Modeling and High-Performance Control of Electric Machines

Chapter 6 Problem 4

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Induction Motor Equations in Statespace Form

The following slides go over Problem 4 of Chapter 6.

State-Space Model of the Induction Motor

Expand the electrical equations

$$
\begin{aligned}
L_S \frac{d}{dt} i_{Sa} + M \frac{d}{dt}\Big(+i_{Ra}\cos(\theta_R) - i_{Rb}\sin(\theta_R)\Big) + R_S i_{Sa} &= u_{Sa}\\
L_S \frac{d}{dt} i_{Sb} + M \frac{d}{dt}\Big(+i_{Ra}\sin(\theta_R) + i_{Rb}\cos(\theta_R)\Big) + R_S i_{Sb} &= u_{Sb}\\
L_R \frac{d}{dt} i_{Ra} + M \frac{d}{dt}\Big(+i_{Sa}\cos(\theta_R) + i_{Sb}\sin(\theta_R)\Big) + R_R i_{Ra} &= 0\\
L_R \frac{d}{dt} i_{Rb} + M \frac{d}{dt}\Big(-i_{Sa}\sin(\theta_R) + i_{Sb}\cos(\theta_R)\Big) + R_R i_{Rb} &= 0
\end{aligned}
$$

to obtain

$$
\begin{aligned}
L_S \frac{d}{dt} i_{Sa} + M\frac{di_{Ra}}{dt}\cos(\theta_R) - M\frac{di_{Rb}}{dt}\sin(\theta_R) - Mi_{Ra}\sin(\theta_R)\omega_R - Mi_{Rb}\cos(\theta_R)\omega_R + R_S i_{Sa} &= u_{Sa}\\
L_S \frac{d}{dt} i_{Sb} + M\frac{di_{Ra}}{dt}\sin(\theta_R) + M\frac{di_{Rb}}{dt}\cos(\theta_R) + Mi_{Ra}\cos(\theta_R)\omega_R - Mi_{Rb}\sin(\theta_R)\omega_R + R_S i_{Sb} &= u_{Sb}\\
L_R \frac{d}{dt} i_{Ra} + M\frac{di_{Sa}}{dt}\cos(\theta_R) + M\frac{di_{Sb}}{dt}\sin(\theta_R) - Mi_{Sa}\sin(\theta_R)\omega_R + Mi_{Sb}\cos(\theta_R)\omega_R + R_R i_{Ra} &= 0\\
L_R \frac{d}{dt} i_{Rb} - M\frac{di_{Sa}}{dt}\sin(\theta_R) + M\frac{di_{Sb}}{dt}\cos(\theta_R) - Mi_{Sa}\cos(\theta_R)\omega_R - Mi_{Sb}\sin(\theta_R)\omega_R + R_R i_{Rb} &= 0.
\end{aligned}
$$

State-Space Model of the Induction Motor

In matrix form this becomes

$$\begin{bmatrix} L_S & 0 & M\cos(\theta_R) & -M\sin(\theta_R) \\ 0 & L_S & M\sin(\theta_R) & M\cos(\theta_R) \\ M\cos(\theta_R) & M\sin(\theta_R) & L_R & 0 \\ -M\sin(\theta_R) & M\cos(\theta_R) & 0 & L_R \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix}$$

$$= - \begin{bmatrix} R_S & 0 & -M\sin(\theta_R)\omega_R & -M\cos(\theta_R)\omega_R \\ 0 & R_S & M\cos(\theta_R)\omega_R & -M\sin(\theta_R)\omega_R \\ -M\sin(\theta_R)\omega_R & M\cos(\theta_R)\omega_R & R_R & 0 \\ -M\cos(\theta_R)\omega_R & -M\sin(\theta_R)\omega_R & 0 & R_R \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix}$$

$$+ \begin{bmatrix} u_{Sa} \\ u_{Sb} \\ 0 \\ 0 \end{bmatrix}.$$

State-Space Model of the Induction Motor

The inverse of the matrix in front of the derivatives is

$$\begin{bmatrix} L_S & 0 & M\cos(\theta_R) & -M\sin(\theta_R) \\ 0 & L_S & M\sin(\theta_R) & M\cos(\theta_R) \\ M\cos(\theta_R) & M\sin(\theta_R) & L_R & 0 \\ -M\sin(\theta_R) & M\cos(\theta_R) & 0 & L_R \end{bmatrix}^{-1}$$

$$= \frac{1}{\sigma L_S L_R}\begin{bmatrix} L_R & 0 & -M\cos(\theta_R) & M\sin(\theta_R) \\ 0 & L_R & -M\sin(\theta_R) & -M\cos(\theta_R) \\ -M\cos(\theta_R) & -M\sin(\theta_R) & L_S & 0 \\ M\sin(\theta_R) & -M\cos(\theta_R) & 0 & L_S \end{bmatrix}$$

State-Space Model of the Induction Motor

$$\frac{d}{dt}\begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix} = \frac{1}{\sigma L_S L_R}\begin{bmatrix} L_R & 0 & -M\cos(\theta_R) & M\sin(\theta_R) \\ 0 & L_R & -M\sin(\theta_R) & -M\cos(\theta_R) \\ -M\cos(\theta_R) & -M\sin(\theta_R) & L_S & 0 \\ M\sin(\theta_R) & -M\cos(\theta_R) & 0 & L_S \end{bmatrix} \times$$

$$\left(- \begin{bmatrix} R_S & 0 & -M\sin(\theta_R)\omega_R & -M\cos(\theta_R)\omega_R \\ 0 & R_S & M\cos(\theta_R)\omega_R & -M\sin(\theta_R)\omega_R \\ -M\sin(\theta_R)\omega_R & M\cos(\theta_R)\omega_R & R_R & 0 \\ -M\cos(\theta_R)\omega_R & -M\sin(\theta_R)\omega_R & 0 & R_R \end{bmatrix}\begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} u_{Sa} \\ u_{Sb} \\ 0 \\ 0 \end{bmatrix}\right).$$

Expanding, this becomes

$$\frac{d}{dt}\begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix}$$

$$= \frac{1}{\sigma L_S L_R}\left[\begin{array}{cc} -L_R R_S & M^2\omega_R \\ -M^2\omega_R & -L_R R_S \\ M\cos(\theta_R)R_S + L_S M\sin(\theta_R)\omega_R & M\sin(\theta_R)R_S - L_S M\cos(\theta_R)\omega_R \\ -M\sin(\theta_R)R_S + L_S M\cos(\theta_R)\omega_R & M\cos(\theta_R)R_S + L_S M\sin(\theta_R)\omega_R \end{array}\right.$$

$$\left.\begin{array}{cc} L_R M\sin(\theta_R)\omega_R + M\cos(\theta_R)R_R & L_R M\cos(\theta_R)\omega_R - M\sin(\theta_R)R_R \\ -L_R M\cos(\theta_R)\omega_R + M\sin(\theta_R)R_R & L_R M\sin(\theta_R)\omega_R + M\cos(\theta_R)R_R \\ -L_S R_R & -M^2\omega_R \\ M^2\omega_R & -L_S R_R \end{array}\right]\begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix}$$

$$+ \frac{1}{\sigma L_S L_R}\begin{bmatrix} L_R & 0 \\ 0 & L_R \\ -M\cos(\theta_R) & -M\sin(\theta_R) \\ M\sin(\theta_R) & -M\cos(\theta_R) \end{bmatrix}\begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix}.$$

State-Space Model of the Induction Motor

Multiplying out the matrices gives

$$\begin{aligned}
\frac{di_{Sa}}{dt} &= \frac{1}{\sigma L_S L_R}\Big(-L_R R_S i_{Sa} + M^2\omega_R i_{Sb} + \left(L_R M \sin(\theta_R)\omega_R + M\cos(\theta_R)R_R\right) i_{Ra} + \\
&\qquad \left(L_R M\cos(\theta_R)\omega_R - M\sin(\theta_R)R_R\right) i_{Rb}\Big) + \frac{1}{\sigma L_S}u_{Sa} \\
\frac{di_{Sb}}{dt} &= \frac{1}{\sigma L_S L_R}\Big(-M^2\omega_R i_{Sa} - L_R R_S i_{Sb} + \left(-L_R M\cos(\theta_R)\omega_R + M\sin(\theta_R)R_R\right) i_{Ra} + \\
&\qquad \left(L_R M\sin(\theta_R)\omega_R + M\cos(\theta_R)R_R\right) i_{Rb}\Big) + \frac{1}{\sigma L_S}u_{Sb} \\
\frac{di_{Ra}}{dt} &= \frac{1}{\sigma L_S L_R}\Big(\left(M\cos(\theta_R)R_S + L_S M\sin(\theta_R)\omega_R\right) i_{Sa} + \left(M\sin(\theta_R)R_S - L_S M\cos(\theta_R)\omega_R\right) i_{Sb} \\
&\qquad - L_S R_R i_{Ra} - M^2\omega_R i_{Rb}\Big) - \frac{M\cos(\theta_R)}{\sigma L_S L_R}u_{Sa} - \frac{M\sin(\theta_R)}{\sigma L_S L_R}u_{Sb} \\
\frac{di_{Rb}}{dt} &= \frac{1}{\sigma L_S L_R}\Big(\left(-M\sin(\theta_R)R_S + L_S M\cos(\theta_R)\omega_R\right) i_{Sa} + \left(M\cos(\theta_R)R_S + L_S M\sin(\theta_R)\omega_R\right) i_{Sb} \\
&\qquad + M^2\omega_R i_{Ra} - L_S R_R i_{Rb}\Big) + \frac{M\sin(\theta_R)}{\sigma L_S L_R}u_{Sa} - \frac{M\cos(\theta_R)}{\sigma L_S L_R}u_{Sb}
\end{aligned}$$

State-Space Model of the Induction Motor

The four differential equations for the currents along with

$$J\frac{d\omega_R}{dt} = M\Big(-i_{Ra}(t)i_{Sa}(t)\sin(\theta_R) + i_{Ra}(t)i_{Sb}(t)\cos(\theta_R)$$

$$- i_{Rb}(t)i_{Sa}(t)\cos(\theta_R) - i_{Rb}(t)i_{Sb}(t)\sin(\theta_R)\Big)$$

$$\frac{d\theta_R}{dt} = \omega_R$$

form a **state-space** model of a two-phase induction motor.

- **State variables** are $i_{Sa}, i_{Sb}, i_{Ra}, i_{Rb}, \omega_R, \theta_R$.
- **State-space model:** Only **first-order** derivatives on the left-hand side and only state variables (**no derivatives**) on the right-hand side.
- A SIMULINK simulation based on this model is given in the **simulation files**.
- An **equivalent** state-space model developed in Problem 6 of Chapter 6 that has no cosines or sines (thus easier to numerically integrate).

The following slides go over Problem 9 of Chapter 6.

Steady-State Solution of the Induction Motor Equations

- Let

$$\underline{u}_S = u_{Sa} + ju_{Sb} = U_S\cos(\omega_S t) + jU_S\sin(\omega_S t) = U_S e^{j\omega_S t}$$
$$\underline{i}_S = i_{Sa} + ji_{Sb} = I_S\cos(\omega_S t + \phi_S) + jI_S\sin(\omega_S t + \phi_S) = I_S e^{j\phi_S} e^{j\omega_S t} = \underline{I}_S e^{j\omega_S t}$$
$$\underline{i}_R = i_{Ra} + ji_{Rb} = I_R\cos((\omega_S - n_p\omega_R)t + \phi_R) + jI_R\sin((\omega_S - n_p\omega_R)t + \phi_R) = \underbrace{I_R e^{j\phi_R} e^{j(\omega_S - n_p\omega_R)t}}_{\underline{I}_R e^{j(\omega_S - n_p\omega_R)t}}$$

- Stator electrical frequency ω_S and rotor speed ω_R are constant.

- The current phasors $\underline{I}_S \triangleq I_S e^{j\phi_S}$ and $\underline{I}_R \triangleq I_R e^{j\phi_R}$ are to be determined.

- Substitute $\underline{i}_S = \underline{I}_S e^{j\omega_S t}$, $\underline{i}_R = \underline{I}_R e^{j(\omega_S - n_p\omega_R)t}$, and $\theta_R(t) = \omega_R t$ into

$$R_S\underline{i}_S + L_S\frac{d}{dt}\underline{i}_S + M\frac{d}{dt}\left(\underline{i}_R e^{jn_p\theta_R}\right) = \underline{u}_S$$
$$R_R\underline{i}_R + L_R\frac{d}{dt}\underline{i}_R + M\frac{d}{dt}\left(\underline{i}_S e^{-jn_p\theta_R}\right) = 0$$
$$n_p M\,\mathrm{Im}\{\underline{i}_S(\underline{i}_R e^{jn_p\theta_R})^*\} - \tau_L = J\frac{d\omega_R}{dt}$$

to obtain

$$(R_S + j\omega_S L_S)\underline{I}_S + j\omega_S M\underline{I}_R = U_S$$
$$(R_R + j(\omega_S - n_p\omega_R)L_R)\underline{I}_R + j(\omega_S - n_p\omega_R)M\underline{I}_S = 0$$
$$M\,\mathrm{Im}\{\underline{I}_S(\underline{I}_R)^*\} - \tau_L = 0.$$

Steady-State Solution of the Induction Motor Equations

Phasor equations from previous slide.

$$\begin{aligned}
(R_S + j\omega_S L_S)\underline{I}_S + j\omega_S M \underline{I}_R &= U_S \\
(R_R + j(\omega_S - n_p\omega_R)L_R)\underline{I}_R + j(\omega_S - n_p\omega_R)M\underline{I}_S &= 0 \\
M\,\mathrm{Im}\{\underline{I}_S(\underline{I}_R)^*\} - \tau_L &= 0.
\end{aligned}$$

Define the normalized slip speed by $S \triangleq \dfrac{\omega_S - n_p\omega_R}{\omega_S}$. Then

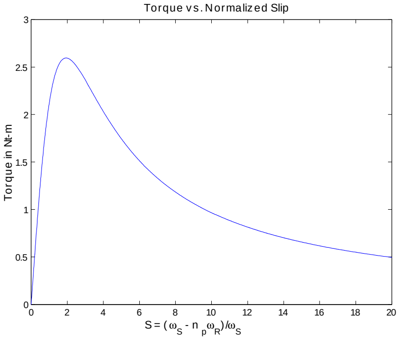
$$\begin{aligned}
\begin{bmatrix} \underline{I}_S \\ \underline{I}_R \end{bmatrix} &= \frac{1}{\det}\begin{bmatrix} R_R/S + j\omega_S L_R & -j\omega_S M \\ -j\omega_S M & R_S + j\omega_S L_S \end{bmatrix}\begin{bmatrix} U_S \\ 0 \end{bmatrix} \\
\det &\triangleq (R_S + j\omega_S L_S)(R_R/S + j\omega_S L_R) - (j\omega_S M)^2
\end{aligned}$$

and

$$\tau = M\,\mathrm{Im}\{\underline{I}_S(\underline{I}_R)^*\} = \mathrm{Im}\left\{\frac{j\omega_S M(R_R/S + j\omega_S L_R)}{|\det|^2}\right\}$$

Steady-State Solution of the Induction Motor Equations

$$
\begin{aligned}
\begin{bmatrix} \underline{I}_S \\ \underline{I}_R \end{bmatrix} &= \frac{1}{\det} \begin{bmatrix} R_R/S + j\omega_S L_R & -j\omega_S M \\ -j\omega_S M & R_S + j\omega_S L_S \end{bmatrix} \begin{bmatrix} U_S \\ 0 \end{bmatrix} \\
\det &\triangleq (R_S + j\omega_S L_S)(R_R/S + j\omega_S L_R) - (j\omega_S M)^2 \\
\tau &= M\,\mathrm{Im}\{\underline{I}_S(\underline{I}_R)^*\} = \mathrm{Im}\left\{ \frac{j\omega_S M (R_R/S + j\omega_S L_R)}{|\det|^2} \right\}
\end{aligned}
$$



If $R_S = 0$ then the torque can be written as

$$\tau = \frac{L_S}{2} \frac{1-\sigma}{\sigma} \left(\frac{U_S}{\omega_S L_S} \right)^2 \frac{2}{S/S_p + S_p/S}, \;\; \sigma = 1 - \frac{M^2}{L_S L_R}, \;\; S_p \triangleq \frac{R_R}{\sigma \omega_S L_R}.$$