^8) \approx \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$. Thus $\pi_7(S^4)$ and $\pi_{15}(S^8)$ contain $\mathbb{Z}$ summands.

Proof

The map $\pi_n(F) \to \pi_n(E)$ in the LES of homotopy groups for a Serre fibration (Serre fibration means: the map $p : E \to B$ satisfies the homotopy lifting property for disks) are induced by the inclusion $F \hookrightarrow E$. So if the inclusion is homotopic to a constant map, then the induced map is 0. Hence LES splits into SES

$$0 \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow 0$$

Since the map $p : E \to B$ has the homotopy lifting property with respect to all disks, we can find a section $\pi_n(B) \to \pi_n(E)$ for the induced map $\pi_n(E) \to \pi_n(B)$, which means that the above short exact sequence splits.

In the Hopf bundle $S^3 \to S^7 \to S^4$, the map $S^3 \hookrightarrow S^7$ is an inclusion. So we have a split SES

$$0 \longrightarrow \pi_7(S^7) \longrightarrow \pi_7(S^4) \longrightarrow \pi_6(S^3) \longrightarrow 0$$

i.e.

$$\pi_7(S^4) \cong \pi_7(S^7) \oplus \pi_6(S^3)$$

Since $\pi_7(S^7) = \mathbb{Z}$, $\pi_7(S^4)$ contains $\mathbb{Z}$ summand. Similarly for $\pi_{15}(S^8)$.

4.2.32 Show that if $S^k \to S^m \to S^n$ is a fiber bundle, then $k = n - 1$ and $m = 2n - 1$. (Look at the long exact sequence of homotopy groups.)

Proof

Note that we have $n \leq m$, $k \leq m$ and $k + n = m$. If $k = m$, then $n = 0$. S^0 is not connected, this contradicts that $S^m \to S^n$ is surjective. So $k < m$, hence the map $S^k \to S^m$ is homotopic to a constant map. By Exercise 4.2.31 we have

$$\pi_i(S^n) \cong \pi_i(S^m) \oplus \pi_{i-1}(S^k)$$

Thus, $n > 0$ and $m > n$. And we see that $\pi_j(S^k) = 0$ for $j < n - 1$ and $\pi_{n-1}(S^k) = \mathbb{Z}$ since

$$\mathbb{Z} = \pi_n(S^n) = \pi_n(S^m) \oplus \pi_{n-1}(S^k)$$

Therefore, $k = n - 1$ and $m = 2n - 1$.