MATH 512: Advanced Topology
Spring 2021 | Homework #2 Solutions

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9. Show that a retract of a contractible space is contractible.

Solution: Let X be a contractible space, then $H(x,0) = \mathbb{1}_X$ for $x \in X$, $H(x,1) = c_{x_0}$, and $\mathbb{1}_X \simeq c_{x_0}$, where c_{x_0} is the constant map $X \mapsto x_0$. Assume that X retracts to A, then $r : X \to A$ is a retract such that $r(X) = A$, $r|_A = \mathbb{1}_A$.

Define $G = r \circ H$, then $G(a,t) = r(H(a,t))$.

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "r"'] & X \arrow[d, "r"] \\ A \times I \arrow[r, "G"] & A \end{tikzcd}$$

Since

$$G(a,0) = r(H(a,0)) = r(a) = a$$

so $G_0 = G(A,0) = \mathbb{1}_A$. Also,

$$G(a,1) = r(H(a,1)) = r(x_0) = x_0$$

so $G_1 = G(A,1) = c_{x_0}$. Since the composition of continuous maps is continuous, G is the desired homotopy, i.e., the retract of X is contractible.

14. Given positive integers v, e, and f satisfying $v - e + f = 2$, construct a cell structure on $\mathbb{S}^2$ having v 0-cells, e 1-cells, and f 2-cells.

Solution: Suppose $v = 1$. Then $f = 1 + e$. Attach the two ends of all 1-cells to this only 0-cell, like a bouquet of circles. Then fill in each circle with 2-cells. Then, attach the last 2-cell to the boundary of the bouquet of circles.

17. (a) Show that the mapping cylinder of every map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a CW complex.

    Solution:

    (b) Construct a 2-dimensional CW complex that contains both an annulus $\mathbb{S} \times I$ and a Möbius band as deformation retracts.

18. Show that $\mathbb{S}^1 * \mathbb{S}^1 = \mathbb{S}^3$, and more generally $\mathbb{S}^m * \mathbb{S}^n = \mathbb{S}^{m+n+1}$.

20. Show that the subspace $X \subseteq \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle is homotopy equivalent to $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$.

Solution: