ECE 697 Modeling and High-Performance Control of Electric Machines
HW 6 Solutions
Spring 2022

Problem 1 *A Two-Phase Stator with Single-Loop Windings*

(a) Applying Ampère's law to the path $a - b - c - d - a$ gives

$$H(\theta_1)g - H(\theta_2)g = 0$$

or

$$H(\theta_1) = H(\theta_2)$$

for $-\pi/2 < \theta_1, \theta_2 < \pi/2$. That is,

$$H(\theta) = H_1 \text{ for } \pi/2 < \theta < \pi/2$$

and similarly

$$H(\theta) = H_2 \text{ for } \pi/2 < \theta < 3\pi/2$$

where H_1 and H_1 are two constants to be determined.

(b) Applying Ampère's law to the path $1 - 2 - 3 - 4 - 1$, it follows that

$$-H_2 g + H_1 g = i. \tag{1}$$

In the air gap, $B_1 = \mu_0 H_1$ and $B_2 = \mu_0 H_2$. Symmetry requires that $B_2 = -B_1 \Rightarrow H_2 = -H_1$. Thus (1) becomes

$$H_1 g + H_1 g = i.$$

$$\Longrightarrow H_1 = +\frac{i}{2g}, H_2 = -\frac{i}{2g} \tag{2}$$

(c) Alternatively, rather than applying Ampère's law to the path $1 - 2 - 3 - 4 - 1$, use Gauss's law applied to the surface in the Figure 1 to obtain $H_2 = -H_1$ (Recall that $B = \mu_0 H$ in the air gap).

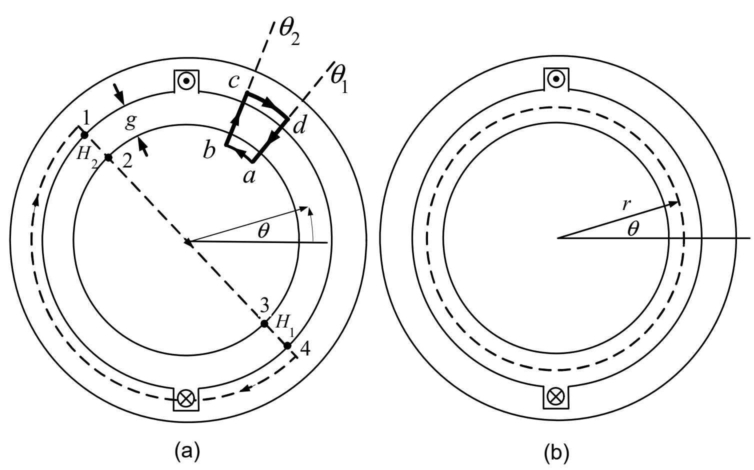


Figure 1: Single loop stator phase. (a) Paths for Ampere's law. (b) Flux surface in the air gap. The closed flux surface completely contains the rotor.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_0^{\ell_1} \int_{-\pi/2}^{\pi/2} \mu_0 H_1 \hat{\mathbf{r}} \cdot (r d\theta d\ell \hat{\mathbf{r}}) + \int_0^{\ell_1} \int_{\pi/2}^{3\pi/2} \mu_0 H_2 \hat{\mathbf{r}} \cdot (r d\theta d\ell \hat{\mathbf{r}}) = 0$$

$$\mu_0 H_1 \pi r \ell_1 + \mu_0 H_2 \pi r \ell_1 = 0$$

$$H_2 = -H_1$$

(d) From (2) it follows that

$$\vec{\mathbf{B}}(\theta) = \begin{cases} +\dfrac{\mu_0 i}{2g}\hat{\mathbf{r}} & \text{for } -\pi/2 < \theta < \pi/2 \\ \\ -\dfrac{\mu_0 i}{2g}\hat{\mathbf{r}} & \text{for } \pi/2 < \theta < 3\pi/2. \end{cases}$$

(e) With

$$\vec{\mathbf{B}}(r,\theta) = \begin{cases} +\dfrac{r_R}{r}\dfrac{\mu_0 i}{2g}\hat{\mathbf{r}} & \text{for } -\pi/2 < \theta < \pi/2 \\ \\ -\dfrac{r_R}{r}\dfrac{\mu_0 i}{2g}\hat{\mathbf{r}} & \text{for } \pi/2 < \theta < 3\pi/2 \end{cases}$$

and a closed flux surface in the air gap as shown in Figure 2, it follows by conservation of flux that

$$\begin{aligned}
\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} &= \int_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \int_{S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \\
&= \int_0^{\ell_1} \int_{\theta_1}^{\theta_2} (B_{Sa}(\theta)\hat{\mathbf{r}}) \cdot (r_1 d\theta d\ell(-\hat{\mathbf{r}})) + \int_0^{\ell_1} \int_{\theta_1}^{\theta_2} (B_{Sa}(\theta)\hat{\mathbf{r}}) \cdot (r_2 d\theta d\ell \hat{\mathbf{r}}) \\
&= \int_0^{\ell_1} \int_{\theta_1}^{\theta_2} \left(\mu_0 \frac{r_R}{r_1}\frac{i_{Sa}}{2g}\hat{\mathbf{r}}\right) \cdot (r_1 d\theta d\ell(-\hat{\mathbf{r}})) + \int_0^{\ell_1} \int_{\theta_1}^{\theta_2} \left(\mu_0 \frac{r_R}{r_2}\frac{i_{Sa}}{2g}\hat{\mathbf{r}}\right) \cdot (r_2 d\theta d\ell \hat{\mathbf{r}}) \\
&= -\mu_0 r_R \frac{i_{Sa}}{2g}(\theta_2 - \theta_1) + \mu_0 r_R \frac{i_{Sa}}{2g}(\theta_2 - \theta_1) \\
&= 0.
\end{aligned}$$

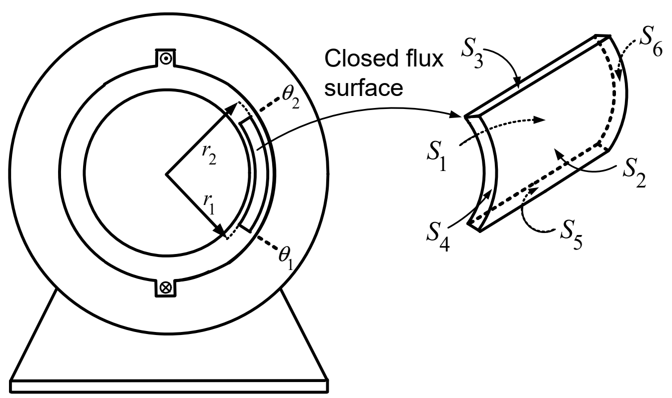


Figure 2: Closed flux surface to show the $1/r$ dependence of the magnetic field in the airgap.

(f) Let the phase currents i_{Sa} and i_{Sb} be periodic with period T as given in Figure 3

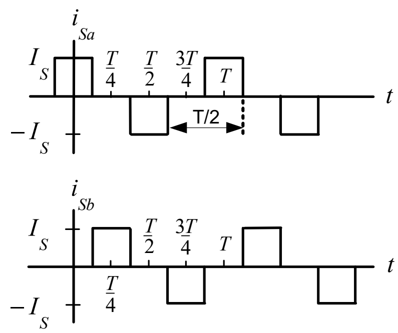


Figure 3: Periodic square wave phase currents.

As shown in the Figure 4 the magnetic axis only rotates to the discrete angular positions $\theta_S = 0$ to $\pi/2$ to π to $3\pi/2$ and then back to 2π.

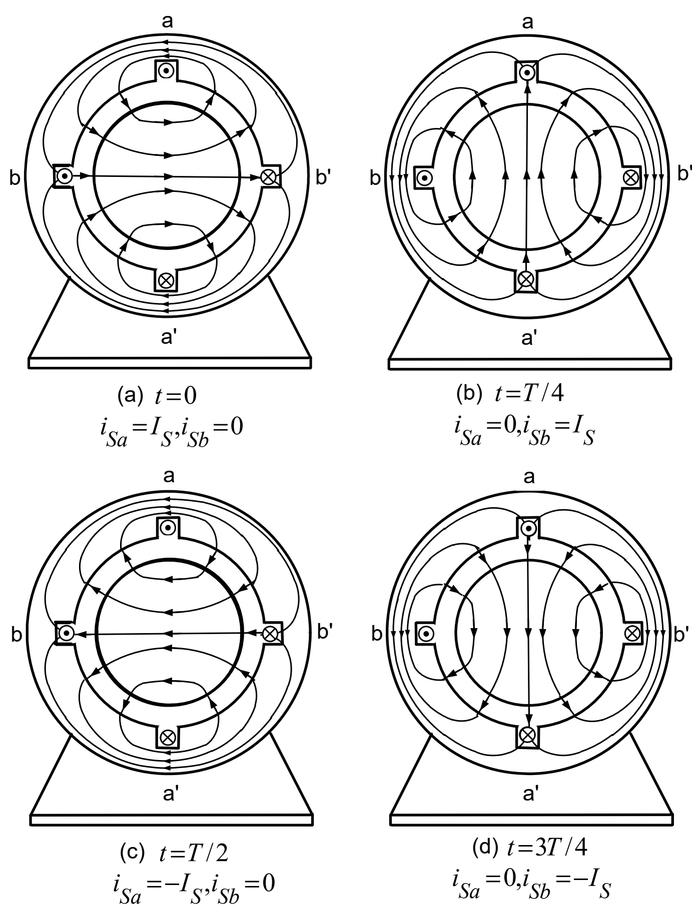


Figure 4: Rotating magnetic field in a machine with a simple two-loop stator.

(g) Let the phase currents i_{Sa} and i_{Sb} be periodic with period T as given in Figure 5.

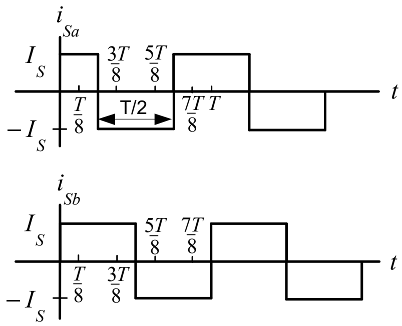


Figure 5: Periodic square wave phase currents.

As shown in the Figure 6 the magnetic axis only rotates to the discrete angular positions $\theta_S = 0$ to $\pi/4$ to $3\pi/4$ to $5\pi/4$ and then back to $\pi/4$.

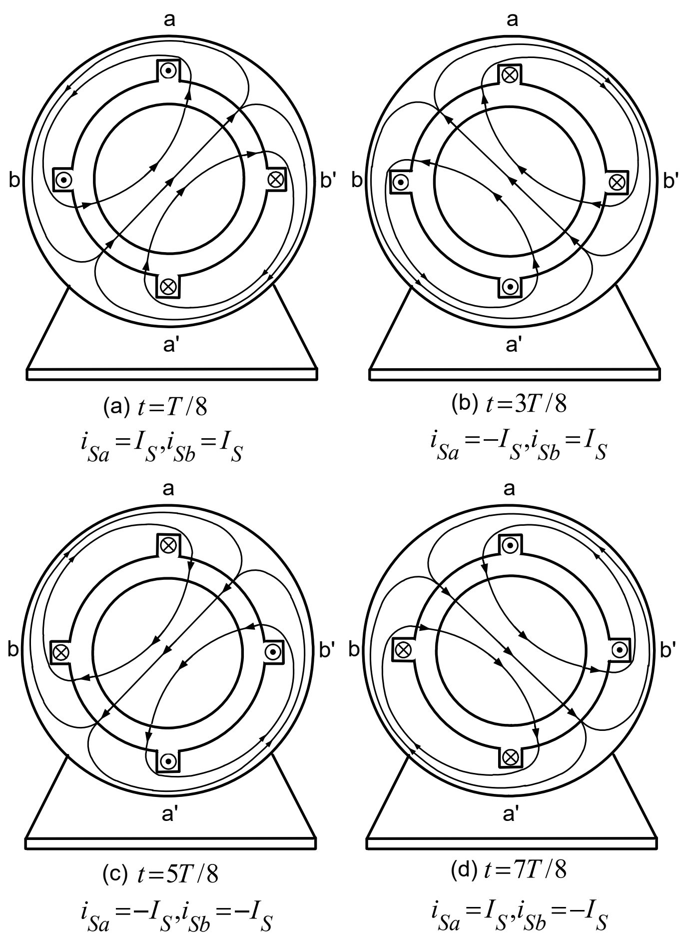


Figure 6: Rotating magnetic field in a machine with a simple two-loop stator.

Problem 2 *Fourier Series Expansion*

(a) As $F'(\theta')$ is an odd function, the Fourier expansion will only have sine terms.

$$F'(\theta') = \sum_{k=1,2,3,\cdots}^{\infty} a_k \sin(k\theta')$$
$$\Longrightarrow \int_0^{2\pi} F'(\theta')\sin(n\theta')d\theta' = \sum_{k=1,2,3,\cdots}^{\infty} a_k \int_0^{2\pi} \sin(n\theta')\sin(k\theta')d\theta'$$
$$\Longrightarrow 2\int_{\theta_1'}^{\pi-\theta_1'} F_1 \sin(n\theta')d\theta' = a_n \int_0^{2\pi} \sin(n\theta')\sin(n\theta')d\theta' = a_n\pi$$

$$a_n = \frac{2}{\pi}\int_{\theta_1'}^{\pi-\theta_1'} F_1 \sin(n\theta')d\theta' = -\frac{2F_1}{\pi n}\cos(n\theta')\Big|_{\theta_1'}^{\pi-\theta_1'} = -\frac{2F_1}{\pi n}\Big(\cos(n\left(\pi-\theta_1'\right) - \cos(n\theta_1')\Big)$$
$$= -\frac{2F_1}{\pi n}\Big((\cos(n\pi)\cos(\theta_1') - \cos(n\theta_1')\Big) = \frac{2F_1}{\pi n}\cos(n\theta_1')(1-(-1)^n)$$

That is,

$$F'(\theta') = \frac{4}{\pi}F_1\left(\sum_{k=1,3,5,\cdots}^{\infty}\frac{1}{k}\cos(k\theta_1')\sin(k\theta')\right).$$

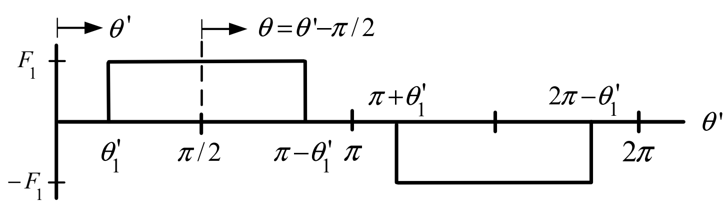


Figure 7: $F'(\theta')$ vs θ'

(b) With respect to the angle $\theta \triangleq \theta' - \pi/2$, this becomes

$$F(\theta) \triangleq F'(\theta+\pi/2) = \sum_{k=1,3,5,\cdots}^{\infty} \frac{4}{\pi}F_1\frac{1}{k}\cos(k\theta_1')\sin(k(\theta+\frac{\pi}{2})).$$

(c) Based on parts (a) and (b), it follows that

$$F(\theta) = \sum_{k=1,3,5,\cdots}^{\infty} \frac{4}{\pi}\frac{1}{k}\left(F_1\cos(k\theta_1') + F_2\cos(k\theta_2') + F_3\cos(k\theta_3')\right)\sin(k(\theta+\frac{\pi}{2})).$$

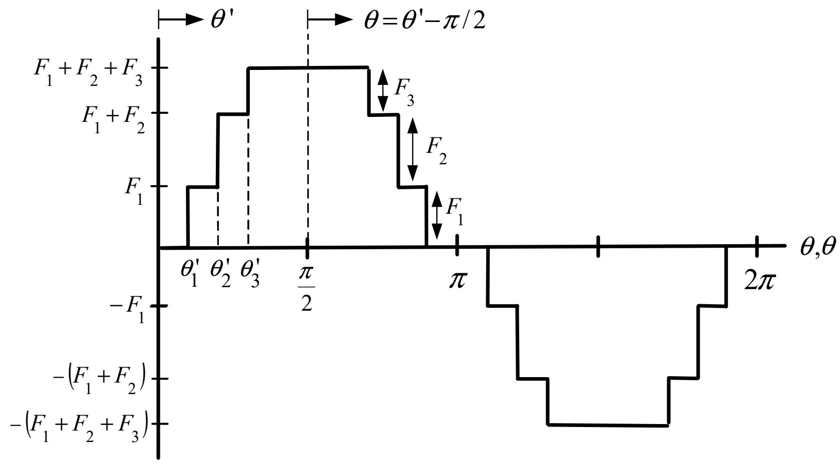


Figure 8: $F(\theta), F'(\theta')$

(d) For $k = 6m - 1$ and using

$$\sin(k(\theta + \pi/2)) = \sin(k\theta)\cos(k\pi/2) + \cos(k\theta)\sin(k\pi/2)$$

we have

$$\begin{aligned}\cos((6m-1)\pi/2) &= \cos(3m\pi)\cos(\pi/2) + \sin(3m\pi)\sin(\pi/2) = \sin(3m\pi) = 0 \\ \sin((6m-1)\pi/2) &= \sin(3m\pi)\cos(\pi/2) - \cos(3m\pi)\sin(\pi/2) = -\cos(3m\pi) = (-1)^{m+1}\end{aligned}$$

so that

$$\sin(k(\theta + \pi/2)) = (-1)^{m+1}\cos(k\theta).$$

Similarly, for $k = 6m + 1$ we have

$$\sin(k(\theta + \pi/2)) = (-1)^{m}\cos(k\theta).$$

Problem 3 *Two-Level Approximate Sinusoidal Winding*

With $F_1 = \mu_0 i_{Sa}/g, F_2 = \mu_0 i_{Sa}/g, \theta_1' = \theta_1 + \pi/2 = -\pi/2 + \pi/2 = 0, \theta_2' = \theta_2 + \pi/2 = -\pi/3 + \pi/2 = \pi/6$, the Fourier expansion is

$$\begin{aligned}F(\theta) &= \frac{4}{\pi}\sum_{k=1,3,5,\cdots}^{\infty}\frac{1}{k}\left(\frac{\mu_0 i_{Sa}}{g}\cos(0) + \frac{\mu_0 i_{Sa}}{g}\cos(k\pi/6)\right)\sin(k(\theta + \frac{\pi}{2})) \\ &= \frac{\mu_0 i_{Sa}}{g}\frac{4}{\pi}\sum_{k=1,3,5,\cdots}^{\infty}\frac{1}{k}\left(1 + \cos(k\pi/6)\right)\sin(k(\theta + \frac{\pi}{2})).\end{aligned}$$

Problem 4 *Three-Level Approximate Sinusoidal Winding*

(a) The enclosed current is

$$i_{\text{enclosed}}(\theta) = \begin{cases} 0 & \text{for} & 0 & < \theta < & \pi/6 \\ i_{Sa} & \text{for} & \pi/6 & < \theta < & \pi/3 \\ 3i_{Sa} & \text{for} & \pi/3 & < \theta < & \pi/2 \\ 6i_{Sa} & \text{for} & \pi/2 & < \theta < & 2\pi/3 \\ 8i_{Sa} & \text{for} & 2\pi/3 & < \theta < & 5\pi/6 \\ 9i_{Sa} & \text{for} & 5\pi/6 & < \theta < & 7\pi/6 \\ 8i_{Sa} & \text{for} & 7\pi/6 & < \theta < & 4\pi/3 \\ 6i_{Sa} & \text{for} & 4\pi/3 & < \theta < & 3\pi/2 \\ 3i_{Sa} & \text{for} & 3\pi/2 & < \theta < & 10\pi/6 \\ i_{Sa} & \text{for} & 10\pi/6 & < \theta < & 11\pi/6 \\ 0 & \text{for} & 11\pi/6 & < \theta < & 2\pi. \end{cases}$$

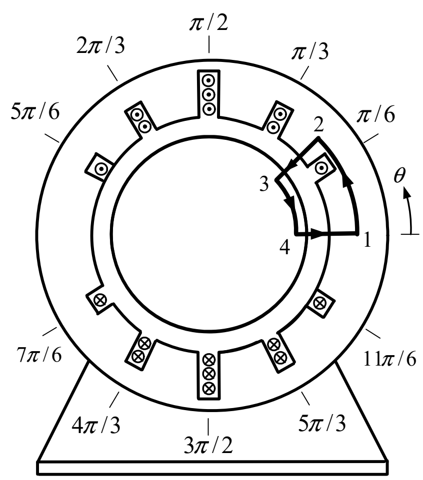


Figure 9: An approximate sinusoidally distributed winding.

(b) The magnetic intensity is then $H_{Sa}(\theta) = H_{Sa}(0) - i_{\text{enclosed}}(\theta)/g$ and conservation of flux requires

$$\int_0^{2\pi} \left(\mu_0 H_{Sa}(0) - \mu_0 \frac{i_{\text{enclosed}}(\theta)}{g} \right) d\theta = 0.$$

As a result,

$$H_{Sa}(0) = \int_0^{2\pi} \frac{i_{\text{enclosed}}(\theta)}{2\pi g} d\theta = \frac{1}{2\pi g} i_{Sa} \left(\frac{\pi}{6} + 3\frac{\pi}{6} + 6\frac{\pi}{6} + 8\frac{\pi}{6} + 9\frac{2\pi}{6} + 8\frac{\pi}{6} + 6\frac{\pi}{6} + 3\frac{\pi}{6} + \frac{\pi}{6} \right) = \frac{i_{Sa}}{g}\frac{9}{2}.$$

The radial magnetic field is then

$$\tilde{\mathbf{B}}_{Sa} = B_{Sa}(r,\theta)\hat{\mathbf{r}} = \mu_0 \frac{r_R}{r} \left(\frac{i_{Sa}}{g}\frac{9}{2} - \frac{i_{\text{enclosed}}(\theta)}{g} \right) \hat{\mathbf{r}}.$$

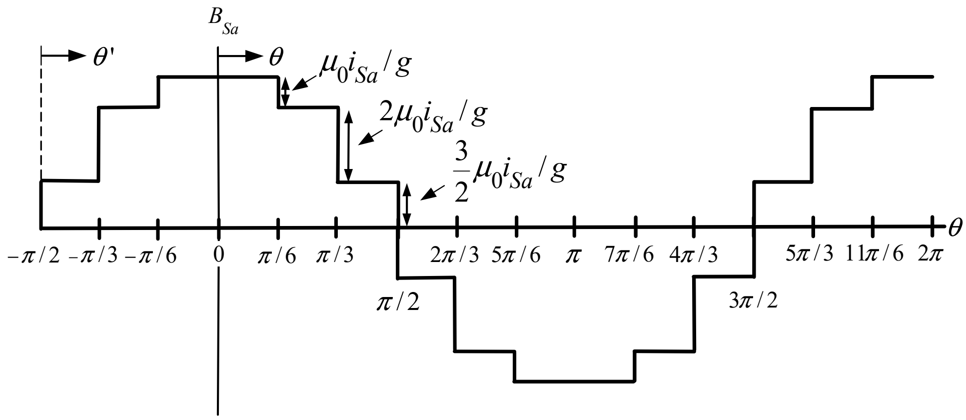


Figure 10: $B_{Sa}(\theta)$ for the three level winding distribution.

(c) With $B_{Sa}(r,\theta) = \mu_0(r_R/r)H_{Sa}(\theta)$, the Fourier series expansion of $B_{Sa}(r,\theta)$ in θ is

$$B_{Sa}(r,\theta) = \mu_0 \frac{i_{Sa}}{g}\frac{4}{\pi}\frac{r_R}{r} \times \sum_{k=1,3,5,\ldots}^{\infty} \frac{1}{k}\left(\frac{3}{2}\cos(0) + 2\cos(k\pi/6) + \cos(k\pi/3)\right)\sin\left(k(\theta+\pi/2)\right).$$

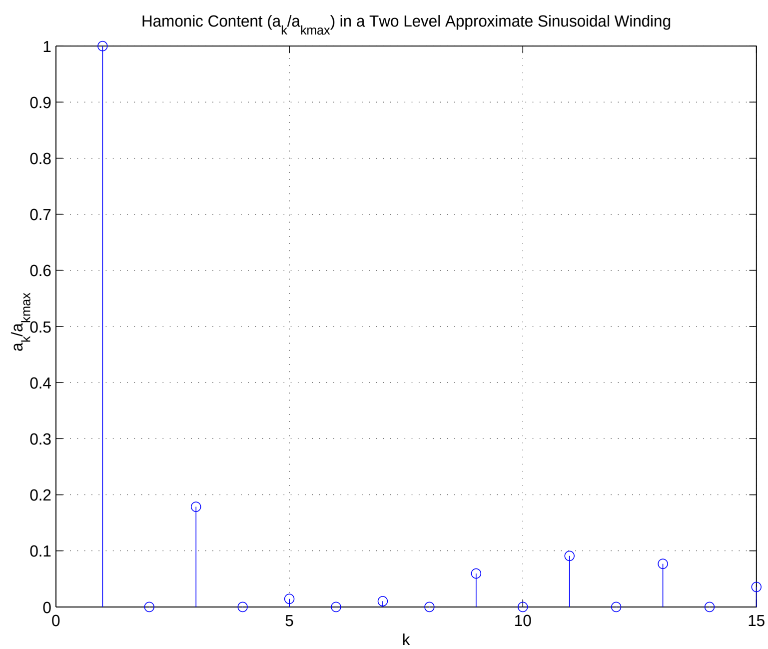
From the text,

$$a_1 = \mu_0 \frac{i_{Sa}}{g}\frac{4}{\pi}\left(1+\cos(\pi/6)\right)$$

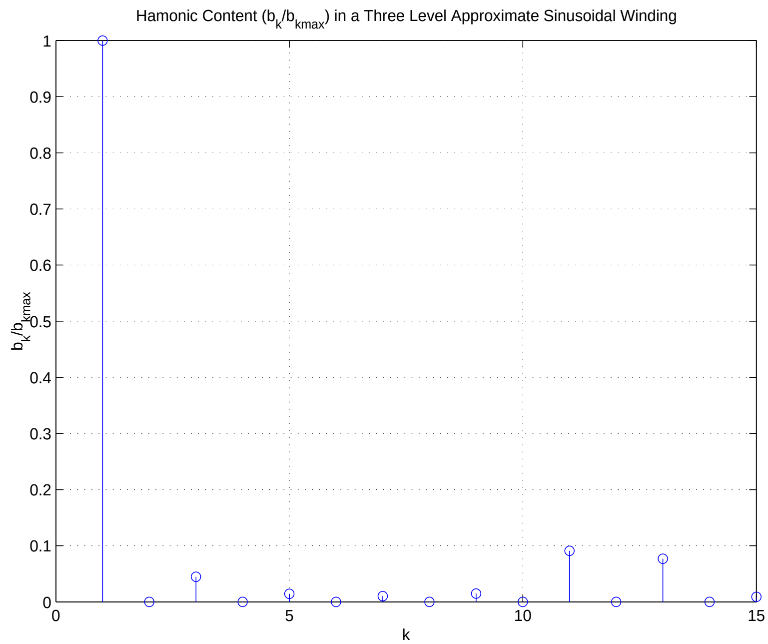
so that

$$\frac{b_1}{a_1} = \frac{\frac{3}{2}\cos(0) + 2\cos(\pi/6) + \cos(\pi/3)}{1+\cos(\pi/6)} = \frac{2+\sqrt{3}}{1+\sqrt{3}/2} = 2.$$

(d)



$a_k/a_{k\,\max}$ versus harmonic number for the two-level winding.



$b_k/b_{k\,\max}$ versus harmonic number for the three-level winding.

(e) As the plot below shows, the three-level winding has smaller triplen harmonics (relative to maximum).

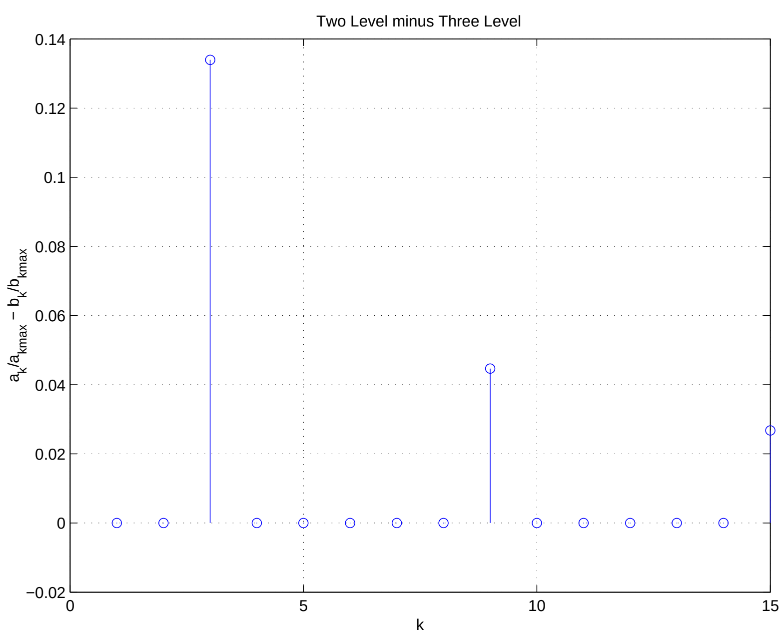


Figure 11: $a_k/a_{k\,\max} - b_k/b_{k\,\max}$ versus harmonic number.

See the simulation files for the .m file to generate the above plots.

(f) Figure 12 shows the stator phase b is identical in structure (assuming the cross sectional area of the windings are negligible) to stator phase a as in Figure 9, but rotated $\pi/2$ radians from it.

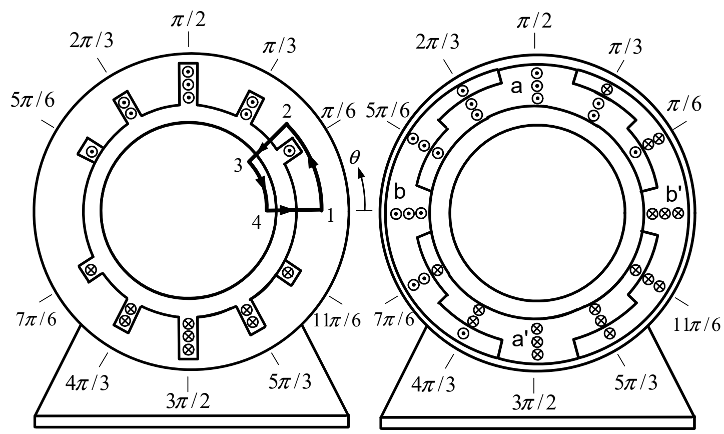


Figure 12: Approximately sinusoidally wound two phase stator.

So, replacing θ by $\theta - \pi/2$ and i_{Sa} by i_{Sb} in the solution given in part (c), the radial magnetic field in the air gap due to the current i_{Sb} in stator phase b is given by

$$B_{Sb}(r,\theta) = \mu_0 \frac{i_{Sb}}{g} \frac{4}{\pi} \frac{r_R}{r} \times \sum_{k=1,3,5,\ldots}^{\infty} \frac{1}{k} \left(\frac{3}{2} \cos(0) + 2\cos(k\pi/6) + \cos(k\pi/3) \right) \sin(k\theta).$$

Problem 5 *Gauss's Law*

The magnetic field due to the current in stator phase a is

$$\vec{\mathbf{B}}_{Sa}(r,\theta) = B_{Sa}(r,\theta)\hat{\mathbf{r}} = \mu_0 \frac{r_R}{r} \left(\frac{2i_{Sa}}{g} - \frac{i_{\text{enclosed}}(\theta)}{g} \right) \hat{\mathbf{r}}.$$

Let the surface normals on the closed surface given by

$$d\vec{\mathbf{S}} = \begin{cases} -r_1 d\theta dz \hat{\mathbf{r}} & \text{for} \quad S_1 \\ r_2 d\theta dz \hat{\mathbf{r}} & \text{for} \quad S_2 \\ dr dz \hat{\boldsymbol{\theta}} & \text{for} \quad S_3 \\ r d\theta dr \hat{\mathbf{z}} & \text{for} \quad S_4 \\ -dr dz \hat{\boldsymbol{\theta}} & \text{for} \quad S_5 \\ -r d\theta dr \hat{\mathbf{z}} & \text{for} \quad S_6. \end{cases}$$

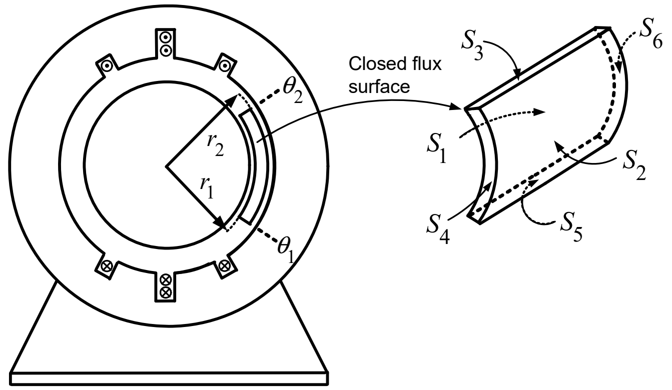


Figure 13: Closed flux surface to show the $1/r$ dependence of the magnetic field in the airgap.

By conservation of flux it follows that

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \int_{S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$$

$$\int_0^{\ell_1} \int_{\theta_1}^{\theta_2} (B_{Sa}(\theta)\hat{\mathbf{r}}) \cdot (r_1 d\theta d\ell(-\hat{\mathbf{r}})) + \int_0^{\ell_1} \int_{\theta_1}^{\theta_2} (B_{Sa}(\theta)\hat{\mathbf{r}}) \cdot (r_2 d\theta d\ell\hat{\mathbf{r}}) = 0.$$

Expanding this becomes

$$0 = \int_0^{\ell_1} \int_{\theta_1}^{\theta_2} \left(\mu_0 \frac{r_R}{r_1} \left(\frac{2i_{Sa}}{g} - \frac{i_{\text{enclosed}}(\theta)}{g}\right) \hat{\mathbf{r}}\right) \cdot (r_1 d\theta d\ell(-\hat{\mathbf{r}})) + \int_0^{\ell_1} \int_{\theta_1}^{\theta_2} \left(\mu_0 \frac{r_R}{r_2} \left(\frac{2i_{Sa}}{g} - \frac{i_{\text{enclosed}}(\theta)}{g}\right) \hat{\mathbf{r}}\right) \cdot (r_2 d\theta d\ell\hat{\mathbf{r}})$$

or, carrying out the integrations,

$$-\mu_0 r_R \left(\frac{2i_{Sa}}{g} - \frac{i_{\text{enclosed}}(\theta)}{g}\right)(\theta_2 - \theta_1) + \mu_0 r_R \left(\frac{2i_{Sa}}{g} - \frac{i_{\text{enclosed}}(\theta)}{g}\right)(\theta_2 - \theta_1) = 0$$

and conservation of flux is satisfied for that surface.

Problem 6 *Three-Phase Machine and Space Harmonics*

Problem 7 *Self-Inductance of a Distributed Winding*

The stator phase winding shown in Figure 14 produces the air gap radial magnetic field due shown in Figure 15.

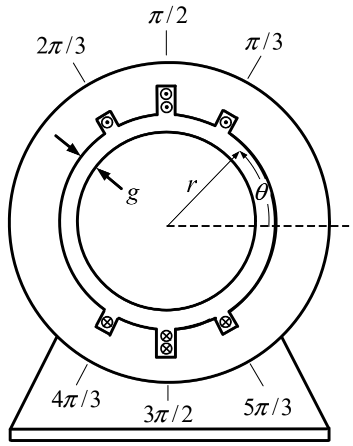


Figure 14: Computation of the self inductance of a distributed winding.

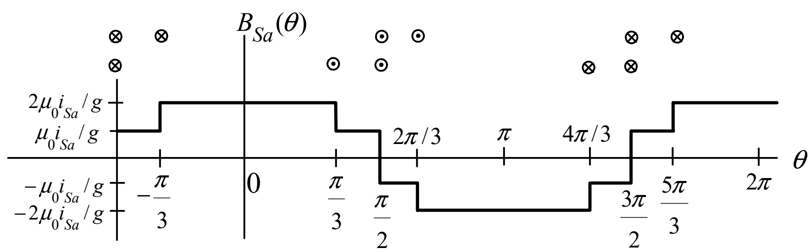


Figure 15: Magnetic field $B_{Sa}(\theta)$ vs. θ due to the current in phase a of a 2-phase machine.

(a) Using the usual half-cylinder flux surface with the outward normal so that the positive direction of travel coincides with the positive direction of current, and referring back to the plot of B_{Sa} versus θ, it follows that

$$\phi_{\pi/3} = \int_0^{\ell_1}\int_{\pi/3-\pi}^{\pi/3} (B_{Sa}(\theta)\hat{\mathbf{r}})\cdot(r_S d\theta dz\hat{\mathbf{r}}) = \frac{\mu_0 \ell_1 r_S}{g}\int_{\pi/3-\pi}^{\pi/3} B_{Sa}(\theta)d\theta = \frac{\mu_0 i_{Sa}}{g}\ell_1 r_S\left(-\frac{\pi}{6}+\frac{\pi}{6}+2\frac{2\pi}{3}\right)$$
$$= \frac{\mu_0 \ell_1 r_S}{g}\frac{4\pi}{3}i_{Sa}$$

$$\phi_{\pi/2} = \int_0^{\ell_1}\int_{\pi/2-\pi}^{\pi/2} (B_{Sa}(\theta)\hat{\mathbf{r}})\cdot(r_S d\theta dz\hat{\mathbf{r}}) = \frac{\mu_0 \ell_1 r_S}{g}\int_{\pi/2-\pi}^{\pi/2} B_{Sa}(\theta)d\theta = \frac{\mu_0 i_{Sa}}{g}\ell_1 r_S\left(\frac{\pi}{6}+2\frac{2\pi}{3}+\frac{\pi}{6}\right)$$
$$= \frac{\mu_0 \ell_1 r_S}{g}\frac{5\pi}{3}i_{Sa}$$

$$\phi_{2\pi/3} = \int_0^{\ell_1}\int_{2\pi/3-\pi}^{2\pi/3} (B_{Sa}(\theta)\hat{\mathbf{r}})\cdot(r_S d\theta dz\hat{\mathbf{r}}) = \frac{\mu_0 \ell_1 r_S}{g}\int_{2\pi/3-\pi}^{2\pi/3} B_{Sa}(\theta)d\theta = \frac{\mu_0 i_{Sa}}{g}\ell_1 r_S\left(2\frac{2\pi}{3}+\frac{\pi}{6}-\frac{\pi}{6}\right)$$
$$= \frac{\mu_0 \ell_1 r_S}{g}\frac{4\pi}{3}i_{Sa}.$$

The flux linkage is then

$$\lambda_{Sa} = \phi_{\pi/3} + 2\phi_{\pi/2} + \phi_{2\pi/3} = \frac{6\pi\mu_0 \ell_1 r_S}{g} i_{Sa}.$$

(b) The self-inductance of the stator phase is then

$$L = \frac{6\pi\mu_0 \ell_1 r_S}{g}.$$

Problem 8 *Flux Linkage*

Figure 16 shows a motor with a permanent magnet rotor and a single stator phase. The magnetic field in the air gap due to the permanent magnet rotor is given by

$$\vec{\mathbf{B}}_R(\theta - \theta_R) = B_{\max} \frac{r_R}{r} \cos(\theta - \theta_R)\hat{\mathbf{r}}.$$

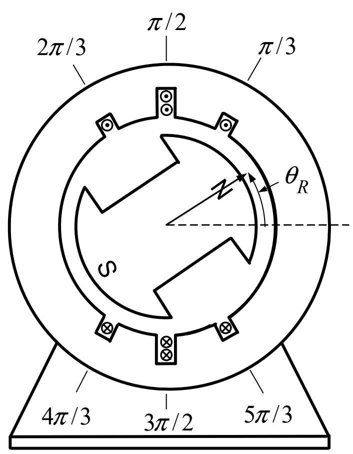


Figure 16: A motor with a PM rotor and a single rotor phase.

(a) Let

$$d\vec{\mathbf{S}} = r_S d\theta d\ell \hat{\mathbf{r}} \tag{3}$$

where r_S is the radius of the inside surface of the stator iron. Note that with this choice of $d\vec{\mathbf{S}}$, the positive direction of travel around the loop coincides with the positive direction chosen for the current in that loop. On the inside surface of the rotor, $r = r_S$ so that the flux in the loop whose sides are in the slots at $\theta = \pi/3$ and $\theta = \pi/3 - \pi$ is

$$\phi_{\pi/3} = \int\limits_{\substack{\text{Loop from} \\ \pi/3-\pi \text{ to } \pi/3}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = \int_0^{\ell_1} \int_{\theta=\pi/3-\pi}^{\theta=\pi/3} B_{\max} \frac{r_R}{r_S} \cos(\theta - \theta_R)\hat{\mathbf{r}} \cdot (r_S d\theta d\ell \hat{\mathbf{r}})$$

$$= \ell_1 r_R \int_{\theta=\pi/3-\pi}^{\theta=\pi/3} B_{\max} \cos(\theta - \theta_R) d\theta = \ell_1 r_R B_{\max} \sin(\theta - \theta_R) d\theta\Big|_{\theta=\pi/3-\pi}^{\theta=\pi/3}$$

$$= 2\ell_1 r_R B_{\max} \sin(\pi/3 - \theta_R).$$

Again using the outward normal so that $d\vec{\mathbf{S}}$ is still given by equation (3) as before, the flux in each of the two loops between $-\pi/2$ to $\pi/2$ is

$$\phi_{\pi/2} = \int\limits_{\substack{\text{Loop from} \\ -\pi/2 \text{ to } \pi/2}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = 2\ell_1 r_R B_{\max} \sin(\pi/2 - \theta_R).$$

Finally, with $d\vec{\mathbf{S}}$ given by (3) once more,

$$\phi_{2\pi/3} = \int\limits_{\substack{\text{Loop from}\\ 2\pi/3-\pi \text{ to } 2\pi/3}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = 2\ell_1 r_R B_{\max} \sin(2\pi/3 - \theta_R).$$

The total flux linkage is then

$$\begin{aligned}
\lambda_{S1} &= \phi_{\pi/3} + 2\phi_{\pi/2} + \phi_{2\pi/3} \\
&= 2\ell_1 r_R B_{\max} \sin(\pi/3 - \theta_R) + 2 \times 2\ell_1 r_R B_{\max} \sin(\pi/2 - \theta_R) + 2\ell_1 r_R B_{\max} \sin(2\pi/3 - \theta_R) \\
&= 2\ell_1 r_R B_{\max} \Big(\sin(\pi/3 - \theta_R) + 2\sin(\pi/2 - \theta_R) + \sin(2\pi/3 - \theta_R)\Big) \\
&= \left(2 + \sqrt{3}\right) 2\ell_1 r_R B_{\max} \cos\theta_R.
\end{aligned}$$

(b) The induced emf in this phase is then

$$-\frac{d\lambda_{S1}}{dt} = \left(2 + \sqrt{3}\right) 2\ell_1 r_R B_{\max} \omega_R \sin(\theta_R).$$

Problem 9 *A Two-Phase Machine*

Problem 10 *Uniformly Distributed Winding*

The magnetic field $\vec{\mathbf{B}}_{S1}$ produced by the stator currents is now derived. As illustrated in Figure 17, the stator winding density of a BLDC is uniform over the inner periphery of the stator. Specifically, stator phase 1 has a winding density given by

$$N_{S1}(\theta) = \begin{cases} \dfrac{N_S}{\pi/3} & \text{for } \dfrac{\pi}{3} \le \theta \le \dfrac{2\pi}{3} \text{ and } \dfrac{4\pi}{3} \le \theta \le \dfrac{5\pi}{3} \\ 0 & \text{elsewhere.} \end{cases}$$

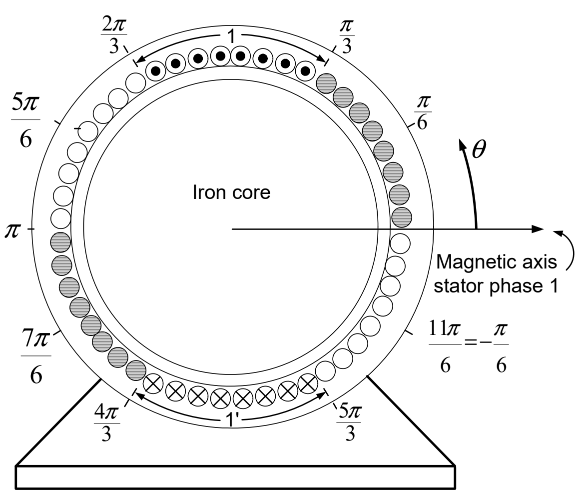


Figure 17: Uniformly distributed windings.

The total number of windings (turns or loops) making up phase 1 is then

$$\int_{\pi/3}^{2\pi/3} \frac{N_S}{\pi/3} d\theta = N_S.$$

Similarly, $N_{S2}(\theta) = N_{S1}(\theta - 2\pi/3)$ and $N_{S3}(\theta) = N_{S1}(\theta - 4\pi/3)$. To determine the radial magnetic field in the air gap produced by the current in stator phase 1, Ampère's law, along with $\vec{\mathbf{H}} \equiv 0$ in the iron, is applied to the path $1-2-3-1$ of Figure 18 to obtain (by symmetry, $H_{S1}(i_{S1}, \pi + \theta) = -H_{S1}(i_{S1}, \theta)$)

$$2gH_{S1}(i_{S1}, \theta) = \begin{cases} N_S i_{S1} & \text{for} \quad -\pi/3 \le \theta \le \pi/3 \\ \displaystyle\int_{\theta}^{2\pi/3} \frac{N_S i_{S1}}{\pi/3} d\theta - \int_{4\pi/3}^{\theta+\pi} \frac{N_S i_{S1}}{\pi/3} d\theta = \frac{N_S i_{S1}}{\pi/3} 2\left(\frac{\pi}{2} - \theta\right) & \text{for} \quad \pi/3 \le \theta \le 2\pi/3 \\ -N_S i_{S1} & \text{for} \quad 2\pi/3 \le \theta \le 4\pi/3 \\ \displaystyle\int_{\theta}^{5\pi/3} -\frac{N_S i_{S1}}{\pi/3} d\theta + \int_{7\pi/3}^{\theta+\pi} \frac{N_S i_{S1}}{\pi/3} d\theta = \frac{N_S i_{S1}}{\pi/3} 2\left(\theta - \frac{3\pi}{2}\right) & \text{for} \quad 4\pi/3 \le \theta \le 5\pi/3. \end{cases}$$

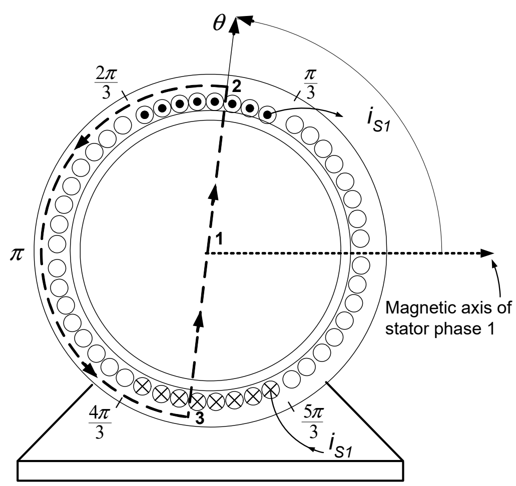


Figure 18: Path $1-2-3-1$ for applying Ampere's Law. Note that for the path drawn, $\pi/3 \le \theta \le 2\pi/3$.

$\vec{\mathbf{H}}_{S1}(i_{S1}, \theta)$ is then given by

$$\vec{\mathbf{H}}_{S1}(i_{S1}, \theta) = \begin{cases} N_S i_{S1} \dfrac{1}{2g} \hat{\mathbf{r}} & \text{for} \quad -\pi/3 \le \theta \le \pi/3 \\ N_S i_{S1} \dfrac{1}{2g} \dfrac{6}{\pi} \left(\dfrac{\pi}{2} - \theta\right) \hat{\mathbf{r}} & \text{for} \quad \pi/3 \le \theta \le 2\pi/3 \\ -N_S i_{S1} \dfrac{1}{2g} \hat{\mathbf{r}} & \text{for} \quad 2\pi/3 \le \theta \le 4\pi/3 \\ N_S i_{S1} \dfrac{1}{2g} \dfrac{6}{\pi} \left(\theta - \dfrac{3\pi}{2}\right) \hat{\mathbf{r}} & \text{for} \quad 4\pi/3 \le \theta \le 5\pi/3. \end{cases}$$

It was assumed that $\vec{\mathbf{H}}_{S1}$ is constant across the air gap. Also, $\vec{\mathbf{B}}_{S1} = \mu_0 \vec{\mathbf{H}}_{S1}$ in the air gap. A factor of r_R/r is now included to ensure that $\vec{\mathbf{B}}_{S1}$ satisfies conservation of flux in the air gap. Consequently, the magnetic field $\vec{\mathbf{B}}_{S1} = B_{S1}(i_{S1}, r, \theta)\hat{\mathbf{r}}$ at any point (r, θ) in the air gap due to the current i_{S1} in phase 1 is given by

$$B_{S1}(i_{S1}, r, \theta)\hat{\mathbf{r}} = \begin{cases} \dfrac{\mu_0 N_S i_{S1}}{2g} \dfrac{r_R}{r} \hat{\mathbf{r}} & \text{for} \quad -\pi/3 \le \theta \le \pi/3 \\ \dfrac{\mu_0 N_S i_{S1}}{2g} \dfrac{r_R}{r} \dfrac{6}{\pi} \left(\dfrac{\pi}{2} - \theta\right) \hat{\mathbf{r}} & \text{for} \quad \pi/3 \le \theta \le 2\pi/3 \\ -\dfrac{\mu_0 N_S i_{S1}}{2g} \dfrac{r_R}{r} \hat{\mathbf{r}} & \text{for} \quad 2\pi/3 \le \theta \le 4\pi/3 \\ \dfrac{\mu_0 N_S i_{S1}}{2g} \dfrac{r_R}{r} \dfrac{6}{\pi} \left(\theta - \dfrac{3\pi}{2}\right) \hat{\mathbf{r}} & \text{for} \quad 4\pi/3 \le \theta \le 5\pi/3. \end{cases} \tag{4}$$

A plot of the normalized radial magnetic field $B_{S1}(i_{S1}, r, \theta)/\left(\frac{\mu_0 N_S i_{S1}}{2g} \frac{r_R}{r}\right)$ due to the current in stator phase 1 for $-\pi/3 \le \theta \le 2\pi$ is shown in Figure 19.

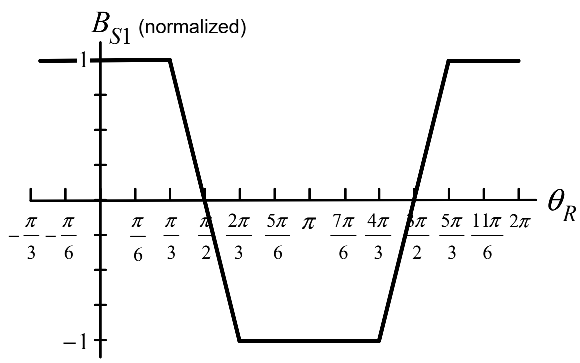


Figure 19: Normalized radial magnetic field $B_{S1}(i_{S1}, r, \theta)/\left(\frac{\mu_0 N_S i_{S1}}{2g} \frac{r_R}{r}\right)$ for $-\pi/3 \le \theta \le 2\pi$