Modeling and High-Performance Control of Electric Machines

Chapter 5 The Physics of AC Machines

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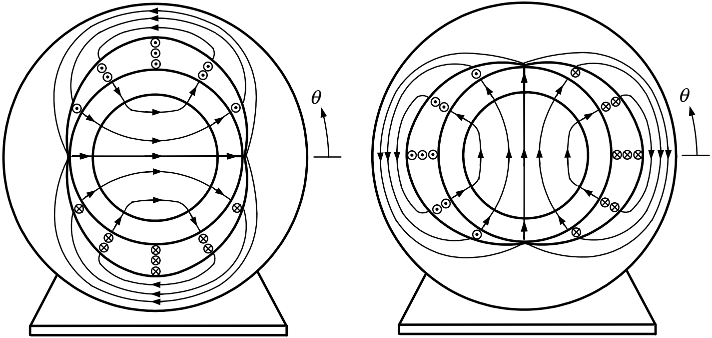
The Physics of AC Machines

- **Rotating Magnetic Field**
- **The Physics of the Induction Machine**
- **The Physics of the Synchronous Machine**
- **Microscopic Viewpoint of AC Machines***
- **Steady-State Analysis of a Squirrel Cage Induction Motor*** (no slides)

Rotating Magnetic Field



Left $\vec{\mathbf{B}}_{Sa}(i_{Sa}, r, \theta) = \dfrac{\mu_0 \ell_2 N_S}{4gr} i_{Sa} \cos(\theta)\hat{\mathbf{r}}.$

Right $\vec{\mathbf{B}}_{Sb}(i_{Sb}, r, \theta) = \dfrac{\mu_0 \ell_2 N_S}{4gr} i_{Sb} \cos(\theta - \pi/2)\hat{\mathbf{r}} = \dfrac{\mu_0 \ell_2 N_S}{4gr} i_{Sb} \sin(\theta)\hat{\mathbf{r}}.$

Rotating Magnetic Field

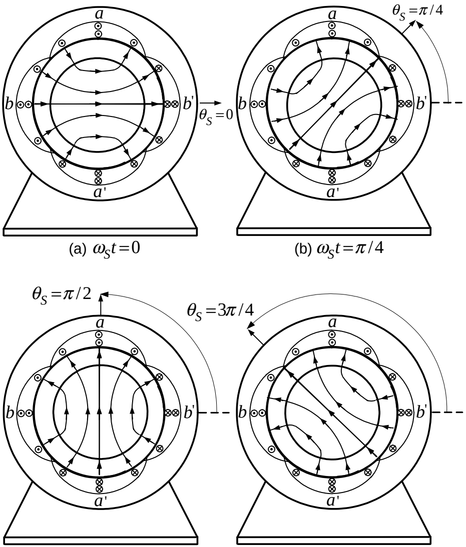
With $i_{Sa}(t) = I_S \cos(\omega_S t)$, $i_{Sb}(t) = I_S \sin(\omega_S t)$ we have

$$\begin{aligned}
\vec{\mathbf{B}}_S &= \frac{\mu_0 \ell_2 N_S I_S}{4gr}\left(\cos(\omega_S t)\cos(\theta) + \sin(\omega_S t)\sin(\theta)\right)\hat{\mathbf{r}} \\
&= \frac{\mu_0 \ell_2 N_S I_S}{4gr}\cos(\theta - \omega_S t)\hat{\mathbf{r}} \\
&= \frac{\mu_0 \ell_2 N_S I_S}{4gr}\cos(\theta - \theta_S(t))\hat{\mathbf{r}}
\end{aligned}$$

where $\theta_S(t) \triangleq \omega_S t$.

- This is a **rotating magnetic field** rotating at the electrical frequency ω_S.
- The center line at $\theta_S(t) \triangleq \omega_S t$ is referred to as the **magnetic axis**.

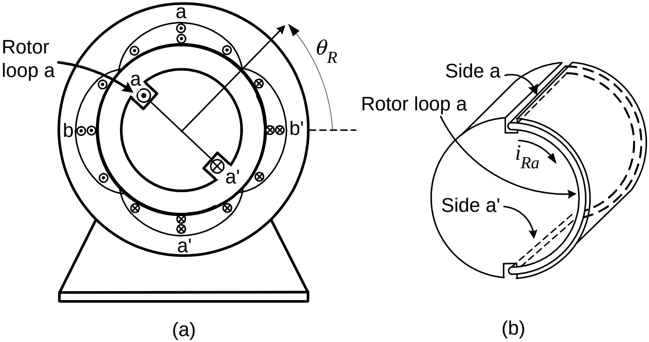
Rotating Magnetic Field $\vec{\mathbf{B}}_S = \dfrac{\mu_0 \ell_2 N_S I_S}{4gr}\cos(\theta - \theta_S(t))\hat{\mathbf{r}}$



(a) $\omega_S t = 0$ (b) $\omega_S t = \pi/4$

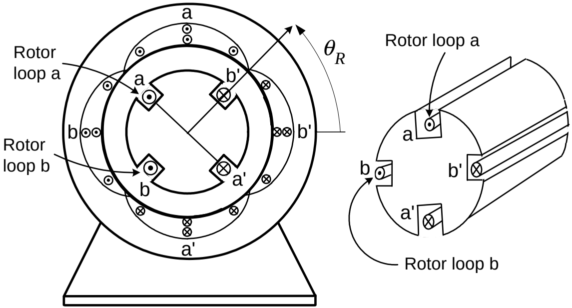
(c) $\omega_S t = \pi/2$ (d) $\omega_S t = 3\pi/4$

The Physics of the Induction Machine



- The stator consists of two sinusoidally wound phases 90° apart.
- A loop wound around the surface of the rotor iron is rotor phase *a*.
- $\theta_R(t)$ is the rotor's angular position defined as the normal to rotor loop *a*.

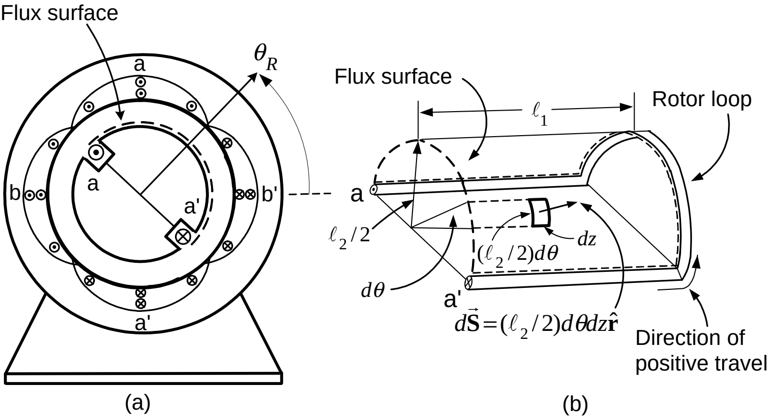
The Physics of the Induction Machine



(a) (b)

- **Rotor loop** *b* is wound around the rotor iron 90° from rotor loop *a*.
- Rotor loop *b* (rotor phase *b*) is **electrically isolated** from rotor loop *a*.
- This results in a simple **two phase induction machine**.
- $\theta_S(t) = \omega_S t$ is the angular position of the magnetic axis of $\vec{\mathbf{B}}_S$.
- (r, θ) denotes the **polar coordinates** of an arbitrary point in the **air gap**.

Induced Emfs in the Rotor Loops



- The rotating magnetic field $\vec{\mathbf{B}}_S$ produces a changing flux in the rotor loops.
- By Faraday's law, an **emf** is induced in each rotor loop.
- $d\vec{\mathbf{S}} = (\ell_2/2)d\theta dz\hat{\mathbf{r}}$

Induced Emfs in the Rotor Loops

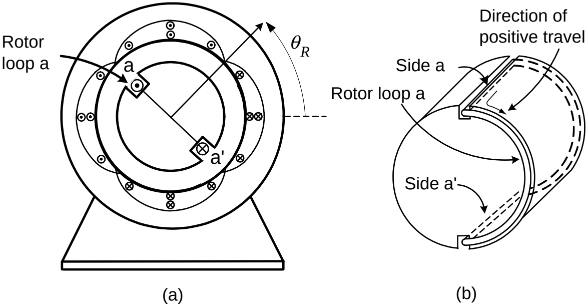
The **flux** λ_{Ra} in rotor loop a due to $\vec{\mathbf{B}}_S$ is

$$
\begin{aligned}
\lambda_{Ra} &= \int_S \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} \\
&= \int_{z=0}^{z=\ell_1} \int_{\theta_R(t)-\pi/2}^{\theta_R(t)+\pi/2} \frac{\mu_0 \ell_2 N_S I_S}{4gr} \Big|_{r=\ell_2/2} \cos(\theta - \theta_S(t))\hat{\mathbf{r}} \cdot ((\ell_2/2) d\theta dz \hat{\mathbf{r}}) \\
&= \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{4g} \sin(\theta - \theta_S(t)) \Big|_{\theta_R(t)-\pi/2}^{\theta_R(t)+\pi/2} \\
&= \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} \cos\Big(\theta_S(t) - \theta_R(t)\Big).
\end{aligned}
$$

By Faraday's law, the induced **emf** in rotor loop a is given by

$$
\begin{aligned}
\xi_{Ra} = -\frac{d\lambda_{Ra}}{dt} &= -\frac{d}{dt} \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} \cos\Big(\theta_S(t) - \theta_R(t)\Big) \\
&= \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} (\omega_S - \omega_R) \sin\Big((\omega_S - \omega_R)t\Big)
\end{aligned}
$$

Induced Emfs in the Rotor Loops



The current i_{Ra} in phase a satisfies

$$L_R \frac{di_{Ra}}{dt} = -R_R i_{Ra} + \xi_{Ra}$$

- L_R is the inductance of each rotor phase.
- R_R is the resistance of each rotor phase.

Neglecting the inductance (i.e., set $L_R = 0$), we have

$$i_{Ra}(t) = \xi_{Ra}/R_R.$$

Induced Emfs in the Rotor Loops

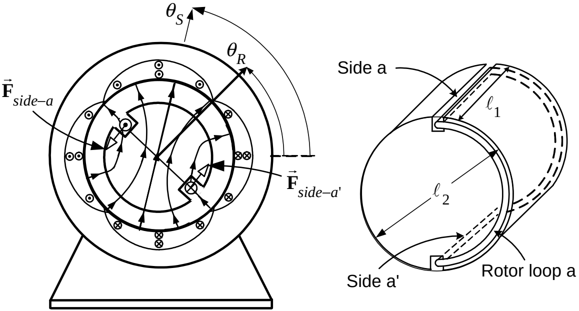
Similarly

$$
\begin{aligned}
\lambda_{Rb} &= \int_S \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} \\
&= \int_{z=0}^{z=\ell_1} \int_{\theta_R(t)}^{\theta_R(t)+\pi} \frac{\mu_0 \ell_2 N_S I_S}{4gr}\Big|_{r=\ell_2/2} \cos\left(\theta - \theta_S(t)\right) \hat{\mathbf{r}} \cdot \left((\ell_2/2)\, dz\, d\theta \hat{\mathbf{r}}\right) \\
&= \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{4g} \sin\left(\theta - \theta_S(t)\right) \Big|_{\theta_R(t)}^{\theta_R(t)+\pi} \\
&= \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} \sin\left(\theta_S(t) - \theta_R(t)\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\xi_{Rb}(t) &= -\frac{d\lambda_{Rb}}{dt} = -\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} (\omega_S - \omega_R) \cos\Big((\omega_S - \omega_R) t\Big) \\
i_{Rb}(t) &= \xi_{Rb}(t) / R_R.
\end{aligned}
$$

Magnetic Forces and Torques on the Rotor



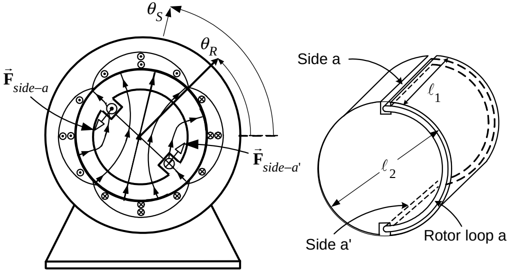
$$\vec{\mathbf{F}}_{side-a} = i_{Ra}\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}_S = i_{Ra}\ell_1\hat{\mathbf{z}} \times \frac{\mu_0\ell_2 N_S I_S}{4gr}\Big|_{r=\ell_2/2}\cos(\theta-\theta_S)_{|\theta=\theta_R+\pi/2}\,\hat{\mathbf{r}}$$

$$= i_{Ra}\frac{\mu_0\ell_1 N_S I_S}{2g}\sin(\theta_S-\theta_R)\hat{\boldsymbol{\theta}}.$$

$$\vec{\boldsymbol{\tau}}_{side-a} = \vec{\mathbf{r}}\times\vec{\mathbf{F}}_{side-a} = (\ell_2/2)\hat{\mathbf{r}} \times i_{Ra}\frac{\mu_0\ell_1 N_S I_S}{2g}\sin(\theta_S-\theta_R)\hat{\boldsymbol{\theta}}$$

$$= \frac{\mu_0\ell_1\ell_2 N_S I_S}{4g}i_{Ra}\sin(\theta_S-\theta_R)\hat{\mathbf{z}}.$$

Magnetic Forces and Torques on the Rotor



$$\vec{\mathbf{F}}_{side-a'} = i_{Ra}\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}_S = i_{Ra}(-\ell_1\hat{\mathbf{z}}) \times \frac{\mu_0 \ell_2 N_S I_S}{4gr}\Big|_{r=\ell_2/2} \cos(\theta - \theta_S)_{|\theta=\theta_R-\pi/2}\hat{\mathbf{r}}$$

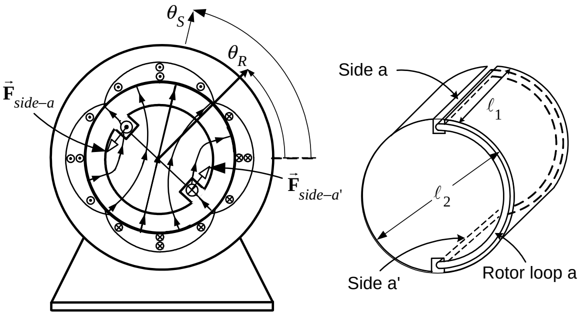
$$= i_{Ra}\frac{\mu_0 \ell_1 N_S I_S}{2g}\sin(\theta_S - \theta_R)\hat{\boldsymbol{\theta}}.$$

$$\vec{\boldsymbol{\tau}}_{side-a'} = (\ell_2/2)\hat{\mathbf{r}} \times \vec{\mathbf{F}}_{side-a'} = \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{4g} i_{Ra}\sin(\theta_S - \theta_R)\hat{\mathbf{z}}.$$

Total torque

$$\vec{\boldsymbol{\tau}}_{Ra} = \tau_{Ra}\hat{\mathbf{z}} = (\tau_{side-a} + \tau_{side-a'})\hat{\mathbf{z}} = \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} i_{Ra}\sin(\theta_S - \theta_R)\hat{\mathbf{z}}.$$

Magnetic Forces and Torques on the Rotor



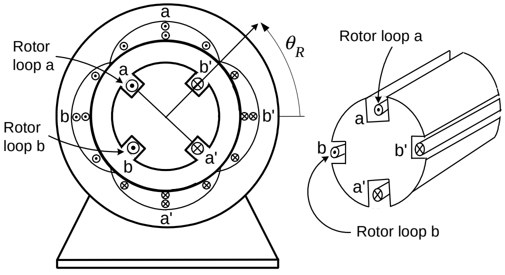
Recall that

$$i_{Ra}(t) = \xi_{Ra}(t)/R_R = \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} \frac{1}{R_R}(\omega_S - \omega_R)\sin\Big((\omega_S - \omega_R)t\Big).$$

so

$$\tau_{Ra} = \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{1}{R_R}(\omega_S - \omega_R)\sin^2\Big((\omega_S - \omega_R)t\Big).$$

Magnetic Forces and Torques on the Rotor



(a) (b)

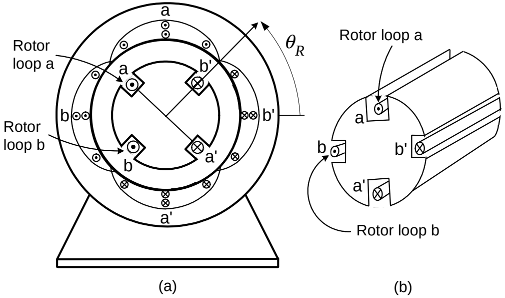
Similarly, for rotor phase b

$$\begin{aligned}\vec{\mathbf{F}}_{side-b} = i_{Rb}\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}_S &= i_{Rb}\ell_1\hat{\mathbf{z}} \times \frac{\mu_0 \ell_2 N_S I_S}{4gr}\Big|_{r=\ell_2/2}\cos(\theta - \theta_S)_{|\theta=\theta_R+\pi}\hat{\mathbf{r}} \\ &= -i_{Rb}\frac{\mu_0 \ell_1 N_S I_S}{2g}\cos(\theta_S - \theta_R)\hat{\boldsymbol{\theta}}.\end{aligned}$$

As in the case of loop a, $F_{side-b'} = F_{side-b}$ and $\tau_{side-b} = \tau_{side-b'}$.

$$\tau_{Rb} = \tau_{side-b} + \tau_{side-b'} = 2\tau_{side-b} = -\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} i_{Rb}\cos\Big((\omega_S - \omega_R)t\Big).$$

Magnetic Forces and Torques on the Rotor



$$i_{Rb}(t) = \xi_{Rb}(t)/R_R = -\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}(\omega_S - \omega_R)\cos\Big((\omega_S - \omega_R)t\Big)/R_R$$

$$\tau_{Rb} = \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{1}{R_R}(\omega_S - \omega_R)\cos^2\Big((\omega_S - \omega_R)t\Big)$$

The **total torque** τ_R is

$$\tau_R = \tau_{Ra} + \tau_{Rb} = \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{1}{R_R}(\omega_S - \omega_R).$$

The torque is **constant** if ω_S and ω_R are constant.

Slip Speed

- The stator currents are $i_{Sa}(t) = I_S \cos(\omega_S t)$, $i_{Sb}(t) = I_S \sin(\omega_S t)$
- Neglecting the inductance of the rotor phases we have

$$\tau_R = \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{1}{R_R}(\omega_S - \omega_R).$$

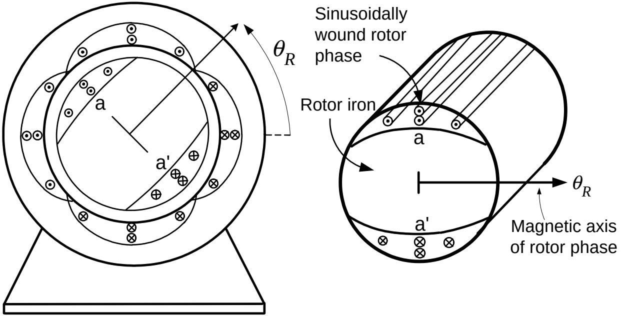
- The torque is proportional to $\omega_S - \omega_R$ called the **slip speed** ω_{slip}, i.e.,

$$\omega_{slip} \triangleq \omega_S - \omega_R.$$

- If the slip speed is **zero**, there is **no** torque. Why?
  Then $\vec{\mathbf{B}}_S$ and the rotor are rotating at the **same** angular rate.
  The fluxes in the rotor loops are then **constant**.
  By Faraday's law **no** voltages or currents are induced in the rotor.
  Thus **no** magnetic force/torque on the rotor.
- The induction machine is also referred to as an **asynchronous** machine as $\omega_S \neq \omega_R$

The Physics of the Synchronous Machine

Sinusoidally Wound Rotor

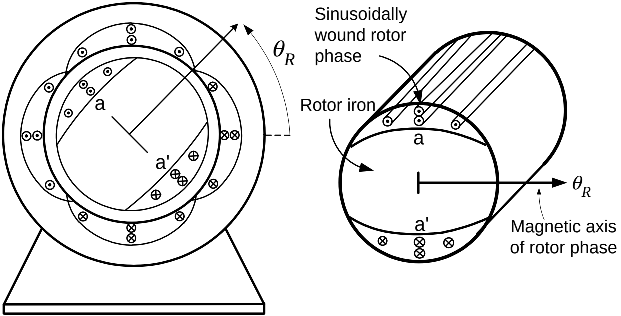


- The stator is the **same** as the induction machine.
- The rotor consists of a cylindrical core of iron with a **single** phase winding.
- The rotor phase has a **sinusoidal turns density** given by

$$N_{RF}(\theta - \theta_R) = \frac{N_F}{2}\left|\sin(\theta - \theta_R)\right|$$

- N_F is the **total number** of windings/turns of the rotor phase.
- The subscript "F" refers to "**field**" so the rotor is also called the **field winding**.

The Physics of the Synchronous Machine

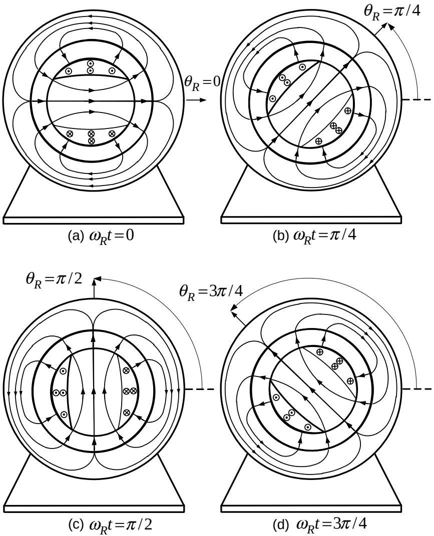


- The magnetic field produced by the current i_F in the rotor phase is

$$\vec{\mathbf{B}}_R = B_R \hat{\mathbf{r}} = \frac{\mu_0 \ell_2 N_F i_F}{4gr} \cos(\theta - \theta_R)\hat{\mathbf{r}}$$

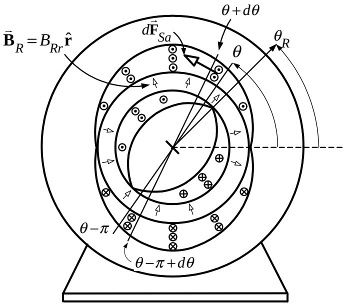
- Synchronous machine operation requires the rotor current to be **constant**.
- We will set $i_F = I_F$ with I_F **constant**.
- The terminology **field winding** refers to its current being held **constant.**

$$\vec{\mathbf{B}}_R = B_R\hat{\mathbf{r}} = \frac{\mu_0 \ell_2 N_F i_F}{4gr}\cos(\theta - \theta_R)\hat{\mathbf{r}}$$



- Stator phases not shown for clarity.

Magnetic Forces/Torques

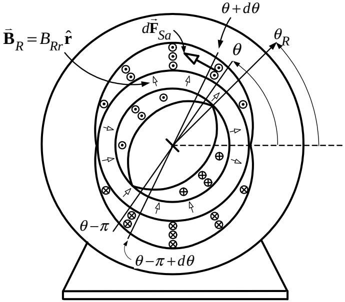


Stator phase *b* not shown for clarity.

$$\begin{aligned}\vec{\mathbf{B}}_S(i_{Sa}, i_{Sb}, r, \theta) &= \frac{\mu_0 \ell_2 N_S}{4gr}\Big(i_{Sa}\cos(\theta) + \ i_{Sb}\sin(\theta)\Big)\hat{\mathbf{r}} \\ \vec{\mathbf{B}}_R(I_R, r, \theta) &= \frac{\mu_0 \ell_2 N_F I_F}{4gr}\cos(\theta - \theta_R)\hat{\mathbf{r}}.\end{aligned}$$

- For the IM we computed the torque **on the rotor** produced by $\vec{\mathbf{B}}_S$.
- We could use the **same** approach for the **synchronous** machine.
- Instead we compute the torque **on the stator** produced by $\vec{\mathbf{B}}_R$.

Magnetic Forces/Torques

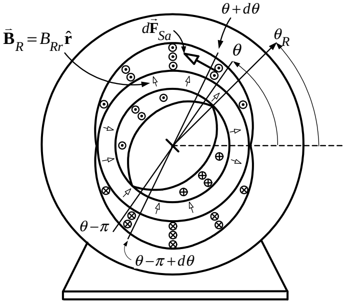


$$d\vec{\mathbf{F}}_{Sa} = \begin{cases} i_{Sa}(t)\dfrac{N_S}{2}\,|\sin(\theta)|d\theta(+\ell_1\hat{\mathbf{z}}) \times \vec{\mathbf{B}}_R, & 0 \le \theta \le \pi \\[2ex] i_{Sa}(t)\dfrac{N_S}{2}\,|\sin(\theta)|d\theta(-\ell_1\hat{\mathbf{z}}) \times \vec{\mathbf{B}}_R, & \pi \le \theta \le 2\pi. \end{cases}$$

As $|\sin(\theta)| = -\sin(\theta)$ in the interval $\pi \le \theta \le 2\pi$ we have ($r_S = \ell_2/2 + g$)

$$\begin{aligned} d\vec{\mathbf{F}}_{Sa} &= \left(i_{Sa}(t)\frac{N_S}{2}\sin(\theta)d\theta\right)(\ell_1\hat{\mathbf{z}}) \times \vec{\mathbf{B}}_R \quad \text{for } 0 \le \theta \le 2\pi \\ &= \left(i_{Sa}(t)\frac{N_S}{2}\sin(\theta)d\theta\right)(\ell_1\hat{\mathbf{z}}) \times \frac{\mu_0 \ell_2 N_F I_F}{4gr}\Big|_{r=r_S}\cos(\theta - \theta_R)\hat{\mathbf{r}} \end{aligned}$$

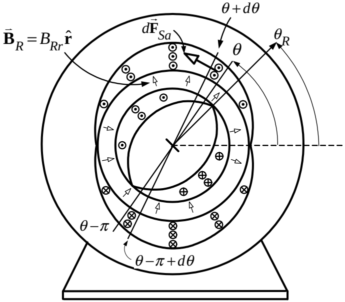
Magnetic Forces/Torques



$$
\begin{aligned}
d\vec{\mathbf{F}}_{Sa} &= \left(i_{Sa}(t)\frac{N_S}{2}\sin(\theta)d\theta\right)(\ell_1\hat{\mathbf{z}}) \times \frac{\mu_0\ell_2 N_F I_F}{4gr}\Big|_{r=r_S}\cos(\theta-\theta_R)\hat{\mathbf{r}} \\
&= \frac{\mu_0\ell_1\ell_2 N_S N_F I_F}{8gr_S} i_{Sa}(t)\sin(\theta)\cos(\theta-\theta_R)d\theta\hat{\boldsymbol{\theta}}
\end{aligned}
$$

$$
\begin{aligned}
d\vec{\boldsymbol{\tau}}_{Sa} = (r_S\hat{\mathbf{r}}) \times d\vec{\mathbf{F}}_{Sa} &= r_S\hat{\mathbf{r}} \times \left(\frac{\mu_0\ell_1\ell_2 N_S N_F I_F}{8gr_S} i_{Sa}(t)\sin(\theta)\cos(\theta-\theta_R)d\theta\right)\hat{\boldsymbol{\theta}} \\
&= \frac{\mu_0\ell_1\ell_2 N_S N_F I_F}{8g} i_{Sa}(t)\sin(\theta)\cos(\theta-\theta_R)d\theta\hat{\mathbf{z}}.
\end{aligned}
$$

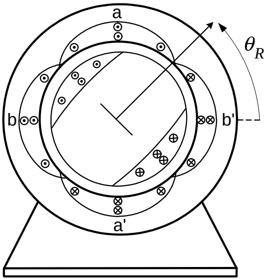
Magnetic Forces/Torques



Total torque on stator phase a

$$
\begin{aligned}
\vec{\boldsymbol{\tau}}_{Sa}(i_{Sa},\theta_R) &= \int_0^{2\pi} \frac{\mu_0 \ell_1 \ell_2 N_S N_F I_F}{8g} i_{Sa}(t)\sin(\theta)\cos(\theta-\theta_R)d\theta\hat{\mathbf{z}} \\
&= \frac{\mu_0 \ell_1 \ell_2 N_S N_F I_F}{8g} i_{Sa}(t)\int_0^{2\pi} \sin(\theta)\Big(\cos(\theta)\cos(\theta_R) + \sin(\theta)\sin(\theta_R)\Big)d\theta\hat{\mathbf{z}} \\
&= \frac{\mu_0 \ell_1 \ell_2 N_S N_F I_F}{8g} i_{Sa}(t)\sin(\theta_R)\int_0^{2\pi} \sin^2(\theta)d\theta\hat{\mathbf{z}} \\
&= \frac{\mu_0 \pi \ell_1 \ell_2 N_S N_F I_F}{8g} i_{Sa}(t)\sin(\theta_R)\hat{\mathbf{z}}.
\end{aligned}
$$

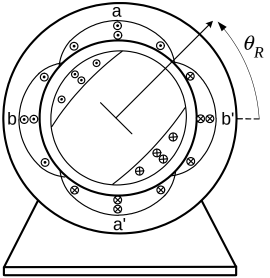
Magnetic Forces/Torques



Phase b winding distribution is $N_{Sb}(\theta) = (N_S/2)|\sin(\theta - \pi/2)|$.

$$\begin{aligned}
d\vec{\mathbf{F}}_{Sb} &= \left(i_{Sb}(t)\frac{N_S}{2}\sin(\theta - \pi/2)d\theta\right)(\ell_1\hat{\mathbf{z}}) \times \vec{\mathbf{B}}_R \\
&= \left(i_{Sb}(t)\frac{N_S}{2}\sin(\theta - \pi/2)d\theta\right)(\ell_1\hat{\mathbf{z}}) \times \frac{\mu_0\ell_2 N_F I_F}{4gr}\Big|_{r=r_S}\cos(\theta - \theta_R)\hat{\mathbf{r}} \\
&= \frac{\mu_0\ell_1\ell_2 N_S N_F I_F}{8gr_S} i_{Sb}(t)\sin(\theta - \pi/2)\cos(\theta - \theta_R)d\theta\ \hat{\mathbf{z}} \times \hat{\mathbf{r}} \\
&= -\frac{\mu_0\ell_1\ell_2 N_S N_F I_F}{8gr_S} i_{Sb}(t)\cos(\theta)\cos(\theta - \theta_R)d\theta\hat{\boldsymbol{\theta}}.
\end{aligned}$$

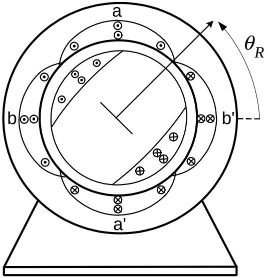
Magnetic Forces/Torques



Incremental torque $d\vec{\boldsymbol{\tau}}_{Sb}$ on stator phase b

$$\begin{aligned} d\vec{\boldsymbol{\tau}}_{Sb} &= (r_S\hat{\mathbf{r}}) \times d\vec{\mathbf{F}}_{Sb} \\ &= (r_S\hat{\mathbf{r}}) \times \left(-\frac{\mu_0 \ell_1 \ell_2 N_S N_F I_F}{8 g r_S} i_{Sb}(t)\cos(\theta)\cos(\theta-\theta_R)d\theta\right)\hat{\boldsymbol{\theta}} \\ &= -\frac{\mu_0 \ell_1 \ell_2 N_S N_F I_F}{8 g} i_{Sb}(t)\cos(\theta)\cos(\theta-\theta_R)d\theta\hat{\mathbf{z}}. \end{aligned}$$

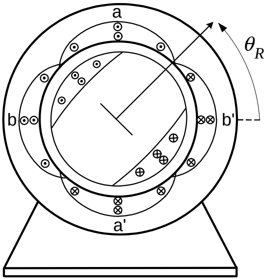
Magnetic Forces/Torques



The **total torque** $\vec{\boldsymbol{\tau}}_{Sb}$ on stator phase b is

$$\begin{aligned}
\vec{\boldsymbol{\tau}}_{Sb}(i_{Sb},\theta_R) &= \int_0^{2\pi} -\frac{\mu_0 \ell_1 \ell_2 N_S N_F I_F}{8g} i_{Sb}(t)\cos(\theta)\cos(\theta-\theta_R)d\theta\hat{\mathbf{z}} \\
&= -\frac{\mu_0 \ell_1 \ell_2 N_S N_F I_F}{8g} i_{Sb}(t)\int_0^{2\pi}\cos(\theta)\Big(\cos(\theta)\cos(\theta_R)+\sin(\theta)\sin(\theta_R)\Big)d\theta\hat{\mathbf{z}} \\
&= -\frac{\mu_0 \ell_1 \ell_2 N_S N_F I_F}{8g} i_{Sb}(t)\cos(\theta_R)\int_0^{2\pi}\cos^2(\theta)d\theta\hat{\mathbf{z}} \\
&= -\frac{\mu_0 \pi \ell_1 \ell_2 N_S N_F I_F}{8g} i_{Sb}(t)\cos(\theta_R)\hat{\mathbf{z}}.
\end{aligned}$$

Magnetic Forces/Torques



The total torque **exerted on the stator** by the rotor is then

$$\vec{\boldsymbol{\tau}}_S = \vec{\boldsymbol{\tau}}_{Sa} + \vec{\boldsymbol{\tau}}_{Sb} = \underbrace{\frac{\mu_0 \pi \ell_1 \ell_2 N_S N_F}{8g}}_{M} I_F \Big(i_{Sa}(t)\sin(\theta_R) - \ i_{Sb}(t)\cos(\theta_R) \Big) \hat{\mathbf{z}}.$$

The torque **exerted on the rotor** by the stator is $\vec{\boldsymbol{\tau}}_R = -\vec{\boldsymbol{\tau}}_S$, i.e.,

$$\vec{\boldsymbol{\tau}}_R = M I_F \left(-i_{Sa}(t)\sin(\theta_R) + i_{Sb}(t)\cos(\theta_R) \right) \hat{\mathbf{z}}.$$

Steady-State Torque

- Let $i_{Sa}(t) = I_S \cos(\omega_S t), \;\; i_{Sb}(t) = I_S \sin(\omega_S t)$
- Let $\theta_R(t) = \omega_R t - \delta$

$$
\begin{aligned}
\tau_R &= MI_F \left(-i_{Sa}(t)\sin(\theta_R) + i_{Sb}(t)\cos(\theta_R)\right) \\
&= MI_F I_S \left(-\cos(\omega_S t)\sin(\omega_R t - \delta) + \sin(\omega_S t)\cos(\omega_R t - \delta)\right) \\
&= MI_F I_S \sin((\omega_S - \omega_R)t + \delta))
\end{aligned}
$$

- **Constant** torque requires $\omega_S = \omega_R$.

  I.e., $\vec{\mathbf{B}}_S$ and the rotor must be rotating **synchronously**. Then

$$
\tau_R = MI_F I_S \sin(\delta).
$$

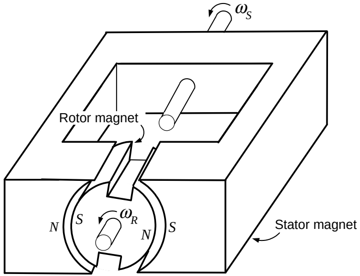
- Stator currents set up a rotating **"stator magnet"** which pulls the **"rotor magnet"**.

Steady-State Torque

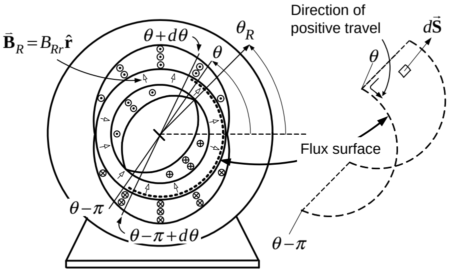
- Let $i_{Sa}(t) = I_S \cos(\omega_S t), \;\; i_{Sb}(t) = I_S \sin(\omega_S t)$
- $\vec{\mathbf{B}}_S$ and $\vec{\mathbf{B}}_R$ rotate **synchronously**.

$$\tau_R = M I_F I_S \sin(\delta).$$

- Stator currents set up a rotating **"stator magnet"** which pulls the **"rotor magnet".**



Emfs and Energy Conversion



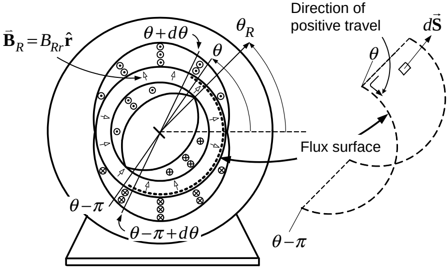
$$\phi_{Sa}(\theta) \triangleq \underbrace{\int}_{\substack{\text{A single loop}\\ \text{at the angle } \theta}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = \int_{\theta'=\theta-\pi}^{\theta'=\theta} \int_{z=0}^{z=\ell_1} \left(\frac{\mu_0 N_F \ell_2 I_F}{4 g r_S} \cos(\theta' - \theta_R(t)) \hat{\mathbf{r}} \right) \cdot \left(r_S \, d\theta' dz \, \hat{\mathbf{r}} \right)$$

$$= \int_{\theta'=\theta-\pi}^{\theta'=\theta} \frac{\mu_0 N_F \ell_1 \ell_2 I_F}{4g} \cos(\theta' - \theta_R(t)) d\theta'$$

$$= \frac{\ell_1 \ell_2}{2} \frac{\mu_0 N_F I_F}{2g} \left(\sin(\theta - \theta_R) - \sin(\theta - \pi - \theta_R) \right)$$

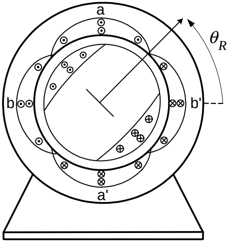
$$= \ell_1 \ell_2 \frac{\mu_0 N_F I_F}{2g} \sin(\theta - \theta_R).$$

Emfs and Energy Conversion



$$
\begin{aligned}
\lambda_{Sa}(\theta_R) \triangleq \int_{\theta=0}^{\theta=\pi} \phi_{Sa}(\theta)\frac{N_S}{2}\sin(\theta)d\theta &= \int_0^{\pi} \ell_1\ell_2\frac{\mu_0 N_F I_F}{2g}\sin(\theta-\theta_R)\frac{N_S}{2}\sin(\theta)d\theta \\
&= \ell_1\ell_2\frac{\mu_0 N_F I_F}{2g}\frac{N_S}{2}\int_0^{\pi}\sin(\theta)\sin(\theta-\theta_R)d\theta \\
&= \frac{\mu_0\ell_1\ell_2 N_S N_F}{4g} I_F \frac{\pi}{2}\cos(\theta_R) \\
&= M I_F\cos(\theta_R), \quad M = \underbrace{(\mu_0\pi\ell_1\ell_2 N_S N_R/8g)}_{\text{coeff of mutual inductance}}.
\end{aligned}
$$

Emfs and Energy Conversion - Stator Phase b

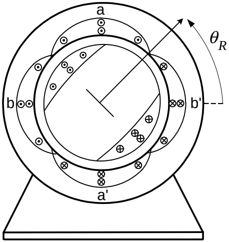


$$\phi_{Sb}(\theta) \triangleq \int_{\substack{\text{A single loop of stator} \\ \text{phase } b \text{ at the angle } \theta}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}}$$

$$= \int_{\theta'=\theta-\pi}^{\theta'=\theta} \int_{z=0}^{z=\ell_1} \left(\frac{\mu_0 N_F \ell_2 I_F}{4 g r_S} \cos(\theta' - \theta_R(t)) \hat{\mathbf{r}} \right) \cdot \left(r_S d\theta' dz \hat{\mathbf{r}} \right)$$

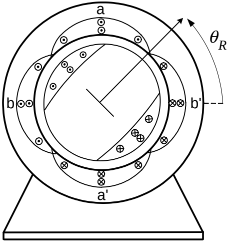
$$= \ell_1 \ell_2 \frac{\mu_0 N_F I_F}{2g} \sin(\theta - \theta_R).$$

Emfs and Energy Conversion - Stator Phase b



$$
\begin{aligned}
\lambda_{Sb}(\theta_R) &= \int_{\theta=\pi/2}^{\theta=3\pi/2} \phi_{Sb}(\theta)\frac{N_S}{2}\sin(\theta-\pi/2)d\theta \\
&= \int_{\pi/2}^{3\pi/2} \ell_1\ell_2\frac{\mu_0 N_F I_F}{2g}\sin(\theta-\theta_R)\frac{N_S}{2}\sin(\theta-\pi/2)d\theta \\
&= -\ell_1\ell_2\frac{\mu_0 N_F I_F}{2g}\frac{N_S}{2}\int_{\pi/2}^{3\pi/2}\cos(\theta)\sin(\theta-\theta_R)d\theta \\
&= -\ell_1\ell_2\ell_1\ell_2\frac{\mu_0 N_F I_F}{2g}\frac{N_S}{2}\left(-\frac{\pi}{2}\sin(\theta_R)\right) \\
&= M I_F\sin(\theta_R), \quad M = \underbrace{(\mu_0\pi\ell_1\ell_2 N_S N_R/8g)}_{\text{coeff of mutual inductance}}.
\end{aligned}
$$

Stator flux linkages and Mutual Inductance



- $\lambda_{Sa}, \lambda_{Sb}$ - **flux linkages** in the stator phases due to the **rotor's** magnetic field.

$$\lambda_{Sa}(\theta_R) \triangleq MI_F \cos(\theta_R),$$

$$\lambda_{Sb}(\theta_R) = MI_F \sin(\theta_R).$$

Induced Emfs and Energy Conversion

$$\begin{aligned}\xi_{Sa} &= -d\lambda_{Sa}/dt = +MI_F\sin(\theta_R)\omega_R \\ \xi_{Sb} &= -d\lambda_{Sb}/dt = -MI_F\cos(\theta_R)\omega_R.\end{aligned}$$

With $i_{Sa}(t) = I_S\cos(\omega_S t)$, $i_{Sb}(t) = I_S\sin(\omega_S t)$, $\theta_R = \omega_R t - \delta$

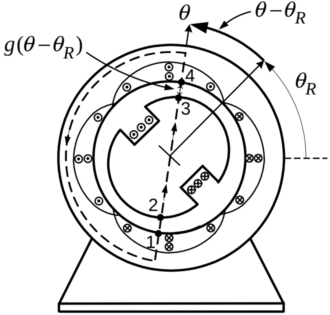
$$\begin{aligned}P_{\text{elec}} &= i_{Sa}(t)\xi_{Sa}(t) + i_{Sb}(t)\xi_{Sb}(t) \\ &= MI_F I_S\Big(\cos(\omega_S t)\sin(\omega_R t - \delta) - \sin(\omega_S t)\cos(\omega_R t - \delta)\Big)\omega_R \\ &= MI_F I_S\omega_R\sin(\omega_R t - \delta - \omega_S t) \\ &= -\underbrace{MI_F I_S\sin(\delta)}_{\tau_R}\omega_R \quad \textbf{with} \quad \omega_S = \omega_R.\end{aligned}$$

Conservation of Energy (Power)

$$P_{\text{elec}} + P_{\text{mech}} = -MI_F I_S\sin(\delta)\omega_R + MI_F I_S\sin(\delta)\omega_R = 0.$$

Electrical power absorbed = mechanical power produced.

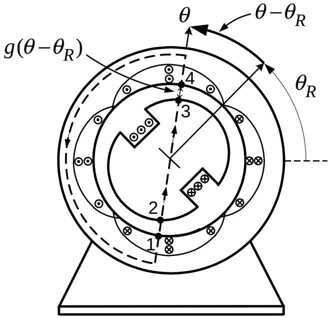
Synchronous Motor with a Salient Rotor



- **Sinusoidally wound** rotor is difficult (expensive) to make.
- Only need a rotor whose magnetic field is **sinusoidally distributed** in $\theta - \theta_R$.
- Put a **uniformly** wound coil on the rotor.
- **Shape** the pole faces of the rotor so that the **air gap length** is

$$g(\theta - \theta_R) \triangleq \frac{g_0}{\cos(\theta - \theta_R)}.$$

Synchronous Motor with a Salient Rotor



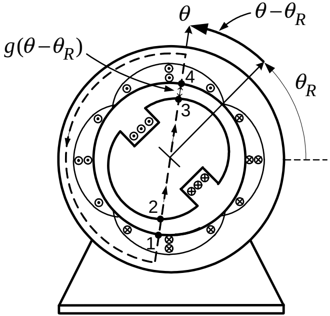
The rotor iron is now wrapped with a standard coil with N_F turns.
$g(\theta - \theta_R) \triangleq g_0/|\cos(\theta - \theta_R)|$, g_0 is the **minimum** air-gap distance.

$$\int_1^2 \vec{\mathbf{H}} \cdot d\vec{\ell} + \int_3^4 \vec{\mathbf{H}} \cdot d\vec{\ell} = N_F i_F, \quad d\vec{\ell} = \begin{cases} -dr\hat{\mathbf{r}} & \text{for path} \quad 1-2 \\ +dr\hat{\mathbf{r}} & \text{for path} \quad 3-4 \end{cases}$$

or with $\vec{\mathbf{H}} = H_r \hat{\mathbf{r}}$

$$-H_r(\theta + \pi) g(\theta + \pi - \theta_R) + H_r(\theta) g(\theta - \theta_R) = N_F i_F.$$

Synchronous Motor with a Salient Rotor



By symmetry, $H_r(\theta) = -H_r(\theta \pm \pi)$ and $g(\theta + \pi - \theta_R) = g(\theta - \theta_R)$ so $2H_r(\theta)g(\theta - \theta_R) = N_F i_F$ or

$$H_r(\theta) = \frac{N_F i_F}{2g(\theta - \theta_R)} = \frac{N_F i_F}{2g_0}\cos(\theta - \theta_R).$$

Multiplying by μ_0 and r_R/r (to satisfy conservation of flux in the air gap)

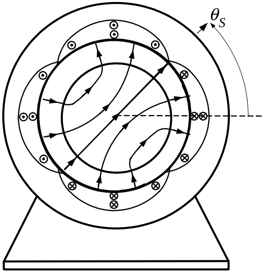
$$\vec{\mathbf{B}}_R(\theta - \theta_R) = \frac{\mu_0 N_F I_F}{2g_0}\frac{r_R}{r}\cos(\theta - \theta_R)\hat{\mathbf{r}}.$$

Same magnetic field as a sinusoidally wound rotor.

Microscopic Viewpoint of AC Machines*

*This is an optional section

Rotating Axial Electric Field Due to the Stator Currents



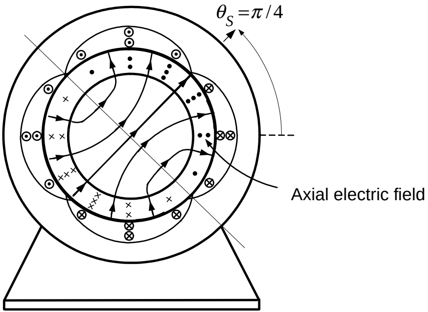
$$\vec{\mathbf{B}}_S(r,\theta,t) = B_{Sr}(r,\theta,t)\hat{\mathbf{r}} = \frac{\mu_0 \ell_2 N_S I_S}{4gr}\cos(\theta - \omega_S t)\hat{\mathbf{r}}.$$

The **axial** electric field $\vec{\mathbf{E}}_S$ in the air gap is the solution to $\nabla \times \vec{\mathbf{E}}_S = -\dfrac{\partial \vec{\mathbf{B}}_S}{\partial t}$ given by

$$\vec{\mathbf{E}}_S(\theta,t) = E_{Sz}(\theta,t)\hat{\mathbf{z}} = \frac{\mu_0 \ell_2 N_S I_S}{4g}\omega_S \cos(\theta - \omega_S t)\hat{\mathbf{z}}.$$

- Verify this by direct substitution!

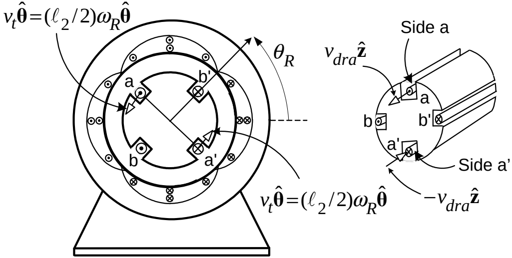
Rotating Axial Electric Field Due to the Stator Currents



$$\vec{\mathbf{E}}_S(\theta,t) = E_{Sz}(\theta,t)\hat{\mathbf{z}} = \frac{\mu_0 \ell_2 N_S I_S}{4g}\omega_S \cos(\theta - \omega_S t)\hat{\mathbf{z}}.$$

- For $\theta_S - \pi/2 \leq \theta \leq \theta_S + \pi/2$, the electric field is **out** of the page $(\cdot)$
- For $\theta_S + \pi/2 \leq \theta \leq \theta_S + 3\pi/2$, the electric field is **into** the page $(\times)$.

Induction Machine in the Stationary Coordinate System



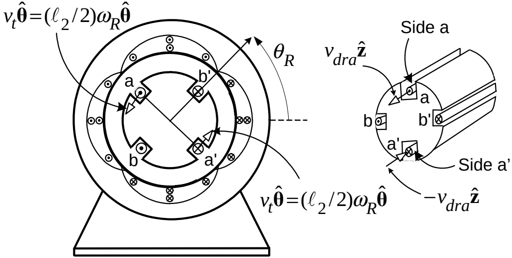
The **total force** on a charge carrier in a **rotor loop** is $\vec{\mathbf{F}} = q\vec{\mathbf{E}}_S + q\vec{\mathbf{v}}\times\vec{\mathbf{B}}_S$.

$$\begin{aligned}
\vec{\mathbf{B}}_S(r,\theta,t) &= B_{Sr}(r,\theta,t)\hat{\mathbf{r}} = (\mu_0\ell_2 N_S I_S/4gr)\cos(\theta-\omega_S t)\hat{\mathbf{r}} \\
\vec{\mathbf{E}}_S(\theta,t) &= E_{Sz}(\theta,t)\hat{\mathbf{z}} = \frac{\mu_0\ell_2 N_S I_S}{4g}\omega_S\cos(\theta-\omega_S t)\hat{\mathbf{z}}.
\end{aligned}$$

- Let v_{dra} be the (drift) speed of the charge carriers along rotor loop a.
- If $v_{dra} > 0$ then the current i_{Ra} is positive.
- Let $v_t = (\ell_2/2)\omega_R$ - the speed of the charge carriers in the $\hat{\boldsymbol{\theta}}$ direction.
- The **total velocity** of the charge carriers along the axial sides of rotor loop a is

$$\vec{\mathbf{v}} = \begin{cases} +v_{dra}\hat{\mathbf{z}} + v_t\hat{\boldsymbol{\theta}} & \text{for side } a \\ -v_{dra}\hat{\mathbf{z}} + v_t\hat{\boldsymbol{\theta}} & \text{for side } a'. \end{cases}$$

Induction Machine in the Stationary Coordinate System



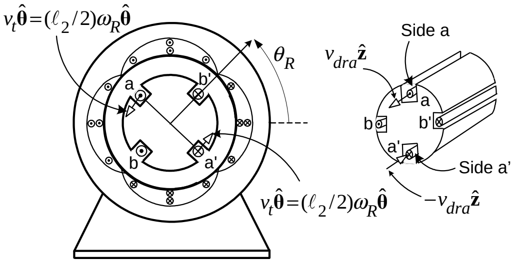
The **magnetic-force/unit-charge** on the charge carriers of rotor loop a is

$$\vec{\mathbf{v}} \times \vec{\mathbf{B}}_S = \begin{cases} +v_{dra}B_{Sr}(\ell_2/2, \theta_R + \pi/2, t)\hat{\boldsymbol{\theta}} - v_t B_{Sr}(\ell_2/2, \theta_R + \pi/2, t)\hat{\mathbf{z}} & \text{for side } a \\ -v_{dra}B_{Sr}(\ell_2/2, \theta_R - \pi/2, t)\hat{\boldsymbol{\theta}} - v_t B_{Sr}(\ell_2/2, \theta_R - \pi/2, t)\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$$

$$B_{Sr}(\ell_2/2, \theta_R + \pi/2, t)\hat{\mathbf{r}} = \frac{\mu_0 N_S I_S}{2g}\cos(\theta_R + \pi/2 - \omega_S t)\hat{\mathbf{r}} = \frac{\mu_0 N_S I_S}{2g}\sin(\omega_S t - \theta_R)\hat{\mathbf{r}}$$

$$B_{Sr}(\ell_2/2, \theta_R - \pi/2, t)\hat{\mathbf{r}} = \frac{\mu_0 N_S I_S}{2g}\cos(\theta_R - \pi/2 - \omega_S t)\hat{\mathbf{r}} = -\frac{\mu_0 N_S I_S}{2g}\sin(\omega_S t - \theta_R)\hat{\mathbf{r}}.$$

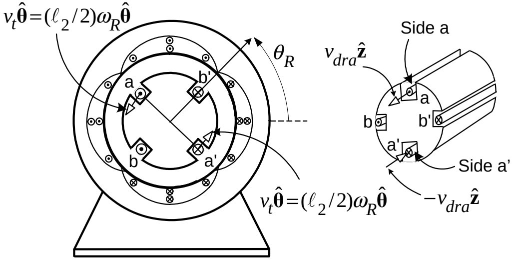
Induction Machine in the Stationary Coordinate System



With $v_t = (\ell_2/2)\omega_R$ and $\theta_R = \omega_R t$

$$\vec{\mathbf{v}} \times \vec{\mathbf{B}}_S = \begin{cases} v_{dra}\dfrac{\mu_0 N_S I_S}{2g}\sin(\omega_S t - \omega_R t)\hat{\boldsymbol{\theta}} - \omega_R\dfrac{\mu_0 \ell_2 N_S I_S}{4g}\sin(\omega_S t - \omega_R t)\hat{\mathbf{z}} & \text{for side } a \\ v_{dra}\dfrac{\mu_0 N_S I_S}{2g}\sin(\omega_S t - \omega_R t)\hat{\boldsymbol{\theta}} + \omega_R\dfrac{\mu_0 \ell_2 N_S I_S}{4g}\sin(\omega_S t - \omega_R t)\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$$

Induction Machine in the Stationary Coordinate System



The electric field $\vec{\mathbf{E}}_S$ in rotor loop a is given by

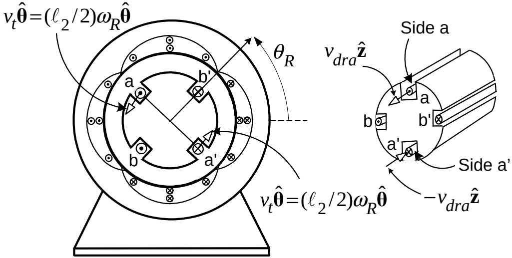
$$\vec{\mathbf{E}}_S = \begin{cases} E_{Sz}(\theta_R + \pi/2, t)\hat{\mathbf{z}} & \text{for side } a \\ E_{Sz}(\theta_R - \pi/2, t)\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$$

With $\theta_R(t) = \omega_R t$, this becomes

$$\vec{\mathbf{E}}_S = \frac{\mu_0 \ell_2 N_S I_S}{4g}\omega_S \cos(\omega_R t + \pi/2 - \omega_S t)\hat{\mathbf{z}} = \frac{\mu_0 \ell_2 N_S I_S}{4g}\omega_S \sin(\omega_S t - \omega_R t)\hat{\mathbf{z}} \quad \text{side } a$$

$$\vec{\mathbf{E}}_S = \frac{\mu_0 \ell_2 N_S I_S}{4g}\omega_S \cos(\omega_R t - \pi/2 - \omega_S t)\hat{\mathbf{z}} = -\frac{\mu_0 \ell_2 N_S I_S}{4g}\omega_S \sin(\omega_S t - \omega_R t)\hat{\mathbf{z}} \quad \text{side } a'$$

Induction Machine in the Stationary Coordinate System

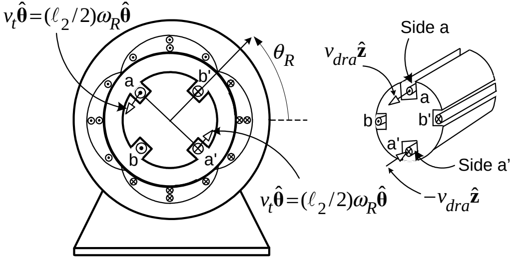


The total force per unit charge $\vec{\mathbf{F}}/q = \vec{\mathbf{E}}_S + \vec{\mathbf{v}}\times\vec{\mathbf{B}}_S$ is then

$$\vec{\mathbf{F}}_{\text{side }a}/q = \frac{\mu_0\ell_2 N_S I_S}{4g}\omega_S \sin(\omega_S t - \omega_R t)\hat{\mathbf{z}} + v_{dra}\frac{\mu_0 N_S I_S}{2g}\sin(\omega_S t - \omega_R t)\hat{\boldsymbol{\theta}}$$
$$-\omega_R\frac{\mu_0\ell_2 N_S I_S}{4g}\sin(\omega_S t - \omega_R t)\hat{\mathbf{z}}$$

$$\vec{\mathbf{F}}_{\text{side }a'}/q = -\frac{\mu_0\ell_2 N_S I_S}{4g}\omega_S \sin(\omega_S t - \omega_R t)\hat{\mathbf{z}} + v_{dra}\frac{\mu_0 N_S I_S}{2g}\sin(\omega_S t - \omega_R t)\hat{\boldsymbol{\theta}}$$
$$+\omega_R\frac{\mu_0\ell_2 N_S I_S}{4g}\sin(\omega_S t - \omega_R t)\hat{\mathbf{z}}.$$

Induction Machine in the Stationary Coordinate System



These last two expressions simplify to

$$\vec{\mathbf{F}}_{\text{side } a}/q = v_{dra}\frac{\mu_0 N_S I_S}{2g}\sin\left((\omega_S-\omega_R)t\right)\hat{\boldsymbol{\theta}} + \underbrace{\frac{\mu_0 \ell_2 N_S I_S}{4g}(\omega_S-\omega_R)\sin\left((\omega_S-\omega_R)t\right)\hat{\mathbf{z}}}_{\vec{\mathbf{E}}_S+(\vec{\mathbf{v}}\times\vec{\mathbf{B}}_S)_z}$$

$$\vec{\mathbf{F}}_{\text{side } a'}/q = v_{dra}\frac{\mu_0 N_S I_S}{2g}\sin\left((\omega_S-\omega_R)t\right)\hat{\boldsymbol{\theta}} \underbrace{-\frac{\mu_0 \ell_2 N_S I_S}{4g}(\omega_S-\omega_R)\sin\left((\omega_S-\omega_R)t\right)\hat{\mathbf{z}}}_{\vec{\mathbf{E}}_S+(\vec{\mathbf{v}}\times\vec{\mathbf{B}}_S)_z}.$$

- The component in $\hat{\boldsymbol{\theta}}$ direction is what produces the **torque**.

Induction Machine in the Stationary Coordinate System

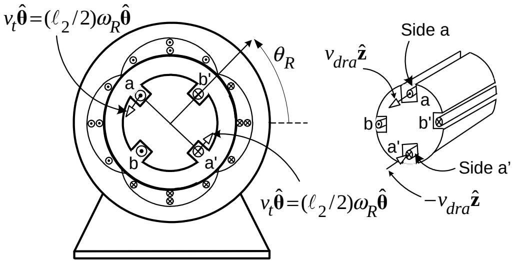
- $qNS\ell_1$ is the **total** number of charge carriers on each axial side of the loop.
- N the number of charge carriers per unit volume in rotor loop a.
- S the cross-sectional area of the rotor loop.
- $i_{Ra} = qNSv_{dra}$ is the rotor current.

The **total tangential magnetic force** on these charge carriers is

$$\begin{aligned}
\vec{\mathbf{F}}_{\hat{\boldsymbol{\theta}}\text{ side }a}/q &= qNS\ell_1 v_{dra}\frac{\mu_0 N_S I_S}{2g}\sin\left((\omega_S-\omega_R)t\right)\hat{\boldsymbol{\theta}} \\
\vec{\mathbf{F}}_{\hat{\boldsymbol{\theta}}\text{ side }a'}/q &= qNS\ell_1 v_{dra}\frac{\mu_0 N_S I_S}{2g}\sin\left((\omega_S-\omega_R)t\right)\hat{\boldsymbol{\theta}}.
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{\tau}}_{Ra} = \left(\frac{\ell_2}{2}\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\hat{\theta}}\right)_{side-a} + \left(\frac{\ell_2}{2}\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\hat{\theta}}\right)_{side-a'} &= 2i_{Ra}\ell_1\frac{\mu_0 N_S I_S}{2g}\sin((\omega_S-\omega_R)t)\frac{\ell_2}{2}\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} \\
&= i_{Ra}\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\sin((\omega_S-\omega_R)t)\hat{\mathbf{z}}.
\end{aligned}$$

Induction Machine in the Stationary Coordinate System



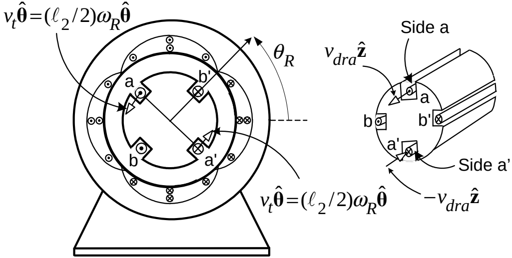
The current i_{Ra} in the rotor loop is produced by the **total emf** in the loop. The total **axial** or (z-component) of the force per unit charge is

$$
\begin{aligned}
(\vec{\mathbf{F}}_{\text{side } a}/q)_z &= \vec{\mathbf{E}}_S + (\vec{\mathbf{v}}\times\vec{\mathbf{B}}_S)_z = \frac{\mu_0\ell_2 N_S I_S}{4g}(\omega_S-\omega_R)\sin((\omega_S-\omega_R)t)\hat{\mathbf{z}} \\
(\vec{\mathbf{F}}_{\text{side } a'}/q)_z &= \vec{\mathbf{E}}_S + (\vec{\mathbf{v}}\times\vec{\mathbf{B}}_S)_z = -\frac{\mu_0\ell_2 N_S I_S}{4g}(\omega_S-\omega_R)\sin((\omega_S-\omega_R)t)\hat{\mathbf{z}}.
\end{aligned}
$$

$$
\xi_{Ra} \triangleq \oint_{\text{rotor loop } a} (\vec{\mathbf{F}}/q)\cdot d\vec{\boldsymbol{\ell}} = \oint_{\text{rotor loop } a} (\vec{\mathbf{E}}_S + \vec{\mathbf{v}}\times\vec{\mathbf{B}}_S)\cdot d\vec{\boldsymbol{\ell}}
$$

where $d\vec{\boldsymbol{\ell}} = \begin{cases} +dz\hat{\mathbf{z}} & \text{for side } a \\ -dz\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$

Induction Machine in the Stationary Coordinate System



Evaluating the line integral we have

$$\xi_{Ra} = \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}(\omega_S - \omega_R)\sin((\omega_S - \omega_R)t).$$

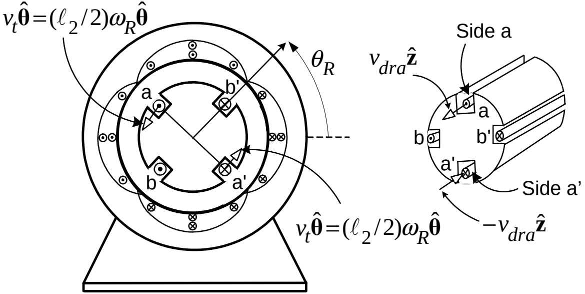
Neglecting the rotor inductance so that $i_{Ra} = \xi_{Ra}/R_R$, the torque on rotor loop a is

$$\tau_{Ra} = \frac{1}{R_R}\left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 (\omega_S - \omega_R)\sin^2\Big((\omega_S - \omega_R)t\Big).$$

A similar analysis shows that

$$\tau_{Rb} = \frac{1}{R_R}\left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 (\omega_S - \omega_R)\cos^2\Big((\omega_S - \omega_R)t\Big).$$

Induction Machine in the Stationary Coordinate System

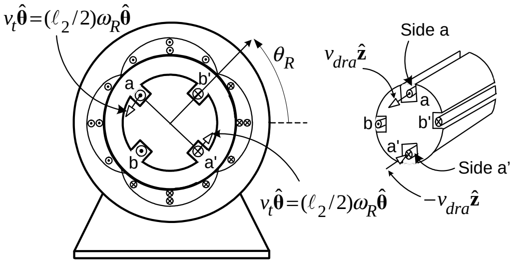


The total torque on the rotor is

$$\tau = \tau_{Ra} + \tau_{Rb} = \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{1}{R_R}(\omega_S - \omega_R)$$

which is the same expression as that computed in the macroscopic case.

Electromechanical Energy Conversion



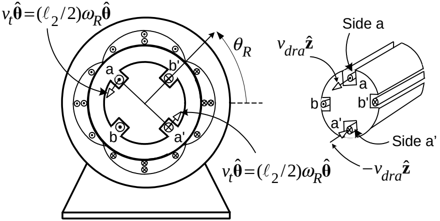
Let V_{Ra} denote the voltage in rotor loop a produced by the stator electric field $\vec{\mathbf{E}}_S$, i.e.,

$$V_{Ra} \triangleq \oint_{\text{rotor loop } a} \vec{\mathbf{E}}_S \cdot d\vec{\boldsymbol{\ell}} = \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\omega_S \sin\Big((\omega_S - \omega_R)t\Big).$$

Define the "back emf" ζ_{Ra} as

$$\begin{aligned}
\zeta_{Ra} \triangleq \oint_{\text{rotor loop } a} \vec{\mathbf{v}} \times \vec{\mathbf{B}}_S \cdot d\vec{\boldsymbol{\ell}} &= \int_0^{\ell_1} \left(-\omega_R \frac{\mu_0 \ell_2 N_S I_S}{4g} \sin(\omega_S t - \omega_R t)\hat{\mathbf{z}}\right) \cdot (d\ell\hat{\mathbf{z}}) \\
&\quad + \int_0^{\ell_1} \left(\omega_R \frac{\mu_0 \ell_2 N_S I_S}{4g} \sin(\omega_S t - \omega_R t)\hat{\mathbf{z}}\right) \cdot (-d\ell\hat{\mathbf{z}}) \\
&= -\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\omega_R \sin((\omega_S - \omega_R)t).
\end{aligned}$$

Electromechanical Energy Conversion

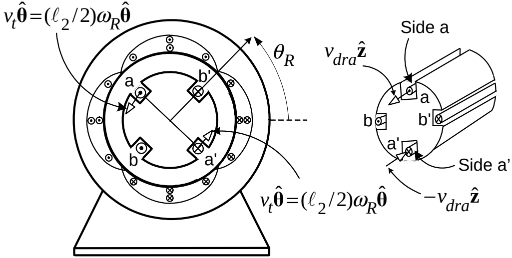


In summary, the **total emf** ξ_{Ra} in rotor loop a is

$$\xi_{Ra} = \oint_{\text{rotor loop } a} (\vec{\mathbf{E}}_S + \vec{\mathbf{v}}\times\vec{\mathbf{B}}_S)\cdot d\vec{\boldsymbol{\ell}} = V_{Ra} + \zeta_{Ra} = \frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}(\omega_S - \omega_R)\sin((\omega_S - \omega_R)t)$$

- V_{Ra} is the voltage in rotor loop a **produced** by $\vec{\mathbf{E}}_S$.
- ζ_{Ra} is the back emf in rotor loop a **produced** by $(\vec{\mathbf{v}}\times\vec{\mathbf{B}}_S)_z$.
- Recall $\tau_{Ra} = i_{Ra}\dfrac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\sin((\omega_S - \omega_R)t)$.
- Recall $\zeta_{Ra} = -\dfrac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\omega_R \sin((\omega_S - \omega_R)t)$.
- By **direct substitution** $\tau_{Ra}\omega_R + i_{Ra}\zeta_{Ra} = 0$.
- The power $i_{Ra}\zeta_{Ra}$ absorbed by the back emf **equals** the mechanical power $\tau_{Ra}\omega_R$.

Electromechanical Energy Conversion



- $\tau_{Ra}\omega_R$ is the **mechanical power** produced on rotor loop *a*.
- $i_{Ra}\zeta_{Ra}$ is the **electrical power** absorbed by the back emf.
- $\xi_{Ra} = V_{Ra} + \zeta_{Ra}$ is the **total emf** in rotor loop *a*.
- $i_{Ra}V_{Ra}$ is the **total electrical power** into rotor loop *a* from the stator.

$$i_{Ra}V_{Ra} = i_{Ra}\xi_{Ra} - i_{Ra}\zeta_{Ra} = i_{Ra}\xi_{Ra} + \tau_{Ra}\omega_R.$$

- The **total power** $i_{Ra}V_{Ra}$ into rotor loop *a* is converted into:
  (1) the **electrical power** $i_{Ra}\xi_{Ra}$ (dissipated as heat in the rotor loop resistance) **and**
  (2) the **mechanical power** $\tau_{Ra}\omega_R$ $(= -i_{Ra}\zeta_{Ra})$.

Faraday's Law and the Integral of Force/Unit-Charge

By Faraday's law the **total emf** ξ_{Ra} in rotor loop a is

$$\xi_{Ra} = -\frac{d}{dt}\left(\int_S \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}}\right).$$

Using the microscopic approach we showed the **total emf** ξ_{Ra} is given by

$$\xi_{Ra} = \oint_C (\vec{\mathbf{E}}_S + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_S) \cdot d\vec{\boldsymbol{\ell}}.$$

In general, with S any surface enclosed by a curve C, we have

$$\xi = -\frac{d}{dt}\left(\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}\right) = \oint_C (\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot d\vec{\boldsymbol{\ell}}.$$

To show this, we calculate

$$\begin{aligned}
\oint_C (\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot d\vec{\boldsymbol{\ell}} &= \int_S \left(\nabla \times \vec{\mathbf{E}}\right) \cdot d\vec{\mathbf{S}} + \oint_C \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \cdot d\vec{\boldsymbol{\ell}} \quad \textbf{Stokes' theorem} \\
&= \int_S \left(-\frac{\partial \vec{\mathbf{B}}}{\partial t}\right) \cdot d\vec{\mathbf{S}} + \oint_C \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \cdot d\vec{\boldsymbol{\ell}} \\
&= -\frac{\partial}{\partial t}\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \oint_C \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \cdot d\vec{\boldsymbol{\ell}}.
\end{aligned}$$

Faraday's Law and the Integral of Force/Unit-Charge

By the previous slide

$$\oint_C (\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot d\vec{\boldsymbol{\ell}} = -\frac{\partial}{\partial t} \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \oint_C \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \cdot d\vec{\boldsymbol{\ell}}.$$

Using the vector identity

$$\left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \cdot d\vec{\boldsymbol{\ell}} = -\vec{\mathbf{B}} \cdot \left(\vec{\mathbf{v}} \times d\vec{\boldsymbol{\ell}}\right)$$

this reduces to

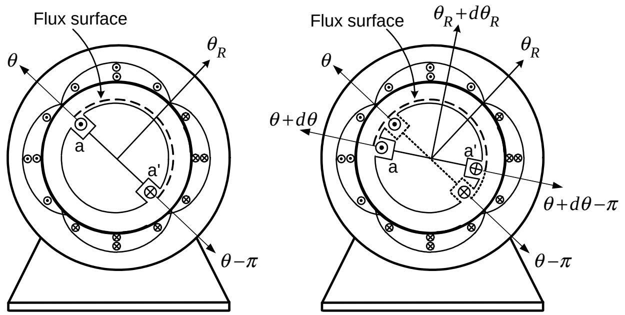
$$\oint_C (\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot d\vec{\boldsymbol{\ell}} = -\frac{\partial}{\partial t} \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} - \oint_C \vec{\mathbf{B}} \cdot \left(\vec{\mathbf{v}} \times d\vec{\boldsymbol{\ell}}\right).$$

- $\vec{\mathbf{v}}$ is the **total velocity** of the charge carriers in the loop C.
- If the loop itself is **not** moving, then $\vec{\mathbf{v}}$ is in the same direction as $d\vec{\boldsymbol{\ell}}$ so that $\vec{\mathbf{v}} \times d\vec{\boldsymbol{\ell}} = 0$.
- In general, with γ the **angle between** $\vec{\mathbf{v}}$ and $d\vec{\boldsymbol{\ell}}$, we have

$$|\,\vec{\mathbf{v}} \times d\vec{\boldsymbol{\ell}}\,| = v\sin(\gamma)d\ell = v_\perp d\ell.$$

- $v_\perp = v\sin(\gamma)$ is the velocity component **perpendicular** to $d\vec{\boldsymbol{\ell}}$.
- The quantity $|\,\vec{\mathbf{v}} \times d\vec{\boldsymbol{\ell}}\,|$ represents a **change** in the **flux surface** as explained next.

Faraday's Law and the Integral of Force/Unit-Charge



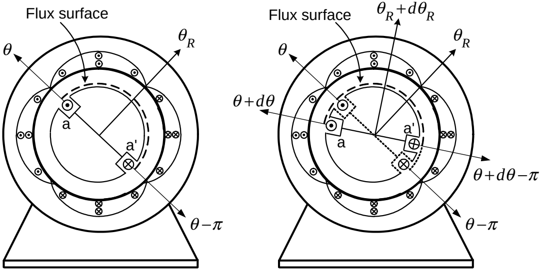
To fix ideas, let's go back to rotor loop a with the fields $\vec{\mathbf{B}}_S$ and $\vec{\mathbf{E}}_S$.

- At time t the rotor is at $\theta_R(t)$ with side a at $\theta(t) = \theta_R(t) + \pi/2$. Side a' is at $\theta(t) + d\theta - \pi$.
- In the time dt, side a of the loop rotates from $\theta(t) = \theta_R(t) + \pi/2$ to

$$\theta(t+dt) = \theta(t) + d\theta = \theta_R(t) + d\theta_R + \pi/2.$$

  Side a' goes to $\theta(t) + d\theta - \pi = \theta_R(t) + d\theta_R - \pi/2$.
- $d\theta = d\theta_R$ and $d\theta/dt = d\theta_R/dt = \omega_R$.

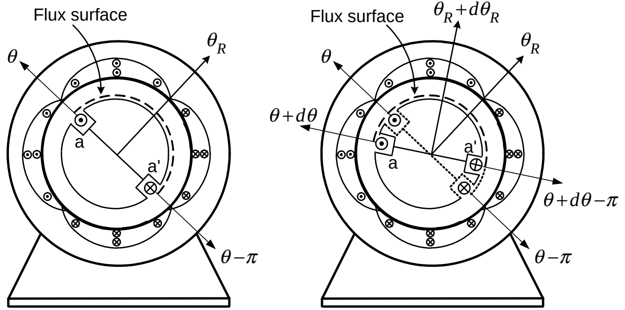
Faraday's Law and the Integral of Force/Unit-Charge



$$
\begin{aligned}
dt\oint_C \vec{\mathbf{B}}_S\cdot\left(\vec{\mathbf{v}}\times d\vec{\boldsymbol{\ell}}\,\right) &= dt\int_{\text{side } a} B_{Sr}(r_R,\theta,t)\hat{\mathbf{r}}\cdot\left(\frac{\ell_2}{2}\omega_R\hat{\boldsymbol{\theta}}\times d\ell\hat{\mathbf{z}}\right)\\
&\quad + dt\int_{\text{side } a\prime} B_{Sr}(r_R,\theta-\pi,t)\hat{\mathbf{r}}\cdot\left(\frac{\ell_2}{2}\omega_R\hat{\boldsymbol{\theta}}\times(-d\ell)\hat{\mathbf{z}}\right)\\
&= \int_{\text{side } a} B_{Sr}(r_R,\theta,t)\hat{\mathbf{r}}\cdot\left(\omega_R dt\frac{\ell_2}{2}d\ell\hat{\mathbf{r}}\right)+\int_{\text{side } a'} B_{Sr}(r_R,\theta-\pi,t)\hat{\mathbf{r}}\cdot\left(-\omega_R dt\frac{\ell_2}{2}d\ell\hat{\mathbf{r}}\right)\\
&= \int_{\text{side } a}\left(B_{Sr}(r_R,\theta,t)\frac{\ell_2}{2}\omega_R dt\right)d\ell+\int_{\text{side } a'}\left(-B_{Sr}(r_R,\theta-\pi,t)\frac{\ell_2}{2}\omega_R dt\right)d\ell\\
&= B_{Sr}(r_R,\theta,t)\ell_1\frac{\ell_2}{2}d\theta - B_{Sr}(r_R,\theta-\pi,t)\ell_1\frac{\ell_2}{2}d\theta
\end{aligned}
$$

where $d\theta=\omega_R dt$ was used in the last line.

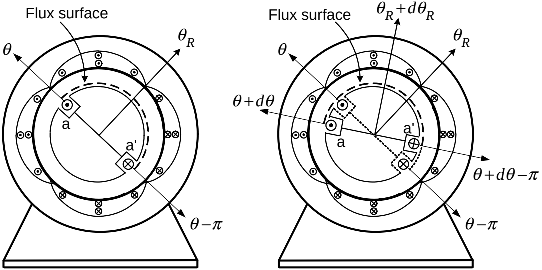
Faraday's Law and the Integral of Force/Unit-Charge



- $B_{Sr}(r_R, \theta, t)\ell_1(\ell_2/2)d\theta$ is the **additional** flux due to the surface area change as side a moved from θ to $\theta + d\theta$.
- $-B_{Sr}(r_R, \theta - \pi, t)\ell_1(\ell_2/2)d\theta$ is the **decrease** in flux due to the surface area change as side a' moved from $\theta - \pi$ to $\theta + d\theta - \pi$.
- I.e., the **change in flux** $d\phi$ due to the **rotation** of the loop through the angle $d\theta$ is

$$dt \oint_C \left(\vec{\mathbf{v}} \times d\vec{\boldsymbol{\ell}}\ \right) \cdot \vec{\mathbf{B}}_S = B_{Sr}(r_R, \theta, t)\ell_1 \frac{\ell_2}{2} d\theta - B_{Sr}(r_R, \theta - \pi, t)\ell_1 \frac{\ell_2}{2} d\theta.$$

Faraday's Law and the Integral of Force/Unit-Charge

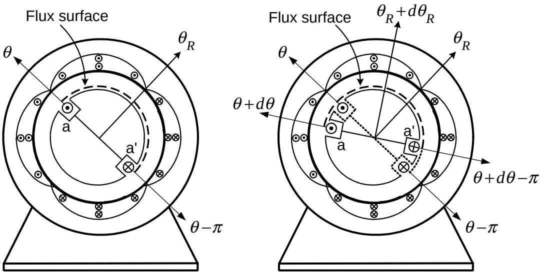


In more detail,

$$\phi(\theta(t), t) \triangleq \int_S \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} = \int_{\theta(t)-\pi}^{\theta(t)} B_r(r_R, \theta', t)\ell_1 \frac{\ell_2}{2} d\theta'$$

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{\partial\phi(\theta(t), t)}{\partial t} + \frac{\partial\phi(\theta(t), t)}{\partial\theta}\frac{d\theta}{dt} \\
&= \frac{\partial\phi(\theta, t)}{\partial t} + \frac{\partial}{\partial\theta}\left(\int_{\theta(t)-\pi}^{\theta(t)} B_r(r_R, \theta', t)\ell_1 \frac{\ell_2}{2} d\theta'\right)\frac{d\theta}{dt} \\
&= \frac{\partial}{\partial t}\int_S \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} + \left(B_r(r_R, \theta, t)\ell_1\frac{\ell_2}{2} - B_r(r_R, \theta-\pi, t)\ell_1\frac{\ell_2}{2}\right)\frac{d\theta}{dt}.
\end{aligned}$$

Faraday's Law and the Integral of Force/Unit-Charge



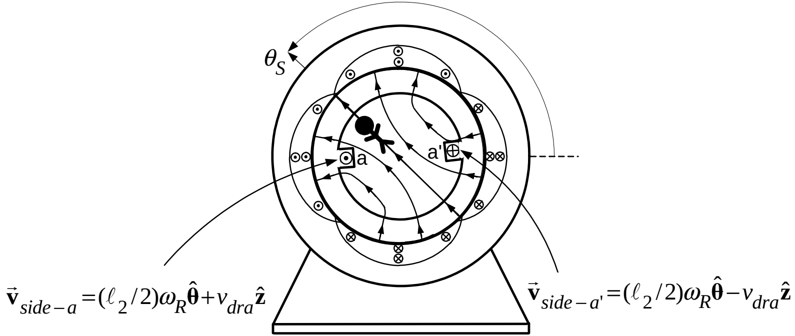
Thus the rate of change of flux due to the motion of the loop is

$$\oint_C \left(\vec{\mathbf{v}} \times d\vec{\ell}\,\right) \cdot \vec{\mathbf{B}} = \frac{\partial\phi(\theta,t)}{\partial\theta}\frac{d\theta}{dt}.$$

In summary

$$\begin{aligned}\xi = -\frac{d\phi}{dt} = -\frac{\partial\phi(\theta,t)}{\partial t} - \frac{\partial\phi(\theta,t)}{\partial\theta}\frac{d\theta}{dt} &= -\frac{\partial}{\partial t}\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} - \oint_C \left(\vec{\mathbf{v}} \times d\vec{\ell}\,\right) \cdot \vec{\mathbf{B}} \\ &= \oint_C (\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot d\vec{\ell}.\end{aligned}$$

Induction Machine in the Synchronous Coordinate System

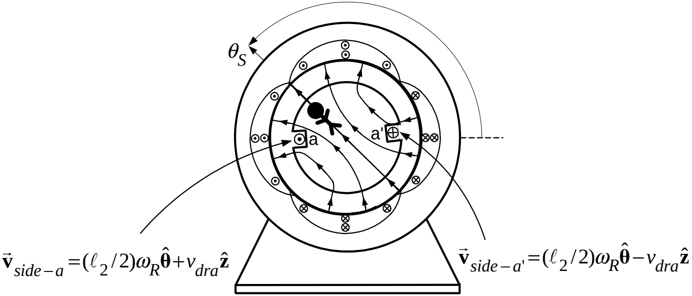


- Go into a reference frame which **rotates** with $\vec{\mathbf{B}}_S$ at angular speed ω_S.
- Electric and magnetic fields **change** from one moving coordinate system to another.
- In any reference/coordinate system, the Lorentz force on a charge carrier is given by

$$\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}).$$

- $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are the electric and magnetic fields **measured** in the particular reference/coordinate system.
- $\vec{\mathbf{v}}$ is the velocity of the charge carrier as **measured** in the same coordinate system.

Induction Machine in the Synchronous Coordinate System



In the coordinate system **fixed to stator**, we have

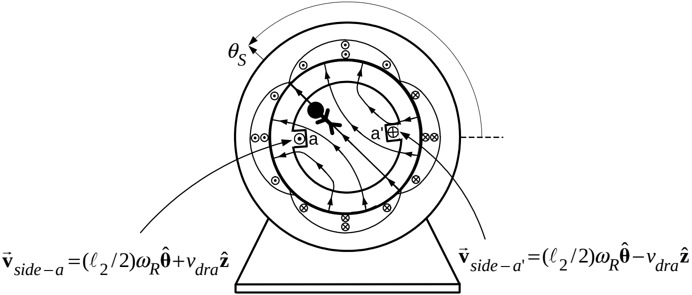
$$\vec{\mathbf{E}}_S(\theta, t) = E_{Sz}(\theta, t)\hat{\mathbf{z}} = \frac{\mu_0 \ell_2 N_S I_S}{4g}\omega_S \cos(\theta - \omega_S t)\hat{\mathbf{z}}$$

$$\vec{\mathbf{B}}_S(r, \theta, t) = B_{Sr}(r, \theta, t)\hat{\mathbf{r}} = \frac{\mu_0 \ell_2 N_S I_S}{4gr}\cos(\theta - \omega_S t)\hat{\mathbf{r}}$$

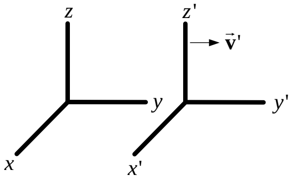
and, with $v_t = (\ell_2/2)\omega_R$,

$$\vec{\mathbf{v}}_{Ra} = \begin{cases} v_{dra}\hat{\mathbf{z}} + v_t\hat{\boldsymbol{\theta}} & \text{side } a \\ -v_{dra}\hat{\mathbf{z}} + v_t\hat{\boldsymbol{\theta}} & \text{side } a'. \end{cases}$$

Induction Machine in the Synchronous Coordinate System



- Go into a coordinate system rotating with $\vec{\mathbf{B}}_S$ at angular speed ω_S.
- This referred to as the **synchronous** coordinate system.
- We will refer to it as the primed ($'$) coordinate system.

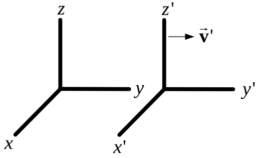


- The primed system has velocity $\vec{\mathbf{v}}' = v'\hat{\mathbf{y}}$ with respect to the unprimed system.
- $\vec{\mathbf{E}}_{||}$ and $\vec{\mathbf{B}}_{||}$ are the electric and magnetic fields parallel to the motion, i.e., in the y direction.
- $\vec{\mathbf{E}}_{\perp}$ and $\vec{\mathbf{B}}_{\perp}$ are the electric and magnetic fields perpendicular to the motion, i.e., in an x-z plane.
- $\gamma \triangleq 1/\sqrt{1-(v'/c)^2}$ where c is the speed of light.

$$\vec{\mathbf{E}}'_{||} = \vec{\mathbf{E}}_{||} \qquad \vec{\mathbf{B}}'_{||} = \vec{\mathbf{B}}_{||}$$

$$\vec{\mathbf{E}}'_{\perp} = \gamma\left(\vec{\mathbf{E}}_{\perp} + \vec{\mathbf{v}}'\times\vec{\mathbf{B}}_{\perp}\right) \quad \vec{\mathbf{B}}'_{\perp} = \gamma\left(\vec{\mathbf{B}}_{\perp} + (v'/c^2)\times\vec{\mathbf{E}}_{\perp}\right)$$

Transformation of the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields

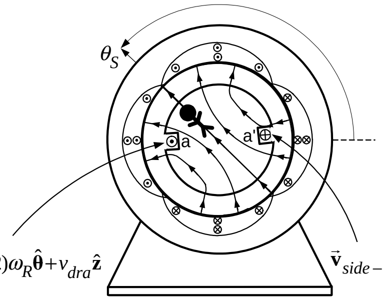


- The primed system has velocity $\vec{\mathbf{v}}' = v'\hat{\mathbf{y}}$ with respect to the unprimed system.
- $\vec{\mathbf{E}}_{||}$ and $\vec{\mathbf{B}}_{||}$ are the electric and magnetic fields parallel to the motion, i.e., in the y direction.
- $\vec{\mathbf{E}}_{\perp}$ and $\vec{\mathbf{B}}_{\perp}$ are the electric and magnetic fields perpendicular to the motion, i.e., in an x-z plane.
- Here $(v'/c)^2 \ll 1$ so we take $\gamma = 1/\sqrt{1-(v'/c)^2} = 1$.

$$\begin{aligned} \vec{\mathbf{E}}'_{||} &= \vec{\mathbf{E}}_{||} & \vec{\mathbf{B}}'_{||} &= \vec{\mathbf{B}}_{||} \\ \vec{\mathbf{E}}'_{\perp} &= \vec{\mathbf{E}}_{\perp} + \vec{\mathbf{v}}' \times \vec{\mathbf{B}}_{\perp} & \vec{\mathbf{B}}'_{\perp} &= \vec{\mathbf{B}}_{\perp}. \end{aligned}$$

- **Only** the component of $\vec{\mathbf{E}}$ perpendicular to the motion changes!

Transformation of the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields



- **Coordinate transformation**

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\omega_S t) & \sin(\omega_S t) & 0 \\ -\sin(\omega_S t) & \cos(\omega_S t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

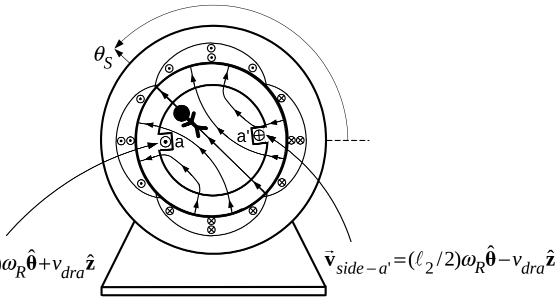
- In **cylindrical coordinates**, $r' = r, \ \ \theta' = \theta - \omega_S t, \ \ z' = z$.
- A **stationary** point (r', θ') in the **rotating** system has velocity

$$\vec{\mathbf{v}}' = r'\omega_S\hat{\boldsymbol{\theta}}$$

in the (original unprimed) **stator** coordinate system.

- In the **rotating** system the **rotor's speed** is $\omega_R' = -(\omega_S - \omega_R) \leq 0$.

Transformation of the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields



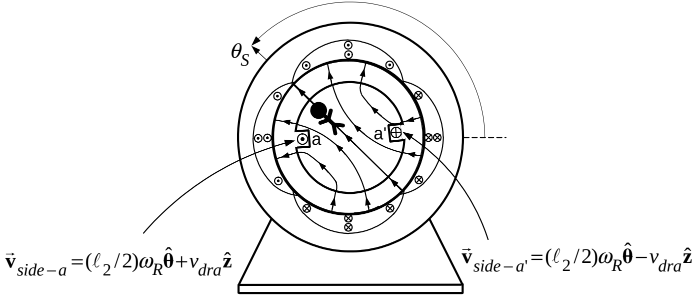
In the air gap of the **stator** coordinate system

$$
\begin{aligned}
\vec{\mathbf{E}}_S(\theta, t) &= E_{Sz}(\theta, t)\hat{\mathbf{z}} = \frac{\mu_0 \ell_2 N_S I_S}{4g}\omega_S \cos(\theta - \omega_S t)\hat{\mathbf{z}} \\
\vec{\mathbf{B}}_S(r, \theta, t) &= B_{Sr}(r, \theta, t)\hat{\mathbf{r}} = \frac{\mu_0 \ell_2 N_S I_S}{4gr}\cos(\theta - \omega_S t)\hat{\mathbf{r}} \\
\vec{\mathbf{v}}' &= r\omega_R\hat{\boldsymbol{\theta}}
\end{aligned}
$$

and

$$
\vec{\mathbf{E}}_\perp = \vec{\mathbf{E}}_S, \quad \vec{\mathbf{E}}_{||} = \mathbf{0}, \quad \vec{\mathbf{B}}_\perp = \vec{\mathbf{B}}_S, \quad \vec{\mathbf{B}}_{||} = \mathbf{0}.
$$

Transformation of the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields

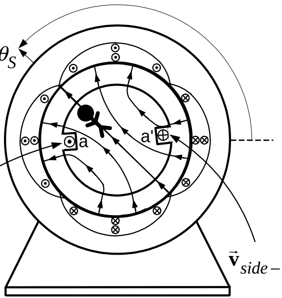


In the air gap of the **rotating** coordinate system

$$\begin{aligned}
\vec{\mathbf{E}}'_S &= \vec{\mathbf{E}}_S + \vec{\mathbf{v}}' \times \vec{\mathbf{B}}_S = E_{Sz}(\theta, t)\hat{\mathbf{z}} + r\omega_S\hat{\boldsymbol{\theta}} \times B_{Sr}\hat{\mathbf{r}} \\
&= \frac{\mu_0 \ell_2 N_S I_S}{4g}\omega_S \cos(\theta - \omega_S t)\hat{\mathbf{z}} - r\omega_S \frac{\mu_0 \ell_2 N_S I_S}{4gr}\cos(\theta - \omega_S t)\hat{\mathbf{z}} \\
&= 0
\end{aligned}$$

$$\vec{\mathbf{B}}'_S = B'_{Sr}(r', \theta')\hat{\mathbf{r}} = \vec{\mathbf{B}}_S\big|_{r=r', \theta=\theta'+\omega_S t} = \frac{\mu_0 \ell_2 N_S I_S}{4gr'}\cos(\theta')\hat{\mathbf{r}}.$$

Transformation of the $\vec{E}$ and $\vec{B}$ fields.



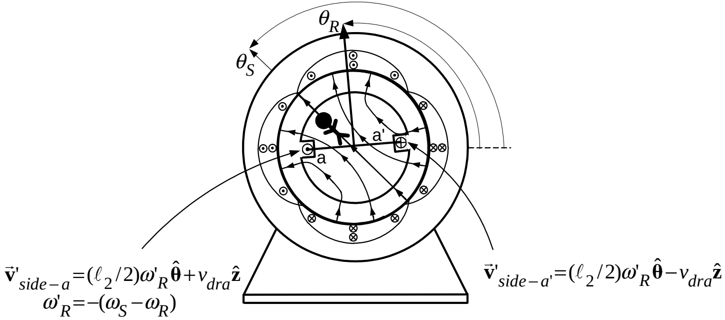
The fact $\vec{\mathbf{E}}'_S = 0$ could have been **anticipated**.

- To an observer in the rotating coordinate system, the magnetic field is given by

$$\vec{\mathbf{B}}'_S = (\mu_0 \ell_2 N_S I_S / 4gr') \cos(\theta') \hat{\mathbf{r}}.$$

- This is **static** in time.
- Faraday's law $\nabla \times \vec{\mathbf{E}}'_S = -\partial \vec{\mathbf{B}}'_S / \partial t = 0$ then gives $\vec{\mathbf{E}}'_S = \mathbf{0}$ as the solution.

Force Per Unit Charge in the Rotating System



In the **rotating** coordinate system the force on rotor loop a is

$$\vec{\mathbf{F}}' = q(\vec{\mathbf{E}}'_S + \vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S) = q\vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S.$$

With $\omega'_R = -(\omega_S - \omega_R)$, **total velocity** of the charge carriers in the **rotating** system is

$$\vec{\mathbf{v}}'_{Ra} = \begin{cases} (\ell_2/2)\omega'_R\hat{\boldsymbol{\theta}} + v_{dra}\hat{\mathbf{z}} & \text{side } a \\ (\ell_2/2)\omega'_R\hat{\boldsymbol{\theta}} - v_{dra}\hat{\mathbf{z}} & \text{side } a' \end{cases}$$

Force Per Unit Charge in the Rotating System

- $\vec{\mathbf{B}}'_S = \vec{\mathbf{B}}_S = B_{Sr}\hat{\mathbf{r}}$
- $\theta' = \theta_R + \pi/2 - \omega_S t = \omega'_R t + \pi/2$ side a
- $\theta' = \theta_R - \pi/2 - \omega_S t = \omega'_R t - \pi/2$ side a'

$$\vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S = \begin{cases} +v_{dra} B_{Sr}(\ell_2/2, \omega'_R t + \pi/2)\hat{\boldsymbol{\theta}} - (\ell_2/2)\omega'_R B_{Sr}(\ell_2/2, \omega'_R t + \pi/2)\hat{\mathbf{z}} & \text{side } a \\ -v_{dra} B_{Sr}(\ell_2/2, \omega'_R t - \pi/2)\hat{\boldsymbol{\theta}} - (\ell_2/2)\omega'_R B_{Sr}(\ell_2/2, \omega'_R t - \pi/2)\hat{\mathbf{z}} & \text{side } a' \end{cases}$$

where

$$\begin{aligned} B_{Sr}(\ell_2/2, \omega'_R t + \pi/2) &= \frac{\mu_0 \ell_2 N_S I_S}{4g(\ell_2/2)} \cos(\omega'_R t + \pi/2) = -\frac{\mu_0 N_S I_S}{2g} \sin(\omega'_R t) \\ B_{Sr}(\ell_2/2, \omega'_R t - \pi/2) &= \frac{\mu_0 \ell_2 N_S I_S}{4g(\ell_2/2)} \cos(\omega'_R t - \pi/2) = +\frac{\mu_0 N_S I_S}{2g} \sin(\omega'_R t). \end{aligned}$$

Finally

$$\begin{aligned} \vec{\mathbf{F}}'_{\text{side } a}/q &= \vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S = -v_{dra} \frac{\mu_0 N_S I_S}{2g} \sin(\omega'_R t)\hat{\boldsymbol{\theta}} + (\ell_2/2)\omega'_R \frac{\mu_0 N_S I_S}{2g} \sin(\omega'_R t)\hat{\mathbf{z}} \\ \vec{\mathbf{F}}'_{\text{side } a'}/q &= \vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S = -v_{dra} \frac{\mu_0 N_S I_S}{2g} \sin(\omega'_R t)\hat{\boldsymbol{\theta}} - (\ell_2/2)\omega'_R \frac{\mu_0 N_S I_S}{2g} \sin(\omega'_R t)\hat{\mathbf{z}}. \end{aligned}$$

- As $\omega'_R = -(\omega_S - \omega_R)$ this is the **same** as in the **stator** coordinate system.

Force Per Unit Charge in the Rotating System

We have shown

$$\vec{\mathbf{F}}'/q = \vec{\mathbf{v}}'_{Ra}\times\vec{\mathbf{B}}'_S = \vec{\mathbf{F}}/q = q(\vec{\mathbf{E}}_S + \vec{\mathbf{v}}_{Ra}\times\vec{\mathbf{B}}_S).$$

The forces in the two coordinate systems must be equal.
The total torque on rotor loop a is then ($i_{Ra} = qNSv_{dra}$)

$$\vec{\boldsymbol{\tau}}'_{Ra} = 2(qNS\ell_1)(-v_{dra})\frac{\mu_0 N_S I_S}{2g}\sin(\omega'_R t)(\ell_2/2)(\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}) = -i_{Ra}\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\sin(\omega'_R t)\hat{\mathbf{z}}$$

or

$$\tau'_{Ra} = i_{Ra}\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\sin\Big((\omega_S-\omega_R)t\Big).$$

Similarly

$$\vec{\boldsymbol{\tau}}'_{Rb} = -i_{Rb}\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\cos(\omega'_R t)\hat{\mathbf{z}} = -i_{Rb}\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\cos\Big((\omega_S-\omega_R)t\Big)\hat{\mathbf{z}}.$$

Force Per Unit Charge in the Rotating System

We showed

$$\left(\vec{\mathbf{v}}'_{Ra}\times\vec{\mathbf{B}}'_S\right)_z = \begin{cases} +\omega'_R\dfrac{\mu_0\ell_2 N_S I_S}{4g}\sin(\omega'_R t)\hat{\mathbf{z}} & \text{for side } a \\ -\omega'_R\dfrac{\mu_0\ell_2 N_S I_S}{4g}\sin(\omega'_R t)\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$$

As $\vec{\mathbf{E}}'_S = \mathbf{0}$, this is the **total axial force** per unit charge $(\vec{\mathbf{F}}/q)_z$. With

$$d\vec{\boldsymbol{\ell}} = \begin{cases} dz\hat{\mathbf{z}} & \text{for side } a \\ -dz\hat{\mathbf{z}} & \text{for side } a' \end{cases}$$

the emf induced in rotor loop a is simply

$$\begin{aligned} \xi'_{Ra} = \oint_{\text{rotor loop } a} (\vec{\mathbf{F}}/q)_z \cdot d\vec{\boldsymbol{\ell}} &= \ell_1\omega'_R\frac{\mu_0\ell_2 N_S I_S}{2g}\sin(\omega'_R t) \\ &= (\omega_S-\omega_R)\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\sin((\omega_S-\omega_R)t). \end{aligned}$$

Similarly,

$$\xi'_{Rb} = \omega'_R\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\cos(\omega'_R t) = -(\omega_S-\omega_R)\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\cos((\omega_S-\omega_R)t).$$

Force Per Unit Charge in the Rotating System

With

$$\begin{aligned} i_{Ra} &= \xi'_{Ra}/R_R \\ i_{Rb} &= \xi'_{Rb}/R_R \end{aligned}$$

the **total torque** on the rotor is

$$\tau' = \tau'_{Ra} + \tau'_{Rb} = -\omega'_R\left(\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\right)^2\frac{1}{R_R} = \frac{1}{R_R}\left(\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\right)^2(\omega_S - \omega_R).$$

This is the **same** as in the **stator** coordinate system.

Electromechanical Energy Conversion

We just showed

$$\begin{aligned} \tau' &= -\omega'_R \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} \right)^2 \frac{1}{R_R} > 0 \\ \omega'_R &= -(\omega_S - \omega_R) < 0. \end{aligned}$$

The mechanical power **delivered** to the rotor is

$$\tau' \omega'_R = -(\omega_S - \omega_R)^2 \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} \right)^2 \frac{1}{R_R} < 0.$$

The **tangential velocity** of the rotor loop is in the $-\hat{\boldsymbol{\theta}}$ direction.

But the **torque** is pushing in the $+\hat{\boldsymbol{\theta}}$ direction, i.e., it **opposes** the rotor's speed!

Where is this mechanical power going?

$$i_{Ra}\xi'_{Ra} + i_{Rb}\xi'_{Rb} = (\xi'_{Ra})^2/R_R + (\xi'_{Rb})^2/R_R = (\omega_S - \omega_R)^2 \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g} \right)^2 \frac{1}{R_R}$$

In the **rotating** system, the induction machine is a **generator** rather than a motor.

Magnetic Force and Work

In the **rotating** system $\vec{\mathbf{E}}'_S = \mathbf{0}$ so

$$\vec{\mathbf{F}}'/q = \vec{\mathbf{E}}'_S + \vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S = \vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S$$

where

$$\vec{\mathbf{v}}'_{Ra} = \begin{cases} (\ell_2/2)\omega'_R\hat{\boldsymbol{\theta}} + v_{dra}\hat{\mathbf{z}} & \text{side } a \\ (\ell_2/2)\omega'_R\hat{\boldsymbol{\theta}} - v_{dra}\hat{\mathbf{z}} & \text{side } a'. \end{cases}$$

Explicitly,

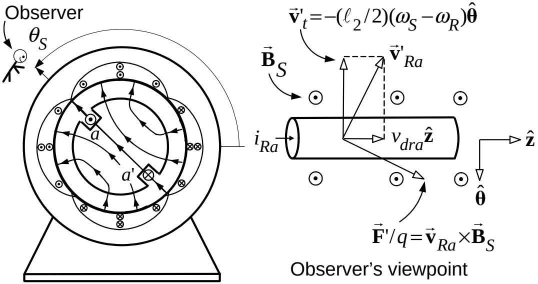
$$\vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S = \begin{cases} -v_{dra}\dfrac{\mu_0 N_S I_S}{2g}\sin(\omega'_R t)\hat{\boldsymbol{\theta}} + \omega'_R\dfrac{\mu_0 \ell_2 N_S I_S}{4g}\sin(\omega'_R t)\hat{\mathbf{z}} & \text{side } a \\ \\ -v_{dra}\dfrac{\mu_0 N_S I_S}{2g}\sin(\omega'_R t)\hat{\boldsymbol{\theta}} - \omega'_R\dfrac{\mu_0 \ell_2 N_S I_S}{4g}\sin(\omega'_R t)\hat{\mathbf{z}} & \text{side } a'. \end{cases}$$

The power per unit charge done by the **magnetic force** $\vec{\mathbf{F}}' = q(\vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S)$ is

$$(\vec{\mathbf{F}}'/q) \cdot \vec{\mathbf{v}}'_{Ra} = (\vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S) \cdot \vec{\mathbf{v}}'_{Ra} \equiv \mathbf{0}.$$

- Magnetic forces **cannot** change the energy of a charged particle as they are always **orthogonal** to the velocity.

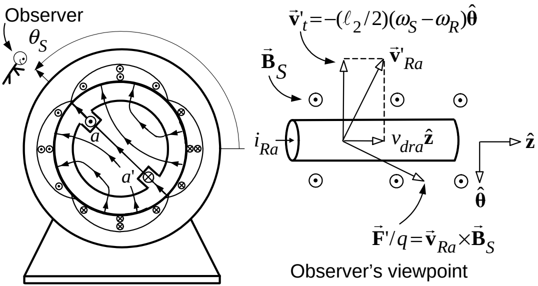
Magnetic Force and Work



- The observer is above side a and **rotating** with $\vec{\mathbf{B}}_S$ at angular speed ω_S.
- The magnetic force **opposes** the rotor velocity to produce the current in the rotor.
- The magnetic force **converts** the mechanical (kinetic) energy into electrical energy.

$$
\begin{aligned}
2(qNS\ell_1)\vec{\mathbf{F}}' \cdot \vec{\mathbf{v}}'_{Ra} &= 2(qNS\ell_1)(\vec{\mathbf{v}}'_{Ra} \times \vec{\mathbf{B}}'_S) \cdot \vec{\mathbf{v}}'_{Ra} \\
&= 2(qNS\ell_1)\left(-v_{dra}\frac{\mu_0 N_S I_S}{2g}\sin(\omega'_R t)\hat{\boldsymbol{\theta}} + \omega'_R \frac{\mu_0 \ell_2 N_S I_S}{4g}\sin(\omega'_R t)\hat{\mathbf{z}}\right) \cdot \left(\frac{\ell_2}{2}\omega'_R\hat{\boldsymbol{\theta}} + v_{dra}\hat{\mathbf{z}}\right) \\
&= 2\left(-qNS\ell_1 v_{dra}\frac{\mu_0 N_S I_S}{2g}\sin(\omega'_R t)\hat{\boldsymbol{\theta}}\right) \cdot \frac{\ell_2}{2}\omega'_R\hat{\boldsymbol{\theta}} + 2\left(\omega'_R\frac{\mu_0\ell_2 N_S I_S}{4g}\sin(\omega'_R t)\hat{\mathbf{z}}\right) \cdot qNS\ell_1 v_{dra}\hat{\mathbf{z}}.
\end{aligned}
$$

Magnetic Force and Work



- By the previous slide

$$\begin{aligned} 2(qNS\ell_1)\vec{\mathbf{F}}' \cdot \vec{\mathbf{v}}'_{Ra} &= -\left(i_{Ra}\frac{\mu_0\ell_1\ell_2N_SI_S}{2g}\sin(\omega'_Rt)\right)\omega'_R\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} + \left(\omega'_R\frac{\mu_0\ell_1\ell_2N_SI_S}{2g}\sin(\omega'_Rt)\right)i_{Ra}\hat{\mathbf{z}}\cdot\hat{\mathbf{z}} \\ &= 0. \end{aligned}$$

- The coefficient of $\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}$ is $\tau'_{Ra}\omega'_R$ - the **mechanical power**.
- The coefficient of $\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}$ is $\xi'_{Ra}i_{Ra}$ - the **electrical power**.
- These terms are equal in magnitude, but opposite in sign.
- The magnetic force **converts** the mechanical energy to electrical energy.