ECE 697 Modeling and High-Performance Control of Electric Machines
HW 5 Solutions
Spring 2022

Problem 1 *Azimuthal Magnetic Field of a Circular Current Loop*

Consider the circular curve C of radius R in the $x-y$ plane (not necessarily at $z=0$) whose origin is at $(0,0,z_0)$. Symmetry requires that the azimuthal component of $\vec{\mathbf{B}}$ be only a function of r and z, but not θ; that is, $B_\theta\hat{\boldsymbol{\theta}} = B_\theta(r,z)\hat{\boldsymbol{\theta}}$. Then

$$\oint \vec{\mathbf{B}}\cdot d\vec{\boldsymbol{\ell}} = \oint \left(B_\theta\hat{\boldsymbol{\theta}}\right)\cdot\left(rd\theta\hat{\boldsymbol{\theta}}\right) = 2\pi R B_\theta(R,z_0) = 0 \implies B_\theta(R,z_0) = 0 \text{ for any } (R,z_0).$$

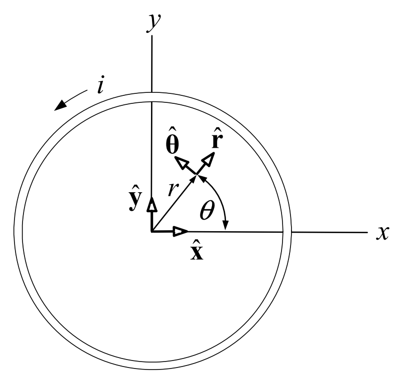

Figure 1: Circular loop carrying the current i.

Problem 2 *Ampère's Law*

(a) Figure 2 shows a long straight conductor carrying a current I. The current is uniformly distributed across the cross section so that the current i enclosed within a circle of radius r is

$$i = \begin{cases} I\dfrac{r^2}{R^2} & \text{for } 0 \le r \le R \\ I & \text{for } r > R. \end{cases}$$

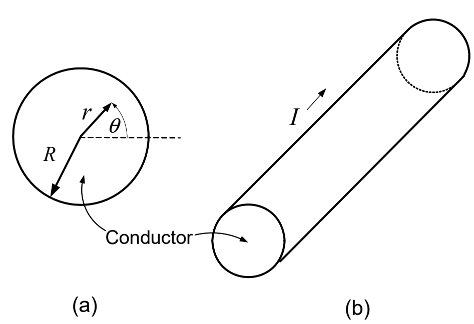

Figure 2: Long straight wire. (a) Cross sectional view. (b) Current in the conductor.

As shown in Figure 3, a cylindrical coordinate system whose axis is along the wire and origin is at the center of the wire is used. It was shown in the text that *outside* the wire

$$\vec{\mathbf{B}}(r) = \frac{\mu_0 I}{2\pi r}.$$

Similar arguments as in the text show that $B_z = B_r = 0$ *inside* the wire.

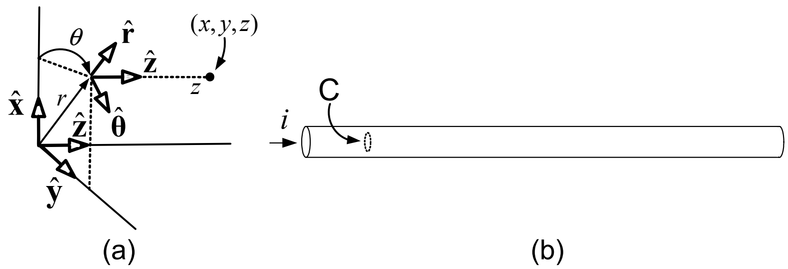

Figure 3: Cylindrical coordinate system for the long straight wire.

Apply Ampère's law to the circular curve C of radius r_1 shown in the figure whose origin coincides with the center of the wire to obtain

$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \oint \left(B_\theta \hat{\boldsymbol{\theta}}\right) \cdot \left(r d\theta \hat{\boldsymbol{\theta}}\right) = 2\pi r_1 B_\theta(r_1) = \mu_0 I \frac{r_1^2}{R^2}$$

or

$$B_\theta(r_1) = \frac{\mu_0 I r_1^2}{2\pi r_1 R^2} = \frac{\mu_0 I}{2\pi R^2} r_1 \text{ for } 0 \le r_1 \le R.$$

(b) A cylindrical conductor with an off axis cylindrical hole carries a total current I. Find the magnetic field in the cylindrical hole. The total current in the conductor is I and is uniformly spread across the cross section of area $\pi\left(R^2 - r_2^2\right)$ so that its density is

$$\frac{I}{\pi\left(R^2 - r_2^2\right)}.$$

Consider the hole to be filled up and carrying current with the same current density so that the total current in the hole is

$$I_{hole} = \frac{r_2^2}{R^2 - r_2^2} I.$$

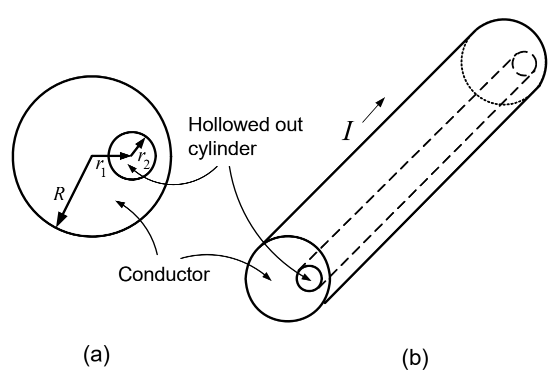

Figure 4: Conductor with a hollowed cylindrical core. (a) Cross sectional view. (b) Current in the conductor.

With the hole filled, the total current in the wire is $I + I_{hole}$ and, by superposition, the magnetic field can be found by computing the B field due to the solid wire (filled hole) carrying the current $I + I_{hole}$ and subtracting the B field due to the current I_{hole} in the filled in hole. Using a second cylindrical coordinate system (ρ, φ, z) centered at the hollowed out hole, it follows that

$$\vec{\mathbf{B}} = \frac{\mu_0 \left(I + I_{hole}\right)}{2\pi R^2} r\hat{\boldsymbol{\theta}} - \frac{\mu_0 I_{hole}}{2\pi r_2^2} \rho\hat{\boldsymbol{\varphi}} \text{ inside the hole.}$$

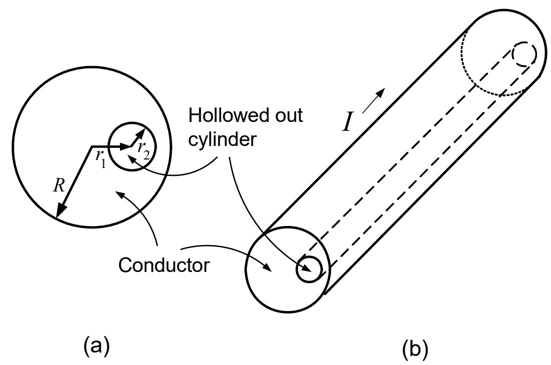

Figure 5: Conductor with a hollowed cylindrical core. (a) Cross sectional view. (b) Current in the conductor.

Problem 3 *Radial Magnetic Field of a Long Straight Wire.*

Cylindrical symmetry requires that the magnetic field can only depend on r, that is, it has the form

$$\vec{\mathbf{B}} = B_r(r)\hat{\mathbf{r}} + B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}}.$$

Applying Gauss's law to the closed surface S (whose radius is R and length is ℓ) in Figure 6, one obtains

$$\begin{aligned}\oint_S \vec{\mathbf{B}} \cdot d\vec{S} &= \int_{S_1} (B_z(r)\hat{\mathbf{z}}) \cdot (-rd\theta dr\hat{\mathbf{z}}) + \int_{S_2} (B_z(r)\hat{\mathbf{z}}) \cdot (rd\theta dr\hat{\mathbf{z}}) + \int_{S_3} (B_r(r)\hat{\mathbf{r}}) \cdot (Rd\theta dz\hat{\mathbf{r}}) \\ &= 2\pi \int_0^R rB_z(r)dr - 2\pi \int_0^R rB_z(r)dr + 2\pi R\ell B_r(R) \\ &= 2\pi R\ell B_r(R) = 0 \Longrightarrow B_r(R) = 0 \text{ for any } R.\end{aligned}$$

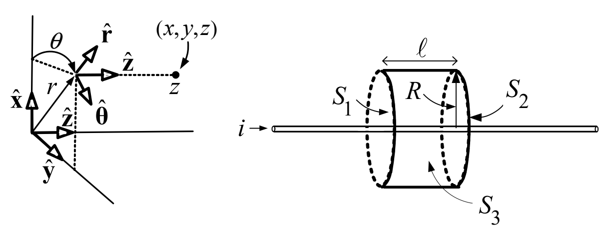

Figure 6: Closed flux surface used to show $B_r \equiv 0$ around an inifintely long straight wire carrying a current.

Problem 4 *Radial Magnetic Field of an Ideal Solenoid*

Using some symmetry arguments when applying Gauss's law to the closed surface shown in Figure 7 whose sides are S_1, S_2, S_3, show that the radial component of $\vec{\mathbf{B}}$ must be zero *inside* an infinitely long solenoidal coil. Using a similar argument, show that the radial component of $\vec{\mathbf{B}}$ *outside* of an infinitely long solenoidal coil must also be zero.

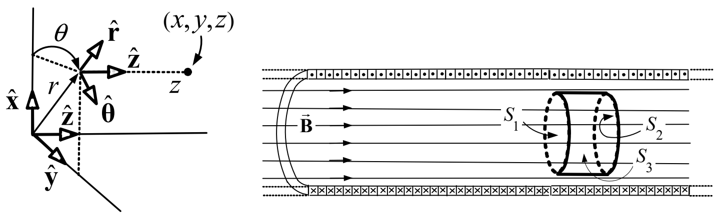

Figure 7:

As this is an infinitely long solenoid, cylindrical symmetry requires that the magnetic field can only depend on r, that is, it has the form

$$\vec{\mathbf{B}} = B_r(r)\hat{\mathbf{r}} + B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}}$$

both inside and outside the solenoid. Applying Gauss's law to the closed surface (whose radius is R and length is ℓ)

$$\begin{aligned}\oint_S \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{S}} &= \int_{S_1} (B_z(r)\hat{\mathbf{z}}) \cdot (-rd\theta dr\hat{\mathbf{z}}) + \int_{S_2} (B_z(r)\hat{\mathbf{z}}) \cdot (rd\theta dr\hat{\mathbf{z}}) + \int_{S_3} (B_r(r)\hat{\mathbf{r}}) \cdot (Rd\theta dz\hat{\mathbf{r}}) \\ &= 2\pi \int_0^R rB_z(r)dr - 2\pi \int_0^R rB_z(r)dr + 2\pi R\ell B_r(R) \\ &= 2\pi R\ell B_r(R) = 0 \Longrightarrow B_r(R) = 0 \text{ for any } R.\end{aligned}$$

This argument works whether the closed surface is completely inside the solenoid or extends outside because the value of the magnetic field does not depend on z so that the two surface integrals over S_1 and S_2 will always cancel each other.

Problem 5 *Azimuthal Magnetic Field of an Ideal Solenoid*

By the symmetry of the problem,

$$\vec{\mathbf{B}} = B_r(r)\hat{\mathbf{r}} + B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}}.$$

As C_3 is inside the solenoid,

$$\oint_{C_3} \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \oint_{C_3} B_\theta(r)\hat{\boldsymbol{\theta}} \cdot \left(rd\theta\hat{\boldsymbol{\theta}}\right) = 2\pi r B_\theta(r) = 0 \implies B_\theta(r) = 0 \text{ for all } r.$$

As C_5 is outside the solenoid,

$$\begin{aligned}\oint_{C_5} \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} &= \oint_{C_5} B_\theta(r)\hat{\boldsymbol{\theta}} \cdot \left(rd\theta\hat{\boldsymbol{\theta}}\right) = \mu_0 i_{\text{enclosed}} \\ 2\pi r B_\theta(r) &= \mu_0 i \\ B_\theta(r) &= \mu_0 i/(2\pi r).\end{aligned}$$

Problem 6 *Magnetic Field in an Ideal Toroidal Coil*

(a) By symmetry, $\vec{\mathbf{B}}$ cannot depend on θ (see Figure 8). Assuming the small diameter d of the solenoid is much smaller than it major radius, we make an approximation that $\vec{\mathbf{B}}$ cannot depend on φ either. Then we may write

$$\vec{\mathbf{B}} = B_\rho(\rho)\hat{\boldsymbol{\rho}} + B_\varphi(\rho)\hat{\boldsymbol{\varphi}} + B_\theta(\rho)\hat{\boldsymbol{\theta}}.$$

By Ampère's law

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \oint_C B_\varphi(\rho)\hat{\boldsymbol{\varphi}} \cdot (\rho d\varphi \hat{\boldsymbol{\varphi}}) = 2\pi\rho B_\varphi(\rho) = 0 \implies B_\varphi(\rho) = 0.$$

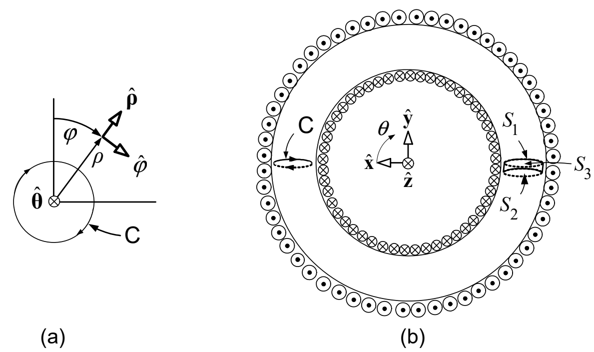

Figure 8: (a) "Toroidal" coordinate system. $\hat{\theta}$ is into the page. (b) Curve and surface for Ampere's and Gauss's law, respectively to show that $B_\rho = B_\varphi = 0$.

Setting $R = (r_1 + r_2)/2$ with r_1, r_2 are the inner and outer radii of the toroid, respectively, Gauss's law gives

$$\begin{aligned}
\oint_S \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{S}} &= \int_{S_1} \left(B_\theta(\rho)\hat{\boldsymbol{\theta}}\right) \cdot \left(-\rho d\varphi d\rho \hat{\boldsymbol{\theta}}\right) + \int_{S_2} \left(B_\theta(\rho)\hat{\boldsymbol{\theta}}\right) \cdot \left(\rho d\varphi d\rho \hat{\boldsymbol{\theta}}\right) + \int_{S_3} (B_\rho(\rho)\hat{\boldsymbol{\rho}}) \cdot (R d\theta \rho d\varphi \hat{\boldsymbol{\rho}}) \\
&= -2\pi \int_{S_1} B_\theta(\rho)\rho d\rho - 2\pi \int_{S_2} B_\theta(\rho)\rho d\rho + 2\pi R\rho d\theta B_\rho(\rho) \\
&= 2\pi R\rho d\theta B_\rho(\rho) \\
&= 0 \\
\implies B_\rho(\rho) &= 0 \text{ for any } \rho.
\end{aligned}$$

(b) Show that the $\hat{\boldsymbol{\theta}}$ component of $\vec{\mathbf{B}}$ must be zero *outside* the toroidal coil. Let C be a circle whose center axis goes through the toroid's center and whose radius is greater than r_2 the outside radius of the toroid. Then no current is enclosed by C and, by symmetry, B_θ can only depend on r and z, but not θ.

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \oint_C B_\theta(r,z)\hat{\boldsymbol{\theta}} \cdot \left(r d\theta \hat{\boldsymbol{\theta}}\right) = 2\pi r B_\theta(r,z) = 0 \implies B_\theta(r,z) = 0.$$