ECE 697 Modeling and High-Performance Control of Electric Machines
HW 1 Solutions
Spring 2022

Problem 1 *Faraday's Law*

Consider the figure below where the magnet is *moving up* into a square planar loop of copper wire.

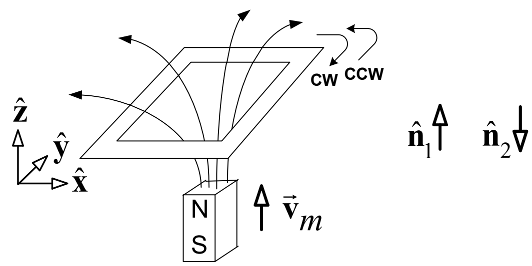

Figure 1: Induced emf in a loop due to a moving magnet.

(a) Using the normal $\hat{\mathbf{n}}_1$, roughly sketch the flux $\phi_1(t) = \int_S \vec{\mathbf{B}} \cdot (\hat{\mathbf{n}}_1 dS)$ as a function of t while the magnet is below the copper loop. Is the flux in the loop produced by the magnet increasing or decreasing?

Using the normal $\vec{\mathbf{n}}_1$, $\phi_1(t)$ is positive and increasing so that $-d\phi_1(t)/dt < 0$.

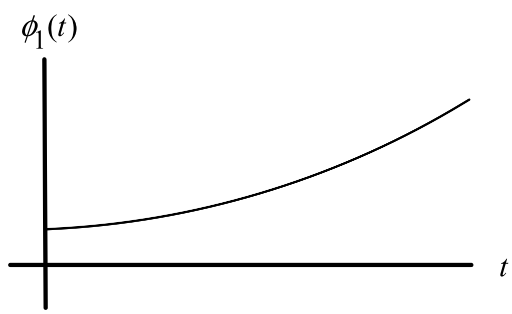

Figure 2: Flux $\phi_1(t)$ versus time.

(b) Using the normal $\hat{\mathbf{n}}_1$, what is the direction of positive travel around the surface whose boundary is the loop (clockwise or counterclockwise)?

The positive direction of travel around the surface is CCW.

(c) What is the direction of the induced current in Figure 1 (clockwise or counterclockwise)? Let $\psi_1(i) = \int_S \vec{\mathbf{B}}_i \cdot (\hat{\mathbf{n}}_1 dS)$ be the flux in the loop due to the induced current. Is $\psi_1(i)$ positive or negative while the magnet is below the loop? Is $\psi_1(i)$ increasing or decreasing while the magnet is below the loop, but moving up?

The induced voltage will cause current to flow in the CW direction as $-d\phi_1(t)/dt < 0$. $\psi_1(i) < 0$. $\psi_1(i)$ is decreasing (becoming more negative) opposing the increasing $\phi_1(t)$ (Lenz's law).

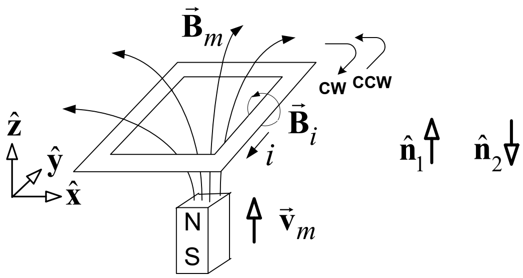

Figure 3: The arrow $\rightarrow$ next to the current i is the actual direction of the current and not a sign convention.

(d) Using the normal $\hat{\mathbf{n}}_2$, roughly sketch the flux $\phi_2(t) = \int_S \vec{\mathbf{B}} \cdot (\hat{\mathbf{n}}_2 dS)$ as a function of t while the magnet is below the copper loop. Is the flux in the loop produced by the magnet increasing or decreasing?

Using the normal $\vec{\mathbf{n}}_2$, $\phi_2(t)$ is negative and decreasing (becoming more negative) so that $-d\phi_2(t)/dt > 0$.

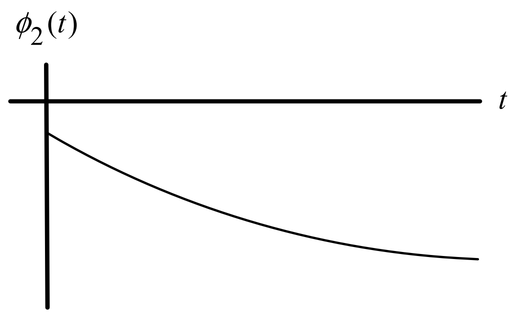

Figure 4: Flux $\phi_2(t)$ versus time.

(e) Using the normal $\hat{\mathbf{n}}_2$, what is the direction of positive travel around the surface whose boundary is the loop (clockwise or counterclockwise)?

The positive direction of travel around the surface is CW.

(f) What is the direction of the induced current in Figure 1 (clockwise or counterclockwise)? Let $\psi_2(i) = \int_S \vec{\mathbf{B}}_i \cdot (\hat{\mathbf{n}}_2 dS)$ be the flux in the loop due to the induced current. Is $\psi_2(i)$ positive or negative while the magnet is below the loop? Is $\psi_2(i)$ increasing or decreasing while the magnet is below the loop, but moving up?

The induced voltage will cause current to flow in the CW direction as $-d\phi_2(t)/dt > 0$. (Note that this current is exactly the same as that computed using the normal $\vec{\mathbf{n}}_1$ in part (*c*) as it must because it is the same physical situation.) $\psi_2(i) > 0$. $\psi_2(i)$ is increasing opposing the decreasing $\phi_2(t)$ (Lenz's law).

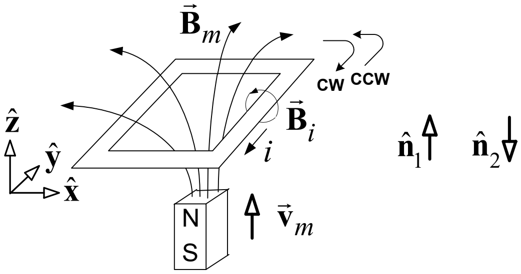

Figure 5: The arrow → next to the current i is the actual direction of the current and not a sign convention.

Problem 2 *Faraday's Law*

The magnet is *moving down* away from a square planar loop of copper wire.

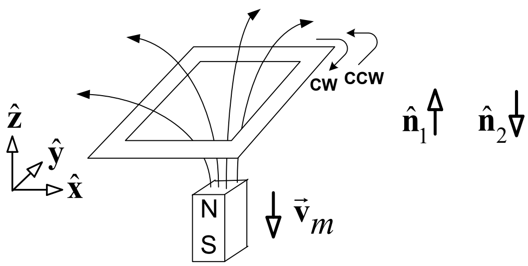

Figure 6: Induced emf in a loop due to a moving magnet.

(a) Using the normal $\hat{\mathbf{n}}_1$, roughly sketch the flux $\phi_1(t) = \int_S \vec{\mathbf{B}} \cdot (\hat{\mathbf{n}}_1 dS)$ as a function of t. Is the flux in the loop produced by the magnet increasing or decreasing?

Using the normal $\vec{\mathbf{n}}_1$, $\phi_1(t)$ is positive and *decreasing* so that $-d\phi_1(t)/dt > 0$.

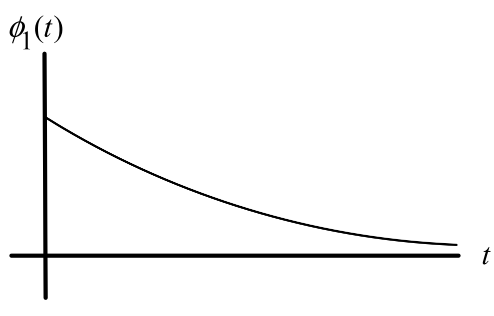

Figure 7: Flux $\phi_1(t)$ versus time.

(b) Using the normal $\hat{\mathbf{n}}_1$, what is the direction of positive travel around the surface whose boundary is the loop (clockwise or counterclockwise)?

The positive direction of travel around the surface is CCW.

(c) What is the direction of the induced current in Figure 6 (clockwise or counterclockwise)? Let $\psi_1(i) = \int_S \vec{\mathbf{B}}_i \cdot (\hat{\mathbf{n}}_1 dS)$ be the flux in the loop due to the induced current. Is $\psi_1(i)$ positive or negative while the magnet is below the loop? Is $\psi_1(i)$ increasing or decreasing while the magnet is below the loop but moving down?

The induced voltage will cause current to flow in the CCW direction as $-d\phi_1(t)/dt > 0$. $\psi_1(i) > 0$. So $\psi_1(i)$ is adding positive flux to the loop opposing the decreasing flux $\phi_1(t)$ (Lenz's law).

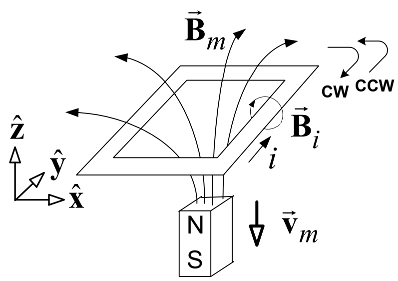

Figure 8: The arrow $\rightarrow$ next to the current i is the actual direction of the current and not a sign convention.

(d) Using the normal $\hat{\mathbf{n}}_2$, roughly sketch the flux $\phi_2(t) = \int_S \mathbf{B} \cdot (\hat{\mathbf{n}}_2 dS)$ as a function of t. Is the flux in the loop produced by the magnet increasing or decreasing?

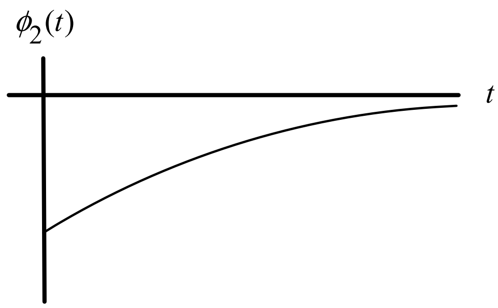

Figure 9: Flux $\phi_2(t)$ versus time.

Using the normal $\vec{\mathbf{n}}_2$, $\phi_2(t)$ is negative and *increasing* (becoming less negative) so that $-d\phi_2(t)/dt < 0$.

(e) Using the normal $\hat{\mathbf{n}}_2$, what is the direction of positive travel around the surface whose boundary is the loop (clockwise or counterclockwise)?

The positive direction of travel around the surface is CW and the induced voltage will cause current to flow in the CCW direction as $-d\phi_2(t)/dt < 0$. (Note that this current is exactly the same as that computed using the normal $\vec{\mathbf{n}}_1$ in part (c) as it must because it is the same physical situation.)

(f) What is the direction of the induced current in Figure 6 (clockwise or counterclockwise)? Let $\psi_2(i) = \int_S \vec{\mathbf{B}}_i \cdot (\hat{\mathbf{n}}_2 dS)$ be the flux in the loop due to the induced current. Is $\psi_2(i)$ positive or negative while the magnet is below the loop? Is $\psi_2(i)$ increasing or decreasing while the magnet is below the loop, but moving down?

The induced voltage will cause current to flow in the counterclockwise direction which is the same direction found in part (c). $\psi_2(i) < 0$. $\psi_2(i)$ is adding negative flux in the loop opposing the increasing $\phi_2(t)$ (Lenz's law).

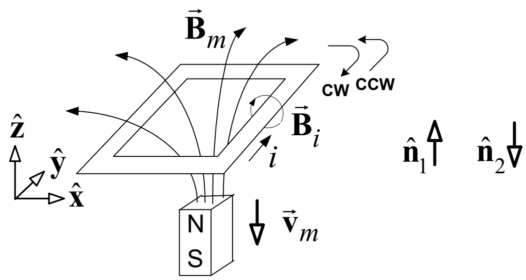

Figure 10: The arrow $\rightarrow$ next to the current i is the actual direction of the current and not a sign convention.

Problem 3 *The Linear DC Motor*

Consider the simple linear DC motor analyzed in the text.

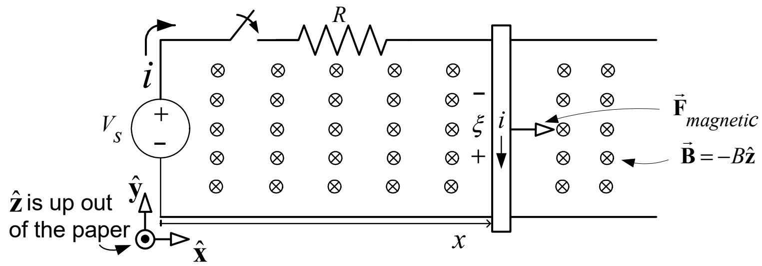

Figure 11:

(a) With the normal to the surface enclosed by the loop taken to be $\vec{\mathbf{n}} = -\hat{\mathbf{z}}$, the flux through the surface is

$$\phi(t) = \int_0^\ell \int_0^x (-B\hat{\mathbf{z}}) \cdot (-dxdy\hat{\mathbf{z}}) = B\ell x.$$

(b) The direction of positive travel around this flux surface is clockwise as viewed from above.

(c) The induced emf ξ in the loop is

$$\xi = -\frac{d\phi}{dt} = -B\ell\frac{dx}{dt} = -B\ell v.$$

(d) The source voltage V_S and ξ have the *same* sign convention so ξ is now negative opposing the source voltage. With ξ drawn next to the sliding bar, there should be a "−" sign above ξ and a "+" sign below it. The equations describing the linear motor are

$$V_S = B\ell v + Ri$$
$$m_\ell\frac{d^2x}{dt^2} = i\ell B - f\omega.$$

Eliminating i gives

$$m_\ell\frac{d^2x}{dt^2} = \ell B\frac{V_S - B\ell dx/dt}{R} - f\omega$$

or

$$m_\ell\frac{d^2x}{dt^2} + (\ell^2B^2/R + f)\frac{dx}{dt} = \frac{\ell B}{R}V_S.$$

Problem 4 *The Linear DC Motor*

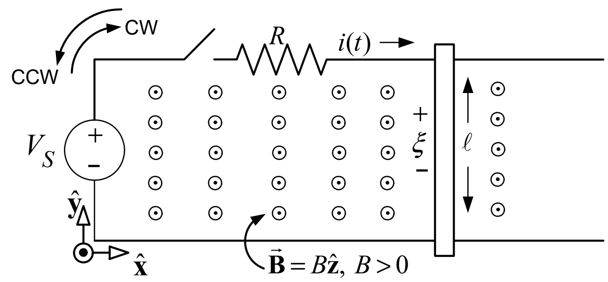

Figure 12:

Consider the simple linear motor in the figure where the magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{z}}$ ($B > 0$) is up out of the page. Closing the switch causes a current to flow in the CW direction.

(a) The magnetic force $\vec{\mathbf{F}}_{magnetic}$ on the sliding bar is $Bi\ell\,(-\hat{x})$ so the bar is moving in the $-\hat{x}$ direction.

(b) With the normal to the surface enclosed by the loop taken to be $\vec{\mathbf{n}} = \hat{\mathbf{z}}$, the flux through the surface is

$$\phi(t) = \int_0^\ell \int_0^x (B\hat{\mathbf{z}}) \cdot (dxdy\hat{\mathbf{z}}) = B\ell x.$$

(c) The direction of positive travel around this flux surface is CCW. The induced emf ξ in the loop is

$$\xi = -\frac{d\phi}{dt} = -B\ell\frac{dx}{dt} = -B\ell v > 0 \text{ as } v < 0.$$

(d) Counterclockwise

(e) The source voltage V_S and ξ have opposite sign conventions. ξ is positive and thus opposing the source voltage. (That is, $\xi > 0$ acts to push current in the counterclockwise direction). There should be a "+" sign above ξ and a "−" sign below it as shown above.

Problem 5 *Back Emf in the Single-Loop Motor*

Consider the single-loop motor with the flux surface as indicated. A voltage source connected to the brushes is forcing current down side a ($\otimes$) and up side a' ($\odot$).

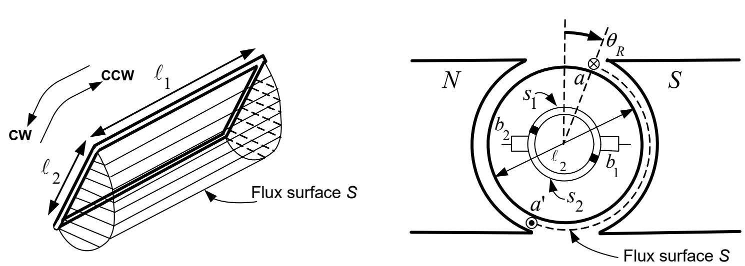

Figure 13:

(a) With the motor at the angular position θ_R such that $0 < \theta_R < \pi$ and, using the **inward** normal (i.e.,

$\vec{\mathbf{n}} = -\hat{\mathbf{r}}$), the flux through the surface is $\left(d\vec{\mathbf{S}} = -(\ell_2/2)d\theta dz\hat{\mathbf{r}}\right)$

$$\begin{aligned}\phi(\theta_R) &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \\ &= \int_0^{\ell_1}\int_{\theta=\theta_R}^{\theta=\pi} (B\hat{\mathbf{r}}) \cdot (-\frac{\ell_2}{2}d\theta dz\hat{\mathbf{r}}) + \int_0^{\ell_1}\int_{\theta=\pi}^{\theta=\pi+\theta_R} (-B\hat{\mathbf{r}}) \cdot (-\frac{\ell_2}{2}d\theta dz\hat{\mathbf{r}}) \\ &= \int_0^{\ell_1}\int_{\theta=\theta_R}^{\theta=\pi} -B\frac{\ell_2}{2}d\theta dz + \int_0^{\ell_1}\int_{\theta=\pi}^{\theta=\pi+\theta_R} B\frac{\ell_2}{2}d\theta dz \\ &= -\frac{\ell_1\ell_2 B}{2}(\pi - \theta_R) + \frac{\ell_1\ell_2 B}{2}\theta_R \\ &= \ell_1\ell_2 B\left(\theta_R - \frac{\pi}{2}\right).\end{aligned}$$

This derivation is based on the fact that the $\vec{\mathbf{B}}$ field is directed radially outward over the length $(\ell_2/2)(\pi - \theta_R)$ and radially inward over the length $(\ell_2/2)\theta_R$. Similarly, it follows that $\phi(\theta_R) = \ell_1\ell_2 B\left(\theta_R - \pi - \frac{\pi}{2}\right)$ for $\pi < \theta_R < 2\pi$. In general,

$$\phi(\theta_R) = \ell_1\ell_2 B\left(\theta_R \bmod \pi - \frac{\pi}{2}\right)$$

is the correct expression for any angle θ_R.

(b) The positive direction of travel around the flux surface is CCW so that if $\xi > 0$, it act to push current in the CCW direction.

(c) The induced emf in the rotor loop is

$$\xi = -\frac{d\phi}{dt} = -(\ell_1\ell_2 B)\frac{d\theta_R}{dt} = -K_b\omega_R.$$

V_S and ξ now have the same sign convention and the total sum of these two voltages in the loop is then $V_S + \xi = V_S - K_b\omega_R$. We must get the same physical result as the example in the text. Thus ξ is now negative because the sign convention in this problem is opposite to that used in the text.

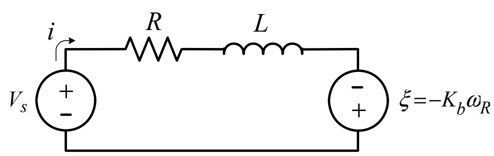

Figure 14:

Problem 6 *Gauss's law and Conservation of Flux*

The closed flux surface encloses a half-cylindrical volume. This surface is composed of the flat planar surface, the half-disk in the front, the half-disk in the back, and the half-cylindrical surface. Applying Gauss's law to the closed surface that encloses the half-cylindrical volume, it follows that (using an *outward* normal on the surface of the half-cylindrical volume).

$$\begin{aligned}\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} &= \int_{\text{flat\_planar\_surface}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \int_{\text{half\_cylindrical\_surface}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \int_{\text{half\_disk\_front}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \int_{\text{half\_disk\_back}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \\ &= \int_{\text{flat\_planar\_surface}} \vec{\mathbf{B}} \cdot (dS\hat{\mathbf{n}}_{\text{outward}}) + \int_{\text{half\_cylindrical\_surface}} \vec{\mathbf{B}} \cdot (rd\theta dz\hat{\mathbf{r}}) \\ &\quad + \int_{\text{half\_disk\_front}} \vec{\mathbf{B}} \cdot (-rd\theta dr\hat{\mathbf{z}}) + \int_{\text{half\_disk\_back}} \vec{\mathbf{B}} \cdot (rd\theta dr\hat{\mathbf{z}}) \\ &= 0.\end{aligned}$$

Rearranging, this becomes

$$-\int_{\text{flat\_planar\_surface}} \vec{\mathbf{B}} \cdot (dS\hat{\mathbf{n}}_{\text{outward}}) = \int_{\text{half\_cylindrical\_surface}} \vec{\mathbf{B}} \cdot (rd\theta dz\hat{\mathbf{r}}) + \int_{\text{half\_disk\_front}} \vec{\mathbf{B}} \cdot (-rd\theta dr\hat{\mathbf{z}}) + \int_{\text{half\_disk\_back}} \vec{\mathbf{B}} \cdot (rd\theta dr\hat{\mathbf{z}})$$

or

$$\int_{\text{flat\_planar\_surface}} \vec{\mathbf{B}} \cdot (dS\hat{\mathbf{n}}_{\text{inward}}) = \int_{\text{half\_cylindrical\_surface}} \vec{\mathbf{B}} \cdot (rd\theta dz\hat{\mathbf{r}}) + \int_{\text{half\_disk\_front}} \vec{\mathbf{B}} \cdot (-rd\theta dr\hat{\mathbf{z}}) + \int_{\text{half\_disk\_back}} \vec{\mathbf{B}} \cdot (rd\theta dr\hat{\mathbf{z}})\,.$$

Taking the $\vec{\mathbf{B}}$ field to be zero on the two half-disks, it follows that

$$\int_{\text{flat\_planar\_surface}} \vec{\mathbf{B}} \cdot (dS\hat{\mathbf{n}}_{\text{inward}}) = \int_{\text{half\_cylindrical\_surface}} \vec{\mathbf{B}} \cdot (rd\theta dz\hat{\mathbf{r}})\,.$$

An important point here is that taking $\hat{\mathbf{n}}_{\text{inward}}$ on the flat planar surface to compute the flux and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ on the half cylindrical surface to compute the flux leads to both choices having the *same* positive direction of travel around the loop.