ECE 697 Modeling and High-Performance Control of Electric Machines
HW 9 Solutions
Spring 2022

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Problem 10 *Mathematical Model of a Multiple Pole-Pair Motor*

Ampère's law $\oint \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{\ell}} = i_{enclosed}$ is applied to the closed-path 1−2−3−4 indicated in Figure 1.

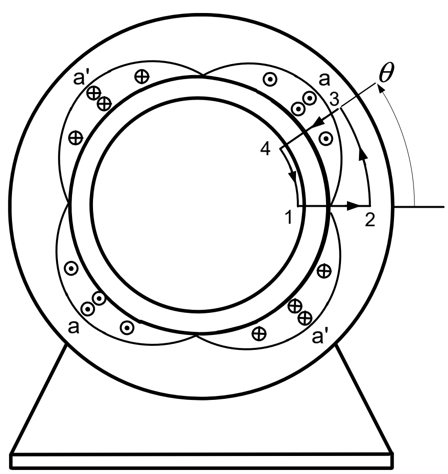

Figure 1: Use of Ampere's law to determine the air gap radial magnetic field produced by i_{Sa} in a n_p pole-pair machine.

The path is traversed in a counterclockwise fashion so that the enclosed current is considered positive if it is coming out of the page. However, the positive direction of current is into the page (i.e., $-i_{Sa}$ is used in Ampère's law) is also where $\sin(n_p\theta) < 0$ so that

$$\oint_{1-2-3-4-1} \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{\ell}} = \int_0^{\theta} i_{Sa}\frac{N_S}{2}\sin(n_p\theta')d\theta'. \tag{1}$$

is correct for *any* angle θ. Going through the usual procedure gives

$$H_{Sa}(i_{Sa},0)g - H_{Sa}(i_{Sa},\theta)g = -i_{Sa}\frac{N_S}{2n_p}\cos(n_p\theta) + i_{Sa}\frac{N_S}{2n_p} \tag{2}$$

where it was assumed that $\vec{\mathbf{H}}_{Sa}$ is *constant* across the air gap. Rearranging,

$$H_{Sa}(i_{Sa},\theta) = i_{Sa}\frac{N_S}{2n_p g}\cos(n_p\theta) + H_{Sa}(i_{Sa},0) - i_{Sa}\frac{N_S}{2n_p g}.$$

In this equation, both $H_{Sa}(i_{Sa},\theta)$ and $H_{Sa}(i_{Sa},0)$ are unknown. Applying the conservation of flux law to a closed cylindrical surface in the air gap which encloses the rotor gives

$$\oint_S \vec{\mathbf{B}}\cdot d\vec{\mathbf{S}} = \int_0^{\ell_1}\int_0^{2\pi} \mu_0 H_{Sa}(i_{Sa},\theta)\hat{\mathbf{r}}\cdot(r_R d\theta dz\hat{\mathbf{r}}) = 0$$

or

$$H_{Sa}(i_{Sa},0) - i_{Sa}\frac{N_S}{2n_p g} \equiv 0.$$

So,

$$\begin{aligned} H_{Sa}(i_{Sa},\theta) &= \frac{N_S}{2n_p g} i_{Sa}\cos(n_p\theta) \\ B_{Sa}(i_{Sa},\theta) &= \frac{\mu_0 N_S}{2n_p g} i_{Sa}\cos(n_p\theta). \end{aligned} \tag{3}$$

In applying Ampère's law, it was assumed that $\vec{\mathbf{B}} = \mu_0\vec{\mathbf{H}}$ was constant across the air gap in the radial direction, that is, $\vec{\mathbf{B}}$ did not depend on the cylindrical coordinate r. However, in order to satisfy the conservation of flux $\oint_S \vec{\mathbf{B}}\cdot d\vec{\mathbf{S}} = 0$ for a closed flux surface in the air gap, it is necessary that $\vec{\mathbf{B}}$ decrease as $1/r$ in the air gap. As in the single pole-pair case, the factor r_R/r is included in the expressions for H_{Sa} and B_{Sa} so that conservation of flux holds in the air gap. Finally then, the magnetic field $\vec{\mathbf{B}}_{Sa}$ *in the air gap* due to i_{Sa} is given by

$$\vec{\mathbf{B}}_{Sa}(i_{Sa},r,\theta) = \frac{\mu_0 N_S r_R}{2n_p g}\frac{i_{Sa}\cos(n_p\theta)}{r}\hat{\mathbf{r}}.$$

Similarly, for phase b which is also sinusoidally wound, but rotated $\pi/(2n_p)$ counterclockwise from phase a, the magnetic field $\vec{\mathbf{B}}_{Sb}$ in the air gap due to i_{Sb} is given by

$$\vec{\mathbf{B}}_{Sb}(i_{Sb},r,\theta) = \frac{\mu_0 N_S r_R}{2n_p g}\frac{i_{Sb}\sin(n_p\theta)}{r}\hat{\mathbf{r}}.$$

Similarly, using the winding distributions for the rotor phases and Ampère's law result in the radial air gap magnetic fields produced by the rotor currents being given by

$$B_{Ra}(i_{Ra},\theta) = \frac{\mu_0 N_R}{2n_p g} i_{Ra}\cos\Big(n_p(\theta-\theta_R)\Big) \tag{4}$$

$$B_{Rb}(i_{Ra},\theta) = \frac{\mu_0 N_R}{2n_p g} i_{Rb}\sin\Big(n_p(\theta-\theta_R)\Big). \tag{5}$$

Again, these expressions are modified to include the factor r_R/r so that $\vec{\mathbf{B}}_R$ satisfies the conservation of flux for an arbitrary closed flux surface in the air gap so that

$$\vec{\mathbf{B}}_{Ra}(i_{Ra},r,\theta-\theta_R) = \frac{\mu_0 N_R}{2n_p g}\frac{r_R}{r} i_{Ra}\cos\Big(n_p(\theta-\theta_R)\Big)\hat{\mathbf{r}} \tag{6}$$

$$\vec{\mathbf{B}}_{Rb}(i_{Rb},r,\theta-\theta_R) = \frac{\mu_0 N_R}{2n_p g}\frac{r_R}{r} i_{Rb}\sin\Big(n_p(\theta-\theta_R)\Big)\hat{\mathbf{r}} \tag{7}$$

where r_R is the rotor radius. In summary, the magnetic fields due to the stator and rotor currents are

$$\begin{aligned}\vec{\mathbf{B}}_S(i_{Sa}, i_{Sb}, r, \theta) &= \vec{\mathbf{B}}_{Sa}(i_{Sa}, r, \theta) + \vec{\mathbf{B}}_{Sb}(i_{Sb}, r, \theta) \\ &= \frac{\mu_0 r_R N_S}{2n_p g}\frac{1}{r}\left(i_{Sa}\cos(n_p\theta) + i_{Sb}\sin(n_p\theta)\right)\hat{\mathbf{r}}\end{aligned} \tag{8}$$

and

$$\begin{aligned}\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, r, \theta-\theta_R) &= \vec{\mathbf{B}}_{Ra}(i_{Ra}, r, \theta-\theta_R) + \vec{\mathbf{B}}_{Rb}(i_{Rb}, r, \theta-\theta_R) \\ &= \frac{\mu_0 N_R r_R}{2n_p g}\frac{1}{r}\Big(i_{Ra}\cos\Big(n_p(\theta-\theta_R)\Big) + i_{Rb}\sin\Big(n_p(\theta-\theta_R)\Big)\Big)\hat{\mathbf{r}},\end{aligned} \tag{9}$$

respectively.

The computation of the total flux linkage in the sinusoidally-wound stator phase a is now considered. The flux in stator phase a is due to the magnetic fields produced by both the stator currents i_{Sa}, i_{Sb} and the rotor currents i_{Ra}, i_{Rb}. That is,

$$\lambda_{Sa}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int_{\substack{\text{All loops of}\\ \text{stator phase } a}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R)\cdot d\vec{\mathbf{S}}.$$

This computation is carried out in two parts by separately considering the flux due to the stator currents and then the flux due to the rotor currents. That is,

$$\lambda_{Sa}(0, 0, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int_{\substack{\text{All loops of}\\ \text{stator phase } a}} \vec{\mathbf{B}}_S\cdot d\vec{\mathbf{S}}$$

and

$$\lambda_{Sa}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) \triangleq \int_{\substack{\text{All loops of}\\ \text{stator phase } a}} \vec{\mathbf{B}}_R\cdot d\vec{\mathbf{S}}$$

so that

$$\lambda_{Sa}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) = \lambda_{Sa}(0, 0, i_{Sa}, i_{Sb}, \theta_R) + \lambda_{Sa}(i_{Ra}, i_{Rb}, 0, 0, \theta_R).$$

Stator Flux Linkage Produced by the Stator Currents

Consider a single turn (loop) of stator phase a at the angular position θ with $0 \le \theta \le \pi/n_p$, that is, one axial side of the loop is at θ and the other axial side is at $\theta - \pi/n_p$. Choose the flux surface S for this single turn to be a half-cylindrical shell that lies just inside the air gap next to the stator surface.

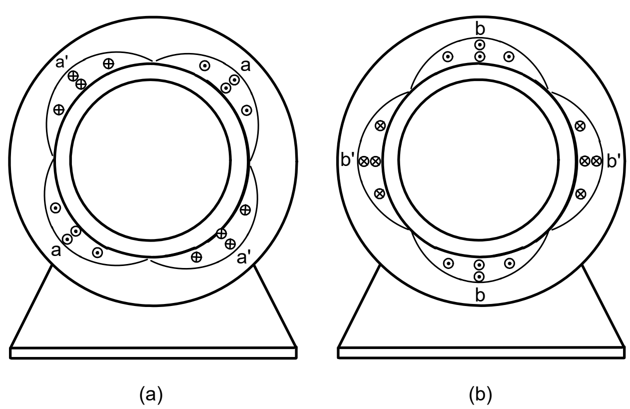

Figure 2: (a) Sinusoidally wound stator phase a with $n_p = 2$ and turns density $N_{Sa}(\theta) = (n_p N_S/2)\,|\sin(n_p\theta)|$. (b) Sinusoidally wound stator phase b with $n_p = 2$ and turns density $N_{Sb}(\theta) = (n_p N_S/2)\,|\sin\left(n_p(\theta - \pi/2)\right)|$.

To compute the flux through the cylindrical part of the surface, the surface element vector is chosen as $d\vec{\mathbf{S}} = r_S d\theta' dz \hat{\mathbf{r}}$. Then, the flux $\phi_{Sa}(i_{Sa}, i_{Sb}, \theta)$ in each turn of stator phase a at the angular position θ of the stator is

$$\begin{aligned}\phi_{Sa}(i_{Sa}, i_{Sb}, \theta) &\triangleq \int_{\theta'=\theta-\pi/n_p}^{\theta'=\theta} \int_{z=0}^{z=\ell_1} \frac{\mu_0 r_R N_S}{2n_p g} \frac{1}{r_S} \left(i_{Sa} \cos(n_p\theta') + i_{Sb} \sin(n_p\theta') \right) \hat{\mathbf{r}} \cdot (r_S d\theta' dz \hat{\mathbf{r}}) \\ &= \frac{\mu_0 r_R \ell_1 N_S}{n_p^2 g} \left(i_{Sa} \sin(n_p\theta) - i_{Sb} \cos(n_p\theta) \right). \end{aligned} \tag{10}$$

Remark Note that by taking the flux surface normal to be $d\vec{\mathbf{S}} = r_S d\theta' dz \hat{\mathbf{r}}$, the positive direction of travel around the turns of stator phase a at the angular position θ is out of the page on side a and into the page on side a', i.e., it is the same direction as positive current flow in stator phase a.

Between the angular positions θ and $\theta + d\theta$ of stator phase a there are $(N_S/2)\sin(n_p\theta)d\theta$ turns each having the flux (10) in them. Thus the flux linkage in all the turns of phase a between θ and $\theta + d\theta$ is then

$$\begin{aligned} d\lambda_{Sa}(0, 0, i_{Sa}, i_{Sb}, \theta_R) &\triangleq \phi_{Sa}(i_{Sa}, i_{Sb}, \theta)(N_S/2)\sin(n_p\theta)d\theta \\ &= \frac{\mu_0 r_R \ell_1 N_S}{n_p^2 g} \left(i_{Sa} \sin(n_p\theta) - i_{Sb} \cos(n_p\theta) \right) \frac{N_S}{2} \sin(n_p\theta) d\theta. \end{aligned} \tag{11}$$

To obtain the flux linkage in all the turns of stator phase a, simply integrate (11) over the pole pairs. The first pole pair has its windings between $-\pi/n_p$ to π/n_p so that flux linkage in the first pole pair is found by integrating the flux (11) from $\theta = 0$ to $\theta = \pi/n_p$, i.e.,

$$\begin{aligned} \int_{\substack{\text{All loops of the } 1^{st} \text{ pole-pair} \\ \text{of stator phase } a}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} &= \int_{\theta=0}^{\theta=\pi/n_p} \frac{\mu_0 r_R \ell_1 N_S}{n_p^2 g} \left(i_{Sa} \sin(n_p\theta) - i_{Sb} \cos(n_p\theta) \right) \frac{N_S}{2} \sin(n_p\theta) d\theta \\ &= \frac{\mu_0 r_R \ell_1 N_S}{n_p^2 g} i_{Sa} \frac{\pi}{2n_p}. \end{aligned} \tag{12}$$

The second pole-pair is from π/n_p to $3\pi/n_p$ so that one would integrate the flux (11) from $\theta = \pi/n_p$ to $\theta = 3\pi/n_p$. However, the calculation just preformed shows that the same result will be obtained. The total flux linkage in all the windings of phase a is found by simply multiplying the flux (12) by n_p. That is,

$$\begin{aligned} \lambda_{Sa}(0, 0, i_{Sa}, i_{Sb}, \theta_R) &= \int_{\substack{\text{All loops of} \\ \text{stator phase } a}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} \\ &= n_p \int_{\theta=0}^{\theta=\pi/n_p} \frac{\mu_0 r_R \ell_1 N_S}{n_p^3 g} \left(i_{Sa} \sin(n_p\theta) - i_{Sb} \cos(n_p\theta) \right) \frac{N_S}{2} \sin(n_p\theta) d\theta \\ &= \frac{\mu_0 r_R \ell_1 N_S^2}{n_p^2 g} i_{Sa} \frac{\pi}{2} = L_S i_{Sa} \end{aligned} \tag{13}$$

where

$$L_S \triangleq \frac{\mu_0 r_R \ell_1 \pi N_S^2}{4n_p^2 g} = \frac{\mu_0 \ell_1 \ell_2 \pi}{8n_p^2 g} N_S^2.$$

Note that for this particular two-phase machine, the magnetic field produced by the current i_{Sb} in phase b does not produce any net flux in phase a. Similarly,

$$\lambda_{Sb}(0, 0, i_{Sa}, i_{Sb}, \theta_R) = \int_{\substack{\text{All loops of} \\ \text{stator phase } b}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} = \frac{\mu_0 r_R \ell_1 \pi N_S^2}{4n_p^2 g} i_{Sb} = L_S i_{Sb}. \tag{14}$$

Stator Flux Linkage Produced by the Rotor Currents

The flux in stator phase a due to magnetic field produced by the rotor currents is now computed from

$$\lambda_{Sa}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) \triangleq \int_{\substack{\text{All loops of}\\ \text{stator phase } a}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}}.$$

Let the rotor be at an arbitrary location θ_R and consider a flux surface S between θ and $\theta - \pi/n_p$ at the inside surface of the stator. To account for spreading of the rotor magnetic field in the air gap through the two ends of the motor, the factor κ is used. With $d\vec{\mathbf{S}} = r_S d\theta' dz \hat{\mathbf{r}}$ as the surface element vector, the flux $\varphi_{Sa}(i_{Ra}, i_{Rb}, \theta - \theta_R)$ in each turn of stator phase a at the angular position θ of the stator is

$$\begin{aligned}\varphi_{Sa}(i_{Ra}, i_{Rb}, \theta - \theta_R) &\triangleq \int_{\theta'=\theta-\pi/n_p}^{\theta'=\theta} \int_{z=0}^{z=\ell_1} \kappa \frac{\mu_0 N_R r_R}{2 n_p g} \frac{1}{r_S} \Big(i_{Ra} \cos\Big(n_p(\theta' - \theta_R) \; + i_{Rb} \sin\big(n_p(\theta' - \theta_R)\big)\, \hat{\mathbf{r}} \cdot (r_S d\theta' dz \hat{\mathbf{r}})\Big)\Big) \\ &= \kappa \frac{\mu_0 r_R \ell_1 N_R}{n_p^2 g} \Big(i_{Ra} \sin\left(n_p(\theta - \theta_R)\right) - i_{Rb} \cos\left(n_p(\theta - \theta_R)\right) \Big). \end{aligned} \tag{15}$$

Between the angular positions θ and $\theta + d\theta$, there are $(N_S/2)\sin(n_p\theta)d\theta$ turns each having the flux (15) in them. Thus the incremental flux $d\lambda_{Sa}$ in the turns of stator phase a between θ and $\theta + d\theta$ produced by the rotor's magnetic field is then

$$\begin{aligned} d\lambda_{Sa} &= \varphi_{Sa}(i_{Ra}, i_{Rb}, \theta - \theta_R) \frac{N_S}{2} \sin(n_p\theta) d\theta \\ &= \kappa \frac{\mu_0 r_R \ell_1 N_R N_S}{2 n_p^2 g} \left(i_{Ra} \sin\left(n_p(\theta - \theta_R)\right) - i_{Rb} \cos\left(n_p(\theta - \theta_R)\right)\right) \sin(n_p\theta) d\theta. \end{aligned} \tag{16}$$

To obtain the total flux linkage due to the rotor's magnetic field in all the turns of stator phase a, integrate (16) over the pole-pairs to obtain

$$\begin{aligned} \lambda_{Sa}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) &= \int_{\substack{\text{All loops of}\\ \text{stator phase } a}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} \\ &= n_p \int_{\theta=0}^{\theta=\pi/n_p} \kappa \frac{\mu_0 r_R \ell_1 N_S N_R}{2 n_p^2 g} \left(i_{Ra} \sin\left(n_p(\theta - \theta_R)\right) - i_{Rb} \cos\left(n_p(\theta - \theta_R)\right)\right) \sin(n_p\theta) d\theta \\ &= \kappa \frac{\mu_0 r_R \ell_1 N_S N_R}{2 n_p g} \Bigg(i_{Ra} \int_{\theta=0}^{\theta=\pi/n_p} \left(\sin(n_p\theta)\cos(n_p\theta_R) - \cos(n_p\theta)\sin(n_p\theta_R)\right) \sin(n_p\theta) d\theta \\ &\quad - i_{Rb} \int_{\theta=0}^{\theta=\pi/n_p} \left(\cos(n_p\theta)\cos(n_p\theta_R) + \sin(n_p\theta)\sin(n_p\theta_R)\right)\sin(n_p\theta) d\theta \Bigg) \\ &= \kappa \frac{\mu_0 r_R \ell_1 N_S N_R}{2 n_p g} \int_{\theta=0}^{\theta=\pi/n_p} \Big(i_{Ra} \sin^2(n_p\theta)\cos(n_p\theta_R) - i_{Rb} \sin^2(n_p\theta) \sin\left(n_p\theta_R\right)\big)\Big) d\theta \\ &= M \left(i_{Ra} \cos(n_p\theta_R) - i_{Rb} \sin(n_p\theta_R)\right) \end{aligned} \tag{17}$$

where

$$M \triangleq \kappa \frac{\mu_0 \pi r_R \ell_1 N_S N_R}{4 n_p^2 g} = \kappa \frac{\mu_0 \pi \ell_1 \ell_2 N_S N_R}{8 n_p^2 g}.$$

Similarly,

$$\begin{aligned}\lambda_{Sb}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) &= \int\limits_{\substack{\text{All loops of}\\ \text{stator phase } b}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} \\ &= M\left(i_{Ra}\sin(n_p\theta_R) + i_{Rb}\cos(n_p\theta_R)\right). \end{aligned} \tag{18}$$

Total Flux Linkage in the Stator Phases

Combining (13), (17), (14), and (18), the total flux linkage in each of the stator phases is then

$$\begin{aligned}\lambda_{Sa}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) &= \int\limits_{\substack{\text{All loops of}\\ \text{stator phase } a}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}} \\ &= L_S i_{Sa} + M\left(i_{Ra}\cos(n_p\theta_R) - i_{Rb}\sin(n_p\theta_R)\right) \\ \lambda_{Sb}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) &= \int\limits_{\substack{\text{All loops of}\\ \text{stator phase } b}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}} \\ &= L_S i_{Sb} + M\left(i_{Ra}\sin(n_p\theta_R) + i_{Rb}\cos(n_p\theta_R)\right). \end{aligned} \tag{19}$$

Flux Linkage in the Rotor Phases

The flux linkages in the two sinusoidally wound phases of the rotor are computed according to

$$\lambda_{Ra}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int\limits_{\substack{\text{All loops of}\\ \text{rotor phase } a}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}}$$

$$\lambda_{Rb}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int\limits_{\substack{\text{All loops of}\\ \text{rotor phase } b}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}}.$$

Starting with rotor phase a and, as in the case of the stator flux linkages, the computation is done in two parts as

$$\lambda_{Ra}(0, 0, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int\limits_{\substack{\text{All loops of}\\ \text{rotor phase } a}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}}$$

and

$$\lambda_{Ra}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) \triangleq \int\limits_{\substack{\text{All loops of}\\ \text{rotor phase } a}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}}.$$

Then

$$\lambda_{Ra}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) = \lambda_{Ra}(0, 0, i_{Sa}, i_{Sb}, \theta_R) + \lambda_{Ra}(i_{Ra}, i_{Rb}, 0, 0, \theta_R).$$

Rotor Flux Linkage Produced by the Stator Currents

The flux linkage in the rotor phase due to the magnetic field established by the stator currents is given by

$$\lambda_{Ra}(0, 0, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int\limits_{\substack{\text{All loops of}\\ \text{rotor phase } a}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}}.$$

To do this computation, consider a single turn (loop) of the rotor phase at the angular position θ with $0 \leq \theta - \theta_R \leq \pi/n_p$ so that one axial side of the loop is at θ and the other axial side is at $\theta - \pi/n_p$. Choose the flux surface S for this single turn to be a half-cylindrical shell with one side at θ and the other at $\theta - \pi/n_p$. To account for the spreading of the stator magnetic field in the air gap through the two ends of the motor, the factor κ is again used.

The surface element vector is chosen as $d\vec{\mathbf{S}} = r_R d\theta' dz \hat{\mathbf{r}}$. Then, the flux $\phi_{Ra}(i_{Sa}, i_{Sb}, \theta)$ in each turn of rotor phase a at the angular position θ of the rotor is

$$\begin{aligned}
\phi_{Ra}(i_{Sa}, i_{Sb}, \theta) &= \int_{\theta'=\theta-\pi/n_p}^{\theta'=\theta} \int_{z=0}^{z=\ell_1} \kappa \frac{\mu_0 r_R N_S}{2n_p g} \frac{1}{r_R} \left(i_{Sa} \cos(n_p\theta') + i_{Sb} \sin(n_p\theta') \right) \hat{\mathbf{r}} \cdot (r_R d\theta' dz \hat{\mathbf{r}}) \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_S}{n_p^2 g} \left(i_{Sa} \sin(n_p\theta) - i_{Sb} \cos(n_p\theta) \right). \qquad (20)
\end{aligned}$$

Remark Note that by taking the flux surface normal to be $d\vec{\mathbf{S}} = r_R d\theta' dz \hat{\mathbf{r}}$, the positive direction of travel around the turns of rotor phase a at the angular position θ is out of the page on side a and into the page on side a', that is, the same direction as positive current flow in rotor phase a.

Between the angular positions θ and $\theta + d\theta$, there are $(N_R/2) \sin(n_p(\theta - \theta_R))\, d\theta$ turns each having the flux (20) in them. Thus the incremental flux linkage $d\lambda_{Ra}$ in the turns of phase a between θ and $\theta + d\theta$ produced by the stator's magnetic field is then

$$\begin{aligned}
d\lambda_{Ra} &\triangleq \phi_{Ra}(i_{Sa}, i_{Sb}, \theta) \frac{N_R}{2} \sin(n_p(\theta - \theta_R))\, d\theta \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_S N_R}{2n_p^2 g} \left(i_{Sa} \sin(n_p\theta) - i_{Sb} \cos(n_p\theta) \right) \sin(n_p(\theta - \theta_R))\, d\theta. \qquad (21)
\end{aligned}$$

To obtain the flux linkage in all the turns of rotor phase a, integrate (21) as θ varies from θ_R to $\theta_R + \pi$, that is,

$$\begin{aligned}
\lambda_{Ra}(0, 0, i_{Sa}, i_{Sb}, \theta_R) &= \int_{\substack{\text{All loops of} \\ \text{rotor phase } a}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} \\
&= n_p \int_{\theta=\theta_R}^{\theta=\theta_R+\pi/n_p} \kappa \frac{\mu_0 r_R \ell_1 N_S N_R}{2n_p^2 g} \Big(i_{Sa} \sin(n_p\theta) - i_{Sb} \cos(n_p\theta) \Big) \sin(n_p(\theta - \theta_R))\, d\theta \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_S N_R}{2n_p g} \int_{\theta=\theta_R}^{\theta=\theta_R+\pi/n_p} \Big(i_{Sa} \sin(n_p(\theta - \theta_R + \theta_R)) \sin(n_p(\theta - \theta_R)) \\
&\quad - i_{Sb} \cos(n_p(\theta - \theta_R + \theta_R)) \sin(n_p(\theta - \theta_R)) \Big)\, d\theta \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_S N_R}{2n_p g} \times \\
&\quad \int_{\theta=\theta_R}^{\theta=\theta_R+\pi/n_p} \Big(i_{Sa} \sin^2(n_p(\theta - \theta_R)) \cos(n_p\theta_R) + i_{Sa} \cos(n_p(\theta - \theta_R)) \sin(n_p\theta_R) \sin(n_p(\theta - \theta_R)) \\
&\quad - i_{Sb} \cos(n_p(\theta - \theta_R)) \cos(n_p\theta_R) \sin(n_p(\theta - \theta_R)) + i_{Sb} \sin^2(n_p(\theta - \theta_R)) \sin(n_p\theta_R) \Big)\, d\theta \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_S N_R}{2n_p g} \left(i_{Sa} \frac{\pi}{2n_p} \cos(n_p\theta_R) + \frac{\pi}{2n_p} i_{Sb} \sin(n_p\theta_R) \right) \\
&= M \left(i_{Sa} \cos(n_p\theta_R) + i_{Sb} \sin(n_p\theta_R) \right) \qquad (22)
\end{aligned}$$

where, as before,

$$M \triangleq \kappa \frac{\mu_0 \pi r_R \ell_1 N_S N_R}{4n_p^2 g} = \kappa \frac{\mu_0 \pi \ell_1 \ell_2 N_S N_R}{8n_p^2 g}.$$

Similarly,

$$\lambda_{Rb}(0,0,i_{Sa},i_{Sb},\theta_R) \triangleq \int_{\substack{\text{All loops of}\\ \text{rotor phase } b}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}}$$
$$= M\left(-i_{Sa}\sin(n_p\theta_R) + i_{Sb}\cos(n_p\theta_R)\right). \tag{23}$$

Rotor Flux Linkage Produced by the Rotor Currents

Consider a single turn (loop) of rotor phase a at the angular position θ with $0 \le \theta - \theta_R \le \pi$ and a half-cylindrical shaped flux surface with one side at θ and the other at $\theta - \pi/n_p$.

To compute flux through the cylindrical part of the surface, choose $d\vec{\mathbf{S}} = r_R d\theta' dz \hat{\mathbf{r}}$ as the surface element vector. Then, the flux $\phi_{Ra}(i_{Ra}, i_{Rb}, \theta - \theta_R)$ in each turn of rotor phase a at the angular position θ of the rotor is

$$\begin{aligned}\phi_{Ra}(i_{Ra},i_{Rb},\theta-\theta_R) &\triangleq \int_{\theta'=\theta-\pi/n_p}^{\theta'=\theta}\int_{z=0}^{z=\ell_1} \frac{\mu_0 r_R N_R}{2n_p g}\frac{1}{r_R}\Big(i_{Ra}\cos\Big(n_p(\theta'-\theta_R) +\ i_{Rb}\sin\big(n_p(\theta'-\theta_R)\big)\Big)\Big)\,\hat{\mathbf{r}}\cdot(r_R d\theta' dz\hat{\mathbf{r}})\\ &= \frac{\mu_0 r_R \ell_1 N_R}{n_p^2 g}\Big(i_{Ra}\sin\left(n_p(\theta-\theta_R)\right) -\ i_{Rb}\cos\left(n_p(\theta-\theta_R)\right)\Big).\end{aligned} \tag{24}$$

Remark Note that by taking the flux surface normal to be $d\vec{\mathbf{S}} = r_R d\theta' dz \hat{\mathbf{r}}$, the positive direction of travel around the turns of rotor phase a at the angular position θ is out of the page on side a of the rotor phase and into the page on side a' of the rotor phase, that is, the same direction as positive current flow in rotor phase a.

Between the angular positions θ and $\theta + d\theta$, there are $(N_R/2)\sin\left(n_p(\theta-\theta_R)\right)d\theta$ turns each having the flux (24) in them. Thus the incremental flux linkage in the turns of phase a between θ and $\theta + d\theta$ is then

$$\begin{aligned}d\lambda_{Ra} &\triangleq \varphi_{Ra}(i_{Ra},i_{Rb},\theta-\theta_R)\frac{N_R}{2}\sin\left(n_p(\theta-\theta_R)\right)d\theta\\ &= \frac{\mu_0 r_R \ell_1 N_R^2}{2n_p^2 g}\left(i_{Ra}\sin\left(n_p(\theta-\theta_R)\right) - i_{Rb}\cos\left(n_p(\theta-\theta_R)\right)\right)\sin\left(n_p(\theta-\theta_R)\right)d\theta.\end{aligned} \tag{25}$$

To obtain the flux linkage in all the turns of stator phase a, integrate (25) over the rotor pole pairs to obtain

$$\begin{aligned}\lambda_{Ra}(i_{Ra},i_{Rb},0,0,\theta_R) &= \int_{\substack{\text{All loops of}\\ \text{rotor phase } a}} \vec{\mathbf{B}}_R\cdot d\vec{\mathbf{S}}\\ &= n_p\int_{\theta=\theta_R}^{\theta=\theta_R+\pi/n_p}\frac{\mu_0 r_R\ell_1 N_R^2}{2n_p^2 g}\Big(i_{Ra}\sin\left(n_p(\theta-\theta_R)\right) -\ i_{Rb}\cos\left(n_p(\theta-\theta_R)\right)\sin\left(n_p(\theta-\theta_R)\right)\Big)d\theta\\ &= \frac{\mu_0 r_R\ell_1 N_R^2}{2n_p g}\int_{\theta=\theta_R}^{\theta=\theta_R+\pi/n_p}\Big(i_{Ra}\sin^2\left(n_p(\theta-\theta_R)\right) -\ i_{Rb}\cos\left(n_p(\theta-\theta_R)\right)\sin\left(n_p(\theta-\theta_R)\right)\Big)d\theta\\ &= \frac{\mu_0 r_R\ell_1 N_R^2}{2n_p g}i_{Ra}\frac{\pi}{2n_p}\\ &= L_R i_{Ra}\end{aligned} \tag{26}$$

where

$$L_R \triangleq \frac{\mu_0 r_R \ell_1 \pi N_R^2}{4n_p^2 g} = \frac{\mu_0\ell_1\ell_2\pi}{8n_p^2 g}N_R^2.$$

Note that for this particular two-phase system, the magnetic field produced by the current i_{Rb} in rotor phase b does not produce any net flux in rotor phase a. Similarly,

$$\lambda_{Rb}(i_{Ra},i_{Rb},0,0,\theta_R) = \int_{\substack{\text{All loops of}\\ \text{rotor phase } b}} \vec{\mathbf{B}}_R\cdot d\vec{\mathbf{S}} = L_R i_{Rb}. \tag{27}$$

Total Flux Linkage in the Rotor Phases

Combining the expressions (22), (26), (23), and (27), the total flux in the sinusoidally wound rotor phases due to both the stator and rotor magnetic fields is then

$$\lambda_{Ra}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) = \int\limits_{\substack{\text{All loops of}\\ \text{rotor phase } a}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}} = L_R i_{Ra} + M\left(i_{Sa}\cos(n_p\theta_R) + i_{Sb}\sin(n_p\theta_R)\right)$$

$$\lambda_{Rb}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) = \int\limits_{\substack{\text{All loops of}\\ \text{rotor phase } b}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}} = L_R i_{Rb} + M\left(-i_{Sa}\sin(n_p\theta_R) + i_{Sb}\cos(n_p\theta_R)\right).$$

Torque Production

The magnetic field at an arbitrary angle θ in the air-gap due to the stator currents is

$$\vec{\mathbf{B}}_S(i_{Sa},\, i_{Sb},\, \theta) = \frac{\mu_0 \ell_2 N_S}{4n_p g r}\left(i_{Sa}(t)\cos(n_p\theta) + i_{Sb}(t)\sin(n_p\theta)\right)\hat{\mathbf{r}}.$$

Let

$$\begin{aligned} i_S(t) &\triangleq \sqrt{i_{Sa}^2(t) + i_{Sb}^2(t)} \\ \xi(t) &\triangleq Tan^{-1}\left(i_{Sb}(t)/i_{Sa}(t)\right) \end{aligned}$$

and $\vec{\mathbf{B}}_S$ may be rewritten as

$$\begin{aligned} \vec{\mathbf{B}}_S &= \frac{\mu_0 \ell_2 N_S}{4n_p g r} i_S(t)\left(\cos(\xi)\cos(n_p\theta) + \sin(\xi)\sin(n_p\theta)\right)\hat{\mathbf{r}} \\ &= \frac{\mu_0 \ell_2 N_S}{4n_p g r} i_S(t)\cos\left(n_p\theta - \xi\right)\hat{\mathbf{r}}. \end{aligned}$$

Similarly, the magnetic field at the angle θ in the air-gap due to the rotor currents is

$$\vec{\mathbf{B}}_R(i_{Ra},\, i_{Rb},\, \theta - \theta_R) = \frac{\mu_0 \ell_2 N_R}{4n_p g r}\left(i_{Ra}(t)\cos\left(n_p(\theta - \theta_R)\right) + i_{Rb}(t)\sin\left(n_p(\theta - \theta_R)\right)\right)\hat{\mathbf{r}}.$$

Let

$$\begin{aligned} i_R(t) &\triangleq \sqrt{i_{Ra}^2(t) + i_{Rb}^2(t)} \\ \zeta(t) &\triangleq Tan^{-1}\left(i_{Rb}(t)/i_{Ra}(t)\right) \end{aligned}$$

and $\vec{\mathbf{B}}_R$ may be rewritten as

$$\begin{aligned} \vec{\mathbf{B}}_R &= \frac{\mu_0 \ell_2 N_R}{4n_p g r} i_R(t)\left(\cos(\zeta)\cos\left(n_p(\theta - \theta_R)\right) + \sin(\zeta)\sin\left(n_p(\theta - \theta_R)\right)\right)\hat{\mathbf{r}} \\ &= \frac{\mu_0 \ell_2 N_R}{4n_p g r} i_R(t)\cos(n_p\left(\theta - \theta_R\right) - \zeta)\hat{\mathbf{r}}. \end{aligned}$$

To compute the torque $\vec{\boldsymbol{\tau}}_R$, the force on the rotor current loops due to the stator magnetic field is found. The differential magnetic-force $d\vec{\mathbf{F}}_{Ra}$ produced on the loops of rotor phase a between θ and $\theta + d\theta$ by $\vec{\mathbf{B}}_S = B_S\hat{\mathbf{r}} = \kappa\frac{\mu_0 \ell_2 N_S}{4n_p g r_R} i_S(t)\cos(n_p\theta - \xi)\hat{\mathbf{r}}$ is[1]

$$d\vec{\mathbf{F}}_{Ra} = \begin{cases} i_{Ra}(t)\frac{N_R}{2}\ \sin\left(n_p(\theta - \theta_R)\right)d\theta(+\ell_1\hat{\mathbf{z}}) \times B_S\hat{\mathbf{r}} & \text{for } \theta_R \le \theta \le \theta_R + \pi/n_p \\ i_{Ra}(t)\frac{N_R}{2}\left|\sin\left(n_p(\theta - \theta_R)\right)\right|d\theta(-\ell_1\hat{\mathbf{z}}) \times B_S\hat{\mathbf{r}} & \text{for } \theta_R + \pi/n_p \le \theta \le \theta_R + 2\pi/n_p \end{cases}$$

[1] Note that the leakage factor κ has been included since this is the stator magnetic field on the rotor side of the airgap.

where $(N_R/2)\sin(n_p(\theta-\theta_R))\,d\theta$ is the number of axial sides of rotor phase a between θ and $\theta+d\theta$ each carrying the current $i_{Ra}(t)$. As $\sin(n_p(\theta-\theta_R))\leq 0$ for $\theta_R+\pi/n_p\leq\theta\leq\theta_R+2\pi/n_p$ while $\sin(n_p(\theta-\theta_R))\geq 0$ for $\theta_R\leq\theta\leq\theta_R+\pi/n_p$, $d\vec{\mathbf{F}}_{Ra}$ can be written more compactly for all $\theta_R\leq\theta\leq\theta_R+2\pi/n_p$ as

$$\begin{aligned}
d\vec{\mathbf{F}}_{Ra} &= i_{Ra}(t)\frac{N_R}{2}\sin(n_p(\theta-\theta_R))d\theta\ell_1\hat{\mathbf{z}}\times B_S\hat{\mathbf{r}}\\
&= i_{Ra}(t)\frac{N_R}{2}\sin(n_p(\theta-\theta_R))\,\ell_1 B_S d\theta\hat{\boldsymbol{\theta}}\\
&= i_{Ra}(t)\frac{N_R}{2}\sin(n_p(\theta-\theta_R))\,\ell_1\kappa\frac{\mu_0\ell_2N_S}{4n_pgr}\,|_{r=\ell_2/2}\,i_S(t)\cos(n_p\theta-\xi)d\theta\hat{\boldsymbol{\theta}}\\
&= \kappa\frac{\mu_0\ell_1N_SN_R}{4n_pg}i_{Ra}(t)i_S(t)\sin(n_p(\theta-\theta_R))\cos(n_p\theta-\xi)d\theta\hat{\boldsymbol{\theta}}.
\end{aligned}$$

The differential torque $d\vec{\boldsymbol{\tau}}_{Ra}$ is then given by

$$\begin{aligned}
d\vec{\boldsymbol{\tau}}_{Ra} &= (\ell_2/2)\hat{\mathbf{r}}\times d\vec{\mathbf{F}}_{Ra}\\
&= (\ell_2/2)\frac{\mu_0\ell_1N_SN_R}{4n_pg}i_{Ra}(t)i_S(t)\sin(n_p(\theta-\theta_R))\cos(n_p\theta-\xi)d\theta\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}\\
&= \kappa\frac{\mu_0\ell_1\ell_2N_SN_R}{8n_pg}i_{Ra}(t)i_S(t)\sin(n_p(\theta-\theta_R))\cos(n_p\theta-\xi)d\theta\hat{\mathbf{z}}\\
&= \frac{n_pM}{\pi}i_{Ra}(t)i_S(t)\sin(n_p(\theta-\theta_R))\cos(n_p\theta-\xi)d\theta\hat{\mathbf{z}}\\
&= \frac{n_pM}{\pi}i_{Ra}(t)i_S(t)\frac{1}{2}\left(\sin(2n_p\theta-n_p\theta_R-\xi)+\sin(\xi-n_p\theta_R)\right)d\theta\hat{\mathbf{z}}
\end{aligned}$$

where $M=\kappa\mu_0\pi\ell_1\ell_2N_SN_R/\left(8n_p^2g\right)$ is the coefficient of mutual inductance as defined above. The torque on rotor phase a is then

$$\begin{aligned}
\vec{\boldsymbol{\tau}}_{Ra} = \int_0^{2\pi} d\vec{\boldsymbol{\tau}}_{Ra} &= \int_0^{2\pi}\frac{n_pM}{2\pi}i_{Ra}(t)i_S(t)\left(\sin(2n_p\theta-n_p\theta_R-\xi)+\sin(\xi-n_p\theta_R)\right)d\theta\hat{\mathbf{z}}\\
&= n_pMi_{Ra}(t)i_S(t)\sin(\xi-n_p\theta_R)\hat{\mathbf{z}}\\
&= n_pMi_{Ra}(t)i_S(t)\left(\sin(\xi)\cos(n_p\theta_R)-\sin(n_p\theta_R)\cos(\xi)\right)\hat{\mathbf{z}}\\
&= n_pMi_{Ra}(t)\left(i_{Sb}(t)\cos(n_p\theta_R)-i_{Sa}(t)\sin(n_p\theta_R)\right)\hat{\mathbf{z}}.
\end{aligned}$$

Similarly,

$$d\vec{\mathbf{F}}_{Rb} = \begin{cases} i_{Rb}(t)\frac{N_R}{2}\sin(n_p\theta-\pi/2-n_p\theta_R)d\theta(\ell_1\hat{\mathbf{z}})\times(B_S\hat{\mathbf{r}}) & \text{for } \theta_R+\frac{\pi}{2n_p}\leq\theta\leq\theta_R+\frac{3\pi}{2n_p}\\ i_{Rb}(t)\frac{N_R}{2}\left|\sin(n_p\theta-\pi/2-n_p\theta_R)\right|d\theta(-\ell_1\hat{\mathbf{z}})\times(B_S\hat{\mathbf{r}}) & \text{for } \theta_R-\frac{\pi}{2n_p}\leq\theta\leq\theta_R+\frac{\pi}{2n_p}. \end{cases}$$

As

$$\begin{aligned}
&\sin(n_p(\theta-\theta_R)-\pi/2)>0 \qquad \text{for } \theta_R+\frac{\pi}{2n_p}\leq\theta\leq\theta_R+\frac{3\pi}{2n_p}\\
&\sin(n_p(\theta-\theta_R)-\pi/2)<0 \qquad \text{for } \theta_R-\frac{\pi}{2n_p}\leq\theta\leq\theta_R+\frac{\pi}{2n_p},
\end{aligned}$$

the above expression for the force can be written more compactly for all $\theta_R \le \theta \le \theta_R + 2\pi/n_p$ as

$$\begin{aligned}
d\vec{\mathbf{F}}_{Rb} &= i_{Rb}(t)\frac{N_R}{2}\sin\left(n_p(\theta-\theta_R)-\pi/2\right)d\theta(\ell_1\hat{\mathbf{z}})\times(B_S\hat{\mathbf{r}})\\
&= i_{Rb}(t)\frac{N_R}{2}\sin\left(n_p(\theta-\theta_R)-\pi/2\right)\ell_1 B_S d\theta\hat{\boldsymbol{\theta}}\\
&= -i_{Rb}(t)\frac{N_R}{2}\cos\left(n_p(\theta-\theta_R)\right)\ell_1\kappa\frac{\mu_0\ell_2 N_S}{4n_p g r}\mid_{r=\ell_2/2} i_S(t)\cos n_p(\theta-\xi)d\theta\hat{\boldsymbol{\theta}}\\
&= -\kappa\frac{\mu_0\ell_1 N_S N_R}{4n_p g}i_{Rb}(t)i_S(t)\cos(n_p\left(\theta-\theta_R\right))\cos(n_p\theta-\xi)d\theta\hat{\boldsymbol{\theta}}
\end{aligned}$$

so that the differential torque $d\vec{\boldsymbol{\tau}}_{Rb}$ is then given by

$$\begin{aligned}
d\vec{\boldsymbol{\tau}}_{Rb} &= (\ell_2/2)\hat{\mathbf{r}}\times d\vec{\mathbf{F}}_{Rb}\\
&= -(\ell_2/2)\kappa\frac{\mu_0\ell_1 N_S N_R}{4n_p g}i_{Rb}(t)i_S(t)\cos\left(n_p(\theta-\theta_R)\right)\cos(n_p\theta-\xi)d\theta\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}\\
&= -\kappa\frac{\mu_0\ell_1\ell_2 N_S N_R}{8n_p g}i_{Rb}(t)i_S(t)\cos\left(n_p(\theta-\theta_R)\right)\cos(n_p\theta-\xi)d\theta\hat{\mathbf{z}}\\
&= -\frac{n_p M}{\pi}i_{Rb}(t)i_S(t)\cos\left(n_p(\theta-\theta_R)\right)\cos(n_p\theta-\xi)d\theta\hat{\mathbf{z}}\\
&= -\frac{n_p M}{\pi}i_{Rb}(t)i_S(t)\frac{1}{2}\left(\cos(2n_p\theta-n_p\theta_R-\xi)+\cos(\xi-n_p\theta_R)\right)d\theta\hat{\mathbf{z}}.
\end{aligned}$$

The torque on rotor phase b is then

$$\begin{aligned}
\vec{\boldsymbol{\tau}}_{Rb} = \int_0^{2\pi} d\vec{\boldsymbol{\tau}}_{Rb} &= -\int_0^{2\pi}\frac{n_p M}{2\pi}i_{Rb}(t)i_S(t)\left(\cos(2n_p\theta-n_p\theta_R-\xi)+\cos(\xi-n_p\theta_R)\right)d\theta\hat{\mathbf{z}}\\
&= -n_p M i_{Rb}(t)i_S(t)\cos(\xi-n_p\theta_R)\hat{\mathbf{z}}\\
&= -n_p M i_{Rb}(t)i_S(t)\left(\cos(\xi)\cos(n_p\theta_R)+\sin(\xi)\sin(n_p\theta_R)\right)\hat{\mathbf{z}}\\
&= -n_p M i_{Rb}(t)\left(i_{Sa}(t)\cos(n_p\theta_R)+i_{Sb}(t)\sin(n_p\theta_R)\right)\hat{\mathbf{z}}.
\end{aligned}$$

The total torque on the rotor is then

$$\begin{aligned}
\tau_R &= \tau_{Ra}+\tau_{Rb}\\
&= n_p M\Big(-i_{Ra}(t)i_{Sa}(t)\sin(n_p\theta_R)+i_{Ra}(t)i_{Sb}(t)\cos(n_p\theta_R)\ -i_{Rb}(t)i_{Sa}(t)\cos(n_p\theta_R)-i_{Rb}(t)i_{Sb}(t)\sin(n_p\theta_R)\Big).
\end{aligned}$$

Problem 11

Problem 12 *Multiple Pole-Pair Wound Rotor Synchronous Machine Model*

This follows directly from the multiple pole-pair model of a two-phase sinusoidally wound induction motor. That is, with $i_F = i_{Ra}$, $N_F = N_R$, and $i_{Rb} = 0$, it follows that

$$\begin{aligned}
\lambda_{Sa}(i_F,i_{Sa},i_{Sb},\theta_R) &= L_S i_{Sa} + M i_F\cos(n_p\theta_R)\\
\lambda_{Sb}(i_F,i_{Sa},i_{Sb},\theta_R) &= L_S i_{Sb} + M i_F\sin(n_p\theta_R)\\
\lambda_F(i_F,i_{Sa},i_{Sb},\theta_R) &= L_F i_{Fa} + M\left(i_{Sa}\cos(n_p\theta_R)+i_{Sb}\sin(n_p\theta_R)\right)\\
\tau_R &= n_p M\Big(-i_F i_{Sa}\sin(n_p\theta_R)+\ i_F i_{Sb}\cos(n_p\theta_R)\Big)
\end{aligned}$$

where $L_S = \frac{\mu_0\ell_1\ell_2\pi}{8g}N_S^2, L_F = \frac{\mu_0\ell_1\ell_2\pi}{8g}N_F^2$, and $M = \kappa\frac{\mu_0\pi\ell_1\ell_2}{8g}N_S N_F$ (N_S is the number of winding per pole-pair in each of the stator phases and N_F is the number of turns in the field winding).

The dynamic equations of this machine are then

$$\begin{aligned}
\frac{d}{dt}\Big(L_S i_{Sa} + M i_F \cos(n_p\theta_R)\Big) &= -R_S i_{Sa} + u_{Sa} \\
\frac{d}{dt}\Big(L_S i_{Sb} + M i_F \sin(n_p\theta_R)\Big) &= -R_S i_{Sb} + u_{Sb} \\
\frac{d}{dt}\Big(L_F i_{Fa} + M\left(i_{Sa}\cos(n_p\theta_R) + i_{Sb}\sin(n_p\theta_R)\right)\Big) &= -R_F i_F + u_F \\
J\frac{d\omega_R}{dt} &= n_p M i_F\Big(-i_{Sa}\sin(n_p\theta_R) + i_{Sb}\cos(n_p\theta_R)\Big) - \tau_L.
\end{aligned}$$