MATH 512: Advanced Topology
Spring 2021 | Homework #1 Solutions

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1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution: Let $I = [-1, 1]$ be an interval on $\mathbb{R}^2$. Let the origin $(0, 0)$ be the point deleted from the torus. Consider the map $f : I^2 - \{0\} \to \mathbb{S}$

$$f(x) = \frac{x}{\|x\|}.$$

Notice that f is a deformation retraction onto $\mathbb{S}$.

Consider $g = f|_{\partial I^2}$, which sends all points on ∂I^2 onto $\mathbb{S}$. With this definition, $g^{-1} \circ f$ is a deformation retraction sending all points $I^2 - \{0\}$ to ∂I^2.

Now, define the homotopy $H : \mathbb{T}^2 - \{*\} \times I \to \partial I^2$ by

$$H(x, t) = (1 - t)x + t(g^{-1} \circ f).$$

This is a deformation retraction of the torus with one point deleted onto ∂I^2, which is a graph of two cricles intersecting at a point.

2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto $\mathbb{S}^{n-1}$.

Solution: consider the following family of functions $f_t : \mathbb{R}^n - \{0\} \to \mathbb{S}^{n-1}$ for all $t \in \mathbb{R}$.

$$f_t(x) = x + \left(\frac{1}{\|x\|} - 1\right) tx.$$

Notice that $f_0(x) = x$, so that $f_0 = \mathbb{1}$. We compute

$$f_t(x) \cdot f_t(x) = \|x\|^2 \left(1 + \left(\frac{1}{\|x\|} - 1\right) t\right)^2.$$

Hence $\|f_1(x)\|^2 = 1$, showing that $f_1(x) \in \mathbb{S}^{n-1}$ for any $x \in \mathbb{R}^n - \{0\}$.

Finally, if $x \in \mathbb{S}^{n-1}$, then $\|x\| = 1$ so f_t reduces to the identity map. ■

3. (a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

Solution:

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

Solution:

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Solution:

4. A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}$, $f_1(X) \subseteq A$, and $f_t(A) \subseteq A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Solution: Let $i : A \hookrightarrow X$ be the inclusion and let $f : X \to A$ be the range restriction of $f_1 : X \to X$. We want to show that

$$if \simeq \mathbb{1}_X \quad \text{and} \quad fi \simeq \mathbb{1}_A.$$

Let $\overline{f}_t = f_t|_A$, range restricted to A. This is a homotopy between $\mathbb{1}_A$ and fi.

Since $if = f_1$ and $f_0 = \mathbb{1}$, f_t itself is a homotopy between $\mathbb{1}_X$ and if.

5. Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of $x \in X$ there exists a neighborhood $V \subseteq U$ of x such that the inclusion $V \hookrightarrow U$ is nullhomotopic.

Solution: By hypothesis, there is a homotopy $f : X \times I \to X$ with $f_0 = \mathbb{1}_X$ and $f(x, t) = x$ for all t. By continuity, $f^{-1}(U)$ is an open set for all open sets $U \subseteq X$. In other words, for each t there is a neighborhood $V_t \ni x$ and an open interval $I_t \ni t$ such that $f(V_t, I_t) \subseteq U$.

I is compact so there is a finite set of intervals $I_1, \ldots, I_n$, covering I, with corresponding sets $V_1, \ldots, V_n$ so that $f(V_i \times I_i) \subseteq U$ for $i = 1, \ldots, n$.

Now, set $V = U \cap V_1 \cap \cdots \cap V_n$ and notice that $f(V \times I) \subseteq U$ and $f|_{V \times I}$ is a homotopy from $V \hookrightarrow U$ to a constant map. ■

6. (a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0, 1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1 - r]$ for r a rational number in $[0, 1]$. Show that X deformation retracts to any point in the segment $[0, 1] \times \{0\}$, but not to any other point.

Solution:

(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure (in the book). Show that Y is contractible but does not deformation retract onto any point.

Solution:

(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show that there is a deformation retraction in the weak sense of Y onto Z, but no true deformation retraction.

Solution: