# Modeling and High-Performance Control of Electric Machines

## Chapter 1 Problems 15, 16 and 17

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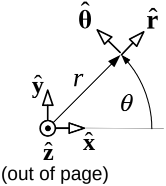
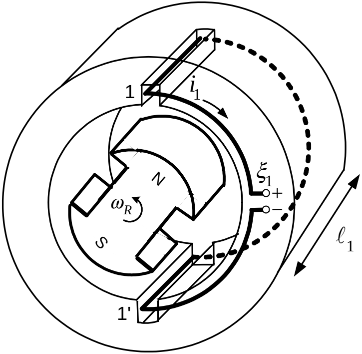
# Three Phase Permanent Magnet Generator

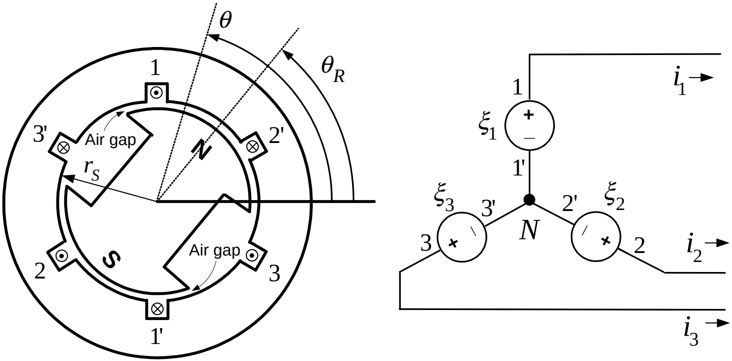
The following slides go over problems 15, 16, and 17 of Chapter 1 and a little more.

# Three Phase Permanent Magnet Generator



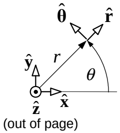



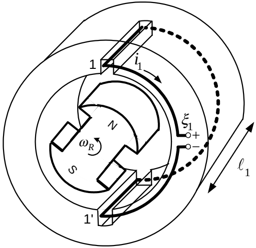

- Use a cylindrical coordinate system as shown.
- The rotor is a single pole-pair permanent magnet.
- Only phase 1 is shown; it has a half-cylindrical shape.
- Note the **sign convention** for current:
  The current is **positive** in phase 1 if it is coming **out** of the $+$ terminal.




- Each phase winding has the same half-cylindrical shape as in the previous slide.
- Phase winding $2 - 2'$ is rotated $2\pi/3$ $(120^\circ)$ from phase $1 - 1'$.
- Phase winding $3 - 3'$ is rotated $2\pi/3$ $(120^\circ)$ from phase $2 - 2'$.
- The **negative** terminals of the three windings are connected **together**.
- The point of connection $N$ of the three windings is called the **neutral**.

# Three Phase Permanent Magnet Generator (continued)




The **magnetic field** in the **air gap** due to the rotor's **permanent magnet** is

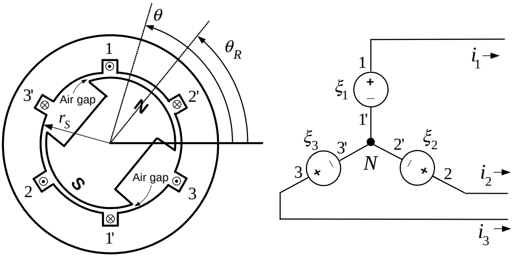
$$\vec{\mathbf{B}}(r,\theta) = B_{R\max}\frac{r_S}{r}\cos(\theta-\theta_R)\hat{\mathbf{r}}.$$

$B_{R\max} > 0$, $r_S$ is the inside radius of the stator iron.

$$\begin{aligned}\phi_{1-1'} = \int_{S_1}\vec{\mathbf{B}}\cdot d\vec{\mathbf{S}} &= \int_0^{\ell_1}\int_{-\pi/2}^{\pi/2} B_{R\max}\cos(\theta-\theta_R)\hat{\mathbf{r}}\cdot\left(r_S\, d\theta\, d\ell\hat{\mathbf{r}}\right)\\ &= r_S\ell_1 B_{R\max}\sin(\theta-\theta_R)\Big|_{-\pi/2}^{\pi/2}\\ &= 2r_S\ell_1 B_{R\max}\cos(\theta_R).\end{aligned}$$

If $\xi_{1-1'} = -\dfrac{d\phi_{1-1'}}{dt} > 0$ will $\xi_{1-1'}$ push current **into** the $+$ terminal or **out** of it?

## Three Phase Permanent Magnet Generator (continued)

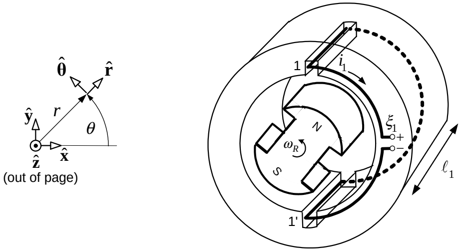



$$\phi_{1-1'} = 2r_S\ell_1 B_{R\,\max}\cos(\theta_R)$$
$$\phi_{2-2'} = 2r_S\ell_1 B_{R\,\max}\cos(\theta_R - 2\pi/3)$$
$$\phi_{3-3'} = 2r_S\ell_1 B_{R\,\max}\cos(\theta_R - 4\pi/3).$$

The induced emfs are then $(\theta_R = \omega_R t)$

$$\xi_{1-1'} = -\frac{d\phi_{1-1'}}{dt} = 2r_S\ell_1 B_{R\,\max}\omega_R\sin(\omega_R t)$$
$$\xi_{2-2'} = -\frac{d\phi_{2-2'}}{dt} = 2r_S\ell_1 B_{R\,\max}\omega_R\sin(\omega_R t - 2\pi/3)$$
$$\xi_{3-3'} = -\frac{d\phi_{3-3'}}{dt} = 2r_S\ell_1 B_{R\,\max}\omega_R\sin(\omega_R t - 4\pi/3).$$

## Three Phase Permanent Magnet Generator (continued)



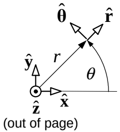

- The magnetic field in the air gap is **changing** in time as the PM rotor is turning.
- By Faraday's law in **differential form** ($\nabla \times \vec{\mathbf{E}} = -\partial \vec{\mathbf{B}}/\partial t$), there is an **electric field** in the air gap given by

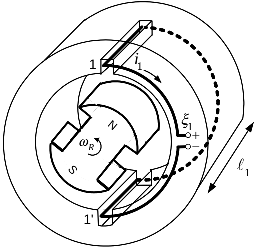
$$\vec{\mathbf{E}}_R\left(\theta - \theta_R\right) = \omega_R B_{R\max} r_S \cos(\theta - \theta_R)\hat{\mathbf{z}}.$$

$$\begin{aligned}\xi_{1-1'} = \int_{1'}^{1} \vec{\mathbf{E}}_R(\theta - \theta_R)\cdot d\vec{\boldsymbol{\ell}} &= \int_{side1} \left(\omega_R B_{R\max} r_S \cos(\pi/2 - \theta_R)\hat{\mathbf{z}}\right)\cdot\left(d\ell\hat{\mathbf{z}}\right) + \\ &\quad \int_{side1'} \left(\omega_R B_{R\max} r_S \cos(-\pi/2 - \theta_R)\hat{\mathbf{z}}\right)\cdot\left(-d\ell\hat{\mathbf{z}}\right) \\ &= 2\omega_R B_{R\max} r_S \ell_1 \sin(\omega_R t)\end{aligned}$$

  where $\theta_R = \omega_R t$, $\cos(\pi/2 - \theta_R) = \sin(\theta_R)$, $\cos(-\pi/2 - \theta_R) = -\sin(\theta_R)$ were used.
- This the **same** result we got using Faraday's law in **integral form**, i.e., $\xi = -d\phi/dt$.

## Three Phase Permanent Magnet Generator - Torque

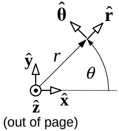


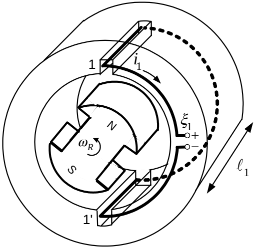




**Magnetic Force $\vec{\mathbf{F}}_{side1}$** on stator side 1.

$$\begin{aligned}
\vec{\mathbf{F}}_{side1} = i_1\vec{\ell}_1 \times \vec{\mathbf{B}} &= i_1\ell_1\hat{\mathbf{z}} \times B_{R\,\max}\frac{r_S}{r}\Big|_{r_S}\cos(\theta-\theta_R)\big|_{\theta=\pi/2}\hat{\mathbf{r}} \\
&= i_1\ell_1 B_{R\,\max}\cos(\pi/2-\theta_R)\big|_{\theta=\pi/2}(\hat{\mathbf{z}}\times\hat{\mathbf{r}}) \\
&= i_1\ell_1 B_{R\,\max}\sin(\theta_R)\hat{\boldsymbol{\theta}}
\end{aligned}$$

# Three Phase Permanent Magnet Generator - Torque



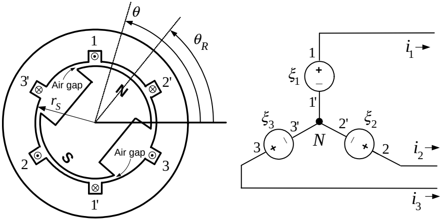

**Magnetic Force $\vec{\mathbf{F}}_{side1'}$** on stator side $1'$.

$$\begin{aligned}
\vec{\mathbf{F}}_{side1'} = i_1\vec{\ell}_1 \times \vec{\mathbf{B}} &= i_1(-\ell_1\hat{\mathbf{z}}) \times B_{R\,\max}\frac{r_S}{r}\Big|_{r_S}\cos(\theta-\theta_R)\Big|_{\theta=3\pi/2}\hat{\mathbf{r}} \\
&= -i_1\ell_1 B_{R\,\max}\cos(3\pi/2-\theta_R)(\hat{\mathbf{z}}\times\hat{\mathbf{r}}) \\
&= i_1\ell_1 B_{R\,\max}\sin(\theta_R)\hat{\boldsymbol{\theta}}
\end{aligned}$$

**Side** $1-1'$ **Total Force and Torque**

$$\begin{aligned}
\vec{\mathbf{F}}_{1-1'} &= \vec{\mathbf{F}}_{side1} + \vec{\mathbf{F}}_{side1'} = 2i_1\ell_1 B_{R\,\max}\sin(\theta_R)\hat{\boldsymbol{\theta}} \\
\vec{\boldsymbol{\tau}}_{1-1'} &= r_S\hat{\mathbf{r}} \times \vec{\mathbf{F}}_{1-1'} = r_S 2i_1\ell_1 B_{R\,\max}\sin(\theta_R)\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = r_S 2i_1\ell_1 B_{R\,\max}\sin(\theta_R)\hat{\mathbf{z}}
\end{aligned}$$

# Three Phase Permanent Magnet Generator - Torque




$$\vec{\boldsymbol{\tau}}_{1-1'} = r_S\hat{\mathbf{r}} \times \vec{\mathbf{F}}_{1-1'} = r_S 2 i_1 \ell_1 B_{R\,\max}\sin(\theta_R)\hat{\mathbf{z}}$$

$$\vec{\boldsymbol{\tau}}_{2-2'} = r_S\hat{\mathbf{r}} \times \vec{\mathbf{F}}_{2-2'} = r_S 2 i_2 \ell_1 B_{R\,\max}\sin(\theta_R - 2\pi/3)\hat{\mathbf{z}}$$

$$\vec{\boldsymbol{\tau}}_{3-3'} = r_S\hat{\mathbf{r}} \times \vec{\mathbf{F}}_{3-3'} = r_S 2 i_3 \ell_1 B_{R\,\max}\sin(\theta_R - 4\pi/3)\hat{\mathbf{z}}$$

$$\vec{\boldsymbol{\tau}}_S \triangleq r_S 2\ell_1 B_{R\,\max}\Big(i_1\sin(\theta_R) + i_2\sin(\theta_R - 2\pi/3) + i_3\sin(\theta_R - 4\pi/3)\Big)\hat{\mathbf{z}}$$

Let $\theta_R = \omega t + \delta$, $i_1 = I_S\cos(\omega t)$, $i_2 = I_S\cos(\omega t - 2\pi/2)$, $i_3 = I_S\cos(\omega t - 4\pi/3)$. Using a trigonometric identity the torque **exerted on the stator** by the PM rotor is

$$\vec{\boldsymbol{\tau}}_S \triangleq r_S 2\ell_1 B_{R\,\max}\frac{3}{2}\sin(\delta)\hat{\mathbf{z}}.$$

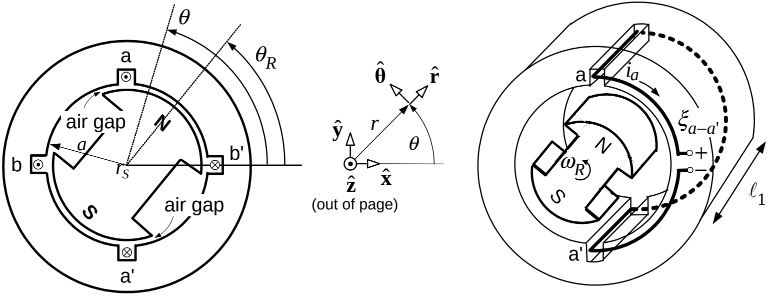
Thus the torque **exerted on the rotor** by the magnetic field of the stator currents is then

$$\vec{\boldsymbol{\tau}}_R \triangleq -\frac{r_R}{r_S}\vec{\boldsymbol{\tau}}_S = -r_R 2\ell_1 B_{R\,\max}\frac{3}{2}\sin(\delta)\hat{\mathbf{z}}.$$

# A Two-Phase Two-Pole Generator.

This example is a precursor to problem 16 of Chapter 1

# A Two-Phase PM Generator

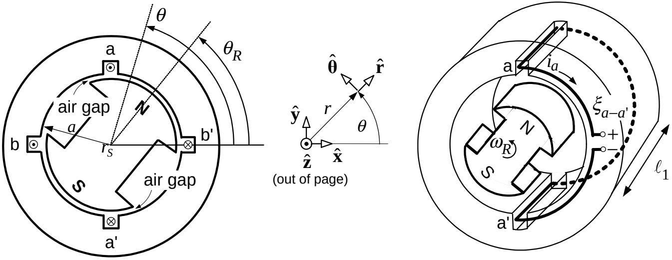



- A **two-pole two-phase** machine.
- The rotor is a two-pole **permanent magnet.**
- The pole faces of the PM are **shaped** so that

$$\vec{\mathbf{B}}(r,\theta) = B_{R\,\max}\frac{r_S}{r}\cos(\theta-\theta_R)\hat{\mathbf{r}}.$$

- $B_{R\,\max} > 0$, $r_S = \ell_2/2 + g$ is the radius of the **inside** surface of the stator iron.
- The stator phases $a - a'$ and $b - b'$ are wound to have a **half-cylindrical** shape.
- Phase $b - b'$ is rotated $\pi/2$ with respect to phase $a - a'$.

# A Two-Phase PM Generator




$$\phi_{a-a'} = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_0^{\ell_1}\int_{-\pi/2}^{\pi/2} B_{R\,\max}\frac{r_S}{r_S}\cos(\theta-\theta_R)\hat{\mathbf{r}}\cdot(r_S d\theta d\ell\hat{\mathbf{r}}) = r_S\ell_1 B_{R\,\max}\sin(\theta-\theta_R)\Big|_{-\pi/2}^{\pi/2}$$

$$= 2r_S\ell_1 B_{R\,\max}\cos(\theta_R).$$

The voltage produced (induced) in phase $a$ is

$$\xi_a = -\frac{d\phi_{a-a'}}{dt} = -2r_S\ell_1 B_{R\,\max}(-\sin(\theta_R))\frac{d\theta_R}{dt} = 2r_S\ell_1 B_{R\,\max}\sin(\theta_R)\omega_R.$$

Similarly,

$$\xi_b = -\frac{d\phi_{b-b'}}{dt} = 2r_S\ell_1 B_{R\,\max}\sin(\theta_R-\pi/2)\omega_R = -2r_S\ell_1 B_{R\,\max}\cos(\theta_R)\omega_R.$$

## A Two-Phase PM Generator

- Assume that each loop has resistance $R_R$ and inductance $L_R = 0$.
- Let the two phases each be connected to an external load resistance $R_L$.

The currents in the loops are given by $(R \triangleq R_R + R_L)$

$$\begin{aligned} i_a &= \frac{\xi_a}{R_R + R_L} = \frac{\xi_a}{R} \\ i_b &= \frac{\xi_b}{R_R + R_L} = \frac{\xi_b}{R}. \end{aligned}$$

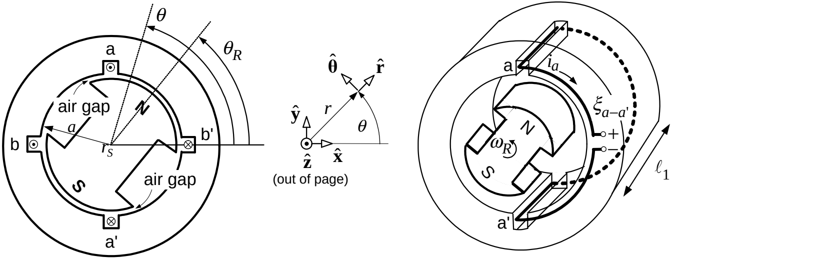
The power produced by the generator is

$$i_a\xi_a + i_b\xi_b = \xi_a^2/R + \xi_b^2/R = \frac{\left(2r_S\ell_1 B_{R\,\max}\sin(\theta_R)\omega_R\right)^2 + \left(-2r_S\ell_1 B_{R\,\max}\cos(\theta_R)\omega_R\right)^2}{R}$$

$$= \frac{\left(2r_S\ell_1 B_{R\,\max}\omega_R\right)^2}{R}.$$

# A Two-Phase PM Generator

## Torque on Stator Windings




$$\begin{aligned}
\tau_{a-a'} &= r_S i_a \ell_1 B_{R\max}\frac{r_S}{r}\cos(\theta-\theta_R)\Big|_{r=r_S,\theta=\pi/2} + r_S(-i_a)\ell_1 B_{R\max}\frac{r_S}{r}\cos(\theta-\theta_R)\Big|_{r=r_S,\theta=-\pi/2}\\
&= r_S i_a \ell_1 B_{R\max}\cos(\pi/2-\theta_R) - r_S i_a \ell_1 B_{R\max}\frac{r_S}{r}\cos(-\pi/2-\theta_R)\\
&= 2 r_S i_a \ell_1 B_{R\max}\sin(\theta_R)\\
\tau_{b-b'} &= r_S i_b \ell_1 B_{R\max}\frac{r_S}{r}\cos(\theta-\theta_R)\Big|_{r=r_S,\theta=\pi} + r_S(-i_b)\ell_1 B_{R\max}\frac{r_S}{r}\cos(\theta-\theta_R)\Big|_{r=r_S,\theta=0}\\
&= r_S i_b \ell_1 B_{R\max}\cos(\pi-\theta_R) - r_S i_b \ell_1 B_{R\max}\frac{r_S}{r}\cos(-\theta_R)\\
&= -2 r_S i_b \ell_1 B_{R\max}\cos(\theta_R)
\end{aligned}$$

# A Two-Phase PM Generator

## Torque on Stator Windings

$$\begin{aligned}
\tau_S = \tau_{a-a'} + \tau_{b-b'} &= 2r_S\frac{\xi_a}{R}\ell_1 B_{R\max}\sin(\theta_R) - 2r_S\frac{\xi_b}{R}\ell_1 B_{R\max}\cos(\theta_R)\\
&= 2r_S\left(\frac{2r_S\ell_1 B_{R\max}\sin(\theta_R)\omega_R}{R}\right)\ell_1 B_{R\max}\sin(\theta_R)\\
&\quad -2r_S\left(\frac{-2r_S\ell_1 B_{R\max}\cos(\theta_R)\omega_R}{R}\right)\ell_1 B_{R\max}\cos(\theta_R)\\
&= \frac{(2r_S\ell_1 B_{R\max})^2}{R}\omega_R.
\end{aligned}$$

The torque exerted by the stator **on the rotor** is the negative of this, i.e.,

$$\tau_R = -\tau_S = -\frac{(2r_S\ell_1 B_{R\max})^2}{R}\omega_R.$$

## Conservation of Energy

$$i_a\xi_a + i_b\xi_b + \tau_R\omega_R = \frac{(2r_S\ell_1 B_{R\max}\omega_R)^2}{R} - \frac{(2r_S\ell_1 B_{R\max})^2}{R}\omega_R^2 = 0.$$

# A Two-Phase Four-Pole Generator.

The following slides go over problem 16 of Chapter 1

# A Two-Phase Four-Pole Generator

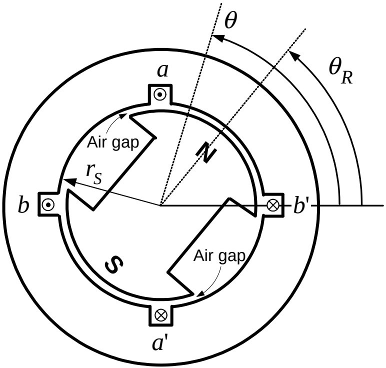
First just consider a two-phase **two-pole** generator.

Its rotor is a two-pole (one north and one south) **permanent magnet**.

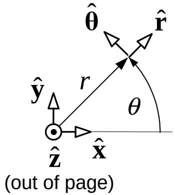
The pole faces are **shaped** so that the magnetic field is given by

$$\vec{\mathbf{B}}(r,\theta) = B_{R\max}\frac{r_S}{r}\cos(\theta-\theta_R)\hat{\mathbf{r}}, \quad B_{R\max} > 0, r_S = \ell_2/2 + g$$

The stator phases are **half-cylinders**.

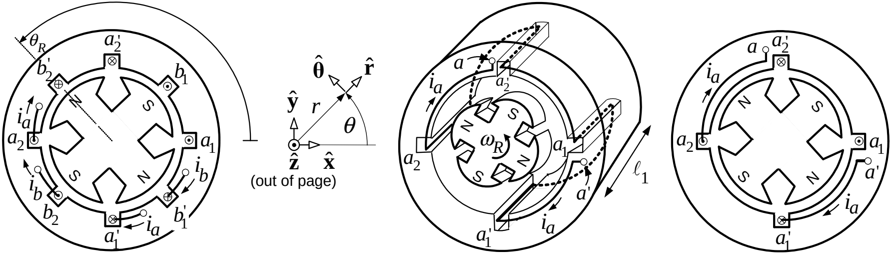



(a)




(b)

# A Two-Phase Four-Pole Generator




(a) (b) (c)

$$\vec{\mathbf{B}}(r,\theta) = B_{R\,\max}\frac{r_S}{r}\cos(\theta-\theta_R)\hat{\mathbf{r}}, \quad B_{R\,\max} > 0,\, r_S = \ell_2/2 + g$$

$$\begin{aligned}
\lambda_a &= \int_0^{\ell_1}\int_{-\pi/2}^{0} B_{R\,\max}\cos\left(n_p(\theta-\theta_R)\right)\hat{\mathbf{r}}\cdot\left(r_S\,d\theta\,d\ell\,\hat{\mathbf{r}}\right) + \int_0^{\ell_1}\int_{\pi/2}^{\pi} B_{R\,\max}\cos\left(n_p(\theta-\theta_R)\right)\hat{\mathbf{r}}\cdot\left(r_S\,d\theta\,d\ell\,\hat{\mathbf{r}}\right)\\
&= \frac{r_S\ell_1 B_{R\,\max}}{n_p}\sin\left(n_p(\theta-\theta_R)\right)\Big|_{-\pi/2}^{0} + \frac{r_S\ell_1 B_{R\,\max}}{n_p}\sin\left(n_p(\theta-\theta_R)\right)\Big|_{\pi/2}^{\pi}\\
&= \frac{r_S\ell_1 B_{R\,\max}}{n_p}\Big(-\sin\left(n_p\theta_R\right) + \sin\left(n_p\left(\pi/2+\theta_R\right)\right) + \sin\left(n_p\left(\pi-\theta_R\right)\right) - \sin\left(n_p\left(\pi/2-\theta_R\right)\right)\Big)\\
&= \frac{r_S\ell_1 B_{R\,\max}}{2}\Big(-\sin\left(2\theta_R\right) + \sin\left(2\left(\pi/2+\theta_R\right)\right) + \sin\left(2\left(\pi-\theta_R\right)\right) - \sin\left(2\left(\pi/2-\theta_R\right)\right)\Big)\\
&= \frac{r_S\ell_1 B_{R\,\max}}{2}\left(-4\sin(2\theta_R)\right) = -2r_S\ell_1 B_{R\,\max}\sin(2\theta_R).
\end{aligned}$$

## A Two-Phase Four-Pole Generator

- Phase $a - a'$ is made up of two windings: $a_1 - a_1'$ and $a_2 - a_2'$.
- The flux surface of each of these windings uses the outward normal $\hat{\mathbf{r}}$.
- The positive direction of travel around each of the two flux surfaces **coincides** with the positive direction of current.

The induced voltage in phase $a - a'$ is

$$\xi_{a-a'} = -\frac{d\lambda_a}{dt} = 4r_S\ell_1 B_{R\,\max}\omega_R\cos(2\theta_R) = 4r_S\ell_1 B_{R\,\max}\omega_R\cos(2\omega_R t)$$

Similarly

$$\lambda_b(\theta_R) = \lambda_a(\theta_R - \pi/4) = -2r_S\ell_1 B_{R\,\max}\sin(2\,(\theta_R - \pi/4))$$

and so the induced voltage in phase $b - b'$ is

$$\begin{aligned}\xi_{b-b'} = -d\lambda_b/dt = 4r_S\ell_1 B_{R\,\max}\omega_R\cos(2\,(\theta_R - \pi/4)) &= 4r_S\ell_1 B_{R\,\max}\omega_R\sin(2\theta_R)\\ &= 4r_S\ell_1 B_{R\,\max}\omega_R\sin(2\omega_R t)\end{aligned}$$

# A Two-Phase Four-Pole Generator

By Faraday's law in **differential form** ($\nabla \times \vec{\mathbf{E}} = -\partial \vec{\mathbf{B}}/\partial t$) there is an **electric field** in the air gap given by

$$\vec{\mathbf{E}}(\theta - \theta_R) = \omega_R B_{R\,\max} r_S \cos\left(n_p(\theta - \theta_R)\right)\hat{\mathbf{z}}.$$

The **voltage** in phase $a - a'$ is computed as

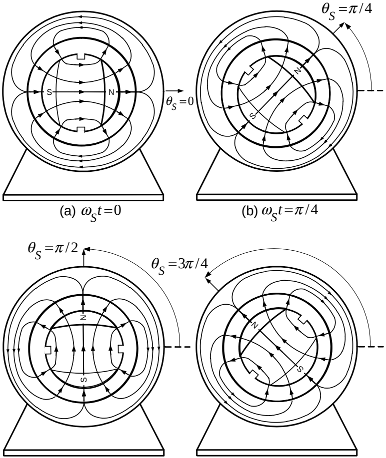
$$\begin{aligned}
\xi_{a-a'} &= \int_{a'}^{a} \vec{\mathbf{E}}(\theta - \theta_R)\cdot d\vec{\boldsymbol{\ell}} \\
&= \int\limits_{side\ a_1'} \omega_R B_{R\,\max} r_S \cos(n_p\left(-\pi/2 - \theta_R\right))\hat{\mathbf{z}}\cdot(-d\ell\hat{\mathbf{z}}) + \int\limits_{side\ a_1} \omega_R B_{R\,\max} r_S \cos(n_p\left(0 - \theta_R\right))\hat{\mathbf{z}}\cdot d\ell\hat{\mathbf{z}} \\
&\quad + \int\limits_{side\ a_2'} \omega_R B_{R\,\max} r_S \cos(n_p\left(\pi/2 - \theta_R\right))\hat{\mathbf{z}}\cdot(-d\ell\hat{\mathbf{z}}) + \int\limits_{side\ a_2} \omega_R B_{R\,\max} r_S \cos(n_p\left(\pi - \theta_R\right))\hat{\mathbf{z}}\cdot d\ell\hat{\mathbf{z}} \\
&= \omega_R B_{R\,\max} r_S \ell_1 \Big(-\cos\left(2\left(-\pi/2 - \theta_R\right)\right) + \cos\left(2\theta_R\right) - \cos\left(2\left(\pi/2 - \theta_R\right)\right) + \cos\left(2\left(\pi - \theta_R\right)\right)\Big) \\
&= \omega_R B_{R\,\max} r_S \ell_1 \Big(-\cos\left(\pi + 2\theta_R\right) + \cos\left(2\theta_R\right) - \cos\left(\pi - 2\theta_R\right) + \cos\left(2\theta_R\right)\Big) \\
&= 4 r_S \ell_1 B_{R\,\max}\omega_R \cos\left(2\theta_R\right) \\
&= 4 r_S \ell_1 B_{R\,\max}\omega_R \cos\left(2\omega_R t\right)
\end{aligned}$$

which is the same result as using Faraday's law.

# A PM-Generator/Induction-Motor Machine.

The following slides go over problem 17 of Chapter 1

## A PM-Generator/Induction-Motor Machine




(a) $\omega_S t = 0$ (b) $\omega_S t = \pi/4$

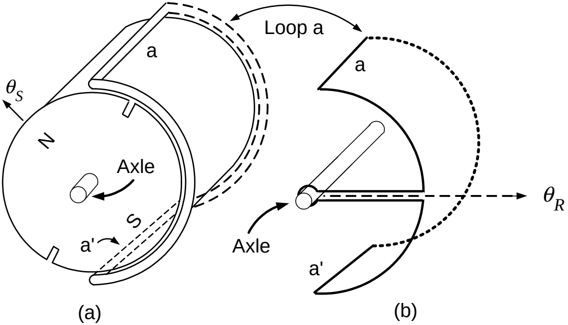
(c) $\omega_S t = \pi/2$ (d) $\omega_S t = 3\pi/4$

The **Permanent Magnet** rotates at constant angular speed $\omega_S$ producing

$$\vec{\mathbf{B}}_S = B_{S\max}\frac{r_R}{r}\cos\left(\theta - \theta_S(t)\right)\hat{\mathbf{r}},\ \ \theta_S(t) = \omega_S t$$

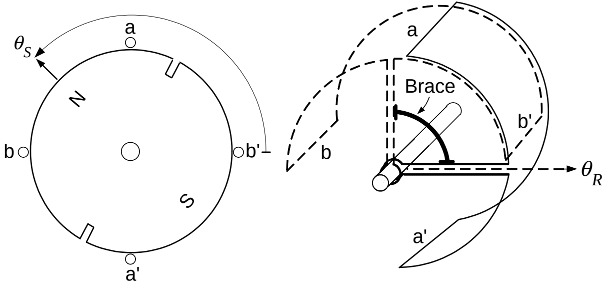
in the air gap.

## A PM-Generator/Induction-Motor Machine




- Added a half-cylindrical shaped loop rotating on the **same axis** as the PM.
- This loop rotates **independently** of the PM.
- As the PM rotates past the loop it will induce voltages and currents in the loop.

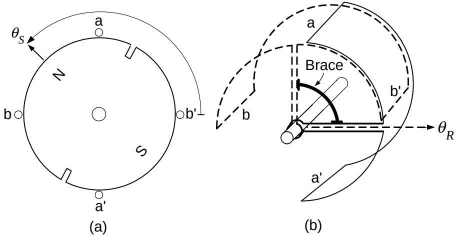
## A PM-Generator/Induction-Motor Machine




(a) (b)

- Added a **second** half-cylindrical shaped loop rotating on the **same axis** as the PM.
- Second loop is 90° from the first loop and **rigidly** attached to it through a brace.
- We will refer to the two rigidly attached loops as the **rotor**.

# A PM-Generator/Induction-Motor Machine




**Given** $\theta_S(t) = \omega_S t$, $\theta_R(t) = \omega_R t$.

**(a)** The **flux** $\lambda_{Ra}$ in rotor loop $a$ due to $\vec{\mathbf{B}}_S = B_S\hat{\mathbf{r}} = B_{S\max}\dfrac{r_R}{r}\cos(\theta - \theta_S)\hat{\mathbf{r}}$ is

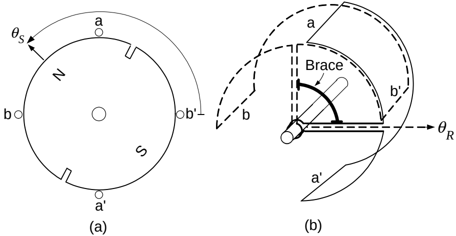
$$\lambda_{Ra} = \int_S \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} = \int_{z=0}^{z=\ell_1}\int_{\theta_R(t)-\pi/2}^{\theta_R(t)+\pi/2} B_{S\max}\frac{r_R}{r}\Big|_{r=r_R}\cos(\theta - \theta_S(t))\hat{\mathbf{r}}\cdot(r_R d\theta dz\hat{\mathbf{r}})$$

$$= r_R\ell_1 B_{S\max}\sin\left(\theta - \theta_S(t)\right)\Big|_{\theta_R(t)-\pi/2}^{\theta_R(t)+\pi/2}$$

$$= 2r_R\ell_1 B_{S\max}\cos\Big(\theta_S(t) - \theta_R(t)\Big)$$

$$= 2r_R\ell_1 B_{S\max}\cos\Big((\omega_S - \omega_R)t\Big).$$

# A PM-Generator/Induction-Motor Machine




**(a)** Similarly,

$$\lambda_{Rb} = \int_S \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} = \int_{z=0}^{z=\ell_1} \int_{\theta_R(t)}^{\theta_R(t)+\pi} B_{S\max} \frac{r_R}{r}\Big|_{r=r_R} \cos\left(\theta - \theta_S(t)\right) \hat{\mathbf{r}} \cdot \left(r_R dz d\theta \hat{\mathbf{r}}\right)$$

$$= r_R \ell_1 B_{S\max} \sin\left(\theta - \theta_S(t)\right) \Big|_{\theta_R(t)}^{\theta_R(t)+\pi}$$

$$= \ell_1 \ell_2 B_{S\max} \sin\left(\theta_S(t) - \theta_R(t)\right)$$

$$= \ell_1 \ell_2 B_{S\max} \sin\Big((\omega_S - \omega_R)t\Big).$$

# A PM-Generator/Induction-Motor Machine

**(b)** Each loop has inductance $L_R$ and resistance $R_R$.

**Induced emf** $\xi_{Ra}$ in rotor loop $a$ is

$$\xi_{Ra} = -\frac{d\lambda_{Ra}}{dt} = -\frac{d}{dt}\ell_1\ell_2 B_{S\max}\cos\Big(\theta_S(t) - \theta_R(t)\Big) = \ell_1\ell_2 B_{S\max}(\omega_S - \omega_R)\sin\Big((\omega_S - \omega_R)t\Big).$$

$i_{Ra}$ is the solution to

$$L_R\frac{di_{Ra}}{dt} = -R_R i_{Ra} + \xi_{Ra}.$$

Similarly, the **induced emf** $\xi_{Rb}$ in rotor loop $b$ is

$$\xi_{Rb} = -\frac{d\lambda_{Rb}}{dt} = -\ell_1\ell_2 B_{S\max}(\omega_S - \omega_R)\cos\Big((\omega_S - \omega_R)t\Big).$$

$i_{Ra}$ is the solution to

$$L_R\frac{di_{Rb}}{dt} = -R_R i_{Rb} + \xi_{Rb}.$$

# A PM-Generator/Induction-Motor Machine

**(b)** Need to solve

$$L_R \frac{di_{Ra}}{dt} + R_R i_{Ra} = \xi_{Ra}, \quad \xi_{Ra} = +\ell_1 \ell_2 B_{S\max}(\omega_S - \omega_R)\sin\left((\omega_S - \omega_R)t\right)$$
$$L_R \frac{di_{Rb}}{dt} + R_R i_{Rb} = \xi_{Rb}, \quad \xi_{Rb} = -\ell_1 \ell_2 B_{S\max}(\omega_S - \omega_R)\cos\left((\omega_S - \omega_R)t\right).$$

The **stable linear time-invariant** system

$$L\frac{di}{dt} + Ri = A\cos(\omega t + \phi)$$

has the **steady-state** solution

$$\begin{aligned} i_{SS}(t) &= |G(j\omega)|\, A\cos\left(\omega t + \phi + \angle G(j\omega)\right), \quad G(j\omega) \triangleq \frac{1}{R + j\omega L} \\ &= \left|\frac{1}{j\omega L + R}\right| A\cos\Big(\omega t + \phi - \tan^{-1}(\omega L/R)\Big). \end{aligned}$$

# A PM-Generator/Induction-Motor Machine

**(b)** $i_{RaSS}$

$$\begin{aligned}
\xi_{Ra} &= +\underbrace{\ell_1\ell_2 B_{S\max}(\omega_S-\omega_R)}_{A}\sin\left((\omega_S-\omega_R)t\right)\\
&= A\cos(\omega t+\phi),\quad A=\ell_1\ell_2 B_{S\max}(\omega_S-\omega_R),\ \omega=\omega_S-\omega_R,\ \phi=-\pi/2
\end{aligned}$$

The **steady-state solution** to

$$L_R\frac{di_{Ra}}{dt}+R_R i_{Ra}=A\cos(\omega t+\phi)$$

is

$$\begin{aligned}
i_{RaSS} &= \frac{1}{\sqrt{R_R^2+(\omega_S-\omega_R)^2L_R^2}}\ell_1\ell_2 B_{S\max}(\omega_S-\omega_R)\times\\
&\quad\cos\left((\omega_S-\omega_R)t-\pi/2-\tan^{-1}\left(\frac{(\omega_S-\omega_R)L_R}{R_R}\right)\right)\\
&= \frac{1}{\sqrt{R_R^2+(\omega_S-\omega_R)^2L_R^2}}\ell_1\ell_2 B_{S\max}(\omega_S-\omega_R)\times\\
&\quad\sin\left((\omega_S-\omega_R)t-\tan^{-1}\left(\frac{(\omega_S-\omega_R)L_R}{R_R}\right)\right)
\end{aligned}$$

## A PM-Generator/Induction-Motor Machine

**(b)** $i_{RbSS}$

$$\begin{aligned}\xi_{Rb} &= -\ell_1\ell_2 B_{S\max}(\omega_S-\omega_R)\cos\left((\omega_S-\omega_R)t\right)\\ &= A\cos(\omega t+\phi),\quad A=-\ell_1\ell_2 B_{S\max}(\omega_S-\omega_R),\ \omega=\omega_S-\omega_R,\ \phi=0\end{aligned}$$

The **steady-state solution** to

$$L_R\frac{di_{Rb}}{dt}+R_R i_{Rb}=A\cos(\omega t+\phi)$$

is

$$\begin{aligned}i_{RbSS} &= -\frac{1}{\sqrt{R_R^2+(\omega_S-\omega_R)^2L_R^2}}\ell_1\ell_2 B_{S\max}(\omega_S-\omega_R)\times\\ &\quad\cos\left((\omega_S-\omega_R)t-\tan^{-1}\left(\frac{(\omega_S-\omega_R)L_R}{R_R}\right)\right).\end{aligned}$$

**(b)** The **steady-state** currents are then

$$i_{RaSS} = \frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} \ell_1 \ell_2 B_{S\max} \times$$

$$\sin\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right)$$

$$i_{RbSS} = -\frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} \ell_1 \ell_2 B_{S\max} \times$$

$$\cos\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right)$$

## $\tau_{Ra}$

**Part (c)** $\vec{\mathbf{B}}_S = B_S\hat{\mathbf{r}} = B_{S\max}\dfrac{r_R}{r}\cos(\theta-\theta_S)\hat{\mathbf{r}}$

$$
\begin{aligned}
\tau_{Ra} &= \frac{\ell_2}{2} i_{Ra}\ell_1\left(B_{S\max}\frac{r_R}{r}\cos(\theta-\theta_S)\right)\Big|_{r=r_R,\theta=\theta_R+\pi/2} + \\
&\quad \frac{\ell_2}{2}(-i_{Ra})\ell_1\left(B_{S\max}\frac{r_R}{r}\cos(\theta-\theta_S)\right)\Big|_{r=r_R,\theta=\theta_R-\pi/2} \\
&= \frac{\ell_2}{2} i_{Ra}\ell_1\left(B_{S\max}\cos(\theta_R-\theta_S+\pi/2)\right) + \\
&\quad \frac{\ell_2}{2}(-i_{Ra})\ell_1\left(B_{S\max}\cos(\theta_R-\theta_S-\pi/2)\right) \\
&= -\frac{\ell_2}{2} i_{Ra}\ell_1\left(B_{S\max}\sin(\theta_R-\theta_S)\right) + \frac{\ell_2}{2}(-i_{Ra})\ell_1\left(B_{S\max}\sin(\theta_R-\theta_S)\right) \\
&= -\ell_1\ell_2 B_{S\max} i_{Ra}\sin\Big((\omega_R-\omega_S)t\Big) \\
&= \ell_1\ell_2 B_{S\max} i_{Ra}\sin\Big((\omega_S-\omega_R)t\Big)
\end{aligned}
$$

$$
\begin{aligned}
\cos(\theta+\pi/2) &= \cos(\theta)\cos(\pi/2)-\sin(\theta)\sin(\pi/2) = -\sin(\theta) \\
\cos(\theta-\pi/2) &= \cos(\theta)\cos(\pi/2)+\sin(\theta)\sin(\pi/2) = \quad \sin(\theta) \\
\sin(-\theta) &= -\sin(\theta)
\end{aligned}
$$

## $\tau_{Rb}$

**Part (c)** $\vec{\mathbf{B}}_S = B_S\hat{\mathbf{r}} = B_{S\max}\dfrac{r_R}{r}\cos(\theta-\theta_S)\hat{\mathbf{r}}$

$$
\begin{aligned}
\tau_{Rb} &= \frac{\ell_2}{2} i_{Rb}\ell_1\left(B_{S\max}\frac{r_R}{r}\cos(\theta-\theta_S)\right)\Big|_{r=r_R,\theta=\theta_R+\pi} + \\
&\quad \frac{\ell_2}{2}(-i_{Rb})\ell_1\left(B_{S\max}\frac{r_R}{r}\cos(\theta-\theta_S)\right)\Big|_{r=r_R,\theta=\theta_R} \\
&= \frac{\ell_2}{2} i_{Rb}\ell_1\left(B_{S\max}\cos(\theta_R-\theta_S+\pi)\right) + \\
&\quad \frac{\ell_2}{2}(-i_{Rb})\ell_1\left(B_{S\max}\cos(\theta_R-\theta_S)\right) \\
&= -\frac{\ell_2}{2} i_{Rb}\ell_1\left(B_{S\max}\cos(\theta_R-\theta_S)\right) + \frac{\ell_2}{2}(-i_{Rb})\ell_1\left(B_{S\max}\cos(\theta_R-\theta_S)\right) \\
&= -\ell_1\ell_2 B_{S\max} i_{Rb}\cos\Big((\omega_R-\omega_S)t\Big) \\
&= -\ell_1\ell_2 B_{S\max} i_{Rb}\cos\Big((\omega_S-\omega_R)t\Big)
\end{aligned}
$$

$\cos(\theta+\pi) = \cos(\theta)\cos(\pi) - \sin(\theta)\sin(\pi) = -\cos(\theta)$
$\cos(-\theta) = \cos(\theta)$

## A PM-Generator/Induction-Motor Machine

The **steady-state torque** $\boldsymbol{\tau}_{Ra}$ on rotor phase $a$ is given by

$$\begin{aligned}
\tau_{Ra} &= \ell_1 \ell_2 B_{S\max} i_{RaSS} \sin\Big((\omega_S - \omega_R)t\Big) \\
&= \underbrace{\frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} (\ell_1 \ell_2 B_{S\max})^2 \times}_{\tau_0} \\
&\qquad \sin\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right) \sin\Big((\omega_S - \omega_R)t\Big).
\end{aligned}$$

The **steady-state torque** $\boldsymbol{\tau}_{Rb}$ on rotor phase $b$ is then

$$\begin{aligned}
\tau_{Rb} &= -\ell_1 \ell_2 B_{S\max} i_{RbSS} \cos\Big((\omega_S - \omega_R)t\Big) \\
&= \underbrace{\frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} (\ell_1 \ell_2 B_{S\max})^2 \times}_{\tau_0} \\
&\qquad \cos\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right) \cos\Big((\omega_S - \omega_R)t\Big)
\end{aligned}$$

## A PM-Generator/Induction-Motor Machine

**(c)** The **steady-state torque** $\tau$ on the rotor is

$$\begin{aligned}\tau &= \tau_{Ra} + \tau_{Rb} \\ &= \tau_0 \sin\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right)\sin\left((\omega_S - \omega_R)t\right) + \\ &\quad \tau_0 \cos\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right)\cos\left((\omega_S - \omega_R)t\right)\end{aligned}$$

Use the trig identity $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

$$\begin{aligned}\tau = \tau_{Ra} + \tau_{Rb} &= \tau_0 \cos\left(-\tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right) \\ &= \tau_0 \cos\left(\tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right) \\ &= \tau_0 \frac{1}{\sqrt{\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)^2 + 1}}\end{aligned}$$

$\cos(-\theta) = \cos(\theta)$ and $\cos(\tan^{-1}(x)) = \dfrac{1}{\sqrt{x^2+1}}$

## A PM-Generator/Induction-Motor Machine

**(d) Steady-State Torque in terms of the Normalized Torque**

$$\tau = \tau_{Ra} + \tau_{Rb} = \overbrace{(\ell_1 \ell_2 B_{S\max})^2 \frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}}}^{\tau_0} \frac{1}{\sqrt{\left(\dfrac{(\omega_S - \omega_R) L_R}{R_R}\right)^2 + 1}}$$

$$= (\ell_1 \ell_2 B_{S\max})^2 \frac{1}{L_R} \frac{\dfrac{(\omega_S - \omega_R) L_R}{R_R}}{\left(\dfrac{(\omega_S - \omega_R) L_R}{R_R}\right)^2 + 1}$$

$$= (\ell_1 \ell_2 B_{S\max})^2 \frac{1}{L_R} \frac{1}{2} \frac{2}{\dfrac{S}{s_p} + \dfrac{s_p}{S}}$$

where $S \triangleq \dfrac{\omega_S - \omega_R}{\omega_S}$ is the **normalized slip** and $s_p \triangleq \dfrac{R_R}{\omega_S L_R}$.

# A PM-Generator/Induction-Motor Machine

## (d) Torque - Slip Curve

$$\tau = \tau_{Ra} + \tau_{Rb} = \underbrace{(\ell_1 \ell_2 B_{S\max})^2 \frac{1}{L_R}\frac{1}{2}}_{\tau_p} \frac{2}{\dfrac{S}{s_p} + \dfrac{s_p}{S}} = \tau_p \frac{2}{\dfrac{S}{s_p} + \dfrac{s_p}{S}}$$

Plot of $\tau/\tau_p$ versus $S/s_p$.