ECE 697 Modeling and High-Performance Control of Electric Machines
HW 8 Solutions
Spring 2022

Chapter 4, Problem 11 *Modifying Ampère's Law to Account for a r_0/r Dependence*

(a) With $H_{Sa}(i_{Sa}, r, \theta) \triangleq \dfrac{r_0}{r} H_{Sa}(i_{Sa}, \theta)$, to have

$$\int_{r=r_R}^{r=r_S} \frac{N_S i_{Sa}}{2g}\frac{r_0}{r}\cos(0)\hat{\mathbf{r}}\cdot(dr\hat{\mathbf{r}}) + \int_{r=r_R}^{r=r_S} \frac{N_S i_{Sa}}{2g}\frac{r_0}{r}\cos(\theta)\hat{\mathbf{r}}\cdot(-dr\hat{\mathbf{r}}) = -\frac{N_S i_{Sa}}{2}\cos(\theta) + \frac{N_S i_{Sa}}{2}$$

reduce to

$$H_{Sa}(i_{Sa}, 0)g - H_{Sa}(i_{Sa}, \theta)g = -i_{Sa}\frac{N_S}{2}\cos(\theta) + i_{Sa}\frac{N_S}{2}.$$

simply requires

$$\int_{r=r_R}^{r=r_S} \frac{r_0}{r} dr = g.$$

(b) The *mean value theorem for integrals* says that there is an r_1, with $r_R < r_1 < r_S$ such that

$$\int_{r=r_R}^{r=r_S} \frac{1}{r} dr = \frac{1}{r_1}(r_S - r_R) = \frac{1}{r_1} g.$$

Consequently, one need only choose $r_0 = r_1$. Another way to see this is to note that r_0 must satisfy

$$\int_{r=r_R}^{r=r_S} \frac{r_0}{r} dr = g$$
$$\Longrightarrow\ r_0 \ln(\frac{r_S}{r_R}) = g.$$

Define $r_0 \triangleq g/\ln(\frac{r_S}{r_R})$. To show that r_0 satisfies $r_R < r_0 < r_S$ use the Taylor series expansion $\ln(1+x) = x - x^2/2 + \cdots$ to write

$$\frac{1}{r_0} = \frac{1}{g}\ln(\frac{r_S}{r_R}) = \frac{1}{g}\ln(\frac{r_R+g}{r_R}) = \frac{1}{g}\ln\left(1 + \frac{g}{r_R}\right) = \frac{1}{g}\left(\frac{g}{r_R} - \left(\frac{g}{r_R}\right)^2 + \cdots\right) < \frac{1}{r_R}$$

and

$$\frac{1}{r_0} < \frac{1}{r_R} \implies r_0 > r_R.$$

On the other hand, one may write

$$\frac{1}{r_0} = -\frac{1}{g}\ln(\frac{r_R}{r_S}) = -\frac{1}{g}\ln(\frac{r_S-g}{r_S}) = -\frac{1}{g}\ln\left(1 - \frac{g}{r_S}\right) = \frac{1}{g}\left(\frac{g}{r_S} + \left(\frac{g}{r_S}\right)^2 + \cdots\right) > \frac{1}{r_S}$$

and

$$\frac{1}{r_S} < \frac{1}{r_0} \implies r_S > r_0.$$

Combining these two computations gives

$$r_R < r_0 < r_S.$$

(c) Note that
$$\int_{r=r_R}^{r=r_S} \frac{r_0}{r} dr = r_0 \left(\ln(r_S) - \ln(r_R)\right) = r_0 \ln\left(\frac{r_S}{r_R}\right)$$
and, as $r_S = r_R + g$, r_0 must satisfy
$$\ln\left(1 + \frac{g}{r_R}\right) = \frac{g}{r_0}.$$
The Taylor series expansion of $\ln(1+x)$ is $\ln(1+x) = x - x^2/2 + \cdots$, so that for small enough $x \triangleq g/r_R$, the approximation $\ln(1+x) = x$ can be used resulting in choosing $r_0 = r_R$. Also, $x = g/r_R$ small means that $r_S = r_R + g \approx r_R$ so that $r_0 = r_S$ is just as valid as choosing $r_0 = r_R$.

Chapter 6

Problem 1

Problem 2

Problem 3

Problem 4 *Statespace Model of the Induction Motor*

Expand the electrical equations
$$\begin{aligned}
L_S \frac{d}{dt} i_{Sa} + M \frac{d}{dt}\Big(+i_{Ra}\cos(\theta_R) - i_{Rb}\sin(\theta_R)\Big) + R_S i_{Sa} &= u_{Sa} \\
L_S \frac{d}{dt} i_{Sb} + M \frac{d}{dt}\Big(+i_{Ra}\sin(\theta_R) + i_{Rb}\cos(\theta_R)\Big) + R_S i_{Sb} &= u_{Sb} \\
L_R \frac{d}{dt} i_{Ra} + M \frac{d}{dt}\Big(+i_{Sa}\cos(\theta_R) + i_{Sb}\sin(\theta_R)\Big) + R_R i_{Ra} &= 0 \\
L_R \frac{d}{dt} i_{Rb} + M \frac{d}{dt}\Big(-i_{Sa}\sin(\theta_R) + i_{Sb}\cos(\theta_R)\Big) + R_R i_{Rb} &= 0
\end{aligned}$$
to obtain
$$\begin{aligned}
L_S \frac{d}{dt} i_{Sa} + M\frac{di_{Ra}}{dt}\cos(\theta_R) - M\frac{di_{Rb}}{dt}\sin(\theta_R) - Mi_{Ra}\sin(\theta_R)\omega_R - Mi_{Rb}\cos(\theta_R)\omega_R + R_S i_{Sa} &= u_{Sa} \\
L_S \frac{d}{dt} i_{Sb} + M\frac{di_{Ra}}{dt}\sin(\theta_R) + M\frac{di_{Rb}}{dt}\cos(\theta_R) + Mi_{Ra}\cos(\theta_R)\omega_R - Mi_{Rb}\sin(\theta_R)\omega_R + R_S i_{Sb} &= u_{Sb} \\
L_R \frac{d}{dt} i_{Ra} + M\frac{di_{Sa}}{dt}\cos(\theta_R) + M\frac{di_{Sb}}{dt}\sin(\theta_R) - Mi_{Sa}\sin(\theta_R)\omega_R + Mi_{Sb}\cos(\theta_R)\omega_R + R_R i_{Ra} &= 0 \\
L_R \frac{d}{dt} i_{Rb} - M\frac{di_{Sa}}{dt}\sin(\theta_R) + M\frac{di_{Sb}}{dt}\cos(\theta_R) - Mi_{Sa}\cos(\theta_R)\omega_R - Mi_{Sb}\sin(\theta_R)\omega_R + R_R i_{Rb} &= 0.
\end{aligned}$$
In matrix form, this becomes
$$\begin{aligned}
&\begin{bmatrix} L_S & 0 & M\cos(\theta_R) & -M\sin(\theta_R) \\ 0 & L_S & M\sin(\theta_R) & M\cos(\theta_R) \\ M\cos(\theta_R) & M\sin(\theta_R) & L_R & 0 \\ -M\sin(\theta_R) & M\cos(\theta_R) & 0 & L_R \end{bmatrix} \frac{d}{dt}\begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix} \\
&= -\begin{bmatrix} R_S & 0 & -M\sin(\theta_R)\omega_R & -M\cos(\theta_R)\omega_R \\ 0 & R_S & M\cos(\theta_R)\omega_R & -M\sin(\theta_R)\omega_R \\ -M\sin(\theta_R)\omega_R & M\cos(\theta_R)\omega_R & R_R & 0 \\ -M\cos(\theta_R)\omega_R & -M\sin(\theta_R)\omega_R & 0 & R_R \end{bmatrix}\begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix} + \begin{bmatrix} u_{Sa} \\ u_{Sb} \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

The inverse of the matrix in front of the derivatives is

$$\begin{bmatrix} L_S & 0 & M\cos(\theta_R) & -M\sin(\theta_R) \\ 0 & L_S & M\sin(\theta_R) & M\cos(\theta_R) \\ M\cos(\theta_R) & M\sin(\theta_R) & L_R & 0 \\ -M\sin(\theta_R) & M\cos(\theta_R) & 0 & L_R \end{bmatrix}^{-1} =$$

$$\frac{1}{\sigma L_S L_R}\begin{bmatrix} L_R & 0 & -M\cos(\theta_R) & M\sin(\theta_R) \\ 0 & L_R & -M\sin(\theta_R) & -M\cos(\theta_R) \\ -M\cos(\theta_R) & -M\sin(\theta_R) & L_S & 0 \\ M\sin(\theta_R) & -M\cos(\theta_R) & 0 & L_S \end{bmatrix}$$

so that

$$\frac{d}{dt}\begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix} = \frac{1}{\sigma L_S L_R}\begin{bmatrix} L_R & 0 & -M\cos(\theta_R) & M\sin(\theta_R) \\ 0 & L_R & -M\sin(\theta_R) & -M\cos(\theta_R) \\ -M\cos(\theta_R) & -M\sin(\theta_R) & L_S & 0 \\ M\sin(\theta_R) & -M\cos(\theta_R) & 0 & L_S \end{bmatrix} \times$$

$$\left(- \begin{bmatrix} R_S & 0 & -M\sin(\theta_R)\omega_R & -M\cos(\theta_R)\omega_R \\ 0 & R_S & M\cos(\theta_R)\omega_R & -M\sin(\theta_R)\omega_R \\ -M\sin(\theta_R)\omega_R & M\cos(\theta_R)\omega_R & R_R & 0 \\ -M\cos(\theta_R)\omega_R & -M\sin(\theta_R)\omega_R & 0 & R_R \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix} + \begin{bmatrix} u_{Sa} \\ u_{Sb} \\ 0 \\ 0 \end{bmatrix} \right).$$

Finally, after expanding, this becomes

$$\frac{d}{dt}\begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix} = \frac{1}{\sigma L_S L_R}\begin{bmatrix} -L_R R_S & M^2\omega_R & L_R M\sin(\theta_R)\omega_R + M\cos(\theta_R)R_R & L_R M\cos(\theta_R)\omega_R - M\sin(\theta_R)R_R \\ -M^2\omega_R & -L_R R_S & -L_R M\cos(\theta_R)\omega_R + M\sin(\theta_R)R_R & L_R M\sin(\theta_R)\omega_R + M\cos(\theta_R)R_R \\ M\cos(\theta_R)R_S + L_S M\sin(\theta_R)\omega_R & M\sin(\theta_R)R_S - L_S M\cos(\theta_R)\omega_R & -L_S R_R & -M^2\omega_R \\ -M\sin(\theta_R)R_S + L_S M\cos(\theta_R)\omega_R & M\cos(\theta_R)R_S + L_S M\sin(\theta_R)\omega_R & M^2\omega_R & -L_S R_R \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix}$$

$$+ \frac{1}{\sigma L_S L_R}\begin{bmatrix} L_R & 0 \\ 0 & L_R \\ -M\cos(\theta_R) & -M\sin(\theta_R) \\ M\sin(\theta_R) & -M\cos(\theta_R) \end{bmatrix}\begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix}.$$

These four differential equations for the currents along with

$$J\frac{d\omega_R}{dt} = M\Big(-i_{Ra}(t)i_{Sa}(t)\sin(\theta_R) + i_{Ra}(t)i_{Sb}(t)\cos(\theta_R) - i_{Rb}(t)i_{Sa}(t)\cos(\theta_R) - i_{Rb}(t)i_{Sb}(t)\sin(\theta_R)\Big)$$
$$\frac{d\theta_R}{dt} = \omega_R$$

form a statespace model of a two-phase induction motor.
A simulation based on this model is given in the Simulation files.

Problem 5 *Space Vector Representation of the Induction Motor*
Along with $\underline{i}_S \triangleq i_{Sa} + ji_{Sb}$, $\underline{i}_R \triangleq i_{Ra} + ji_{Rb}$, $\underline{u}_S \triangleq u_{Sa} + ju_{Sb}$, substitute

$$\begin{aligned}
\underline{i}_R e^{jn_p\theta_R} &= \Big(i_{Ra}\cos(n_p\theta_R) - i_{Rb}\sin(n_p\theta_R)\Big) + j\Big(i_{Ra}\sin(n_p\theta_R) + i_{Rb}\cos(n_p\theta_R)\Big) \\
\underline{i}_S e^{-jn_p\theta_R} &= \Big(i_{Sa}\cos(n_p\theta_R) + i_{Sb}\sin(n_p\theta_R)\Big) + j\Big(-i_{Sa}\sin(n_p\theta_R) + i_{Sb}\cos(n_p\theta_R)\Big)
\end{aligned}$$

into

$$\begin{aligned}
R_S\underline{i}_S + \ L_S\frac{d}{dt}\underline{i}_S + \ M\frac{d}{dt}\left(\underline{i}_R e^{jn_p\theta_R}\right) &= \underline{u}_S \\
R_R\underline{i}_R + \ L_R\frac{d}{dt}\underline{i}_R + \ M\frac{d}{dt}\left(\underline{i}_S e^{-jn_p\theta_R}\right) &= 0 \\
n_pM\,\mathrm{Im}\{\underline{i}_S(\underline{i}_R e^{jn_p\theta_R})^*\} - \tau_L &= J\frac{d\omega_R}{dt}.
\end{aligned}$$

Equating real and imaginary parts gives

$$\begin{aligned}
L_S\frac{d}{dt}i_{Sa} + M\frac{d}{dt}\Big(+i_{Ra}\cos(n_p\theta_R) - i_{Rb}\sin(n_p\theta_R)\Big) + R_Si_{Sa} &= u_{Sa} \\
L_S\frac{d}{dt}i_{Sb} + M\frac{d}{dt}\Big(+i_{Ra}\sin(n_p\theta_R) + i_{Rb}\cos(n_p\theta_R)\Big) + R_Si_{Sb} &= u_{Sb}
\end{aligned}$$

$$\begin{aligned}
L_R\frac{d}{dt}i_{Ra} + M\frac{d}{dt}\Big(+i_{Sa}\cos(n_p\theta_R) + i_{Sb}\sin(n_p\theta_R)\Big) + R_Ri_{Ra} &= 0 \\
L_R\frac{d}{dt}i_{Rb} + M\frac{d}{dt}\Big(-i_{Sa}\sin(n_p\theta_R) + i_{Sb}\cos(n_p\theta_R)\Big) + R_Ri_{Rb} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\tau_R = \ n_pM\Big(&-i_{Ra}(t)i_{Sa}(t)\sin(n_p\theta_R) + i_{Ra}(t)i_{Sb}(t)\cos(n_p\theta_R) \\
&- i_{Rb}(t)i_{Sa}(t)\cos(n_p\theta_R) - i_{Rb}(t)i_{Sb}(t)\sin(n_p\theta_R)\Big).
\end{aligned}$$

Problem 6 *A Standard Model of the Induction Motor*

(a) Define new (fictitious) flux linkages as

$$\underline{\psi}_R \triangleq \psi_{Ra} + j\psi_{Rb} \triangleq \ \underline{\lambda}_R e^{jn_p\theta_R} = L_R\underline{i}_R e^{jn_p\theta_R} + M\underline{i}_S$$

and substitute into

$$R_S\underline{i}_S + \ L_S\frac{d}{dt}\underline{i}_S + \ M\frac{d}{dt}\left(\underline{i}_R e^{jn_p\theta_R}\right) = \underline{u}_S$$

to obtain

$$\begin{aligned}
R_S\underline{i}_S + \ L_S\frac{d}{dt}\underline{i}_S + \ \frac{M}{L_R}\frac{d}{dt}\Big(\underline{\psi}_R - M\underline{i}_S\Big) &= \ \underline{u}_S \\
\implies \frac{M}{L_R}\frac{d}{dt}\underline{\psi}_R + \ \sigma L_S\frac{d}{dt}\underline{i}_S + \ R_S\underline{i}_S &= \ \underline{u}_S
\end{aligned}$$

where $\sigma \triangleq 1 - M^2/L_SL_R$ is the leakage factor. Next, multiply both sides of

$$R_R\underline{i}_R + \ L_R\frac{d}{dt}\underline{i}_R + \ M\frac{d}{dt}\left(\underline{i}_S e^{-jn_p\theta_R}\right) = 0$$

by $e^{jn_p\theta_R}$ to obtain

$$
\begin{aligned}
R_R\underline{i}_R e^{jn_p\theta_R} + \ L_R\frac{d}{dt}\left(\underline{i}_R e^{jn_p\theta_R}\right) - \ jn_p\omega_R L_R\underline{i}_R e^{jn_p\theta_R} + \ M\frac{d}{dt}\underline{i}_S - \ jn_p\omega_R M\underline{i}_S = 0 \\
\implies \frac{R_R}{L_R}\left(\underline{\psi}_R - M\underline{i}_S\right) + \ \frac{d}{dt}\left(\underline{\psi}_R - M\underline{i}_S\right) - \ jn_p\omega_R\left(\underline{\psi}_R - M\underline{i}_S\right) + \ M\frac{d}{dt}\underline{i}_S - \ jn_p\omega_R M\underline{i}_S = 0 \\
\implies \frac{R_R}{L_R}\left(\underline{\psi}_R - M\underline{i}_S\right) + \ \frac{d}{dt}\left(\underline{\psi}_R - M\underline{i}_S\right) - \ jn_p\omega_R\underline{\psi}_R + \ M\frac{d}{dt}\underline{i}_S = 0 \\
\implies \frac{R_R}{L_R}\left(\underline{\psi}_R - M\underline{i}_S\right) + \ \frac{d}{dt}\underline{\psi}_R - \ jn_p\omega_R\underline{\psi}_R = 0 \\
\implies \left(-\frac{1}{T_R} + jn_p\omega_R\right)\underline{\psi}_R + \ \frac{M}{T_R}\underline{i}_S = \frac{d}{dt}\underline{\psi}_R
\end{aligned}
$$

where $T_R \triangleq L_R/R_R$ is the rotor time constant. Finally, rewrite the torque equation

$$
n_p M\,\mathrm{Im}\{\underline{i}_S(\underline{i}_R e^{jn_p\theta_R})^*\} - \tau_L = J\frac{d\omega_R}{dt}
$$

as

$$
\begin{aligned}
n_p\frac{M}{L_R}\,\mathrm{Im}\left\{\underline{i}_S\left(\underline{\psi}_R - M\underline{i}_S\right)^*\right\} - \tau_L = J\frac{d\omega_R}{dt} \\
\implies n_p\frac{M}{L_R}\,\mathrm{Im}\left\{\underline{i}_S\underline{\psi}_R^* - M\underline{i}_S\underline{i}_S^*\right\} - \tau_L = J\frac{d\omega_R}{dt} \\
\implies \frac{n_p M}{L_R}\,\mathrm{Im}\left\{\underline{i}_S\underline{\psi}_R^*\right\} - \tau_L = J\frac{d\omega_R}{dt} \\
\implies \frac{n_p M}{JL_R}\,\mathrm{Im}\left\{\underline{i}_S\underline{\psi}_R^*\right\} - \frac{\tau_L}{J} = \frac{d\omega_R}{dt}.
\end{aligned}
$$

Combining the above gives

$$
\begin{aligned}
\frac{d}{dt}\underline{\psi}_R &= \left(-\frac{1}{T_R} + jn_p\omega_R\right)\underline{\psi}_R + \ \frac{M}{T_R}\underline{i}_S \\
\underline{u}_S &= \frac{M}{L_R}\frac{d}{dt}\underline{\psi}_R + \ \sigma L_S\frac{d}{dt}\underline{i}_S + \ R_S\underline{i}_S \\
\frac{d\omega_R}{dt} &= \mu\,\mathrm{Im}\left\{\underline{i}_S(\underline{\psi}_R)^*\right\} - \frac{\tau_L}{J}
\end{aligned}
$$

where $\mu \triangleq n_p M/(JL_R)$.

(b) Equating the real and imaginary parts in part (a), a differential equation model is given by

$$
\begin{aligned}
\frac{d\psi_{Ra}}{dt} &= -\frac{R_R}{L_R}\psi_{Ra} - n_p\omega_R\psi_{Rb} + \frac{MR_R}{L_R}i_{Sa} \\
\frac{d\psi_{Rb}}{dt} &= -\frac{R_R}{L_R}\psi_{Rb} + n_p\omega_R\psi_{Ra} + \frac{MR_R}{L_R}i_{Sb} \\
u_{Sa} &= R_S i_{Sa} + \sigma L_S\frac{di_{Sa}}{dt} + \frac{M}{L_R}\frac{d\psi_{Ra}}{dt} \\
u_{Sb} &= R_S i_{Sb} + \sigma L_S\frac{di_{Sb}}{dt} + \frac{M}{L_R}\frac{d\psi_{Rb}}{dt} \\
\frac{d\omega_R}{dt} &= \frac{n_p M}{JL_R}\left(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}\right) - \frac{\tau_L}{J}.
\end{aligned}
\tag{1}
$$

(c) Substitute the expressions for $d\psi_{Ra}/dt$ and $d\psi_{Rb}/dt$ from the first two equations of (1) into the 3rd and 4th equations of (1) to obtain

$$
\begin{aligned}
u_{Sa} &= R_S i_{Sa} + \sigma L_S \frac{di_{Sa}}{dt} + \frac{M}{L_R}\left(-\frac{R_R}{L_R}\psi_{Ra} - n_p\omega_R\psi_{Rb} + \frac{MR_R}{L_R}i_{Sa}\right) \\
u_{Sb} &= R_S i_{Sb} + \sigma L_S \frac{di_{Sb}}{dt} + \frac{M}{L_R}\left(-\frac{R_R}{L_R}\psi_{Rb} + n_p\omega_R\psi_{Ra} + \frac{MR_R}{L_R}i_{Sb}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
u_{Sa} &= \left(R_S + \frac{M^2R_R}{L_R^2}\right) i_{Sa} + \sigma L_S \frac{di_{Sa}}{dt} - \frac{MR_R}{L_R^2}\psi_{Ra} - \frac{M}{L_R}n_p\omega_R\psi_{Rb} \\
u_{Sb} &= \left(R_S + \frac{M^2R_R}{L_R^2}\right) i_{Sb} + \sigma L_S \frac{di_{Sb}}{dt} - \frac{MR_R}{L_R^2}\psi_{Rb} + \frac{M}{L_R}n_p\omega_R\psi_{Ra}
\end{aligned}
$$

or

$$
\begin{aligned}
u_{Sa} - \left(\frac{R_S}{\sigma L_S} + \frac{M^2R_R}{\sigma L_S L_R^2}\right) i_{Sa} + \frac{MR_R}{\sigma L_S L_R^2}\psi_{Ra} + \frac{M}{\sigma L_S L_R}n_p\omega_R\psi_{Rb} &= \frac{di_{Sa}}{dt} \\
u_{Sb} - \left(\frac{R_S}{\sigma L_S} + \frac{M^2R_R}{\sigma L_S L_R^2}\right) i_{Sb} + \frac{MR_R}{\sigma L_S L_R^2}\psi_{Rb} - \frac{M}{\sigma L_S L_R}n_p\omega_R\psi_{Ra} &= \frac{di_{Sb}}{dt}.
\end{aligned}
$$

A a statespace form is then

$$
\begin{aligned}
\frac{d\psi_{Ra}}{dt} &= -\frac{R_R}{L_R}\psi_{Ra} - n_p\omega_R\psi_{Rb} + \frac{MR_R}{L_R}i_{Sa} \\
\frac{d\psi_{Rb}}{dt} &= -\frac{R_R}{L_R}\psi_{Rb} + n_p\omega_R\psi_{Ra} + \frac{MR_R}{L_R}i_{Sb} \\
\frac{di_{Sa}}{dt} &= -\left(\frac{R_S}{\sigma L_S} + \frac{M^2R_R}{\sigma L_S L_R^2}\right) i_{Sa} + \frac{MR_R}{\sigma L_S L_R^2}\psi_{Ra} + \frac{M}{\sigma L_S L_R}n_p\omega_R\psi_{Rb} + u_{Sa} \\
\frac{di_{Sb}}{dt} &= -\left(\frac{R_S}{\sigma L_S} + \frac{M^2R_R}{\sigma L_S L_R^2}\right) i_{Sb} + \frac{MR_R}{\sigma L_S L_R^2}\psi_{Rb} - \frac{M}{\sigma L_S L_R}n_p\omega_R\psi_{Ra} + u_{Sb} \\
\frac{d\omega_R}{dt} &= \frac{n_pM}{JL_R}\left(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}\right) - \frac{\tau_L}{J}.
\end{aligned}
$$