Modeling and High-Performance Control of Electric Machines

Chapter 10 Trapezoidal Back-Emf PM Synchronous Machines

John Chiasson

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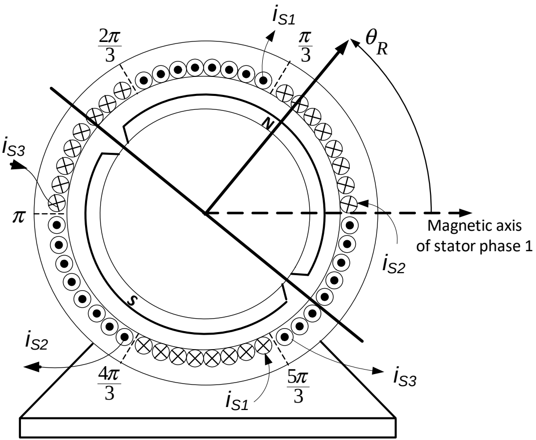
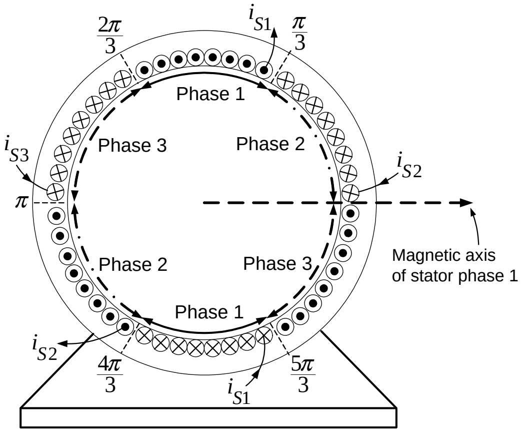
Construction of the BLDC

- Three stator phase windings **uniformly** wound.
- I.e., the number of turns between θ and $\theta + d\theta$ is constant independent of θ.



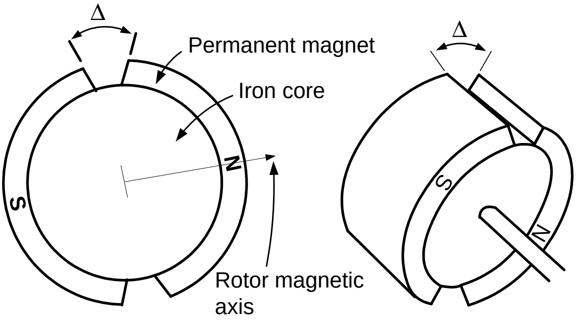
Construction of the BLDC

- i_{S1} - Phase 1 is uniformly wound between $\pi/3$ & $2\pi/3$ and $4\pi/3$ & $5\pi/3$.
- i_{S2} - Phase 2 is uniformly wound between π & $4\pi/3$ and 0 & $\pi/3$.
- i_{S3} - Phase 3 is uniformly wound between $2\pi/3$ & π and $5\pi/3$ & 2π.

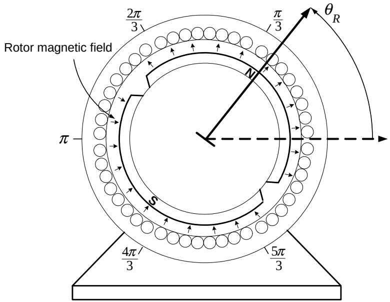


Construction of the BLDC

- Permanent magnet is bound to a cylindrical core of soft iron.
- The magnetic axis of the rotor as well as the angle Δ between the north and south poles are as shown.
- The rotor position is taken to be along the magnetic axis of the rotor.



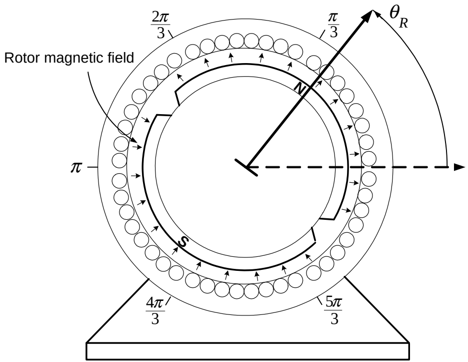
Rotor Magnetic Field in the Air Gap



$$\vec{\mathbf{B}}_R(r,\theta-\theta_R) =$$

$$\begin{cases} B_{R0}\dfrac{r_R}{r}\hat{\mathbf{r}} & \text{for} \quad -\dfrac{\pi}{2}+\dfrac{\Delta}{2} \le \theta-\theta_R \le \dfrac{\pi}{2}-\dfrac{\Delta}{2} \\[2ex] -B_{R0}\dfrac{r_R}{r}\dfrac{\theta-\theta_R-\pi/2}{\Delta/2}\hat{\mathbf{r}} & \text{for} \quad \dfrac{\pi}{2}-\dfrac{\Delta}{2} \le \theta-\theta_R \le \dfrac{\pi}{2}+\dfrac{\Delta}{2} \\[2ex] -B_{R0}\dfrac{r_R}{r}\hat{\mathbf{r}} & \text{for} \quad \dfrac{\pi}{2}+\dfrac{\Delta}{2} \le \theta-\theta_R \le \dfrac{3\pi}{2}-\dfrac{\Delta}{2} \\[2ex] B_{R0}\dfrac{r_R}{r}\dfrac{\theta-\theta_R-3\pi/2}{\Delta/2}\hat{\mathbf{r}} & \text{for} \quad \dfrac{3\pi}{2}-\dfrac{\Delta}{2} \le \theta-\theta_R \le \dfrac{3\pi}{2}+\dfrac{\Delta}{2} \end{cases}$$

Rotor Magnetic Field in the Air Gap



As an approximation take $\Delta = 0$ so that $\vec{\mathbf{B}}_R$ simplifies to

$$\vec{\mathbf{B}}_R(r,\theta-\theta_R)=\begin{cases} B_{R0}\dfrac{r_R}{r}\hat{\mathbf{r}} & \text{for} \quad -\pi/2 \le \theta-\theta_R \le \pi/2 \\ -B_{R0}\dfrac{r_R}{r}\hat{\mathbf{r}} & \text{for} \quad +\pi/2 \le \theta-\theta_R \le 3\pi/2. \end{cases}$$

Stator Magnetic Field in the Air Gap

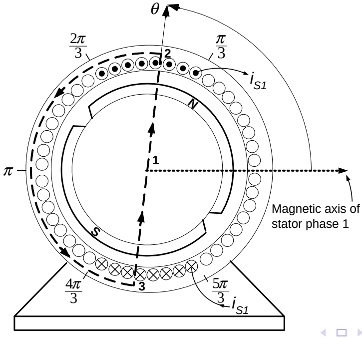
Stator phase 1 has a winding density given by

$$N_{S1}(\theta) = \begin{cases} \dfrac{N_S}{\pi/3} & \text{for } \dfrac{\pi}{3} \le \theta \le \dfrac{2\pi}{3} \text{ and } \dfrac{4\pi}{3} \le \theta \le \dfrac{5\pi}{3} \\ 0 & \text{elsewhere.} \end{cases}$$

Phase 1 has a total of $\displaystyle\int_{\pi/3}^{2\pi/3} \frac{N_S}{\pi/3} d\theta = N_S$ windings (turns/loops).

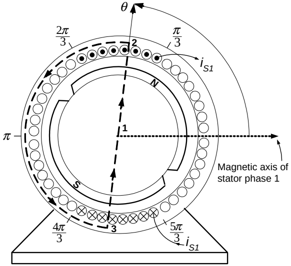
Similarly, $N_{S2}(\theta) = N_{S1}(\theta - 2\pi/3)$ and $N_{S3}(\theta) = N_{S1}(\theta - 4\pi/3)$.

Apply Ampère's law with $\vec{\mathbf{H}} \equiv 0$ in the iron to the path 1−2−3−1.



Stator Magnetic Field in the Air Gap

Applying Ampère's law to the path 1–2–3–1



gives $\left(H_{S1}(i_{S1}, \pi+\theta) = -H_{S1}(i_{S1}, \theta)\right)$

$$2gH_{S1}(i_{S1}, \theta) = \begin{cases} N_S i_{S1}, & -\pi/3 \le \theta \le \pi/3 \\ \displaystyle\int_\theta^{2\pi/3} \frac{N_S i_{S1}}{\pi/3} d\theta - \int_{4\pi/3}^{\theta+\pi} \frac{N_S i_{S1}}{\pi/3} d\theta, & \pi/3 \le \theta \le 2\pi/3 \\ -N_S i_{S1}, & 2\pi/3 \le \theta \le 4\pi/3 \\ \displaystyle\int_\theta^{5\pi/3} - \frac{N_S i_{S1}}{\pi/3} d\theta + \int_{7\pi/3}^{\theta+\pi} \frac{N_S i_{S1}}{\pi/3} d\theta, & 4\pi/3 \le \theta \le 5\pi/3 \end{cases}$$

Stator Magnetic Field in the Air Gap

Carrying out the simple computations results in

$$\vec{\mathbf{H}}_{S1}(i_{S1},\theta)=\begin{cases}\dfrac{N_S i_{S1}}{2g}\hat{\mathbf{r}} & \text{for} \quad -\pi/3\le\theta\le\pi/3\\[2ex] \dfrac{N_S i_{S1}}{2g}\dfrac{6}{\pi}\left(\dfrac{\pi}{2}-\theta\right)\hat{\mathbf{r}} & \text{for} \quad \pi/3\le\theta\le 2\pi/3\\[2ex] -\dfrac{N_S i_{S1}}{2g}\hat{\mathbf{r}} & \text{for} \quad 2\pi/3\le\theta\le 4\pi/3\\[2ex] \dfrac{N_S i_{S1}}{2g}\dfrac{6}{\pi}\left(\theta-\dfrac{3\pi}{2}\right)\hat{\mathbf{r}} & \text{for} \quad 4\pi/3\le\theta\le 5\pi/3.\end{cases}$$

Stator Magnetic Field in the Air Gap

- $\vec{\mathbf{B}}_{S1} = \mu_0 \vec{\mathbf{H}}_{S1}$ in the air gap.
- A factor of r_R/r is included so that $\vec{\mathbf{B}}_{S1}$ satisfies conservation of flux in the air gap.
- The magnetic field $\vec{\mathbf{B}}_{S1} = B_{S1}(i_{S1}, r, \theta)\hat{\mathbf{r}}$ in the air gap due i_{S1} is given by

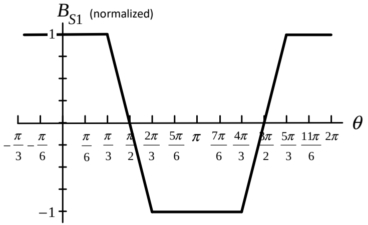
$$B_{S1}(i_{S1}, r, \theta)\hat{\mathbf{r}} = \begin{cases} \dfrac{\mu_0 N_S i_{S1}}{2g}\dfrac{r_R}{r}\hat{\mathbf{r}} & \text{for} \quad -\pi/3 \le \theta \le \pi/3 \\[2ex] \dfrac{\mu_0 N_S i_{S1}}{2g}\dfrac{r_R}{r}\dfrac{6}{\pi}\left(\dfrac{\pi}{2} - \theta\right)\hat{\mathbf{r}} & \text{for} \quad \pi/3 \le \theta \le 2\pi/3 \\[2ex] -\dfrac{\mu_0 N_S i_{S1}}{2g}\dfrac{r_R}{r}\hat{\mathbf{r}} & \text{for} \quad 2\pi/3 \le \theta \le 4\pi/3 \\[2ex] \dfrac{\mu_0 N_S i_{S1}}{2g}\dfrac{r_R}{r}\dfrac{6}{\pi}\left(\theta - \dfrac{3\pi}{2}\right)\hat{\mathbf{r}} & \text{for} \quad 4\pi/3 \le \theta \le 5\pi/3. \end{cases}$$

Similarly,

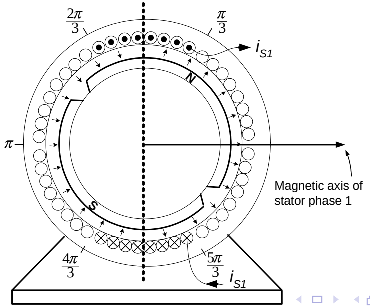
$$\begin{aligned} B_{S2}(i_{S2}, r, \theta) &= B_{S1}(i_{S2}, r, \theta - 2\pi/3) \\ B_{S3}(i_{S3}, r, \theta) &= B_{S1}(i_{S3}, r, \theta - 4\pi/3). \end{aligned}$$

Stator Magnetic Field in the Air Gap

- Plot of $B_{S1}(i_{S1}, r, \theta) / \left(\dfrac{\mu_0 N_S i_{S1}}{2g} \dfrac{r_R}{r} \right)$ for $-\pi/3 \leq \theta \leq 2\pi$.

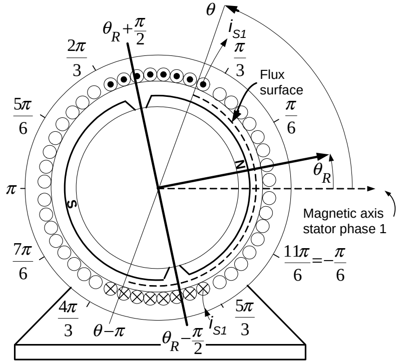


- $\vec{\mathbf{B}}_{S1}$ is **radially out** to the right of the dashed vertical line.
- $\vec{\mathbf{B}}_{S1}$ is **radially in** to the left of the vertical line.



Stator Flux Linkage Produced by $\vec{\mathbf{B}}_S$

- Compute the flux $\phi_{11}(i_{S1}, \theta)$ in a winding of stator phase 1 at the angle θ where $\pi/3 \le \theta \le 2\pi/3$.



$$
\begin{aligned}
\phi_{11}(i_{S1}, \theta) &= \int_0^{\ell_1} \int_{\theta-\pi}^{\theta} \vec{\mathbf{B}}_{S1}(i_{S1}, r_S, \theta') \cdot (r_S d\theta' d\ell \hat{\mathbf{r}}) \\
&= r_S \ell_1 \int_{-\pi/3}^{\pi/3} \frac{\mu_0 N_S i_{S1}}{2g} \frac{r_R}{r_S} d\theta' + r_S \ell_1 \int_{\pi/3}^{\theta} \frac{\mu_0 N_S i_{S1}}{g\pi/3} \frac{r_R}{r_S} \left(\frac{\pi}{2} - \theta' \right) d\theta' \\
&\quad + r_S \ell_1 \int_{\theta+\pi}^{5\pi/3} \frac{\mu_0 N_S i_{S1}}{g\pi/3} \frac{r_R}{r_S} \left(\theta' - \frac{3\pi}{2} \right) d\theta'.
\end{aligned}
$$

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_S$

$$
\begin{aligned}
\phi_{11}(i_{S1},\theta) &= \int_0^{\ell_1}\int_{\theta-\pi}^{\theta}\vec{\mathbf{B}}_{S1}(i_{S1},r_S,\theta')\cdot\left(r_S d\theta' d\ell\hat{\mathbf{r}}\right)\\
&= r_S\ell_1\int_{-\pi/3}^{\pi/3}\frac{\mu_0 N_S i_{S1}}{2g}\frac{r_R}{r_S}d\theta' + r_S\ell_1\int_{\pi/3}^{\theta}\frac{\mu_0 N_S i_{S1}}{g\pi/3}\frac{r_R}{r_S}\left(\frac{\pi}{2}-\theta'\right)d\theta'\\
&\quad + r_S\ell_1\int_{\theta+\pi}^{5\pi/3}\frac{\mu_0 N_S i_{S1}}{g\pi/3}\frac{r_R}{r_S}\left(\theta'-\frac{3\pi}{2}\right)d\theta'\\
&= \frac{\mu_0 r_R\ell_1 N_S i_{S1}}{2g}\left(\frac{2\pi}{3}-\frac{1}{2}\frac{6}{\pi}\left(\frac{\pi}{2}-\theta'\right)^2\Big|_{\pi/3}^{\theta}+\frac{1}{2}\frac{6}{\pi}\left(\theta'-\frac{3\pi}{2}\right)^2\Big|_{\theta+\pi}^{5\pi/3}\right)\\
&= \frac{\mu_0 r_R\ell_1 N_S i_{S1}}{2g}\left(\frac{2\pi}{3}-\frac{3}{\pi}\left(\left(\frac{\pi}{2}-\theta\right)^2-\left(\frac{\pi}{6}\right)^2\right)+\frac{3}{\pi}\left(\left(\frac{\pi}{6}\right)^2-\left(\theta-\frac{\pi}{2}\right)^2\right)\right)\\
&= \frac{\mu_0 r_R\ell_1 N_S i_{S1}}{2g}\left(-\frac{6}{\pi}\left(\theta-\frac{\pi}{2}\right)^2+\frac{5\pi}{6}\right).
\end{aligned}
$$

- The outward normal was used to compute the flux.
- So, if $-d\phi_{11}/dt > 0$, the induced emf in stator phase 1 will push current in the same direction as that chosen for positive current flow in stator phase 1.

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_S$

The total flux linkage $\lambda_{S1}(i_{S1}, i_{S2} = 0, i_{S3} = 0)$ in stator phase 1 due to i_{S1}

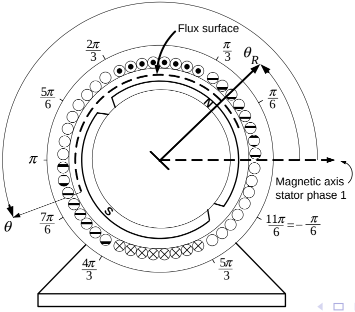
$$\begin{aligned}
\lambda_{S1}(i_{S1},0,0) = \int_{\pi/3}^{2\pi/3} \phi_{11}(i_{S1},\theta)\frac{N_S}{\pi/3}d\theta \quad &= \frac{\mu_0 r_R \ell_1 N_S^2 i_{S1}}{2g}\frac{3}{\pi}\int_{\pi/3}^{2\pi/3}\left(-\frac{6}{\pi}\left(\theta-\frac{\pi}{2}\right)^2+\frac{5\pi}{6}\right)d\theta \\
&= \frac{\mu_0 r_R \ell_1 N_S^2 i_{S1}}{2g}\left(-\frac{18}{\pi^2}\frac{1}{3}\left(\theta-\frac{\pi}{2}\right)^3\Big|_{\pi/3}^{2\pi/3}+\frac{5}{2}\left(\frac{2\pi}{3}-\frac{\pi}{3}\right)\right) \\
&= \frac{\mu_0 r_R \ell_1 N_S^2 i_{S1}}{2g}\left(-\frac{\pi}{18}+\frac{5}{2}\frac{\pi}{3}\right) \\
&= \frac{\mu_0 r_R \ell_1 \pi N_S^2 i_{S1}}{2g}\left(\frac{14}{18}\right) \\
&= L_S i_{S1}
\end{aligned}$$

where $L_S \triangleq \left(\frac{7}{3}\right)\frac{\mu_0 \ell_1 \ell_2 \pi}{12g}N_S^2$ and $r_R = \ell_2/2$.

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_S$

- $\phi_{21}(i_{S1}, \theta)$ is the flux in a single winding of stator phase 2 at an angle θ due i_{S1}.
- With $\pi \le \theta \le 4\pi/3$ this is computed as follows.

$$\begin{aligned}
\phi_{21}(i_{S1}, \theta) &= \int_0^{\ell_1} \int_{\theta-\pi}^{\theta} \vec{\mathbf{B}}_{S1}(i_{S1}, r_S, \theta') \cdot \left(r_S \, d\theta' \, d\ell \hat{\mathbf{r}}\right) \\
&= r_S \ell_1 \int_{\theta-\pi}^{\pi/3} \frac{\mu_0 N_S i_{S1}}{2g} \frac{r_R}{r_S} d\theta' + r_S \ell_1 \int_{\pi/3}^{2\pi/3} \frac{\mu_0 N_S i_{S1}}{g\pi/3} \frac{r_R}{r_S} \left(\frac{\pi}{2} - \theta'\right) d\theta' + r_S \ell_1 \int_{2\pi/3}^{\theta} -\frac{\mu_0 N_S i_{S1}}{2g} \frac{r_R}{r_S} d\theta' \\
&= \frac{\mu_0 r_R \ell_1 N_S i_{S1}}{2g} \left(\frac{4\pi}{3} - \theta - \frac{6}{\pi} \frac{1}{2} \left(\theta' - \frac{\pi}{2}\right)^2 \Big|_{\pi/3}^{2\pi/3} - \left(\theta - \frac{2\pi}{3}\right) \right) \\
&= \frac{\mu_0 r_R \ell_1 N_S i_{S1}}{g} (\pi - \theta).
\end{aligned}$$



Stator Flux Linkage Produced by $\vec{\mathbf{B}}_S$

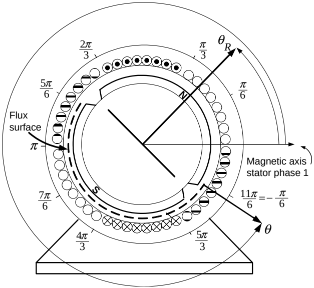
The total flux linkage $\lambda_{S2}(i_{S1}, 0, 0)$ in stator phase 2 produced by the current in stator phase 1 is then

$$
\begin{aligned}
\lambda_{S2}(i_{S1}, 0, 0) &= \int_{\pi}^{4\pi/3} \phi_{21}(i_{S1}, \theta) \frac{N_S}{\pi/3} d\theta \\
&= \int_{\pi}^{4\pi/3} \frac{\mu_0 r_R \ell_1 N_S i_{S1}}{g} (\pi - \theta) \frac{N_S}{\pi/3} d\theta \\
&= -\frac{3}{\pi} \frac{\mu_0 r_R \ell_1 N_S^2}{g} i_{S1} \frac{1}{2} (\pi - \theta)^2 \Big|_{\pi}^{4\pi/3} \\
&= -\frac{\mu_0 r_R \ell_1 \pi N_S^2}{6g} i_{S1} \\
&= -M i_{S1}
\end{aligned}
$$

where $M \triangleq \dfrac{\mu_0 \ell_1 \ell_2 \pi}{12g} N_S^2$ and $r_R = \ell_2/2$.

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_S$

With $5\pi/3 \le \theta \le 2\pi$ we compute the flux $\phi_{31}(i_{S1}, \theta)$ in a single winding of stator phase 3 at an angle θ due to i_{S1}.



$$
\begin{aligned}
\phi_{31}(i_{S1}, \theta) &= \int_0^{\ell_1} \int_{\theta-\pi}^{\theta} \vec{\mathbf{B}}_{S1}(i_{S1}, r_S, \theta') \cdot \left(r_S \, d\theta' \, d\ell \hat{\mathbf{r}}\right) \\
&= r_S \ell_1 \int_{\theta-\pi}^{4\pi/3} -\frac{\mu_0 N_S i_{S1}}{2g} \frac{r_R}{r_S} d\theta' + r_S \ell_1 \int_{4\pi/3}^{5\pi/3} \frac{\mu_0 N_S i_{S1}}{g\pi/3} \frac{r_R}{r} \left(\theta' - \frac{3\pi}{2}\right) d\theta' + r_S \ell_1 \int_{5\pi/3}^{\theta} \frac{\mu_0 N_S i_{S1}}{2g} \frac{r_R}{r_S} d\theta' \\
&= \frac{\mu_0 r_R \ell_1 N_S i_{S1}}{2g} \left(-\left(\frac{7\pi}{3} - \theta\right) + \frac{6}{\pi} \frac{1}{2} \left(\theta' - \frac{3\pi}{2}\right)^2 \Bigg|_{4\pi/3}^{5\pi/3} + \left(\theta - \frac{5\pi}{3}\right) \right) \\
&= \frac{\mu_0 r_R \ell_1 N_S i_{S1}}{g} (\theta - 2\pi).
\end{aligned}
$$

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_S$

The total flux linkage in stator phase 3 produced by i_{S1} is then

$$\begin{aligned}\lambda_{S3}(i_{S1},0,0) = \int_{5\pi/3}^{2\pi}\phi_{31}(i_{S1},\theta)\frac{N_S}{\pi/3}d\theta &= \int_{5\pi/3}^{2\pi}\frac{\mu_0 r_R \ell_1 N_S i_{S1}}{g}(\theta-2\pi)\frac{N_S}{\pi/3}d\theta\\ &= \frac{3}{\pi}\frac{\mu_0 r_R \ell_1 N_S^2}{g}i_{S1}\frac{1}{2}(\theta-2\pi)^2\Big|_{5\pi/3}^{2\pi}\\ &= -\frac{\mu_0 r_R \ell_1 \pi N_S^2}{6g}i_{S1}\\ &= -Mi_{S1}\end{aligned}$$

where $M \triangleq \dfrac{\mu_0 \ell_1 \ell_2 \pi}{12g}N_S^2$. We have shown

$$\begin{aligned}\lambda_{S1}(i_{S1},0,0) &= +L_S i_{S1}\\ \lambda_{S2}(i_{S1},0,0) &= -Mi_{S1}\\ \lambda_{S3}(i_{S1},0,0) &= -Mi_{S1}\end{aligned}$$

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_S$

- The other phases are computed similarly.
- In summary, the flux linkages in the stator phases due to the stator currents are

$$\begin{aligned}
\lambda_{S1}(i_{S1}, i_{S2}, i_{S3}) &= +L_S i_{S1} - M i_{S2} - M i_{S3} \\
\lambda_{S2}(i_{S1}, i_{S2}, i_{S3}) &= -M i_{S1} + L_S i_{S2} - M i_{S3} \\
\lambda_{S3}(i_{S1}, i_{S2}, i_{S3}) &= -M i_{S1} - M i_{S2} + L_S i_{S3}
\end{aligned}$$

where

$$L_S = \frac{7}{3}\frac{\mu_0 \ell_1 \ell_2 \pi}{12g} N_S^2, \qquad M = \frac{\mu_0 \ell_1 \ell_2 \pi}{12g} N_S^2.$$

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_S$

In matrix form the flux linkages are given by

$$\begin{bmatrix} \lambda_{S1} \\ \lambda_{S2} \\ \lambda_{S3} \end{bmatrix} = \begin{bmatrix} L_S & -M & -M \\ -M & L_S & -M \\ -M & -M & L_S \end{bmatrix} \begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix}.$$

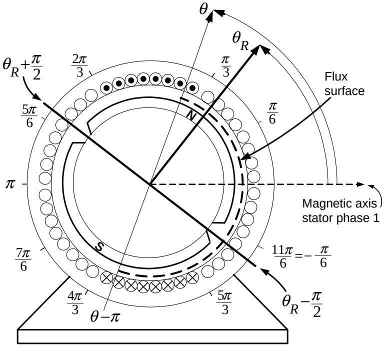
The inverse of the inductance matrix on the right is

$$\frac{1}{(L_S - 2M)(M + L_S)} \begin{bmatrix} L_S - M & M & M \\ M & L_S - M & M \\ M & M & L_S - M \end{bmatrix}.$$

- As $L_S = (7/3)M$ we have $(L_S - 2M)(M + L_S) > 0$.

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_R$

Compute the flux linkage $\lambda_{S1\_R}(\theta_R)$ in phase 1 due to $\vec{\mathbf{B}}_R$.
The top side of phase 1 is located in the interval $\pi/3 \le \theta \le 2\pi/3$.
Case 1. $\pi/6 \le \theta_R \le 5\pi/6$.

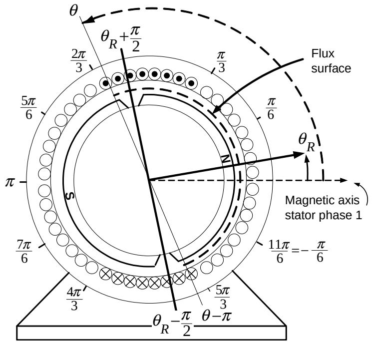


$$
\begin{aligned}
\phi_{S1\_R}(\theta) &= \int_0^{\ell_1}\int_{\theta-\pi}^{\theta} \vec{\mathbf{B}}_R(r_S,\theta'-\theta_R)\cdot\left(r_S\,d\theta'\,d\ell\,\hat{\mathbf{r}}\right)\\
&= \int_0^{\ell_1}\int_{\theta_R-\pi/2}^{\theta} B_{R0}r_R\,d\theta'\,d\ell + \int_0^{\ell_1}\int_{\theta-\pi}^{\theta_R-\pi/2} -B_{R0}r_R\,d\theta'\,d\ell\\
&= 2r_R\ell_1 B_{R0}(\theta-\theta_R).
\end{aligned}
$$

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_R$

Compute the flux linkage $\lambda_{S1\_R}(\theta_R)$ in phase 1 due to $\vec{\mathbf{B}}_R$.

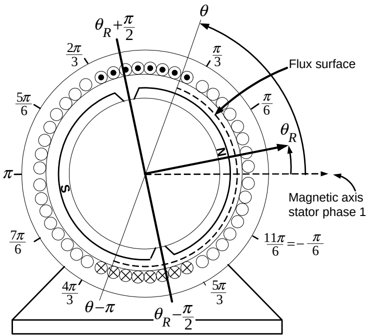
Case 2. $-\pi/6 \le \theta_R \le \pi/6$ and $\theta_R + \pi/2 \le \theta \le 2\pi/3$.



$$
\begin{aligned}
\phi_{S1\_R}(\theta) &= \int_0^{\ell_1}\int_{\theta-\pi}^{\theta} \vec{\mathbf{B}}_R(r_S,\theta'-\theta_R)\cdot\left(r_S d\theta' d\ell \hat{\mathbf{r}}\right) \\
&= \int_0^{\ell_1}\int_{\theta_R+\pi/2}^{\theta} -B_{R0} r_R d\theta' d\ell + \int_0^{\ell_1}\int_{\theta-\pi}^{\theta_R+\pi/2} B_{R0} r_R d\theta' d\ell \\
&= -2 r_R \ell_1 B_{R0}(\theta-\theta_R-\pi).
\end{aligned}
$$

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_R$

Case 3. $-\pi/6 \le \theta_R \le \pi/6$ and $\pi/3 \le \theta \le \theta_R + \pi/2$.



$$
\begin{aligned}
\phi_{S1\_R}(\theta) &= \int_0^{\ell_1}\int_{\theta-\pi}^{\theta} \vec{\mathbf{B}}_R(r_S,\theta'-\theta_R)\cdot\left(r_S d\theta' d\ell \hat{\mathbf{r}}\right) \\
&= \int_0^{\ell_1}\int_{\theta_R-\pi/2}^{\theta} B_{R0} r_R d\theta' d\ell + \int_0^{\ell_1}\int_{\theta-\pi}^{\theta_R-\pi/2} -B_{R0} r_R d\theta' d\ell \\
&= 2 r_R \ell_1 B_{R0}(\theta-\theta_R).
\end{aligned}
$$

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_R$

We have just shown that

$$\phi_{S1\_R}(\theta) = \begin{cases} 2r_R\ell_1 B_{R0}(\theta - \theta_R) & \text{for} \quad \pi/6 \le \theta_R \le 5\pi/6 \\[2ex] -2r_R\ell_1 B_{R0}(\theta - \theta_R - \pi) & \text{for} \quad \theta_R + \pi/2 \le \theta \le 2\pi/3 \\ & \text{and} \quad -\pi/6 \le \theta_R \le \pi/6 \\[2ex] 2r_R\ell_1 B_{R0}(\theta - \theta_R) & \text{for} \quad \pi/3 \le \theta \le \theta_R + \pi/2 \\ & \text{and} \quad -\pi/6 \le \theta_R \le \pi/6. \end{cases}$$

Total flux linkage $\lambda_{S1\_R}(\theta_R)$

$$\lambda_{S1\_R}(\theta_R) = \int_{\pi/3}^{2\pi/3} \phi_{S1\_R}(\theta)\frac{N_S}{\pi/3}d\theta$$

$$= \begin{cases} \displaystyle\int_{\pi/3}^{2\pi/3} 2r_R\ell_1 B_{R0}(\theta - \theta_R)\frac{N_S}{\pi/3}d\theta & \text{for} \ +\pi/6 \le \theta_R \le 5\pi/6 \\[3ex] \displaystyle\int_{\pi/3}^{\theta_R+\pi/2} 2r_R\ell_1 B_{R0}(\theta - \theta_R)\frac{N_S}{\pi/3}d\theta \\ \displaystyle\quad + \int_{\theta_R+\pi/2}^{2\pi/3} -2r_R\ell_1 B_{R0}(\theta - \theta_R - \pi)\frac{N_S}{\pi/3}d\theta & \text{for} \ -\pi/6 \le \theta_R \le \pi/6. \end{cases}$$

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_R$

Simplifying, this becomes

$$\lambda_{S1\_R}(\theta_R) = \begin{cases} \dfrac{3}{\pi} r_R \ell_1 N_S B_{R0} \left. (\theta - \theta_R)^2 \right|_{\pi/3}^{2\pi/3} & \text{for } +\pi/6 \le \theta_R \le 5\pi/6 \\[2ex] r_R \ell_1 B_{R0} \dfrac{N_S}{\pi/3} \left(\left. (\theta - \theta_R)^2 \right|_{\pi/3}^{\theta_R + \pi/2} - \left. (\theta - \theta_R - \pi)^2 \right|_{\theta_R + \pi/2}^{2\pi/3} \right) & \\ & \text{for } -\pi/6 \le \theta_R \le \pi/6 \end{cases}$$

or

$$\lambda_{S1\_R}(\theta_R) = \begin{cases} r_R \ell_1 N_S B_{R0} \left(2(\dfrac{\pi}{3} - \theta_R) + \dfrac{\pi}{3} \right) & \text{for } \quad +\pi/6 \le \theta_R \le 5\pi/6 \\[2ex] r_R \ell_1 N_S B_{R0} \left(-\dfrac{6}{\pi} \theta_R^2 + \dfrac{5\pi}{6} \right) & \text{for } \quad -\pi/6 \le \theta_R \le \pi/6. \end{cases}$$

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_R$

By symmetry, $\lambda_{S1\_R}(\theta_R \pm \pi) = -\lambda_{S1\_R}(\theta_R)$ so that the total flux linkage in stator phase 1 due to the rotor's magnetic field may be written as

$$\lambda_{S1\_R}(\theta_R) = \begin{cases} +r_R\ell_1 N_S B_{R0}\left(-\dfrac{6}{\pi}\theta_R^2 + \dfrac{5\pi}{6}\right), & -\dfrac{\pi}{6} \le \theta_R \le \dfrac{\pi}{6} \\ +r_R\ell_1 N_S B_{R0}\left(2(\dfrac{\pi}{3} - \theta_R) + \dfrac{\pi}{3}\right), & \dfrac{\pi}{6} \le \theta_R \le \dfrac{5\pi}{6} \\ -r_R\ell_1 N_S B_{R0}\left(-\dfrac{6}{\pi}\left(\theta_R - \pi\right)^2 + \dfrac{5\pi}{6}\right), & \dfrac{5\pi}{6} \le \theta_R \le \dfrac{7\pi}{6} \\ -r_R\ell_1 N_S B_{R0}\left(2\left(\dfrac{\pi}{3} - (\theta_R - \pi)\right) + \dfrac{\pi}{3}\right), & \dfrac{7\pi}{6} \le \theta_R \le \dfrac{11\pi}{6}. \end{cases}$$

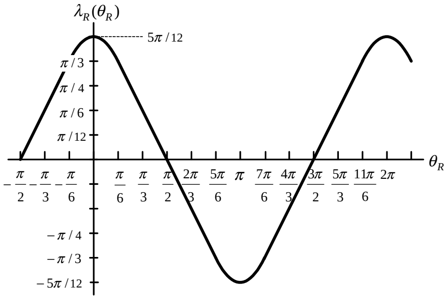
Define $\lambda_R(\theta_R)$ as

$$\lambda_R(\theta_R) \triangleq \lambda_{S1\_R}(\theta_R)/M_{SR}$$

where $M_{SR} \triangleq 2r_R\ell_1 N_S B_{R0}$ is the *coefficient of mutual inductance* between the stator and the rotor.

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_R$

Plot of $\lambda_R(\theta_R)$ versus θ_R.



A simple computation shows that

$$-1 \leq \frac{\partial \lambda_R(\theta_R)}{\partial \theta_R} \leq 1.$$

Finally

$$\begin{aligned}
\lambda_{S1_R}(\theta_R) &= M_{SR}\lambda_R(\theta_R) \\
\lambda_{S2_R}(\theta_R) &= M_{SR}\lambda_R(\theta_R - 2\pi/3) \\
\lambda_{S3_R}(\theta_R) &= M_{SR}\lambda_R(\theta_R - 4\pi/3).
\end{aligned}$$

Emf in the Stator Windings Produced by $\vec{\mathbf{B}}_R$

By Faraday's law, the back emf induced in the windings of phase 1 by rotor's magnetic field is given by

$$e_{S1} \triangleq -\frac{d}{dt}M_{SR}\lambda_R(\theta_R) = \begin{cases} +M_{SR}\dfrac{6\theta_R}{\pi}\omega_R, & -\pi/6 \le \theta_R \le \pi/6 \\ +M_{SR}\omega_R, & \pi/6 \le \theta_R \le 5\pi/6 \\ -M_{SR}\dfrac{6(\theta_R-\pi)}{\pi}\omega_R, & 5\pi/6 \le \theta_R \le 7\pi/6 \\ -M_{SR}\omega_R, & 7\pi/6 \le \theta_R \le 11\pi/6 \end{cases} \quad (1)$$

where $\omega_R = d\theta_R/dt$. With $e_p \triangleq M_{SR} = 2r_R\ell_1 N_S B_{R0}$ and

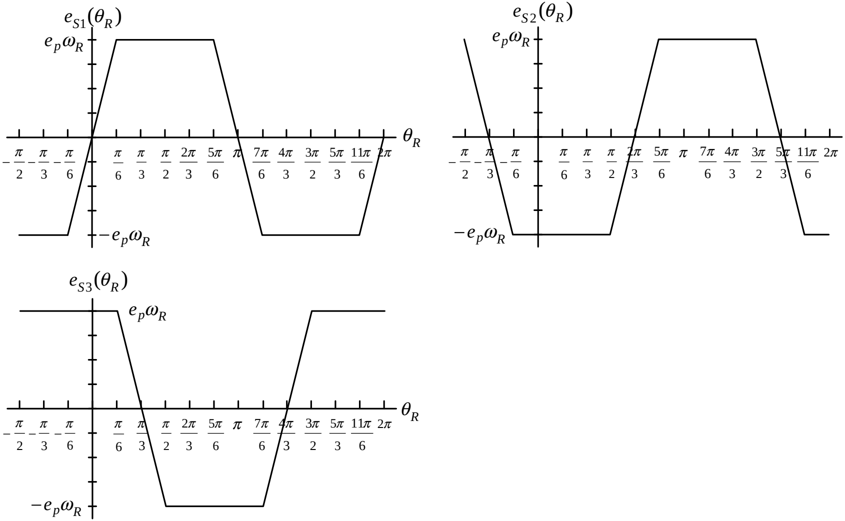
$$e(\theta_R) \triangleq \frac{e_{S1}}{M_{SR}\omega_R} = -\frac{\partial\lambda_R(\theta_R)}{\partial\theta_R}, \quad (2)$$

the back emf in each stator phase may now be written succinctly as

$$\begin{aligned} e_{S1} &= e_p e(\theta_R)\omega_R \\ e_{S2} &= e_p e(\theta_R - 2\pi/3)\omega_R \\ e_{S2} &= e_p e(\theta_R - 4\pi/3)\omega_R. \end{aligned}$$

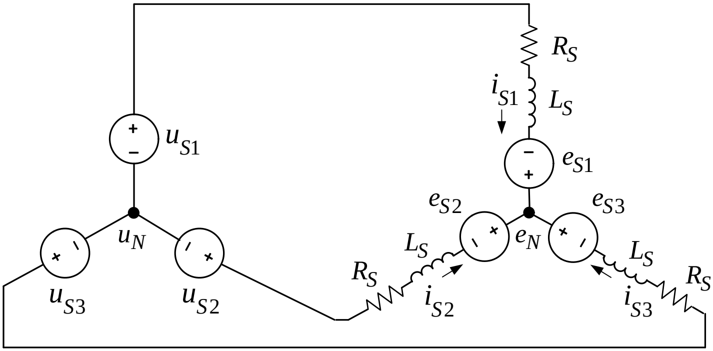
Trapezoidal Back Emf in the Stator Windings

As $-1 \leq e(\theta_R) \leq 1$ the factor $e_p\omega_R$ is the peak value of the back emf.



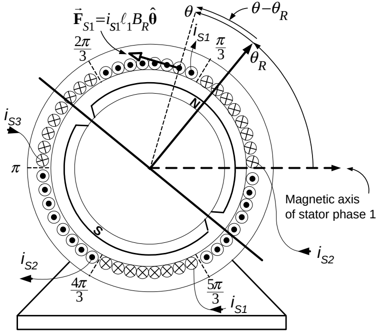
Equivalent Circuit

- Note the sign convention for the back emfs.
- If $e_{S1} > 0$ then it "wants" to force current in the positive direction of i_{S1}.
- The back emfs are not balanced, i.e., $e_{S1} + e_{S2} + e_{S3} \neq 0$.



Torque

- The top sides of the windings of phase 1 are at $\pi/3 \leq \theta \leq 2\pi/3$.



- The force $\vec{\mathbf{F}}_{S1}$ exerted by $\vec{\mathbf{B}}_R(r_S, \theta - \theta_R)$ on the axial side of a winding at θ is

$$\begin{aligned}\vec{\mathbf{F}}_{S1}(r, \theta - \theta_R) = i_{S1}\,(\ell_1 \hat{\mathbf{z}}) \times \vec{\mathbf{B}}_R(r_S, \theta - \theta_R) \quad &= \quad i_{S1}\ell_1 B_R(r_S, \theta - \theta_R)\hat{\mathbf{z}} \times \hat{\mathbf{r}} \\ &= \quad i_{S1}\ell_1 B_R(r_S, \theta - \theta_R)\hat{\boldsymbol{\theta}}.\end{aligned}$$

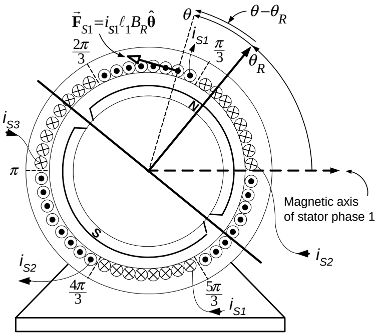
- On the bottom side where $4\pi/3 \leq \theta \leq 5\pi/3$, the force is

$$\begin{aligned}\vec{\mathbf{F}}_{S1}(r, \theta - \theta_R) = i_{S1}\,(-\ell_1 \hat{\mathbf{z}}) \times \vec{\mathbf{B}}_R(r_S, \theta - \theta_R) \quad &= \quad -i_{S1}\ell_1 B_R(r_S, \theta - \theta_R)\hat{\mathbf{z}} \times \hat{\mathbf{r}} \\ &= \quad -i_{S1}\ell_1 B_R(r_S, \theta - \theta_R)\hat{\boldsymbol{\theta}}.\end{aligned}$$

Torque τ_{S1} on stator phase 1

$$
\begin{aligned}
\tau_{S1} &= \int_{\pi/3}^{2\pi/3} \vec{\mathbf{r}} \times \vec{\mathbf{F}}_{S1} \frac{N_S}{\pi/3} d\theta + \int_{4\pi/3}^{5\pi/3} \vec{\mathbf{r}} \times \vec{\mathbf{F}}_{S1} \frac{N_S}{\pi/3} d\theta \\
&= \int_{\pi/3}^{2\pi/3} i_{S1}\ell_1 B_R(r_S, \theta - \theta_R) r_S \frac{N_S}{\pi/3} d\theta \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} - \int_{4\pi/3}^{5\pi/3} i_{S1}\ell_1 B_R(r_S, \theta - \theta_R) r_S \frac{N_S}{\pi/3} d\theta \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \\
&= i_{S1}\ell_1 \frac{N_S}{\pi/3} \left(\int_{\pi/3}^{2\pi/3} r_S B_R(r_S, \theta - \theta_R) d\theta \; - \int_{4\pi/3}^{5\pi/3} r_S B_R(r_S, \theta - \theta_R) d\theta \right) \hat{\mathbf{z}} \\
&= 2 i_{S1}\ell_1 \frac{N_S}{\pi/3} \int_{\pi/3}^{2\pi/3} r_S B_R(r_S, \theta - \theta_R) d\theta \hat{\mathbf{z}}
\end{aligned}
$$

using $B_R(r_S, \theta - \theta_R) = -B_R(r_S, \theta - \theta_R \pm \pi)$.



Torque

Using the expression

$$\vec{\mathbf{B}}_R(r,\theta-\theta_R)=\begin{cases} B_{R0}\dfrac{r_R}{r}\hat{\mathbf{r}} & \text{for} \quad -\pi/2 \le \theta-\theta_R \le \pi/2 \\ -B_{R0}\dfrac{r_R}{r}\hat{\mathbf{r}} & \text{for} \quad +\pi/2 \le \theta-\theta_R \le 3\pi/2. \end{cases}$$

we have

$$\int_{\pi/3}^{2\pi/3} r_S B_R(r_S,\theta-\theta_R)d\theta$$

$$= \begin{cases} \displaystyle\int_{\pi/3}^{\theta_R+\pi/2} r_R B_{R0}d\theta + \int_{\theta_R+\pi/2}^{2\pi/3} -r_R B_{R0}d\theta & \text{for} \quad -\dfrac{\pi}{6} \le \theta_R \le \dfrac{\pi}{6} \\ \displaystyle\int_{\pi/3}^{2\pi/3} r_R B_{R0}d\theta & \text{for} \quad \dfrac{\pi}{6} \le \theta_R \le \dfrac{5\pi}{6} \\ \displaystyle\int_{\pi/3}^{\theta_R-\pi/2} -r_R B_{R0}d\theta + \int_{\theta_R-\pi/2}^{2\pi/3} r_R B_{R0}d\theta & \text{for} \quad \dfrac{5\pi}{6} \le \theta_R \le \dfrac{7\pi}{6} \\ \displaystyle\int_{\pi/3}^{2\pi/3} -r_R B_{R0}d\theta & \text{for} \quad \dfrac{7\pi}{6} \le \theta_R \le \dfrac{11\pi}{6} \end{cases}$$

Torque

Doing the computations this becomes

$$\int_{\pi/3}^{2\pi/3} r_S B_R(r_S, \theta - \theta_R) d\theta$$

$$= \begin{cases} +r_R B_{R0}(\theta_R + \pi/6) - r_R B_{R0}(\pi/6 - \theta_R), & -\dfrac{\pi}{6} \leq \theta_R \leq \dfrac{\pi}{6} \\ +r_R B_{R0}\pi/3, & \dfrac{\pi}{6} \leq \theta_R \leq \dfrac{5\pi}{6} \\ -r_R B_{R0}(\theta_R - 5\pi/6) + r_R B_{R0}(7\pi/6 - \theta_R), & \dfrac{5\pi}{6} \leq \theta_R \leq \dfrac{7\pi}{6} \\ -r_R B_{R0}\pi/3, & \dfrac{7\pi}{6} \leq \theta_R \leq \dfrac{11\pi}{6} \end{cases}$$

Torque

Combining terms results in

$$\int_{\pi/3}^{2\pi/3} r_S B_R(r_S, \theta - \theta_R) d\theta = \begin{cases} +2 r_R B_{R0} \theta_R, & -\pi/6 \le \theta_R \le \pi/6 \\ +r_R B_{R0} \pi/3, & \pi/6 \le \theta_R \le 5\pi/6 \\ -2 r_R B_{R0} (\theta_R - \pi), & 5\pi/6 \le \theta_R \le 7\pi/6 \\ -r_R B_{R0} \pi/3, & 7\pi/6 \le \theta_R \le 11\pi/6. \end{cases}$$

As $\tau_{S1} = 2 i_{S1} \ell_1 \dfrac{N_S}{\pi/3} \displaystyle\int_{\pi/3}^{2\pi/3} r_S B_R(r_S, \theta - \theta_R) d\theta$ we have

$$\tau_{S1} = \begin{cases} +2\ell_1 r_R N_S B_{R0} \dfrac{6\theta_R}{\pi} i_{S1} & \text{for} \quad -\pi/6 \le \theta_R \le \pi/6 \\ +2\ell_1 r_R N_S B_{R0} i_{S1} & \text{for} \quad \pi/6 \le \theta_R \le 5\pi/6 \\ -2\ell_1 r_R N_S B_{R0} \dfrac{6(\theta_R - \pi)}{\pi} i_{S1} & \text{for} \quad 5\pi/6 \le \theta_R \le 7\pi/6 \\ -2\ell_1 r_R N_S B_{R0} i_{S1} & \text{for} \quad 7\pi/6 \le \theta_R \le 11\pi/6. \end{cases}$$

Torque

The torque τ_{R1} exerted on the rotor by the stator magnetic field τ_{R1} is

$$\tau_{R1} = -\tau_{S1} = \begin{cases} -2\ell_1 r_R N_S B_{R0} \dfrac{6\theta_R}{\pi} i_{S1} & \text{for} \quad -\pi/6 \le \theta_R \le \pi/6 \\ -2\ell_1 r_R N_S B_{R0} i_{S1} & \text{for} \quad \pi/6 \le \theta_R \le 5\pi/6 \\ +2\ell_1 r_R N_S B_{R0} \dfrac{6(\theta_R - \pi)}{\pi} i_{S1} & \text{for} \quad 5\pi/6 \le \theta_R \le 7\pi/6 \\ +2\ell_1 r_R N_S B_{R0} i_{S1} & \text{for} \quad 7\pi/6 \le \theta_R \le 11\pi/6. \end{cases}$$

With $\tau_p = 2\ell_1 r_R N_S B_{R0}$ this may be rewritten as

$$\tau_{R1}(\theta_R, i_{S1}) = \begin{cases} -\tau_p \dfrac{6\theta_R}{\pi} i_{S1} & \text{for} \quad -\pi/6 \le \theta_R \le \pi/6 \\ -\tau_p i_{S1} & \text{for} \quad \pi/6 \le \theta_R \le 5\pi/6 \\ +\tau_p \dfrac{6(\theta_R - \pi)}{\pi} i_{S1} & \text{for} \quad 5\pi/6 \le \theta_R \le 7\pi/6 \\ +\tau_p i_{S1} & \text{for} \quad 7\pi/6 \le \theta_R \le 11\pi/6. \end{cases}$$

Torque

Recall

$$e_{S1} \triangleq -\frac{d}{dt}M_{SR}\lambda_R(\theta_R) = \begin{cases} +M_{SR}\dfrac{6\theta_R}{\pi}\omega_R, & -\pi/6 \le \theta_R \le \pi/6 \\ +M_{SR}\omega_R, & \pi/6 \le \theta_R \le 5\pi/6 \\ -M_{SR}\dfrac{6(\theta_R-\pi)}{\pi}\omega_R, & 5\pi/6 \le \theta_R \le 7\pi/6 \\ -M_{SR}\omega_R, & 7\pi/6 \le \theta_R \le 11\pi/6 \end{cases}$$

$$e(\theta_R) \triangleq \frac{e_{S1}}{M_{SR}\omega_R} = -\frac{\partial\lambda_R(\theta_R)}{\partial\theta_R}$$

so we may write

$$\begin{aligned} \tau_{R1}(\theta_R, i_{S1}) &= -\tau_p e(\theta_R) i_{S1} \\ \tau_{R2}(\theta_R, i_{S2}) &= -\tau_p e(\theta_R - 2\pi/3) i_{S2} \\ \tau_{R3}(\theta_R, i_{S3}) &= -\tau_p e(\theta_R - 4\pi/3) i_{S3}. \end{aligned}$$

Mathematical Model

Stator flux linkages

$$\begin{aligned}\lambda_1(i_{S1}, i_{S2}, i_{S3}) &= +L_S i_{S1} - M i_{S2} - M i_{S3} + e_p \lambda_R(\theta_R)\\ \lambda_2(i_{S1}, i_{S2}, i_{S3}) &= -M i_{S1} + L_S i_{S2} - M i_{S3} + e_p \lambda_R(\theta_R - 2\pi/3)\\ \lambda_3(i_{S1}, i_{S2}, i_{S3}) &= -M i_{S1} - M i_{S2} + L_S i_{S3} + e_p \lambda_R(\theta_R - 4\pi/3)\end{aligned}$$

Phase torques

$$\begin{aligned}\tau_{R1}(\theta_R, i_{S1}) &= -\tau_p e(\theta_R)\, i_{S1}\\ \tau_{R2}(\theta_R, i_{S2}) &= -\tau_p e(\theta_R - 2\pi/3) i_{S2}\\ \tau_{R3}(\theta_R, i_{S3}) &= -\tau_p e(\theta_R - 4\pi/3) i_{S3}\end{aligned}$$

Phase voltages u_{S1}, u_{S2}, u_{S3} and stator phase resistance R_S.

By Faraday's law and the magnetic force law:

$$\begin{aligned}\frac{d}{dt}\lambda_1(i_{S1}, i_{S2}, i_{S3}) &= -R_S i_{S1} + u_{S1} - e_N\\ \frac{d}{dt}\lambda_2(i_{S1}, i_{S2}, i_{S3}) &= -R_S i_{S2} + u_{S2} - e_N\\ \frac{d}{dt}\lambda_3(i_{S1}, i_{S2}, i_{S3}) &= -R_S i_{S3} + u_{S3} - e_N\\ J\frac{d\omega}{dt} &= \tau_{R1}(\theta_R, i_{S1}) + \tau_{R2}(\theta_R, i_{S2}) + \tau_{R3}(\theta_R, i_{S3}) - \tau_L\\ \frac{d\theta_R}{dt} &= \omega_R\end{aligned}$$

Mathematical Model

$$
\begin{aligned}
\frac{d}{dt}\Big(L_S i_{S1} - M i_{S2} - M i_{S3} + e_p \lambda_R(\theta_R)\Big) &= -R_S i_{S1} + u_{S1} - e_N \\
\frac{d}{dt}\Big(-M i_{S1} + L_S i_{S2} - M i_{S3} + e_p \lambda_R(\theta_R - 2\pi/3)\Big) &= -R_S i_{S2} + u_{S2} - e_N \\
\frac{d}{dt}\Big(-M i_{S1} - M i_{S2} + L_S i_{S3} + e_p \lambda_R(\theta_R - 4\pi/3)\Big) &= -R_S i_{S3} + u_{S3} - e_N \\
J\frac{d\omega_R}{dt} &= \tau - \tau_L \\
\frac{d\theta_R}{dt} &= \omega_R
\end{aligned}
$$

with

$$
\tau \triangleq -\tau_p e(\theta_R) i_{S1} - \tau_p e(\theta_R - 2\pi/3) i_{S2} - \tau_p e(\theta_R - 4\pi/3) i_{S3}.
$$

Equivalently

$$
\begin{bmatrix} u_{S1} - e_N \\ u_{S2} - e_N \\ u_{S3} - e_N \end{bmatrix} = \begin{bmatrix} L_S & -M & -M \\ -M & L_S & -M \\ -M & -M & L_S \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} + R_S \begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} - e_p \begin{bmatrix} e(\theta_R) \\ e(\theta_R - 2\pi/3) \\ e(\theta_R - 4\pi/3) \end{bmatrix} \omega_R
$$

$$
J\frac{d\omega_R}{dt} = -\tau_p e(\theta_R) i_{S1} - \tau_p e(\theta_R - 2\pi/3) i_{S2} - \tau_p e(\theta_R - 4\pi/3) i_{S3} - \tau_L
$$

$$
\frac{d\theta_R}{dt} = \omega_R.
$$

Mathematical Model

As the stator currents are *balanced* the model may be rewritten as

$$\begin{bmatrix} u_{S1} \\ u_{S2} \\ u_{S3} \end{bmatrix} = \begin{bmatrix} L_S + M & 0 & 0 \\ 0 & L_S + M & 0 \\ 0 & 0 & L_S + M \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} + R_S \begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} - e_p \begin{bmatrix} e(\theta_R) \\ e(\theta_R - 2\pi/3) \\ e(\theta_R - 4\pi/3) \end{bmatrix} \omega_R$$

$$J\frac{d\omega_R}{dt} = -\tau_p e(\theta_R) i_{S1} - \tau_p e(\theta_R - 2\pi/3) i_{S2} - \tau_p e(\theta_R - 4\pi/3) i_{S3} - \tau_L$$

$$\frac{d\theta_R}{dt} = \omega_R.$$

In state-space form this becomes

$$\begin{aligned}
\frac{di_{S1}}{dt} &= \frac{e_p}{L_S + M}\omega_R e(\theta_R) - \frac{R_S}{L_S + M} i_{S1} + \frac{1}{L_S + M}(u_{S1} - e_N) \\
\frac{di_{S2}}{dt} &= \frac{e_p}{L_S + M}\omega_R e(\theta_R - 2\pi/3) - \frac{R_S}{L_S + M} i_{S2} + \frac{1}{L_S + M}(u_{S2} - e_N) \\
\frac{di_{S3}}{dt} &= \frac{e_p}{L_S + M}\omega_R e(\theta_R - 4\pi/3) - \frac{R_S}{L_S + M} i_{S3} + \frac{1}{L_S + M}(u_{S3} - e_N) \\
\frac{d\omega_R}{dt} &= -(\tau_p/J) e(\theta_R) i_{S1} - (\tau_p/J) e(\theta_R - 2\pi/3) i_{S2} - (\tau_p/J) e(\theta_R - 4\pi/3) i_{S3} - \tau_L/J \\
\frac{d\theta_R}{dt} &= \omega_R
\end{aligned}$$

where $e_p = \tau_p$.

Operation and Control

The back-emf voltages are

$$\begin{aligned} e_{S1} &= e_p e(\theta_R)\omega_R \\ e_{S2} &= e_p e(\theta_R - 2\pi/3)\omega_R \\ e_{S3} &= e_p e(\theta_R - 4\pi/3)\omega_R. \end{aligned}$$

The power in each phase absorbed in each phase by the back emfs are

$$p_{S1} \triangleq e_{S1} i_{S1}, \quad p_{S2} \triangleq e_{S2} i_{S2}, \quad p_{S3} \triangleq e_{S3} i_{S3}.$$

The total power absorbed by the back emfs

$$p \triangleq e_{S1} i_{S1} + e_{S2} i_{S2} + e_{S3} i_{S3}.$$

Set the current reference i_{S1r} for the current in phase 1 as

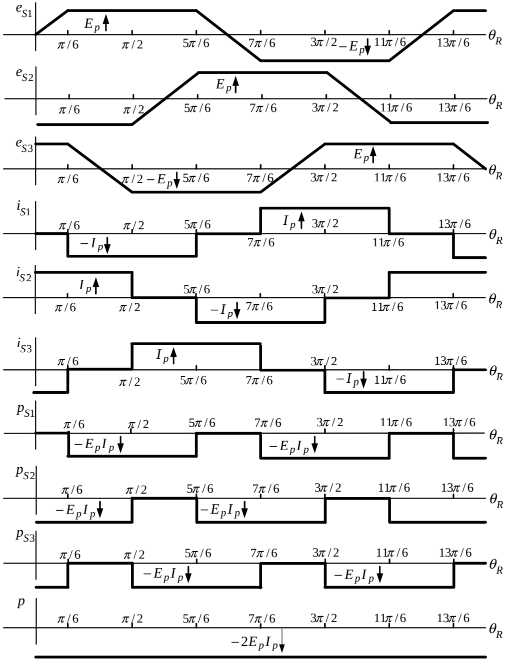
$$i_{S1r} = \begin{cases} -I_p & \text{if} \quad e_{S1} = +E_p = +e_p\omega_R \\ +I_p & \text{if} \quad e_{S1} = -E_p = -e_p\omega_R \\ 0 & \text{otherwise} \end{cases}$$

Equivalently we can write

$$i_{S1r}(\theta_R) = I_p i_S(\theta_R)$$

where

$$i_S(\theta_R) \triangleq \begin{cases} 0 & \text{for} \quad -\pi/6 \le \theta_R \le \pi/6 \\ -1 & \text{for} \quad \pi/6 \le \theta_R \le 5\pi/6 \\ 0 & \text{for} \quad 5\pi/6 \le \theta_R \le 7\pi/6 \\ 1 & \text{for} \quad 7\pi/6 \le \theta_R \le 11\pi/6. \end{cases}$$


e_{S1}
E_p
$-E_p$
e_{S2}
E_p
e_{S3}
E_p
$-E_p$
i_{S1}
I_p
$-I_p$
i_{S2}
I_p
$-I_p$
i_{S3}
I_p
$-I_p$
p_{S1}
$-E_pI_p$
p_{S2}
$-E_pI_p$
p_{S3}
$-E_pI_p$
p
$-2E_pI_p$
$\pi/6$ $\pi/2$ $5\pi/6$ $7\pi/6$ $3\pi/2$ $11\pi/6$ $13\pi/6$
θ_R

Operation and Control

- Though $e_N \neq 0$ the usual case is to ignore this and take e_N to be zero.
- Use a PI current controller

$$u_{S1} = K_P(i_{S1r} - i_{S1}) + K_I \int_0^t (i_{S1r}(\tau) - i_{S1}(\tau))d\tau$$

  to force $i_{S1} \to i_{S1r}$.

- Similarly, choose $i_{S2r} = I_p i_S(\theta_R - 2\pi/3)$, $i_{S3r} = I_p i_S(\theta_R - 4\pi/3)$ and PI controllers to force $i_{S2} \to i_{S2r}$, $i_{S3} \to i_{S3r}$.
- Current is commutated every $\pi/3$ radians or $60°$.
- Commutation is done at $\theta_R = 30°, 90°, 150°, 210°, 270°$, and $330°$ to implement $i_{S1r}, i_{S2r}, i_{S3r}$.
- Hall effect sensors are used to detect these discrete rotor positions for current commutation.
- Power absorbed by the back emf is

$$\begin{aligned} e_{S1}i_{S1} + e_{S2}i_{S2} + e_{S3}i_{S3} &= e_p e(\theta_R)\omega_R i_{S1r} + e_p e(\theta_R - 2\pi/3)\omega_R i_{S2r} + e_p e(\theta_R - 4\pi/3)\omega_R i_{S3r} \\ &= -2e_p\omega_R I_p \end{aligned}$$

Operation and Control

Recall that the torque is given by

$$\tau \triangleq -\tau_p e\left(\theta_R\right) i_{S1} - \tau_p e\left(\theta_R - 2\pi/3\right) i_{S2} - \tau_p e\left(\theta_R - 4\pi/3\right) i_{S3}.$$

With the currents chosen as above, the mechanical power is

$$\begin{aligned}
\tau\omega_R &= -\tau_p e\left(\theta_R\right)\omega_R i_{S1} - \tau_p e(\theta_R - 2\pi/3)\omega_R i_{S2} - \tau_p e(\theta_R - 4\pi/3)\omega_R i_{S3} \\
&= -e_p e\left(\theta_R\right)\omega_R i_{S1r} - e_p e(\theta_R - 2\pi/3)\omega_R i_{S2r} - e_p e(\theta_R - 4\pi/3)\omega_R i_{S3r} \\
&= 2 e_p \omega_R I_p \\
&= 2\tau_p I_p \omega_R .
\end{aligned}$$

That is, the torque is simply given by

$$\tau = 2\tau_p I_p .$$

In summary, choose the current references as

$$\begin{aligned}
i_{S1r} &= I_p i_S(\theta_R) \\
i_{S2r} &= I_p i_S(\theta_R - 2\pi/3) \\
i_{S3r} &= I_p i_S(\theta_R - 4\pi/3)
\end{aligned}$$

and PI controllers as

$$\begin{aligned}
u_{S1} &= K_P(i_{S1r} - i_{S1}) + K_I \int_0^t (i_{S1r}(\tau) - i_{S1}(\tau))d\tau \\
u_{S2} &= K_P(i_{S2r} - i_{S2}) + K_I \int_0^t (i_{S2r}(\tau) - i_{S2}(\tau))d\tau \\
u_{S1} &= K_P(i_{S3r} - i_{S3}) + K_I \int_0^t (i_{S3r}(\tau) - i_{S3}(\tau))d\tau .
\end{aligned}$$

Operation and Control

With α_r denoting the reference angular acceleration set

$$\begin{aligned} I_p &= \frac{J}{2\tau_p}\alpha_r \\ \alpha_r &= K_I\int_0^t \Big(\theta_{ref}(\tau)-\theta(\tau)\Big)d\tau + K_P\Big(\theta_{ref}(t)-\theta(t)\Big) + \ K_D\Big(\omega_{ref}(t)-\omega(t)\Big) + \alpha_{ref}. \end{aligned}$$

Define $e_0(t) \triangleq \int_0^t(\theta_{ref}(\tau)-\theta(\tau))d\tau$, $e_1(t) \triangleq \theta_{ref}(t)-\theta(t)$, and $e_2(t) \triangleq \omega_{ref}(t)-\omega(t)$ so that

$$\frac{d}{dt}\begin{bmatrix} e_0 \\ e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_I & -K_P & -K_D \end{bmatrix}\begin{bmatrix} e_0 \\ e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/J \end{bmatrix}\tau_L.$$

With $r_1 > 0, r_2 > 0$, and $r_3 > 0$ set the gains as

$$\begin{aligned} K_D &= r_1+r_2+r_3 \\ K_P &= r_1r_2+r_1r_3+r_2r_3 \\ K_I &= r_1r_2r_3 \end{aligned}$$

so that the closed-loop characteristic polynomial is

$$\begin{aligned} a(s) = \det\left(sI - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_I & -K_P & -K_D \end{bmatrix}\right) &= s^3+K_Is^2+K_Ps+K_I \\ &= s^3+(r_1+r_2+r_3)s^2+(r_1r_2+r_1r_3+r_2r_3)s+r_1r_2r_3 \\ &= (s+r_1)(s+r_2)(s+r_3). \end{aligned}$$

Operation and Control

Balanced Currents

- The currents

$$i_{S1r} = I_p i_S(\theta_R), \quad i_{S2r} = I_p i_S(\theta_R - 2\pi/3), \quad i_{S3r} = I_p i_S(\theta_R - 4\pi/3)$$

are balanced, that is,

$$I_p i_S(\theta_R) + I_p i_S(\theta_R - 2\pi/3) + I_p i_S(\theta_R - 4\pi/3) \equiv 0.$$

- The back-emf voltages $e_{S1}(\theta_R)$, $e_{S2}(\theta_R)$, and $e_{S3}(\theta_R)$ are **not** balanced as

$$\begin{aligned} e_{S1}(\theta_R) + e_{S2}(\theta_R) + e_{S3}(\theta_R) &= -e_p\omega_R\Big(e(\theta_R) + (e\theta_R - 2\pi/3) + e(\theta_R - 4\pi/3)\Big) \\ &\neq 0. \end{aligned}$$

- In contrast, three-phase PM synchronous machines with sinusoidally distributed windings have balanced back-emf voltages.

The Terminology "Brushless DC Motor"

The stator current references are

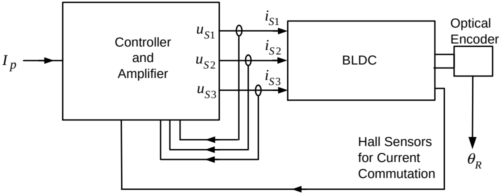
$$
\begin{aligned}
i_{S1r}(\theta_R) &= I_p i_S(\theta_R) \\
i_{S2r}(\theta_R) &= I_p i_S(\theta_R - 2\pi/3) \\
i_{S3r}(\theta_R) &= I_p i_S(\theta_R - 4\pi/3)
\end{aligned}
$$

$$
i_S(\theta_R) \triangleq \begin{cases} 0 & \text{for} \quad -\pi/6 \le \theta_R \le \pi/6 \\ -1 & \text{for} \quad \pi/6 \le \theta_R \le 5\pi/6 \\ 0 & \text{for} \quad 5\pi/6 \le \theta_R \le 7\pi/6 \\ 1 & \text{for} \quad 7\pi/6 \le \theta_R \le 11\pi/6. \end{cases}
$$

The input to the controller is simply I_p so that

$$
\begin{aligned}
J d\omega/dt &= 2\tau_p I_p - \tau_L \\
d\theta/dt &= \omega.
\end{aligned}
$$

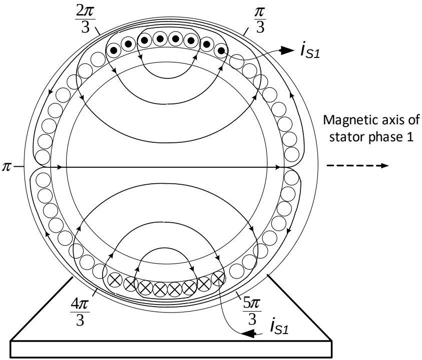
Same form as a current command DC motor with torque constant $K_T = 2\tau_p$.

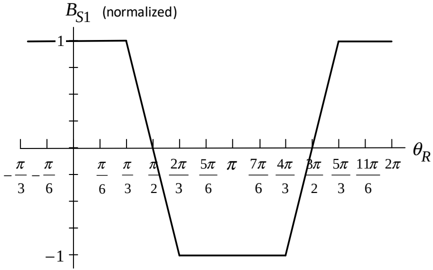


This feedback system is referred to as a "brushless DC motor".

Stator and Rotor Magnetic Axes during Operation

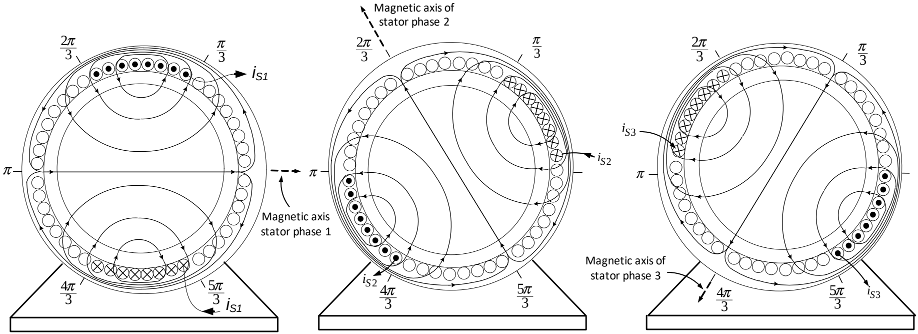
- Magnetic axis of stator phase 1 with $i_{S1} > 0$.





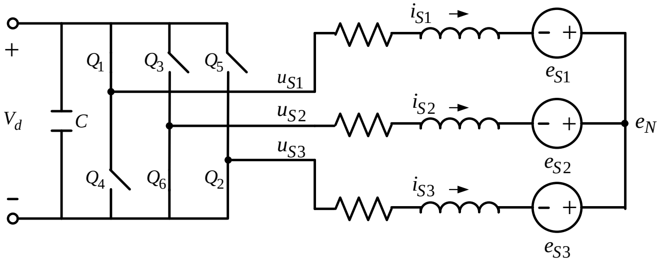
Stator and Rotor Magnetic Axes during Operation

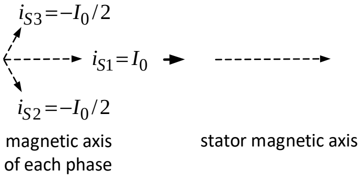
- Magnetic Axes of the three stator phases.



- With $i_{S1} = -I_p, i_{S2} = I_p, i_{S3} = 0$ the stator magnetic axis is at $5\pi/6$ or $150°$.
- With $i_{S1} = -I_p, i_{S2} = 0, i_{S3} = I_p$ the stator magnetic axis is at $7\pi/6$ or $210°$.

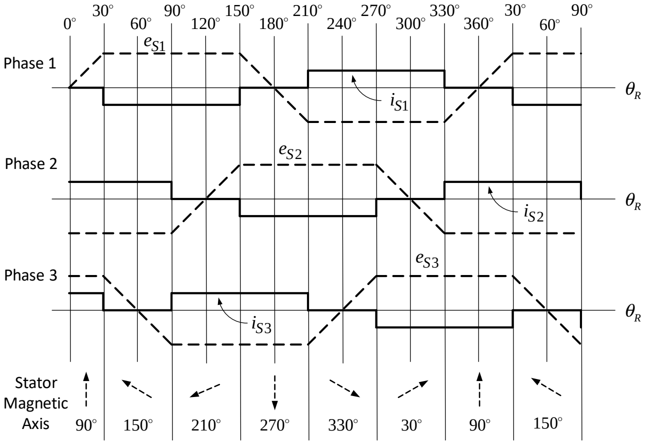
- Set $i_{S1} = I_0, i_{S2} = -I_0/2, i_{S3} = -I_0/2$ to line up the rotor to $\theta_R = 0$.





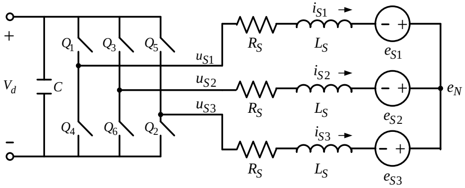
Stator and Rotor Magnetic Axes during Operation

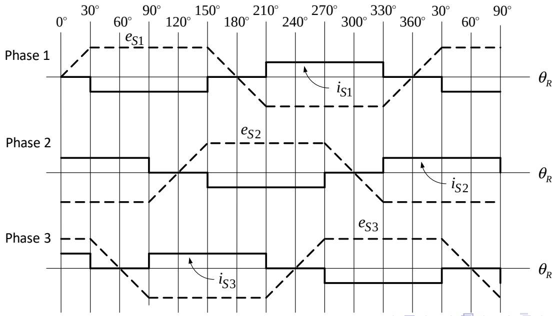
- During operation one current is at I_p, another at $-I_p$, and the third at 0.
- The magnetic axis of rotor is θ_R.
- The magnetic axis of the stator is 60° to 120° ahead of the rotor.



Sensorless Speed Control - Hall Sensors

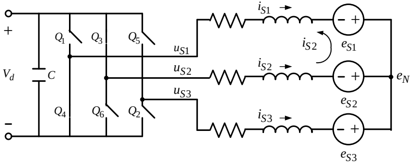
- If $e_{S1} > 0$ it "wants" to force current in the positive direction of i_{S1}.
- Commutate current every 60°.

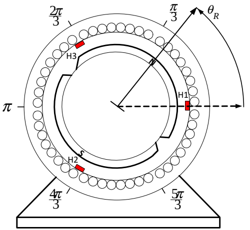


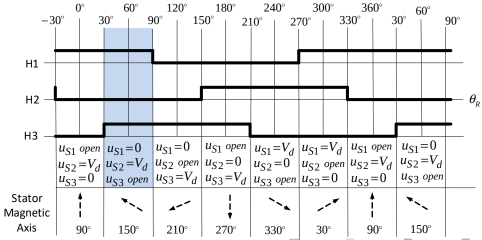


Sensorless Speed Control - Hall Sensors

- To initialize set $u_{S1} = V_d, u_{S2} = 0, u_{S3} = 0$ so $i_{S1} = V_d/R_S, i_{S2} = i_{S3} = -(1/2)V_d/R_S$.
- $\vec{\mathbf{B}}_S$ is at 0° so rotor moves to $\theta_R = 0$.
- Then set $u_{S1} = V_d, u_{S2} = 0, u_{S3}$ open so $\vec{\mathbf{B}}_S$ is at 60° and rotor turns clockwise.
- Use Hall sensors H_1, H_2, H_3 to detect rotor position every 60°.
- Close switches $Q_1 \& Q_6$ if $30^\circ \le \theta_R \le 60^\circ$ so $i_{S2} = -i_{S1}$ and $i_{S3} = 0$.
- $\omega_R = \pi/\Delta t$ where Δt is the time for a Hall sensor to go high and then low.

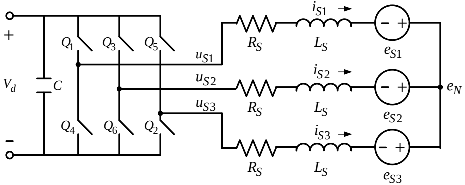


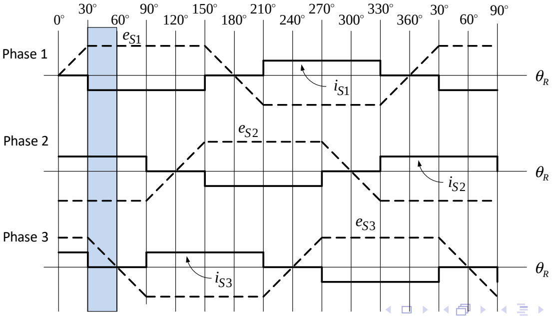




Sensorless Speed Control - Back EMF

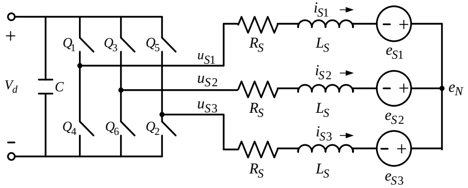
- Close switches $Q_1 \& Q_6$ if $30^\circ \leq \theta_R \leq 60^\circ$ so $i_{S2} = -i_{S1}$ with $i_{S3} = 0$.
- $e_{S3} = u_{S3} - e_N = 0$ at $\theta_R = 60^\circ$. Detect this zero crossing of e_{S3}.
- Commutate current from phase 2 to phase 3 at $\theta_R = 90^\circ$.





Sensorless Speed Control - Back EMF

- Close switches $Q_1 \& Q_6$ if $30^\circ \leq \theta_R \leq 60^\circ$ and $i_{S2} = -i_{S1}$ with $i_{S3} = 0$.
- $e_{S3} = u_{S3} - e_N = 0$ at $\theta_R = 60^\circ$. Detect this zero crossing of e_{S3}.
- Commutate current from phase 2 to phase 3 at $\theta_R = 90^\circ$.



$$\begin{bmatrix} u_{S1} - e_N \\ u_{S2} - e_N \\ u_{S3} - e_N \end{bmatrix} = \begin{bmatrix} L_S + M & 0 & 0 \\ 0 & L_S + M & 0 \\ 0 & 0 & L_S + M \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} + R_S \begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} - \begin{bmatrix} e_{S1} \\ e_{S2} \\ e_{S3} \end{bmatrix} \omega_R$$

Add the first two eqns to get

$$u_{S1} - e_N + u_{S2} - e_N = e_{S1} + e_{S2} = 0 \quad \text{for} \quad \pi/6 \leq \theta_R \leq \pi/2$$

or

$$e_N = (u_{S1} + u_{S2})/2 \quad \text{for} \quad \pi/6 \leq \theta_R \leq \pi/2.$$

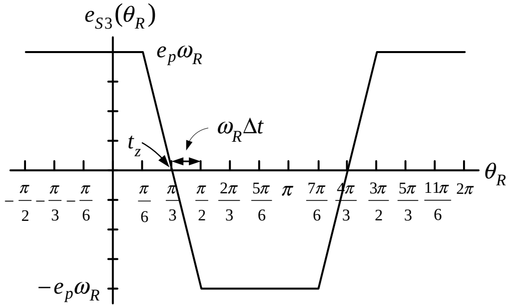
Then

$$e_{S3} = u_{S3} - e_N = u_{S3} - (u_{S1} + u_{S2})/2.$$

Sensorless Speed Control - Back EMF

- $e_{S3} = u_{S3} - e_N = u_{S3} - (u_{S1} + u_{S2})/2$ for $\pi/6 \le \theta_R \le \pi/2$.
- Δt denotes the time from the zero crossing t_z of e_{S3} to the time $\Delta\theta_R = \pi/6$ radians.

$$\int_{t_z}^{t_z+\Delta t} |e_{S3}(t')|dt' = \int_{t_z}^{t_z+\Delta t} \frac{e_p\omega_R}{\Delta t}(t'-t_z)dt' = \frac{e_p\omega_R}{\Delta t}\frac{(t'-t_z)^2}{2}\bigg|_{t_z}^{t_z+\Delta t} = \frac{1}{2}e_p\omega_R\Delta t = \frac{1}{2}e_p\frac{\pi}{6}.$$



- When $e_{S3} = 0$ start computing $U(t) \triangleq \int_{t_z}^{t} |e_{S3}(t')|dt'$.
- When $U(t) = U_{\text{threshold}} \triangleq \frac{1}{2}e_p\frac{\pi}{6}$ commutate the current.
- Must start open loop.