Modeling and High-Performance Control of Electric Machines

Chapter 2 Feedback Control

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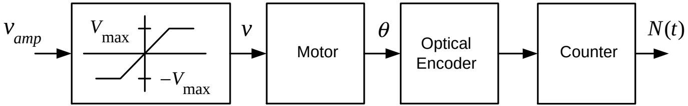
Feedback Control

- **Model of a DC Motor Servo System**
- **Speed Estimation**
- **Trajectory Generation**
- **Design of a State Feedback Tracking Controller**
- **Nested Loop Control Structure** (no slides)
- **Identification of the DC Motor Parameters**
- **Filtering of Noisy Signals** (no slides)

Model of a DC Motor Servo System

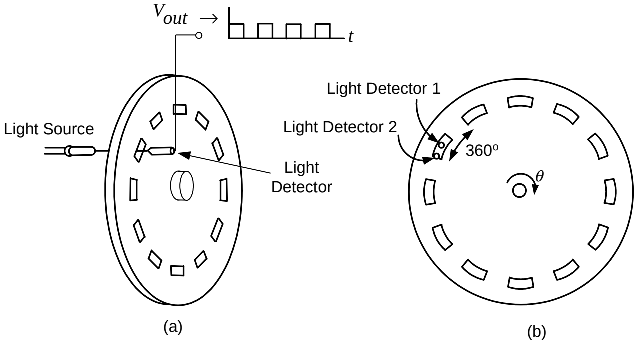
Differential Equation Model of a DC Motor

$$
\begin{aligned}
L\frac{di}{dt} &= -Ri(t) - K_b\omega(t) + v(t) \\
J\frac{d\omega}{dt} &= -f\omega(t) + K_T i(t) - \tau_L(t) \\
\frac{d\theta}{dt} &= \omega(t)
\end{aligned}
$$

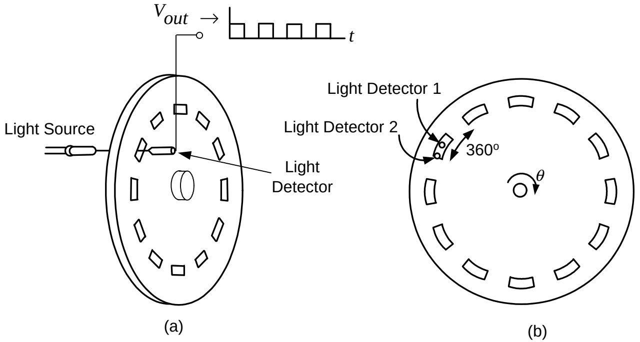


Optical Encoder

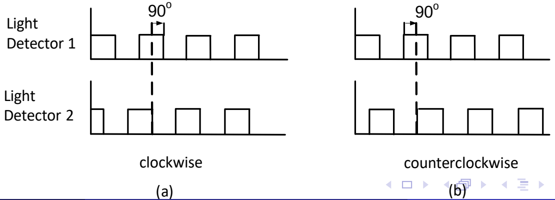
- 12 windows (lines or slots).
- Digital electronic circuitry can detect a pulse going high or low.
- With 12 pulses there are a total of 24 times a pulse went either high or low.
- The resolution is $2\pi/24$ radians or $360^\circ/24 = 15^\circ$.
- By counting the number N of rising and falling edges of the pulse the position is known to within 15°.



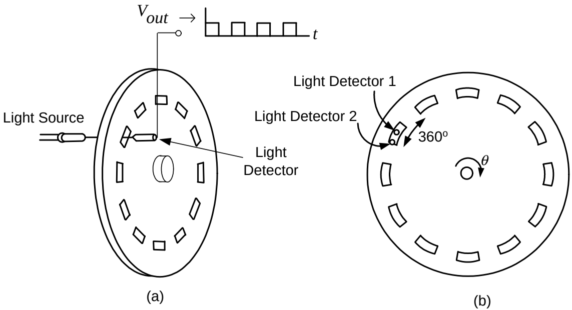
Optical Encoder



- Voltage waveforms out of the encoder.

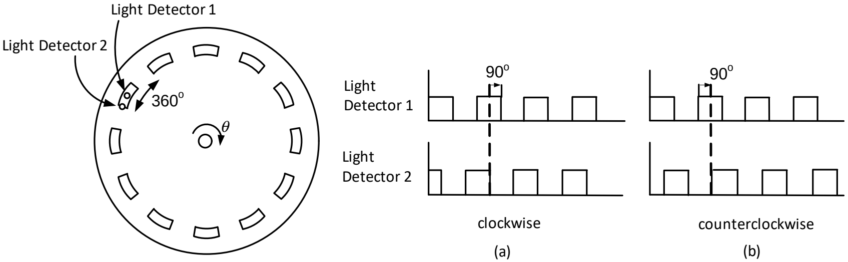


Optical Encoder



- The length of the windows is the same as the length of the distance between windows.
- The two light detectors are placed a distance apart equal to 1/2 of a window length.
- One period of the voltage waveform corresponds to the distance from the beginning of one window to the next.
- This voltage waveform is considered to be 360°.
- The two light detectors are considered to be 90° apart (*quadrature*).

Optical Encoder



Clockwise Rotation

- Voltage of light detector 1 is 90° behind that of light detector 2.

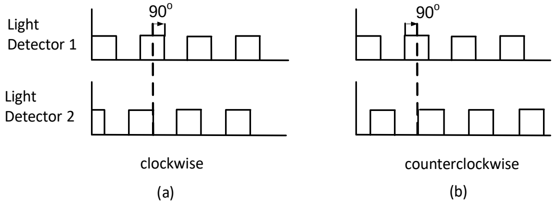
Counterclockwise Rotation

- The voltage of light detector 2 is 90° behind that of light detector 1.

Direction of Rotation

- Electronic circuitry detects the relative phase to determine the direction of rotation.

Optical Encoder



Encoder Resolution

- If an optical encoder has N_w windows (lines/slots), then there are $2N_w$ rising and falling edges per revolution.

  The resolution is then $2\pi/(2N_w)$ radians.

- Count the voltage pulses from *both* light detectors.
  There are then $4N_w$ (equally spaced) rising and falling edges per revolution.

  The resolution is then $2\pi/(4N_w)$ radians.

- $N_w = 500$, the resolution of the encoder is $2\pi/2000$ radians or $360^\circ/2000 = 0.18^\circ$.

Encoder model

- Let N_{enc} denote the number of counts (rising & falling edges from both detectors) **per revolution**.
- Let $N(t)$ denote the pulses out of the encoder at time t.
- The angular position of the shaft in radians is given by

$$\theta_m(t) = \frac{2\pi}{N_{enc}} N(t) \text{ radians.}$$

Backward Difference Estimation of Speed

- The *backward difference* estimate of speed is

$$\omega_{bd}(kT) \triangleq \frac{2\pi}{N_{enc}} \left(\frac{N(kT) - N(kT - T)}{T} \right),$$

- T is the time between samples.
- $N(kT)$ is the optical encoder count at time kT.

Encoder Resolution

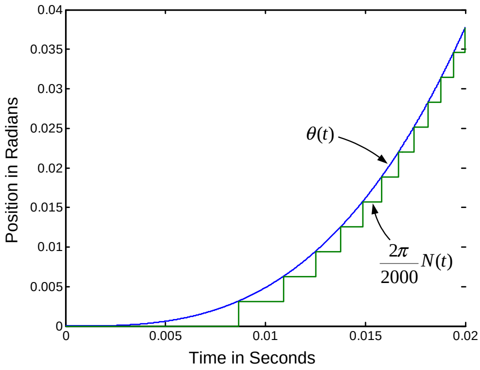


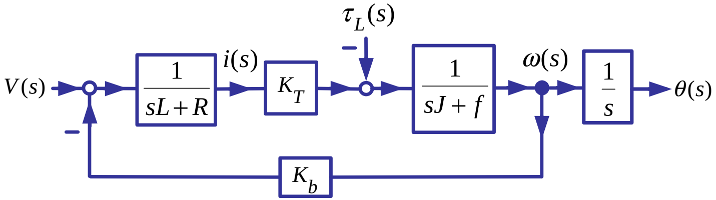
Figure: Plot of $\theta(t)$ and the encoder output $(2\pi/N_{enc})N(t)$.

Thus, with $\theta(kT)$ the true position in radians, we have

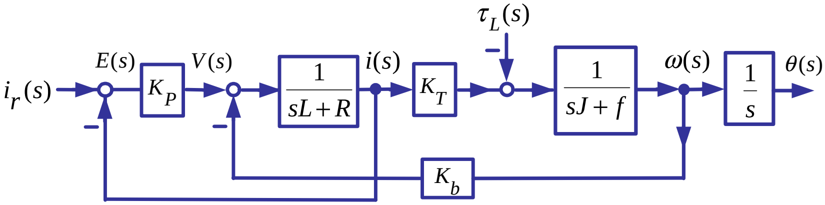
$$\theta(kT) = \frac{2\pi}{N_{enc}}N(kT) + \frac{2\pi}{N_{enc}}e(kT).$$

Current Command

$$
\begin{aligned}
L\frac{di}{dt} &= -Ri(t) - K_b\omega(t) + v(t) &&\Longleftrightarrow\ i(s) = \frac{-K_b\omega(s) + V(s)}{sL + R} \\
J\frac{d\omega}{dt} &= -f\omega(t) + K_T i(t) - \tau_L(t) &&\Longleftrightarrow\ \omega(s) = \frac{K_T i(s) - \tau_L(s)}{sJ + f} \\
\frac{d\theta}{dt} &= \omega(t) &&\Longleftrightarrow\ \theta(s) = \frac{1}{s}\omega(s).
\end{aligned}
$$



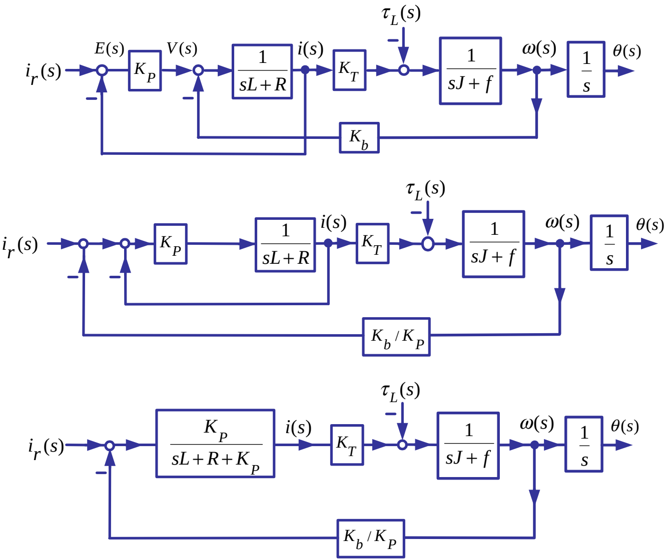
Current Command



- The torque is $K_T i$.
- If we can control the motor current then we control the motor torque.
- Let $i_r(t)$ denote the **desired** current we want in the motor.
- **Measure** the current $i(t)$.
- **Command** the voltage $K_p\left(i_r(t) - i(t)\right)$ to the amplifier.

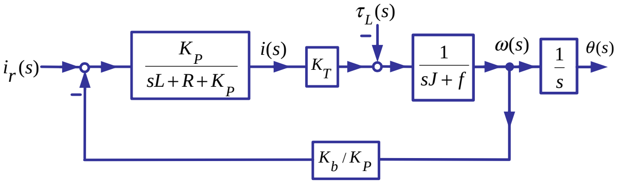
Current Command

Block diagram reduction



Current Command

Block diagram reduction



With $\tau_L(s) \equiv 0$,

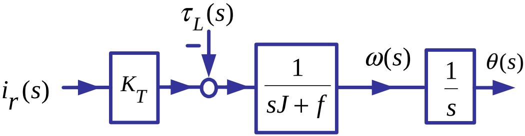
$$G(s) \triangleq \omega(s)/i_r(s) = \frac{\dfrac{K_P}{sL+R+K_P}\dfrac{K_T}{sJ+f}}{1+\dfrac{K_b}{K_p}\dfrac{K_P}{sL+R+K_P}\dfrac{K_T}{sJ+f}} = \frac{K_P K_T}{(sL+R+K_P)(sJ+f)+K_T K_b}$$

$$= \frac{K_T}{(\frac{sL+R}{K_P}+1)(sJ+f)+K_T K_b/K_P}.$$

Let $K_P \to \infty$ to obtain

$$G(s) = \omega(s)/i_r(s) = \frac{K_T}{sJ+f}.$$

Current Command



- $v(t) = K_P(i_r(t) - i(t))$
- Let K_p be large.
- $i_r(t) \rightarrow i(t)$

The reduced order model is now

$$\begin{aligned} \frac{d\omega}{dt} &= (K_T/J)i_r(t) - (f/J)\omega(t) - \tau_L/J \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

- With a good current controller, the voltage $v(t)$ is automatically adjusted to force $i(t) \rightarrow i_r(t)$.
- We can then treat $i_r(t) \approx i(t)$ as the input.
- Problem 6 of Chapter 2 asks for a simulation of this current command controller.

Backward Difference Estimation of Speed

- The optical encoder gives the position measurement, but not the speed of the motor.
- Let T be the time between samples.
- Let $N(kT)$ be the optical encoder count at time kT.

The **backward difference** estimate of angular velocity is

$$\hat{\omega}_{bd}(kT) \triangleq \frac{2\pi}{2000}\left(\frac{N(kT) - N(kT - T)}{T}\right).$$

Error in the Backward Difference Estimate

- $\hat{\omega}_{bd}(kT) \triangleq \dfrac{2\pi}{2000}\left(\dfrac{N(kT) - N(kT - T)}{T}\right)$
- At any discrete time kT, $N(kT)$ is in error by **at most** one encoder count.
- $N(kT)$ can only be **too small** by **at most** one encoder count.
- $N(kT)$ is **never** too large because of the way the encoder works.
- With $\theta(kT)$ the "**true" position** of the motor, we have

$$\theta(kT) = \frac{2\pi}{2000}N(kT) + \frac{2\pi}{2000}e(kT).$$

- $0 \le e(kT) < 1$ is the **positive fractional** count that the encoder cannot sense.

$$\begin{aligned}\omega(kT) &= \left(\frac{\theta(kT) - \theta(kT - T)}{T}\right)\\ &= \frac{2\pi}{2000}\left(\frac{N(kT) - N(kT - T)}{T}\right) + \frac{2\pi}{2000}\left(\frac{e(kT) - e(kT - T)}{T}\right)\end{aligned}$$

- $0 \le e(kT) \le 1$ **and** $0 \le e((k-1)T) \le 1 \implies |e(kT) - e(kT - T)| \le 1.$

Error in the Backward Difference Estimate of Speed

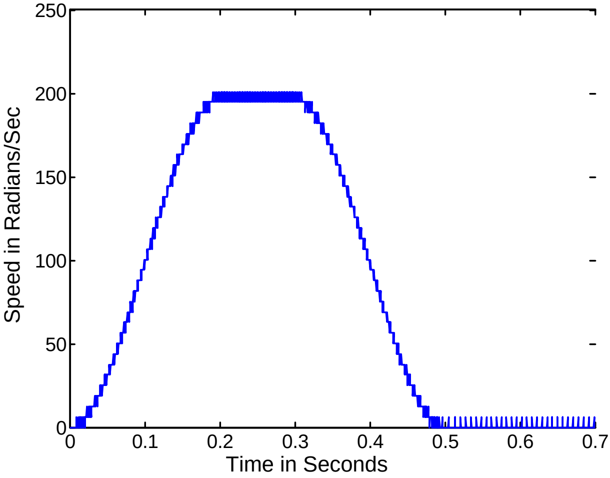
$$
\begin{aligned}
\omega(kT) &= \left(\frac{\theta(kT)-\theta(kT-T)}{T}\right) \\
&= \frac{2\pi}{2000}\left(\frac{N(kT)-N(kT-T)}{T}\right)+\frac{2\pi}{2000}\left(\frac{e(kT)-e(kT-T)}{T}\right) \\
|\omega(kT)-\hat{\omega}_{bd}(kT)| &= \frac{2\pi}{2000}\left|\frac{e(kT)-e(kT-T)}{T}\right| \le \frac{2\pi}{2000}\frac{1}{T}.
\end{aligned}
$$

- As T becomes smaller (sample rate $\uparrow$), the error gets larger.
- As T becomes larger (sample rate $\downarrow$), the backward difference approximation

$$\omega(kT) \approx \left(\frac{\theta(kT)-\theta(kT-T)}{T}\right)$$

  becomes less valid.
- T is a trade-off between encoder error and accuracy of the backward difference.

Backward Difference Speed Estimation



- $T = 0.5$ msec; Encoder resolution is $2\pi/2000$.
- Error bound is $|\omega(kT) - \hat{\omega}_{bd}(kT)| \leq \dfrac{2\pi}{2000}\dfrac{1}{T} = 6.28$ radians/sec.

Estimation of Speed Using an Observer

Take the load torque to be constant, i.e., $\tau_L = \tau_{L0} u_s(t)$.
The system equations for the DC motor are then

$$\begin{aligned}
\frac{d\theta}{dt} &= \omega \\
\frac{d\omega}{dt} &= -(f/J)\omega + (K_T/J)i(t) - \tau_L/J \\
\frac{d\tau_L/J}{dt} &= 0.
\end{aligned}$$

Define an **observer** or **state estimator** by

$$\begin{aligned}
\frac{d\hat{\theta}}{dt} &= \hat{\omega} + \ell_1(\theta - \hat{\theta}) \\
\frac{d\hat{\omega}}{dt} &= (K_T/J)i(t) - (f/J)\hat{\omega} - \hat{\tau}_L/J + \ell_2(\theta - \hat{\theta}) \\
\frac{d\hat{\tau}_L/J}{dt} &= 0 + \ell_3(\theta - \hat{\theta})
\end{aligned}$$

- $\theta(t) = (2\pi/2000)N(t)$ is the (discretized) position measurement from the encoder.
- $\hat{\theta}, \hat{\omega}, \hat{\tau}_L/J$ are the **estimates** of the position, speed, and load torque, respectively.
- We show how to choose ℓ_1, ℓ_2, ℓ_3 so that $\hat{\omega}(t) \to \omega(t)$ and $\tau_L(t)/J \to \hat{\tau}_L/J$.

Estimation of Speed Using an Observer

First let $\ell_1 = \ell_2 = \ell_3 = 0$.

$$\begin{aligned}
\frac{d\hat{\theta}}{dt} &= \hat{\omega} \\
\frac{d\hat{\omega}}{dt} &= (K_T/J)i(t) - (f/J)\hat{\omega} - \hat{\tau}_L/J \\
\frac{d\hat{\tau}_L/J}{dt} &= 0
\end{aligned}$$

- This is just a real-time **simulation** of the motor.
- The motor current is sampled and brought into the control computer.
- The above equations are integrated in **real time** by the computer.
- This will not work in practice as
  The initial conditions of the simulation must be the **same** as the motor.
  The value of the load torque τ_{L0} is almost never known and usually changes.

Estimation of Speed Using an Observer

Motor:

$$\begin{aligned}\frac{d\theta}{dt} &= \omega\\ \frac{d\omega}{dt} &= -(f/J)\omega + (K_T/J)i(t) - \tau_L/J\\ \frac{d\tau_L/J}{dt} &= 0.\end{aligned}$$

Observer:

$$\begin{aligned}\frac{d\hat{\theta}}{dt} &= \hat{\omega} + \ell_1(\theta - \hat{\theta})\\ \frac{d\hat{\omega}}{dt} &= (K_T/J)i(t) - (f/J)\hat{\omega} - \hat{\tau}_L/J + \ell_2(\theta - \hat{\theta})\\ \frac{d\hat{\tau}_L/J}{dt} &= 0 + \ell_3(\theta - \hat{\theta})\end{aligned}$$

- Define $e_1(t) \triangleq \theta(t) - \hat{\theta}(t)$, $e_2(t) \triangleq \omega(t) - \hat{\omega}(t)$, $e_3(t) = \tau_L(t)/J - \hat{\tau}_L(t)/J$.
- Subtract the observer equations **from** the motor equations to obtain

$$\begin{aligned}\frac{de_1}{dt} &= e_2 - \ell_1 e_1\\ \frac{de_2}{dt} &= -(f/J)e_2 - e_3 - \ell_2 e_1\\ \frac{de_3}{dt} &= -\ell_3 e_3\end{aligned}$$

Estimation of Speed Using an Observer

Observer error equations

$$\frac{de_1}{dt} = e_2 - \ell_1 e_1$$
$$\frac{de_2}{dt} = -(f/J)e_2 - e_3 - \ell_2 e_1$$
$$\frac{de_3}{dt} = -\ell_3 e_3$$

Laplace transform of the error equations

$$\begin{bmatrix} s+\ell_1 & -1 & 0 \\ \ell_2 & s+f/J & 1 \\ \ell_3 & 0 & s \end{bmatrix} \begin{bmatrix} E_1(s) \\ E_2(s) \\ E_3(s) \end{bmatrix} = \begin{bmatrix} e_1(0) \\ e_2(0) \\ e_3(0) \end{bmatrix}.$$

or

$$\begin{bmatrix} E_1(s) \\ E_2(s) \\ E_3(s) \end{bmatrix} = \begin{bmatrix} s+\ell_1 & -1 & 0 \\ \ell_2 & s+f/J & 1 \\ \ell_3 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} e_1(0) \\ e_2(0) \\ e_3(0) \end{bmatrix}$$
$$= \begin{bmatrix} \dfrac{s(s+f/J)e_1(0) + se_2(0) - e_1(0)}{s^3 + s^2(\ell_1 + f/J) + s(\ell_1 f/J + \ell_2) - \ell_3} \\ \dfrac{(\ell_3 - s\ell_2)e_1(0) + s(s+\ell_1)e_2(0) - (s+\ell_1)e_3(0)}{s^3 + s^2(\ell_1 + f/J) + s(\ell_1 f/J + \ell_2) - \ell_3} \\ \dfrac{-\ell_3(s+f/J)e_1(0) - \ell_3 e_2(0) + (\ell_2 + (s+\ell_1)(s+f/J))e_3(0)}{s^3 + s^2(\ell_1 + f/J) + s(\ell_1 f/J + \ell_2) - \ell_3} \end{bmatrix}.$$

Estimation of Speed Using an Observer

Choose the gains ℓ_1, ℓ_2, and ℓ_3 so that $e_1(t) \to 0$, $e_2(t) \to 0$, and $e_3(t) \to 0$.

With $p_1 > 0, p_2 > 0, p_3 > 0$ set

$$\begin{aligned} s^3 + s^2(\ell_1 + f/J) + s(\ell_1 f/J + \ell_2) - \ell_3 &= (s + p_1)(s + p_2)(s + p_3) \\ &= s^3 + (p_1 + p_2 + p_3)s + (p_1 p_2 + p_1 p_2 + p_2 p_3)s + p_1 p_2 p_3. \end{aligned}$$

Solving the gains are chosen to be

$$\begin{aligned} \ell_1 &= p_1 + p_2 + p_3 - f/J \\ \ell_2 &= p_1 p_2 + p_1 p_2 + p_2 p_3 - \ell_1 f/J \\ \ell_3 &= -p_1 p_2 p_3. \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} E_1(s) \\ E_2(s) \\ E_3(s) \end{bmatrix} &= \begin{bmatrix} \dfrac{s(s + f/J)e_1(0) + se_2(0) - e_1(0)}{(s + p_1)(s + p_2)(s + p_3)} \\ \dfrac{(\ell_3 - s\ell_2)e_1(0) + s(s + \ell_1)e_2(0) - (s + \ell_1)e_3(0)}{(s + p_1)(s + p_2)(s + p_3)} \\ \dfrac{-\ell_3(s + f/J)e_1(0) - \ell_3 e_2(0) + (\ell_2 + (s + \ell_1)(s + f/J))e_3(0)}{(s + p_1)(s + p_2)(s + p_3)} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{A_1}{s + p_1} + \dfrac{B_1}{s + p_2} + \dfrac{C_1}{s + p_3} \\ \dfrac{A_2}{s + p_1} + \dfrac{B_2}{s + p_2} + \dfrac{C_2}{s + p_3} \\ \dfrac{A_3}{s + p_1} + \dfrac{B_3}{s + p_2} + \dfrac{C_3}{s + p_3} \end{bmatrix}. \end{aligned}$$

Estimation of Speed Using an Observer

As $t \longrightarrow 0$ we have

$$
\begin{aligned}
\theta(t) - \hat{\theta}(t) &= e_1(t) = A_1 e^{-p_1 t} + B_1 e^{-p_2 t} + C_1 e^{-p_3 t} \longrightarrow 0 \\
\omega(t) - \hat{\omega}(t) &= e_2(t) = A_2 e^{-p_1 t} + B_2 e^{-p_2 t} + C_2 e^{-p_3 t} \longrightarrow 0 \\
\tau_L(t)/J - \hat{\tau}_L(t)/J &= e_3(t) = A_3 e^{-p_1 t} + B_3 e^{-p_2 t} + C_3 e^{-p_3 t} \longrightarrow 0.
\end{aligned}
$$

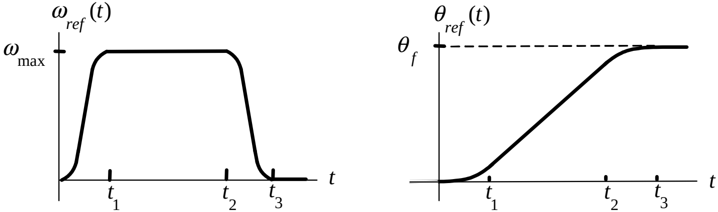
- Problem 9 of Chapter 2 asks for a simulation of this observer.

Trajectory Generation

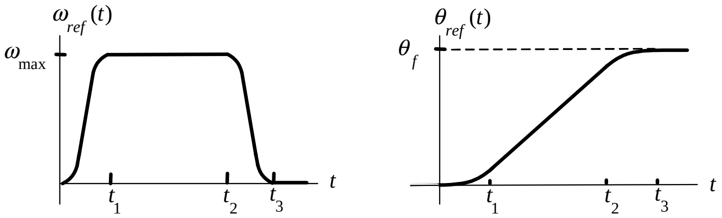
Current Command Motor Model:

$$\begin{aligned}\frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -(f/J)\omega + (K_T/J)i_r.\end{aligned}$$

- Design a **position** and **speed reference trajectory** for a **point-to-point** move.
- Point-to-point move: $\theta_{ref}(t)$ satisfies $\theta_{ref}(0) = 0$ and $\theta_{ref}(t_f) = \theta_f$.
  t_f is the final time.
  θ_f is the final desired position.
  Motor angle goes from the "point" 0 to the "point" θ_f.
- Simple symmetric trajectory:



Trajectory Generation

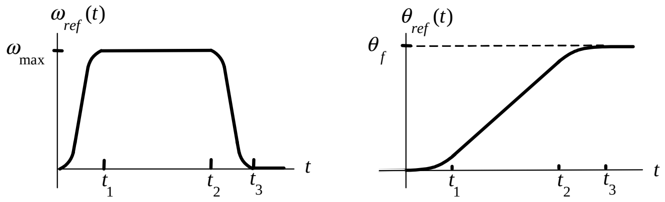


Smooth trajectory requires:

$$
\begin{array}{rcl}
\omega_{ref}(0) = 0 & & \dot{\omega}_{ref}(0) = 0 \\
\omega_{ref}(t_1) = \omega_{\max} & & \dot{\omega}_{ref}(t_1) = 0 \\
\omega_{ref}(t) = \omega_{\max} & & t_1 \le t \le t_2 \\
\omega_{ref}(t_2) = \omega_{\max} & & \dot{\omega}_{ref}(t_2) = 0 \\
\omega_{ref}(t_3) = 0 & & \dot{\omega}_{ref}(t_3) = 0.
\end{array}
$$

- Set $t_3 - t_2 = t_1$, i.e., $t_3 = t_1 + t_2$.
- Set $\omega_{ref}(t) = \omega_{ref}(t_3 - t)$ for $t_2 \le t \le t_3$.
- Must have $\int_0^{t_f} \omega_{ref}(\tau) d\tau = \theta_f$.

Trajectory Generation



Try

$$\omega_{ref}(t) = c_1 t^2 + c_2 t^3 \quad \text{for } 0 \le t \le t_1.$$

- Satisfies $\omega_{ref}(0) = 0, \dot{\omega}_{ref}(0) = 0$.
- The conditions at t_1 become

$$\begin{aligned} \omega_{ref}(t_1) &= c_1 t_1^2 + c_2 t_1^3 &= \omega_{max} \\ \dot{\omega}_{ref}(t_1) &= 2c_1 t_1 + 3c_2 t_1^2 &= 0 \end{aligned}$$

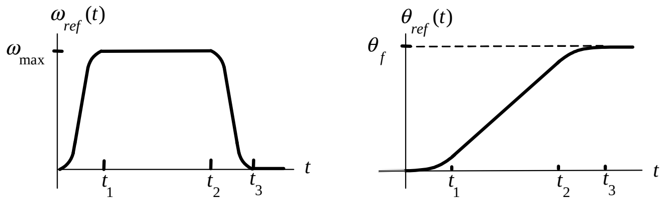
or

$$\begin{bmatrix} t_1^2 & t_1^3 \\ 2t_1 & 3t_1^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \omega_{max} \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{t_1^4} \begin{bmatrix} 3t_1^2 & -t_1^3 \\ -2t_1 & t_1^2 \end{bmatrix} \begin{bmatrix} \omega_{max} \\ 0 \end{bmatrix} = \begin{bmatrix} +3\omega_{max}/t_1^2 \\ -2\omega_{max}/t_1^3 \end{bmatrix}.$$

Trajectory Generation



$$\omega_{ref}(t) = \begin{cases} c_1 t^2 + c_2 t^3 & 0 \le t \le t_1 \\ \omega_{\max} & t_1 \le t \le t_2 \\ c_1(t_3 - t)^2 + c_2(t_3 - t)^3 & t_2 \le t \le t_3. \end{cases}$$

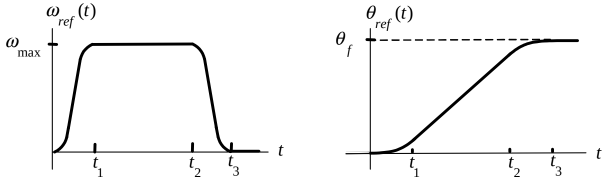
The distance traveled at time t_1 is

$$\theta_{ref}(t_1) = \int_0^{t_1} \omega_{ref}(\tau)d\tau = c_1 t_1^3/3 + c_2 t_1^4/4 = \frac{3\omega_{\max}}{t_1^2}\frac{t_1^3}{3} - \frac{2\omega_{\max}}{t_1^3}\frac{t_1^4}{4} = \frac{\omega_{\max} t_1}{2}.$$

At time t_3 we require

$$\begin{aligned}\theta_f = \int_0^{t_3} \omega_{ref}(\tau)d\tau = 2\theta_{ref}(t_1) + \omega_{\max}(t_2 - t_1) &= 2\frac{\omega_{\max} t_1}{2} + \omega_{\max}(t_2 - t_1) \\ &= \omega_{\max} t_2.\end{aligned}$$

Trajectory Generation



Constraint on $\omega_{\max}$ and τ_2 given by $\theta_f = \omega_{\max} t_2$.

Position reference:

$$\theta_{ref}(t) = \int_0^t \omega_{ref}(\tau) d\tau = \begin{cases} c_1 t^3/3 + c_2 t^4/4 & 0 \leq t \leq t_1 \\ \omega_{\max} t_1/2 + \omega_{\max}(t - t_1) & t_1 \leq t \leq t_2 \\ \omega_{\max} t_2 - c_1(t_3 - t)^3/3 - c_2(t_3 - t)^4/4 & t_2 \leq t \leq t_3. \end{cases}$$

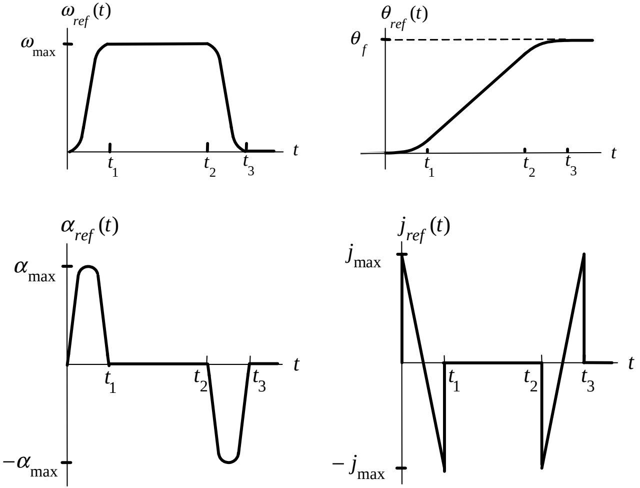
Acceleration reference:

$$\alpha_{ref}(t) = \frac{d\omega_{ref}(t)}{dt} = \begin{cases} 2c_1 t + 3c_2 t^2 & 0 \leq t \leq t_1 \\ 0 & t_1 \leq t \leq t_2 \\ -2c_1(t_3 - t) - 3c_2(t_3 - t)^2 & t_2 \leq t \leq t_3. \end{cases}$$

Current reference:

$$i_{ref}(t) \triangleq \frac{J\dfrac{d\omega_{ref}(t)}{dt} + f\omega_{ref}(t)}{K_T}.$$

Trajectory Generation



- $j_{ref}(t) \triangleq \dfrac{d\alpha_{ref}(t)}{dt}$

Specifying a Reference Trajectory

Typical scenario:

- θ_f is given and one chooses t_1 and t_2 with $t_1 < t_2$.
- t_3 and $\omega_{\max}$ are specified by

$$t_3 = t_2 + t_1 \;\text{ and }\; \omega_{\max} = \theta_f / t_1.$$

- These choices specify the **mechanical reference trajectory**:

$$\Big(\theta_{ref}(t),\; \omega_{ref}(t),\; \alpha_{ref}(t),\; j_{ref}(t)\Big).$$

- The **electrical reference trajectory** is then

$$\begin{aligned} i_{ref}(t) &\triangleq \frac{J\dfrac{d\omega_{ref}(t)}{dt} + f\omega_{ref}(t)}{K_T} \\ v_{ref}(t) &\triangleq L\frac{di_{ref}(t)}{dt} + Ri_{ref}(t) + K_b\omega_{ref}(t). \end{aligned}$$

- How does one choose t_1, t_2?
  Fast point-to-point move requires t_1 and t_2 to be **small**.
  This results in **larger peak values** of $\omega_{ref}(t), \alpha_{ref}(t)$ and thus of $i_{ref}(t), v_{ref}(t)$.
  By **trial and error**, t_1 and t_2 are chosen so that $|i_{ref}(t)| \le I_{\max}$, $|v_{ref}(t)| \le V_{\max}$.

Design of a State Feedback Tracking Controller

State Space Model:

$$\begin{aligned} d\theta/dt &= \omega \\ d\omega/dt &= (K_T/J)i_r - (f/J)\omega - \tau_L/J \end{aligned}$$

State variables θ, ω **Input** i_r **Disturbance** τ_L

The **reference trajectory** and **reference input** satisfy

$$\begin{aligned} d\theta_{ref}/dt &= \omega_{ref} \\ d\omega_{ref}/dt &= (K_T/J)i_{ref} - (f/J)\omega_{ref}. \end{aligned}$$

Error System: $e_1(t) \triangleq \theta_{ref}(t) - \theta(t)$, $e_2(t) \triangleq \omega_{ref}(t) - \omega(t)$.

$$\begin{aligned} de_1/dt &= e_2 \\ de_2/dt &= -(f/J)e_2 + \tau_L/J + w. \end{aligned}$$

where $w \triangleq \dfrac{K_T}{J}(i_{ref} - i_r)$.

Design of a State Feedback Tracking Controller

Choose w to force $e_1(t) \to 0$ and $e_2(t) \to 0$ as $t \to \infty$.
Then $i_r = i_{ref} - \dfrac{J}{K_T} w$.

Specify w as

$$w = -\left(K_0 \int_0^t e_1(\tau) d\tau + K_1 e_1(t) + K_2 e_2(t) \right).$$

Then

$$i_r = i_{ref} + \frac{J}{K_T} \left(K_0 \int_0^t e_1(\tau) d\tau + K_1 e_1(t) + K_2 e_2(t) \right).$$

Define $e_0(t) \triangleq \int_0^t e_1(\tau) d\tau$. The error system is now

$$\begin{aligned}
\frac{de_0}{dt} &= e_1 \\
\frac{de_1}{dt} &= e_2 \\
\frac{de_2}{dt} &= -(f/J)e_2 - K_0 e_0 - K_1 e_1 - K_2 e_2 + \tau_L / J.
\end{aligned}$$

Design of a State Feedback Tracking Controller

Laplace transform of the error system

$$
\begin{aligned}
sE_0(s) - e_0(0) &= E_1(s) \\
sE_1(s) - e_1(0) &= E_2(s) \\
sE_2(s) - e_2(0) &= -K_0 E_0(s) - K_1 E_1(s) - (K_2 + f/J)E_2(s) + \tau_L(s)/J.
\end{aligned}
$$

or

$$
\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ K_0 & K_1 & s + f/J + K_2 \end{bmatrix}
\begin{bmatrix} E_0(s) \\ E_1(s) \\ E_2(s) \end{bmatrix}
=
\begin{bmatrix} e_0(0) \\ e_1(0) \\ e_2(0) \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ \tau_L(s)/J \end{bmatrix}.
$$

The **inverse** of the 3×3 matrix on the left-hand side is

$$
\frac{1}{\underbrace{s^3 + (K_2 + f/J)s^2 + K_1 s + K_0}_{\text{characteristic polynomial}}}
\begin{bmatrix} s^2 + (K_2 + f/J)s + K_1 & s + K_2 + f/J & 1 \\ -K_0 & s^2 + (K_2 + f/J)s & s \\ -K_0 s & -(K_1 s + K_0) & s^2 \end{bmatrix}.
$$

Design of a State Feedback Tracking Controller

Let the load torque be constant, i.e., $\tau_L = \tau_{L0} u_s(t)$.

$$\begin{aligned}
E_0(s) &= \frac{(s^2 + (K_2 + f/J)s + K_1)e_0(0) + (s + K_2 + f/J)e_1(0) + e_2(0) + (\tau_{L0}/J)/s}{s^3 + (K_2 + f/J)s^2 + K_1 s + K_0} \\
E_1(s) &= \frac{-K_0 e_0(0) + (s^2 + (K_2 + f/J)s)e_1(0) + s e_2(0) + \tau_{L0}/J}{s^3 + (K_2 + f/J)s^2 + K_1 s + K_0} \\
E_2(s) &= \frac{-K_0 s e_0(0) - (K_1 s + K_0)e_1(0) + s^2 e_2(0) + s(\tau_{L0}/J)}{s^3 + (K_2 + f/J)s^2 + K_1 s + K_0}.
\end{aligned}$$

All have the same denominator (characteristic polynomial)

$$a(s) \triangleq s^3 + (K_2 + f/J)s^2 + K_1 s + K_0.$$

Let $r_1, r_2, r_3 > 0$; choose the gains K_0, K_1, K_2 so

$$a(s) = (s + r_1)(s + r_2)(s + r_3) = s^3 + (r_1 + r_2 + r_3)s^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)s + r_1 r_2 r_3.$$

$$\begin{aligned}
K_2 &= r_1 + r_2 + r_3 - f/J \\
K_1 &= r_1 r_2 + r_1 r_3 + r_2 r_3 \\
K_0 &= r_1 r_2 r_3.
\end{aligned}$$

Design of a State Feedback Tracking Controller

The **closed-loop poles** are $-r_1$, $-r_2$, $-r_3$.

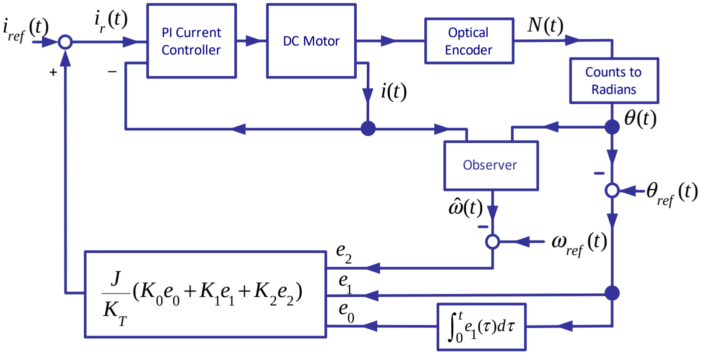
$$\begin{aligned}\theta_{ref}(s) - \theta(s) = E_1(s) &= \frac{-K_0 e_0(0) + (s^2 + (K_2 + f/J)s)e_1(0) + se_2(0) + \tau_{L0}/J}{(s+r_1)(s+r_2)(s+r_3)} \\ &= \frac{A}{s+r_1} + \frac{B}{s+r_2} + \frac{C}{s+r_3}.\end{aligned}$$

Then

$$\theta_{ref}(t) - \theta(t) = e_1(t) = Ae^{-r_1 t} + Be^{-r_2 t} + Ce^{-r_3 t} \to 0 \text{ as } t \to \infty.$$

- The further the closed-loop poles are in the left-half plane, the faster
$$e_1(t) = \theta_{ref}(t) - \theta(t) \to 0.$$
- However, the larger the values of r_1, r_2, r_3 the larger the gains
$$\begin{aligned}K_2 &= r_1 + r_2 + r_3 - f/J \\ K_1 &= r_1 r_2 + r_1 r_3 + r_2 r_3 \\ K_0 &= r_1 r_2 r_3.\end{aligned}$$
- Then
$$i_r = i_{ref} + \frac{J}{K_T}\left(K_0 \int_0^t e_1(\tau)d\tau + K_1 e_1(t) + K_2 e_2(t)\right)$$
can be quite large causing the amplifier to saturate.

Design of a State Feedback Tracking Controller



- Vary the location of the **closed-loop poles** $-r_1, -r_2, -r_3$ to obtain a fast response without saturating the amplifier.
- This is called "**tuning the system**".
- The control designer also must choose the **observer** poles $-p_1, -p_2, -p_3$.
  Desire $\hat{\omega}(t) \to \omega(t)$ faster than the rate at which $\omega_{ref}(t) - \hat{\omega}(t) \to 0$.
  $\hat{\omega}(t)$ is then a good estimate of $\omega(t)$ for the feedback $K_1(\omega_{ref}(t) - \hat{\omega}(t))$.

Design of a State Feedback Tracking Controller

$\lim_{t\to\infty} e_2(t) = \lim_{t\to\infty} (\omega_{ref}(t) - \omega(t))$.

$$\begin{aligned}
E_2(s) &= \frac{-K_0 s e_0(0) - (K_1 s + K_0)e_1(0) + s^2 e_2(0) + s(\tau_{L0}/J)}{(s+r_1)(s+r_2)(s+r_3)} \\
&= \frac{A_2}{s+r_1} + \frac{B_2}{s+r_2} + \frac{C_2}{s+r_3}. \\
\Longrightarrow e_2(t) &= \omega(t) - \omega_{ref}(t) = A_2 e^{-r_1 t} + B_2 e^{-r_2 t} + C_2 e^{-r_3 t} \to 0
\end{aligned}$$

$\lim_{t\to\infty} e_0(t) = \lim_{t\to\infty} \int_0^t (\theta_{ref}(\tau) - \theta(\tau) d\tau$.

$$\begin{aligned}
E_0(s) &= \frac{(s^2 + (K_2 + f/J)s + K_1)e_0(0) + (s + K_2 + f/J)e_1(0) + e_2(0) + (\tau_{L0}/J)/s}{(s+r_1)(s+r_2)(s+r_3)} \\
&= \frac{(s^2 + (K_2 + f/J)s + K_1)e_0(0) + (s + K_2 + f/J)e_1(0) + e_2(0)}{(s+r_1)(s+r_2)(s+r_3)} \\
&\quad + \frac{1}{(s+r_1)(s+r_2)(s+r_3)} \frac{\tau_{L0}/J}{s}
\end{aligned}$$

$sE_0(s)$ is stable so by the final value theorem

$$\lim_{t\to\infty} e_0(t) = \lim_{t\to\infty} \int_0^t (\theta_{ref}(\tau) - \theta(\tau) d\tau = \lim_{s\to 0} sE_0(s) = \frac{\tau_{L0}}{r_1 r_2 r_3} = \frac{\tau_{L0}/J}{K_0}.$$

Design of a State Feedback Tracking Controller

Finally look at $i_r(\infty) = \lim_{t\to\infty} i_r(t)$.

$$i_r = i_{ref} + \frac{J}{K_T}\left(K_0 \int_0^t e_1(\tau)d\tau + K_1 e_1(t) + K_2 e_2(t)\right)$$

The reference current $i_{ref}(t)$ is zero for $t \geq t_3$.

Also $e_1(t) \to 0, e_2(t) \to 0$ as $t \to \infty$.

We have

$$i_r(\infty) = \lim_{t\to\infty} i_r(t) = \frac{J}{K_T} K_0 \lim_{t\to\infty} e_0(t) = \frac{\tau_{L0}}{K_T}.$$

- The motor current $i(t)$ goes to $\dfrac{\tau_{L0}}{K_T}$.
- The motor torque $K_T i(t) \to \tau_{L0}$ which cancels out the load torque!
- This is the reason for the integrator term $\int_0^t e_1(\tau)d\tau$ in the feedback.
- Problem 11 of Chapter 2 asks for a simulation of this state feedback trajectory tracking controller.

Identification of the DC Motor Parameters

Mathematical model of the DC motor

$$\begin{aligned} L\frac{di}{dt} &= -Ri(t) - K_b\omega(t) + v(t) \\ J\frac{d\omega}{dt} &= -f\omega(t) + K_T i(t) \\ \frac{d\theta}{dt} &= \omega(t). \end{aligned}$$

- Determine the values of the motor parameters $L, R, K_b = K_T, J$, and f.
- Rewrite the first two model equations as

$$\begin{bmatrix} di/dt & i(t) & \omega(t) & 0 & 0 \\ 0 & 0 & -i(t) & d\omega/dt & \omega(t) \end{bmatrix} \begin{bmatrix} L \\ R \\ K_T \\ J \\ f \end{bmatrix} = \begin{bmatrix} v(t) \\ 0 \end{bmatrix}.$$

- Two linear algebraic equations in the unknowns L, R, K_T, J, and f.
- The coefficients are found from the measured/calculated data

$$\theta(t), \omega(t), d\omega/dt, i(t), di/dt, v.$$

- Find the values of L, R, K_T, J, and f that satisfy these two equations for all t.

Identification of the DC Motor Parameters

At time nT we have

$$v(nT), \omega(nT), \frac{d\omega(nT)}{dt} = \frac{\omega(nT) - \omega(nT)}{T} i(nT), \frac{di(nT)}{dt} = \frac{i(nT) - i(nT)}{T}.$$

Write the above two linear equations as

$$\underbrace{\begin{bmatrix} v(nT) \\ 0 \end{bmatrix}}_{y(nT)} = \underbrace{\begin{bmatrix} \frac{di}{dt}(nT) & i(nT) & \omega(nT) & 0 & 0 \\ 0 & 0 & -i(nT) & \frac{d\omega}{dt}(nT) & \omega(nT) \end{bmatrix}}_{W(nT)} \underbrace{\begin{bmatrix} L \\ R \\ K_T \\ J \\ f \end{bmatrix}}_{K}$$

or

$$y(nT) = W(nT)K.$$

- W is referred to as the *regressor* matrix.

Identification of the DC Motor Parameters

- Determine the constant vector K that for all n satisfies

$$y(nT) = W(nT)K.$$

First step is multiply both sides by $W^T(nT)$ to obtain

$$W^T(nT)y(nT) = W^T(nT)W(nT)K.$$

$$W^T(nT)W(nT) = \begin{bmatrix} di(nT)/dt & 0 \\ i(nT) & 0 \\ \omega(nT) & -i(nT) \\ 0 & d\omega(nT)/dt \\ 0 & \omega(nT) \end{bmatrix} \begin{bmatrix} di(nT)/dt & i(nT) & \omega(nT) & 0 & 0 \\ 0 & 0 & -i(nT) & d\omega(nT)/dt & \omega(nT) \end{bmatrix}$$

$$= \begin{bmatrix} (di/dt)^2 & idi/dt & \omega di/dt & 0 & 0 \\ idi/dt & i^2 & \omega i & 0 & 0 \\ \omega di/dt & \omega i & \omega^2 + i^2 & -id\omega/dt & -\omega i \\ 0 & 0 & -id\omega/dt & (d\omega/dt)^2 & \omega d\omega/dt \\ 0 & 0 & -\omega i & \omega d\omega/dt & \omega^2 \end{bmatrix}_{|\, t = nT}$$

$$W^T(nT)y(nT) = \begin{bmatrix} di(nT)/dt & 0 \\ i(nT) & 0 \\ \omega(nT) & -i(nT) \\ 0 & d\omega(nT)/dt \\ 0 & \omega(nT) \end{bmatrix} \begin{bmatrix} v(nT) \\ 0 \end{bmatrix} = \begin{bmatrix} v(nT)di(nT)/dt \\ v(nT)i(nT) \\ v(nT)\omega(nT) \\ 0 \\ 0 \end{bmatrix}.$$

Identification of the DC Motor Parameters

- $W^T(nT)W(nT) \in \mathbb{R}^{5\times 5}$; that is, it is a square matrix.
- $W^T(nT)W(nT)$ is not invertible for all n as

$$W^T(nT)W(nT)\begin{bmatrix} -i(nT) \\ di(nT)/dt \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} di(nT)/dt & 0 \\ i(nT) & 0 \\ \omega(nT) & -i(nT) \\ 0 & d\omega(nT)/dt \\ 0 & \omega(nT) \end{bmatrix} \begin{bmatrix} di(nT)/dt & i(nT) & \omega(nT) & 0 & 0 \\ 0 & 0 & -i(nT) & d\omega(nT)/dt & \omega(nT) \end{bmatrix} \begin{bmatrix} -i(nT) \\ di(nT)/dt \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} di(nT)/dt & 0 \\ i(nT) & 0 \\ \omega(nT) & -i(nT) \\ 0 & d\omega(nT)/dt \\ 0 & \omega(nT) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Identification of the DC Motor Parameters

Sum up over n

$$W^T(nT)y(nT) = W^T(nT)W(nT)K$$

to obtain

$$\underbrace{\left(\sum_{n=1}^{N} W^T(nT)W(nT)\right)}_{R_W} K = \underbrace{\sum_{n=1}^{N} W^T(nT)y(nT)}_{R_{Wy}}.$$

- If the matrix sum $R_W \triangleq \sum_{n=1}^{N} W^T(nT)W(nT)$ is **invertible** then

$$K = R_W^{-1} R_{Wy}.$$

- Must choose a voltage input $v(t)$ so that R_W is invertible.

- If $i(t)$ or $\omega(t)$ is constant then $R_W \triangleq \sum_{n=1}^{N} W^T(nT)W(nT)$ is **not** invertible.

Identification of the DC Motor Parameters

Least-Squares Approximation

Recall that

$$
\begin{aligned}
W(nT) &\triangleq \begin{bmatrix} di(nT)/dt & i(nT) & \omega(nT) & 0 & 0 \\ 0 & 0 & -i(nT) & d\omega(nT)/dt & \omega(nT) \end{bmatrix} \in \mathbb{R}^{2\times 5} \\
y(nT) &\triangleq \begin{bmatrix} v(nT) \\ 0 \end{bmatrix} \in \mathbb{R}^2
\end{aligned}
$$

- We have assumed thre is a K such that $y(nT) = W(nT)K$ is true for all n.
- However, the motor model is not exact and the measurements $\imath(t), \omega(t)$ are noisy.
- There will **not** be a parameter vector K that satisfies $y(nT) = W(nT)K$ for all n.

Least-Squares Approximation

- Reformulate to finding the value of K that "best" fits $y(nT) = W(nT)K$ for all n.

To do this define the error

$$e(nT) \triangleq y(nT) - W(nT)K \in \mathbb{R}^2.$$

Take the "best" fit to be the value of K that minimizes the **squared error** given by

$$\begin{aligned}
E^2(K) &\triangleq \sum_{n=1}^{N} (y(nT) - W(nT)K)^T (y(nT) - W(nT)K) \\
&= \sum_{n=1}^{N} (y(nT) - \hat{y}(nT))^T (y(nT) - \hat{y}(nT)) \\
&= \sum_{n=1}^{N} (y_1(nT) - \hat{y}_1(nT))^2 + (y_2(nT) - \hat{y}_2(nT))^2 \\
&= \sum_{n=1}^{N} e_1(nT)^2 + e_2(nT)^2
\end{aligned}$$

where $e_1(nT) \triangleq y_1(nT) - \hat{y}_1(nT)$, $e_2(nT) \triangleq y_2(nT) - \hat{y}_2(nT)$.

Least-Squares Approximation

In the equation

$$y(nT) = W(nT)K$$

- $y(nT)$ is considered the *output*.
- $\hat{y}(nT) = W(nT)K$ is the *predicted output* based on K as the estimate of the parameters.
- The error at time nT is

$$e(nT) \triangleq y(nT) - W(nT)K = y(nT) - \hat{y}(nT) \in \mathbb{R}^2.$$

- The total *squared error* is

$$E^2(K) \triangleq \sum_{n=1}^{N} \left(y(nT) - W(nT)K\right)^T \left(y(nT) - W(nT)K\right)$$

- The K that minimizes the total squared error is the *least squares* estimate.

Identification of the DC Motor Parameters

Least-Squares Approximation

- We now show the least-squares estimate is $R_W^{-1} R_{Wy}$.

To proceed (recall $(AB)^T = B^T A^T$)

$$
\begin{aligned}
E^2(K) &\triangleq \sum_{n=1}^{N} (y(nT) - W(nT)K)^T (y(nT) - W(nT)K) \\
&= \sum_{n=1}^{N} \Big(y^T(nT)y(nT) - y^T(nT)W(nT)K - K^T W^T(nT)y(nT) + K^T W^T(nT)W(nT)K \Big) \\
&= \sum_{n=1}^{N} y^T(nT)y(nT) - \left(\sum_{n=1}^{N} y^T(nT)W(nT) \right) K - K^T \left(\sum_{n=1}^{N} W^T(nT)y(nT) \right) + \\
&\quad K^T \left(\sum_{n=1}^{N} W^T(nT)W(nT) \right) K.
\end{aligned}
$$

Or

$$
\begin{aligned}
E^2(K) &= R_y - R_{yW}K - K^T R_{Wy} + K^T R_W K \\
&= R_y - R_{yW} R_W^{-1} R_{Wy} + (K - R_W^{-1} R_{Wy})^T R_W (K - R_W^{-1} R_{Wy})
\end{aligned}
$$

where the last line assume R_W is invertible.

Digression: Symmetric, Positive Semidefinite and Positive Definite Matrices

- $Q \in \mathbb{R}^{m\times m}$ is *symmetric* if $Q^T = Q$.
- $Q = Q^T \in \mathbb{R}^{m\times m}$ is *positive semidefinite* if for all $x \in \mathbb{R}^m$, $x^T Qx \geq 0$.
- $Q = Q^T \in \mathbb{R}^{m\times m}$ is *positive definite* if $x^T Qx \geq 0$ for all $x \in \mathbb{R}^m$ *and* $x^T Qx = 0$ if and only if $x = 0$.

Example

$$
\begin{aligned}
Q_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \; Q_1 = Q_1^T \\
x^T Q_1 x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 2x_2^2 \geq 0 \;\text{ for all }\; x \in \mathbb{R}^2 \\
x^T Q_1 x &= 0 \;\text{ if and only if } x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Q_1 is positive definite.

Digression: Symmetric, Positive Semidefinite and Positive Definite Matrices

Example

$$Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q_2 = Q_2^T$$

$$x^T Q_2 x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_2^2 \geq 0 \text{ for all } x \in \mathbb{R}^2$$

$$x^T Q_2 x = 0 \text{ with } x = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$

Q_2 is positive semidefinite, but Q_2 is not positive definite.

Example

$$Q_3 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q_3 = Q_3^T.$$

$$x^T Q_3 x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -x_1^2 + 2x_2^2$$

$$x^T Q_3 x < 0 \text{ if } x = [0\ 1]^T \text{ and } x^T Q_3 x > 0 \text{ if } x = [1\ 0]^T$$

Q_3 is neither positive definite nor positive semidefinite.

Identification of the DC Motor Parameters

R_W is a **symmetric** matrix as

$$R_W^T = \left(\sum_{n=1}^{N} W^T(nT)W(nT)\right)^T = \sum_{n=1}^{N} \left(W^T(nT)W(nT)\right)^T = \sum_{n=1}^{N} W^T(nT)\left(W^T(nT)\right)^T = R_W.$$

R_W is a **positive semidefinite** matrix as

$$x^T R_W x = x^T \left(\sum_{n=1}^{N} W^T(nT)W(nT)\right) x = \sum_{n=1}^{N} x^T W^T(nT)W(nT)x = ||W(nT)x||^2 \geq 0.$$

- The controls engineer designs an input to the motor such that

$$R_W = \sum_{n=1}^{N} W^T(nT)W(nT)$$

  is **invertible**.
- Fact: A *positive semidefinite invertible* matrix is **positive definite**.

Identification of the DC Motor Parameters

- Recall the expression for the squared error.

$$\begin{aligned} E^2(K) &= R_y - R_{yW}K - K^T R_{Wy} + K^T R_W K \\ &= R_y - R_{yW}R_W^{-1}R_{Wy} + (K - R_W^{-1}R_{Wy})^T R_W (K - R_W^{-1}R_{Wy}). \end{aligned}$$

- Assume R_W is invertible. Then R_W is positive definite so that

$$(K - R_W^{-1}R_{Wy})^T R_W (K - R_W^{-1}R_{Wy}) \geq 0 \text{ for all } K \in \mathbb{R}^5.$$

- Thus $E^2(K)$ is minimized for

$$K = K^* \triangleq R_W^{-1}R_{Wy}.$$

Identification of the DC Motor Parameters

- How good is the least squares estimate K^*?
- K^* minimizes

$$\begin{aligned} E^2(K) &= R_y - R_{yW}K - K^T R_{Wy} + K^T R_W K \\ &= R_y - R_{yW}R_W^{-1}R_{Wy} + (K - R_W^{-1}R_{Wy})^T R_W (K - R_W^{-1}R_{Wy}). \end{aligned}$$

- However, the exact value of the parameters are unknown so the error is unknown.
- Compare K^* with a known value of K, specifically, with $K = 0$.

$$E^2(K)_{|K=0} = R_y.$$

- The *residual error* is

$$E^2(K)_{|K=K^* \triangleq R_W^{-1}R_{Wy}} = R_y - R_{yW}R_W^{-1}R_{Wy} \geq 0.$$

- We have

$$\frac{E^2(K^*)}{E^2(0)} = \frac{R_y - R_{yW}R_W^{-1}R_{Wy}}{R_y} \leq 1.$$

- Define the error index by

$$\text{Error Index} \triangleq \sqrt{\frac{E^2(K^*)}{E^2(0)}} = \sqrt{\frac{R_y - R_{yW}R_W^{-1}R_{Wy}}{R_y}} \leq 1.$$

Identification of the DC Motor Parameters

From previous slide

$$\text{Error Index} \triangleq \sqrt{\frac{E^2(K^*)}{E^2(0)}} = \sqrt{\frac{R_y - R_{yW} R_W^{-1} R_{Wy}}{R_y}} \leq 1.$$

- $E^2(K^*)/E^2(0)$ minimum squared error relative to the squared error with $K = 0$.
- Taking the square root gives the relative error rather than squared error.
- If the error index is close to 1, then the estimate is not much better than taking all the parameter values equal to zero!
- If the error index is close to one, then one would suspect that the original model of the system is incorrect.
- The error index must be much less than one for the estimate to be of any value.

Identification of the DC Motor Parameters

Parametric Error Indices

- How sensitive is the error $E^2(K^*)$ to each parameter.
- Consider a change δK_1 in the parameter K_1^*. With

$$K = K_1^* + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \delta K_1$$

  is

$$E^2(K) >> E^2(K^*) \text{ or } E^2(K) \approx E^2(K^*)?$$

- If $E^2(K) \approx E^2(K^*)$ then we don't know if $K_1^* + \delta K_1$ or K_1^* is the better estimate.
  - The accuracy of the parameter estimate of K_1^* would be in doubt.
- If $E^2(K) >> E^2(K^*)$ so the residual error is very sensitive to changes in K_1^*.
  - In this case we consider the estimate K_1^* to be relatively accurate.

Identification of the DC Motor Parameters

Parametric Error Indices

We have

$$\left.\frac{\partial E^2(K)}{\partial K}\right|_{K=K^*} = \left. 2R_W\left(K - R_W^{-1}R_{Wy}\right)\right|_{K=K^*} = 0_{1\times 5}$$

- Not possible to use the derivative $\left.\dfrac{\partial E^2(K)}{\partial K_i}\right|_{K=K^*}$ as a measure of how sensitive the error is with respect to K_i as it is always zero.
- Need an alternative apporach. To proceed, consider $E^2(K^* + \delta K)$ given by

$$\begin{aligned}
E^2(K)|_{K=K^*+\delta K} &= R_y - R_{yW}R_W^{-1}R_{Wy} + (K - R_W^{-1}R_{Wy})^T R_W (K - R_W^{-1}R_{Wy})|_{K=K^*+\delta K} \\
&= R_y - R_{yW}R_W^{-1}R_{Wy} + (K^* + \delta K - R_W^{-1}R_{Wy})^T R_W (K^* + \delta K - R_W^{-1}R_{Wy}) \\
&= E^2(K^*) + \delta K^T R_W \delta K
\end{aligned}$$

using the fact that $K^* = R_W^{-1}R_{Wy}$.

Parametric Error Indices

In the previous slide we showed

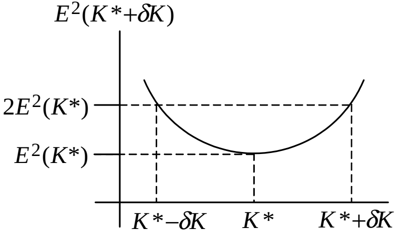
$$E^2(K^* + \delta K) = E^2(K^*) + \delta K^T R_W \delta K$$

Now consider only perturbations δK that double the residual error, i.e.,

$$E^2(K^* + \delta K) = E^2(K^*) + \delta K^T R_W \delta K = 2E^2(K^*).$$

or

$$\delta K^T R_W \delta K = E^2(K^*).$$



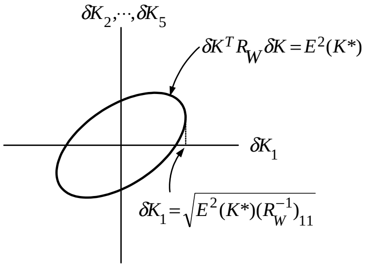
Identification of the DC Motor Parameters

Parametric Error Indices

The set of points $\delta K \in \mathbb{R}^5$ that satisfy

$$\delta K^T R_W \delta K = E^2(K^*)$$

define an *ellipsoid* as illustrated in the figure.



The *parametric error index* for K_i^* is the *maximum* value of δK_i that satisfies

$$\delta K^T R_W \delta K = E^2(K^*).$$

This is the largest possible value for δK_i that results in $E^2(K^* + \delta K) = 2E^2(K^*)$.

Identification of the DC Motor Parameters

Parametric Error Indices

Solve the constrained maximization problem using Lagrange multipliers. We have

$$C(\delta K, \lambda) \triangleq \delta K_i + \lambda\Big(E^2(K^*) - \delta K^T R_W \delta K\Big).$$

To fix ideas let $i = 1$, and compute

$$\begin{aligned}
\frac{\partial C(\delta K, \lambda)}{\partial \delta K_1} &= 1 - 2\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} R_W \delta K = 0 \\
\frac{\partial C(\delta K, \lambda)}{\partial \delta K_2} &= -2\lambda \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} R_W \delta K = 0 \\
\frac{\partial C(\delta K, \lambda)}{\partial \delta K_3} &= -2\lambda \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} R_W \delta K = 0 \\
\frac{\partial C(\delta K, \lambda)}{\partial \delta K_4} &= -2\lambda \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} R_W \delta K = 0 \\
\frac{\partial C(\delta K, \lambda)}{\partial \delta K_5} &= -2\lambda \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} R_{W_w} \delta K = 0 \\
\frac{\partial C(\delta K, \lambda)}{\partial \lambda} &= E^2(K^*) - \delta K^T R_W \delta K = 0.
\end{aligned}$$

Identification of the DC Motor Parameters

The first four equations from above can be rewritten as

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 2\lambda R_W \delta K = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving for δK gives

$$\delta K = \frac{1}{2\lambda} R_W^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

To compute λ, multiply both sides of $E^2(K^*) = \delta K^T R_W \delta K$ by δK^T to obtain

$$\delta K_1 = 2\lambda \delta K^T R_W \delta K = 2\lambda E^2(K^*).$$

Rearrange to get

$$2\lambda = \delta K_1 / E^2(K^*).$$

Substitute this expression $2\lambda = \delta K_1 / E^2(K^*)$ into

$$\delta K = \frac{1}{2\lambda} R_W^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

to obtain

$$\delta K = \frac{E^2(K^*)}{\delta K_1} R_W^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

With $\left(R_W^{-1}\right)_{11}$ denoting the $(1,1)$ element of the matrix R_W^{-1} we have

$$\delta K_1 = \frac{E^2(K^*)}{\delta K_1} \left(R_W^{-1}\right)_{11}$$

or

$$\delta K_1 = \sqrt{E^2(K^*)(R_W^{-1})_{11}}$$

Identification of the DC Motor Parameters

In general

$$\delta K_i = \sqrt{E^2(K^*)(R_W^{-1})_{ii}}.$$

- δK_i is the max amount the estimate can change to double the residual error.
- A large δK_i means the parameter estimate could vary greatly without a large change in the residual error.
  - In this case the accuracy of the parameter estimate is suspect.
- A small δK_i means the residual error is very sensitive to the changes in the parameter estimates
  - In this case the parameter estimates may be considered to be more accurate.
- In any case, the error indices should not be considered as actual errors, but rather as orders of magnitude of the errors to be expected, to guide the identification process and to warn about unreliable results.
- The choice of a parametric error index as corresponding to a doubling of the residual error is arbitrary. A different level of residual error would lead to a scaling of all the components of the parametric error index by a common factor.