Modeling and High-Performance Control of Electric Machines

Chapter 6 Mathematical Models of AC Machines

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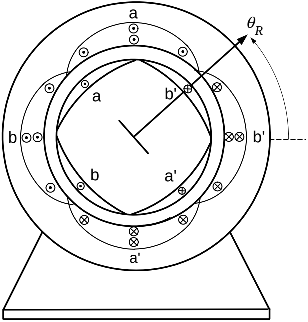
Mathematical Models of AC Machines

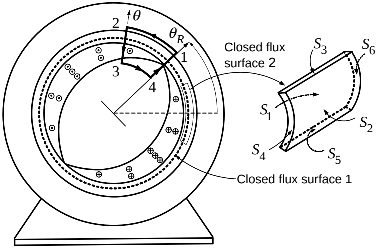
- **The Magnetic Field** $\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, r, \theta - \theta_R)$
- **Leakage**
- **Flux Linkages in AC Machines**
- **Torque Production in AC Machines**
- **Mathematical Model of a Sinusoidally Wound Induction Machine**
- **Total Leakage Factor**
- **The Squirrel Cage Rotor**
- **Induction Machine With Multiple Pole Pairs**
- **Mathematical Model of a Wound Rotor Synchronous Machine**
- **Mathematical Model of a PM Synchronous Machine**
- **The Stator and Rotor Magnetic Fields of an IM Rotate Synchronously***
- **Torque, Energy, and Co-energy*** (no slides)

Two-Pole Two-Phase Sinusoidally Wound Machine



- Symmetric two-pole two-phase machine.
- Two sinusoidally wound stator phases with N_S turns each and 90° apart.
- R_S denotes the resistance of each stator phase.
- Two sinusoidally wound rotor phases with N_R turns each and 90° apart.
- R_R denotes the resistance of each rotor phase.
- Find expressions for $\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, r, \theta)$ and $\vec{\mathbf{B}}_S(i_{Sa}, i_{Sb}, r, \theta)$ in the air gap.
- Compute the flux linkages and torques to obtain the mathematical model.

The Magnetic Field $\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, r, \theta - \theta_R)$



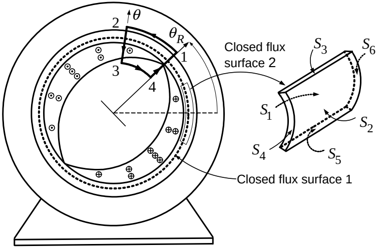
- Compute $\vec{\mathbf{B}}_{Ra}$ in the air gap produced by **just** i_{Ra}.
- The number of turns of rotor phase a between θ and $\theta + d\theta$ is $\dfrac{N_R}{2}\left|\sin(\theta - \theta_R)\right| d\theta$.

$$\begin{aligned}
\int_4^1 \vec{\mathbf{H}}_{Ra} \cdot d\vec{\boldsymbol{\ell}} + \int_2^3 \vec{\mathbf{H}}_{Ra} \cdot d\vec{\boldsymbol{\ell}} &= \int_{\theta'=\theta_R}^{\theta'=\theta} i_{Ra} \frac{N_R}{2} \sin(\theta' - \theta_R) d\theta' \\
\int_{\ell=0}^{\ell=g} H_{Ra}(i_{Ra}, \theta_R)\hat{\mathbf{r}} \cdot (d\ell\hat{\mathbf{r}}) + \int_{\ell=0}^{\ell=g} H_{Ra}(i_{Ra}, \theta)\hat{\mathbf{r}} \cdot (-d\ell\hat{\mathbf{r}}) &= -i_{Ra}\frac{N_R}{2}\cos(\theta - \theta_R) + i_{Ra}\frac{N_R}{2} \\
H_{Ra}(i_{Ra}, \theta_R)g - H_{Ra}(i_{Ra}, \theta)g &= -i_{Ra}\frac{N_R}{2}\cos(\theta - \theta_R) + i_{Ra}\frac{N_R}{2}.
\end{aligned}$$

or

$$H_{Ra}(i_{Ra}, \theta) = i_{Ra}\frac{N_R}{2g}\cos(\theta - \theta_R) + H_{Ra}(i_{Ra}, \theta_R) - i_{Ra}\frac{N_R}{2g}.$$

The Magnetic Field $\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, r, \theta - \theta_R)$



- $H_{Ra}(i_{Ra}, \theta) = i_{Ra}\frac{N_R}{2g}\cos(\theta - \theta_R) + H_{Ra}(i_{Ra}, \theta_R) - i_{Ra}\frac{N_R}{2g}$.
- $H_{Ra}(i_{Ra}, \theta), H_{Ra}(i_{Ra}, \theta_R)$ are **unknown**.
- Applying $\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$ to the **closed surface** 1 gives $H_{Ra}(i_{Ra}, \theta_R) = i_{Ra}(N_R/2g)$.

$$H_{Ra}(i_{Ra}, \theta) = \frac{N_R}{2g} i_{Ra}\cos(\theta - \theta_R) \text{ and } B_{Ra}(i_{Ra}, \theta) = \frac{\mu_0 N_R}{2g} i_{Ra}\cos(\theta - \theta_R).$$

- Applying Ampère's law, we assumed $\vec{\mathbf{H}}_{Ra}$ was **constant** across the air gap.
- Applying $\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$ to the **closed surface** 2 requires the factor r_R/r, i.e.

$$\vec{\mathbf{B}}_{Ra}(i_{Ra}, r, \theta - \theta_R) = B_{Ra}(i_{Ra}, r, \theta - \theta_R)\hat{\mathbf{r}} \triangleq \frac{\mu_0 N_R}{2g}\frac{r_R}{r} i_{Ra}\cos(\theta - \theta_R)\hat{\mathbf{r}}.$$

The Magnetic Field $\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, r, \theta - \theta_R)$

Similarly

$$\begin{aligned}
\vec{\mathbf{B}}_{Rb}(i_{Rb}, r, \theta - \theta_R) = B_{Rb}(i_{Rb}, r, \theta - \theta_R)\hat{\mathbf{r}} &= \frac{\mu_0 N_R}{2g}\frac{r_R}{r} i_{Rb}\cos(\theta - \pi/2 - \theta_R)\hat{\mathbf{r}} \\
&= \frac{\mu_0 N_R}{2g}\frac{r_R}{r} i_{Rb}\sin(\theta - \theta_R)\hat{\mathbf{r}}.
\end{aligned}$$

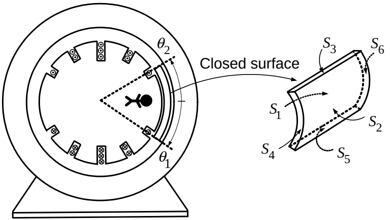
The **total** rotor magnetic field is

$$\begin{aligned}
\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, r, \theta - \theta_R) &= \vec{\mathbf{B}}_{Ra}(i_{Ra}, r, \theta - \theta_R) + \vec{\mathbf{B}}_{Rb}(i_{Rb}, r, \theta - \theta_R) \\
&= \frac{\mu_0 N_R r_R}{2g}\frac{1}{r}\Big(i_{Ra}\cos(\theta - \theta_R) + \ i_{Rb}\sin(\theta - \theta_R)\Big)\hat{\mathbf{r}}.
\end{aligned}$$

The **total** stator magnetic field is

$$\vec{\mathbf{B}}_S(i_{Sa}, i_{Sb}, r, \theta) = \vec{\mathbf{B}}_{Sa}(i_{Sa}, r, \theta) + \vec{\mathbf{B}}_{Sb}(i_{Sb}, r, \theta) = \frac{\mu_0 N_S r_R}{2g}\frac{1}{r}\Big(i_{Sa}\cos(\theta) + \ i_{Sb}\sin(\theta)\Big)\hat{\mathbf{r}}.$$

Leakage

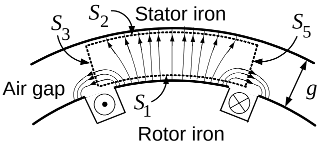


- We have assumed the magnetic field is **radially** directed in the air gap.
- On the closed surface we defined **surface element vectors** by

$$d\vec{\mathbf{S}} = \begin{cases} -r_R\, d\theta dz\hat{\mathbf{r}} & \text{on } S_1 \\ r_S\, d\theta dz\hat{\mathbf{r}} & \text{on } S_2 \\ drdz\hat{\boldsymbol{\theta}} & \text{on } S_3 \\ rd\theta dr\hat{\mathbf{z}} & \text{on } S_4 \\ -drdz\hat{\boldsymbol{\theta}} & \text{on } S_5 \\ -rd\theta dr\hat{\mathbf{z}} & \text{on } S_6. \end{cases}$$

- We took $\vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} \equiv 0$ on the surfaces S_3, S_4, S_5, and S_6.
- Then $\oint_S \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = 0$ **required** a $1/r$ dependence by $\vec{\mathbf{B}}_R$ in the air gap.

Leakage



In a real machine:

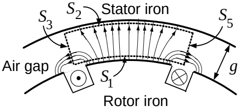
- The slots have finite dimensions.
- The air gap is of finite size.
- The machine has a finite length ℓ_1.

Effect on $\vec{\mathbf{B}}_R$: The lines of $\vec{\mathbf{B}}_R$ due to the rotor currents are shown.
Close to the slot opening the lines of $\vec{\mathbf{B}}_R$ tend to circle around the rotor winding.
$\vec{\mathbf{B}}_R$ spreads out in the $\hat{\boldsymbol{\theta}}$ direction due to the finite length g.
$\vec{\mathbf{B}}_R$ spreads out in the $\hat{\mathbf{z}}$ direction at the rotor ends due to the finite axial length ℓ_1.

Consequence:

- The radial $\vec{\mathbf{B}}_R$ field on the surface S_2 actually **decreases** a little more than $1/r$.

Leakage



- To simplify the discussion, assume **no** spreading in axial (z) direction.
- This means $\int_{S_4} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = \int_{S_6} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = 0$.
- $\oint \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = \int_{S_1} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} + \int_{S_2} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} + \int_{S_3} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} + \int_{S_5} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = 0$.
- Rearranging: $\int_{S_1} \vec{\mathbf{B}}_R \cdot \left(-d\vec{\mathbf{S}}\right) = \int_{S_2} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} + \int_{S_3} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} + \int_{S_5} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}}$ or

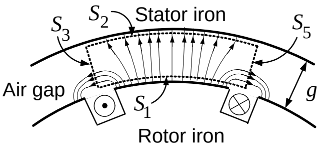
$$\int_{S_1} B_R(r_R, \theta, t)\hat{\mathbf{r}} \cdot (r_R d\theta dz \hat{\mathbf{r}}) = \int_{S_2} B_R(r_S, \theta, t)\hat{\mathbf{r}} \cdot (r_S d\theta dz \hat{\mathbf{r}}) + \underbrace{\int_{S_3} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} + \int_{S_5} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}}}_{\text{Leakage flux}}.$$

- S_1 is at the surface of the rotor and $-d\vec{\mathbf{S}} = r_R d\theta dz \hat{\mathbf{r}}$ on S_1.
- **If** there was no spreading in the $\hat{\boldsymbol{\theta}}$ direction, then

$$\int_{S_1} B_R(r_R, \theta, t)\hat{\mathbf{r}} \cdot (r_R d\theta dz \hat{\mathbf{r}}) = \int_{S_2} B_R(r_S, \theta, t)\hat{\mathbf{r}} \cdot (r_S d\theta dz \hat{\mathbf{r}}) .$$

- $\vec{\mathbf{B}}_R$ **does spread out** so the fluxes through S_3 and S_5 are **positive**.
- Thus $\vec{\mathbf{B}}_R$ on S_2 must be **less** than it would be with no spreading in the $\hat{\boldsymbol{\theta}}$ direction.

Leakage



Account for this flux leakage as follows:

- Let $\kappa \in \mathbb{R}$ with $0 < \kappa < 1$. At the **stator side** of the air gap modify $\vec{\mathbf{B}}_R$ to be

$$\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, r, \theta - \theta_R)\Big|_{r=r_S} = \kappa \frac{\mu_0 N_R}{2g} \frac{r_R}{r_S} \left(i_{Ra} \cos(\theta - \theta_R) + i_{Rb} \sin(\theta - \theta_R) \right) \hat{\mathbf{r}}.$$

  The flux $\int_{S_2} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}}$ is a factor κ less than the flux $\int_{S_1} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}}$.

- Similarly, on the **rotor side** of the air gap, $\vec{\mathbf{B}}_S$ is modified to be

$$\vec{\mathbf{B}}_S(i_{Sa}, i_{Sb}, r, \theta)\Big|_{r=r_R} = \kappa \frac{\mu_0 N_S}{2g} \frac{r_R}{r_R} \left(i_{Sa} \cos(\theta) + i_{Sb} \sin(\theta) \right) \hat{\mathbf{r}}.$$

  The flux $\int_{S_1} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}}$ is a factor κ less than the flux $\int_{S_2} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}}$.

Flux Linkages in AC Machines

Flux Linkages in the Stator Phases

$$\lambda_{Sa}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int_{\substack{\text{All loops of} \\ \text{stator phase } a}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}}.$$

Stator Flux linkage due to $\vec{\mathbf{B}}_S$

$$\lambda_{Sa}(0, 0, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int_{\substack{\text{All loops of} \\ \text{stator phase } a}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}}$$

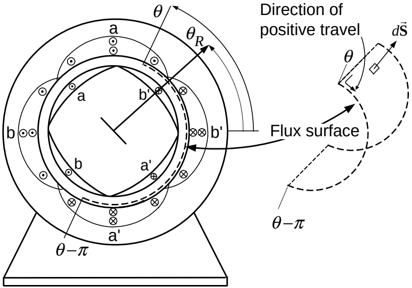
Stator Flux linkage due to $\vec{\mathbf{B}}_R$

$$\lambda_{Sa}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) \triangleq \int_{\substack{\text{All loops of} \\ \text{stator phase } a}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}}$$

Then

$$\lambda_{Sa}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) = \lambda_{Sa}(0, 0, i_{Sa}, i_{Sb}, \theta_R) + \lambda_{Sa}(i_{Ra}, i_{Rb}, 0, 0, \theta_R).$$

Stator Flux linkage due to $\vec{\mathbf{B}}_S$



$$
\begin{aligned}
\phi_{Sa}(i_{Sa}, i_{Sb}, \theta) &\triangleq \int_{z=0}^{z=\ell_1} \int_{\theta'=\theta-\pi}^{\theta'=\theta} \frac{\mu_0 r_R N_S}{2g} \frac{1}{r_S} \Big(i_{Sa}\cos(\theta') + i_{Sb}\sin(\theta') \Big) \hat{\mathbf{r}} \cdot (r_S\, d\theta'\, dz \hat{\mathbf{r}}) \\
&= \frac{\mu_0 r_R \ell_1 N_S}{2g} \int_{\theta'=\theta-\pi}^{\theta'=\theta} \Big(i_{Sa}\cos(\theta') + i_{Sb}\sin(\theta') \Big)\, d\theta' \\
&= \frac{\mu_0 r_R \ell_1 N_S}{2g} \Big(i_{Sa}\sin(\theta') - i_{Sb}\cos(\theta') \Big)\Big|_{\theta'=\theta-\pi}^{\theta'=\theta} \\
&= \frac{\mu_0 r_R \ell_1 N_S}{2g} \Big(i_{Sa}\big(\sin(\theta) - \sin(\theta-\pi)\big) - i_{Sb}\big(\cos(\theta) - \cos(\theta-\pi)\big)\Big) \\
&= \frac{\mu_0 r_R \ell_1 N_S}{g} \Big(i_{Sa}\sin(\theta) - i_{Sb}\cos(\theta) \Big).
\end{aligned}
$$

Stator Flux linkage due to $\vec{\mathbf{B}}_S$

Between θ and $\theta + d\theta$ there are $(N_S/2)\sin(\theta)d\theta$ turns **each** having the flux ϕ_{Sa}. The **incremental flux linkage** $d\lambda_{Sa}$ in the turns between θ and $\theta + d\theta$ is

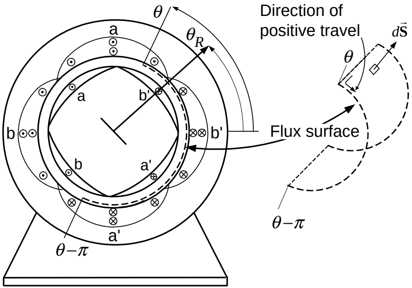
$$d\lambda_{Sa} \triangleq \phi_{Sa}(i_{Sa}, i_{Sb}, \theta)\frac{N_S}{2}\sin(\theta)d\theta = \frac{\mu_0 r_R \ell_1 N_S}{g}\Big(i_{Sa}\sin(\theta) - i_{Sb}\cos(\theta)\Big)\frac{N_S}{2}\sin(\theta)d\theta.$$

$$\begin{aligned}
\lambda_{Sa}(0,0,i_{Sa},i_{Sb},\theta_R) &= \int_{\substack{\text{All loops of}\\ \text{stator phase } a}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} \\
&= \int_{\theta=0}^{\theta=\pi} \frac{\mu_0 r_R \ell_1 N_S}{g}\Big(i_{Sa}\sin(\theta) - i_{Sb}\cos(\theta)\Big)\frac{N_S}{2}\sin(\theta)d\theta \\
&= \frac{\mu_0 r_R \ell_1 N_S^2}{2g}\int_{\theta=0}^{\theta=\pi}\Big(i_{Sa}\sin^2(\theta) - i_{Sb}\cos(\theta)\sin(\theta)\Big)d\theta \\
&= \frac{\mu_0 r_R \ell_1 N_S^2}{2g} i_{Sa}\frac{\pi}{2} \\
&= L_S i_{Sa}
\end{aligned}$$

where

$$L_S \triangleq \frac{\mu_0 r_R \ell_1 \pi N_S^2}{4g} = \frac{\mu_0 \ell_1 \ell_2 \pi}{8g} N_S^2.$$

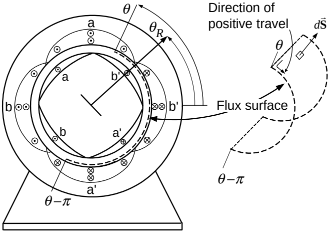
Stator Flux linkage due to $\vec{\mathbf{B}}_S$



Similarly,

$$\lambda_{Sb}(0,0,i_{Sa},i_{Sb},\theta_R) = \int_{\substack{\text{All loops of}\\ \text{stator phase } b}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} = \frac{\mu_0 r_R \ell_1 \pi N_S^2}{4g} i_{Sb} = L_S i_{Sb}.$$

Stator Flux Linkage due to $\vec{\mathbf{B}}_R$



$$\lambda_{Sa}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) \triangleq \int_{\substack{\text{All loops of} \\ \text{stator phase } a}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}}.$$

The flux $\phi_{Sa}(i_{Ra}, i_{Rb}, \theta - \theta_R)$ in **each turn** of stator phase a at θ due to $\vec{\mathbf{B}}_R$ is

$$\begin{aligned}
\phi_{Sa}(i_{Ra}, i_{Rb}, \theta - \theta_R) &\triangleq \int_{z=0}^{z=\ell_1} \int_{\theta'=\theta-\pi}^{\theta'=\theta} \kappa \frac{\mu_0 N_R r_R}{2g} \frac{1}{r_S} \Big(i_{Ra}\cos(\theta' - \theta_R) + i_{Rb}\sin(\theta' - \theta_R)\Big) \hat{\mathbf{r}} \cdot (r_S\, d\theta' dz \hat{\mathbf{r}}) \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_R}{2g} \int_{\theta'=\theta-\pi}^{\theta'=\theta} \Big(i_{Ra}\cos(\theta' - \theta_R) + i_{Rb}\sin(\theta' - \theta_R)\Big)\, d\theta' \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_R}{g} \Big(i_{Ra}\sin(\theta - \theta_R) - i_{Rb}\cos(\theta - \theta_R)\Big).
\end{aligned}$$

- Note the factor κ has been **included** in $\vec{\mathbf{B}}_R$ because $r = r_S$.

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_R$

Between θ and $\theta + d\theta$, there are $(N_S/2)\sin(\theta)d\theta$ turns each having the flux ϕ_{Sa}. The **incremental flux linkage** $d\lambda_{Sa}$ in the turns between θ and $\theta + d\theta$ is

$$d\lambda_{Sa} = \phi_{Sa}(i_{Ra}, i_{Rb}, \theta - \theta_R)\frac{N_S}{2}\sin(\theta)d\theta = \kappa\frac{\mu_0 r_R \ell_1 N_R N_S}{2g}\Big(i_{Ra}\sin(\theta - \theta_R) - i_{Rb}\cos(\theta - \theta_R)\Big)\sin(\theta)d\theta.$$

$$\begin{aligned}
\lambda_{Sa}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) &= \int_{\text{stator phase } a} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} \\
&= \int_{\theta=0}^{\theta=\pi} \kappa\frac{\mu_0 r_R \ell_1 N_S N_R}{2g}\left(i_{Ra}\sin(\theta - \theta_R) - i_{Rb}\cos(\theta - \theta_R)\right)\sin(\theta)d\theta \\
&= \kappa\frac{\mu_0 r_R \ell_1 N_S N_R}{2g}\Bigg(i_{Ra}\int_{\theta=0}^{\theta=\pi}\Big(\sin(\theta)\cos(\theta_R) - \cos(\theta)\sin(\theta_R)\Big)\sin(\theta)d\theta \\
&\quad - i_{Rb}\int_{\theta=0}^{\theta=\pi}\Big(\cos(\theta)\cos(\theta_R) + \sin(\theta)\sin(\theta_R)\Big)\sin(\theta)d\theta\Bigg) \\
&= \kappa\frac{\mu_0 r_R \ell_1 N_S N_R}{2g}\int_{\theta=0}^{\theta=\pi}\Big(i_{Ra}\sin^2(\theta)\cos(\theta_R) - i_{Rb}\sin^2(\theta)\sin\theta_R\Big)\,d\theta \\
&= M\Big(i_{Ra}\cos(\theta_R) - i_{Rb}\sin(\theta_R)\Big)
\end{aligned}$$

where

$$M \triangleq \kappa\frac{\mu_0 \pi r_R \ell_1 N_S N_R}{4g} = \kappa\frac{\mu_0 \pi \ell_1 \ell_2 N_S N_R}{8g}.$$

Stator Flux Linkage Produced by $\vec{\mathbf{B}}_R$

Similarly,

$$\lambda_{Sb}(i_{Ra}, i_{Rb}, , 0, 0, \theta_R) = \int_{\substack{\text{All loops of} \\ \text{stator phase } b}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = M\Big(i_{Ra}\sin(\theta_R) + i_{Rb}\cos(\theta_R)\Big).$$

Total Flux Linkage in the Stator Phases

$$\begin{aligned}
\lambda_{Sa}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) &= \int_{\substack{\text{All loops of} \\ \text{stator phase } a}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}} \\
&= L_S i_{Sa} + M\Big(i_{Ra}\cos(\theta_R) - i_{Rb}\sin(\theta_R)\Big) \\
\lambda_{Sb}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) &= \int_{\substack{\text{All loops of} \\ \text{stator phase } b}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}} \\
&= L_S i_{Sb} + M\Big(i_{Ra}\sin(\theta_R) + i_{Rb}\cos(\theta_R)\Big).
\end{aligned}$$

Flux Linkages in the Rotor Phases

$$\lambda_{Ra}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int_{\substack{\text{All loops of} \\ \text{rotor phase } a}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}}$$

$$\lambda_{Rb}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int_{\substack{\text{All loops of} \\ \text{rotor phase } b}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}}.$$

Rotor Flux linkage due to $\vec{\mathbf{B}}_S$

$$\lambda_{Ra}(0, 0, i_{Sa}, i_{Sb}, \theta_R) \triangleq \int_{\substack{\text{All loops of} \\ \text{rotor phase } a}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}}$$

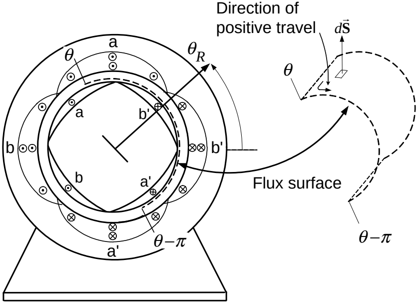
Rotor Flux linkage due to $\vec{\mathbf{B}}_R$

$$\lambda_{Ra}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) \triangleq \int_{\substack{\text{All loops of} \\ \text{rotor phase } a}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}}.$$

Then

$$\lambda_{Ra}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) = \lambda_{Ra}(0, 0, i_{Sa}, i_{Sb}, \theta_R) + \lambda_{Ra}(i_{Ra}, i_{Rb}, 0, 0, \theta_R).$$

Rotor Flux Linkage Produced by $\vec{\mathbf{B}}_S$



$$
\begin{aligned}
\phi_{Ra}(i_{Sa}, i_{Sb}, \theta) &= \int_{z=0}^{z=\ell_1} \int_{\theta'=\theta-\pi}^{\theta'=\theta} \kappa \frac{\mu_0 r_R N_S}{2g} \frac{1}{r_R} \Big(i_{Sa}\cos(\theta') + i_{Sb}\sin(\theta') \Big) \hat{\mathbf{r}} \cdot (r_R d\theta' dz \hat{\mathbf{r}}) \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_S}{2g} \int_{\theta'=\theta-\pi}^{\theta'=\theta} \Big(i_{Sa}\cos(\theta') + i_{Sb}\sin(\theta') \Big) d\theta' \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_S}{2g} \Big(i_{Sa}\sin(\theta') - i_{Sb}\cos(\theta') \Big) \Big|_{\theta'=\theta-\pi}^{\theta'=\theta} \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_S}{2g} \Big(i_{Sa}(\sin(\theta) - \sin(\theta-\pi)) - i_{Sb}(\cos(\theta) - \cos(\theta-\pi) \Big) \\
&= \kappa \frac{\mu_0 r_R \ell_1 N_S}{g} \Big(i_{Sa}\sin(\theta) - i_{Sb}\cos(\theta) \Big).
\end{aligned}
$$

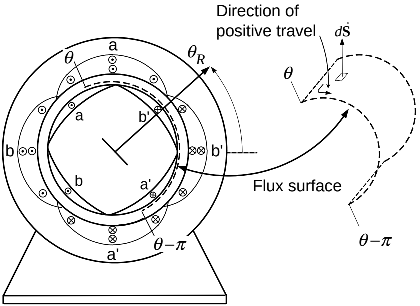
Rotor Flux Linkage Produced by $\vec{\mathbf{B}}_S$

Between θ and $\theta + d\theta$, there are $(N_R/2)\sin(\theta-\theta_R)d\theta$ turns **each** having the flux ϕ_{Ra}. The **incremental flux linkage** $d\lambda_{Ra}$ in the turns θ and $\theta + d\theta$ is

$$d\lambda_{Ra} \triangleq \phi_{Ra}(i_{Sa}, i_{Sb}, \theta)\frac{N_R}{2}\sin(\theta-\theta_R)d\theta = \kappa\frac{\mu_0 r_R \ell_1 N_S N_R}{2g}\Big(i_{Sa}\sin(\theta) - \; i_{Sb}\cos(\theta)\Big)\sin(\theta-\theta_R)d\theta.$$

$$\begin{aligned}
\lambda_{Ra}(0,0,i_{Sa},i_{Sb},\theta_R) &= \int_{\text{rotor phase } a}\vec{\mathbf{B}}_S\cdot d\vec{\mathbf{S}}\\
&= \int_{\theta=\theta_R}^{\theta=\theta_R+\pi}\kappa\frac{\mu_0 r_R \ell_1 N_S N_R}{2g}\Big(i_{Sa}\sin(\theta) - \; i_{Sb}\cos(\theta)\Big)\sin(\theta-\theta_R)d\theta\\
&= \kappa\frac{\mu_0 r_R \ell_1 N_S N_R}{2g}\int_{\theta=\theta_R}^{\theta=\theta_R+\pi}\Big(i_{Sa}\sin(\theta-\theta_R+\theta_R)\sin(\theta-\theta_R)\\
&\qquad\qquad - i_{Sb}\cos(\theta-\theta_R+\theta_R)\sin(\theta-\theta_R)\Big)\,d\theta\\
&= \kappa\frac{\mu_0 r_R \ell_1 N_S N_R}{2g}\\
&\quad\times\int_{\theta=\theta_R}^{\theta=\theta_R+\pi}\Big(i_{Sa}\sin^2(\theta-\theta_R)\cos(\theta_R) + i_{Sa}\cos(\theta-\theta_R)\sin(\theta_R)\sin(\theta-\theta_R)\\
&\qquad - \; i_{Sb}\cos(\theta-\theta_R)\cos(\theta_R)\sin(\theta-\theta_R) + i_{Sb}\sin^2(\theta-\theta_R)\sin(\theta_R)\Big)d\theta\\
&= \kappa\frac{\mu_0 r_R \ell_1 N_S N_R}{2g}\Big(i_{Sa}\frac{\pi}{2}\cos(\theta_R) + \frac{\pi}{2}i_{Sb}\sin(\theta_R)\Big)\\
&= M\Big(i_{Sa}\cos(\theta_R) + i_{Sb}\sin(\theta_R)\Big)
\end{aligned}$$

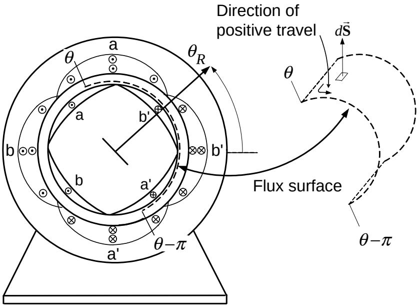
Rotor Flux Linkage Produced by $\vec{\mathbf{B}}_S$



Similarly,

$$\lambda_{Rb}(0,0,i_{Sa},i_{Sb},\theta_R) \triangleq \int_{\substack{\text{All loops of}\\ \text{rotor phase } b}} \vec{\mathbf{B}}_S \cdot d\vec{\mathbf{S}} = M\Big(-i_{Sa}\sin(\theta_R) + i_{Sb}\cos(\theta_R)\Big).$$

Rotor Flux Linkage Produced by $\vec{\mathbf{B}}_R$



$$
\begin{aligned}
\phi_{Ra}(i_{Ra}, i_{Rb}, \theta-\theta_R) &\triangleq \int_{z=0}^{z=\ell_1}\int_{\theta'=\theta-\pi}^{\theta'=\theta} \frac{\mu_0 r_R N_R}{2g}\frac{1}{r_R}\Big(i_{Ra}\cos(\theta'-\theta_R)+i_{Rb}\sin(\theta'-\theta_R)\Big)\hat{\mathbf{r}}\cdot(r_R\, d\theta'\, dz\,\hat{\mathbf{r}}) \\
&= \frac{\mu_0 r_R \ell_1 N_R}{2g}\int_{\theta'=\theta-\pi}^{\theta'=\theta}\Big(i_{Ra}\cos(\theta'-\theta_R)+i_{Rb}\sin(\theta'-\theta_R)\Big)d\theta' \\
&= \frac{\mu_0 r_R \ell_1 N_R}{2g}\Big(i_{Ra}\sin(\theta'-\theta_R) \;-i_{Rb}\cos(\theta'-\theta_R)\Big)\Big|_{\theta'=\theta-\pi}^{\theta'=\theta} \\
&= \frac{\mu_0 r_R \ell_1 N_R}{2g}\Big(i_{Ra}\big(\sin(\theta-\theta_R)-\sin(\theta-\theta_R-\pi)\big)-i_{Rb}\big(\cos(\theta-\theta_R)-\cos(\theta-\theta_R-\pi)\big)\Big) \\
&= \frac{\mu_0 r_R \ell_1 N_R}{g}\Big(i_{Ra}\sin(\theta-\theta_R)-i_{Rb}\cos(\theta-\theta_R)\Big).
\end{aligned}
$$

Rotor Flux Linkage Produced by $\vec{\mathbf{B}}_R$

Between θ and $\theta + d\theta$, there are $(N_R/2)\sin(\theta - \theta_R)d\theta$ turns **each** having the flux ϕ_{Ra}. The **incremental flux linkage** in the turns between θ and $\theta + d\theta$ is

$$\begin{aligned} d\lambda_{Ra} &\triangleq \phi_{Ra}(i_{Ra}, i_{Rb}, \theta - \theta_R)\frac{N_R}{2}\sin(\theta - \theta_R)d\theta \\ &= \frac{\mu_0 r_R \ell_1 N_R^2}{2g}\left(i_{Ra}\sin(\theta - \theta_R) - i_{Rb}\cos(\theta - \theta_R)\right)\sin(\theta - \theta_R)d\theta. \end{aligned}$$

$$\begin{aligned} \lambda_{Ra}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) &= \int_{\substack{\text{All loops of} \\ \text{rotor phase } a}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} \\ &= \int_{\theta=\theta_R}^{\theta=\theta_R+\pi} \frac{\mu_0 r_R \ell_1 N_R^2}{2g}\Big(i_{Ra}\sin(\theta - \theta_R) - \ i_{Rb}\cos(\theta - \theta_R)\Big)\sin(\theta - \theta_R)d\theta \\ &= \frac{\mu_0 r_R \ell_1 N_R^2}{2g}\int_{\theta=\theta_R}^{\theta=\theta_R+\pi}\Big(i_{Ra}\sin^2(\theta - \theta_R) - \ i_{Rb}\cos(\theta - \theta_R)\sin(\theta - \theta_R)\Big)\,d\theta \\ &= \frac{\mu_0 r_R \ell_1 N_R^2}{2g} i_{Ra}\frac{\pi}{2} \\ &= L_R i_{Ra} \end{aligned}$$

where

$$L_R \triangleq \frac{\mu_0 r_R \ell_1 \pi N_R^2}{4g} = \frac{\mu_0 \ell_1 \ell_2 \pi}{8g} N_R^2.$$

Rotor Flux Linkage Produced by $\vec{\mathbf{B}}_R$

Similarly,

$$\lambda_{Rb}(i_{Ra}, i_{Rb}, 0, 0, \theta_R) = \int_{\substack{\text{All loops of} \\ \text{rotor phase } b}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = L_R i_{Rb}.$$

Total Flux Linkage in the Rotor Phases

$$\begin{aligned}
\lambda_{Ra}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) &= \int_{\substack{\text{All loops of} \\ \text{rotor phase } a}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}} \\
&= L_R i_{Ra} + M\Big(i_{Sa}\cos(\theta_R) + i_{Sb}\sin(\theta_R)\Big) \\
\lambda_{Rb}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) &= \int_{\substack{\text{All loops of} \\ \text{rotor phase } b}} (\vec{\mathbf{B}}_S + \vec{\mathbf{B}}_R) \cdot d\vec{\mathbf{S}} \\
&= L_R i_{Rb} + M\Big(-i_{Sa}\sin(\theta_R) + i_{Sb}\cos(\theta_R)\Big).
\end{aligned}$$

Torque Production in AC Machines

We showed

$$\vec{\mathbf{B}}_S(i_{Sa}, i_{Sb}, \theta) = \frac{\mu_0 r_R N_S}{2 g r_R}\Big(i_{Sa}(t)\cos(\theta) + \ i_{Sb}(t)\sin(\theta)\Big)\hat{\mathbf{r}}.$$

Let $i_S(t) \triangleq \sqrt{i_{Sa}^2(t) + i_{Sb}^2(t)}$, $\xi(t) \triangleq \tan^{-1}\left(i_{Sb}(t)/i_{Sa}(t)\right)$ so that

$$\vec{\mathbf{B}}_S = \frac{\mu_0 \ell_2 N_S}{4gr} i_S(t)\Big(\cos(\xi)\cos(\theta) + \ \sin(\xi)\sin(\theta)\Big)\hat{\mathbf{r}} = \frac{\mu_0 \ell_2 N_S}{4gr} i_S(t)\cos(\theta - \xi)\hat{\mathbf{r}}.$$

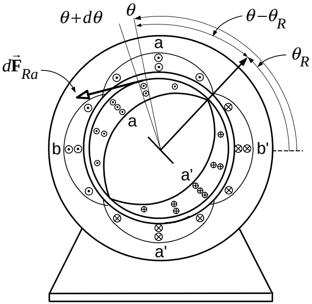
We also showed

$$\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, \theta - \theta_R) = \frac{\mu_0 \ell_2 N_R}{4gr}\Big(i_{Ra}(t)\cos(\theta - \theta_R) + \ i_{Rb}(t)\sin(\theta - \theta_R)\Big)\hat{\mathbf{r}}.$$

Let $i_R(t) \triangleq \sqrt{i_{Ra}^2(t) + i_{Rb}^2(t)}$, $\zeta(t) \triangleq \tan^{-1}\left(i_{Rb}(t)/i_{Ra}(t)\right)$ so that

$$\begin{aligned}\vec{\mathbf{B}}_R &= \frac{\mu_0 \ell_2 N_R}{4gr} i_R(t)\Big(\cos(\zeta)\cos(\theta - \theta_R) + \sin(\zeta)\sin(\theta - \theta_R)\Big)\hat{\mathbf{r}} \\ &= \frac{\mu_0 \ell_2 N_R}{4gr} i_R(t)\cos(\theta - \theta_R - \zeta)\hat{\mathbf{r}}.\end{aligned}$$

Torque Production in AC Machines

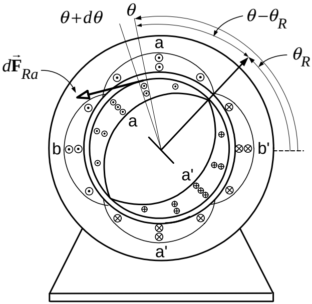


The force $d\vec{\mathbf{F}}_{Ra}$ on the axial sides of rotor phase a between θ and $\theta + d\theta$ by $\vec{\mathbf{B}}_S$ is

$$\begin{aligned} d\vec{\mathbf{F}}_{Ra} &= \begin{cases} i_{Ra}(t)\dfrac{N_R}{2}\sin(\theta-\theta_R)d\theta(+\ell_1\hat{\mathbf{z}})\times B_S\hat{\mathbf{r}}, & \theta_R \le \theta \le \theta_R+\pi \\[2ex] i_{Ra}(t)\dfrac{N_R}{2}|\sin(\theta-\theta_R)|d\theta(-\ell_1\hat{\mathbf{z}})\times B_S\hat{\mathbf{r}}, & \theta_R+\pi \le \theta \le \theta_R+2\pi. \end{cases} \\ &= i_{Ra}(t)\frac{N_R}{2}\sin(\theta-\theta_R)d\theta\ell_1\hat{\mathbf{z}}\times B_S\hat{\mathbf{r}} \ \text{ for all } \theta_R \le \theta \le \theta_R+2\pi \end{aligned}$$

as $|\sin(\theta-\theta_R)| = -\sin(\theta-\theta_R)$ for $\theta_R+\pi \le \theta \le \theta_R+2\pi$.

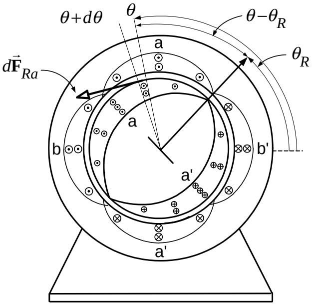
Torque Production in AC Machines



$$
\begin{aligned}
d\vec{\mathbf{F}}_{Ra} &= i_{Ra}(t)\frac{N_R}{2}\sin(\theta-\theta_R)d\theta\ell_1\hat{\mathbf{z}}\times B_S\hat{\mathbf{r}} \\
&= i_{Ra}(t)\frac{N_R}{2}\sin(\theta-\theta_R)\ell_1 B_S d\theta\hat{\boldsymbol{\theta}} \\
&= i_{Ra}(t)\frac{N_R}{2}\sin(\theta-\theta_R)\ell_1\kappa\frac{\mu_0 r_R N_S}{2gr_R}i_S(t)\cos(\theta-\xi)d\theta\hat{\boldsymbol{\theta}} \\
&= \kappa\frac{\mu_0\ell_1 N_S N_R}{4g}i_{Ra}(t)i_S(t)\sin(\theta-\theta_R)\cos(\theta-\xi)d\theta\hat{\boldsymbol{\theta}}
\end{aligned}
$$

- Note that we included the leakage factor κ in $\vec{\mathbf{B}}_S$ (why?)

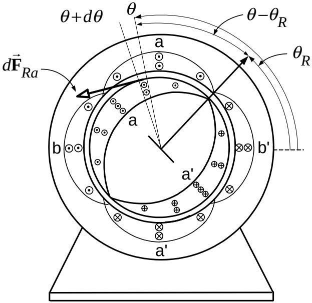
Torque Production in AC Machines



The differential torque $d\vec{\boldsymbol{\tau}}_{Ra}$ is then

$$
\begin{aligned}
d\vec{\boldsymbol{\tau}}_{Ra} = (\ell_2/2)\hat{\mathbf{r}} \times d\vec{\mathbf{F}}_{Ra} &= (\ell_2/2)\frac{\mu_0 \ell_1 N_S N_R}{4g} i_{Ra}(t) i_S(t)\sin(\theta-\theta_R)\cos(\theta-\xi)d\theta\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} \\
&= \kappa\frac{\mu_0 \ell_1 \ell_2 N_S N_R}{8g} i_{Ra}(t) i_S(t)\sin(\theta-\theta_R)\cos(\theta-\xi)d\theta\hat{\mathbf{z}} \\
&= \frac{M}{\pi} i_{Ra}(t) i_S(t)\sin(\theta-\theta_R)\cos(\theta-\xi)d\theta\hat{\mathbf{z}} \\
&= \frac{M}{\pi} i_{Ra}(t) i_S(t)\frac{1}{2}\left(\sin(2\theta-\theta_R-\xi)+\sin(\xi-\theta_R)\right)d\theta\hat{\mathbf{z}}.
\end{aligned}
$$

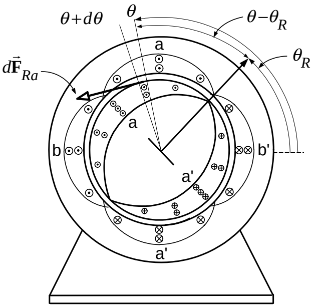
Torque Production in AC Machines



The torque on rotor phase a is then

$$
\begin{aligned}
\vec{\boldsymbol{\tau}}_{Ra} = \int_0^{2\pi} d\vec{\boldsymbol{\tau}}_{Ra} &= \int_0^{2\pi} \frac{M}{2\pi} i_{Ra}(t) i_S(t) \Big(\sin(2\theta - \theta_R - \xi) + \sin(\xi - \theta_R) \Big) \, d\theta \hat{\mathbf{z}} \\
&= M i_{Ra}(t) i_S(t) \sin(\xi - \theta_R) \hat{\mathbf{z}} \\
&= M i_{Ra}(t) i_S(t) \Big(\sin(\xi)\cos(\theta_R) - \cos(\xi)\sin(\theta_R) \Big) \hat{\mathbf{z}} \\
&= M i_{Ra}(t) \Big(i_{Sb}(t)\cos(\theta_R) - i_{Sa}(t)\sin(\theta_R) \Big) \hat{\mathbf{z}}.
\end{aligned}
$$

Torque Production in AC Machines



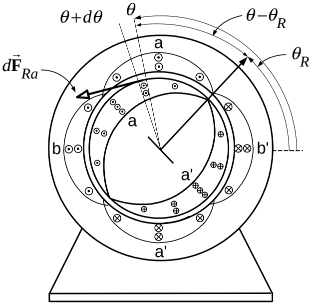
Similarly,

$$
d\vec{\mathbf{F}}_{Rb} = \begin{cases} i_{Rb}(t)\dfrac{N_R}{2}\sin(\theta-\pi/2-\theta_R)d\theta(\ell_1\hat{\mathbf{z}})\times B_S\hat{\mathbf{r}}, & \theta_R+\dfrac{\pi}{2}\le\theta\le\theta_R+\dfrac{3\pi}{2} \\ i_{Rb}(t)\dfrac{N_R}{2}\left|\sin(\theta-\pi/2-\theta_R)\right|d\theta(-\ell_1\hat{\mathbf{z}})\times B_S\hat{\mathbf{r}}, & \theta_R-\dfrac{\pi}{2}\le\theta\le\theta_R+\dfrac{\pi}{2} \end{cases}
$$

$$
= i_{Rb}(t)\frac{N_R}{2}\sin(\theta-\pi/2-\theta_R)d\theta(\ell_1\hat{\mathbf{z}})\times(B_S\hat{\mathbf{r}}) \text{ for all } \theta_R\le\theta\le\theta_R+2\pi
$$

as $\left|\sin(\theta-\frac{\pi}{2}-\theta_R)\right| = -\sin(\theta-\pi/2-\theta_R)$ for $\theta_R-\pi/2\le\theta\le\theta_R+\pi/2<0$.

Torque Production in AC Machines



$$
\begin{aligned}
d\vec{\mathbf{F}}_{Rb} &= i_{Rb}(t)\frac{N_R}{2}\sin(\theta-\pi/2-\theta_R)d\theta(\ell_1\hat{\mathbf{z}})\times(B_S\hat{\mathbf{r}}) \\
&= i_{Rb}(t)\frac{N_R}{2}\sin(\theta-\pi/2-\theta_R)\ell_1 B_S d\theta\hat{\boldsymbol{\theta}} \\
&= -i_{Rb}(t)\frac{N_R}{2}\cos(\theta-\theta_R)\ell_1\kappa\frac{\mu_0 r_R N_S}{2gr_R}i_S(t)\cos(\theta-\xi)d\theta\hat{\boldsymbol{\theta}} \\
&= -\kappa\frac{\mu_0\ell_1 N_S N_R}{4g}i_{Rb}(t)i_S(t)\cos(\theta-\theta_R)\cos(\theta-\xi)d\theta\hat{\boldsymbol{\theta}}.
\end{aligned}
$$

Torque Production in AC Machines

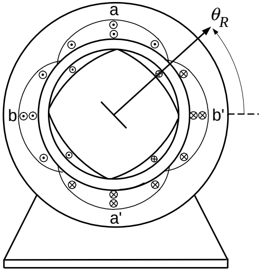
The differential torque $d\vec{\boldsymbol{\tau}}_{Rb}$ is then given by

$$\begin{aligned}
d\vec{\boldsymbol{\tau}}_{Rb} &= (\ell_2/2)\hat{\mathbf{r}} \times d\vec{\mathbf{F}}_{Rb} \\
&= -(\ell_2/2)\kappa\frac{\mu_0\ell_1 N_S N_R}{4g} i_{Rb}(t) i_S(t)\cos(\theta-\theta_R)\cos(\theta-\xi)d\theta\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} \\
&= -\kappa\frac{\mu_0\ell_1\ell_2 N_S N_R}{8g} i_{Rb}(t) i_S(t)\cos(\theta-\theta_R)\cos(\theta-\xi)d\theta\hat{\mathbf{z}} \\
&= -\frac{M}{\pi} i_{Rb}(t) i_S(t)\cos(\theta-\theta_R)\cos(\theta-\xi)d\theta\hat{\mathbf{z}} \\
&= -\frac{M}{\pi} i_{Rb}(t) i_S(t)\frac{1}{2}\left(\cos(2\theta-\theta_R-\xi)+\cos(\xi-\theta_R)\right)d\theta\hat{\mathbf{z}}.
\end{aligned}$$

The torque on rotor phase b is then

$$\begin{aligned}
\vec{\boldsymbol{\tau}}_{Rb} = \int_0^{2\pi} d\vec{\boldsymbol{\tau}}_{Rb} &= -\int_0^{2\pi}\frac{M}{2\pi} i_{Rb}(t) i_S(t)\left(\cos(2\theta-\theta_R-\xi)+\cos(\xi-\theta_R)\right)d\theta\hat{\mathbf{z}} \\
&= -M i_{Rb}(t) i_S(t)\cos(\xi-\theta_R)\hat{\mathbf{z}} \\
&= -M i_{Rb}(t) i_S(t)\Big(\cos(\xi)\cos(\theta_R)+\sin(\xi)\sin(\theta_R)\Big)\hat{\mathbf{z}} \\
&= -M i_{Rb}(t)\Big(i_{Sa}(t)\cos(\theta_R)+i_{Sb}(t)\sin(\theta_R)\Big)\hat{\mathbf{z}}.
\end{aligned}$$

Torque Production in AC Machines



The **total torque** on the rotor is then

$$
\begin{aligned}
\tau_R &= \tau_{Ra} + \tau_{Rb} \\
&= M\Big(-i_{Ra}(t)i_{Sa}(t)\sin(\theta_R) + i_{Ra}(t)i_{Sb}(t)\cos(\theta_R) - i_{Rb}(t)i_{Sa}(t)\cos(\theta_R) - i_{Rb}(t)i_{Sb}(t)\sin(\theta_R)\Big).
\end{aligned}
$$

Mathematical Model of a Sinusoidally Wound Induction Machine

- Rotor phases do **not** have voltage sources (rotor windings are shorted).

$$
\begin{aligned}
\lambda_{Sa}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) &= L_S i_{Sa} + M\Big(+i_{Ra}\cos(\theta_R) - i_{Rb}\sin(\theta_R)\Big)\\
\lambda_{Sb}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) &= L_S i_{Sb} + M\Big(+i_{Ra}\sin(\theta_R) + i_{Rb}\cos(\theta_R)\Big)\\
\lambda_{Ra}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) &= L_R i_{Ra} + M\Big(+i_{Sa}\cos(\theta_R) + i_{Sb}\sin(\theta_R)\Big)\\
\lambda_{Rb}(i_{Ra}, i_{Rb}, i_{Sa}, i_{Sb}, \theta_R) &= L_R i_{Rb} + M\Big(-i_{Sa}\sin(\theta_R) + i_{Sb}\cos(\theta_R)\Big)
\end{aligned}
$$

By **Faraday's and Ohm's laws**

$$
\begin{aligned}
-\frac{d\lambda_{Sa}}{dt} - R_S i_{Sa} + u_{Sa} &= 0\\
-\frac{d\lambda_{Sb}}{dt} - R_S i_{Sb} + u_{Sb} &= 0\\
-\frac{d\lambda_{Ra}}{dt} - R_R i_{Ra} &= 0\\
-\frac{d\lambda_{Rb}}{dt} - R_R i_{Rb} &= 0
\end{aligned}
$$

Mathematical Model of a Sinusoidally Wound Induction Machine

Explicitly

$$
\begin{aligned}
L_S \frac{d}{dt} i_{Sa} + M \frac{d}{dt}\Big(+i_{Ra}\cos(\theta_R) - i_{Rb}\sin(\theta_R)\Big) + R_S i_{Sa} &= u_{Sa} \\
L_S \frac{d}{dt} i_{Sb} + M \frac{d}{dt}\Big(+i_{Ra}\sin(\theta_R) + i_{Rb}\cos(\theta_R)\Big) + R_S i_{Sb} &= u_{Sb} \\
L_R \frac{d}{dt} i_{Ra} + M \frac{d}{dt}\Big(+i_{Sa}\cos(\theta_R) + i_{Sb}\sin(\theta_R)\Big) + R_R i_{Ra} &= 0 \\
L_R \frac{d}{dt} i_{Rb} + M \frac{d}{dt}\Big(-i_{Sa}\sin(\theta_R) + i_{Sb}\cos(\theta_R)\Big) + R_R i_{Rb} &= 0 \\
J\frac{d\omega_R}{dt} &= \tau_R - \tau_L \\
\frac{d\theta_R}{dt} &= \omega_R
\end{aligned}
$$

where τ_L is the load torque and

$$
\begin{aligned}
\tau_R = {} & M\Big(-i_{Ra}(t)i_{Sa}(t)\sin(\theta_R) + i_{Ra}(t)i_{Sb}(t)\cos(\theta_R) \\
& - i_{Rb}(t)i_{Sa}(t)\cos(\theta_R) - i_{Rb}(t)i_{Sb}(t)\sin(\theta_R)\Big).
\end{aligned}
$$

Total Leakage Factor

The **total leakage factor** is defined as

$$\sigma \triangleq 1 - \frac{M^2}{L_S L_R}.$$

Substituting the expressions

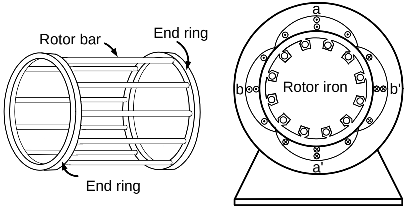
$$\begin{aligned} L_R &= \frac{\mu_0 \ell_1 \ell_2 \pi}{8g} N_R^2 \\ L_S &= \frac{\mu_0 \ell_1 \ell_2 \pi}{8g} N_S^2 \\ M &= \frac{\kappa \mu_0 \pi \ell_1 \ell_2 N_S N_R}{8g} \end{aligned}$$

results in

$$\sigma = 1 - \kappa^2.$$

- Note that **if** $\kappa = 1$ (no leakage), then $\sigma = 0$.
- In an actual motor $\sigma > 0$ and is typically between 0.05 and 0.20.

The Squirrel Cage Rotor



- The above model is **also used** for an induction motor with a squirrel cage rotor.
- There is a **different** current in each rotor bar of the squirrel cage.
- The above model works **remarkably well** to predict the stator currents and torque of a squirrel cage motor.
- The motor parameters R_R, L_R, M, R_S, L_S can be interpreted as those values that **best fit** the above mathematical model to the measured data i_{Sa}, i_{Sb}, ω_R of an actual squirrel cage motor.

Induction Machine With Multiple Pole Pairs

Suppose the machine has n_p pole pairs ($n_p = 1$ in the above model). Then

$$\begin{aligned}
\lambda_{Sa} &= L_S i_{Sa} + M\Big(i_{Ra}\cos(n_p\theta_R) - i_{Rb}\sin(n_p\theta_R)\Big)\\
\lambda_{Sb} &= L_S i_{Sb} + M\Big(i_{Ra}\sin(n_p\theta_R) + i_{Rb}\cos(n_p\theta_R)\Big)\\
\lambda_{Ra} &= L_R i_{Ra} + M\Big(+i_{Sa}\cos(n_p\theta_R) + i_{Sb}\sin(n_p\theta_R)\Big)\\
\lambda_{Rb} &= L_R i_{Rb} + M\Big(-i_{Sa}\sin(n_p\theta_R) + i_{Sb}\cos(n_p\theta_R)\Big)
\end{aligned}$$

where $L_S \triangleq \dfrac{\mu_0\pi\ell_1\ell_2 N_S^2}{8g}$, $L_R \triangleq \dfrac{\mu_0\pi\ell_1\ell_2 N_R^2}{8g}$, $M \triangleq \dfrac{\kappa\mu_0\pi\ell_1\ell_2 N_S N_R}{8g}$.

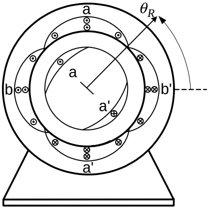
- N_S and N_R are the number of stator and rotor windings, respectively, **per pole pair**.

Induction Machine With Multiple Pole Pairs

$$
\begin{aligned}
u_{Sa} &= R_S i_{Sa} + \frac{d}{dt}\Big(L_S i_{Sa} + M\big(i_{Ra}\cos(n_p\theta_R) - i_{Rb}\sin(n_p\theta_R)\big)\Big)\\
u_{Sb} &= R_S i_{Sb} + \frac{d}{dt}\Big(L_S i_{Sb} + M\big(i_{Ra}\sin(n_p\theta_R) + i_{Rb}\cos(n_p\theta_R)\big)\Big)\\
0 &= R_R i_{Ra} + \frac{d}{dt}\Big(L_R i_{Ra} + M\big(+i_{Sa}\cos(n_p\theta_R) + i_{Sb}\sin(n_p\theta_R)\big)\Big)\\
0 &= R_R i_{Rb} + \frac{d}{dt}\Big(L_R i_{Rb} + M\big(-i_{Sa}\sin(n_p\theta_R) + i_{Sb}\cos(n_p\theta_R)\big)\Big)\\
J\frac{d\omega_R}{dt} &= n_p M\Big(i_{Sb}(i_{Ra}\cos(n_p\theta_R) - i_{Rb}\sin(n_p\theta_R) - i_{Sa}(i_{Ra}\sin(n_p\theta_R) + i_{Rb}\cos(n_p\theta_R))\Big) - \tau_L\\
\frac{d\theta_R}{dt} &= \omega_R.
\end{aligned}
$$

- Consider two induction machines one with $n_p = 1$ while the other has $n_p > 1$.
- Both have the **same** geometrical construction (ℓ_1, ℓ_2, and g are the same).
- Both have the same number N_S of stator windings **per pole pair**.
- Both have the same number N_R of rotor windings **per pole pair**.
- The torque of the machine with $n_p > 1$ pole pairs will be a factor n_p **greater** than the $n_p = 1$ machine.
- This is clear from considering the **coefficient** $n_p M$ in the torque expression.
- The n_p pole pair machine has $n_p > 1$ **more** windings in each phase than the $n_p = 1$ machine.

Mathematical Model of a Wound Rotor Synchronous Machine



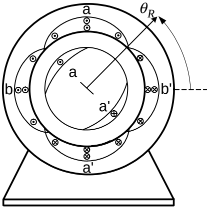
- The stator is the **same** as the induction machine.
- The rotor has only a **single** phase that is sinusoidally wound - the **field winding**.
- **Source voltage** u_F applied to the field winding which has **field current** i_F.
- Use the induction motor **flux expressions** with $L_F \triangleq L_R$ and

$$i_{Ra} = i_F, \quad \lambda_{Ra} = \lambda_F, \quad i_{Rb} \equiv 0, \quad \lambda_{Rb} \equiv 0$$

Synchronous Machine Flux Linkages

$$\begin{aligned}
\lambda_{Sa}(i_F, i_{Sa}, i_{Sb}, \theta_R) &= L_S i_{Sa} + M i_F \cos(\theta_R) \\
\lambda_{Sb}(i_f, i_{Sa}, i_{Sb}, \theta_R) &= L_S i_{Sb} + M i_F \sin(\theta_R) \\
\lambda_F(i_F, i_{Sa}, i_{Sb}, \theta_R) &= L_F i_F + M\Big(i_{Sa}\cos(\theta_R) + i_{Sb}\sin(\theta_R)\Big)
\end{aligned}$$

Mathematical Model of a Wound Rotor Synchronous Machine



With R_F the resistance of the field winding, Faraday's and Ohm's laws give

$$\begin{aligned}
-\frac{d\lambda_{Sa}}{dt} - R_S i_{Sa} + u_{Sa} &= 0 \\
-\frac{d\lambda_{Sb}}{dt} - R_S i_{Sb} + u_{Sb} &= 0 \\
-\frac{d\lambda_F}{dt} - R_F i_F + u_F &= 0.
\end{aligned}$$

- Use the induction motor torque expression with $i_{Ra} = i_F$ and $i_{Rb} \equiv 0$.

$$\tau_R = M i_F \Big(-i_{Sa} \sin(\theta_R) + i_{Sb} \cos(\theta_R) \Big).$$

Mathematical Model of a Wound Rotor Synchronous Machine

$$-\frac{d}{dt}\Big(L_S i_{Sa} + M i_F \cos(\theta_R)\Big) - R_S i_{Sa} + u_{Sa} = 0$$

$$-\frac{d}{dt}\Big(L_S i_{Sb} + M i_F \sin(\theta_R)\Big) - R_S i_{Sb} + u_{Sb} = 0$$

$$-\frac{d}{dt}\Big(L_F i_F + M\left(i_{Sa}\cos(\theta_R) + i_{Sb}\sin(\theta_R)\right)\Big) - R_F i_F + u_F = 0$$

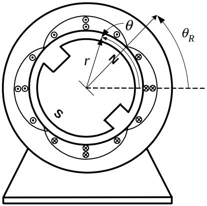
$$M i_F\Big(-i_{Sa}\sin(\theta_R) + i_{Sb}\cos(\theta_R)\Big) - \tau_L = J\frac{d\omega_R}{dt}$$

$$\frac{d\theta_R}{dt} = \omega_R$$

- The rotor current is usually kept constant, that is, $i_F = I_F$ (constant). Do this by setting $u_F = K_P(I_F - i_F) + K_I\int_0^t (I_F - i_F)dt$.

$$\begin{aligned}
L_S\frac{di_{Sa}}{dt} &= -R_S i_{Sa} + M I_F \sin(\theta_R)\omega_R + u_{Sa} \\
L_S\frac{di_{Sb}}{dt} &= -R_S i_{Sb} - M I_F \cos(\theta_R)\omega_R + u_{Sb} \\
J\frac{d\omega_R}{dt} &= M I_F\Big(-i_{Sa}\sin(\theta_R) + i_{Sb}\cos(\theta_R)\Big) - \tau_L \\
\frac{d\theta_R}{dt} &= \omega_R.
\end{aligned}$$

Mathematical Model of a Permanent Magnet Synchronous Machine



- $\vec{\mathbf{B}}_R(r, \theta - \theta_R) = B_m \dfrac{r_R}{r} \cos(\theta - \theta_R)\hat{\mathbf{r}}$.

  Same case as a wound rotor synchronous machine with a constant field current.

$$
\begin{aligned}
L_S \frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m \sin(\theta_R)\omega_R + u_{Sa} \\
L_S \frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m \cos(\theta_R)\omega_R + u_{Sb} \\
J \frac{d\omega_R}{dt} &= K_m \Big(-i_{Sa} \sin(\theta_R) + i_{Sb} \cos(\theta_R)\Big) - \tau_L \\
\frac{d\theta_R}{dt} &= \omega_R .
\end{aligned}
$$

- $K_m = \kappa B_m \dfrac{\pi \ell_1 \ell_2 N_S}{4}$. (Taking $B_m = \dfrac{\mu_0 N_R I_F}{2g}$ gives $K_m = M I_F$ as for wound rotor.)
- $L_S = \dfrac{\mu_0 \ell_1 \ell_2 \pi}{8g} N_S^2$.

The Stator and Rotor Magnetic Fields of an Induction Machine Rotate Synchronously*

*This is an optional section.

$\vec{\mathbf{B}}_S$ and $\vec{\mathbf{B}}_R$ of an Induction Machine Rotate Synchronously*

$$\begin{aligned}
\vec{\mathbf{B}}_S(i_{Sa}, i_{Sb}, r, \theta) &= \frac{\mu_0 r_R N_S}{2g}\frac{1}{r}\Big(i_{Sa}\cos(\theta) + i_{Sb}\sin(\theta)\Big)\hat{\mathbf{r}} \\
\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, r, \theta - \theta_R) &= \frac{\mu_0 r_R N_R}{2g}\frac{1}{r}\Big(i_{Ra}\cos(\theta - \theta_R) + i_{Rb}\sin(\theta - \theta_R)\Big)\hat{\mathbf{r}}.
\end{aligned}$$

Apply $u_{Sa} = U_S\cos(\omega_S t)$, $u_{Sb} = U_S\sin(\omega_S t)$ to obtain steady-state solutions

$$\begin{aligned}
i_{Sa} &= I_S\cos(\omega_S t + \phi_S) \\
i_{Sb} &= I_S\sin(\omega_S t + \phi_S) \\
i_{Ra} &= I_R\cos((\omega_S - \omega_R)t + \phi_R) \\
i_{Rb} &= I_R\sin((\omega_S - \omega_R)t + \phi_R) \\
\theta_R &= \omega_R t.
\end{aligned}$$

- ϕ_S, ϕ_R are functions of ω_S and $(\omega_S - \omega_R)/\omega_S$ (see book)
- ϕ_S, ϕ_R are thus **constant** in steady-state.

$\vec{\mathbf{B}}_S$ and $\vec{\mathbf{B}}_R$ of an Induction Machine Rotate Synchronously

Substitute $i_{Sa} = I_S\cos(\omega_S t + \phi_S)$ and $i_{Sb} = I_S\sin(\omega_S t + \phi_S)$ into

$$\vec{\mathbf{B}}_S(i_{Sa}, i_{Sb}, r, \theta) = \frac{\mu_0 r_R N_S}{2g}\frac{1}{r}\Big(i_{Sa}\cos(\theta) + i_{Sb}\sin(\theta)\Big)\hat{\mathbf{r}}$$

to obtain

$$\vec{\mathbf{B}}_S(r,\theta,t) = \frac{\mu_0 r_R N_S I_S}{2g}\frac{1}{r}\cos(\theta - (\omega_S t + \phi_S))\hat{\mathbf{r}}, \quad \theta_{B_S}(t) \triangleq \omega_S t + \phi_S$$

- $\theta_{B_S}(t) \triangleq \omega_S t + \phi_S$ is the **magnetic axis** of $\vec{\mathbf{B}}_S$.

Substitute $i_{Ra} = I_R\cos((\omega_S - \omega_R)t + \phi_R)$ and $i_{Rb} = I_R\sin((\omega_S - \omega_R)t + \phi_R)$ into

$$\vec{\mathbf{B}}_R(i_{Ra}, i_{Rb}, r, \theta - \theta_R) = \frac{\mu_0 r_R N_R}{2g}\frac{1}{r}\Big(i_{Ra}\cos(\theta - \theta_R) + i_{Rb}\sin(\theta - \theta_R)\Big)\hat{\mathbf{r}}.$$

to obtain

$$\begin{aligned}\vec{\mathbf{B}}_R(r,\theta,t) &= \frac{\mu_0 r_R N_R I_R}{2g}\frac{1}{r}\Big(\cos((\omega_S - \omega_R)t + \phi_R)\cos(\theta - \omega_R t) + \\ &\qquad \sin((\omega_S - \omega_R)t + \phi_R)\sin(\theta - \omega_R t)\Big)\hat{\mathbf{r}} \\ &= \frac{\mu_0 r_R N_R I_R}{2g}\frac{1}{r}\cos\Big(\theta - \omega_R t - \Big((\omega_S - \omega_R)t + \phi_R\Big)\Big)\hat{\mathbf{r}} \\ &= \frac{\mu_0 r_R N_R I_R}{2g}\frac{1}{r}\cos\Big(\theta - (\omega_S t + \phi_R)\Big)\hat{\mathbf{r}}, \quad \theta_{B_R}(t) \triangleq \omega_S t + \phi_R\end{aligned}$$

- $\theta_{B_R}(t)$ is the **magnetic axis** of $\vec{\mathbf{B}}_R$.

$\vec{\mathbf{B}}_S$ and $\vec{\mathbf{B}}_R$ of an Induction Machine Rotate Synchronously

$$\begin{aligned}
\vec{\mathbf{B}}_S(r,\theta,t) &= \frac{\mu_0 r_R N_S I_S}{2g}\frac{1}{r}\cos\Big(\theta - (\omega_S t + \phi_S)\Big)\hat{\mathbf{r}} \\
\theta_{B_S}(t) &\triangleq \omega_S t + \phi_S
\end{aligned}$$

$$\begin{aligned}
\vec{\mathbf{B}}_R(r,\theta,t) &= \frac{\mu_0 r_R N_R I_R}{2g}\frac{1}{r}\cos\Big(\theta - (\omega_S t + \phi_R)\Big)\hat{\mathbf{r}} \\
\theta_{B_R}(t) &\triangleq \omega_S t + \phi_R
\end{aligned}$$

- Both of these magnetic fields (magnets) rotate at the **same** angular rate ω_S.
- The angle between the two **"magnets"** $\theta_{B_S}(t) - \theta_{B_R}(t) = \phi_S - \phi_R$ is **constant**.
- The two magnetic fields rotate **synchronously** together!
- i_{Sa} and i_{Sb} have frequency ω_S and produce $\vec{\mathbf{B}}_S$ rotating at ω_S.
- i_{Ra} and i_{Rb} have frequency $\omega_S - \omega_R$ and produce $\vec{\mathbf{B}}_R$ rotating at $\omega_S - \omega_R$ **with respect to the rotor**.
- The rotor has angular speed ω_R so that $\vec{\mathbf{B}}_R$ has speed $\omega_S - \omega_R + \omega_R = \omega_S$ **with respect to the stator**.

$\vec{\mathbf{B}}_S$ and $\vec{\mathbf{B}}_R$ of an Induction Machine Rotate Synchronously

The steady-state torque is (see book problems)

$$\begin{aligned}
\tau_R = MI_S I_R \sin(\phi_S - \phi_R) &= g2\pi(\ell_1/2)\ell_2 \frac{1}{2\mu_0}\left(\kappa\frac{\mu_0 N_S I_S}{2g}\right)\left(\frac{\mu_0 N_R I_R}{2g}\right)\sin(\phi_S - \phi_R) \\
&= V_{\text{airgap}}\frac{1}{2\mu_0}B_{S\,\text{max}}B_{R\,\text{max}}\sin(\phi_S - \phi_R)
\end{aligned}$$

where $\phi_S - \phi_R$ is the angle between the stator and rotor magnetic axes and

$$\begin{aligned}
M &\triangleq \kappa\mu_0\pi\ell_1\ell_2 N_S N_R/(8g) \\
B_{S\,\text{max}} &\triangleq \kappa\mu_0 N_S I_S/(2g) \\
B_{R\,\text{max}} &\triangleq \mu_0 N_R I_R/(2g) \\
V_{\text{airgap}} &\triangleq g2\pi(\ell_1/2)\ell_2 \ (\approx \text{ vol of the air gap})
\end{aligned}$$

- There is an expression identical **in form** for the torque of a **synchronous** motor! (see book)