ECE 697 Modeling and High-Performance Control of Electric Machines
HW 10 Solutions
Spring 2022

Problem 1 *State Space Form*
Starting with the model

$$
\begin{aligned}
\frac{d\omega}{dt} &= \frac{n_p M}{J L_R}\left(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}\right) - \frac{f}{J}\omega - \frac{\tau_L}{J} \\
\frac{d\psi_{Ra}}{dt} &= -\frac{R_R}{L_R}\psi_{Ra} - n_p\omega\psi_{Rb} + \frac{M R_R}{L_R} i_{Sa} \\
\frac{d\psi_{Rb}}{dt} &= -\frac{R_R}{L_R}\psi_{Rb} + n_p\omega\psi_{Ra} + \frac{M R_R}{L_R} i_{Sb} \\
u_{Sa} &= R_S i_{Sa} + \sigma L_S \frac{di_{Sa}}{dt} + \frac{M}{L_R}\frac{d\psi_{Ra}}{dt} \\
u_{Sb} &= R_S i_{Sb} + \sigma L_S \frac{di_{Sb}}{dt} + \frac{M}{L_R}\frac{d\psi_{Rb}}{dt}
\end{aligned}
$$

substitute

$$
\begin{aligned}
\frac{d\psi_{Ra}}{dt} &= -\frac{R_R}{L_R}\psi_{Ra} - n_p\omega\psi_{Rb} + \frac{M R_R}{L_R} i_{Sa} \\
\frac{d\psi_{Rb}}{dt} &= -\frac{R_R}{L_R}\psi_{Rb} + n_p\omega\psi_{Ra} + \frac{M R_R}{L_R} i_{Sb}
\end{aligned}
$$

into

$$
\begin{aligned}
u_{Sa} &= R_S i_{Sa} + \sigma L_S \frac{di_{Sa}}{dt} + \frac{M}{L_R}\frac{d\psi_{Ra}}{dt} \\
u_{Sb} &= R_S i_{Sb} + \sigma L_S \frac{di_{Sb}}{dt} + \frac{M}{L_R}\frac{d\psi_{Rb}}{dt}
\end{aligned}
$$

to obtain

$$
\begin{aligned}
u_{Sa} &= R_S i_{Sa} + \sigma L_S \frac{di_{Sa}}{dt} + \frac{M}{L_R}\left(-\frac{R_R}{L_R}\psi_{Ra} - n_p\omega\psi_{Rb} + \frac{M R_R}{L_R} i_{Sa}\right) \\
u_{Sb} &= R_S i_{Sb} + \sigma L_S \frac{di_{Sb}}{dt} + \frac{M}{L_R}\left(-\frac{R_R}{L_R}\psi_{Rb} + n_p\omega\psi_{Ra} + \frac{M R_R}{L_R} i_{Sb}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
u_{Sa} &= R_S i_{Sa} + \sigma L_S \frac{di_{Sa}}{dt} - \frac{M R_R}{L_R^2}\psi_{Ra} - \frac{M}{L_R} n_p\omega\psi_{Rb} + \frac{M^2 R_R}{L_R^2} i_{Sa} \\
u_{Sb} &= R_S i_{Sb} + \sigma L_S \frac{di_{Sb}}{dt} - \frac{M R_R}{L_R^2}\psi_{Rb} + \frac{M}{L_R} n_p\omega\psi_{Ra} + \frac{M^2 R_R}{L_R^2} i_{Sb}.
\end{aligned}
$$

Solving for the derivative of the currents gives

$$
\begin{aligned}
\frac{di_{Sa}}{dt} &= \eta\beta\psi_{Ra} + \beta n_p\omega\psi_{Rb} - \gamma i_{Sa} + u_{Sa}/\sigma L_S \\
\frac{di_{Sb}}{dt} &= \eta\beta\psi_{Rb} - \beta n_p\omega\psi_{Ra} - \gamma i_{Sb} + u_{Sb}/\sigma L_S
\end{aligned}
$$

where

$$
\eta \triangleq \frac{R_R}{L_R}, \beta \triangleq \frac{M}{\sigma L_R L_S}, \mu \triangleq \frac{n_p M}{J L_R}, \text{ and } \gamma \triangleq \frac{M^2 R_R}{\sigma L_R^2 L_S} + \frac{R_S}{\sigma L_S}.
$$

Problem 2 *Induction Motor Model in the Space Vector Formulation*

A space vector representation of the induction motor is given by

$$
\begin{aligned}
R_S\underline{i}_S + L_S\frac{d}{dt}\underline{i}_S + M\frac{d}{dt}\left(\underline{i}_R e^{jn_p\theta_R}\right) &= \underline{u}_S \\
R_R\underline{i}_R + L_R\frac{d}{dt}\underline{i}_R + M\frac{d}{dt}\left(\underline{i}_S e^{-jn_p\theta_R}\right) &= 0 \\
n_p M\,\mathrm{Im}\{\underline{i}_S(\underline{i}_R e^{j\theta_R})^*\} - \tau_L &= J\frac{d\omega_R}{dt}
\end{aligned}
$$

Define new (fictitious) flux linkages as

$$\underline{\psi}_R \triangleq \psi_{Ra} + j\psi_{Rb} \triangleq \underline{\lambda}_R e^{jn_p\theta_R} = L_R\underline{i}_R e^{jn_p\theta_R} + M\underline{i}_S$$

and substitute into

$$R_S\underline{i}_S + \; L_S\frac{d}{dt}\underline{i}_S + M\frac{d}{dt}\left(\underline{i}_R e^{jn_p\theta_R}\right) = \underline{u}_S$$

to obtain

$$
\begin{aligned}
R_S\underline{i}_S + L_S\frac{d}{dt}\underline{i}_S + \frac{M}{L_R}\frac{d}{dt}\left(\underline{\psi}_R - M\underline{i}_S\right) &= \underline{u}_S \\
\Longrightarrow \frac{M}{L_R}\frac{d}{dt}\underline{\psi}_R + \sigma L_S\frac{d}{dt}\underline{i}_S + R_S\underline{i}_S &= \underline{u}_S
\end{aligned}
$$

where $\sigma \triangleq 1 - M^2/L_SL_R$ is the leakage factor.

Next, multiply both sides of

$$R_R\underline{i}_R + L_R\frac{d}{dt}\underline{i}_R + \; M\frac{d}{dt}\left(\underline{i}_S e^{-jn_p\theta_R}\right) = 0$$

by $e^{jn_p\theta_R}$ to obtain

$$
\begin{aligned}
R_R\underline{i}_R e^{jn_p\theta_R} + L_R\frac{d}{dt}\left(\underline{i}_R e^{jn_p\theta_R}\right) - jn_p\omega_R L_R\underline{i}_R e^{jn_p\theta_R} + \; M\frac{d}{dt}\underline{i}_S + jn_p\omega_R M\underline{i}_S &= 0 \\
\Longrightarrow \frac{R_R}{L_R}\left(\underline{\psi}_R - M\underline{i}_S\right) + \frac{d}{dt}\left(\underline{\psi}_R - M\underline{i}_S\right) - jn_p\omega_R\left(\underline{\psi}_R - M\underline{i}_S\right) + \; M\frac{d}{dt}\underline{i}_S + jn_p\omega_R M\underline{i}_S &= 0 \\
\Longrightarrow \frac{R_R}{L_R}\left(\underline{\psi}_R - M\underline{i}_S\right) + \frac{d}{dt}\left(\underline{\psi}_R - M\underline{i}_S\right) - jn_p\omega_R\underline{\psi}_R + M\frac{d}{dt}\underline{i}_S &= 0 \\
\Longrightarrow \frac{R_R}{L_R}\left(\underline{\psi}_R - M\underline{i}_S\right) + \frac{d}{dt}\underline{\psi}_R - jn_p\omega_R\underline{\psi}_R &= 0 \\
\Longrightarrow \left(-\frac{1}{T_R} + jn_p\omega_R\right)\underline{\psi}_R + \frac{M}{T_R}\underline{i}_S &= \frac{d}{dt}\underline{\psi}_R
\end{aligned}
$$

where $T_R \triangleq L_R/R_R$ is the rotor time constant. Finally, rewrite the torque equation

$$n_p M\,\mathrm{Im}\{\underline{i}_S(\underline{i}_R e^{jn_p\theta_R})^*\} - \tau_L = J\frac{d\omega_R}{dt}$$

as

$$
\begin{aligned}
n_p\frac{M}{L_R}\,\mathrm{Im}\left\{\underline{i}_S\left(\underline{\psi}_R - M\underline{i}_S\right)^*\right\} - \tau_L &= J\frac{d\omega_R}{dt} \\
\Longrightarrow n_p\frac{M}{L_R}\,\mathrm{Im}\left\{\underline{i}_S\underline{\psi}_R^* - M\underline{i}_S\underline{i}_S^*\right\} - \tau_L &= J\frac{d\omega_R}{dt} \\
\Longrightarrow \frac{n_pM}{L_R}\,\mathrm{Im}\left\{\underline{i}_S\underline{\psi}_R^*\right\} - \tau_L &= J\frac{d\omega_R}{dt} \\
\Longrightarrow \frac{n_pM}{JL_R}\,\mathrm{Im}\left\{\underline{i}_S\underline{\psi}_R^*\right\} - \frac{\tau_L}{J} &= \frac{d\omega_R}{dt}.
\end{aligned}
$$

Combining the above gives

$$\begin{aligned}
\frac{d}{dt}\underline{\psi}_R &= \left(-\frac{1}{T_R} + jn_p\omega_R\right)\underline{\psi}_R + \frac{M}{T_R}\underline{i}_S \\
\underline{u}_S &= \frac{M}{L_R}\frac{d}{dt}\underline{\psi}_R + \sigma L_S \frac{d}{dt}\underline{i}_S + R_S\underline{i}_S \\
\frac{d\omega_R}{dt} &= \mu\,\mathrm{Im}\left\{\underline{i}_S(\underline{\psi}_R)^*\right\} - \frac{\tau_L}{J}
\end{aligned}$$

where $\mu \triangleq n_p M/(JL_R)$.

Equating real and imaginary parts, the model can be rewritten as the five differential equations

$$\begin{aligned}
\frac{d\psi_{Ra}}{dt} &= -\frac{R_R}{L_R}\psi_{Ra} - n_p\omega_R\psi_{Rb} + \frac{MR_R}{L_R}i_{Sa} \\
\frac{d\psi_{Rb}}{dt} &= -\frac{R_R}{L_R}\psi_{Rb} + n_p\omega_R\psi_{Ra} + \frac{MR_R}{L_R}i_{Sb} \\
u_{Sa} &= R_S i_{Sa} + \sigma L_S\frac{di_{Sa}}{dt} + \frac{M}{L_R}\frac{d\psi_{Ra}}{dt} \\
u_{Sb} &= R_S i_{Sb} + \sigma L_S\frac{di_{Sb}}{dt} + \frac{M}{L_R}\frac{d\psi_{Rb}}{dt} \\
\frac{d\omega_R}{dt} &= \frac{n_p M}{JL_R}\left(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}\right) - \frac{\tau_L}{J}.
\end{aligned}$$

Problem 3 *Induction Motor Model in Terms of the Flux Linkages*

With λ_{Sa} and λ_{Sb} the total flux linkage in stator phases a and b, respectively, define

$$\underline{\lambda}_S \triangleq \lambda_{Sa} + j\lambda_{Sb} \triangleq L_S\underline{i}_S + M\underline{i}_R e^{+jn_p\theta_R}.$$

Similarly, let λ_{Ra} and λ_{Rb} be the total flux linkage in rotor phases a and b, respectively, and define

$$\underline{\lambda}_R \triangleq \lambda_{Ra} + j\lambda_{Rb} \triangleq L_R\underline{i}_R + M\underline{i}_S e^{-jn_p\theta_R}.$$

(a) By Faraday's law, the electrical equations of the induction motor may be written as

$$\begin{aligned}
R_S\underline{i}_S + \frac{d}{dt}\underline{\lambda}_S &= \underline{u}_S \\
R_R\underline{i}_R + \frac{d}{dt}\underline{\lambda}_R &= 0.
\end{aligned}$$

(b) Inverting the relationship

$$\begin{bmatrix} \underline{\lambda}_S \\ \underline{\lambda}_R e^{+jn_p\theta_R} \end{bmatrix} = \begin{bmatrix} L_S & M \\ M & L_R \end{bmatrix}\begin{bmatrix} \underline{i}_S \\ \underline{i}_R e^{+jn_p\theta_R} \end{bmatrix}$$

gives

$$\begin{bmatrix} \underline{i}_S \\ \underline{i}_R e^{+jn_p\theta_R} \end{bmatrix} = \frac{1}{L_SL_R - M^2}\begin{bmatrix} L_R & -M \\ -M & L_S \end{bmatrix}\begin{bmatrix} \underline{\lambda}_S \\ \underline{\lambda}_R e^{+jn_p\theta_R} \end{bmatrix}.$$

Substituting these expressions for $\underline{i}_S$ and $\underline{i}_R e^{+jn_p\theta_R}$ into

$$\begin{aligned}
R_S\underline{i}_S + \frac{d}{dt}\left(L_S\frac{d}{dt}\underline{i}_S + M\underline{i}_R e^{jn_p\theta_R}\right) &= \underline{u}_S \\
R_R\underline{i}_R + \frac{d}{dt}\left(L_R\underline{i}_R + M\underline{i}_S e^{-jn_p\theta_R}\right) &= 0 \\
n_pM\,\mathrm{Im}\{\underline{i}_S(\underline{i}_R e^{jn_p\theta_R})^*\} - \tau_L &= J\frac{d\omega_R}{dt}
\end{aligned}$$

gives

$$
\begin{aligned}
\frac{R_S}{L_S L_R - M^2}\left(L_R \underline{\lambda}_S - M \underline{\lambda}_R e^{+jn_p\theta_R}\right) + \frac{d}{dt}\underline{\lambda}_S &= \underline{u}_S \\
\frac{R_R}{L_S L_R - M^2}\left(-M \underline{\lambda}_S e^{-jn_p\theta_R} + L_S \underline{\lambda}_R\right) + \frac{d}{dt}\underline{\lambda}_R &= 0 \\
\frac{n_p M}{\left(L_S L_R - M^2\right)^2} \operatorname{Im}\left\{\left(L_R \underline{\lambda}_S - M \underline{\lambda}_R e^{+jn_p\theta_R}\right)\left(-M \underline{\lambda}_S + L_S \underline{\lambda}_R e^{+jn_p\theta_R}\right)^*\right\} - \tau_L &= J\frac{d\omega_R}{dt}.
\end{aligned}
$$

The last equation is simplified by

$$
\begin{aligned}
\frac{n_p M}{\left(L_S L_R - M^2\right)^2} \operatorname{Im}\left\{\left(L_S L_R \underline{\lambda}_S (\underline{\lambda}_R e^{+jn_p\theta_R})^* + M^2 (\underline{\lambda}_R e^{+jn_p\theta_R}) \underline{\lambda}_S^*\right)\right\} - \tau_L &= J\frac{d\omega_R}{dt} \\
\frac{n_p M}{\left(L_S L_R - M^2\right)^2} \operatorname{Im}\left\{\left(L_S L_R \underline{\lambda}_S (\underline{\lambda}_R e^{+jn_p\theta_R})^* - M^2 \underline{\lambda}_S (\underline{\lambda}_R e^{+jn_p\theta_R})^*\right)\right\} - \tau_L &= J\frac{d\omega_R}{dt} \\
\frac{n_p M}{\left(L_S L_R - M^2\right)^2} \operatorname{Im}\left\{\left(\left(L_S L_R - M^2\right) \underline{\lambda}_S (\underline{\lambda}_R e^{+jn_p\theta_R})^*\right)\right\} - \tau_L &= J\frac{d\omega_R}{dt}
\end{aligned}
$$

so that finally

$$
\begin{aligned}
\frac{R_S}{L_S L_R - M^2}\left(L_R \underline{\lambda}_S - M \underline{\lambda}_R e^{+jn_p\theta_R}\right) + \frac{d}{dt}\underline{\lambda}_S &= \underline{u}_S \\
\frac{R_R}{L_S L_R - M^2}\left(-M \underline{\lambda}_S e^{-jn_p\theta_R} + L_S \underline{\lambda}_R\right) + \frac{d}{dt}\underline{\lambda}_R &= 0 \\
\frac{M}{L_S L_R - M^2} \operatorname{Im}\{\underline{\lambda}_S (\underline{\lambda}_R e^{+jn_p\theta_R})^*\} - \tau_L &= J\frac{d\omega_R}{dt} \\
\frac{d\theta_R}{dt} &= \omega_R
\end{aligned}
$$

is a representation of the induction motor in terms $\underline{\lambda}_S \triangleq \lambda_{Sa} + j\lambda_{Sb}, \underline{\lambda}_R \triangleq \lambda_{Ra} + j\lambda_{Rb}, \omega_R$, and θ_R.

Note that by equating real and imaginary parts of the answer to part (b), a statespace representation of the induction motor in terms of the state variables $\lambda_{Sa}, \lambda_{Sb}, \lambda_{Ra}, \lambda_{Rb}, \omega_R$, and θ_R is

$$
\begin{aligned}
\frac{d\lambda_{Sa}}{dt} &= -\frac{R_S L_R}{L_S L_R - M^2}\lambda_{Sa} + \frac{R_S M}{L_S L_R - M^2}\Big(\lambda_{Ra}\cos(n_p\theta_R) - \lambda_{Rb}\sin(n_p\theta_R)\Big) + u_{Sa} \\
\frac{d\lambda_{Sb}}{dt} &= -\frac{R_S L_R}{L_S L_R - M^2}\lambda_{Sa} + \frac{R_S M}{L_S L_R - M^2}\Big(\lambda_{Ra}\sin(n_p\theta_R) + \lambda_{Rb}\cos(n_p\theta_R)\Big) + u_{Sb} \\
\frac{d\lambda_{Ra}}{dt} &= -\frac{R_R L_S}{L_S L_R - M^2}\lambda_{Ra} + \frac{M R_R}{L_S L_R - M^2}\Big(\lambda_{Sa}\cos(n_p\theta_R) + \lambda_{Sb}\sin(n_p\theta_R)\Big) \\
\frac{d\lambda_{Rb}}{dt} &= -\frac{R_R L_S}{L_S L_R - M^2}\lambda_{Rb} + \frac{M R_R}{L_S L_R - M^2}\Big(-\lambda_{Sa}\sin(n_p\theta_R) + \lambda_{Sb}\cos(n_p\theta_R)\Big) \\
J\frac{d\omega_R}{dt} &= \frac{M}{L_S L_R - M^2}\Big(-\lambda_{Sa}\Big(\lambda_{Ra}\sin(n_p\theta_R) + \lambda_{Rb}\cos(n_p\theta_R)\Big) + \lambda_{Sb}\Big(\lambda_{Ra}\cos(n_p\theta_R) - \lambda_{Rb}\sin(n_p\theta_R)\Big)\Big) - \tau_L \\
\frac{d\theta_R}{dt} &= \omega_R.
\end{aligned}
$$

Problem 4 *Field-Oriented Induction Motor Model*

We have

$$\begin{aligned}
\frac{d}{dt}\underline{\psi}_R &= \left(-\frac{1}{T_R} + jn_p\omega\right)\underline{\psi}_R + \frac{M}{T_R}\underline{i}_S \\
\underline{u}_S &= \frac{M}{L_R}\frac{d}{dt}\underline{\psi}_R + \sigma L_S \frac{d}{dt}\underline{i}_S + R_S\underline{i}_S \\
\frac{d\omega}{dt} &= \mu \operatorname{Im}\left\{\underline{i}_S(\underline{\psi}_R)^*\right\} - \frac{\tau_L}{J}
\end{aligned}$$

where

$$\begin{aligned}
\underline{\psi}_R &\triangleq \psi_{Ra} + j\psi_{Rb} = \psi_d e^{j\rho} \\
\psi_d &\triangleq \sqrt{\psi_{Ra}^2 + \psi_{Rb}^2} \\
\rho &\triangleq \tan^{-1}(\psi_{Rb}/\psi_{Ra}) \\
\underline{i}_{dq} &\triangleq i_d + ji_q = \underline{i}_S e^{-j\rho} \\
\underline{u}_{dq} &\triangleq u_d + ju_q = \underline{u}_S e^{-j\rho}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d}{dt}\left(\psi_d e^{j\rho}\right) &= \left(-\frac{1}{T_R} + jn_p\omega\right)\psi_d e^{j\rho} + \frac{M}{T_R}\underline{i}_{dq}e^{j\rho} \\
\underline{u}_{dq}e^{j\rho} &= \frac{M}{L_R}\frac{d}{dt}\left(\psi_d e^{j\rho}\right) + \sigma L_S \frac{d}{dt}\left(\underline{i}_{dq}e^{j\rho}\right) + R_S\underline{i}_{dq}e^{j\rho} \\
\frac{d\omega}{dt} &= \mu \operatorname{Im}\left\{\underline{i}_{dq}e^{j\rho}(\psi_d e^{j\rho})^*\right\} - \frac{\tau_L}{J}
\end{aligned}$$

or

$$\begin{aligned}
\psi_d e^{j\rho} j\frac{d\rho}{dt} + e^{j\rho}\frac{d}{dt}\psi_d &= \left(-\frac{1}{T_R} + jn_p\omega\right)\psi_d e^{j\rho} + \frac{M}{T_R}\underline{i}_{dq}e^{j\rho} \\
\underline{u}_{dq}e^{j\rho} &= \frac{M}{L_R}\left(\psi_d e^{j\rho} j\frac{d\rho}{dt} + e^{j\rho}\frac{d}{dt}\psi_d\right) + \sigma L_S\left(e^{j\rho}\frac{d}{dt}\underline{i}_{dq} + \underline{i}_{dq}e^{j\rho}j\frac{d\rho}{dt}\right) + R_S\underline{i}_{dq}e^{j\rho} \\
\frac{d\omega}{dt} &= \mu \operatorname{Im}\left\{\underline{i}_{dq}\psi_d\right\} - \frac{\tau_L}{J}.
\end{aligned}$$

After some rearrangement, one obtains

$$\begin{aligned}
j\psi_d\frac{d\rho}{dt} + \frac{d}{dt}\psi_d &= \left(-\frac{1}{T_R} + jn_p\omega\right)\psi_d + \frac{M}{T_R}\underline{i}_{dq} \\
\underline{u}_{dq} &= \frac{M}{L_R}\left(\psi_d j\frac{d\rho}{dt} + \frac{d}{dt}\psi_d\right) + \sigma L_S\left(\frac{d}{dt}\underline{i}_{dq} + \underline{i}_{dq}j\frac{d\rho}{dt}\right) + R_S\underline{i}_{dq} \\
\frac{d\omega}{dt} &= \mu i_q\psi_d - \frac{\tau_L}{J}.
\end{aligned}$$

Equating real and imaginary parts one obtains

$$
\begin{aligned}
\frac{d}{dt}\psi_d &= -\frac{1}{T_R}\psi_d + \frac{M}{T_R}i_d \\
\psi_d\frac{d\rho}{dt} &= n_p\omega\psi_d + \frac{M}{T_R}i_d \\
u_d &= \frac{M}{L_R}\frac{d}{dt}\psi_d + \sigma L_S\left(\frac{di_d}{dt} - i_q\frac{d\rho}{dt}\right) + R_S i_d \\
u_q &= \frac{M}{L_R}\psi_d\frac{d\rho}{dt} + \sigma L_S\left(\frac{di_q}{dt} + i_d\frac{d\rho}{dt}\right) + R_S i_q \\
\frac{d\omega}{dt} &= \mu i_q\psi_d - \frac{\tau_L}{J}.
\end{aligned}
$$

Using $d\rho/dt = n_p\omega + Mi_q/(T_R\psi_d)$ and $d\psi_d/dt = -(1/T_R) + (M/T_R)i_d$, this becomes

$$
\begin{aligned}
\frac{d}{dt}\psi_d &= -\frac{1}{T_R}\psi_d + \frac{M}{T_R}i_d \\
\frac{d\rho}{dt} &= n_p\omega + \frac{M}{T_R}\frac{i_q}{\psi_d} \\
u_d &= \frac{M}{L_R}\left(-\frac{1}{T_R}\psi_d + \frac{M}{T_R}i_d\right) + \sigma L_S\frac{di_d}{dt} - \sigma L_S i_q\left(n_p\omega + \frac{M}{T_R}\frac{i_q}{\psi_d}\right) + \ R_S i_d \\
u_q &= \left(\frac{M}{L_R}\psi_d + \sigma L_S i_d\right)\left(n_p\omega + \frac{M}{T_R}\frac{i_q}{\psi_d}\right) + \sigma L_S\frac{di_q}{dt} + R_S i_q \\
\frac{d\omega}{dt} &= \mu i_q\psi_d - \frac{\tau_L}{J}.
\end{aligned}
$$

Finally, rearranging gives

$$
\begin{aligned}
\frac{d\theta}{dt} &= \omega \\
\frac{d\omega}{dt} &= \mu\psi_d i_q - \tau_L/J \\
\frac{d\psi_d}{dt} &= -\eta\psi_d + \eta M i_d \\
\frac{di_d}{dt} &= -\gamma i_d + (\eta M/\sigma L_R L_S)\psi_d + n_p\omega i_q + \eta M i_q^2/\psi_d + u_d/\sigma L_S \\
\frac{di_q}{dt} &= -\gamma i_q - (M/\sigma L_R L_S)n_p\omega\psi_d - n_p\omega i_d - \eta M i_q i_d/\psi_d + u_q/\sigma L_S \\
\frac{d\rho}{dt} &= n_p\omega + \eta M i_q/\psi_d
\end{aligned}
$$

where

$$
\eta \triangleq \frac{R_R}{L_R}, \beta \triangleq \frac{M}{\sigma L_R L_S}, \mu \triangleq \frac{n_p M}{J L_R}, \text{ and } \gamma \triangleq \frac{M^2 R_R}{\sigma L_R^2 L_S} + \frac{R_S}{\sigma L_S}.
$$

Problem 5 *Stator Flux Induction Motor Model*

(a) The stator flux linkage is given by

$$
\underline{\lambda}_S = L_S\underline{i}_S + M\left(\underline{i}_R e^{jn_p\theta_R}\right) = \lambda_S e^{j\rho_S}.
$$

Rewrite the system model

$$
\begin{aligned}
R_S \underline{i}_S + L_S \frac{d}{dt}\underline{i}_S + M\frac{d}{dt}\left(\underline{i}_R e^{jn_p\theta_R}\right) &= \underline{u}_S \\
R_R \underline{i}_R + L_R \frac{d}{dt}\underline{i}_R + M\frac{d}{dt}\left(\underline{i}_S e^{-jn_p\theta_R}\right) &= 0 \\
n_p M \operatorname{Im}\left\{\underline{i}_S \left(\underline{i}_R e^{jn_p\theta_R}\right)^* \right\} - \tau_L &= J\frac{d\omega_R}{dt}
\end{aligned}
$$

in terms of $\underline{\lambda}_S, \underline{i}_S$, and ω_R. The first equation becomes

$$\frac{d}{dt}\underline{\lambda}_S = -R_S\underline{i}_S + \underline{u}_S.$$

Letting $\underline{i}_r \triangleq \underline{i}_R e^{jn_p\theta_R}$, the second equation becomes

$$R_R\underline{i}_r + L_R\frac{d}{dt}\underline{i}_r + M\frac{d}{dt}\underline{i}_S - jn_p\omega_R\left(L_R\underline{i}_r + M\underline{i}_S\right) = 0.$$

Substituting

$$\underline{i}_r = \frac{\underline{\lambda}_S - L_S\underline{i}_S}{M},$$

this can be rewritten as

$$R_R\left(\frac{\underline{\lambda}_S - L_S\underline{i}_S}{M}\right) + L_R\frac{d}{dt}\left(\frac{\underline{\lambda}_S - L_S\underline{i}_S}{M}\right) + M\frac{d}{dt}\underline{i}_S - jn_p\omega_R\left(L_R\left(\frac{\underline{\lambda}_S - L_S\underline{i}_S}{M}\right) + M\underline{i}_S\right) = 0$$

or

$$\frac{R_R}{M}\underline{\lambda}_S + \frac{L_R}{M}\frac{d}{dt}\underline{\lambda}_S - jn_p\omega_R\frac{L_R}{M}\underline{\lambda}_S - R_R\frac{L_S}{M}\underline{i}_S - \frac{L_SL_R}{M}\frac{d}{dt}\underline{i}_S + M\frac{d}{dt}\underline{i}_S + jn_p\omega_R\left(\frac{L_SL_R}{M}\underline{i}_S - M\underline{i}_S\right) = 0$$

or

$$\frac{R_R}{L_R}\underline{\lambda}_S + \frac{d}{dt}\underline{\lambda}_S - jn_p\omega_R\underline{\lambda}_S - \frac{R_R}{L_R}L_S\underline{i}_S - L_S\frac{d}{dt}\underline{i}_S + \frac{M^2}{L_R}\frac{d}{dt}\underline{i}_S + jn_p\omega_R\left(L_S\underline{i}_S - \frac{M^2}{L_R}\underline{i}_S\right) = 0$$

or

$$\frac{R_R}{L_R}\underline{\lambda}_S + \frac{d}{dt}\underline{\lambda}_S - jn_p\omega_R\underline{\lambda}_S - \frac{R_R}{L_R}L_S\underline{i}_S - \sigma L_S\frac{d}{dt}\underline{i}_S + jn_p\omega_R\sigma L_S\underline{i}_S = 0$$

or

$$\frac{d}{dt}\underline{\lambda}_S + \left(\frac{1}{T_R} - jn_p\omega_R\right)\underline{\lambda}_S - \sigma L_S\frac{d}{dt}\underline{i}_S - L_S\left(\frac{1}{T_R} - jn_p\omega_R\sigma\right)\underline{i}_S = 0.$$

The equation for torque is

$$n_pM\operatorname{Im}\left\{\underline{i}_S\left(\underline{i}_Re^{jn_p\theta_R}\right)^* \right\} = n_pM\operatorname{Im}\left\{\underline{i}_S\left(\frac{\underline{\lambda}_S - L_S\underline{i}_S}{M}\right)^* \right\} = n_p\operatorname{Im}\left\{\underline{i}_S\left(\underline{\lambda}_S\right)^* \right\}.$$

Collecting the equations together, we have

$$
\begin{aligned}
\frac{d}{dt}\underline{\lambda}_S &= -R_S\underline{i}_S + \underline{u}_S \\
\frac{d}{dt}\underline{\lambda}_S + \left(\frac{1}{T_R} - jn_p\omega_R\right)\underline{\lambda}_S - \sigma L_S\frac{d}{dt}\underline{i}_S - L_S\left(\frac{1}{T_R} - jn_p\omega_R\sigma\right)\underline{i}_S &= 0 \\
n_p\operatorname{Im}\left\{\underline{i}_S\left(\underline{\lambda}_S\right)^*\right\} - \tau_L &= J\frac{d\omega_R}{dt}
\end{aligned}
$$

or

$$\begin{aligned}
\frac{d}{dt}\underline{\lambda}_S &= -R_S\underline{i}_S + \underline{u}_S \\
\sigma L_S \frac{d}{dt}\underline{i}_S + L_S\left(\frac{1}{T_R} - jn_p\omega_R\sigma\right)\underline{i}_S &= -R_S\underline{i}_S + \underline{u}_S + \left(\frac{1}{T_R} - jn_p\omega_R\right)\underline{\lambda}_S \\
n_p \operatorname{Im}\left\{\underline{i}_S\left(\underline{\lambda}_S\right)^*\right\} - \tau_L &= J\frac{d\omega_R}{dt}.
\end{aligned}$$

(b) In statespace form, this becomes

$$\begin{aligned}
\frac{d\lambda_{Sa}}{dt} &= -R_S i_{Sa} + u_{Sa} \\
\frac{d\lambda_{Sb}}{dt} &= -R_S i_{Sb} + u_{Sb} \\
\frac{di_{Sa}}{dt} &= \left(-R_S i_{Sa} + u_{Sa} - \frac{L_S}{T_R} i_{Sa} - n_p\omega_R\sigma L_S i_{Sb} + \frac{1}{T_R}\lambda_{Sa} + n_p\omega_R\lambda_{Sb}\right)/\sigma L_S \\
\frac{di_{Sb}}{dt} &= \left(-R_S i_{Sb} + u_{Sb} - \frac{L_S}{T_R} i_{Sb} + n_p\omega_R\sigma L_S i_{Sa} + \frac{1}{T_R}\lambda_{Sa} - n_p\omega_R\lambda_{Sa}\right)/\sigma L_S \\
J\frac{d\omega_R}{dt} &= n_p\left(i_{Sb}\lambda_{Sa} - i_{Sa}\lambda_{Sb}\right) - \tau_L.
\end{aligned}$$

Problem 6 *ψ_d and ψ_q*

Now

$$\begin{aligned}
\rho &= \tan^{-1}\left(\psi_{Rb}/\psi_{Ra}\right) \\
\implies \tan(\rho) &= \psi_{Rb}/\psi_{Ra} \\
\implies \tan^2(\rho) + 1 &= \left(\psi_{Rb}/\psi_{Ra}\right)^2 + 1 = \sec^2\left(\psi_{Rb}/\psi_{Ra}\right) \\
\implies \cos^2(\rho) &= \frac{1}{\left(\psi_{Rb}/\psi_{Ra}\right)^2 + 1} = \frac{\psi_{Ra}^2}{\psi_{Ra}^2 + \psi_{Rb}^2} \\
\implies \cos(\rho) &= \frac{\psi_{Ra}}{\sqrt{\psi_{Ra}^2 + \psi_{Rb}^2}} = \frac{\psi_{Ra}}{\psi_d}
\end{aligned}$$

and also

$$\begin{aligned}
\sin^2(\rho) &= 1 - \cos^2(\rho) = 1 - \frac{\psi_{Ra}^2}{\psi_{Ra}^2 + \psi_{Rb}^2} = \frac{\psi_{Rb}^2}{\psi_{Ra}^2 + \psi_{Rb}^2} \\
\sin(\rho) &= \frac{\psi_{Rb}}{\sqrt{\psi_{Ra}^2 + \psi_{Rb}^2}} = \frac{\psi_{Rb}}{\psi_d}.
\end{aligned}$$

Finally,

$$\psi_q = -\sin(\rho)\psi_{Ra} + \cos(\rho)\psi_{Rb} = -\frac{\psi_{Rb}}{\psi_d}\psi_{Ra} + \frac{\psi_{Ra}}{\psi_d}\psi_{Rb} \equiv 0.$$

Feedback Control and Reference Trajectories

Problem 7 *Tracking a Constant Flux Reference*

Substitute

$$i_{dr} = K_{\psi I}\int_0^t (\psi_{d0} - \psi_d)dt + K_{\psi P}(\psi_{d0} - \psi_d) + i_{d0}$$

into

$$\frac{d\psi_d}{dt} = -\eta\psi_d + \eta M i_{dr}$$

to obtain

$$\frac{d\psi_d}{dt} = -\eta\psi_d + \eta M\left(K_{\psi I}\int_0^t(\psi_{d0}-\psi_d)dt + K_{\psi P}(\psi_{d0}-\psi_d) + i_{d0}\right).$$

Then, with $e_0 = \int_0^t(\psi_{d0}-\psi_d)dt$, $e_1 = \psi_{d0}-\psi_d$ and $\psi_{d0} = M i_{d0}$, we have

$$\begin{aligned}
\frac{de_0}{dt} &= e_1\\
\frac{de_1}{dt} &= \eta\psi_d - \eta M K_{\psi I}\int_0^t(\psi_{d0}-\psi_d)dt - \eta M K_{\psi P}(\psi_{d0}-\psi_d) - \eta M i_{d0}\\
&= -\eta\left(\psi_{d0}-\psi_d\right) - \eta M K_{\psi I}\int_0^t(\psi_{d0}-\psi_d)dt - \eta M K_{\psi P}(\psi_{d0}-\psi_d)\\
&= -\eta M K_{\psi I} e_0 - \eta\left(M K_{\psi P}+1\right)e_1.
\end{aligned}$$

Setting

$$\begin{aligned}
\eta\left(M K_{\psi P}+1\right) &= r_1 + r_2\\
\eta M K_{\psi I} &= r_1 r_2
\end{aligned}$$

or

$$\begin{aligned}
K_{\psi P} &= \frac{\left(r_1+r_2\right)/\eta - 1}{M}\\
K_{\psi I} &= \frac{r_1 r_2}{\eta M}
\end{aligned}$$

this becomes

$$\begin{aligned}
\frac{de_0}{dt} &= e_1\\
\frac{de_1}{dt} &= -r_1 r_2 e_0 - (r_1+r_2)e_1.
\end{aligned}$$

Taking the Laplace transform gives

$$\begin{aligned}
sE_0(s) - e_0(0) &= E_1(s)\\
sE_1(s) - e_1(0) &= -r_1 r_2 E_0(s) - (r_1+r_2)E_1(s)
\end{aligned}$$

or

$$\begin{bmatrix} s & -1 \\ r_1 r_2 & s+r_1+r_2 \end{bmatrix}\begin{bmatrix} E_0(s) \\ E_1(s) \end{bmatrix} = \begin{bmatrix} e_0(0) \\ e_1(0) \end{bmatrix}.$$

Solving for $E_0(s)$ and $E_1(s)$ gives

$$\begin{aligned}
\begin{bmatrix} E_0(s) \\ E_1(s) \end{bmatrix} &= \frac{1}{(s+r_1)(s+r_2)}\begin{bmatrix} s+r_1+r_2 & 1 \\ -r_1 r_2 & s \end{bmatrix}\begin{bmatrix} e_0(0) \\ e_1(0) \end{bmatrix}\\
&= \frac{1}{(s+r_1)(s+r_2)}\begin{bmatrix} \left(s+r_1+r_2\right)e_0(0) + e_1(0) \\ -r_1 r_2 e_0(0) + s e_1(0) \end{bmatrix}
\end{aligned}$$

where it follows by the method of partial fractions (r_1, r_2 distinct)

$$E_1(s) = \frac{-r_1 r_2 e_0(0) + s e_1(0)}{(s+r_1)(s+r_2)} = \frac{A}{s+r_1} + \frac{B}{s+r_2}$$

so that $e_1(t) \to 0$ with $r_1 > 0, r_2 > 0$.

Problem 8 *Trajectory Tracking via Field-Oriented Control*

Substitute

$$i_{qr} = \left(K_0 \int_0^t (\theta_{ref} - \theta) dt + K_1(\theta_{ref} - \theta) + K_2(\omega_{ref} - \omega) + (f/J)\omega + \alpha_{ref} \right) / \mu\psi_{d0}$$

into

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= \mu\psi_{d0} i_{qr} - (f/J)\omega - \tau_L/J \end{aligned}$$

to obtain

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= K_0 \int_0^t (\theta_{ref} - \theta) dt + K_1(\theta_{ref} - \theta) + K_2(\omega_{ref} - \omega) + \alpha_{ref} - \tau_L/J. \end{aligned}$$

With $e_0 = \int_0^t (\theta_{ref} - \theta) dt$, $e_1 = \theta_{ref} - \theta$, and $e_2 = \omega_{ref} - \omega$, and assuming the reference trajectory satisfies

$$\begin{aligned} \frac{d\theta_{ref}(t)}{dt} &= \omega_{ref}(t) \\ \frac{d\omega_{ref}(t)}{dt} &= \alpha_{ref}(t), \end{aligned}$$

one obtains

$$\begin{aligned} \frac{de_0}{dt} &= e_1 \\ \frac{de_1}{dt} &= e_2 \\ \frac{de_2}{dt} &= -K_0 e_0 - K_1 e_1 - K_2 e_2 + \tau_L/J. \end{aligned}$$

The Laplace transform of this system is

$$\begin{aligned} sE_0(s) - e_0(0) &= E_1(s) \\ sE_1(s) - e_1(0) &= E_2(s) \\ sE_2(s) - e_2(0) &= -K_0 E_0(s) - K_1 E_1(s) - K_2 E_2(s) + \tau_L(s)/J \end{aligned}$$

or

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ K_0 & K_1 & s + K_2 \end{bmatrix} \begin{bmatrix} E_0(s) \\ E_1(s) \\ E_2(s) \end{bmatrix} = \begin{bmatrix} e_0(0) \\ e_1(0) \\ e_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tau_L(s)/J \end{bmatrix}.$$

Then

$$\begin{bmatrix} E_0(s) \\ E_1(s) \\ E_2(s) \end{bmatrix} = \frac{1}{s^3 + K_2 s^2 + K_1 s + K_0} \begin{bmatrix} s^2 + K_2 s + K_1 & s + K_2 & 1 \\ -K_0 & s^2 + K_2 s & s \\ -K_0 s & -(K_1 s + K_0) & s^2 \end{bmatrix} \left(\begin{bmatrix} e_0(0) \\ e_1(0) \\ e_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tau_L(s)/J \end{bmatrix} \right).$$

With $\tau_L(s) = \tau_{L0}/s$ and

$$\begin{aligned} K_2 &= r_1 + r_2 + r_3 \\ K_1 &= r_1 r_2 + r_1 r_3 + r_2 r_3 \\ K_0 &= r_1 r_2 r_3, \end{aligned}$$

the characteristic polynomial may be written as

$$a(s) = s^3 + K_2 s^2 + K_1 s + K_0 = s^3 + (r_1 + r_2 + r_3)s^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)s + r_1 r_2 r_3$$
$$= (s + r_1)(s + r_2)(s + r_3).$$

The errors can then be written in the form

$$E_0(s) = \frac{\left(s^2 + K_2 s + K_1\right) e_0(0) + (s + K_2)\, e_1(0) + e_2(0)}{(s + r_1)(s + r_2)(s + r_3)} + \frac{\tau_{L0}/J}{(s + r_1)(s + r_2)(s + r_3)} \frac{1}{s}$$

$$E_1(s) = \frac{-K_0 e_0(0) + \left(s^2 + K_2 s\right) e_1(0) + s e_2(0)}{(s + r_1)(s + r_2)(s + r_3)} + \frac{s\,(\tau_{L0}/J)}{(s + r_1)(s + r_2)(s + r_3)} \frac{1}{s}$$

$$E_2(s) = \frac{-K_0 s e_0(0) - (K_1 s + K_0)\, e_1(0) + s^2 e_2(0)}{(s + r_1)(s + r_2)(s + r_3)} + \frac{s^2\,(\tau_{L0}/J)}{(s + r_1)(s + r_2)(s + r_3)} \frac{1}{s}.$$

The closed-loop poles are $p_1 = -r_1$, $p_2 = -r_2$, $p_3 = -r_3$ and assuming these poles are distinct, $E_1(s)$ becomes

$$E_1(s) = \frac{-K_0 e_0(0) + \left(s^2 + (K_2 + f/J)\, s\right) e_1(0) + s e_2(0)}{(s + r_1)(s + r_2)(s + r_3)} + \frac{\tau_{L0}/J}{(s + r_1)(s + r_2)(s + r_3)} \frac{1}{s}$$
$$= \frac{A}{s + r_1} + \frac{B}{s + r_2} + \frac{C}{s + r_3}$$

where A, B, and C are constants. The inverse Laplace transform of $E_1(s)$ is

$$e_1(t) = Ae^{-r_1 t} + Be^{-r_2 t} + Ce^{-r_3 t} \to 0.$$

Similarly, $e_2(t) \to 0$.

Problem 9 *Back Emf*

The field-oriented model of the induction motor is given by

$$\frac{d\omega}{dt} = \mu \psi_d i_q - (f/J)\omega - \tau_L/J$$
$$\frac{d\psi_d}{dt} = -\eta \psi_d + \eta M i_d$$
$$\frac{di_d}{dt} = -\gamma i_d + (\eta M/\sigma L_R L_S)\psi_d + n_p \omega i_q + \eta M i_q^2/\psi_d + u_d/\sigma L_S$$
$$\frac{di_q}{dt} = -\gamma i_q - (M/\sigma L_R L_S) n_p \omega \psi_d - n_p \omega i_d - \eta M i_q i_d/\psi_d + u_q/\sigma L_S$$
$$\frac{d\rho}{dt} = n_p \omega + \eta M i_q/\psi_d.$$

The mechanical power produced is

$$\tau\omega = J\mu\psi_d i_q \omega = \frac{n_p M}{L_R} \psi_d i_q \omega$$

and the electrical power into the q axis is $u_q i_q$. In particular, the voltage term $-(M/L_R) n_p \omega \psi_d$ in the q axis absorbs the electrical power

$$\left(-\frac{n_p M}{L_R} \omega \psi_d\right) i_q$$

which is the negative of the mechanical power produced. The back-emf term in the voltage is the term that when multiplied by the current in that phase gives the (negative of the) mechanical power produced. Thus the back emf in the field-oriented coordinate system is

$$-\frac{n_p M}{L_R}\omega\psi_d.$$

Problem 10 *Tracking a Time-Varying Flux Reference*

Substitute

$$i_{dr} = K_{\psi P}(\psi_{dref} - \psi_d) + K_{\psi I}\int_0^t (\psi_{dref} - \psi_d)dt + i_{dref}$$

into

$$\frac{d\psi_d}{dt} = -\eta\psi_d + \eta M i_{dr}$$

to obtain

$$\frac{d\psi_d}{dt} = -\eta\psi_d + \eta M\left(K_{\psi I}\int_0^t (\psi_{dref} - \psi_d)dt + K_{\psi P}(\psi_{dref} - \psi_d) + i_{dref}\right).$$

Then, with $e_0 = \int_0^t (\psi_{dref} - \psi_d)dt$, $e_1 = \psi_{dref} - \psi_d$, and (i_{dref}, ψ_{dref}) satisfying

$$d\psi_{dref}/dt = -\eta\psi_{dref} + \eta M i_{dref},$$

it follows that

$$\begin{aligned}
\frac{de_0}{dt} &= e_1 \\
\frac{de_1}{dt} &= \frac{d\psi_{dref}}{dt} + \eta\psi_d - \eta M K_{\psi I}\int_0^t (\psi_{dref} - \psi_d)dt - \eta M K_{\psi P}(\psi_{dref} - \psi_d) - \eta M i_{dref} \\
&= -\eta\left(\psi_{dref} - \psi_d\right) - \eta M K_{\psi I}\int_0^t (\psi_{dref} - \psi_d)dt - \eta M K_{\psi P}(\psi_{dref} - \psi_d) \\
&= -\eta M K_{\psi I} e_0 - \eta\left(M K_{\psi P} + 1\right) e_1.
\end{aligned}$$

Setting

$$\begin{aligned}
\eta\left(M K_{\psi P} + 1\right) &= r_1 + r_2 \\
\eta M K_{\psi I} &= r_1 r_2
\end{aligned}$$

or

$$\begin{aligned}
K_{\psi P} &= \frac{(r_1 + r_2)/\eta - 1}{M} \\
K_{\psi I} &= \frac{r_1 r_2}{\eta M}
\end{aligned}$$

this becomes

$$\begin{aligned}
\frac{de_0}{dt} &= e_1 \\
\frac{de_1}{dt} &= -r_1 r_2 e_0 - (r_1 + r_2)e_1.
\end{aligned}$$

Taking the Laplace transform gives

$$\begin{aligned}
sE_0(s) - e_0(0) &= E_1(s) \\
sE_1(s) - e_1(0) &= -r_1 r_2 E_0(s) - (r_1 + r_2)E_1(s).
\end{aligned}$$

or

$$\begin{bmatrix} s & -1 \\ r_1 r_2 & s + r_1 + r_2 \end{bmatrix} \begin{bmatrix} E_0(s) \\ E_1(s) \end{bmatrix} = \begin{bmatrix} e_0(0) \\ e_1(0) \end{bmatrix}.$$

Solving gives

$$\begin{bmatrix} E_0(s) \\ E_1(s) \end{bmatrix} = \frac{1}{(s+r_1)(s+r_2)} \begin{bmatrix} s + r_1 + r_2 & 1 \\ -r_1 r_2 & s \end{bmatrix} \begin{bmatrix} e_0(0) \\ e_1(0) \end{bmatrix} = \frac{1}{(s+r_1)(s+r_2)} \begin{bmatrix} (s + r_1 + r_2)\, e_0(0) + e_1(0) \\ -r_1 r_2 e_0(0) + s e_1(0) \end{bmatrix}$$

where it follows by the method of partial fractions (assuming r_1 and r_2 are distinct)

$$E_1(s) = \frac{-r_1 r_2 e_0(0) + s e_1(0)}{(s+r_1)(s+r_2)} = \frac{A}{s+r_1} + \frac{B}{s+r_2}$$

so that $e_1(t) \to 0$ with $r_1 > 0$ and $r_2 > 0$.

Problem 11 *Trajectory Tracking Via Input-Output Linearization*
Substitute

$$i_{qr} = \left(K_0 \int_0^t (\theta_{ref} - \theta) dt + K_1(\theta_{ref} - \theta) + K_2(\omega_{ref} - \omega) + (f/J)\omega + \alpha_{ref} \,/ \mu\psi_d \right)$$

into

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= \mu\psi_d i_{qr} - (f/J)\omega - \tau_L/J \end{aligned}$$

to obtain

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= K_0 \int_0^t (\theta_{ref} - \theta) dt + K_1(\theta_{ref} - \theta) + K_2(\omega_{ref} - \omega) + \alpha_{ref} - \tau_L/J. \end{aligned}$$

With $e_0 = \int_0^t (\theta_{ref} - \theta) dt$, $e_1 = \theta_{ref} - \theta$, and $e_2 = \omega_{ref} - \omega$, and assuming the reference trajectory satisfies

$$\begin{aligned} \frac{d\theta_{ref}(t)}{dt} &= \omega_{ref}(t) \\ \frac{d\omega_{ref}(t)}{dt} &= \alpha_{ref}(t), \end{aligned}$$

one obtains

$$\begin{aligned} \frac{de_0}{dt} &= e_1 \\ \frac{de_1}{dt} &= e_2 \\ \frac{de_2}{dt} &= -K_0 e_0 - K_1 e_1 - K_2 e_2 + \tau_L/J. \end{aligned}$$

The Laplace transform of this system is then

$$\begin{aligned} sE_0(s) - e_0(0) &= E_1(s) \\ sE_1(s) - e_1(0) &= E_2(s) \\ sE_2(s) - e_2(0) &= -K_0 E_0(s) - K_1 E_1(s) - K_2 E_2(s) + \tau_L(s)/J \end{aligned}$$

or

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ K_0 & K_1 & s+K_2 \end{bmatrix} \begin{bmatrix} E_0(s) \\ E_1(s) \\ E_2(s) \end{bmatrix} = \begin{bmatrix} e_0(0) \\ e_1(0) \\ e_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tau_L(s)/J \end{bmatrix}.$$

Then

$$\begin{bmatrix} E_0(s) \\ E_1(s) \\ E_2(s) \end{bmatrix} = \frac{1}{s^3+K_2s^2+K_1s+K_0} \begin{bmatrix} s^2+K_2s+K_1 & s+K_2 & 1 \\ -K_0 & s^2+K_2s & s \\ -K_0s & -(K_1s+K_0) & s^2 \end{bmatrix} \left(\begin{bmatrix} e_0(0) \\ e_1(0) \\ e_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tau_L(s)/J \end{bmatrix} \right).$$

With $\tau_L(s) = \tau_{L0}/s$ and

$$\begin{aligned} K_2 &= r_1 + r_2 + r_3 \\ K_1 &= r_1r_2 + r_1r_3 + r_2r_3 \\ K_0 &= r_1r_2r_3, \end{aligned}$$

the characteristic polynomial may then be written as

$$\begin{aligned} a(s) = s^3 + K_2s^2 + K_1s + K_0 &= s^3 + (r_1+r_2+r_3)s^2 + (r_1r_2+r_1r_3+r_2r_3)s + r_1r_2r_3 \\ &= (s+r_1)(s+r_2)(s+r_3). \end{aligned}$$

Then

$$E_0(s) = \frac{\left(s^2+K_2s+K_1\right)e_0(0) + (s+K_2)\,e_1(0) + e_2(0)}{(s+r_1)(s+r_2)(s+r_3)} + \frac{\tau_{L0}/J}{(s+r_1)(s+r_2)(s+r_3)}\frac{1}{s}$$

$$E_1(s) = \frac{-K_0e_0(0) + \left(s^2+K_2s\right)e_1(0) + se_2(0)}{(s+r_1)(s+r_2)(s+r_3)} + \frac{s\left(\tau_{L0}/J\right)}{(s+r_1)(s+r_2)(s+r_3)}\frac{1}{s}$$

$$E_2(s) = \frac{-K_0se_0(0) - (K_1s+K_0)\,e_1(0) + s^2e_2(0)}{(s+r_1)(s+r_2)(s+r_3)} + \frac{s^2\left(\tau_{L0}/J\right)}{(s+r_1)(s+r_2)(s+r_3)}\frac{1}{s}.$$

The closed-loop poles are $p_1 = -r_1$, $p_2 = -r_2$, $p_3 = -r_3$ and, assuming these poles are distinct, $E_1(s)$ has the form

$$\begin{aligned} E_1(s) &= \frac{-K_0e_0(0) + \left(s^2+(K_2+f/J)\,s\right)e_1(0) + se_2(0)}{(s+r_1)(s+r_2)(s+r_3)} + \frac{\tau_{L0}/J}{(s+r_1)(s+r_2)(s+r_3)}\frac{1}{s} \\ &= \frac{A}{s+r_1} + \frac{B}{s+r_2} + \frac{C}{s+r_3} \end{aligned}$$

where A, B, and C are constants. The inverse Laplace transform of $E_1(s)$ is then

$$e_1(t) = Ae^{-r_1t} + Be^{-r_2t} + Ce^{-r_3t} \to 0.$$

Similarly, $e_2(t) \to 0$.

Problem 12 *Field Weakening*

When the motor is decelerating, it is generating voltage rather than consuming voltage from the source. Consequently, the limits of the supply voltage are not a constraint in this mode. However, the reverse blocking voltage of the switches of a PWM amplifier are a constraint on the voltage.

Problem 13 *Maximum Torque*

$$\tau = J\mu M i_{d0} i_{q0} = J\mu M \sqrt{I_{\max}^2 - i_{q0}^2}\, i_{q0}$$

or

$$\tau^2 = (J\mu M)^2 \left(I_{\max}^2 - i_{q0}^2\right) i_{q0}^2.$$

To maximize with respect to i_{q0}, one proceeds as follows:

$$\begin{aligned}
\frac{\partial}{\partial i_{q0}}\tau^2 &= 0 \\
\Longrightarrow\ (-2i_{q0})\left(i_{q0}^2\right) + \left(I_{\max}^2 - i_{q0}^2\right) 2i_{q0} &= 0 \\
\Longrightarrow\ I_{\max}^2 &= 2i_{q0}^2 \\
\Longrightarrow\ i_{q0} &= I_{\max}/\sqrt{2} \\
\Longrightarrow\ i_{d0} &= I_{\max}/\sqrt{2}.
\end{aligned}$$

Problem 14 *Closed-Loop Poles of the Input-Output Controller*

(a) *Field-Oriented Controller*

The poles of the closed-loop mechanical system are the roots of

$$s^3 + K_2 s^2 + K_1 s + K_0 = s^3 + 1.07\times 10^3 s^2 + 7.04\times 10^5 s + 1.07\times 10^6 = 0$$

which are

$$-1.5234, -534.24 \pm j645.73$$

The poles of the closed-loop flux tracking controller are the roots of

$$\begin{aligned}
s^2 + \eta\left(MK_{\psi P} + 1\right)s + \eta M K_{\psi I} &= s^2 + \frac{R_R}{L_R}\left(MK_{\psi P} + 1\right)s + \frac{R_R}{L_R} M K_{\psi I} \\
&= s^2 + \frac{3.9}{0.014}\left(0.0011\times 1600 + 1\right)s + \frac{3.9}{0.014} 0.0011\times 23000 \\
&= s^2 + 768.86 s + 7047.9
\end{aligned}$$

which are

$$-759.58, -9.2787$$

(b) *Input-Output Linearization Controller*

The poles of the closed-loop mechanical system are the roots of

$$s^3 + K_2 s^2 + K_1 s + K_0 = s^3 + 125 s^2 + 5.5\times 10^4 s + 3\times 10^5 = 0$$

which are

$$-5.5208, -59.740 \pm j225.33$$

The poles of the closed-loop flux tracking controller are the roots of

$$\begin{aligned}
s^2 + \eta\left(MK_{\psi P} + 1\right)s + \eta M K_{\psi I} &= s^2 + \frac{R_R}{L_R}\left(MK_{\psi P} + 1\right)s + \frac{R_R}{L_R} M K_{\psi I} \\
&= s^2 + \frac{3.9}{0.014}\left(0.0011\times 10000 + 1\right)s + \frac{3.9}{0.014} 0.0011\times 420000 \\
&= s^2 + 3342.9 s + 1.287\times 10^5
\end{aligned}$$

which are

$$-3303.9, -38.953$$