Modeling and High-Performance Control of Electric Machines

Chapter 8 Induction Motor Control

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Induction Motor Control

- **Dynamic Equations of the Induction Motor**
- **Field-Oriented and Input-Output Linearization Control of an Induction Motor**
- **Observers**
- **Optimal Field Weakening** (no slides)
- **Identification of T_R and R_S**

Dynamic Equations of the Induction Motor

Two phase equivalent model

$$
\begin{aligned}
u_{Sa} &= R_S i_{Sa} + \frac{d}{dt}\underbrace{\Big(L_S i_{Sa} + M\big(i_{Ra}\cos(n_p\theta) - i_{Rb}\sin(n_p\theta)\big)\Big)}_{\lambda_{Sa}} \\
u_{Sb} &= R_S i_{Sb} + \frac{d}{dt}\underbrace{\Big(L_S i_{Sb} + M\big(i_{Ra}\sin(n_p\theta) + i_{Rb}\cos(n_p\theta)\big)\Big)}_{\lambda_{Sb}} \\
0 &= R_R i_{Ra} + \frac{d}{dt}\underbrace{\Big(L_R i_{Ra} + M\big(+i_{Sa}\cos(n_p\theta) + i_{Sb}\sin(n_p\theta)\big)\Big)}_{\lambda_{Ra}} \\
0 &= R_R i_{Rb} + \frac{d}{dt}\underbrace{\Big(L_R i_{Rb} + M\big(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)\big)\Big)}_{\lambda_{Rb}} \\
J\frac{d\omega}{dt} &= n_p M\Big(i_{Sb}\big(i_{Ra}\cos(n_p\theta) - i_{Rb}\sin(n_p\theta)\big) - i_{Sa}\big(i_{Ra}\sin(n_p\theta) + i_{Rb}\cos(n_p\theta)\big)\Big) - \tau_L \\
\frac{d\theta}{dt} &= \omega
\end{aligned}
$$

where $L_S \triangleq \dfrac{\mu_0\pi\ell_1\ell_2 N_S^2}{8g}$, $L_R \triangleq \dfrac{\mu_0\pi\ell_1\ell_2 N_R^2}{8g}$, $M \triangleq \dfrac{\kappa\mu_0\pi\ell_1\ell_2 N_S N_R}{8g}$.

Dynamic Equations of the Induction Motor

The Control Problem

- Choose u_{Sa}, u_{Sb} force ω/θ to track a given reference trajectory.
- Measurements of i_{Sa}, i_{Sb}, and θ are usually available for feedback control.
- Measurements of i_{Ra}, i_{Rb} are typically **not** available.
- Our nonlinear differential equation model is **complicated.**
- **Transform** the model to a coordinate system where a control strategy becomes clear.
- We do two transformations to get our desired coordinate system.

Dynamic Equations of the Induction Motor

First transformation - Eliminate sines and cosines
Define an equivalent set of rotor flux linkages as

$$\begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix} = \begin{bmatrix} \cos(n_p\theta) & -\sin(n_p\theta) \\ \sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} \lambda_{Ra} \\ \lambda_{Rb} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(n_p\theta) & -\sin(n_p\theta) \\ \sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} M\cos(n_p\theta) & M\sin(n_p\theta) & L_R & 0 \\ -M\sin(n_p\theta) & M\cos(n_p\theta) & 0 & L_R \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Ra} \\ i_{Rb} \end{bmatrix}$$

$$= \begin{bmatrix} L_R(i_{Ra}\cos(n_p\theta) - i_{Rb}\sin(n_p\theta)) + Mi_{Sa} \\ L_R(i_{Ra}\sin(n_p\theta) + i_{Rb}\cos(n_p\theta)) + Mi_{Sb} \end{bmatrix}$$

We then have

$$\begin{bmatrix} \dfrac{\psi_{Ra} - Mi_{Sa}}{L_R} \\ \\ \dfrac{\psi_{Rb} - Mi_{Sb}}{L_R} \end{bmatrix} = \begin{bmatrix} i_{Ra}\cos(n_p\theta) - i_{Rb}\sin(n_p\theta) \\ \\ i_{Ra}\sin(n_p\theta) + i_{Rb}\cos(n_p\theta) \end{bmatrix}.$$

Use the left side to **eliminate** the cosine/sine expressions on the right side.

Dynamic Equations of the Induction Motor

First transformation (continued)

Using

$$\begin{bmatrix} (\psi_{Ra} - Mi_{Sa})/L_R \\ (\psi_{Rb} - Mi_{Sb})/L_R \end{bmatrix} = \begin{bmatrix} i_{Ra}\cos(n_p\theta) - i_{Rb}\sin(n_p\theta) \\ i_{Ra}\sin(n_p\theta) + i_{Rb}\cos(n_p\theta) \end{bmatrix},$$

the stator equations

$$\begin{aligned} u_{Sa} &= R_S i_{Sa} + \frac{d}{dt}L_S i_{Sa} + \frac{d}{dt}M\Big(i_{Ra}\cos(n_p\theta) - i_{Rb}\sin(n_p\theta)\Big) \\ u_{Sb} &= R_S i_{Sb} + \frac{d}{dt}L_S i_{Sb} + \frac{d}{dt}M\Big(i_{Ra}\sin(n_p\theta) + i_{Rb}\cos(n_p\theta)\Big) \end{aligned}$$

become

$$\begin{aligned} u_{Sa} &= R_S i_{Sa} + L_S\frac{d}{dt}i_{Sa} + M\frac{d}{dt}(\psi_{Ra} - Mi_{Sa})/L_R \\ u_{Sb} &= R_S i_{Sb} + L_S\frac{d}{dt}i_{Sb} + M\frac{d}{dt}(\psi_{Rb} - Mi_{Sb})/L_R \end{aligned}$$

or

$$\begin{aligned} u_{Sa} &= R_S i_{Sa} + L_S\left(1 - \frac{M^2}{L_R L_S}\right)\frac{d}{dt}i_{Sa} + \frac{M}{L_R}\frac{d}{dt}\psi_{Ra} \\ u_{Sb} &= R_S i_{Sb} + L_S\left(1 - \frac{M^2}{L_R L_S}\right)\frac{d}{dt}i_{Sb} + \frac{M}{L_R}\frac{d}{dt}\psi_{Rb}. \end{aligned}$$

Dynamic Equations of the Induction Motor

First transformation (continued)
From

$$\begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix} = \begin{bmatrix} \cos(n_p\theta) & -\sin(n_p\theta) \\ \sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} \lambda_{Ra} \\ \lambda_{Rb} \end{bmatrix}$$

we have

$$\begin{bmatrix} \lambda_{Ra} \\ \lambda_{Rb} \end{bmatrix} = \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix}.$$

Substitute into the rotor equations

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_R i_{Ra} \\ R_R i_{Rb} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_{Ra} \\ \lambda_{Rb} \end{bmatrix}$$

to obtain

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_R i_{Ra} \\ R_R i_{Rb} \end{bmatrix} + \frac{d}{dt} \left(\begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix} \right).$$

Expanding

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_R i_{Ra} \\ R_R i_{Rb} \end{bmatrix} &+ \frac{d}{dt} \left(\begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \right) \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix} \\ &+ \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix} \end{aligned}$$

Dynamic Equations of the Induction Motor

First transformation (continued)

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_R i_{Ra} \\ R_R i_{Rb} \end{bmatrix} + n_p\omega \begin{bmatrix} -\sin(n_p\theta) & \cos(n_p\theta) \\ -\cos(n_p\theta) & -\sin(n_p\theta) \end{bmatrix} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix} + \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix}$$

Multiply on the left by $\begin{bmatrix} \cos(n_p\theta) & -\sin(n_p\theta) \\ \sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix}$ to obtain

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \cos(n_p\theta) & -\sin(n_p\theta) \\ \sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} R_R i_{Ra} \\ R_R i_{Rb} \end{bmatrix} \\ &+ \begin{bmatrix} \cos(n_p\theta) & -\sin(n_p\theta) \\ \sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} n_p\omega \begin{bmatrix} -\sin(n_p\theta) & \cos(n_p\theta) \\ -\cos(n_p\theta) & -\sin(n_p\theta) \end{bmatrix} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix} \\ &+ \frac{d}{dt} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix} \end{aligned}$$

or

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = R_R \begin{bmatrix} i_{Ra}\cos(n_p\theta) - i_{Rb}\sin(n_p\theta) \\ i_{Ra}\sin(n_p\theta) + i_{Rb}\cos(n_p\theta) \end{bmatrix} - n_p\omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix}$$

Dynamic Equations of the Induction Motor

First transformation (continued)
Again using

$$\begin{bmatrix} i_{Ra}\cos(n_p\theta) - i_{Rb}\sin(n_p\theta) \\ i_{Ra}\sin(n_p\theta) + i_{Rb}\cos(n_p\theta) \end{bmatrix} = \begin{bmatrix} (\psi_{Ra} - Mi_{Sa})/L_R \\ (\psi_{Rb} - Mi_{Sb})/L_R \end{bmatrix}$$

we obtain

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{R_R}{L_R}\begin{bmatrix} \psi_{Ra} - Mi_{Sa} \\ \psi_{Rb} - Mi_{Sb} \end{bmatrix} - n_p\omega\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix}.$$

Finally, the **torque equation** is transformed as

$$\begin{aligned} J\frac{d\omega}{dt} &= n_p M\Big(i_{Sb}\big(i_{Ra}\cos(n_p\theta) - i_{Rb}\sin(n_p\theta)\big) - i_{Sa}\big(i_{Ra}\sin(n_p\theta) + i_{Rb}\cos(n_p\theta)\big)\Big) - \tau_L \\ &= n_p M\left(i_{Sb}\frac{\psi_{Ra} - Mi_{Sa}}{L_R} - i_{Sa}\frac{\psi_{Rb} - Mi_{Sb}}{L_R}\right) - \tau_L \\ &= n_p\frac{M}{L_R}(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}) - \tau_L. \end{aligned}$$

Dynamic Equations of the Induction Motor

Collecting the above equations together:

$$
\begin{aligned}
\frac{d\theta}{dt} &= \omega \\
\frac{d\omega}{dt} &= \frac{n_p M}{J L_R}(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}) - \frac{\tau_L}{J} \\
\frac{d\psi_{Ra}}{dt} &= -\frac{R_R}{L_R}\psi_{Ra} - n_p\omega\psi_{Rb} + \frac{M R_R}{L_R} i_{Sa} \\
\frac{d\psi_{Rb}}{dt} &= -\frac{R_R}{L_R}\psi_{Rb} + n_p\omega\psi_{Ra} + \frac{M R_R}{L_R} i_{Sb} \\
u_{Sa} &= R_S i_{Sa} + \sigma L_S \frac{di_{Sa}}{dt} + \frac{M}{L_R}\frac{d\psi_{Ra}}{dt} \\
u_{Sb} &= R_S i_{Sb} + \sigma L_S \frac{di_{Sb}}{dt} + \frac{M}{L_R}\frac{d\psi_{Rb}}{dt}
\end{aligned}
$$

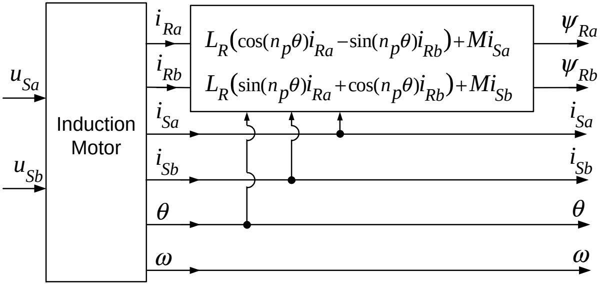
where $\sigma \triangleq 1 - \dfrac{M^2}{L_R L_S}$ (leakage parameter).

- Eliminate $\dfrac{d\psi_{Ra}}{dt}, \dfrac{d\psi_{Rb}}{dt}$ from the 5^{th} and 6^{th} eqns using the 3^{rd} and 4^{th} eqns.

Statespace Model of the Induction Motor

$$
\begin{aligned}
d\theta/dt &= \omega \\
d\omega/dt &= \mu(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}) - \tau_L/J \\
d\psi_{Ra}/dt &= -\eta\psi_{Ra} - n_p\omega\psi_{Rb} + \eta M i_{Sa} \\
d\psi_{Rb}/dt &= -\eta\psi_{Rb} + n_p\omega\psi_{Ra} + \eta M i_{Sb} \\
di_{Sa}/dt &= \eta\beta\psi_{Ra} + \beta n_p\omega\psi_{Rb} - \gamma i_{Sa} + u_{Sa}/\sigma L_S \\
di_{Sb}/dt &= \eta\beta\psi_{Rb} - \beta n_p\omega\psi_{Ra} - \gamma i_{Sb} + u_{Sb}/\sigma L_S
\end{aligned}
$$

with $\eta \triangleq \dfrac{R_R}{L_R}$, $\beta \triangleq \dfrac{M}{\sigma L_R L_S}$, $\mu \triangleq \dfrac{n_p M}{J L_R}$, $\gamma \triangleq \dfrac{M^2 R_R}{\sigma L_R^2 L_S} + \dfrac{R_S}{\sigma L_S}$.



Field-Oriented Control of an Induction Motor

$$
\begin{aligned}
\frac{d\theta}{dt} &= \omega \\
\frac{d\omega}{dt} &= \mu(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}) - (f/J)\omega - \tau_L/J \\
\frac{d\psi_{Ra}}{dt} &= -\eta\psi_{Ra} - n_p\omega\psi_{Rb} + \eta M i_{Sa} \\
\frac{d\psi_{Rb}}{dt} &= -\eta\psi_{Rb} + n_p\omega\psi_{Ra} + \eta M i_{Sb} \\
\frac{di_{Sa}}{dt} &= \eta\beta\psi_{Ra} + \beta n_p\omega\psi_{Rb} - \gamma i_{Sa} + u_{Sa}/\sigma L_S \\
\frac{di_{Sb}}{dt} &= \eta\beta\psi_{Rb} - \beta n_p\omega\psi_{Ra} - \gamma i_{Sb} + u_{Sb}/\sigma L_S
\end{aligned}
$$

- This model is simpler in that there are **no** cosine or sine functions.
- Easier to **numerically integrate**, i.e., a larger step size obtains the same accuracy.
- It is still **nonlinear.**
- **Not clear** how to choose the inputs u_{Sa}, u_{Sb} to track a position/speed trajectory.

Rotor-Flux Field-Oriented Coordinate System

$$
\begin{aligned}
\theta &= \theta \\
\omega &= \omega \\
\psi_d &= \sqrt{\psi_{Ra}^2 + \psi_{Rb}^2} \\
\rho &= \tan^{-1}(\psi_{Rb}/\psi_{Ra}) \\
i_d &= \quad \cos(\rho) i_{Sa} + \sin(\rho) i_{Sb} \\
i_q &= -\sin(\rho) i_{Sa} + \cos(\rho) i_{Sb}
\end{aligned}
$$

and

$$
\begin{aligned}
u_d &= \quad \cos(\rho) u_{Sa} + \sin(\rho) u_{Sb} \\
u_q &= -\sin(\rho) u_{Sa} + \cos(\rho) u_{Sb}
\end{aligned}
$$

- We compute the differential equation model in these new coordinates.
- This coordinate system is the key to high-performance control of the IM.

Field-Oriented Control of an Induction Motor

Equation for ω

As $\cos(\rho) = \psi_{Ra}/\psi_d, \sin(\rho) = \psi_{Rb}/\psi_d$ we have

$$\begin{aligned}
\frac{d\omega}{dt} &= \mu(\psi_{Ra} i_{Sb} - \psi_{Rb} i_{Sa}) - (f/J)\omega - \tau_L/J \\
&= \mu\psi_d \left((\psi_{Ra}/\psi_d) i_{Sb} - (\psi_{Rb}/\psi_d) i_{Sa} \right) - (f/J)\omega - \tau_L/J \\
&= \mu\psi_d \left(-\sin(\rho) i_{Sa} + \cos(\rho) i_{Sb} \right) - (f/J)\omega - \tau_L/J \\
&= \mu\psi_d i_q - (f/J)\omega - \tau_L/J.
\end{aligned}$$

Equation for ψ_d

$$\begin{aligned}
\frac{d\psi_d}{dt} &= \frac{d}{dt}\sqrt{\psi_{Ra}^2 + \psi_{Rb}^2} = \frac{1/2}{\sqrt{\psi_{Ra}^2 + \psi_{Rb}^2}} \left(2\psi_{Ra} \frac{d}{dt}\psi_{Ra} + 2\psi_{Rb} \frac{d}{dt}\psi_{Rb} \right) \\
&= \cos(\rho)(-\eta\psi_{Ra} - n_p\omega\psi_{Rb} + \eta M i_{Sa}) + \sin(\rho)(-\eta\psi_{Rb} + n_p\omega\psi_{Ra} + \eta M i_{Sb}) \\
&= -\eta(\cos(\rho)\psi_{Ra} + \sin(\rho)\psi_{Rb}) + n_p\omega \underbrace{(-\cos(\rho)\psi_{Rb} + \sin(\rho)\psi_{Ra})}_{0} \\
&\quad + \eta M(\cos(\rho) i_{Sa} + \sin(\rho) i_{Sb}) \\
&= -\eta\psi_d + \eta M i_d.
\end{aligned}$$

Field-Oriented Control of an Induction Motor

Equation for ρ

$$
\begin{aligned}
\frac{d\rho}{dt} &= \frac{d}{dt}\tan^{-1}\left(\frac{\psi_{Rb}}{\psi_{Ra}}\right) \\
&= \frac{1}{1+(\psi_{Rb}/\psi_{Ra})^2}\frac{\dot{\psi}_{Rb}\psi_{Ra}-\psi_{Rb}\dot{\psi}_{Ra}}{\psi_{Ra}^2} \\
&= \frac{1}{\psi_{Ra}^2+\psi_{Rb}^2}(\psi_{Ra}(-\eta\psi_{Rb}+n_p\omega\psi_{Ra}+\eta M i_{Sb})-\psi_{Rb}(-\eta\psi_{Ra}-n_p\omega\psi_{Rb}+\eta M i_{Sa})) \\
&= \frac{1}{\psi_d^2}n_p\omega(\psi_{Ra}^2+\psi_{Rb}^2)+\eta M\frac{1}{\psi_d}((\psi_{Ra}/\psi_d)i_{Sb}-(\psi_{Rb}/\psi_d)i_{Sa}) \\
&= n_p\omega+\eta M\frac{1}{\psi_d}(\cos(\rho)i_{Sb}-\sin(\rho)i_{Sa}) \\
&= n_p\omega+\eta M\frac{i_q}{\psi_d}.
\end{aligned}
$$

Field-Oriented Control of an Induction Motor

Equation for i_d

$$
\begin{aligned}
\frac{di_d}{dt} &= \frac{d}{dt}\left(\cos(\rho) i_{Sa} + \sin(\rho) i_{Sb}\right) \\
&= \left(-\sin(\rho) i_{Sa} + \cos(\rho) i_{Sb}\right)\frac{d\rho}{dt} + \cos(\rho)\frac{di_{Sa}}{dt} + \sin(\rho)\frac{di_{Sb}}{dt} \\
&= i_q \frac{d\rho}{dt} + \cos(\rho)(\eta\beta\psi_{Ra} + \beta n_p \omega\psi_{Rb} - \gamma i_{Sa} + u_{Sa}/\sigma L_S) \\
&\quad + \sin(\rho)\left(\eta\beta\psi_{Rb} - \beta n_p \omega\psi_{Ra} - \gamma i_{Sb} + u_{Sb}/\sigma L_S\right) \\
&= i_q(n_p\omega + \eta M i_q/\psi_d) + \eta\beta(\cos(\rho)\psi_{Ra} + \sin(\rho)\psi_{Rb}) + n_p\omega\beta(\cos(\rho)\psi_{Rb} - \sin(\rho)\psi_{Ra}) \\
&\quad - \gamma(\cos(\rho) i_{Sa} + \sin(\rho) i_{Sb}) + (\cos(\rho) u_{Sa} + \sin(\rho) u_{Sb})/\sigma L_S \\
&= i_q n_p \omega + \eta M i_q^2/\psi_d + \eta\beta\psi_d - \gamma i_d + u_d/\sigma L_S.
\end{aligned}
$$

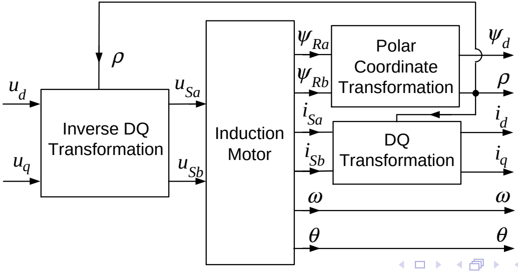
Field-Oriented Control of an Induction Motor

Equation for i_q

$$\begin{aligned}
\frac{di_q}{dt} &= \frac{d}{dt}(-\sin(\rho)i_{Sa} + \cos(\rho)i_{Sb}) \\
&= -(\cos(\rho)i_{Sa} + \sin(\rho)i_{Sb})\frac{d\rho}{dt} - \sin(\rho)\frac{di_{Sa}}{dt} + \cos(\rho)\frac{di_{Sb}}{dt} \\
&= -i_d\frac{d\rho}{dt} - \sin(\rho)(\eta\beta\psi_{Ra} + \beta n_p\omega\psi_{Rb} - \gamma i_{Sa} + u_{Sa}/\sigma L_S) \\
&\quad + \cos(\rho)(\eta\beta\psi_{Rb} - \beta n_p\omega\psi_{Ra} - \gamma i_{Sb} + u_{Sb}/\sigma L_S) \\
&= -i_d(n_p\omega + \eta M i_q/\psi_d) + \eta\beta(-\sin(\rho)\psi_{Ra} + \cos(\rho)\psi_{Rb}) \\
&\quad -n_p\omega\beta(\sin(\rho)\psi_{Rb} + \cos(\rho)\psi_{Ra}) - \gamma(-\sin(\rho)i_{Sa} + \cos(\rho)i_{Sb}) \\
&\quad + (-\sin(\rho)u_{Sa} + \cos(\rho)u_{Sb})/\sigma L_S \\
&= -i_d n_p\omega - \eta M i_d i_q/\psi_d - n_p\omega\beta\psi_d - \gamma i_q + u_q/\sigma L_S.
\end{aligned}$$

Induction Motor in the Field-Oriented Coordinate System

$$
\begin{aligned}
\frac{d\theta}{dt} &= \omega \\
\frac{d\omega}{dt} &= \mu\psi_d i_q - \tau_L/J \\
\frac{d\psi_d}{dt} &= -\eta\psi_d + \eta M i_d \\
\frac{di_d}{dt} &= -\gamma i_d + (\eta M/\sigma L_R L_S)\psi_d + n_p\omega i_q + \eta M i_q^2/\psi_d + u_d/\sigma L_S \\
\frac{di_q}{dt} &= -\gamma i_q - (M/\sigma L_R L_S)n_p\omega\psi_d - n_p\omega i_d - \eta M i_q i_d/\psi_d + u_q/\sigma L_S \\
\frac{d\rho}{dt} &= n_p\omega + \eta M i_q/\psi_d .
\end{aligned}
$$



Current-Command Field-Oriented Control

- The torque is now simply $\tau = J\mu\psi_d i_q$.
- The equations for i_d and i_q are still quite complicated

Let i_{dr}, i_{qr} be the desired (reference) currents for the machine.
In practice it is known that applying the **PI Current Controllers**

$$\begin{aligned} u_d &= K_{dI}\int_0^t (i_{dr} - i_d)dt' + K_{dP}(i_{dr} - i_d) \\ u_q &= K_{qI}\int_0^t (i_{qr} - i_q)dt' + K_{qP}(i_{qr} - i_q), \end{aligned}$$

the gains $K_{dI}, K_{dP}, K_{qI}, K_{qP}$ can by chosen so that

$$\begin{aligned} i_d &\to i_{dr} \\ i_q &\to i_{qr}. \end{aligned}$$

For all practical purposes i_{dr}, i_{qr} are essentially equal to i_d, i_q respectively.

Current-Command Field-Oriented Control

- With the input voltages given by

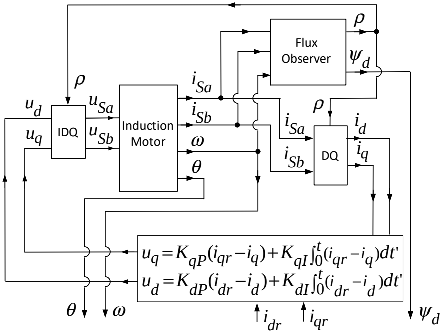
$$
\begin{aligned}
u_d &= K_{dI} \int_0^t (i_{dr} - i_d) dt' + K_{dP}(i_{dr} - i_d) \\
u_q &= K_{qI} \int_0^t (i_{qr} - i_q) dt' + K_{qP}(i_{qr} - i_q),
\end{aligned}
$$

  we can consider i_{dr}, i_{qr} as **new inputs** and the field-oriented model reduces to

$$
\begin{aligned}
d\theta/dt &= \omega \\
d\omega/dt &= \mu\psi_d i_{qr} - \tau_L/J \\
d\psi_d/dt &= -\eta\psi_d + \eta M i_{dr} \\
d\rho/dt &= n_p\omega + \eta M i_q/\psi_d.
\end{aligned}
$$

- Note we did **not** replace i_q with i_{qr} in the $d\rho/dt$ equation as we do not **control** ρ.
- We show below that we will need to **estimate** ρ and ψ_d.

Current-Command Field-Oriented Control



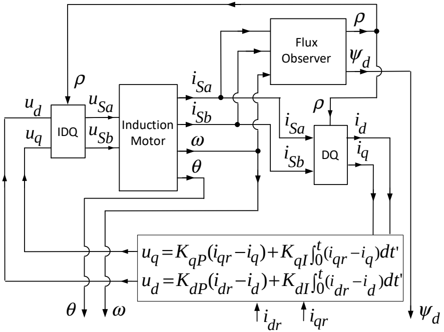
- The **flux observer** is a way to estimate ρ, ψ_d from i_{Sa}, i_{Sb}, ω (see slide 47).
- The **direct-quadrature (DQ)** transformation for i_{Sa}, i_{Sb} is

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} \cos(\rho) & \sin(\rho) \\ -\sin(\rho) & \cos(\rho) \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix}$$

- The **inverse direct-quadrature (IDQ)** transformation for u_{Sa}, u_{Sb} is

$$\begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix} = \begin{bmatrix} \cos(\rho) & -\sin(\rho) \\ \sin(\rho) & \cos(\rho) \end{bmatrix} \begin{bmatrix} u_d \\ u_q \end{bmatrix}$$

Current-Command Field-Oriented Control



- The PI current controller forces $i_d \to i_{dr}$ and $i_q \to i_{qr}$ **very fast**.
- Consequently, we can consider i_{dr} and i_{qr} as the **inputs**.
- The field-oriented induction motor model simplifies to

$$
\begin{aligned}
d\theta/dt &= \omega \\
d\omega/dt &= \mu\psi_d i_{qr} - \tau_L/J \\
d\psi_d/dt &= -\eta\psi_d + \eta M i_{dr} \\
d\rho/dt &= n_p\omega + \eta M i_q/\psi_d.
\end{aligned}
$$

Current-Command Field-Oriented Control

$$
\begin{aligned}
d\theta/dt &= \omega \\
d\omega/dt &= \mu\psi_d i_{qr} - \tau_L/J \\
d\psi_d/dt &= -\eta\psi_d + \eta M i_{dr} \\
d\rho/dt &= n_p\omega + \eta M i_q/\psi_d.
\end{aligned}
$$

- We will control θ, ω using i_{qr}.
- We will control ψ_d using i_{dr}.
- We will need to **estimate** ρ and ψ_d to implement the dq transformations.

Current-Command Field-Oriented Control

$$\begin{aligned} d\theta/dt &= \omega \\ d\omega/dt &= \mu\psi_d i_{qr} - \tau_L/J \\ d\psi_d/dt &= -\eta\psi_d + \eta M i_{dr} \\ d\rho/dt &= n_p\omega + \eta M i_q/\psi_d. \end{aligned}$$

- The flux ψ_d is regulated to a **constant value** $\psi_{d0} = M i_{d0}$ by

$$i_{dr} = K_{\psi I}\int_0^t (\psi_{d0} - \psi_d)dt' + K_{\psi P}(\psi_{d0} - \psi_d) + i_{d0}.$$

With proper choice of the feedback gains $\psi_d \to \psi_{d0} = M i_{d0}$.

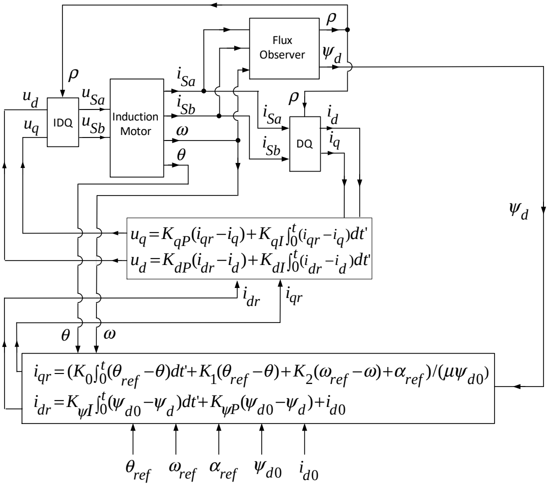
- With a properly working flux regulator, we can rewrite the equations for θ and ω as

$$\begin{aligned} d\theta/dt &= \omega \\ d\omega/dt &= \mu\psi_{d0} i_{qr} - \tau_L/J. \end{aligned}$$

Same form as a DC motor! Then $(\theta_{ref}(t), \omega_{ref}(t), \alpha_{ref}(t))$ is tracked using

$$i_{qr} = \left(K_0\int_0^t (\theta_{ref} - \theta)dt' + K_1(\theta_{ref} - \theta) + K_2(\omega_{ref} - \omega) + \alpha_{ref}\right)/\mu\psi_{d0}.$$

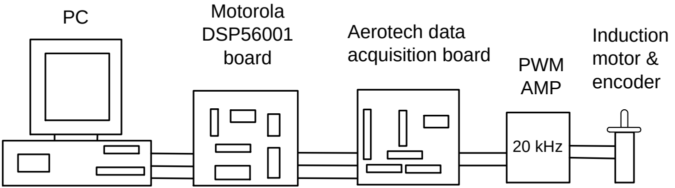
Current-Command Field-Oriented Control



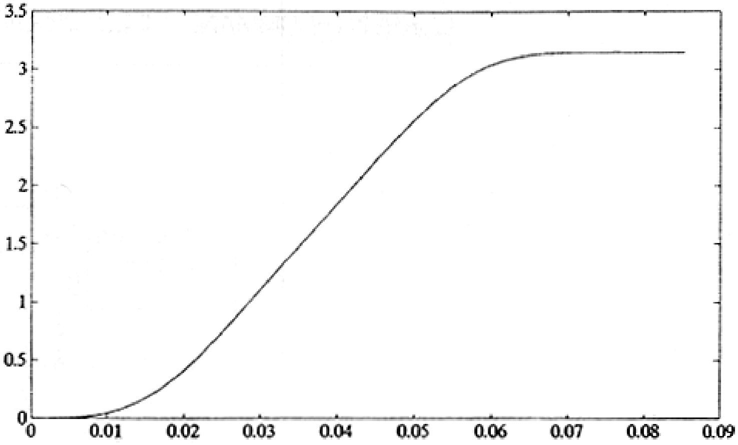
- The currents i_{Sa}, i_{Sb} and θ are measured.
- $\omega, \rho, \psi_d, i_d, i_q, i_{dr}, i_{qr}, u_d, u_q, u_{Sa}, u_{Sb}$ are computed in the microprocessor.
- The values of u_{Sa}, u_{Sb} are the commanded values sent to the amplifier.

Experimental Results Using a Field-Oriented Controller

- 6-pole ($n_p = 3$) 1/12-HP two-phase induction motor with a squirrel cage rotor.
- Motor rated for 2.4 A (continuous) and 60 V.
- A Motorola *DSP*56001 DSP was used to implement the control algorithm.
- Two PWM amplifiers (± 80 V and ± 6 A).
- Two analog-to-digital (A/D) converters to sample the stator currents.
- Two digital-to-analog (D/A) converters to command voltage to the amplifiers.
- A 2000 pulse/rev encoder (resolution of $360^\circ/2000 = 0.18^\circ$) to sense θ.
- The sample rate was 8 kHz.
- $M = 0.0117$ H, $R_R = 3.9\ \Omega$, $R_S = 1.7\ \Omega$, $L_R = 0.014$ H, $L_S = 0.014\ H$, $f = 0.00014$ N-m/rad/sec and $J = 0.00011$ kg-m^2.
- $\psi_{d0} = Mi_{d0} = 0.0117(5.5/\sqrt{2}) = 0.0455$ Wb.

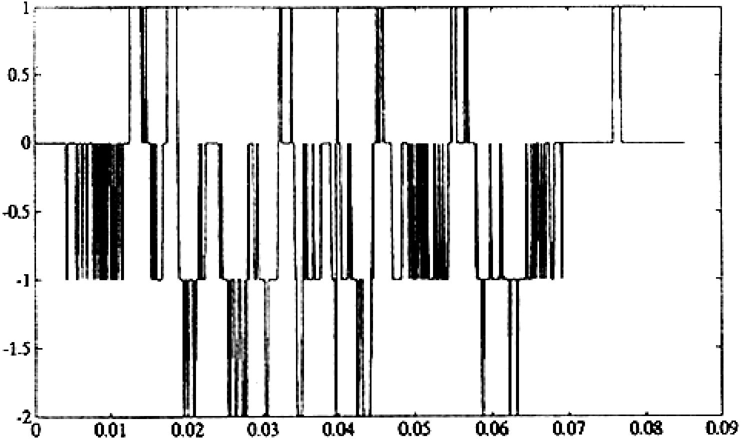


Position Response



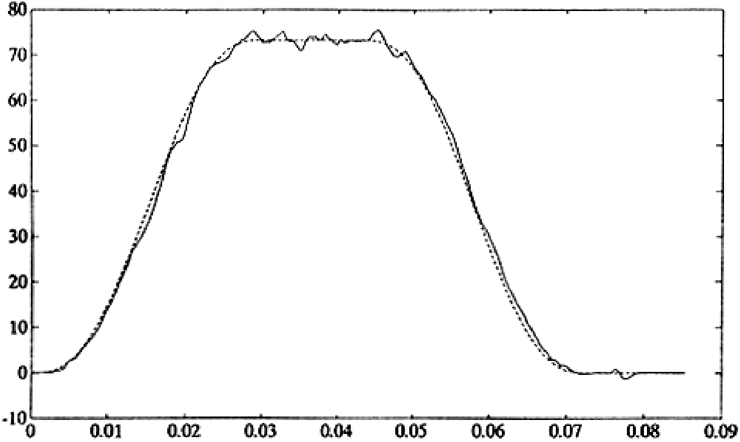
- Motor rotates 180° in 73 msec.

Position Error $\theta_{ref} - \theta$ in Encoder Counts

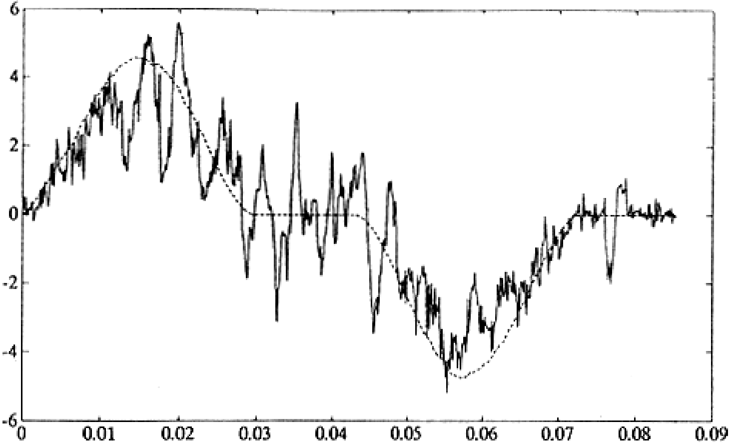


- The measured error is less than **two counts** for the entire move.
- Two encoder counts is $2(2\pi/2000) = 0.00314$ radians.

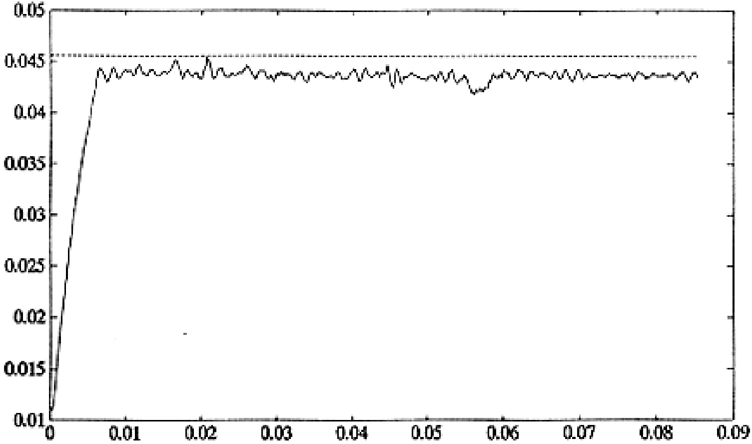
Speed Response: ω_{ref} and ω



Quadrature Current: i_q and $i_{qref} = \alpha_{ref}/\psi_{d0}$



Flux Response: ψ_d and $\psi_{dref} = \psi_{d0}$



Gain Values

The PI current gains:

$$\begin{aligned} K_{dI} &= 740 \\ K_{dP} &= 6.4 \\ K_{qI} &= 740 \\ K_{qP} &= 6.4. \end{aligned}$$

Trajectory tracking controller gains:

$$\begin{aligned} K_0 &= 1.07 \times 10^6 \\ K_1 &= 7.04 \times 10^5 \\ K_2 &= 1.07 \times 10^3. \end{aligned}$$

The flux regulator gains:

$$\begin{aligned} K_{\psi P} &= 1600 \\ K_{\psi I} &= 23000. \end{aligned}$$

Field Weakening

- The torque is controlled through the **quadrature current** i_q.
- To maintain or increase i_q requires $di_q/dt \geq 0$, that is,

$$\frac{di_q}{dt} = -\gamma i_q - (M/\sigma L_R L_S) n_p \omega \psi_d - n_p \omega i_d - \eta M i_q i_d/\psi_d + u_q/\sigma L_S \geq 0$$
$$\Longrightarrow u_q \geq (M^2 R_R/L_R^2 + R_S) i_q + (M/L_R) n_p \omega \psi_d + \sigma L_S n_p \omega i_d + \sigma L_S T_R i_q M i_d/\psi_d.$$

- In steady state $\psi_d = M i_d$ so that this inequality reduces to

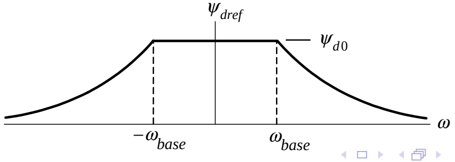
$$u_q \geq (M^2 R_R/L_R^2 + R_S) i_q + (M/L_R) n_p \omega \psi_d + \sigma n_p \omega \psi_d + \sigma L_S T_R i_q.$$

- At higher speeds the **dominant term** on the rhs of this inequality is the back-emf

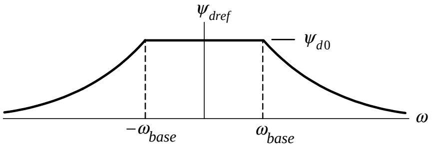
$$(M/L_R) n_p \omega \psi_d.$$

- The **base speed** ω_{base} is often chosen as the speed where $(M/L_R) n_p \omega \psi_{d0}$ equals $V_{\max}$.
- To go to higher speeds one **decreases** the flux ψ_d as follows:

$$\psi_{dref}(\omega) = \begin{cases} \psi_{d0} & \text{for} \quad |\omega| < \omega_{base} \\ \psi_{d0}\omega_{base}/|\omega| & \text{for} \quad |\omega| \geq \omega_{base} \end{cases}$$



Field Weakening



- The direct current i_{dr} is chosen to force ψ_d to track ψ_{dref} by

$$i_{dr} = K_{\psi I} \int_0^t (\psi_{dref} - \psi_d) dt' + K_{\psi P}(\psi_{dref} - \psi_d) + i_{d0}.$$

- This reduction of ψ_d above ω_{base} is referred to as **field weakening**.

Input-Output Linearization

- In field-oriented control ω and ψ_d are only **asymptotically** decoupled. I.e., the $d\omega/dt$ equation is **linear** only **after** ψ_d is constant.
- **Input-output linearization** lets us control ω and ψ_d **independently** of each other.

Field-Oriented Current Command Model

$$\begin{aligned} d\theta/dt &= \omega \\ d\omega/dt &= \mu\psi_d i_{qr} - \tau_L/J \\ d\psi_d/dt &= -\eta\psi_d + \eta M i_{dr} \\ d\rho/dt &= n_p\omega + \eta M i_q/\psi_d. \end{aligned}$$

i_{dr}, i_{qr} are the inputs. Set

$$i_{dr} = u_1 \text{ and } i_{qr} = \frac{u_2}{\mu\psi_d}$$

so that the system becomes

$$\begin{aligned} d\theta/dt &= \omega \\ d\omega/dt &= u_2 - \tau_L/J \\ d\psi_d/dt &= -\eta\psi_d + \eta M u_1 \\ d\rho/dt &= n_p\omega + \eta M i_q/\psi_d. \end{aligned}$$

- The equations for θ, ω, ψ_d are now **linear!**

Input-Output Linearization

- The input u_1 is chosen to force the **linear** system

$$d\psi_d/dt = -\eta\psi_d + \eta M u_1$$

  to track a given flux reference trajectory ψ_{dr}.

- The input u_2 is used to force the **linear** system

$$\begin{aligned} d\theta/dt &= \omega \\ d\omega/dt &= u_2 - \tau_L/J \end{aligned}$$

  to track a given mechanical reference trajectory $(\theta_{ref}(t), \omega_{ref}(t), \alpha_{ref}(t))$. The current reference i_{qr} is then simply chosen to be $i_{qr} = u_2/(\mu\psi_d)$.

- The system is **linear** from the **inputs** u_1, u_2 to the **outputs** ω, ψ.
- Hence the designation as an **input-output linearization** controller.

Input-Output Linearization

- The system is **nonlinear** as the equation for ρ is nonlinear.
- The boundedness (stability) of ρ is guaranteed as it is reset to 0 every 2π radians.

Let the **flux reference** $\psi_{dref}(t)$ and the **direct current reference** i_{dref} satisfy

$$d\psi_{dref}/dt = -\eta\psi_{dref} + \eta M i_{dref}.$$

With proper choice of $K_{\psi P}, K_{\psi I}$ we have $\psi_d \to \psi_{dref}$ if

$$u_1 = i_{dr} = K_{\psi P}(\psi_{dref} - \psi_d) + K_{\psi I}\int_0^t (\psi_{dref} - \psi_d)dt' + i_{dref}.$$

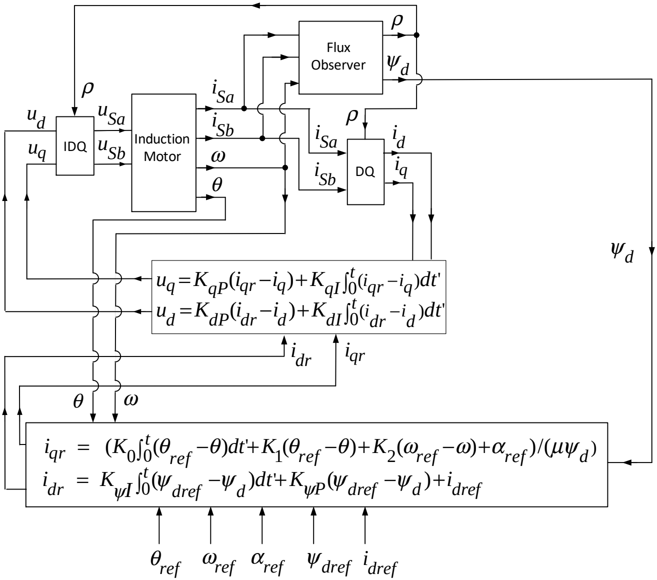
Let $\left(\theta_{ref}(t), \omega_{ref}(t), \alpha_{ref}(t)\right)$ be the **mechanical reference** trajectory.

With proper choice of K_0, K_1, K_2 we have $\theta \to \theta_{ref}$ and $\omega \to \omega_{ref}$ if

$$u_2 = K_0\int_0^t (\theta_{ref} - \theta)dt' + K_1(\theta_{ref} - \theta) + K_2(\omega_{ref} - \omega) + \alpha_{ref}$$

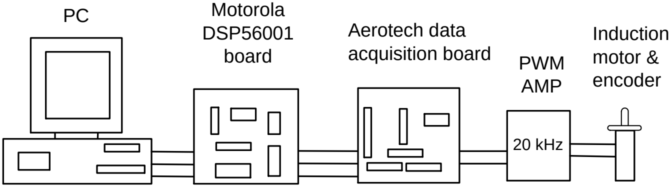
even with a constant load torque τ_L.

Input-Output Linearization

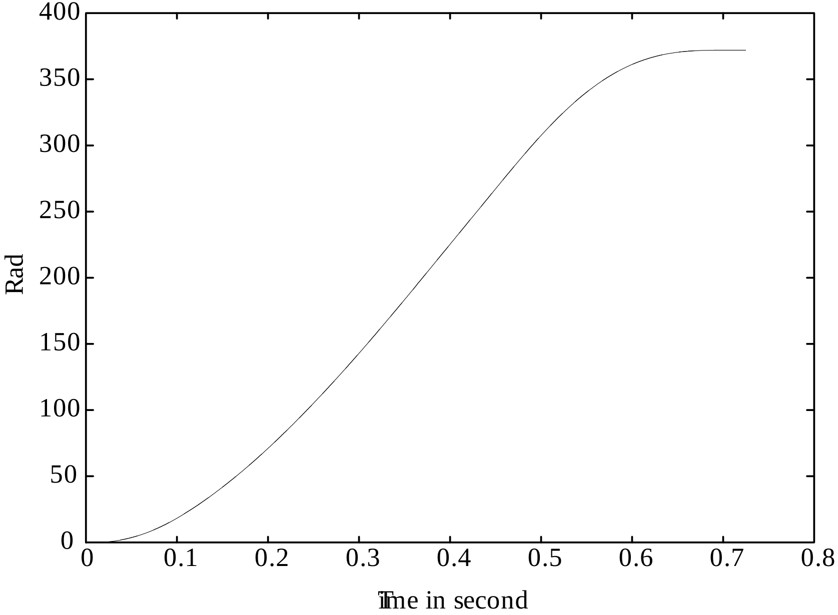


Experimental Results Using an Input-Output Controller

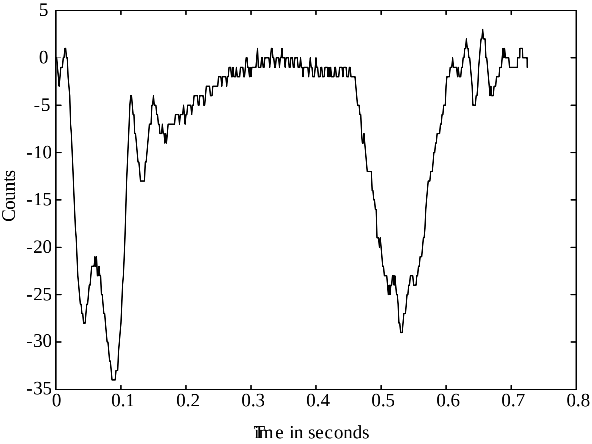
- Same setup as for the field-oriented controller.
- Trajectory is a point-to-point position move.
- The motor is brought up to a speed of 8000 rev/min in 0.38 seconds.
- The motor is brought down from 8000 rev/min to 0 rev/min in 0.265 sec.
- The PI current gains are $K_{dI} = 9000$, $K_{dP} = 15$, $K_{qI} = 9000$, $K_{qP} = 15$.
- The PI gains flux tracking gains are $K_{\psi P} = 10,000$, $K_{\psi I} = 420,000$.
- The PID gains for tracking the mechanical trajectory are $K_0 = 3.0 \times 10^5$, $K_1 = 5.5 \times 10^4$, $K_2 = 125$.
- The sample rate was 4 kHz.



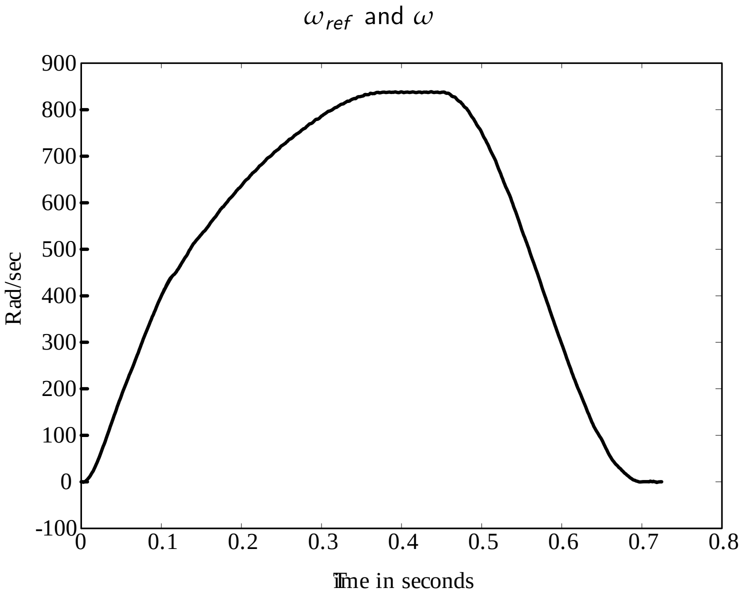
θ **and** θ_{ref}



$\theta_{ref} - \theta$ in encoder counts

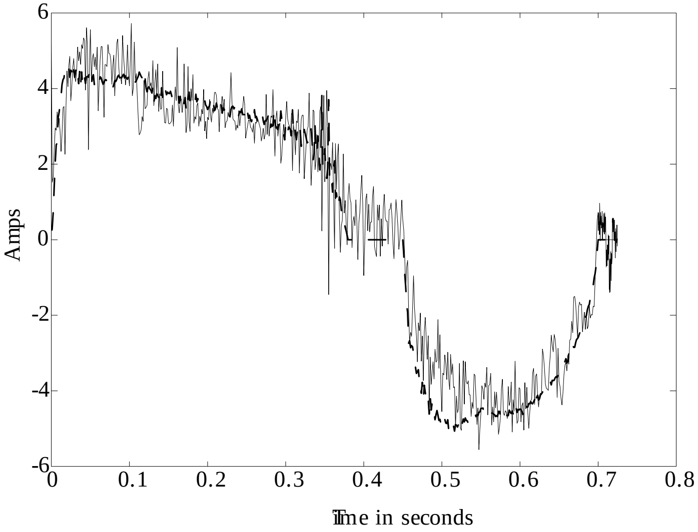


- The **maximum error** is 34 encoder counts.
- At the **end** of the run the final position error is **zero**.

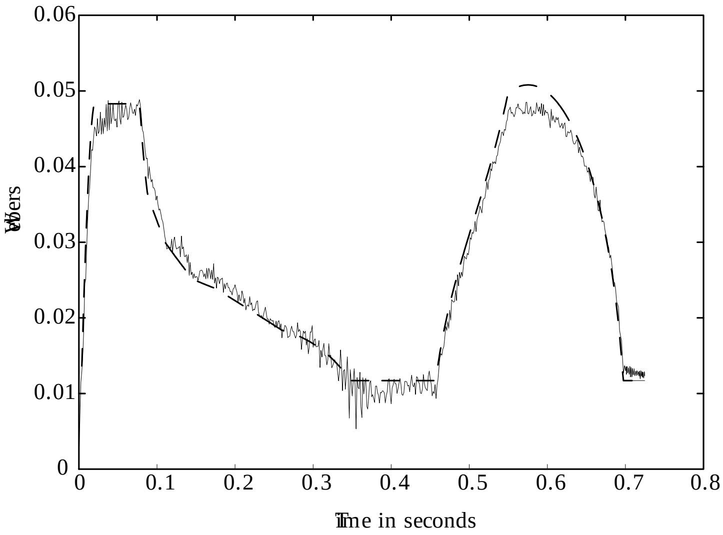


- Note the excellent speed tracking despite the **time-varying flux**.

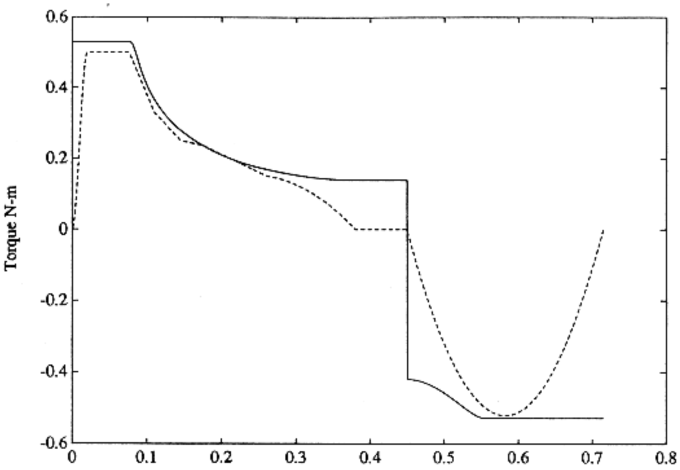
i_q and $i_{qref} \triangleq \alpha_{ref}/(\mu\psi_{dref})$



ψ_d and ψ_{dref}

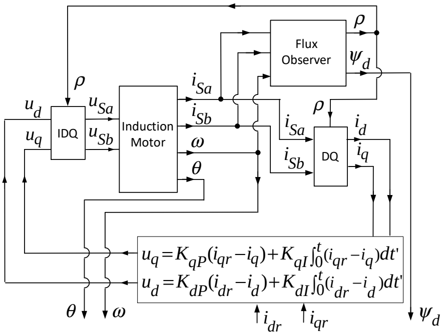


$\tau_{optimum}$ and τ_{ref}



- $\tau = J\mu\psi_d i_q$ (dashed curve).
- $\tau_{optimum}$ - Optimum torque given the voltage and current constraints.

Observers



- Field-oriented control **requires** the values of ρ and ψ_d (equivalently ψ_{Ra} and ψ_{Rb}). We will present a flux observer as a way to **estimate** the rotor fluxes.

- When a position sensor is used, a speed sensor is typically **not** available. One usually **numerically differentiates** the position measurement to get the speed. We show a **smoother estimate** of ω can be found using a **speed observer**.

Flux Observer

Recall the equations for ψ_{Ra}, ψ_{Rb} given by $(\eta = 1/T_R = R_R/L_R)$

$$\begin{aligned} d\psi_{Ra}/dt &= -\eta\psi_{Ra} - n_p\omega\psi_{Rb} + \eta M i_{Sa} \\ d\psi_{Rb}/dt &= -\eta\psi_{Rb} + n_p\omega\psi_{Ra} + \eta M i_{Sb}. \end{aligned}$$

- Let T be the sample period and measure i_{Sa}, i_{Sb}, θ at times $t = kT, k = 0, 1, 2, \ldots$
- Compute $\omega(kT) = \dfrac{\theta(kT) - \theta((k-1)T)}{T}$.

Estimate the fluxes ψ_{Ra}, ψ_{Rb} by **real-time** integration of

$$\begin{aligned} d\hat{\psi}_{Ra}/dt &= -\eta\hat{\psi}_{Ra} - n_p\omega\hat{\psi}_{Rb} + \eta M i_{Sa} \\ d\hat{\psi}_{Rb}/dt &= -\eta\hat{\psi}_{Rb} + n_p\omega\hat{\psi}_{Ra} + \eta M i_{Sb}. \end{aligned}$$

I.e., the real-time solution $\hat{\psi}_{Ra}, \hat{\psi}_{Rb}$ is our estimate of the fluxes.

Assumptions:

- The equations for the flux linkages are an accurate model.
- The parameters η and M are known.
- The currents and speed are measured/computed precisely.
- The numerical integration of the flux equations is done accurately.

Flux Observer

The initial conditions $\psi_{Ra}(0)$ and $\psi_{Rb}(0)$ are **unknown**.

We now show $\hat{\psi}_{Ra}(t) \to \psi_{Ra}(t), \hat{\psi}_{Rb}(t) \to \psi_{Rb}(t)$ though $\psi_{Ra}(0), \psi_{Rb}(0)$ are unknown.

With

$$\varepsilon_{Ra} \triangleq \psi_{Ra} - \hat{\psi}_{Ra}, \quad \varepsilon_{Rb} \triangleq \psi_{Rb} - \hat{\psi}_{Rb}$$

subtract

$$\begin{aligned} d\hat{\psi}_{Ra}/dt &= -\eta\hat{\psi}_{Ra} - n_p\omega\hat{\psi}_{Rb} + \eta M i_{Sa} \\ d\hat{\psi}_{Rb}/dt &= -\eta\hat{\psi}_{Rb} + n_p\omega\hat{\psi}_{Ra} + \eta M i_{Sb} \end{aligned}$$

from

$$\begin{aligned} d\psi_{Ra}/dt &= -\eta\psi_{Ra} - n_p\omega\psi_{Rb} + \eta M i_{Sa} \\ d\psi_{Rb}/dt &= -\eta\psi_{Rb} + n_p\omega\psi_{Ra} + \eta M i_{Sb} \end{aligned}$$

to obtain

$$\begin{aligned} \dot{\varepsilon}_{Ra} &= -\eta\varepsilon_{Ra} - n_p\omega\varepsilon_{Rb} \\ \dot{\varepsilon}_{Rb} &= -\eta\varepsilon_{Rb} + n_p\omega\varepsilon_{Ra}. \end{aligned}$$

Flux Observer

The flux error satisfies

$$\begin{aligned}\dot{\varepsilon}_{Ra} &= -\eta\varepsilon_{Ra} - n_p\omega\varepsilon_{Rb}\\ \dot{\varepsilon}_{Rb} &= -\eta\varepsilon_{Rb} + n_p\omega\varepsilon_{Ra}.\end{aligned}$$

To show $\hat{\psi}_{Ra}(t) \to \psi_{Ra}(t), \hat{\psi}_{Rb}(t) \to \psi_{Rb}(t)$ define V by

$$V(t) \triangleq \left(\psi_{Ra}(t) - \hat{\psi}_{Ra}(t)\right)^2 + \left(\psi_{Rb}(t) - \hat{\psi}_{Rb}(t)\right)^2 = \varepsilon_{Ra}^2(t) + \varepsilon_{Rb}^2(t).$$

Then

$$\begin{aligned}dV/dt = 2\varepsilon_{Ra}\dot{\varepsilon}_{Ra} + 2\varepsilon_{Rb}\dot{\varepsilon}_{Rb} &= 2\varepsilon_{Ra}\left(-\eta\varepsilon_{Ra} - n_p\omega\varepsilon_{Rb}\right) + 2\varepsilon_{Rb}\left(-\eta\varepsilon_{Rb} + n_p\omega\varepsilon_{Ra}\right)\\ &= -2\eta\left(\varepsilon_{Ra}^2 + \varepsilon_{Rb}^2\right)\\ &= -2\eta V.\end{aligned}$$

That is,

$$dV/dt = -2\eta V \quad \textbf{with solution} \quad V(t) = V(0)e^{-2\eta t} \to 0.$$

$V(0) = \left(\psi_{Ra}(0) - \hat{\psi}_{Ra}(0)\right)^2 + \left(\psi_{Rb}(0) - \hat{\psi}_{Rb}(0)\right)^2$ is **unknown.**

However $V(t) \to 0 \Longrightarrow \hat{\psi}_{Ra}(t) \to \psi_{Ra}(t),\ \hat{\psi}_{Rb}(t) \to \psi_{Rb}(t).$

Flux Observer and Noise

- Suppose starting at time t_1 there is **noise** on the measurement of $\omega(t_1)$.
- Suppose further that at (say) time $t_1 + \delta$ the noise is **no longer present** on the speed measurement.
- For $t_1 < t < t_1 + \delta$, the speed is measured as $\omega(t) + n(t)$ rather than $\omega(t)$.
- This **incorrect** measurement is used to calculate the estimates $\hat{\psi}_{Ra}(t), \hat{\psi}_{Rb}(t)$.
- For $t \geq t_1 + \delta$ the measured speed is again $\omega(t)$, that is, the correct value.
- So for $t \geq t_1 + \delta$ we have

$$\begin{aligned} d\hat{\psi}_{Ra}/dt &= -\eta\hat{\psi}_{Ra} - n_p\omega\hat{\psi}_{Rb} + \eta M i_{Sa} \\ d\hat{\psi}_{Rb}/dt &= -\eta\hat{\psi}_{Rb} + n_p\omega\hat{\psi}_{Ra} + \eta M i_{Sb} \end{aligned}$$

but the initial conditions $\hat{\psi}_{Ra}(t_1 + \delta), \hat{\psi}_{Rb}(t_1 + \delta)$ are **unknown**.

By the above we still have $\hat{\psi}_{Ra}(t) \to \psi_{Ra}(t),\ \hat{\psi}_{Rb}(t) \to \psi_{Rb}(t)$.

Flux Observer in Field-Oriented Coordinates

- $\hat{\psi}_{Ra}(t) \to \psi_{Ra}(t),\ \hat{\psi}_{Rb}(t) \to \psi_{Rb}(t)$ irrespective of the initial conditions.
- We also showed the estimator recovers from measurement disturbances.

It is more convenient to implement

$$\begin{aligned} d\hat{\psi}_{Ra}/dt &= -\eta\hat{\psi}_{Ra} - n_p\omega\hat{\psi}_{Rb} + \eta M i_{Sa} \\ d\hat{\psi}_{Rb}/dt &= -\eta\hat{\psi}_{Rb} + n_p\omega\hat{\psi}_{Ra} + \eta M i_{Sb} \end{aligned}$$

in field-oriented coordinates.

Flux Observer in Field-Oriented Coordinates

$$\rho \triangleq \tan^{-1}(\psi_{Rb}/\psi_{Ra}), \qquad \psi_d \triangleq \sqrt{\psi_{Ra}^2 + \psi_{Rb}^2}$$

Then

$$\begin{aligned} d\hat{\rho}/dt &= n_p\omega + \eta M\hat{\imath}_q/\hat{\psi}_d = n_p\omega + \eta M\Big(-i_{Sa}\sin(\hat{\rho}) + i_{Sb}\cos(\hat{\rho})\Big)/\hat{\psi}_d \\ d\hat{\psi}_d/dt &= -\eta\hat{\psi}_d + \eta M\hat{\imath}_d = -\eta\hat{\psi}_d + \eta M\Big(i_{Sa}\cos(\hat{\rho}) + i_{Sb}\sin(\hat{\rho})\Big). \end{aligned}$$

Flux Observer in Field-Oriented Coordinates

We have two flux observers:

$$\begin{aligned} d\hat{\psi}_{Ra}/dt &= -\eta\hat{\psi}_{Ra} - n_p\omega\hat{\psi}_{Rb} + \eta M i_{Sa} \\ d\hat{\psi}_{Rb}/dt &= -\eta\hat{\psi}_{Rb} + n_p\omega\hat{\psi}_{Ra} + \eta M i_{Sb} \end{aligned}$$

$$\begin{aligned} d\hat{\rho}/dt &= n_p\omega + \eta M\hat{\imath}_q/\hat{\psi}_d = n_p\omega + \eta M\Big(-i_{Sa}\sin(\hat{\rho}) + i_{Sb}\cos(\hat{\rho})\Big)/\hat{\psi}_d \\ d\hat{\psi}_d/dt &= -\eta\hat{\psi}_d + \eta M\hat{\imath}_d = -\eta\hat{\psi}_d + \eta M\Big(i_{Sa}\cos(\hat{\rho}) + i_{Sb}\sin(\hat{\rho})\Big). \end{aligned}$$

- ψ_d, i_d, and i_q vary much **slower** than $\psi_{Ra}(t)$ and $\psi_{Rb}(t)$.
- If the motor is running at a constant speed, then ψ_d, i_d, i_q are **constant**.
- $\psi_{Ra}(t), \psi_{Rb}(t)$ vary at the **stator frequency** which at high speeds is **large**. This requires a **small time step** to accurately integrate $d\hat{\psi}_{Ra}/dt, d\hat{\psi}_{Rb}/dt$.
- Both observers require the values of η and M. $\eta = 1/T_R = R_R/L_R$. R_R can vary by 100% due to ohmic heating.

Backward Difference Speed Estimate

If a position sensor is available then a speed sensor is typically not used.

Backward Difference

- Often ω is computed from θ as

$$\hat{\omega}_{bd}(kT) = \frac{\theta(kT) - \theta((k-1)T)}{T}.$$

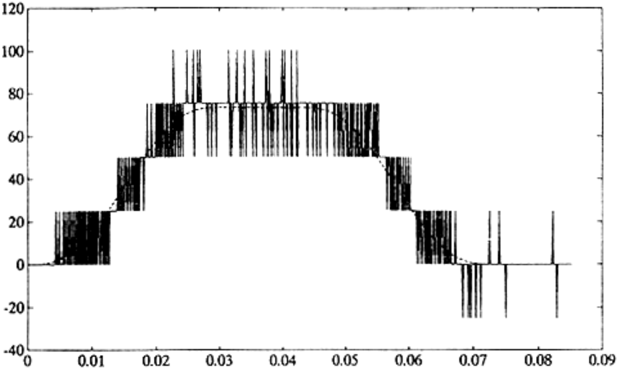
- T is the sample period.
- The difference $\theta(kT) - \theta((k-1)T)$ can be in error by no more than one count.
- For a 2000 pulse/rev encoder, one count is $2\pi/2000$ radians.
- Then the speed error $|\omega - \hat{\omega}_{bd}|$ is bounded by (see Chapter 2)

$$|\omega - \hat{\omega}_{bd}| \leq \frac{2\pi}{2000}\frac{1}{T}.$$

- The noise is significant at high sample rates (T small) and moderate to low speeds. (Less encoder counts are detected per sample period at lower speeds.)

Backward Difference - Experimental Results

- Consider the speed response for the field-oriented control experiment (see slide 29).
- The motor turned 180° in 73 msec.
- The top speed of the motor was less than 75 rad/sec.
- The sample rate was 8 kHz.
- $|\omega - \hat{\omega}_{bd}| \leq \dfrac{2\pi}{2000} 8000 = 25.13$ rad/sec.



Speed Observer

- We now obtain a smoother estimate of speed using an observer.

Consider the load torque to be constant (slowly varying). Then

$$\begin{aligned} d\theta/dt &= \omega \\ d\omega/dt &= \mu\psi_d i_q - \tau_L \\ d\tau_L/dt &= 0 \end{aligned}$$

The quantities ψ_d and i_q are not known, but can be estimated as shown above.
Define

$$\begin{aligned} d\hat{\theta}/dt &= \hat{\omega} + \ell_1(\theta - \hat{\theta}) \\ d\hat{\omega}/dt &= \mu\hat{\psi}_d\hat{\imath}_q + \ell_2(\theta - \hat{\theta}) \\ d\hat{\tau}_L/dt &= 0 + \ell_3(\theta - \hat{\theta}). \end{aligned}$$

With $e_1 = \theta - \hat{\theta}$, $e_2 = \omega - \hat{\omega}$, $e_3 = \tau_L - \hat{\tau}_L$ and $\hat{\psi}_d\hat{\imath}_q \to \psi_d i_q$ fast enough we have

$$\begin{aligned} de_1/dt &= e_2 - \ell_1 e_1 \\ de_2/dt &= -e_3 - \ell_2 e_1 \\ de_3/dt &= -\ell_3 e_3. \end{aligned}$$

We want to show $e_2(t) = \omega(t) - \hat{\omega}(t) \to 0$.

Speed Observer

$$\begin{aligned} de_1/dt &= e_2 - \ell_1 e_1 \\ de_2/dt &= -e_3 - \ell_2 e_1 \\ de_3/dt &= -\ell_3 e_3. \end{aligned}$$

Taking the Laplace transform gives

$$\begin{bmatrix} s+\ell_1 & -1 & 0 \\ \ell_2 & s & 1 \\ \ell_3 & 0 & s \end{bmatrix} \begin{bmatrix} E_1(s) \\ E_2(s) \\ E_3(s) \end{bmatrix} = \begin{bmatrix} e_1(0) \\ e_2(0) \\ e_3(0) \end{bmatrix}$$

or

$$\begin{bmatrix} E_1(s) \\ E_2(s) \\ E_3(s) \end{bmatrix} = \begin{bmatrix} s+\ell_1 & -1 & 0 \\ \ell_2 & s & 1 \\ \ell_3 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} e_1(0) \\ e_2(0) \\ e_3(0) \end{bmatrix}$$

$$= \frac{1}{s^3 + \ell_1 s^2 + \ell_2 s - \ell_3} \begin{bmatrix} s^2 & s & -1 \\ -(\ell_2 s - \ell_3) & s(s+\ell_1) & -(s+\ell_1) \\ -\ell_3 s & -\ell_3 & (s+\ell_1)s + \ell_2 \end{bmatrix} \begin{bmatrix} e_1(0) \\ e_2(0) \\ e_3(0) \end{bmatrix}.$$

Speed Observer

$$\mathcal{L}\{e_2(t)\} = E_2(s) = \frac{-(\ell_2 s - \ell_3)\, e_1(0) + s\,(s+\ell_1)\, e_2(0) - (s+\ell_1)\, e_3(0)}{s^3 + \ell_1 s^2 + \ell_2 s - \ell_3}$$

With $r_1 > 0, r_2 > 0, r_3 > 0$ choose ℓ_1, ℓ_2, ℓ_3 so that

$$\begin{aligned} s^3 + \ell_1 s^2 + \ell_2 s - \ell_3 &= (s+r_1)(s+r_2)(s+r_3) \\ &= s^3 + (r_1 + r_2 + r_3)s + (r_1 r_2 + r_1 r_2 + r_2 r_3)s + r_1 r_2 r_3. \end{aligned}$$

That is, set

$$\ell_1 = r_1 + r_2 + r_3, \quad \ell_2 = r_1 r_2 + r_1 r_2 + r_2 r_3, \quad \ell_3 = -r_1 r_2 r_3.$$

By partial fractions (assuming r_1, r_2, and r_3 are distinct)

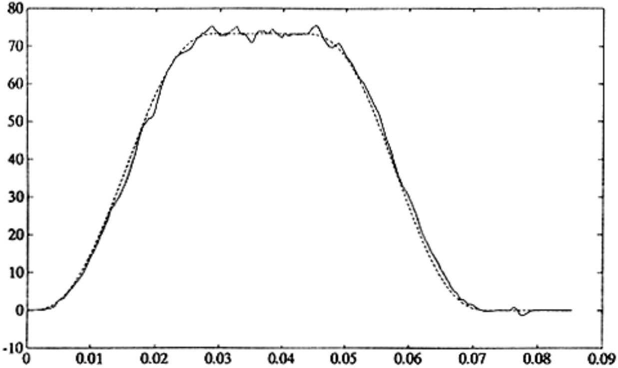
$$\begin{aligned} E_2(s) &= \frac{-(\ell_2 s - \ell_3)\, e_1(0) + s\,(s+\ell_1)\, e_2(0) - (s+\ell_1)\, e_3(0)}{s^3 + \ell_1 s^2 + \ell_2 s - \ell_3} \\ &= \frac{-(\ell_2 s - \ell_3)\, e_1(0) + s\,(s+\ell_1)\, e_2(0) - (s+\ell_1)\, e_3(0)}{(s+r_1)(s+r_2)(s+r_3)} \\ &= \frac{A_1}{s+r_1} + \frac{B_1}{s+r_2} + \frac{C_1}{s+r_3} \text{ for some constants } A_1, B_1, C_1. \\ \Longrightarrow e_2(t) &= A_1 e^{-r_1 t} + B_1 e^{-r_2 t} + C_1 e^{-r_3 t} \to 0 \text{ as } t \to \infty. \end{aligned}$$

Speed Observer - Experimental Results

- In this experiment $\tau_L = 0$.
- The observer reduces to

$$\begin{aligned} d\hat{\theta}/dt &= \hat{\omega} + \ell_1(\theta - \hat{\theta}) \\ d\hat{\omega}/dt &= \mu\hat{\psi}_d\hat{\imath}_q + \ell_2(\theta - \hat{\theta}). \end{aligned}$$

- The gains are set as $\ell_1 = 1.8 \times 10^3$ and $\ell_2 = 8 \times 10^5$.



Speed and Flux Observer

- The speed observer requires an accurate value of $\mu = \dfrac{n_p M}{J L_R}$.
- The flux observer requires accurate values of $\eta = 1/T_R, M$.
- The flux estimator and speed estimator are **coupled**.

That is, to estimate the flux and speed requires integrating

$$
\begin{aligned}
\frac{d\hat{\rho}}{dt} &= n_p\hat{\omega} + \eta M(-i_{Sa}\sin(\hat{\rho}) + i_{Sb}\cos(\hat{\rho}))/\hat{\psi}_d \\
\frac{d\hat{\psi}_d}{dt} &= -\eta\hat{\psi}_d + \eta M(i_{Sa}\cos(\hat{\rho}) + i_{Sb}\sin(\hat{\rho})) \\
\frac{d\hat{\theta}}{dt} &= \hat{\omega} + \ell_1(\theta - \hat{\theta}) \\
\frac{d\hat{\omega}}{dt} &= \mu\hat{\psi}_d(-i_{Sa}\sin(\hat{\rho}) + i_{Sb}\cos(\hat{\rho})) - (f/J)\hat{\omega} + \ell_2(\theta - \hat{\theta}) \\
\frac{d\hat{\tau}_L}{dt} &= 0 + \ell_3(\theta - \hat{\theta})
\end{aligned}
$$

where i_{Sa}, i_{Sb}, and θ are the measured "inputs" to this observer.

Online Identification of T_R and R_S

- Induction motor parameters: M, L_S, L_R, R_S, R_R, J, f, τ_L.
- Standard methods for the estimation of induction motor parameters include (1) locked rotor test (2) no-load test and (3) the standstill freq response test.
- T_R and R_S change due to Ohmic heating. Need to estimate them online.
- Mathematical Model

$$\begin{aligned}
\frac{d\omega}{dt} &= \frac{n_p M}{J L_R}(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}) - \frac{f}{J}\omega - \frac{\tau_L}{J} \\
\frac{d\psi_{Ra}}{dt} &= -\frac{1}{T_R}\psi_{Ra} - n_p\omega\psi_{Rb} + \frac{M}{T_R} i_{Sa} \\
\frac{d\psi_{Rb}}{dt} &= -\frac{1}{T_R}\psi_{Rb} + n_p\omega\psi_{Ra} + \frac{M}{T_R} i_{Sb} \\
\frac{di_{Sa}}{dt} &= \frac{\beta}{T_R}\psi_{Ra} + \beta n_p\omega\psi_{Rb} - \gamma i_{Sa} + \frac{1}{\sigma L_S} u_{Sa} \\
\frac{di_{Sb}}{dt} &= \frac{\beta}{T_R}\psi_{Rb} - \beta n_p\omega\psi_{Ra} - \gamma i_{Sb} + \frac{1}{\sigma L_S} u_{Sb}
\end{aligned}$$

$$\begin{aligned}
&T_R = L_R/R_R, && \sigma = 1 - M^2/(L_S L_R) \\
&\beta = M/(\sigma L_S L_R), && \gamma = R_S/(\sigma L_S) + M^2 R_R/(\sigma L_S L_R^2)
\end{aligned}$$

Online Identification of T_R and R_S

Using the transformations

$$\begin{bmatrix} i_{Sx} \\ i_{Sy} \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix},$$

$$\begin{bmatrix} u_{Sx} \\ u_{Sy} \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix},$$

$$\begin{bmatrix} \psi_{Rx} \\ \psi_{Ry} \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix}.$$

to obtain the model

$$\begin{aligned}
\frac{di_{Sx}}{dt} &= \frac{1}{\sigma L_S} u_{Sx} - \gamma i_{Sx} + \frac{\beta}{T_R}\psi_{Rx} + n_p\beta\omega\psi_{Ry} + n_p\omega i_{Sy} \\
\frac{di_{Sy}}{dt} &= \frac{1}{\sigma L_S} u_{Sy} - \gamma i_{Sy} + \frac{\beta}{T_R}\psi_{Ry} - n_p\beta\omega\psi_{Rx} - n_p\omega i_{Sx} \\
\frac{d\psi_{Rx}}{dt} &= \frac{M}{T_R} i_{Sx} - \frac{1}{T_R}\psi_{Rx} \\
\frac{d\psi_{Ry}}{dt} &= \frac{M}{T_R} i_{Sy} - \frac{1}{T_R}\psi_{Ry} \\
\frac{d\omega}{dt} &= \frac{n_p M}{J L_R}(i_{Sy}\psi_{Rx} - i_{Sx}\psi_{Ry}) - \frac{f}{J}\omega - \frac{\tau_L}{J}.
\end{aligned}$$

- These variables vary at the slip frequency rather than the stator frequency.

Online Identification of T_R and R_S

Differentiate the first two equations of

$$
\begin{aligned}
\frac{di_{Sx}}{dt} &= \frac{1}{\sigma L_S}u_{Sx} - \gamma i_{Sx} + \frac{\beta}{T_R}\psi_{Rx} + n_p\beta\omega\psi_{Ry} + n_p\omega i_{Sy} \\
\frac{di_{Sy}}{dt} &= \frac{1}{\sigma L_S}u_{Sy} - \gamma i_{Sy} + \frac{\beta}{T_R}\psi_{Ry} - n_p\beta\omega\psi_{Rx} - n_p\omega i_{Sx} \\
\frac{d\psi_{Rx}}{dt} &= \frac{M}{T_R}i_{Sx} - \frac{1}{T_R}\psi_{Rx} \\
\frac{d\psi_{Ry}}{dt} &= \frac{M}{T_R}i_{Sy} - \frac{1}{T_R}\psi_{Ry} \\
\frac{d\omega}{dt} &= \frac{n_pM}{JL_R}(i_{Sy}\psi_{Rx} - i_{Sx}\psi_{Ry}) - \frac{f}{J}\omega - \frac{\tau_L}{J}.
\end{aligned}
$$

to obtain

$$
\frac{1}{\sigma L_s}\frac{du_{Sx}}{dt} = \frac{d^2 i_{Sx}}{dt^2} + \gamma\frac{di_{Sx}}{dt} - \frac{\beta}{T_R}\frac{d\psi_{Rx}}{dt} - n_p\beta\omega\frac{d\psi_{Ry}}{dt} - n_p\beta\psi_{Ry}\frac{d\omega}{dt} - n_p\omega\frac{di_{Sy}}{dt} - n_p i_{Sy}\frac{d\omega}{dt} \tag{1}
$$

$$
\frac{1}{\sigma L_s}\frac{du_{Sy}}{dt} = \frac{d^2 i_{Sy}}{dt^2} + \gamma\frac{di_{Sy}}{dt} - \frac{\beta}{T_R}\frac{d\psi_{Ry}}{dt} + n_p\beta\omega\frac{d\psi_{Rx}}{dt} + n_p\beta\psi_{Rx}\frac{d\omega}{dt} + n_p\omega\frac{di_{Sx}}{dt} + n_p i_{Sx}\frac{d\omega}{dt}. \tag{2}
$$

- Solve the first four equations of the model to obtain $\psi_{Rx}, \psi_{Ry}, d\psi_{Rx}/dt, d\psi_{Ry}/dt$.
- Substitute these expressions for $\psi_{Rx}, \psi_{Ry}, d\psi_{Rx}/dt, d\psi_{Ry}/dt$ into (1) and (2).

Online Identification of T_R and R_S

$$0 = -\frac{d^2 i_{Sx}}{dt^2} + \frac{di_{Sy}}{dt} n_p\omega + \frac{1}{\sigma L_S}\frac{du_{Sx}}{dt} - (\gamma + \frac{1}{T_R})\frac{di_{Sx}}{dt} - i_{Sx}(-\frac{\beta M}{T_R^2} + \frac{\gamma}{T_R}) + i_{Sy} n_p\omega(\frac{1}{T_R} + \frac{\beta M}{T_R}) + \frac{u_{Sx}}{\sigma L_S T_R}$$
$$+ n_p\frac{d\omega}{dt} i_{Sy} - n_p\frac{d\omega}{dt}\frac{1}{\sigma L_S(1 + n_p^2\omega^2 T_R^2)}\Big(-\sigma L_S T_R\frac{di_{Sy}}{dt} - \gamma i_{Sy}\sigma L_S T_R - i_{Sx} n_p\omega\sigma L_S T_R - \frac{di_{Sx}}{dt} n_p\omega\sigma L_S T_R^2$$
$$-\gamma i_{Sx} n_p\omega\sigma L_S T_R^2 + i_{Sy} n_p^2\omega^2\sigma L_S T_R^2 + n_p\omega T_R^2 u_{Sx} + T_R u_{Sy}\Big)$$

$$0 = -\frac{d^2 i_{Sy}}{dt^2} - \frac{di_{Sx}}{dt} n_p\omega + \frac{1}{\sigma L_S}\frac{du_{Sy}}{dt} - (\gamma + \frac{1}{T_R})\frac{di_{Sy}}{dt} - i_{Sy}(-\frac{\beta M}{T_R^2} + \frac{\gamma}{T_R}) - i_{Sx} n_p\omega(\frac{1}{T_R} + \frac{\beta M}{T_R}) + \frac{u_{Sy}}{\sigma L_S T_R}$$
$$- n_p\frac{d\omega}{dt} i_{Sx} + n_p\frac{d\omega}{dt}\frac{1}{\sigma L_S(1 + n_p^2\omega^2 T_R^2)}\Big(-\sigma L_S T_R\frac{di_{Sx}}{dt} - \gamma i_{Sx}\sigma L_S T_R + i_{Sy} n_p\omega\sigma L_S T_R + \frac{di_{Sy}}{dt} n_p\omega\sigma L_S T_R^2$$
$$+\gamma i_{Sy} n_p\omega\sigma L_S T_R^2 + i_{Sx} n_p^2\omega^2\sigma L_S T_R^2 - n_p\omega T_R^2 u_{Sy} + T_R u_{Sx}\Big).$$

Assuming *constant* speed these reduce to

$$0 = -\frac{d^2 i_{Sx}}{dt^2} + \frac{di_{Sy}}{dt} n_p\omega + \frac{1}{\sigma L_S}\frac{du_{Sx}}{dt} - (\gamma + \frac{1}{T_R})\frac{di_{Sx}}{dt} - i_{Sx}(-\frac{\beta M}{T_R^2} + \frac{\gamma}{T_R}) + i_{Sy} n_p\omega(\frac{1}{T_R} + \frac{\beta M}{T_R}) + \frac{u_{Sx}}{\sigma L_S T_R}$$

$$0 = -\frac{d^2 i_{Sy}}{dt^2} - \frac{di_{Sx}}{dt} n_p\omega + \frac{1}{\sigma L_S}\frac{du_{Sy}}{dt} - (\gamma + \frac{1}{T_R})\frac{di_{Sy}}{dt} - i_{Sy}(-\frac{\beta M}{T_R^2} + \frac{\gamma}{T_R}) - i_{Sx} n_p\omega(\frac{1}{T_R} + \frac{\beta M}{T_R}) + \frac{u_{Sy}}{\sigma L_S T_R}.$$

Online Identification of T_R and R_S

Substitute $\gamma = R_S/(\sigma L_S) + \beta M/T_R$ into

$$0 = -\frac{d^2 i_{Sx}}{dt^2} + \frac{di_{Sy}}{dt} n_p\omega + \frac{1}{\sigma L_S}\frac{du_{Sx}}{dt} - (\gamma + \frac{1}{T_R})\frac{di_{Sx}}{dt} - i_{Sx}(-\frac{\beta M}{T_R^2} + \frac{\gamma}{T_R}) + i_{Sy} n_p\omega(\frac{1}{T_R} + \frac{\beta M}{T_R}) + \frac{u_{Sx}}{\sigma L_S T_R}$$

$$0 = -\frac{d^2 i_{Sy}}{dt^2} - \frac{di_{Sx}}{dt} n_p\omega + \frac{1}{\sigma L_S}\frac{du_{Sy}}{dt} - (\gamma + \frac{1}{T_R})\frac{di_{Sy}}{dt} - i_{Sy}(-\frac{\beta M}{T_R^2} + \frac{\gamma}{T_R}) - i_{Sx} n_p\omega(\frac{1}{T_R} + \frac{\beta M}{T_R}) + \frac{u_{Sy}}{\sigma L_S T_R}$$

to obtain

$$0 = -\frac{d^2 i_{Sx}}{dt^2} + \frac{di_{Sy}}{dt} n_p\omega + \frac{1}{\sigma L_S}\frac{du_{Sx}}{dt} - \left(\frac{R_S}{\sigma L_S} + \left(\frac{\beta M + 1}{T_R}\right)\right)\frac{di_{Sx}}{dt} - i_{Sx}\left(\frac{R_S}{T_R}\frac{1}{\sigma L_S}\right) + i_{Sy} n_p\omega\left(\frac{\beta M + 1}{T_R}\right) + \frac{u_{Sx}}{\sigma L_S T_R}$$

$$0 = -\frac{d^2 i_{Sy}}{dt^2} - \frac{di_{Sx}}{dt} n_p\omega + \frac{1}{\sigma L_S}\frac{du_{Sy}}{dt} - \left(\frac{R_S}{\sigma L_S} + \left(\frac{\beta M + 1}{T_R}\right)\right)\frac{di_{Sy}}{dt} - i_{Sy}\left(\frac{R_S}{T_R}\frac{1}{\sigma L_S}\right) - i_{Sx} n_p\omega\left(\frac{\beta M + 1}{T_R}\right) + \frac{u_{Sy}}{\sigma L_S T_R}.$$

Regressor Model

Rewrite

$$0=-\frac{d^2 i_{Sx}}{dt^2}+\frac{di_{Sy}}{dt}n_p\omega+\frac{1}{\sigma L_S}\frac{du_{Sx}}{dt}-\left(R_S/(\sigma L_S)+(\beta M+1)/T_R\right)\frac{di_{Sx}}{dt}-i_{Sx}\left(\frac{R_S}{T_R}\frac{1}{\sigma L_S}\right)$$
$$+i_{Sy}n_p\omega((\beta M+1)/T_R)+\frac{u_{Sx}}{\sigma L_S T_R}$$
$$0=-\frac{d^2 i_{Sy}}{dt^2}-\frac{di_{Sx}}{dt}n_p\omega+\frac{1}{\sigma L_S}\frac{du_{Sy}}{dt}-\left(R_S/(\sigma L_S)+(\beta M+1)/T_R\right)\frac{di_{Sy}}{dt}-i_{Sy}\left(\frac{R_S}{T_R}\frac{1}{\sigma L_S}\right)$$
$$-i_{Sx}n_p\omega(\beta M+1)/T_R+\frac{u_{Sy}}{\sigma L_S T_R}.$$

in regressor form as

$$\underbrace{\begin{bmatrix}\dfrac{d^2 i_{Sx}}{dt^2}-\dfrac{di_{Sy}}{dt}n_p\omega-\dfrac{1}{\sigma L_S}\dfrac{du_{Sx}}{dt}\\ \dfrac{d^2 i_{Sy}}{dt^2}+\dfrac{di_{Sx}}{dt}n_p\omega-\dfrac{1}{\sigma L_S}\dfrac{du_{Sy}}{dt}\end{bmatrix}}_{y(t)}=$$

$$\underbrace{\begin{bmatrix}-\dfrac{di_{Sx}}{dt}\dfrac{1}{\sigma L_S} & (\beta M+1)\left(-\dfrac{di_{Sx}}{dt}+i_{Sy}n_p\omega\right)+\dfrac{u_{Sx}}{\sigma L_S} & -\dfrac{i_{Sx}}{\sigma L_S}\\ -\dfrac{di_{Sy}}{dt}\dfrac{1}{\sigma L_S} & (\beta M+1)\left(-\dfrac{di_{Sy}}{dt}-i_{Sx}n_p\omega\right)+\dfrac{u_{Sy}}{\sigma L_S} & -\dfrac{i_{Sy}}{\sigma L_S}\end{bmatrix}}_{W(t)}\underbrace{\begin{bmatrix}R_S\\ 1/T_R\\ R_S/T_R\end{bmatrix}}_{K}$$

Regressor Model

From previous slide

$$\underbrace{\begin{bmatrix} \frac{d^2 i_{Sx}}{dt^2} - \frac{di_{Sy}}{dt} n_p\omega - \frac{1}{\sigma L_S}\frac{du_{Sx}}{dt} \\ \frac{d^2 i_{Sy}}{dt^2} + \frac{di_{Sx}}{dt} n_p\omega - \frac{1}{\sigma L_S}\frac{du_{Sy}}{dt} \end{bmatrix}}_{y(t)} =$$

$$\underbrace{\begin{bmatrix} -\frac{di_{Sx}}{dt}\frac{1}{\sigma L_S} & (\beta M + 1)\left(-\frac{di_{Sx}}{dt} + i_{Sy} n_p\omega\right) + \frac{u_{Sx}}{\sigma L_S} & -\frac{i_{Sx}}{\sigma L_S} \\ -\frac{di_{Sy}}{dt}\frac{1}{\sigma L_S} & (\beta M + 1)\left(-\frac{di_{Sy}}{dt} - i_{Sx} n_p\omega\right) + \frac{u_{Sy}}{\sigma L_S} & -\frac{i_{Sy}}{\sigma L_S} \end{bmatrix}}_{W(t)} \underbrace{\begin{bmatrix} R_S \\ 1/T_R \\ R_S/T_R \end{bmatrix}}_{K}$$

or

$$y(t) = W(t)K$$

where

$$K = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} \triangleq \begin{bmatrix} R_S \\ 1/T_R \\ R_S/T_R \end{bmatrix},$$

- This is *overparameterized* as

$$K_3 = K_1 K_2.$$

Nonlinear Least-Squares Identification

- T the sample period, nT the time of the n-th measurement.
- Write

$$y(nT) = W(nT)K$$

  where

$$K = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} \triangleq \begin{bmatrix} R_S \\ 1/T_R \\ R_S/T_R \end{bmatrix},$$

- The *residual error* associated to a parameter vector K by

$$E^2(K) = \sum_{n=1}^{N} \left| y(nT) - W(nT)K \right|^2.$$

- The least-squares estimate K^* is that value that minimizes $E^2(K)$.
- Ignoring the *overparameterization* problem, K^* is given by .

$$K^* = \left[\sum_{n=1}^{N} W^T(nT)W(nT) \right]^{-1} \left[\sum_{n=1}^{N} W^T(nT)y(nT) \right].$$

- This turns out to be *ill-conditioned* in that small changes in data $W(nT), y(nT)$ can cause large changes in K^*.

Nonlinear Least-Squares Identification

- Deal with the overparameterization problem.
- The error

$$E^2(K) = \sum_{n=1}^{N} \left| y(nT) - W(nT)K \right|^2 = R_y - 2R_{Wy}^T K + K^T R_W K$$

where

$$R_y \triangleq \sum_{n=1}^{N} y^T(nT)y(nT), \quad R_{Wy} \triangleq \sum_{n=1}^{N} W^T(nT)y(nT), \quad R_W \triangleq \sum_{n=1}^{N} W^T(nT)W(nT).$$

- Define the new error function $E_p^2(K_1, K_2)$ as

$$E_p^2(K_1, K_2) \triangleq \sum_{n=1}^{N} \left| y(nT) - W(nT)K \right|^2_{K_3=K_1K_2} = R_y - 2R_{Wy}^T K\Big|_{K_3=K_1K_2} + K^T R_W K\Big|_{K_3=K_1K_2}.$$

- The minimum occurs at $K_p^* = \begin{bmatrix} K_1^* & K_2^* \end{bmatrix}$ which is a solution of the two extrema polynomial equations

$$\begin{aligned} p_1(K_1, K_2) &\triangleq \frac{\partial E_p^2(K_1, K_2)}{\partial K_1} = 0 \\ p_2(K_1, K_2) &\triangleq \frac{\partial E_p^2(K_1, K_2)}{\partial K_2} = 0. \end{aligned}$$

Nonlinear Least-Squares Identification

The degrees of the polynomials $p_1(K_1, K_2), p_2(K_1, K_2)$

$$p_1(K_1, K_2) \triangleq \frac{\partial E_p^2(K_1, K_2)}{\partial K_1} = 0$$

$$p_2(K_1, K_2) \triangleq \frac{\partial E_p^2(K_1, K_2)}{\partial K_2} = 0.$$

are

	deg K_1	deg K_2
$p_1(K_1, K_2)$	1	2
$p_2(K_1, K_2)$	2	1

Rewrite the two polynomials as

$$p_1(K_1, K_2) = a_1(K_2)K_1 + a_0(K_2)$$

$$p_2(K_1, K_2) = b_2(K_2)K_1^2 + b_1(K_2)K_1 + b_0(K_2).$$

Nonlinear Least-Squares Identification

Need to solve

$$p_1(K_1, K_2) = a_1(K_2)K_1 + a_0(K_2) = 0$$
$$p_2(K_1, K_2) = b_2(K_2)K_1^2 + b_1(K_2)K_1 + b_0(K_2) = 0.$$

From $p_1(K_1, K_2) = 0$ we have

$$K_1 = -a_0(K_2)/a_1(K_2).$$

Substitute this into $p_2(K_1, K_2) = 0$ to get

$$b_2(K_2)a_0^2(K_2)/a_1^2(K_2) - b_1(K_2)a_0(K_2)/a_1(K_2) + b_0(K_2) = 0.$$

Multiply through $a_1^2(K_2)$ to get the *resultant polynomial*

$$r(K_2) = a_0^2(K_2)b_2(K_2) - a_0(K_2)a_1(K_2)b_1(K_2) + a_1^2(K_2)b_0(K_2)$$

It turns out that $\deg_{K_2}\{r\} = 5$.

Nonlinear Least-Squares Identification

- Let $K_2^{(1)}, ..., K_2^{(5)}$ be the five roots of

$$r(K_2) = a_0^2(K_2)b_2(K_2) - a_0(K_2)a_1(K_2)b_1(K_2) + a_1^2(K_2)b_0(K_2) = 0.$$

- For each $K_2^{(i)}$ the corresponding value for K_1 is

$$K_1^{(i)} = -a_0(K_2^{(i)})/a_1(K_2^{(i)}).$$

- Check which of the five pairs $(K_1^{(i)}, K_2^{(i)})$ gives the minimum value of $E_p^2(K_1, K_2)$.
- This is then

$$K_p^* = \begin{bmatrix} K_1^* \\ K_2^* \end{bmatrix}.$$