ECE 697 Modeling and High-Performance Control of Electric Machines
HW 4 Solutions
Spring 2022

Problem 1 *A PM-Generator/Induction-Motor Machine.*

(a) The *flux* λ_{Ra} in rotor loop a due to the stator magnetic field

$$\vec{\mathbf{B}}_S = B_S\hat{\mathbf{r}} = B_{S\max}\frac{r_R}{r}\cos(\theta-\theta_S)\hat{\mathbf{r}}$$

is calculated as

$$\begin{aligned}
\lambda_{Ra} = \int_S \vec{\mathbf{B}}_S\cdot d\vec{\mathbf{S}} &= \int_{z=0}^{z=\ell_1}\int_{\theta_R(t)-\pi/2}^{\theta_R(t)+\pi/2} B_{S\max}\frac{r_R}{r}|_{r=r_R}\cos(\theta-\theta_S(t))\hat{\mathbf{r}}\cdot(r_R d\theta dz\hat{\mathbf{r}})\\
&= r_R\ell_1 B_{S\max}\sin(\theta-\theta_S(t))\Big|_{\theta_R(t)-\pi/2}^{\theta_R(t)+\pi/2}\\
&= 2r_R\ell_1 B_{S\max}\cos\Big(\theta_R(t)-\theta_S(t)\Big)\\
&= \ell_1\ell_2 B_{S\max}\cos\Big(\theta_S(t)-\theta_R(t)\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lambda_{Rb} = \int_S \vec{\mathbf{B}}_S\cdot d\vec{\mathbf{S}} &= \int_{z=0}^{z=\ell_1}\int_{\theta_R(t)}^{\theta_R(t)+\pi} B_{S\max}\frac{r_R}{r}|_{r=r_R}\cos(\theta-\theta_S(t))\,\hat{\mathbf{r}}\cdot(r_R dz d\theta\hat{\mathbf{r}})\\
&= r_R\ell_1 B_{S\max}\sin(\theta-\theta_S(t))\Big|_{\theta_R(t)}^{\theta_R(t)+\pi}\\
&= -2r_R B_{S\max}\sin(\theta_R(t)-\theta_S(t))\\
&= \ell_1\ell_2 B_{S\max}\sin(\theta_S(t)-\theta_R(t)).
\end{aligned}$$

(b) By Faraday's law, the induced *electromotive force* or *emf* in rotor loop a is given by

$$\xi_{Ra} = -\frac{d\lambda_{Ra}}{dt} = -\frac{d}{dt}\ell_1\ell_2 B_{S\max}\cos\Big(\theta_S(t)-\theta_R(t)\Big) = \ell_1\ell_2 B_{S\max}(\omega_S-\omega_R)\sin\Big((\omega_S-\omega_R)t\Big)$$

where $\omega_R = d\theta_R/dt$.

Similarly the induced *emf* in rotor loop b is then

$$\xi_{Rb}(t) = -\frac{d\lambda_{Rb}}{dt} = -\ell_1\ell_2 B_{S\max}(\omega_S-\omega_R)\cos\Big((\omega_S-\omega_R)t\Big).$$

(c) With $\theta_S-\theta_R = (\omega_S-\omega_R)t$ and R_R, L_R denoting the resistance and inductance of each rotor loop, the equations describing the current dynamics are

$$\begin{aligned}
L_R\frac{di_{Ra}}{dt} + R_R i_{Ra} = \xi_{Ra},\quad \xi_{Ra} &= +\ell_1\ell_2 B_{S\max}(\omega_S-\omega_R)\sin((\omega_S-\omega_R)t)\\
L_R\frac{di_{Rb}}{dt} + R_R i_{Rb} = \xi_{Rb},\quad \xi_{Rb} &= -\ell_1\ell_2 B_{S\max}(\omega_S-\omega_R)\cos((\omega_S-\omega_R)t).
\end{aligned}$$

The *stable linear time-invariant* system

$$L\frac{di}{dt} + Ri = A\cos(\omega t+\phi)$$

has the steady-state solution

$$i_{ss}(t) = |G(j\omega)|\, A\cos(\omega t + \phi + \angle G(j\omega))$$
$$G(j\omega) \triangleq \frac{1}{R + j\omega L}.$$

The *steady-state* solution for the currents are then

$$i_{RaSS} = \frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} \ell_1 \ell_2 B_{S\max} \sin\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right)$$
$$i_{RbSS} = -\frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} \ell_1 \ell_2 B_{S\max} \cos\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right)$$

(d) The total steady-state torque τ_{Ra} on rotor phase a is given by

$$\begin{aligned}\tau_{Ra} &= \ell_1 \ell_2 B_{S\max} i_{RaSS} \sin\Big((\omega_S - \omega_R)t\Big)\\ &= \frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} (\ell_1 \ell_2 B_{S\max})^2 \sin\left((\omega_S - \omega_R)t - \tan^{-1}(\frac{(\omega_S - \omega_R)L_R}{R_R})\right) \sin\Big((\omega_S - \omega_R)t\Big).\end{aligned}$$

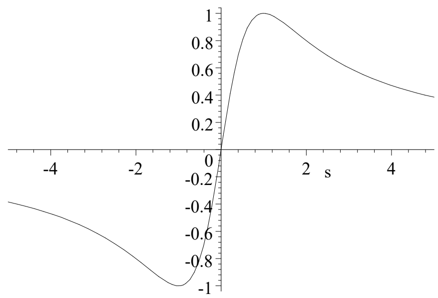
Similarly, the steady-state torque τ_{Rb} on rotor phase b is then

$$\begin{aligned}\tau_{Rb} &= -\ell_1 \ell_2 B_{S\max} i_{RbSS} \cos\Big((\omega_S - \omega_R)t\Big)\\ &= \frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} (\ell_1 \ell_2 B_{S\max})^2 \cos\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right) \cos\Big((\omega_S - \omega_R)t\Big)\end{aligned}$$

(e) Combining the results of part(c), the *total torque* is given by

$$\begin{aligned}\tau = \tau_{Ra} + \tau_{Rb} &= (\ell_1 \ell_2 B_{S\max})^2 \frac{\omega_S - \omega_R}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} \cos\left(\tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right)\\ &= (\ell_1 \ell_2 B_{S\max})^2 \frac{\omega_S - \omega_R}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} \frac{1}{\sqrt{(\frac{(\omega_S - \omega_R)L_R}{R_R})^2 + 1}}\\ &= (\ell_1 \ell_2 B_{S\max})^2 \frac{1}{L_R} \frac{(\omega_S - \omega_R)L_R/R_R}{((\omega_S - \omega_R)L_R/R_R)^2 + 1}\\ &= (\ell_1 \ell_2 B_{S\max})^2 \frac{1}{L_R} \frac{1}{2} \frac{2}{S/s_p + s_p/S}\\ &= \tau_p \frac{2}{S/s_p + s_p/S}\end{aligned}$$

where $S \triangleq \dfrac{\omega_S - \omega_R}{\omega_S}$, $s_p \triangleq \dfrac{R_R}{\omega_S L_R}$, and $\tau_p = \dfrac{(\ell_1 \ell_2 B_{S\max})^2}{2L_R}$.



Plot of the normalized torque $\dfrac{2}{S/s_p + s_p/S}$ versus S/s_p

The quantity $S \triangleq \dfrac{\omega_S - \omega_R}{\omega_S}$ is called the *normalized slip*. The quantity $s_p \triangleq \dfrac{R_R}{\omega_S L_R}$ is the value of S where the peak torque τ_p is achieved.