Modeling and High-Performance Control of Electric Machines

Chapter 4 Rotating Magnetic Fields

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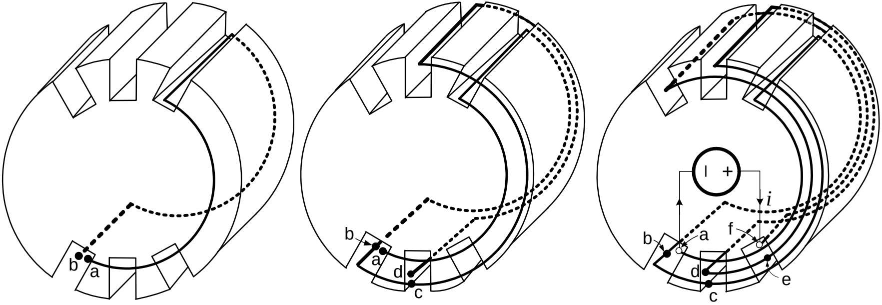
Rotating Magnetic Fields

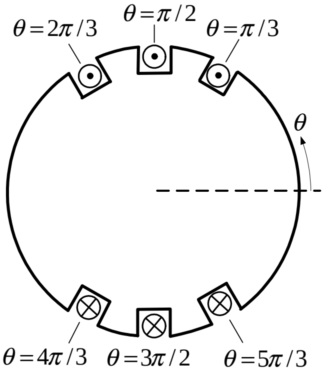
- **Distributed Windings**
- **Approximate Sinusoidally Distributed $\vec{\mathbf{B}}$ Field**
- **Sinusoidally Wound Phases**
- **Sinusoidally Distributed Magnetic Fields**
- **Magnetomotive Force (mmf)**
- **Flux Linkage**
- **Azimuthal Magnetic Field in the Air Gap***

Distributed Windings



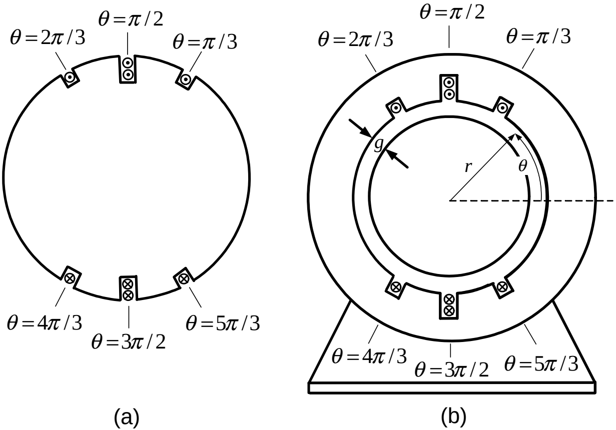
- A **single** length of wire is wound around the core.
- This single wire is wound to make **3 loops (windings/turns/coils)**.
- The 1^{st} (half-cylindrical shape) loop is from a to b.
- The 2^{nd} loop is from c to d.
- The 3^{rd} loop is form e to f.
- This single wire with 3 loops is called a **phase winding**.
- The semi-circular sides of each loop are referred to as **end turns.**
- This is a **distributed** winding as the loops are not all in a single pair of slots.

Distributed Windings - Cross-Sectional View



- The top three slots are at $\theta = \pi/3$, $\theta = \pi/2$, and $\theta = 2\pi/3$.
- The bottom three slots are at $\theta = 4\pi/3$, $\theta = 3\pi/2$, and $\theta = 5\pi/3$.
- $i > 0$ if it is coming out of the top 3 slots and into the bottom 3 slots.

Distributed Windings - Cross-Sectional View

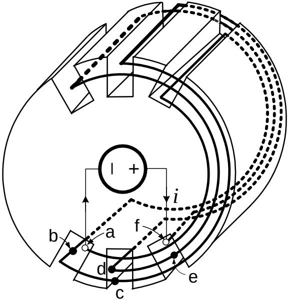


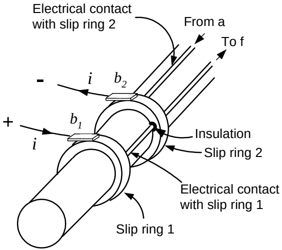
(a) A single **rotor** phase similar to before except that **two loops** are wound in the middle slots.

(b) A single **stator** phase with the slots in the inside surface of the stator iron.

- The radial air gap distance is denoted as g.
- An arbitrary point is located using polar coordinates (r, θ).

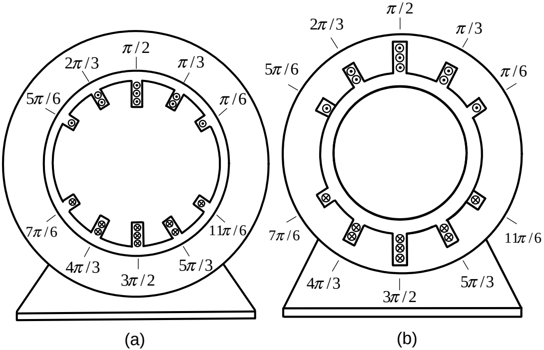
Slip Rings to Bring Electrical Power into the Rotor





- Slip rings 1 and 2 are conducting material and rigidly connected to the rotor.
- The brushes b_1 and b_2 are **fixed** in space.
- The brushes b_1 and b_2 make sliding contact with the slip rings.
- Slip ring 1 is electrically connected point f of the rotor phase.
- Slip ring 2 is electrically connected to point a of the rotor phase.
- Voltage source connected to brushes b_1 and b_2.

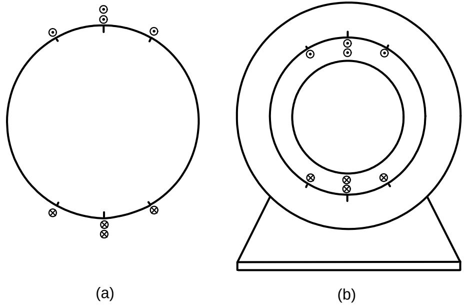
More Distributed Windings



(a) A single **rotor** phase with a distributed winding.
(b) A single **stator** phase with a distributed winding.

- The point of using distributed windings is so that their currents create a **radial** magnetic field in the air gap that is **sinusoidally** distributed in θ.
- This is explained next!

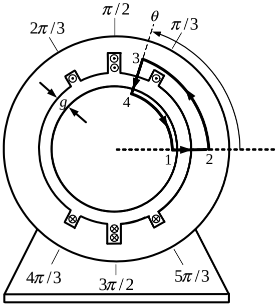
Approximate Sinusoidally-Distributed $\vec{B}$ Field

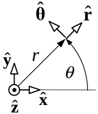


(a) Idealized rotor windings: Slots and wire inside has **zero** cross-section.
(b) Idealized stator windings: Slots and wire inside has **zero** cross-section.

- Compute the **radial $\vec{B}$** field **in the air gap** created by the current in a distributed winding.
- Ampère's law $\oint \vec{\mathbf{H}} \cdot d\vec{\ell} = i_{\text{enclosed}}$ is the **key tool** to do this.
- Our first idealization is that the slots and wire inside them have **zero** cross-section.

Approximate Sinusoidally-Distributed $\vec{\mathbf{B}}$ Field





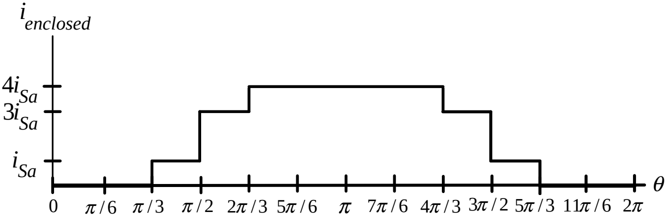
Stator phase a has current i_{Sa}.
Take $\vec{\mathbf{H}} \equiv 0$ in the **iron**.

$$\begin{aligned}
\int_1^2 \vec{\mathbf{H}}_{Sa} \cdot d\vec{\ell} + \int_2^3 \vec{\mathbf{H}}_{Sa} \cdot d\vec{\ell} + \int_3^4 \vec{\mathbf{H}}_{Sa} \cdot d\vec{\ell} + \int_4^1 \vec{\mathbf{H}}_{Sa} \cdot d\vec{\ell} &= i_{\text{enclosed}} \\
\int_1^2 \vec{\mathbf{H}}_{Sa} \cdot d\vec{\ell} + \int_3^4 \vec{\mathbf{H}}_{Sa} \cdot d\vec{\ell} &= i_{\text{enclosed}} \\
\int_1^2 H_{Sa}(0)\hat{\mathbf{r}} \cdot d\ell\hat{\mathbf{r}} + \int_3^4 H_{Sa}(\theta)\hat{\mathbf{r}} \cdot (-d\ell\hat{\mathbf{r}}) &= i_{\text{enclosed}} \\
gH_{Sa}(0) - gH_{Sa}(\theta) &= i_{\text{enclosed}}.
\end{aligned}$$

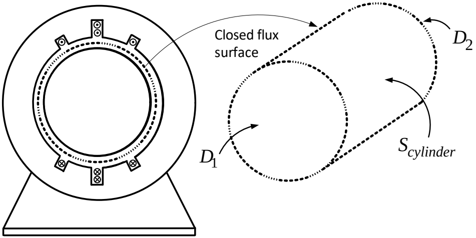
Approximate Sinusoidally-Distributed $\vec{B}$ Field

$$H_{Sa}(\theta) = H_{Sa}(0) - \frac{i_{\text{enclosed}}(\theta)}{g}.$$

$$i_{\text{enclosed}} = \begin{cases} 0 & \text{for} \quad 0 < \theta < \pi/3 \\ i_{Sa} & \text{for} \quad \pi/3 < \theta < \pi/2 \\ 3i_{Sa} & \text{for} \quad \pi/2 < \theta < 2\pi/3 \\ 4i_{Sa} & \text{for} \quad 2\pi/3 < \theta < 4\pi/3 \\ 3i_{Sa} & \text{for} \quad 4\pi/3 < \theta < 3\pi/2 \\ i_{Sa} & \text{for} \quad 3\pi/2 < \theta < 5\pi/3 \\ 0 & \text{for} \quad 5\pi/3 < \theta < 2\pi \end{cases}$$



Compute $H_{Sa}(0)$ using Conservation of Flux



$\vec{\mathbf{B}}_{Sa} = \mu_0 \vec{\mathbf{H}}_{Sa}$ in air.
$\vec{\mathbf{B}}_{Sa} = B_{Sa}\hat{\mathbf{r}} = \mu_0 (H_{Sa}(0) - i_{\text{enclosed}}(\theta)/g)\hat{\mathbf{r}}$ on the cylindrical surface.

$$\begin{aligned} 0 = \oint \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}} &= \underbrace{\int_{D_1} \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}} + \int_{D_2} \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}}}_{=0} + \int_{S_{cylinder}} \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}} \\ &= \int_{S_{cylinder}} \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}}. \end{aligned}$$

Then

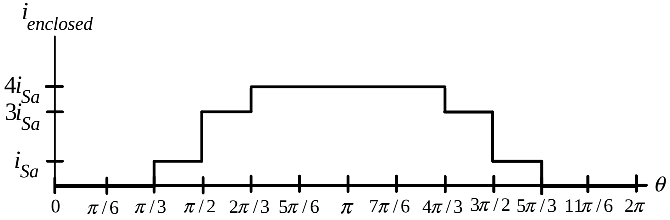
$$\int_{S_{cylinder}} \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}} = \int_0^{\ell_1} \int_0^{2\pi} (B_{Sa}(\theta)\hat{\mathbf{r}}) \cdot (rd\theta dz\hat{\mathbf{r}}) = r\ell_1 \int_0^{2\pi} B_{Sa}(\theta) d\theta = 0$$

or

$$\int_0^{2\pi} \mu_0 \underbrace{(H_{Sa}(0) - i_{\text{enclosed}}(\theta)/g)} d\theta = 0.$$

Compute $H_{Sa}(0)$ using Conservation of Flux

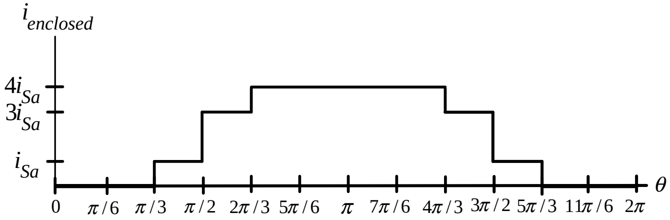
$$\begin{aligned} H_{Sa}(0) = \int_0^{2\pi} \frac{i_{\text{enclosed}}(\theta)}{2\pi g} d\theta &= \frac{1}{2\pi g}\left(i_{Sa}\frac{\pi}{6} + 3i_{Sa}\frac{\pi}{6} + 4i_{Sa}\frac{2\pi}{3} + 3i_{Sa}\ \frac{\pi}{6} + i_{Sa}\frac{\pi}{6}\right) \\ &= \frac{2i_{Sa}}{g}. \end{aligned}$$



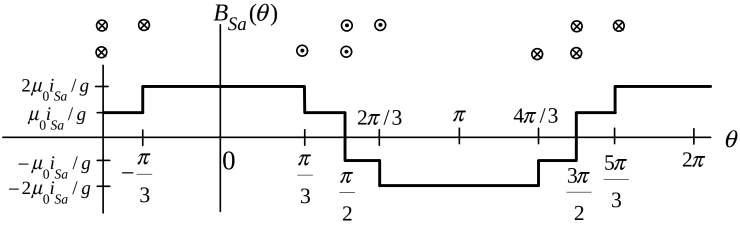
So

$$\vec{\mathbf{B}}_{Sa} = B_{Sa}(\theta)\hat{\mathbf{r}} = \mu_0\left(\frac{2i_{Sa}}{g} - \frac{i_{\text{enclosed}}(\theta)}{g}\right)\hat{\mathbf{r}}.$$

$$\vec{\mathbf{B}}_{Sa} = B_{Sa}(\theta)\hat{\mathbf{r}}$$



$$\vec{\mathbf{B}}_{Sa} = B_{Sa}(\theta)\hat{\mathbf{r}} = \mu_0\left(\frac{2i_{Sa}}{g} - \frac{i_{\text{enclosed}}(\theta)}{g}\right)\hat{\mathbf{r}}.$$



Approximate Sinusoidally-Distributed $\vec{\mathrm{B}}$ Field

Expanding $B_{Sa}(\theta)$ in a **Fourier series**

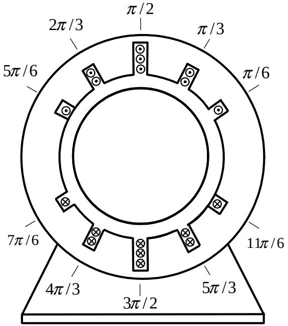
$$
\begin{aligned}
B_{Sa}(\theta) &= \mu_0 \frac{i_{Sa}}{g}\frac{4}{\pi} \times \sum_{k=1,3,5,\ldots}^{\infty} \frac{\left(1+\cos\left(\frac{k\pi}{6}\right)\right)}{k} \sin\left(k\left(\theta+\frac{\pi}{2}\right)\right) \\
&= \mu_0 \frac{i_{Sa}}{g}\frac{4}{\pi}\left(\frac{2+\sqrt{3}}{2}\sin\left(\theta+\frac{\pi}{2}\right)+\frac{1}{3}\sin\left(3\left(\theta+\frac{\pi}{2}\right)\right)+\frac{1-\sqrt{3}/2}{5}\sin\left(5\left(\theta+\frac{\pi}{2}\right)\right)+\cdots\right. \\
&= \mu_0 \frac{i_{Sa}}{g}\frac{4}{\pi}\left(1.866\cos(\theta) - 0.333\cos(3\theta) + 0.0268\cos(5\theta) \mp \cdots\right.
\end{aligned}
$$

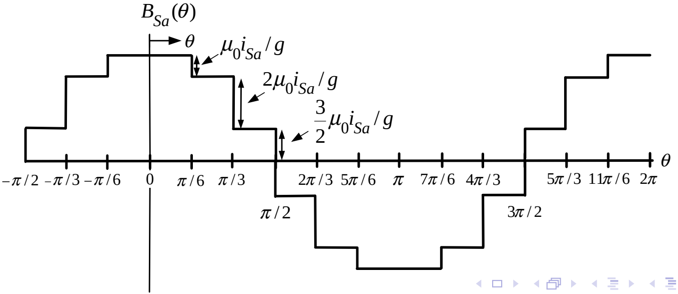
To a first approximation,

$$
\vec{\mathbf{B}}_{Sa}(\theta) = B_{Sa}(\theta)\hat{\mathbf{r}} \approx 1.866\mu_0 \frac{i_{Sa}}{g}\frac{4}{\pi}\cos(\theta)\hat{\mathbf{r}}
$$

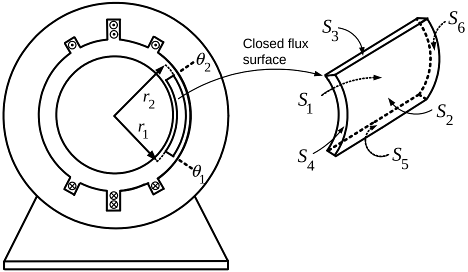
- This is a **sinusoidal distribution** in θ.

2^{nd} Example: Approximate Sinusoidally-Distributed $\vec{B}$ Field





Conservation of Flux and $1/r$ Dependence of $\vec{\mathbf{B}}$



$\vec{\mathbf{B}}$ has only a **radial** component so $\vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$ on S_3, S_4, S_5, S_6.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \int_{S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$$

$$\int_0^{\ell_1} \int_{\theta_1}^{\theta_2} (B_{Sa}(\theta)\hat{\mathbf{r}}) \cdot (r_1 d\theta dz(-\hat{\mathbf{r}})) + \int_0^{\ell_1} \int_{\theta_1}^{\theta_2} (B_{Sa}(\theta)\hat{\mathbf{r}}) \cdot (r_2 d\theta dz\hat{\mathbf{r}}) = 0$$

$$-\ell_1 r_1 \int_{\theta_1}^{\theta_2} B_{Sa}(\theta) d\theta + \ell_1 r_2 \int_{\theta_1}^{\theta_2} B_{Sa}(\theta) d\theta = 0$$

$$\ell_1 (r_2 - r_1) \int_{\theta_1}^{\theta_2} B_{Sa}(\theta) d\theta = 0$$

As $r_1 \neq r_2$, conservation of flux does **no**t hold.

Conservation of Flux and $1/r$ Dependence of $\vec{\mathbf{B}}$

- B_{Sa} was assumed **constant** across the air gap.
- In fact B_{Sa} **must vary** as $1/r$ to satisfy conservation of flux.
- To satisfy $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \equiv 0$, **replace** $B_{Sa}(\theta)$ by

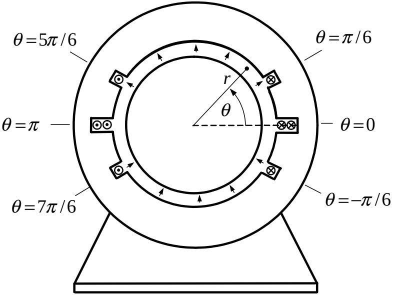
$$B_{Sa}(r,\theta) \triangleq \frac{r_R}{r} B_{Sa}(\theta) = \mu_0 \frac{r_R}{r}\left(\frac{2i_{Sa}}{g} - \frac{i_{\text{enclosed}}(\theta)}{g}\right)\hat{\mathbf{r}}.$$

- The air gap g is **assumed** to be **small** so

$$\frac{r_R}{r} \approx 1 \text{ for } r_R \le r \le r_S = r_R + g.$$

- The factor $\frac{r_R}{r}$ does not really change the value of $\vec{\mathbf{B}}$ in the air gap.

Magnetic Field Distribution due to i_{Sa} and i_{Sb}



Stator phase b is rotated 90° from phase a.
Compute $\vec{\mathbf{B}}_{Sb}$ from $\vec{\mathbf{B}}_{Sa}$ by **replacing** i_{Sa} by i_{Sb} and θ by $\theta - \pi/2$ (See slide 14).

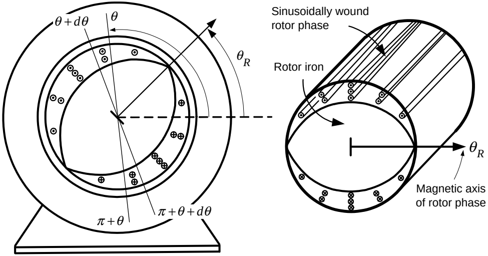
$$\vec{\mathbf{B}}_{Sb}(r,\theta) = B_{Sb}(r,\theta)\hat{\mathbf{r}} = \mu_0 \frac{i_{Sb}}{g}\frac{r_R}{r}\frac{4}{\pi}\sum_{k=1,3,5,\ldots}^{\infty}\frac{(1+\cos(k\pi/6))}{k}\sin(k\theta)\hat{\mathbf{r}} \approx 1.866\mu_0\frac{i_{Sb}}{g}\frac{r_R}{r}\frac{4}{\pi}\sin(\theta)\hat{\mathbf{r}}.$$

Summarizing:

$$\vec{\mathbf{B}}_{Sa}(r,\theta) = B_{Sa}(r,\theta)\hat{\mathbf{r}} \approx 1.866\mu_0\frac{i_{Sa}}{g}\frac{4}{\pi}\frac{r_R}{r}\cos(\theta)\hat{\mathbf{r}}$$

$$\vec{\mathbf{B}}_{Sb}(r,\theta) = B_{Sb}(r,\theta)\hat{\mathbf{r}} \approx 1.866\mu_0\frac{i_{Sb}}{g}\frac{4}{\pi}\frac{r_R}{r}\sin(\theta)\hat{\mathbf{r}}.$$

Sinusoidally Wound Phase



(a) (b)

Sinusoidal turns density

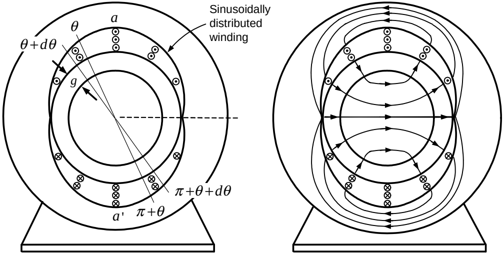
- A **single** strand of wire (**phase winding**) is wrapped around the cylindrical iron core.
- The number of **coils/loops/turns** per angular distance is

$$N_{Ra}(\theta-\theta_R)=\frac{N_R}{2}\sin(\theta-\theta_R)\ \text{ for } \theta_R<\theta<\pi+\theta_R.$$

  I.e., the number of the loops (turns) between θ and $\theta+d\theta$ is $N_{Ra}(\theta-\theta_R)d\theta$.
- The **total number** of turns is

$$\int_{\theta_R}^{\theta_R+\pi}N_{Ra}(\theta-\theta_R)d\theta=N_R.$$

Schematic Representation of a Sinusoidally Wound Phase



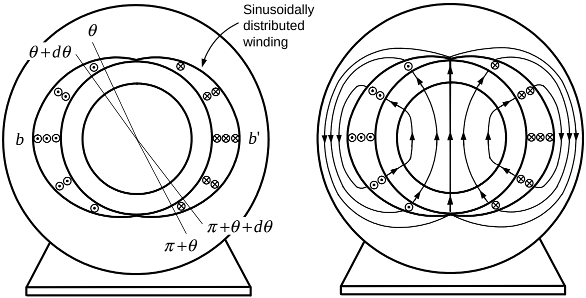
- Slots are not shown and the turns are shown enclosed in a **sine curve envelope**.
- **Sinusoidal turns density**

$$N_{Sa}(\theta) = \frac{N_S}{2}\sin(\theta) \text{ for } 0 < \theta < \pi$$

I.e., the number of the turns between θ and $\theta + d\theta$ is $N_{Sa}(\theta)d\theta$.

- The **total** number of turns making up stator phase a is $\int_0^\pi N_{Sa}(\theta)d\theta = N_S$.
- The **cross-sectional area** of the turns is taken to be **zero**.

Stator Phase b

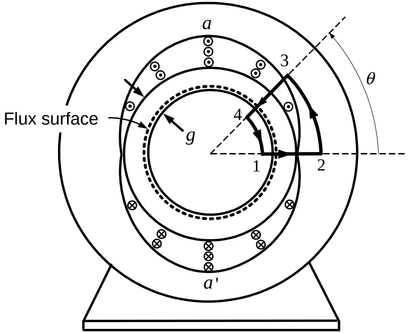


- **Stator phase** b is identical in structure to phase a and rotated 90° with respect to phase a.
- **Sinusoidal turns density**

$$N_{Sb}(\theta) = \frac{N_S}{2}\sin(\theta - \pi/2) \quad \text{for } \pi/2 < \theta < 3\pi/2.$$

- The number of turns between θ and $\theta + d\theta$ is $N_{Sb}(\theta)d\theta = (N_S/2)\sin(\theta - \pi/2)d\theta$.
- The **total** number of turns making up stator phase b is $\int_{\pi/2}^{3\pi/2} N_{Sb}(\theta)d\theta = N_S$.
- The **cross-sectional area** of the turns is taken to be **zero**.

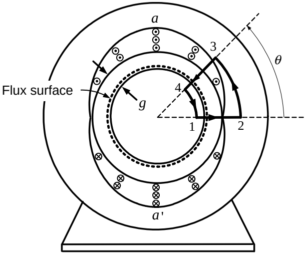
Sinusoidally Distributed Magnetic Fields



Compute $\vec{\mathbf{B}}_{Sa}$ **in the air gap** created by the current i_{Sa}.

$$\begin{aligned}
\oint \vec{\mathbf{H}}_{Sa} \cdot d\vec{\boldsymbol{\ell}} &= \int_0^{\theta} i_{Sa}(N_S/2)\sin(\theta')d\theta' \\
\int_1^2 \vec{\mathbf{H}}_{Sa} \cdot d\vec{\boldsymbol{\ell}} + \int_3^4 \vec{\mathbf{H}}_{Sa} \cdot d\vec{\boldsymbol{\ell}} &= \int_0^{\theta} i_{Sa}(N_S/2)\sin(\theta')d\theta' \\
\int_{\ell=0}^{\ell=g} H_{Sa}(i_{Sa},0)\hat{\mathbf{r}} \cdot (d\ell\hat{\mathbf{r}}) + \int_{\ell=0}^{\ell=g} H_{Sa}(i_{Sa},\theta)\hat{\mathbf{r}} \cdot (-d\ell\hat{\mathbf{r}}) &= -i_{Sa}\frac{N_S}{2}\cos(\theta) + i_{Sa}\frac{N_S}{2}
\end{aligned}$$

Sinusoidally Distributed Magnetic Fields



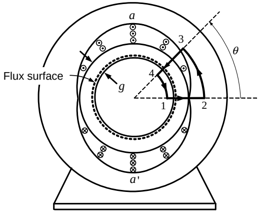
$$H_{Sa}(i_{Sa}, 0)g - H_{Sa}(i_{Sa}, \theta)g = -i_{Sa}\frac{N_S}{2}\cos(\theta) + i_{Sa}\frac{N_S}{2}$$

or

$$H_{Sa}(i_{Sa}, \theta) = i_{Sa}\frac{N_S}{2g}\cos(\theta) + H_{Sa}(i_{Sa}, 0) - i_{Sa}\frac{N_S}{2g}.$$

- Both $H_{Sa}(i_{Sa}, \theta)$ and $H_{Sa}(i_{Sa}, 0)$ are unknown.
- Using conservation of flux to compute $H_{Sa}(i_{Sa}, 0)$.

Sinusoidally Distributed Magnetic Fields



As

$$0 = \oint_S \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}} = \underbrace{\int_{D_1} \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}} + \int_{D_2} \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}}}_{=0} + \int_{S_{cylinder}} \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}} = \int_{S_{cylinder}} \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}}$$

we have

$$\int_{S_{cylinder}} \vec{\mathbf{B}}_{Sa} \cdot d\vec{\mathbf{S}} = \int_0^{\ell_1} \int_0^{2\pi} B_{Sa}(i_{Sa},\theta)\hat{\mathbf{r}} \cdot (r_R d\theta dz \hat{\mathbf{r}}) = \ell_1 r_R \int_0^{2\pi} B_{Sa}(i_{Sa},\theta) d\theta = 0.$$

$B_{Sa}(i_{Sa},\theta) = \mu_0 H_{Sa}(i_{Sa},\theta)$ in the **air gap.**

$$\begin{aligned} 0 = \int_0^{2\pi} B_{Sa}(i_{Sa},\theta) d\theta &= \int_0^{2\pi} \mu_0 \left(i_{Sa} \frac{N_S}{2g} \cos(\theta) + H_{Sa}(i_{Sa},0) - i_{Sa} \frac{N_S}{2g} \right) d\theta \\ &= 2\pi\mu_0 \left(H_{Sa}(i_{Sa},0) - i_{Sa} \frac{N_S}{2g} \right). \end{aligned}$$

Sinusoidally Distributed Magnetic Fields

$$
\begin{aligned}
H_{Sa}(i_{Sa},\theta) &= \frac{N_S}{2g} i_{Sa}\cos(\theta) \\
B_{Sa}(i_{Sa},\theta) &= \frac{\mu_0 N_S}{2g} i_{Sa}\cos(\theta).
\end{aligned}
$$

- Applying Ampère's law, we assumed $\vec{\mathbf{B}} = \mu_0 \vec{\mathbf{H}}$ was **constant** across the air gap. I.e., $\vec{\mathbf{B}}$ did **not** depend on the cylindrical coordinate r.
- To satisfy $\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$, $\vec{\mathbf{B}}$ **must** decrease as $1/r$ in the air gap.
- H_{Sa}, B_{Sa} are modified by the factor r_R/r so that conservation of flux holds.

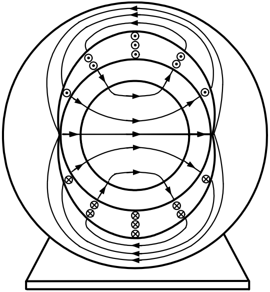
$\vec{\mathbf{B}}_{Sa}$ in the air gap due to i_{Sa} is

$$
\vec{\mathbf{B}}_{Sa}(i_{Sa}, r, \theta) = \frac{\mu_0 N_S r_R}{2g} \frac{i_{Sa}\cos(\theta)}{r}\hat{\mathbf{r}}.
$$

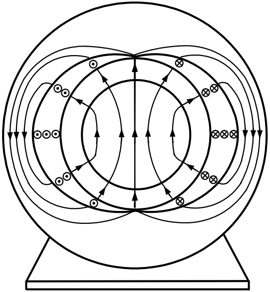
Similarly, for **stator phase** b

$$
\vec{\mathbf{B}}_{Sb}(i_{Sb}, r, \theta) = \frac{\mu_0 N_S r_R}{2g} \frac{i_{Sb}\cos(\theta - \pi/2)}{r}\hat{\mathbf{r}} = \frac{\mu_0 N_S r_R}{2g} \frac{i_{Sb}\sin(\theta)}{r}\hat{\mathbf{r}}.
$$

Sinusoidally Distributed Rotating Magnetic Field



(a) Phase a



(b) Phase b

(a) $\vec{\mathbf{B}}_{Sa}$ field lines due to the current i_{Sa} (drawn with $i_{Sa} > 0$).
(b) $\vec{\mathbf{B}}_{Sb}$ field lines due to the current i_{Sb} (drawn with $i_{Sb} > 0$).

Total magnetic field in the air gap:

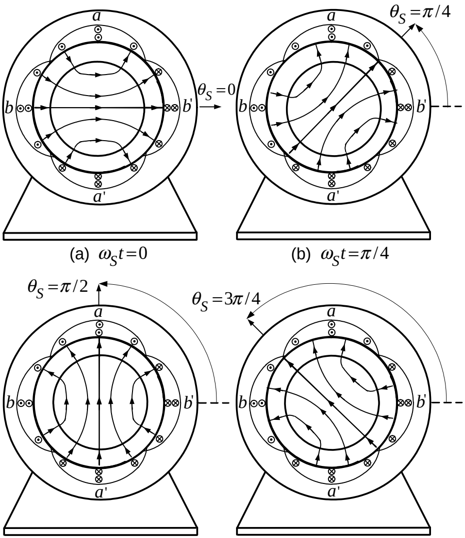
$$\vec{\mathbf{B}}_S(i_{Sa}, i_{Sb}, r, \theta) = \vec{\mathbf{B}}_{Sa}(i_{Sa}, r, \theta) + \vec{\mathbf{B}}_{Sb}(i_{Sb}, r, \theta) = \frac{\mu_0 r_R N_S}{2g}\frac{1}{r}\Big(i_{Sa}\cos(\theta) + i_{Sb}\sin(\theta)\Big)\hat{\mathbf{r}}.$$

Sinusoidally Distributed Rotating Magnetic Field

With $i_{Sa}(t) = I_S\cos(\omega_S t)$, $i_{Sb}(t) = I_S\sin(\omega_S t)$ and $\theta_S(t) \triangleq \omega_S t$

$$\begin{aligned}
\vec{\mathbf{B}}_S(r,\theta,t) &= \frac{\mu_0 r_R N_S I_S}{2g}\frac{1}{r}\Big(\cos(\omega_S t)\cos(\theta) + \sin(\omega_S t)\sin(\theta)\Big)\hat{\mathbf{r}} \\
&= \frac{\mu_0 r_R N_S I_S}{2g}\frac{1}{r}\cos(\theta - \omega_S t)\hat{\mathbf{r}} \\
&= \frac{\mu_0 r_R N_S I_S}{2g}\frac{1}{r}\cos\Big(\theta - \theta_S(t)\Big)\hat{\mathbf{r}}.
\end{aligned}$$

$$\vec{\mathbf{B}}_S(r,\theta,t) = \frac{\mu_0 r_R N_S I_S}{2g}\frac{1}{r}\cos\Big(\theta - \theta_S(t)\Big)\hat{\mathbf{r}} \quad \text{with} \quad \theta_S(t) \triangleq \omega_S t$$



- θ_S is the **magnetic axis** of $\vec{\mathbf{B}}_S(r,\theta,t)$.

Magnetomotive Force (mmf)

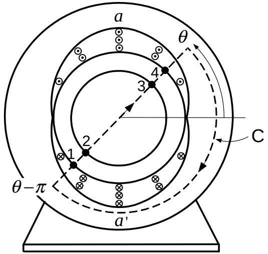
Definition Magnetomotive force (mmf)

The **magnetomotive force (mmf)** is defined to be

$$\Im \triangleq \oint_C \vec{\mathbf{H}} \cdot d\vec{\ell}.$$

- I.e., the **mmf** is the integral of $\vec{\mathbf{H}}$ around a **closed** curve.
- The value of $\Im$ depends on the particular closed-curve C.
- Of course, by Ampère's law, $\Im = \oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = i_{\text{enclosed}}$.
- Many books **incorrectly** consider $\Im$ to be a scalar field, i.e., it has a value at each point in space (like temperature).
- $\vec{\mathbf{H}}$ is a vector field as it has a value at each point in space.

Magnetomotive Force (mmf)



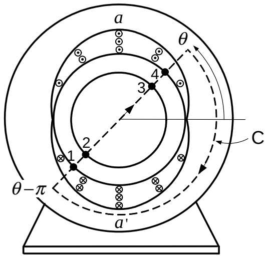
$$\Im \triangleq \oint_C \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} = \int_1^2 \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} + \int_3^4 \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} = \int_1^2 \underbrace{H(\theta - \pi)}_{-H(\theta)}\hat{\mathbf{r}} \cdot (-d\ell\hat{\mathbf{r}}) + \int_3^4 (H(\theta)\hat{\mathbf{r}})\cdot(d\ell\hat{\mathbf{r}})$$

$$= 2\int_3^4 (H(\theta)\hat{\mathbf{r}}) \cdot (d\ell\hat{\mathbf{r}})$$

$$= 2H(\theta)g.$$

- $i_{\text{enclosed}} = -\int_{\theta-\pi}^{\theta} i_{Sa}(N_S/2)\sin(\theta')d\theta' = i_{Sa}N_S\cos(\theta).$
- By Ampère's $\Im(\theta) = 2H(\theta)g = i_{Sa}N_S\cos(\theta).$

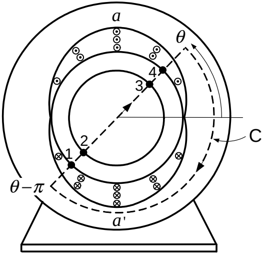
Magnetomotive Force (mmf)



The usual "interpretation" of mmf

- The mmf $\Im = 2H(\theta)g = i_{Sa}N_S\cos(\theta)$ is **"dropped"** across the air gap.
- The amount $\Im_1 = H(\theta)g$ is **"dropped"** across each of the two diametrically opposite sides of the air gap.
- It is then said that an **mmf** $\Im_1(\theta) = H(\theta)g$ is **"set up"** in the air gap.

Magnetomotive Force (mmf)



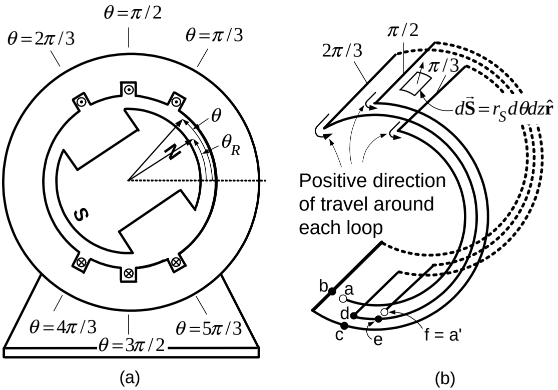
The correct interpretation

- Ampère's law is used to find $\vec{\mathbf{H}}$ in the air gap by using $\vec{\mathbf{H}} \equiv 0$ in the iron.
- $\vec{\mathbf{B}}$ in the air gap is found from $\vec{\mathbf{B}} = \mu_0 \vec{\mathbf{H}}$.
- The mmf is **only** used as a way to compute $\vec{\mathbf{B}}$ in the air gap.
- Noble Laureate Melvin Schwartz:

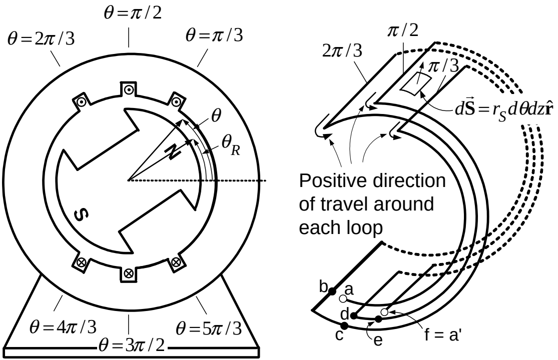
... we must interject a small bit of philosophy. It is customary to call $\vec{\mathbf{B}}$ *the magnetic induction and* $\vec{\mathbf{H}}$ *the magnetic field strength. We reject this custom inasmuch as* $\vec{\mathbf{B}}$ *is the truly fundamental field and* $\vec{\mathbf{H}}$ *is a subsidiary artifact. We shall call* $\vec{\mathbf{B}}$ *the magnetic field and leave the reader to deal with* $\vec{\mathbf{H}}$ *as he pleases.*

Flux Linkage

- $\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}$ is defined for a surface whose boundary is a **closed curve**.
- Faraday's law $\xi = -d\phi/dt$ then gives the induced emf (voltage) in the loop.
- **Phase windings** are comprised of **turns** distributed around the iron core surface.
- We want the **total emf** induced in the **phase winding**.
- **Flux linkage** is a convenient way to do this.



Flux Linkage



(a) (b)

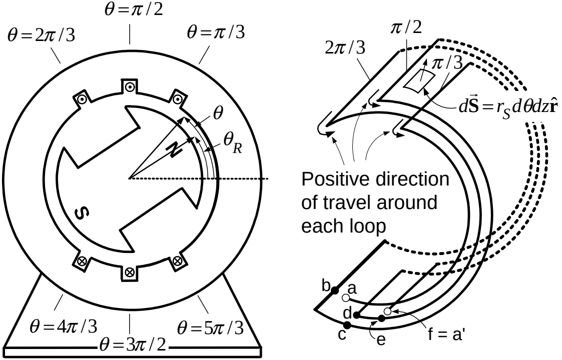
Phase $a - a'$ consists of 3 loops:

Loop 1: Path from a to b and placed in slots at $\theta = \pi/3$ and $\theta = \pi/3 - \pi$.

Loop 2: Path from c to d and place in slots $\theta = \pi/2$ and $\theta = 3\pi/2$.

Loop 3: Path from e to $f = a'$ and placed in slots $\theta = 2\pi/3$ and $\theta = 5\pi/3$.

Flux Linkage



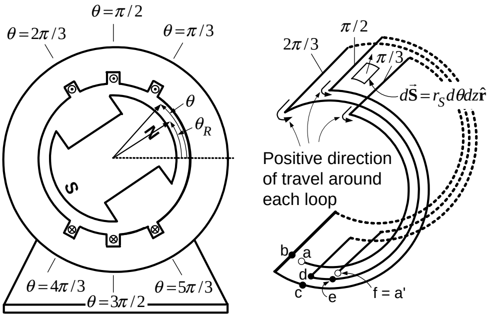
(a) (b)

A PM rotor produces a magnetic field in the air gap given by

$$\vec{\mathbf{B}}_R(\theta - \theta_R) = B_{\max}\frac{r_R}{r}\cos(\theta - \theta_R)\hat{\mathbf{r}}.$$

Compute the **total emf** induced in phase $a - a'$ as the PM rotates.
At any time the emfs will be **different** in each of the three loops.

Flux Linkage



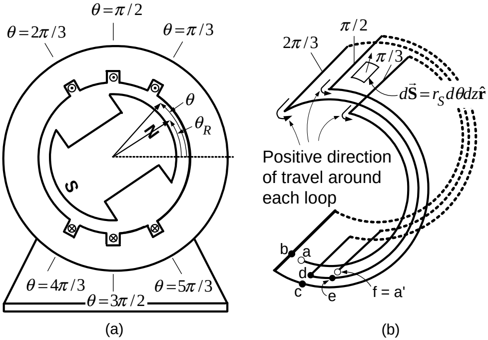
(a) (b)

$d\vec{\mathbf{S}} = r_S d\theta dz\hat{\mathbf{r}}$, r_S is the radius of the inside surface of the stator iron.

Flux in Loop 1:

$$\begin{aligned}
\phi_{\pi/3} = \int_0^{\ell_1}\int_{\theta=\pi/3-\pi}^{\theta=\pi/3} B_{\max}\frac{r_R}{r_S}\cos(\theta-\theta_R)\hat{\mathbf{r}}\cdot(r_S d\theta dz\hat{\mathbf{r}}) &= \ell_1 r_R \int_{\theta=\pi/3-\pi}^{\theta=\pi/3} B_{\max}\cos(\theta-\theta_R)d\theta \\
&= \ell_1 r_R B_{\max}\sin(\theta-\theta_R)d\theta\,\Big|_{\theta=\pi/3-\pi}^{\theta=\pi/3} \\
&= 2\ell_1 r_R B_{\max}\sin(\pi/3-\theta_R).
\end{aligned}$$

Flux Linkage

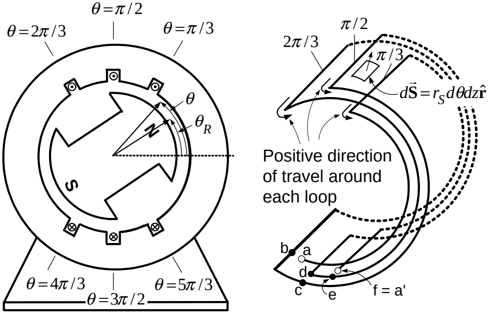


Emf induced in loop 1:

$$\xi_{\pi/3} = -\frac{d\phi_{\pi/3}}{dt} = 2\ell_1 r_R B_{\max}\omega_R \cos(\theta_R - \pi/3)$$

- If $\xi_{\pi/3} > 0$, this emf will force current to go in the **positive direction of travel**.
- This **coincides** with the positive direction of **current** in that loop.

Flux Linkage



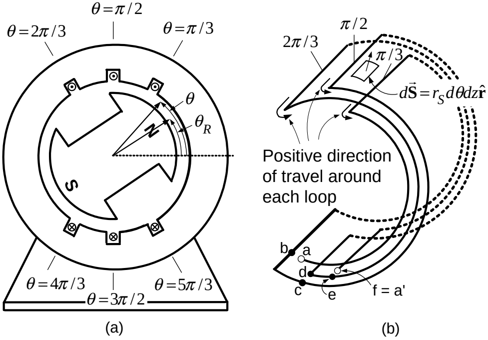
Similarly, for **loop 2** (sides between $-\pi/2$ to $\pi/2$)

$$\phi_{\pi/2} = \int_{\substack{\text{Loop from} \\ -\pi/2 \text{ to } \pi/2}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = 2\ell_1 r_R B_{\max} \sin(\pi/2 - \theta_R)$$

$$\xi_{\pi/2} = -\frac{d\phi_{\pi/2}}{dt} = 2\ell_1 r_R B_{\max} \omega_R \cos(\theta_R - \pi/2).$$

The positive direction of travel around the loop is the positive direction of current.

Flux Linkage



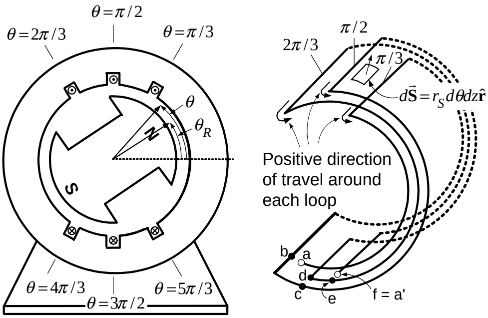
Finally, for **loop 3** (sides between $2\pi/3 - \pi$ and $2\pi/3$)

$$\phi_{2\pi/3} = \int_{\substack{\text{Loop from} \\ 2\pi/3-\pi \text{ to } 2\pi/3}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = 2\ell_1 r_R B_{\max} \sin(2\pi/3 - \theta_R)$$

$$\xi_{2\pi/3} = -\frac{d\phi_{2\pi/3}}{dt} = 2\ell_1 r_R B_{\max}\omega_R \cos(\theta_R - 2\pi/3).$$

The positive direction of travel around the loop is the positive direction of current.

Flux Linkage



(a)

(b)

- All three loops are connected in **series** to make up the phase winding.
- All three loops have the **same** sign convention for positive travel.

$$\begin{aligned}\xi_{a-a'} &= \xi_{\pi/3}+\xi_{\pi/2}+\xi_{2\pi/3}\\ &= 2\ell_1 r_R B_{\max}\omega_R\cos(\theta_R-\pi/3)+2\ell_1 r_R B_{\max}\omega_R\cos(\theta_R-\pi/2)+\\ &\quad 2\ell_1 r_R B_{\max}\omega_R\cos(\theta_R-2\pi/3)\\ &= \left(1+\sqrt{3}\right)2\ell_1 r_R B_{\max}\omega_R\sin\left(\theta_R\right).\end{aligned}$$

Flux Linkage

$$\begin{aligned}
\xi_{a-a'} = \xi_{\pi/3} + \xi_{\pi/2} + \xi_{2\pi/3} &= -\left(\frac{d\phi_{\pi/3}}{dt} + \frac{d\phi_{\pi/2}}{dt} + \frac{d\phi_{2\pi/3}}{dt}\right) \\
&= -\frac{d}{dt}\left(\phi_{\pi/3} + \phi_{\pi/2} + \phi_{2\pi/3}\right) \\
&= -\frac{d}{dt}\lambda_{a-a'}
\end{aligned}$$

where the **flux linkage** $\lambda_{a-a'}$ is defined as

$$\begin{aligned}
\lambda_{a-a'} \triangleq \phi_{\pi/3} + \phi_{\pi/2} + \phi_{2\pi/3} &= 2\ell_1 r_R B_{\max} \sin(\pi/3 - \theta_R) + 2\ell_1 r_R B_{\max} \sin(\pi/2 - \theta_R) \\
&\quad + 2\ell_1 r_R B_{\max} \sin(2\pi/3 - \theta_R) \\
&= \left(1 + \sqrt{3}\right) 2\ell_1 r_R B_{\max} \cos\left(\theta_R\right)
\end{aligned}$$

Then the **total induced emf** $\xi_{a-a'}$ in the phase $a - a'$ is

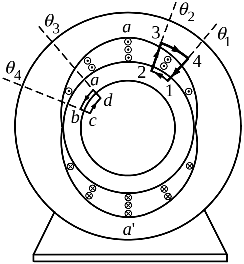
$$\xi_{a-a'} = -\frac{d\lambda_{a-a'}}{dt} = \left(1 + \sqrt{3}\right) 2\ell_1 r_R B_{\max} \omega_R \sin\left(\theta_R\right).$$

- One can first sum the loop fluxes, i.e., compute $\lambda_{a-a'}$.
- Apply Faraday's law to the flux linkage to obtain the **total emf** $\xi_{a-a'}$.
- **Be careful** to have **consistent** sign conventions in each loop of the phase.

Azimuthal Magnetic Field in the Air Gap*

*This is an optional section.

Azimuthal Magnetic Field in the Air Gap



- $\vec{\mathbf{B}}_{Sa}(i_{Sa}, r, \theta) = \dfrac{\mu_0 r_R N_S}{2gr} i_{Sa} \cos(\theta)\hat{\mathbf{r}}$ at a point (r, θ) in the air gap.
- We now show there **must** be a component of the magnetic field in the $\hat{\boldsymbol{\theta}}$ **direction!**

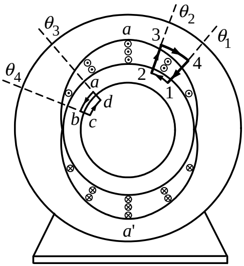
$$\oint_{1-2-3-4-1} \vec{\mathbf{H}}_{Sa} \cdot d\vec{\boldsymbol{\ell}} = \int_1^2 \vec{\mathbf{H}}_{Sa} \cdot d\vec{\boldsymbol{\ell}} = \int_{\theta_1}^{\theta_2} (-i_{Sa}) \frac{N_S}{2} \sin(\theta) d\theta.$$

As $d\vec{\boldsymbol{\ell}} = r_S d\theta \hat{\boldsymbol{\theta}}$ $(r_S = r_R + g)$ we have

$$\int_{\theta_1}^{\theta_2} (H_{Sa\theta} \hat{\boldsymbol{\theta}}) \cdot (r_S d\theta \hat{\boldsymbol{\theta}}) = -\int_{\theta_1}^{\theta_2} i_{Sa} \frac{N_S}{2} \sin(\theta) d\theta \quad \text{for} \quad 0 \leq \theta_1 \leq \theta \leq \theta_2 \leq \pi.$$

This must hold for **any** such θ_1, θ_2!

Azimuthal Magnetic Field in the Air Gap



- We have

$$H_{Sa\theta}(i_{Sa}, r_S, \theta) = -\frac{N_S}{2r_S} i_{Sa} \sin(\theta) \quad \text{for} \quad 0 \le \theta \le \pi.$$

- A similar argument shows that

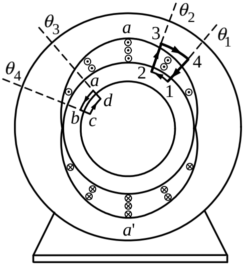
$$H_{Sa\theta}(i_{Sa}, r_S, \theta) = -\frac{N_S}{2r_S} i_{Sa} \sin(\theta) \quad \text{for} \quad \pi \le \theta \le 2\pi.$$

- Thus

$$B_{Sa\theta}(i_{Sa}, r_S, \theta) = -\frac{\mu_0 N_S}{2r_S} i_{Sa} \sin(\theta) \quad \text{for} \quad 0 \le \theta \le 2\pi.$$

- This is the **tangential** magnetic field at the **inside surface** of the stator.

Azimuthal Magnetic Field in the Air Gap



$$\oint_{a\text{-}b\text{-}c\text{-}d\text{-}a} \vec{\mathbf{H}}_{Sa} \cdot d\vec{\boldsymbol{\ell}} = \int_a^b \vec{\mathbf{H}}_{Sa} \cdot d\vec{\boldsymbol{\ell}} = \int_{\theta_3}^{\theta_4} H_{Sa\theta}(i_{Sa}, r_R, \theta) r_R \, d\theta \equiv 0$$

- As θ_3, θ_4 are arbitrary, it follows that

$$H_{Sa\theta}(i_{Sa}, r_R, \theta) \equiv 0 \;\; \text{and} \;\; B_{Sa\theta}(i_{Sa}, r_R, \theta) \equiv 0.$$

- What about $r_R < r < r_S$? Write

$$B_{Sa\theta}(i_{Sa}, r, \theta)\hat{\boldsymbol{\theta}} = -\alpha(r)\frac{\mu_0 N_S}{2 r_S} i_{Sa} \sin(\theta)\hat{\boldsymbol{\theta}}$$

where $\alpha(r_S) = 1$ and $\alpha(r_R) = 0$.

- Need to find $\alpha(r)$!

Azimuthal Magnetic Field in the Air Gap

- $\vec{\mathbf{B}}_{Sa} = B_{Sar}\hat{\mathbf{r}} + B_{Sa\theta}\hat{\boldsymbol{\theta}} + B_{Saz}\hat{\mathbf{z}}$
- $B_{Sar} = \dfrac{\mu_0 \ell_2 N_S}{4g} i_{Sa} \dfrac{\cos(\theta)}{r}, \quad B_{Sa\theta} = -\alpha(r)\dfrac{\mu_0 N_S}{2r_S} i_{Sa}\sin(\theta), \quad B_{Saz} = 0$
- $\nabla \cdot \vec{\mathbf{B}} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(rB_r) + \dfrac{1}{r}\dfrac{\partial B_\theta}{\partial \theta} + \dfrac{\partial B_z}{\partial z} \equiv 0.$
- $\vec{\mathbf{B}}_{Sa} = B_{Sar}\hat{\mathbf{r}} + B_{Sa\theta}\hat{\boldsymbol{\theta}} = B_{Sar}\hat{\mathbf{r}} - \alpha(r)\dfrac{\mu_0 N_S}{2r_S} i_{Sa}\sin(\theta)\hat{\boldsymbol{\theta}}$

$\nabla \cdot \vec{\mathbf{B}}_{Sa} \equiv 0$ gives

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_{Sar}) - \frac{1}{r}\alpha(r)\frac{\mu_0 N_S}{2r_S} i_{Sa}\frac{\partial}{\partial \theta}\sin(\theta) \equiv 0$$

or

$$\begin{aligned}
\frac{\partial}{\partial r}(rB_{Sar}) &= \alpha(r)\frac{\mu_0 N_S}{2r_S} i_{Sa}\cos(\theta) \\
rB_{Sar} &= \frac{\mu_0 N_S}{2r_S} i_{Sa}\cos(\theta)\int_{r_R}^{r}\alpha(r')dr' + f(\theta)^1 \\
B_{Sar} &= \frac{\mu_0 N_S}{2r_S}\frac{i_{Sa}}{r}\cos(\theta)\int_{r_R}^{r}\alpha(r')dr' + \frac{f(\theta)}{r}.
\end{aligned}$$

[1] $f(\theta)$ is the "constant of integration".

Azimuthal Magnetic Field in the Air Gap

- $B_{Sar}(i_{Sa}, r, \theta) = \dfrac{\mu_0 N_S}{2r_S}\dfrac{i_{Sa}}{r}\cos(\theta)\int_{r_R}^{r}\alpha(r')dr' + \dfrac{f(\theta)}{r}$
- To include $B_{Sa\theta}$ we **modified** B_{Sar} in order to satisfy Gauss's law.
- $B_{Sar}(i_{Sa}, r, \theta)|_{r=r_R} = \dfrac{f(\theta)}{r_R}$ - value of B_{Sar} on the **rotor surface**.
- Choose $f(\theta)$ to make B_{Sar} the **same value** we got before considering $B_{Sa\theta}$.

  I.e., **set** $\dfrac{f(\theta)}{r} = \dfrac{\mu_0 r_R N_S}{2gr} i_{Sa}\cos(\theta)$
- We modify B_{Sar} to be

$$B_{Sar}(i_{Sa}, r, \theta) = \frac{\mu_0 r_R N_S}{2g} i_{Sa}\frac{\cos(\theta)}{r}\left(1 + \frac{g}{r_S r_R}\int_{r_R}^{r}\alpha(r')dr'\right).$$

- By this choice of $f(\theta)$, the torque on the rotor will **not** change.

  The torque depends only on the value of B_{Sar} **at** $r = r_R$.
- $0 \le \int_{r_R}^{r}\alpha(r')dr' \le g/2$ so the percent **change** in B_{Sar} is bounded by

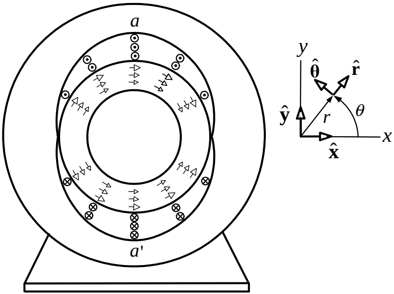
$$\frac{g^2}{2r_S r_R} << 1.$$

Azimuthal Magnetic Field in the Air Gap

Summarizing:

$$\vec{\mathbf{B}}_{Sa} = \frac{\mu_0 r_R N_S}{2g} i_{Sa} \frac{\cos(\theta)}{r}\left(1 + \frac{g}{r_S r_R}\int_{r_R}^{r}\alpha(r')dr'\right)\hat{\mathbf{r}} \underbrace{-\frac{\mu_0 r_R N_S}{2g}\frac{g}{r_S r_R}\alpha(r) i_{Sa}\sin(\theta)}_{B_{Sa\theta}}\hat{\boldsymbol{\theta}}$$

with $\alpha(r_S) = 1$ and $\alpha(r_R) = 0$.



- We next determine $\alpha(r)$.

Azimuthal Magnetic Field in the Air Gap

Determination of $\alpha(r)$

- Ampère's law in differential form $\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_{\text{free}}$ is used.
- In the air gap $\vec{\mathbf{J}}_{\text{free}} = 0$ and $\vec{\mathbf{B}} = \mu_0 \vec{\mathbf{H}}$ so we have $\nabla \times \vec{\mathbf{B}} = \mathbf{0}$.
- In cylindrical coordinates,

$$\nabla \times \vec{\mathbf{B}} = \left(\frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial \theta}\right)\hat{\mathbf{z}} = \mathbf{0}.$$

- $B_{Saz} \equiv 0$ while B_{Sar} and $B_{Sa\theta}$ do not depend on the coordinate z. Need only worry about the z component of $\nabla \times \vec{\mathbf{B}} = \mathbf{0}$, that is,

$$\frac{\partial}{\partial r}(rB_{Sa\theta}) = \frac{\partial B_{Sar}}{\partial \theta}.$$

This becomes

$$\begin{aligned}
-\frac{g}{r_S r_R}\frac{\partial}{\partial r}\Big(r\alpha(r)\Big)\, i_{Sa}\sin(\theta) &= -i_{Sa}\frac{\sin(\theta)}{r}\left(1 + \frac{g}{r_S r_R}\int_{r_R}^{r}\alpha(r')dr'\right) \\
\frac{g}{r_S r_R}\frac{d}{dr}\Big(r\alpha(r)\Big) &= \frac{1}{r}\left(1 + \frac{g}{r_S r_R}\int_{r_R}^{r}\alpha(r')dr'\right) \\
\frac{d}{dr}(r\alpha) &= \frac{1}{r}\frac{r_S r_R}{g} + \frac{\int_{r_R}^{r}\alpha(r')dr'}{r} \\
r\alpha + r^2\frac{d\alpha}{dr} &= \frac{r_S r_R}{g} + \int_{r_R}^{r}\alpha(r')dr'.
\end{aligned}$$

Azimuthal Magnetic Field in the Air Gap

From the previous slide: $r\alpha + r^2 \dfrac{d\alpha}{dr} = \dfrac{r_S r_R}{g} + \displaystyle\int_{r_R}^{r} \alpha(r')dr'$.
Differentiating w.r.t. r and rearranging:

$$\alpha + r\frac{d\alpha}{dr} + 2r\frac{d\alpha}{dr} + r^2\frac{d^2\alpha}{dr^2} = \alpha$$

or

$$\frac{d^2\alpha}{dr^2} + \frac{3}{r}\frac{d\alpha}{dr} = 0.$$

Then $\dfrac{d\alpha}{dr} = c_1 e^{-\int_{r_R}^{r}(3/r')dr'} = c_1 e^{-3\ln(r/r_R)}$ and thus

$$\alpha(r) = c_1 \int_{r_R}^{r} e^{-3\ln(r'/r_R)}dr' + c_2.$$

$\alpha(r_R) = 0 \Longrightarrow c_2 = 0$ and $\alpha(r_S) = 1$ requires that

$$c_1 = \frac{1}{\int_{r_R}^{r_S} e^{-3\ln(r'/r_R)}dr'} \approx \frac{1}{g} \text{ using } \ln(r/r_R) \approx \ln(1) = 0 \text{ for } r_R < r < r_S.$$

Finally

$$\alpha(r) = \frac{1}{g}\int_{r_R}^{r} e^{-3\ln(r'/r_R)}dr' \approx \frac{1}{g}\int_{r_R}^{r} 1dr' = \frac{r - r_R}{g}.$$

Electric Field $\vec{\mathbf{E}}_{Sa}$

$$\vec{\mathbf{B}}_{Sa} = \frac{\mu_0 \ell_2 N_S}{4g} i_{Sa} \frac{\cos(\theta)}{r}\left(1 + \frac{g}{r_S r_R}\int_{r_R}^{r}\alpha(r')dr'\right)\hat{\mathbf{r}} - \frac{\mu_0 \ell_2 N_S}{4g}\frac{g}{r_S r_R}\alpha(r) i_{Sa}\sin(\theta)\hat{\boldsymbol{\theta}}.$$

$\vec{\mathbf{E}}_{Sa}$ is a solution to $\nabla \times \vec{\mathbf{E}}_{Sa} = -\dfrac{\partial \vec{\mathbf{B}}_{Sa}}{\partial t}$ where the curl of $\vec{\mathbf{E}}$ is given by

$$\nabla \times \vec{\mathbf{E}} = \left(\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

$B_{Saz} \equiv 0$ and by symmetry $\dfrac{\partial E_{Sa\theta}}{\partial z} = 0$ and $\dfrac{\partial E_{Sar}}{\partial z} = 0$.

Try a solution of the form $\vec{\mathbf{E}}_{Sa} = E_{Saz}\hat{\mathbf{z}}$, i.e., $E_{Sar} = E_{Sa\theta} \equiv 0$ and solve

$$\nabla \times \vec{\mathbf{E}}_{Sa} = \frac{1}{r}\frac{\partial E_{Saz}}{\partial \theta}\hat{\mathbf{r}} - \frac{\partial E_{Saz}}{\partial r}\hat{\boldsymbol{\theta}} = -\frac{\partial B_{Sar}}{\partial t}\hat{\mathbf{r}} - \frac{\partial B_{Sa\theta}}{\partial t}\hat{\boldsymbol{\theta}}.$$

$$\begin{aligned}
\vec{\mathbf{E}}_{Sa} = E_{Saz}\hat{\mathbf{z}} &= -\frac{\mu_0 \ell_2 N_S}{4g}\frac{di_{Sa}}{dt}\sin(\theta)\left(1 + \frac{g}{r_S r_R}\int_{r_R}^{r}\alpha(r')dr'\right)\hat{\mathbf{z}} \\
&\approx -\frac{\mu_0 \ell_2 N_S}{4g}\frac{di_{Sa}}{dt}\sin(\theta)\left(1 + \frac{(r - r_R)^2}{2 r_S r_R}\right)\hat{\mathbf{z}} \\
&\approx -\frac{\mu_0 \ell_2 N_S}{4g}\frac{di_{Sa}}{dt}\sin(\theta)\hat{\mathbf{z}}.
\end{aligned}$$

The Magnetic and Electric Fields $\vec{\mathbf{B}}_{Sa}, \vec{\mathbf{E}}_{Sa}, \vec{\mathbf{B}}_{Sb}, \vec{\mathbf{E}}_{Sb}$

$$\begin{aligned}
\vec{\mathbf{B}}_{Sa} &= \frac{\mu_0 \ell_2 N_S}{4g} i_{Sa} \frac{\cos(\theta)}{r}\left(1 + \frac{g}{r_S r_R}\int_{r_R}^{r} \alpha(r')dr'\right)\hat{\mathbf{r}} - \frac{\mu_0 \ell_2 N_S}{4g}\frac{g}{r_S r_R}\alpha(r) i_{Sa}\sin(\theta)\hat{\boldsymbol{\theta}} \\
\vec{\mathbf{E}}_{Sa} &= -\frac{\mu_0 r_R N_S}{2g}\frac{di_{Sa}}{dt}\sin(\theta)\left(1 + \frac{g}{r_S r_R}\int_{r_R}^{r} \alpha(r')dr'\right)\hat{\mathbf{z}} \\
\vec{\mathbf{B}}_{Sb} &= \frac{\mu_0 r_R N_S}{2g} i_{Sb} \frac{\sin(\theta)}{r}\left(1 + \frac{g}{r_S r_R}\int_{r_R}^{r} \alpha(r')dr'\right)\hat{\mathbf{r}} + \frac{\mu_0 r_R N_S}{2g}\frac{g}{r_S r_R}\alpha(r) i_{Sb}\cos(\theta)\hat{\boldsymbol{\theta}} \\
\vec{\mathbf{E}}_{Sb} &= \frac{\mu_0 r_R N_S}{2g}\frac{di_{Sb}}{dt}\cos(\theta)\left(1 + \frac{g}{r_S r_R}\int_{r_R}^{r} \alpha(r')dr'\right)\hat{\mathbf{z}}.
\end{aligned}$$

With $i_{Sa}(t) = I_S\cos(\omega_S t)$, $i_{Sb}(t) = I_S\sin(\omega_S t)$ and $\alpha(r) \equiv 0$,

$$\begin{aligned}
\vec{\mathbf{B}}_S(r,\theta,t) &= \frac{\mu_0 r_R N_S I_S}{2g}\frac{1}{r}\cos(\theta - \omega_S t)\hat{\mathbf{r}} \\
\vec{\mathbf{E}}_S(\theta,t) &= \omega_S \frac{\mu_0 r_R N_S I_S}{2g}\cos(\theta - \omega_S t)\hat{\mathbf{z}}.
\end{aligned}$$

- At $r = r_R$, the expressions for $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are the same as taking $\alpha(r) \equiv 0$.
- Thus the **induced emfs** in the rotor loops are **not** affected by neglecting $B_{S\theta}$.