Modeling and High-Performance Control of Electric Machines

Chapter 7 Symmetric Balanced Three-Phase AC Machines

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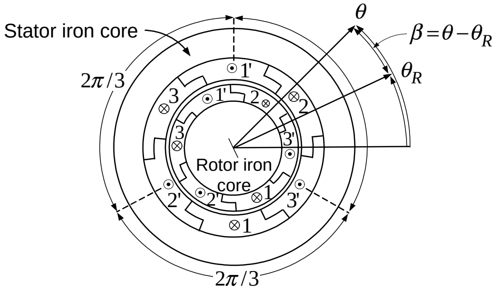
Symmetric Balanced Three-Phase AC Machines

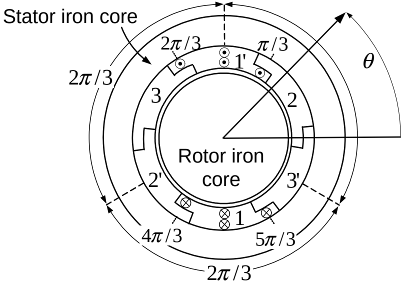
- **Mathematical Model of a Three-Phase Induction Motor**
- **Steady-State Analysis of an Induction Motor**
- **Mathematical Model of a Three-Phase PM Synchronous Motor**
- **Three-Phase, Sinusoidal, 60-Hz Voltages*** (no slides)

Mathematical Model of a Three-Phase Induction Motor



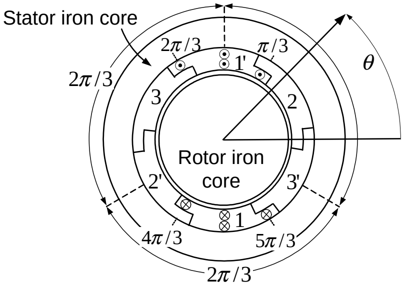
- There are three stator phases wound $2\pi/3$ radians apart.
- There are three rotor phases also wound $2\pi/3$ radians apart.
- The rotor angle is aligned down the middle of rotor phase $1 - 1'$.
- $\beta = \theta - \theta_R$ locates angular position in the air gap with respect to the rotor.

Mathematical Model of a Three-Phase Induction Motor

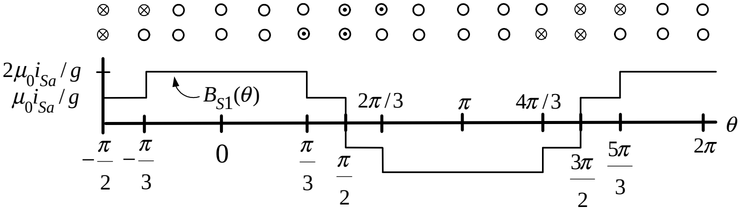


- **Stator phase 1:** Figure shows
  One loop is wound at $\pi/3$ and $4\pi/3$.
  Two loops are wound at $\pi/2$ and $3\pi/2$.
  One loop is wound $2\pi/3$ and $5\pi/3$.
- The loops of the three phases are wound in **two** layers.

Mathematical Model of a Three-Phase Induction Motor



- Using Ampère's law, B_{S1} in the air gap due to i_{S1} is a staircase function.
- It is approximately sinusoidal in θ.



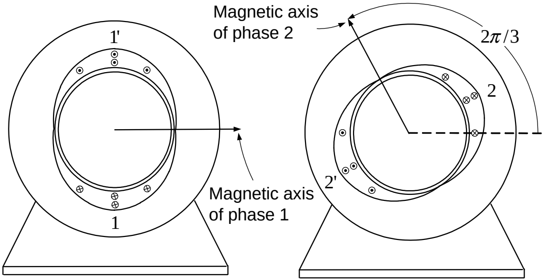
Mathematical Model of a Three-Phase Induction Motor

Stator Sinusoidal Windings

We now idealize the model to have sinusoidal windings.
The stator turns densities are given by

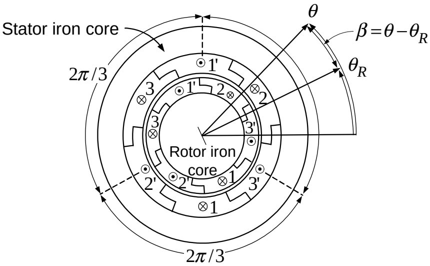
$$\begin{aligned} N_{S1}(\theta) &= \frac{N_S}{2}\left|\sin(\theta)\right| \\ N_{S2}(\theta) &= \frac{N_S}{2}\left|\sin(\theta - 2\pi/3)\right| \\ N_{S3}(\theta) &= \frac{N_S}{2}\left|\sin(\theta - 4\pi/3)\right|. \end{aligned}$$

Stator phases 1 and 2 are illustrated.



Mathematical Model of a Three-Phase Induction Motor

Rotor Sinusoidal Windings



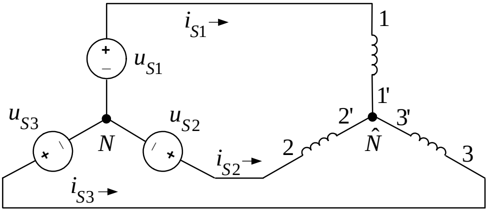
The rotor turn densities are

$$
\begin{aligned}
N_{R1}(\theta-\theta_R) &= \frac{N_R}{2}\left|\sin(\theta-\theta_R)\right| \\
N_{R2}(\theta-\theta_R) &= \frac{N_R}{2}\left|\sin(\theta-\theta_R-2\pi/3)\right| \\
N_{R3}(\theta-\theta_R) &= \frac{N_R}{2}\left|\sin(\theta-\theta_R-4\pi/3)\right|.
\end{aligned}
$$

- Stator windings are **identical** in construction and shifted by $2\pi/3$ from each other.
- Rotor windings are **identical** in construction and shifted by $2\pi/3$ from each other.
- For these reasons we say it is a **symmetric** machine.

Mathematical Model of a Three-Phase Induction Motor

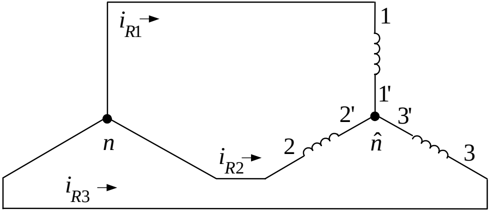
Wye Connected Stator Windings



- The stator end windings $1', 2', 3'$ are tied together to form the **motor neutral** $\hat{N}$.
- The stator end windings $1, 2, 3$ are connected to the **source voltages** u_{S1}, u_{S2}, u_{S3}.
- The other ends of the voltages are tied together to form the **source neutral** N.
- This is called a **wye-connected** motor.
- Due to the wye connection the currents are **balanced**, i.e., $i_{S1} + i_{S2} + i_{S3} \equiv 0$.

Mathematical Model of a Three-Phase Induction Motor

Wye Connected Rotor Windings



- Similarly, the rotor end windings $1, 2, 3$ are **shorted** (connected) together.
- The other rotor end windings $1', 2', 3'$ are also **shorted** together.
- Due to the wye connection the rotor currents are **balanced**, i.e., $i_{R1} + i_{R2} + i_{R3} \equiv 0$.

Mathematical Model of a Three-Phase Induction Motor

The stator magnetic fields are

$$\begin{aligned}
\vec{\mathbf{B}}_{S1}(i_{S1}, r, \theta) &= \frac{\mu_0 N_S}{2g}\frac{r_R}{r} i_{S1}\cos(\theta)\hat{\mathbf{r}} \\
\vec{\mathbf{B}}_{S2}(i_{S2}, r, \theta) &= \frac{\mu_0 N_S}{2g}\frac{r_R}{r} i_{S2}\cos(\theta - 2\pi/3)\hat{\mathbf{r}} \\
\vec{\mathbf{B}}_{S3}(i_{S3}, r, \theta) &= \frac{\mu_0 N_S}{2g}\frac{r_R}{r} i_{S3}\cos(\theta - 4\pi/3)\hat{\mathbf{r}}
\end{aligned}$$

The total stator magnetic field is

$$\vec{\mathbf{B}}_S(i_{S1}, i_{S2}, i_{S3}, r, \theta) = \frac{\mu_0 N_S}{2g}\frac{r_R}{r}\Big(i_{S1}\cos(\theta) + i_{S2}\cos(\theta - 2\pi/3) + i_{S3}\cos(\theta - 4\pi/3)\Big)\hat{\mathbf{r}}.$$

Mathematical Model of a Three-Phase Induction Motor

Suppose the stator phases carry the balanced three-phase set of currents

$$\begin{aligned} i_{S1} &= I_S \cos(\omega_S t) \\ i_{S2} &= I_S \cos(\omega_S t - 2\pi/3) \\ i_{S3} &= I_S \cos(\omega_S t - 4\pi/3). \end{aligned}$$

Then $\vec{\mathbf{B}}_S$ is given by

$$\begin{aligned} \vec{\mathbf{B}}_S(I_S, r, \theta, t) &= \frac{\mu_0 N_S I_S}{2g}\frac{r_R}{r}\Big(\cos(\omega_S t)\cos(\theta) + \cos(\omega_S t - 2\pi/3)\cos(\theta - 2\pi/3) \\ &\qquad + \cos(\omega_S t - 4\pi/3)\cos(\theta - 4\pi/3)\Big)\hat{\mathbf{r}} \\ &= \frac{\mu_0 N_S I_S}{2g}\frac{r_R}{r}\frac{3}{2}\cos(\theta - \omega_S t)\hat{\mathbf{r}}. \end{aligned}$$

With balanced three-phase currents, a radial rotating magnetic field is established.

Mathematical Model of a Three-Phase Induction Motor

Similarly, the rotor magnetic fields are given by

$$\begin{aligned}
\vec{\mathbf{B}}_{R1}(i_{R1}, r, \theta) &= \frac{\mu_0 N_R}{2g}\frac{r_R}{r} i_{R1}\cos(\theta - \theta_R)\hat{\mathbf{r}} \\
\vec{\mathbf{B}}_{R2}(i_{R2}, r, \theta) &= \frac{\mu_0 N_R}{2g}\frac{r_R}{r} i_{R2}\cos(\theta - \theta_R - 2\pi/3)\hat{\mathbf{r}} \\
\vec{\mathbf{B}}_{R3}(i_{R3}, r, \theta) &= \frac{\mu_0 N_R}{2g}\frac{r_R}{r} i_{R3}\cos(\theta - \theta_R - 4\pi/3)\hat{\mathbf{r}}.
\end{aligned}$$

With $\beta = \theta - \theta_R$,

$$\begin{aligned}
\vec{\mathbf{B}}_R(i_{R1}, i_{R2}, i_{R3}, r, \beta) = \frac{\mu_0 N_R}{2g}\frac{r_R}{r}\Big(&i_{R1}\cos(\theta - \theta_R) + i_{R2}\cos(\theta - \theta_R - 2\pi/3) \\
&+ i_{R3}\cos(\theta - \theta_R - 4\pi/3)\Big)\hat{\mathbf{r}} \\
= \frac{\mu_0 N_R}{2g}\frac{r_R}{r}\Big(&i_{R1}\cos(\beta) + i_{R2}\cos(\beta - 2\pi/3) + i_{R3}\cos(\beta - 4\pi/3)\Big)\hat{\mathbf{r}}.
\end{aligned}$$

Mathematical Model of a Three-Phase Induction Motor

Total Magnetic Field on the Stator Side

$$\vec{\mathbf{B}}(i_{S1}, i_{S2}, i_{S3}, i_{R1}, i_{R2}, i_{R3}, r_S, \theta, \theta_R) \triangleq \vec{\mathbf{B}}_S(i_{S1}, i_{S2}, i_{S3}, r_S, \theta) + \kappa \vec{\mathbf{B}}_R(i_{R1}, i_{R2}, i_{R3}, r_S, \theta - \theta_R).$$

Total Magnetic Field on the Rotor Side

On the rotor side of the air gap, the total radial magnetic field $\vec{\mathbf{B}}$ is taken as

$$\vec{\mathbf{B}}(i_{S1}, i_{S2}, i_{S3}, i_{R1}, i_{R2}, i_{R3}, r_R, \theta, \theta_R) \triangleq \kappa \vec{\mathbf{B}}_S(i_{S1}, i_{S2}, i_{S3}, r_R, \theta) + \vec{\mathbf{B}}_R(i_{R1}, i_{R2}, i_{R3}, r_R, \theta - \theta_R)$$

In terms of $\beta = \theta - \theta_R$ (and an abuse of notation),

$$\vec{\mathbf{B}}(i_{S1}, i_{S2}, i_{S3}, i_{R1}, i_{R2}, i_{R3}, r_R, \beta, \theta_R) \triangleq \kappa \vec{\mathbf{B}}_S(i_{S1}, i_{S2}, i_{S3}, r_R, \beta + \theta_R) + \vec{\mathbf{B}}_R(i_{R1}, i_{R2}, i_{R3}, r_R, \beta).$$

Stator Flux Linkages

Using $\vec{\mathbf{B}}$ on the **stator side** of the air gap we have

$$\begin{aligned}
\psi_{S1}(t) &= \int_0^{\pi} \frac{N_S}{2}\sin(\theta)\left(\int_{\theta-\pi}^{\theta} \ell_1 r_S B(i_{S1}, i_{S2}, i_{S3}, i_{R1}, i_{R2}, i_{R3}, r_S, \theta', \theta_R)d\theta'\right)d\theta \\
&= \frac{2}{3}L_S(i_{S1} + i_{S2}\cos(2\pi/3) + i_{S3}\cos(4\pi/3)) \\
&\quad + \frac{2}{3}M\Big(i_{R1}\cos(\theta_R) + i_{R2}\cos(\theta_R + 2\pi/3) + i_{R3}\cos(\theta_R + 4\pi/3)\Big) \\
\psi_{S2}(t) &= \int_{2\pi/3}^{2\pi/3+\pi} \frac{N_S}{2}\sin(\theta - 2\pi/3)\left(\int_{\theta-\pi}^{\theta} \ell_1 r_S B(i_{S1}, i_{S2}, i_{S3}, i_{R1}, i_{R2}, i_{R3}, r_S, \theta', \theta_R)d\theta'\right)d\theta \\
&= \frac{2}{3}L_S(i_{S1}\cos(2\pi/3) + i_{S2} + i_{S3}\cos(2\pi/3)) \\
&\quad + \frac{2}{3}M\Big(i_{R1}\cos(\theta_R - 2\pi/3) + i_{R2}\cos(\theta_R) + i_{R3}\cos(\theta_R + 2\pi/3)\Big) \\
\psi_{S3}(t) &= \int_{4\pi/3}^{4\pi/3+\pi} \frac{N_S}{2}\sin(\theta - 4\pi/3)\left(\int_{\theta-\pi}^{\theta} \ell_1 r_S B(i_{S1}, i_{S2}, i_{S3}, i_{R1}, i_{R2}, i_{R3}, r_S, \theta', \theta_R)d\theta'\right)d\theta \\
&= \frac{2}{3}L_S(i_{S1}\cos(4\pi/3) + i_{S2}\cos(2\pi/3) + i_{S3}) \\
&\quad + \frac{2}{3}M\Big(i_{R1}\cos(\theta_R - 4\pi/3) + i_{R2}\cos(\theta_R - 2\pi/3) + i_{R3}\cos(\theta_R)\Big)
\end{aligned}$$

$$L_S \triangleq \frac{3}{2}\frac{\pi\mu_0\ell_1\ell_2N_S^2}{8g}, \quad M \triangleq \frac{3}{2}\kappa\frac{\pi\mu_0\ell_1\ell_2N_SN_R}{8g}, \quad L_R \triangleq \frac{3}{2}\frac{\pi\mu_0\ell_1\ell_2N_R^2}{8g}.$$

Stator Flux Linkages

- There is a factor 2/3 in flux linkage equations.
- There is a factor 3/2 in the expressions for L_S, M, and L_R.
- Done so L_S, M, L_R represent the **two-phase equivalent** parameter values.

In matrix form, we have

$$\begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix} = \frac{2}{3}L_S\begin{bmatrix} 1 & \cos(2\pi/3) & \cos(4\pi/3) \\ \cos(2\pi/3) & 1 & \cos(2\pi/3) \\ \cos(4\pi/3) & \cos(2\pi/3) & 1 \end{bmatrix}\begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix}$$

$$+\frac{2}{3}M\begin{bmatrix} \cos(\theta_R) & \cos(\theta_R+2\pi/3) & \cos(\theta_R+4\pi/3) \\ \cos(\theta_R-2\pi/3) & \cos(\theta_R) & \cos(\theta_R+2\pi/3) \\ \cos(\theta_R-4\pi/3) & \cos(\theta_R-2\pi/3) & \cos(\theta_R) \end{bmatrix}\begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix}.$$

Note that $\cos(\theta_R-4\pi/3)=\cos(\theta_R+2\pi/3)$.
More compactly

$$\begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix} = C_1\begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} + C_2(\theta_R)\begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix}.$$

- $\psi_{S1}(t)+\psi_{S2}(t)+\psi_{S3}(t)\equiv 0.$

Rotor Flux Linkages

Using $\vec{\mathbf{B}}$ on the **rotor side** of the air gap, the rotor flux linkages are

$$\begin{aligned}\psi_{R1}(t) &= \int_0^{\pi} \frac{N_R}{2}\sin(\beta)\left(\int_{\beta-\pi}^{\beta} \ell_1 r_R B(i_{S1}, i_{S2}, i_{S3}, i_{R1}, i_{R2}, i_{R3}, r_R, \beta', \theta_R)d\beta'\right)d\beta \\ &= \frac{2}{3}L_R(i_{R1} + i_{SR2}\cos(2\pi/3) + i_{R1}\cos(4\pi/3)) \\ &\quad + \frac{2}{3}M\Big(i_{S1}\cos(\theta_R) + i_{S2}\cos(\theta_R - 2\pi/3) + i_{S3}\cos(\theta_R - 4\pi/3)\Big)\end{aligned}$$

$$\begin{aligned}\psi_{R2}(t) &= \int_{2\pi/3}^{2\pi/3+\pi} \frac{N_R}{2}\sin(\beta - 2\pi/3)\left(\int_{\beta-\pi}^{\beta} \ell_1 r_R B(i_{S1}, i_{S2}, i_{S3}, i_{R1}, i_{R2}, i_{R3}, r_R, \beta', \theta_R)d\beta'\right)d\beta \\ &= \frac{2}{3}L_R(i_{R1}\cos(2\pi/3) + i_{R2} + i_{R3}\cos(2\pi/3)) \\ &\quad + \frac{2}{3}M\Big(i_{S1}\cos(\theta_R + 2\pi/3) + i_{S2}\cos(\theta_R) + i_{S3}\cos(\theta_R - 2\pi/3)\Big)\end{aligned}$$

$$\begin{aligned}\psi_{R3}(t) &= \int_{4\pi/3}^{4\pi/3+\pi} \frac{N_R}{2}\sin(\beta - 4\pi/3)\left(\int_{\beta-\pi}^{\beta} \ell_1 r_R B(i_{S1}, i_{S2}, i_{S3}, i_{R1}, i_{R2}, i_{R3}, r_R, \beta', \theta_R)d\beta'\right)d\beta \\ &= \frac{2}{3}L_R(i_{R1}\cos(4\pi/3) + i_{R2}\cos(2\pi/3) + i_{R3}) \\ &\quad + \frac{2}{3}M\Big(i_{S1}\cos(\theta_R + 4\pi/3) + i_{S2}\cos(\theta_R + 2\pi/3) + i_{S3}\cos(\theta_R)\Big)\end{aligned}$$

Rotor Flux Linkages

In matrix form, the rotor flux linkages may be written as

$$\begin{bmatrix} \psi_{R1}(t) \\ \psi_{R2}(t) \\ \psi_{R3}(t) \end{bmatrix} = \frac{2}{3}L_R \begin{bmatrix} 1 & \cos(2\pi/3) & \cos(4\pi/3) \\ \cos(2\pi/3) & 1 & \cos(2\pi/3) \\ \cos(4\pi/3) & \cos(2\pi/3) & 1 \end{bmatrix} \begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix}$$

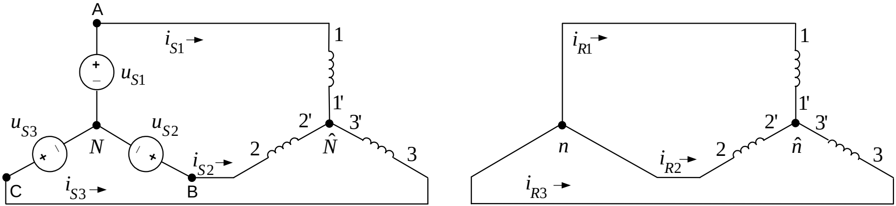
$$+ \frac{2}{3}M \begin{bmatrix} \cos(\theta_R) & \cos(\theta_R - 2\pi/3) & \cos(\theta_R - 4\pi/3) \\ \cos(\theta_R + 2\pi/3) & \cos(\theta_R) & \cos(\theta_R - 2\pi/3) \\ \cos(\theta_R + 4\pi/3) & \cos(\theta_R + 2\pi/3) & \cos(\theta_R) \end{bmatrix} \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix}$$

or, more compactly,

$$\begin{bmatrix} \psi_{R1}(t) \\ \psi_{R2}(t) \\ \psi_{R3}(t) \end{bmatrix} = C_1 \begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix} + C_2(-\theta_R) \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix}$$

- $\psi_{R1}(t) + \psi_{R2}(t) + \psi_{R3}(t) \equiv 0.$

Balanced Conditions



- The source voltages $u_{S1}(t), u_{S2}(t)$, and $u_{S3}(t)$ are wye connected.
- Due to the wye connection the stator and rotor currents are **balanced**, i.e.,

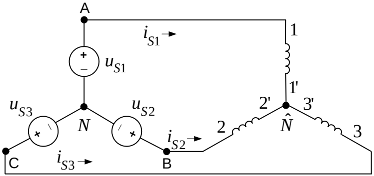
$$i_{S1}(t) + i_{S2}(t) + i_{S3}(t) \equiv 0$$

$$i_{R1}(t) + i_{R2}(t) + i_{R3}(t) \equiv 0.$$

- Whether or **not** the currents are balanced we have

$$\begin{aligned} \psi_{S1}(t) + \psi_{S2}(t) + \psi_{S3}(t) &\equiv 0 \\ \psi_{R1}(t) + \psi_{R2}(t) + \psi_{R3}(t) &\equiv 0. \end{aligned}$$

Balanced Conditions



Phase to **motor neutral** voltages: $v_{A\hat{N}} = v_A - v_{\hat{N}}$, $v_{B\hat{N}}(t) = v_B - v_{\hat{N}}$, $v_{C\hat{N}}(t) = v_C - v_{\hat{N}}$. Faraday's and Ohm's laws give

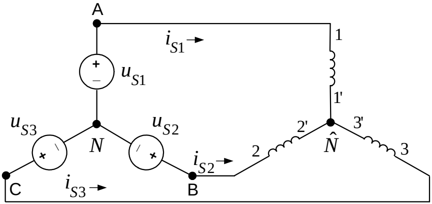
$$\begin{aligned} v_{A\hat{N}} &= R_S i_{S1} + \frac{d\psi_{S1}(t)}{dt} \\ v_{B\hat{N}} &= R_S i_{S2} + \frac{d\psi_{S2}(t)}{dt} \\ v_{C\hat{N}} &= R_S i_{S3} + \frac{d\psi_{S3}(t)}{dt}. \end{aligned}$$

Adding these three equations results in

$$v_{A\hat{N}} + v_{B\hat{N}} + v_{C\hat{N}} = R_S\left(i_{S1} + i_{S2} + i_{S3}\right) + \frac{d}{dt}\left(\psi_{S1}(t) + \psi_{S2}(t) + \psi_{S3}(t)\right) \equiv 0.$$

- The phase to motor neutral voltages are **always** balanced.
- The voltages applied to the **motor** are the phase to **source neutral** voltages $u_{S1}(t), u_{S2}(t), u_{S3}(t)$.

Balanced Conditions



Lemma Let $v_{\hat{N}N} \triangleq v_{\hat{N}} - v_N$, then

$$v_{\hat{N}N} = \frac{u_{S1}(t) + u_{S2}(t) + u_{S3}(t)}{3}.$$

Proof

$$\begin{aligned} u_{S1}(t) &= v_A - v_N = v_{A\hat{N}} + v_{\hat{N}N} \\ u_{S2}(t) &= v_B - v_N = v_{B\hat{N}} + v_{\hat{N}N} \\ u_{S3}(t) &= v_C - v_N = v_{C\hat{N}} + v_{\hat{N}N} \end{aligned}$$

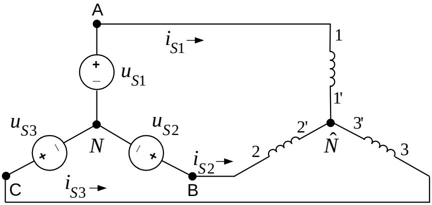
So

$$u_{S1}(t) + u_{S2}(t) + u_{S3}(t) = v_{A\hat{N}} + v_{B\hat{N}}(t) + v_{C\hat{N}}(t) + 3v_{\hat{N}N} = 3v_{\hat{N}N}$$

or

$$v_{\hat{N}N} = \frac{u_{S1}(t) + u_{S2}(t) + u_{S3}(t)}{3}.$$

Balanced Conditions



Corollary If the source voltages are balanced, i.e., $u_{S1}(t) + u_{S2}(t) + u_{S3}(t) \equiv 0$, **then** $v_{\hat{N}N} \equiv 0$ and

$$\begin{aligned} u_{S1}(t) &= v_{A\hat{N}} \\ u_{S2}(t) &= v_{B\hat{N}} \\ u_{S3}(t) &= v_{C\hat{N}}. \end{aligned}$$

Proof By the previous lemma we have

$$v_{\hat{N}N} = \frac{u_{S1}(t) + u_{S2}(t) + u_{S3}(t)}{3} = 0$$

$\Longrightarrow u_{S1}(t) = v_{A\hat{N}} + v_{\hat{N}N} = v_{A\hat{N}}$, etc.

Three-Phase to Two-Phase Transformation

Define a **three-phase to two-phase** transformation of the voltages by

$$\begin{bmatrix} u_{Sa}(t) \\ u_{Sb}(t) \\ u_{S0}(t) \end{bmatrix} \triangleq \sqrt{\frac{2}{3}}\begin{bmatrix} 1 & \cos(2\pi/3) & \cos(4\pi/3) \\ 0 & \sin(2\pi/3) & \sin(4\pi/3) \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} u_{S1}(t) \\ u_{S2}(t) \\ u_{S3}(t) \end{bmatrix}$$
$$= \underbrace{\sqrt{\frac{2}{3}}\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{Q}\begin{bmatrix} u_{S1}(t) \\ u_{S2}(t) \\ u_{S3}(t) \end{bmatrix}.$$

With **balanced** source voltages $u_{S0}(t) \equiv 0$.

Inverse transformation

$$\begin{bmatrix} u_{S1}(t) \\ u_{S2}(t) \\ u_{S3}(t) \end{bmatrix} = \underbrace{\sqrt{\frac{3}{2}}\begin{bmatrix} 2/3 & 0 & \sqrt{2}/3 \\ -1/3 & 1/\sqrt{3} & \sqrt{2}/3 \\ -1/3 & -1/\sqrt{3} & \sqrt{2}/3 \end{bmatrix}}_{Q^{-1}}\begin{bmatrix} u_{Sa}(t) \\ u_{Sb}(t) \\ u_{S0}(t) \end{bmatrix}.$$

- Q is an **orthogonal** matrix, that is, $Q^{-1} = Q^T$.

Three-Phase to Two-Phase Transformation

$$\begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} \triangleq Q\begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix}, \quad \begin{bmatrix} i_{Ra}(t) \\ i_{Rb}(t) \\ i_{R0}(t) \end{bmatrix} \triangleq Q\begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix}$$

$$\begin{bmatrix} \lambda_{Sa}(t) \\ \lambda_{Sb}(t) \\ \lambda_{S0}(t) \end{bmatrix} \triangleq Q\begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix}, \quad \begin{bmatrix} \lambda_{Ra}(t) \\ \lambda_{Rb}(t) \\ \lambda_{R0}(t) \end{bmatrix} \triangleq Q\begin{bmatrix} \psi_{R1}(t) \\ \psi_{R2}(t) \\ \psi_{R3}(t) \end{bmatrix}$$

We **always** have

$$\begin{aligned} \lambda_{S0}(t) &= \frac{1}{\sqrt{3}}\left(\psi_{S1}(t)+\psi_{S2}(t)+\psi_{S3}(t)\right) \equiv 0 \\ \lambda_{R0}(t) &= \frac{1}{\sqrt{3}}\left(\psi_{R1}(t)+\psi_{R2}(t)+\psi_{R3}(t)\right) \equiv 0. \end{aligned}$$

As the phases are **wye connected**

$$\begin{aligned} i_{S0}(t) &= \frac{1}{\sqrt{3}}\left(i_{S1}(t)+i_{S2}(t)+i_{S3}(t)\right) \equiv 0 \\ i_{R0}(t) &= \frac{1}{\sqrt{3}}\left(i_{R1}(t)+i_{R2}(t)+i_{R3}(t)\right) \equiv 0. \end{aligned}$$

If the source voltages are **balanced**

$$u_{S0}(t) = \frac{1}{\sqrt{3}}\left(u_{S1}(t)+u_{S2}(t)+u_{S3}(t)\right) \equiv 0.$$

Three-Phase to Two-Phase Transformation

With balanced source voltages, the **zero sequence** components are zero.

$$\text{I.e., } i_{S0}(t) = i_{R0}(t) = \lambda_{S0}(t) = \lambda_{R0}(t) = u_{S0}(t) \equiv 0.$$

Equivalent Two-Phase Stator Equations

$$\begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix} = C_1 \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} + C_2(\theta_R) \begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix}$$

so that

$$\begin{aligned} \begin{bmatrix} \lambda_{Sa}(t) \\ \lambda_{Sb}(t) \\ \lambda_{S0}(t) \end{bmatrix} &= QC_1Q^{-1} \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} + QC_2(\theta_R)Q^{-1} \begin{bmatrix} i_{Ra}(t) \\ i_{Rb}(t) \\ i_{R0}(t) \end{bmatrix} \\ &= \begin{bmatrix} L_S & 0 & 0 \\ 0 & L_S & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} + \begin{bmatrix} M\cos(\theta_R) & -M\sin(\theta_R) & 0 \\ M\sin(\theta_R) & M\cos(\theta_R) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{Ra}(t) \\ i_{Rb}(t) \\ i_{R0}(t) \end{bmatrix} \end{aligned}$$

or

$$\begin{aligned} \lambda_{Sa}(t) &= L_S i_{Sa}(t) + M\Big(i_{Ra}(t)\cos(\theta_R) - i_{Rb}(t)\sin(\theta_R)\Big) \\ \lambda_{Sb}(t) &= L_S i_{Sb}(t) + M\Big(i_{Ra}(t)\sin(\theta_R) + i_{Rb}(t)\cos(\theta_R)\Big) \\ \lambda_{S0} &\equiv 0. \end{aligned}$$

Equivalent Two-Phase Stator Equations

Recall the stator equations are

$$\begin{bmatrix} u_{S1}(t) \\ u_{S2}(t) \\ u_{S3}(t) \end{bmatrix} = \begin{bmatrix} v_{A\hat{N}} \\ v_{B\hat{N}} \\ v_{C\hat{N}} \end{bmatrix} = R_S \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix}$$

Multiply through by Q

$$\begin{aligned} Q \begin{bmatrix} u_{S1}(t) \\ u_{S2}(t) \\ u_{S3}(t) \end{bmatrix} &= R_S Q \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} + \frac{d}{dt} Q \begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix} \\ \begin{bmatrix} u_{Sa}(t) \\ u_{Sb}(t) \\ u_{S0}(t) \end{bmatrix} &= R_S \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_{Sa}(t) \\ \lambda_{Sb}(t) \\ \lambda_{S0}(t) \end{bmatrix} \end{aligned}$$

or

$$\begin{aligned} u_{Sa} &= R_S i_{Sa} + \frac{d}{dt}\Big(L_S i_{Sa} + M\big(+i_{Ra}\cos(\theta_R) - i_{Rb}\sin(\theta_R)\big)\Big) \\ u_{Sb} &= R_S i_{Sb} + \frac{d}{dt}\Big(L_S i_{Sb} + M\big(+i_{Ra}\sin(\theta_R) + i_{Rb}\cos(\theta_R)\big)\Big) \end{aligned}$$

Equivalent Two-Phase Rotor Equations

$$\begin{bmatrix} \psi_{R1}(t) \\ \psi_{R2}(t) \\ \psi_{R3}(t) \end{bmatrix} = C_1 \begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix} + C_2(-\theta_R) \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} \lambda_{Ra}(t) \\ \lambda_{Rb}(t) \\ \lambda_{R0}(t) \end{bmatrix} &= QC_1Q^{-1} \begin{bmatrix} i_{Ra}(t) \\ i_{Rb}(t) \\ i_{R0}(t) \end{bmatrix} + QC_2(-\theta_R)Q^{-1} \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} \\ &= \begin{bmatrix} L_R & 0 & 0 \\ 0 & L_R & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{Ra}(t) \\ i_{Rb}(t) \\ i_{R0}(t) \end{bmatrix} + \begin{bmatrix} M\cos(\theta_R) & M\sin(\theta_R) & 0 \\ -M\sin(\theta_R) & M\cos(\theta_R) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} \end{aligned}$$

or

$$\begin{aligned} \lambda_{Ra}(t) &= L_R i_{Ra}(t) + M\cos(\theta_R) i_{Sa}(t) + M\sin(\theta_R) i_{Sb}(t) \\ \lambda_{Rb}(t) &= L_R i_{Rb}(t) - M\sin(\theta_R) i_{Sa}(t) + M\cos(\theta_R) i_{Sb}(t) \end{aligned}$$

Equivalent Two-Phase Rotor Equations

Rotor dynamic equations are

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = R_S \begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \psi_{R1}(t) \\ \psi_{R2}(t) \\ \psi_{R3}(t) \end{bmatrix}$$

Multiply through by Q

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} &= R_S Q \begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix} + \frac{d}{dt} Q \begin{bmatrix} \psi_{R1}(t) \\ \psi_{R2}(t) \\ \psi_{R3}(t) \end{bmatrix} \\ &= R_S \begin{bmatrix} i_{Ra}(t) \\ i_{Rb}(t) \\ i_{R0}(t) \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_{Ra}(t) \\ \lambda_{Rb}(t) \\ \lambda_{R0}(t) \end{bmatrix} \end{aligned}$$

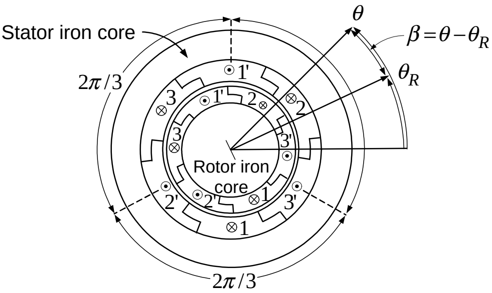
or

$$\begin{aligned} 0 &= R_R i_{Ra} + \frac{d}{dt}\Big(L_R i_{Ra}(t) + M\Big(+i_{Sa}(t)\cos(\theta_R) + i_{Sb}(t)\sin(\theta_R)\Big)\Big) \\ 0 &= R_R i_{Rb} + \frac{d}{dt}\Big(L_R i_{Rb}(t) + M\Big(-i_{Sa}(t)\sin(\theta_R) + i_{Sb}(t)\cos(\theta_R)\Big)\Big). \end{aligned}$$

Equivalent Two-Phase Electrical Equations

$$
\begin{aligned}
u_{Sa} &= L_S \frac{d}{dt} i_{Sa} + M \frac{d}{dt}\Big(+i_{Ra}\cos(\theta_R) - i_{Rb}\sin(\theta_R)\Big) + R_S i_{Sa} \\
u_{Sb} &= L_S \frac{d}{dt} i_{Sb} + M \frac{d}{dt}\Big(+i_{Ra}\sin(\theta_R) + i_{Rb}\cos(\theta_R)\Big) + R_S i_{Sb} \\
0 &= L_R \frac{d}{dt} i_{Ra} + M \frac{d}{dt}\Big(+i_{Sa}\cos(\theta_R) + i_{Sb}\sin(\theta_R)\Big) + R_R i_{Ra} \\
0 &= L_R \frac{d}{dt} i_{Rb} + M \frac{d}{dt}\Big(-i_{Sa}\sin(\theta_R) + i_{Sb}\cos(\theta_R)\Big) + R_R i_{Rb}
\end{aligned}
$$

Equivalent Two-Phase Torque



With $\beta = \theta - \theta_R$ and $r = r_R$ we have

$$\vec{\mathbf{B}}_S(i_{S1}, i_{S2}, i_{S3}, r, \beta, \theta_R)\Big|_{r=r_R} =$$

$$\kappa \frac{\mu_0 N_S}{2g} \frac{r_R}{r_R} \Big(i_{S1} \cos(\beta + \theta_R) + i_{S2} \cos(\beta + \theta_R - 2\pi/3) + i_{S3} \cos(\beta + \theta_R - 4\pi/3) \Big) \hat{\mathbf{r}}.$$

We next compute magnetic force/torque exerted by $\vec{\mathbf{B}}_S$ on the rotor currents.

Equivalent Two-Phase Torque

The torque on **rotor phase 1** is then

$$\vec{\boldsymbol{\tau}}_{R1} = \int_{\beta=0}^{2\pi} (r_R \hat{\mathbf{r}}) \times \left(i_{R1}(t) \frac{N_R}{2} \sin(\beta) d\beta (+\ell_1 \hat{\mathbf{z}}) \times \left(B_{S|_{r=r_R}} \hat{\mathbf{r}} \right) \right)$$

$$= \int_{\beta=0}^{2\pi} r_R i_{R1}(t) \frac{\ell_1 N_R}{2} \sin(\beta) \left(\kappa \frac{\mu_0 N_S}{2g} \right) \Big(i_{S1} \cos(\beta + \theta_R) + i_{S2} \cos(\beta + \theta_R - 2\pi/3) + i_{S3} \cos(\beta + \theta_R - 4\pi/3) \Big) d\beta \; \hat{\mathbf{z}}$$

$$= \kappa \frac{r_R \ell_1 \mu_0 N_R N_S}{4g} i_{R1} \Big(i_{S1}(-\pi \sin(\theta_R)) + i_{S2} \pi \cos(\theta_R - \pi/6) + i_{S3}(-\pi \cos(\theta_R + \pi/6)) \Big) \hat{\mathbf{z}}$$

$$= \frac{2}{3} M i_{R1} (-i_{S1} \sin(\theta_R) + i_{S2} \cos(\theta_R - \pi/6) - i_{S3} \cos(\theta_R + \pi/6)) \hat{\mathbf{z}}$$

where $M \triangleq (3/2)\kappa\pi\mu_0 \ell_1 \ell_2 N_S N_R/(8g)$.

Equivalent Two-Phase Torque

The torque on **rotor phase 2** is

$$\vec{\boldsymbol{\tau}}_{R2} = \int_{\beta=0}^{2\pi} (r_R\hat{\mathbf{r}}) \times \left(i_{R2}\frac{N_R}{2}\sin(\beta - 2\pi/3)d\beta(+\ell_1\hat{\mathbf{z}}) \times \left(B_{S|_{r=r_R}}\hat{\mathbf{r}} \right) \right)$$

$$= \int_{\beta=0}^{2\pi} r_R i_{R2}(t)\frac{\ell_1 N_R}{2}\sin(\beta - 2\pi/3)\left(\kappa\frac{\mu_0 N_S}{2g}\right)\Big(i_{S1}\cos(\beta+\theta_R) + i_{S2}\cos(\beta+\theta_R - 2\pi/3) + i_{S3}\cos(\beta+\theta_R - 4\pi/3)\Big) d\beta\ \hat{\mathbf{z}}$$

$$= \kappa\frac{r_R\ell_1\mu_0 N_R N_S}{4g} i_{R2}\Big(i_{S1}(-\pi\cos(\theta_R+\pi/6)) + i_{S2}(-\pi\sin(\theta_R)) + i_{S3}\pi\sin(\theta_R+\pi/3)\Big)\hat{\mathbf{z}}$$

$$= \frac{2}{3}Mi_{R2}(t)(-i_{S1}\cos(\theta_R+\pi/6) - i_{S2}\sin(\theta_R) + i_{S3}\sin(\theta_R+\pi/3))\hat{\mathbf{z}}.$$

Equivalent Two-Phase Torque

The torque on **rotor phase 3** is

$$\vec{\boldsymbol{\tau}}_{R3} = \int_{\beta=0}^{2\pi} (r_R \hat{\mathbf{r}}) \times \left(i_{R3}(t) \frac{N_R}{2} \sin(\beta - 4\pi/3) d\beta (+\ell_1 \hat{\mathbf{z}}) \times \left(B_{S|_{r=r_R}} \hat{\mathbf{r}} \right) \right)$$

$$= \int_{\beta=0}^{2\pi} r_R i_{R3}(t) \frac{\ell_1 N_R}{2} \sin(\beta - 4\pi/3) \left(\kappa \frac{\mu_0 N_S}{2g} \right) \Big(i_{S1} \cos(\beta + \theta_R)$$
$$+ i_{S2} \cos(\beta + \theta_R - 2\pi/3) + i_{S3} \cos(\beta + \theta_R - 4\pi/3) \Big) d\beta \; \hat{\mathbf{z}}$$

$$= \kappa \frac{r_R \ell_1 \mu_0 N_R N_S}{4g} i_{R3} \Big(i_{S1} \pi \cos(\theta_R - \pi/6) + i_{S2} \pi \sin(\theta_R - \pi/3) + i_{S3} \left(-\pi \sin(\theta_R) \right) \Big) \hat{\mathbf{z}}$$

$$= \frac{2}{3} M i_{R3}(t) (i_{S1} \cos(\theta_R - \pi/6) + i_{S2} \sin(\theta_R - \pi/3) - i_{S3} \sin(\theta_R)) \hat{\mathbf{z}}.$$

Equivalent Two-Phase Torque

The **total torque** on the rotor is then

$$
\begin{aligned}
\tau_R &= \tau_{R1} + \tau_{R2} + \tau_{R3} \\
&= \frac{2}{3} M \Big(i_{R1}(t)(-i_{S1}\sin(\theta_R) + i_{S2}\cos(\theta_R - \pi/6) - i_{S3}\cos(\theta_R + \pi/6)) + \\
&\qquad i_{R2}(t)(-i_{S1}\cos(\theta_R + \pi/6) - i_{S2}\sin(\theta_R) + i_{S3}\sin(\theta_R + \pi/3)) + \\
&\qquad i_{R3}(t)(+i_{S1}\cos(\theta_R - \pi/6) + i_{S2}\sin(\theta_R - \pi/3) - i_{S3}\sin(\theta_R)) \Big).
\end{aligned}
$$

With $i_{S0}(t) \equiv 0$, $i_{R0}(t) \equiv 0$, substitute

$$
\begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} \triangleq Q^{-1} \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix}, \qquad \begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix} \triangleq Q^{-1} \begin{bmatrix} i_{Ra}(t) \\ i_{Rb}(t) \\ i_{R0}(t) \end{bmatrix}
$$

to obtain

$$
\begin{aligned}
\tau_R &= M\Big(-i_{Ra}(t) i_{Sa}(t)\sin(\theta_R) + i_{Ra}(t) i_{Sb}(t)\cos(\theta_R) \\
&\qquad - i_{Rb}(t) i_{Sa}(t)\cos(\theta_R) - i_{Rb}(t) i_{Sb}(t)\sin(\theta_R) \Big).
\end{aligned}
$$

Equivalent Two-Phase Induction Motor Model

$$
\begin{aligned}
u_{Sa} &= L_S \frac{d}{dt} i_{Sa} + M \frac{d}{dt}\Big(+i_{Ra}\cos(\theta_R) - i_{Rb}\sin(\theta_R) \Big) + R_S i_{Sa} \\
u_{Sb} &= L_S \frac{d}{dt} i_{Sb} + M \frac{d}{dt}\Big(+i_{Ra}\sin(\theta_R) + i_{Rb}\cos(\theta_R) \Big) + R_S i_{Sb} \\
0 &= L_R \frac{d}{dt} i_{Ra} + M \frac{d}{dt}\Big(+i_{Sa}\cos(\theta_R) + i_{Sb}\sin(\theta_R) \Big) + R_R i_{Ra} \\
0 &= L_R \frac{d}{dt} i_{Rb} + M \frac{d}{dt}\Big(-i_{Sa}\sin(\theta_R) + i_{Sb}\cos(\theta_R) \Big) + R_R i_{Rb} \\
J\frac{d\omega_R}{dt} &= M\Big(-i_{Ra}(t) i_{Sa}(t)\sin(\theta_R) + i_{Ra}(t) i_{Sb}(t)\cos(\theta_R) \\
&\qquad - i_{Rb}(t) i_{Sa}(t)\cos(\theta_R) - i_{Rb}(t) i_{Sb}(t)\sin(\theta_R) \Big) - \tau_L \\
\frac{d\theta_R}{dt} &= \omega_R.
\end{aligned}
$$

- This is **identical** in form to the two-phase model derived in Chapter 6.
- L_S, L_R, M are the **two-phase equivalent** inductance values for the 3-phase machine.
- The **actual three phase** inductance values are $2L_S/3$, $2L_R/3$, and $2M/3$.
- Both models have the **same** resistances R_S and R_R.

Equivalent Two-Phase Induction Motor Model

Space Vector Model

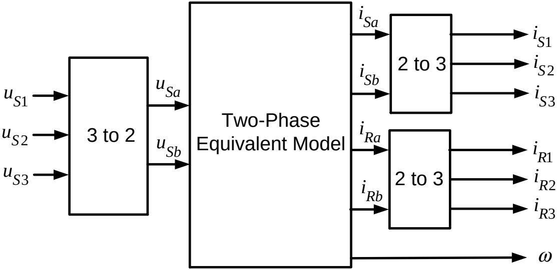
Define

$$
\begin{aligned}
\underline{u}_S &\triangleq u_{Sa} + j u_{Sb} \\
\underline{i}_S &\triangleq i_{Sa} + j i_{Sb} \\
\underline{i}_R &\triangleq i_{Ra} + j i_{Rb}.
\end{aligned}
$$

Then **you** may show that

$$
\begin{aligned}
R_S \underline{i}_S + L_S \frac{d}{dt}\underline{i}_S + M \frac{d}{dt}\left(\underline{i}_R e^{j\theta_R}\right) &= \underline{u}_S \\
R_R \underline{i}_R + L_R \frac{d}{dt}\underline{i}_R + M \frac{d}{dt}\left(\underline{i}_S e^{-j\theta_R}\right) &= 0 \\
M\,\mathrm{Im}\{\underline{i}_S (\underline{i}_R e^{j\theta_R})^*\} - \tau_L &= J\frac{d\omega_R}{dt} \\
\frac{d\theta_R}{dt} &= \omega_R.
\end{aligned}
$$

Simulation of the Three-Phase Machine

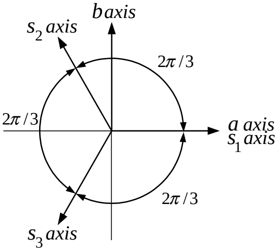


$$\begin{bmatrix} u_{Sa}(t) \\ u_{Sb}(t) \end{bmatrix} \triangleq \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} u_{S1}(t) \\ u_{S2}(t) \\ u_{S3}(t) \end{bmatrix}$$

$$\begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} \triangleq \sqrt{\frac{3}{2}} \begin{bmatrix} 2/3 & 0 \\ -1/3 & 1/\sqrt{3} \\ -1/3 & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \end{bmatrix}$$

$$\begin{bmatrix} i_{R1}(t) \\ i_{R2}(t) \\ i_{R3}(t) \end{bmatrix} \triangleq \sqrt{\frac{3}{2}} \begin{bmatrix} 2/3 & 0 \\ -1/3 & 1/\sqrt{3} \\ -1/3 & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} i_{Ra}(t) \\ i_{Rb}(t) \end{bmatrix}$$

Zero Component



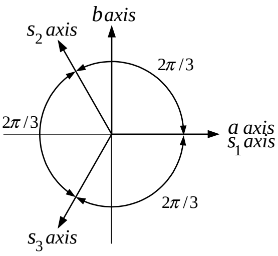
$$\begin{bmatrix} u_{Sa}(t) \\ u_{Sb}(t) \\ u_{S0}(t) \end{bmatrix} \triangleq \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & \cos(2\pi/3) & \cos(4\pi/3) \\ 0 & \sin(2\pi/3) & \sin(4\pi/3) \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} u_{S1}(t) \\ u_{S2}(t) \\ u_{S3}(t) \end{bmatrix}.$$

There are three **magnetic axes** s_1, s_2, and s_3 for the three stator phases.

There are two **orthogonal** axes denoted a and b, respectively.

The s_1, s_2, and s_3 axes are **not** orthogonal.

Zero Component



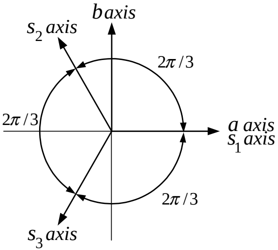
An interpretation of the 3 to 2 transformation is that $u_{S1}(t)$, $u_{S2}(t)$, $u_{S3}(t)$ are the **components** of the vector

$$\vec{\mathbf{u}}_S \triangleq u_{S1}(t)\hat{\mathbf{e}}_{S1} + u_{S2}(t)\hat{\mathbf{e}}_{S2} + u_{S3}(t)\hat{\mathbf{e}}_{S3}$$

with respect to the basis of **orthogonal unit vectors**

$$\hat{\mathbf{e}}_{S1} = \sqrt{\frac{2}{3}}\begin{bmatrix} 1 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \ \hat{\mathbf{e}}_{S2} = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos(2\pi/3) \\ \sin(2\pi/3) \\ 1/\sqrt{2} \end{bmatrix}, \ \hat{\mathbf{e}}_{S3} = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos(4\pi/3) \\ \sin(4\pi/3) \\ 1/\sqrt{2} \end{bmatrix}.$$

Zero Component



On the other hand, $u_{Sa}(t), u_{Sb}(t), u_{S0}(t)$ are the **components** of this **same** vector

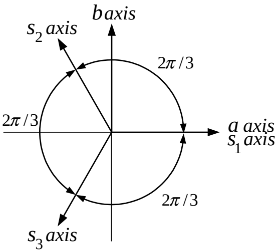
$$\vec{\mathbf{u}}_S = u_{Sa}(t)\hat{\mathbf{e}}_{Sa} + u_{Sb}(t)\hat{\mathbf{e}}_{Sb} + u_{S0}\hat{\mathbf{e}}_{S0}$$

with respect to the basis of **orthogonal unit vectors**

$$\hat{\mathbf{e}}_{Sa} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \hat{\mathbf{e}}_{Sb} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \hat{\mathbf{e}}_{S0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

u_{S0} is called the **zero component** because in a **balanced** three-phase system it is **zero**.

Zero Component



- The 3 to 2 transformation is also referred to as **Clarke's transformation**.
- The three-phase windings with magnetic axes s_1, s_2, s_3 are **magnetically** coupled.
  E.g., i_{S1} will produce a **nonzero** flux linkage in the phases 2 and 3.
- The 3 to 2 transformation **replaces** the 3-phase windings with a 2-phase winding.
  The 2-phase windings are **not** magnetically coupled.
  E.g., i_{Sa} will **not** produce a non zero flux linkage in phase b.
- Phase a is often referred to as the **direct** or **d axis.**
  Phase b is often referred to as the **quadrature** or **q axis.**
- However, **in this book**, the dq notation is reserved for the **field-oriented** coordinate system (see Chapters 8 and 9).

Steady-State Analysis of the Induction Motor

- $\underline{U}_S \triangleq |\underline{U}_S|\, e^{j\angle \underline{U}_S} = U_S e^{j\angle \underline{U}_S}$
- $U_S \triangleq |\underline{U}_S|$ is a **root-mean-square (rms)** voltage rather than a peak voltage. This is to keep the notation in this section consistent with standard practice.

$$
\begin{aligned}
u_{S1} &= \sqrt{2}U_S \cos(\omega_S t + \angle \underline{U}_S) = \frac{\sqrt{2}}{2}\left(\underline{U}_S e^{j\omega_S t} + \underline{U}_S^* e^{-j\omega_S t}\right) \\
u_{S2} &= \sqrt{2}U_S \cos(\omega_S t + \angle \underline{U}_S - 2\pi/3) = \frac{\sqrt{2}}{2}\left(\underline{U}_S e^{j(\omega_S t - 2\pi/3)} + \underline{U}_S^* e^{-j(\omega_S t - 2\pi/3)}\right) \\
u_{S3} &= \sqrt{2}U_S \cos(\omega_S t + \angle \underline{U}_S - 4\pi/3) = \frac{\sqrt{2}}{2}\left(\underline{U}_S e^{j(\omega_S t - 4\pi/3)} + \underline{U}_S^* e^{-j(\omega_S t - 4\pi/3)}\right).
\end{aligned}
$$

- Apply these to the motor and find the resulting steady-state currents and torque.

Steady-State Analysis of the Induction Motor

Steady-State Voltages

The three-phase to two-phase transformation in space vector form is

$$\underline{u}_S = \sqrt{\frac{2}{3}}\left(u_{S1} + u_{S2}e^{j2\pi/3} + u_{S3}e^{j4\pi/3}\right)$$

$$= \sqrt{\frac{2}{3}}\Big(u_{S1} + u_{S2}\cos(2\pi/3) + u_{S3}\cos(4\pi/3)\Big) + j\sqrt{\frac{2}{3}}\Big(u_{S2}\sin(2\pi/3) + u_{S3}\sin(4\pi/3)\Big)$$

$$= u_{Sa} + ju_{Sb}.$$

Using the fact that $1 + e^{j4\pi/3} + e^{j8\pi/3} = 0$ we have

$$\begin{aligned}
\underline{u}_S \quad &= \quad \frac{1}{\sqrt{3}}\left(\underline{U}_S e^{j\omega_S t} + \underline{U}_S^* e^{-j\omega_S t}\right) + \frac{1}{\sqrt{3}}\left(\underline{U}_S e^{j(\omega_S t - 2\pi/3)} + \underline{U}_S^* e^{-j(\omega_S t - 2\pi/3)}\right)e^{j2\pi/3} \\
&\quad + \frac{1}{\sqrt{3}}\left(\underline{U}_S e^{j(\omega_S t - 4\pi/3)} + \underline{U}_S^* e^{-j(\omega_S t - 4\pi/3)}\right)e^{j4\pi/3} \\
&= \quad \sqrt{3}\underline{U}_S e^{j\omega_S t}.
\end{aligned}$$

This is the **steady-state voltage** applied to the induction motor.

Steady-State Analysis of the Induction Motor

Steady-State Stator Currents

- Let $\underline{I}_S \triangleq |\underline{I}_S| e^{j\angle \underline{I}_S} = I_S e^{j\angle \underline{I}_S}$ be an **rms current phasor**.
- Look for **steady-state** stator currents of the form

$$
\begin{aligned}
i_{S1} &= \sqrt{2} I_S \cos(\omega_S t + \angle \underline{I}_S) = \frac{\sqrt{2}}{2}\left(\underline{I}_S e^{j\omega_S t} + \underline{I}_S^* e^{-j\omega_S t}\right) \\
i_{S2} &= \sqrt{2} I_S \cos(\omega_S t + \angle \underline{I}_S - 2\pi/3) = \frac{\sqrt{2}}{2}\left(\underline{I}_S e^{j(\omega_S t - 2\pi/3)} + \underline{I}_S^* e^{-j(\omega_S t - 2\pi/3)}\right) \\
i_{S3} &= \sqrt{2} I_S \cos(\omega_S t + \angle \underline{I}_S - 4\pi/3) = \frac{\sqrt{2}}{2}\left(\underline{I}_S e^{j(\omega_S t - 4\pi/3)} + \underline{I}_S^* e^{-j(\omega_S t - 4\pi/3)}\right)
\end{aligned}
$$

- This is a **balanced** set of currents.
- The three-phase to two-phase transformation results in

$$
\underline{i}_S = \sqrt{\frac{2}{3}}\left(i_{S1} + i_{S2} e^{j2\pi/3} + i_{S3} e^{j4\pi/3}\right) = \sqrt{3}\,\underline{I}_S e^{j\omega_S t}.
$$

Steady-State Analysis of the Induction Motor

Steady-State Rotor Currents

- Let $\underline{I}_R = |\underline{I}_R|\, e^{j\angle \underline{I}_R} = I_R e^{j\angle \underline{I}_R}$.
- Look for **steady-state** rotor currents of the form

$$\begin{aligned}
i_{R1} &= \frac{\sqrt{2}}{2}\left(\underline{I}_R e^{j(\omega_S-\omega_R)t} + \underline{I}_R^* e^{-j(\omega_S-\omega_R)t}\right)\\
i_{R2} &= \frac{\sqrt{2}}{2}\left(\underline{I}_R e^{j((\omega_S-\omega_R)t-2\pi/3)} + \underline{I}_R^* e^{-j((\omega_S-\omega_R)t-2\pi/3)}\right)\\
i_{R3} &= \frac{\sqrt{2}}{2}\left(\underline{I}_R e^{j((\omega_S-\omega_R)t-4\pi/3)} + \underline{I}_R^* e^{-j((\omega_S-\omega_R)t-4\pi/3)}\right).
\end{aligned}$$

- This is a **balanced** set of currents.
- The three-phase to two-phase transformation of the rotor currents results in

$$\underline{i}_R = \sqrt{\frac{2}{3}}\left(i_{R1} + i_{R2}e^{j2\pi/3} + i_{R3}e^{j4\pi/3}\right) = \sqrt{3}\underline{I}_R e^{j(\omega_S-\omega_R)t}.$$

Finally, with $\theta_R(t) = \omega_R t$ we write

$$\underline{i}_R(t)e^{j\theta_R(t)} = \sqrt{3}\underline{I}_R e^{j\omega_S t}.$$

Steady-State Equivalent Circuit Model

Substitute

$$\underline{u}_S = \sqrt{3}\underline{U}_S e^{j\omega_S t}, \quad \underline{i}_S = \sqrt{3}\underline{I}_S e^{j\omega_S t}, \quad \underline{i}_R = \sqrt{3}\underline{I}_R e^{j(\omega_S - \omega_R)t}, \quad \underline{i}_R e^{j\theta_R(t)} = \sqrt{3}\underline{I}_R e^{j\omega_S t}$$

into

$$\begin{aligned} R_S \underline{i}_S + L_S \frac{d}{dt}\underline{i}_S + M\frac{d}{dt}\left(\underline{i}_R e^{j\theta_R}\right) &= \underline{u}_S \\ R_R \underline{i}_R + L_R \frac{d}{dt}\underline{i}_R + M\frac{d}{dt}\left(\underline{i}_S e^{-j\theta_R}\right) &= 0 \end{aligned}$$

to obtain

$$R_S\sqrt{3}\underline{I}_S e^{j\omega_S t} + j\omega_S L_S\sqrt{3}\underline{I}_S e^{j\omega_S t} + j\omega_S M\sqrt{3}\underline{I}_R e^{j\omega_S t} = \sqrt{3}\underline{U}_S e^{j\omega_S t}$$

$$R_R\sqrt{3}\underline{I}_R e^{j(\omega_S-\omega_R)t} + j(\omega_S-\omega_R)L_R\sqrt{3}\underline{I}_R e^{j(\omega_S-\omega_R)t} + j(\omega_S-\omega_R)M\sqrt{3}\underline{I}_S e^{j(\omega_S-\omega_R)t} = 0$$

or

$$\begin{aligned} (R_S + j\omega_S L_S)\underline{I}_S + j\omega_S M\underline{I}_R &= \underline{U}_S \\ (R_R + j(\omega_S-\omega_R)L_R)\underline{I}_R + j(\omega_S-\omega_R)M\underline{I}_S &= 0. \end{aligned}$$

Steady-State Equivalent Circuit Model

With $\omega_{slip} \triangleq \omega_S - \omega_R$, define the **normalized slip**

$$S = \frac{\omega_{slip}}{\omega_S} = \frac{\omega_S - \omega_R}{\omega_S}.$$

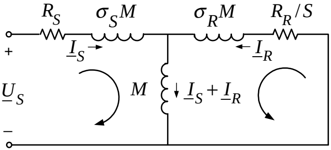
Replace $\omega_{slip} = \omega_S - \omega_R$ by $S\omega_S$

$$\begin{aligned}(R_S + j\omega_S L_S)\underline{I}_S + j\omega_S M \underline{I}_R &= \underline{U}_S \\ (R_R/S + j\omega_S L_R)\underline{I}_R + j\omega_S M \underline{I}_S &= 0.\end{aligned}$$

- $\sigma = 1 - M^2/(L_S L_R) > 0 \Longrightarrow L_S, L_R$ are slightly greater than M.
- Set $L_S = (1+\sigma_S)M, L_R = (1+\sigma_R)M$.

$$\begin{aligned}(R_S + j\omega_S \sigma_S M)\underline{I}_S + j\omega_S M(\underline{I}_S + \underline{I}_R) &= \underline{U}_S \\ (R_R/S + j\omega_S \sigma_R M)\underline{I}_R + j\omega_S M(\underline{I}_S + \underline{I}_R) &= 0.\end{aligned}$$

Equivalent circuit



Steady-State Equivalent Circuit Model

Rewrite

$$\begin{aligned}(R_S + j\omega_S L_S)\underline{I}_S + j\omega_S M \underline{I}_R &= \underline{U}_S \\ (R_R/S + j\omega_S L_R)\underline{I}_R + j\omega_S M \underline{I}_S &= 0\end{aligned}$$

in **matrix form** as

$$\begin{bmatrix} R_S + j\omega_S L_S & j\omega_S M \\ j\omega_S M & R_R/S + j\omega_S L_R \end{bmatrix}\begin{bmatrix} \underline{I}_S \\ \underline{I}_R \end{bmatrix} = \begin{bmatrix} \underline{U}_S \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} \underline{I}_S \\ \underline{I}_R \end{bmatrix} = \frac{1}{(R_S + j\omega_S L_S)(R_R/S + j\omega_S L_R) - (j\omega_S M)^2}\begin{bmatrix} R_R/S + j\omega_S L_R & -j\omega_S M \\ -j\omega_S M & R_S + j\omega_S L_S \end{bmatrix}\begin{bmatrix} \underline{U}_S \\ 0 \end{bmatrix}.$$

The **input impedance** is then

$$\begin{aligned} Z_S = \frac{\underline{U}_S}{\underline{I}_S} &= \frac{(R_S + j\omega_S L_S)(R_R/S + j\omega_S L_R) - (j\omega_S M)^2}{R_R/S + j\omega_S L_R} \\ &= R_S + j\omega_S L_S - \frac{(j\omega_S M)^2}{R_R/S + j\omega_S L_R}.\end{aligned}$$

Stator Impedance

With $M = \dfrac{L_S}{1+\sigma_S} = \dfrac{L_R}{1+\sigma_R}$ and $\sigma = 1 - \dfrac{1}{(1+\sigma_S)(1+\sigma_R)}$ we have

$$
\begin{aligned}
Z_S &= R_S + j\omega_S L_S - \frac{(j\omega_S M)^2}{R_R/S + j\omega_S L_R} \\
&= R_S + j\omega_S L_S \left(1 - \frac{\dfrac{S}{R_R} j\omega_S L_R \dfrac{1}{(1+\sigma_S)(1+\sigma_R)}}{1 + \dfrac{j\omega_S L_R S}{R_R}} \right) \\
&= R_S + j\omega_S L_S \left(\frac{1 + \dfrac{j\omega_S L_R S}{R_R}\left(1 - \dfrac{1}{(1+\sigma_S)(1+\sigma_R)}\right)}{1 + \dfrac{j\omega_S L_R S}{R_R}} \right) \\
&= R_S + j\omega_S L_S \left(\frac{1 + \dfrac{j\sigma\omega_S L_R S}{R_R}}{1 + \dfrac{j\omega_S L_R S}{R_R}} \right).
\end{aligned}
$$

Stator Impedance

Finally, defining the **pull out slip** S_p as

$$S_p \triangleq \frac{R_R}{\sigma\omega_S L_R},$$

the input impedance

$$Z_S = R_S + j\omega_S L_S\left(\frac{1+\dfrac{j\sigma\omega_S L_R S}{R_R}}{1+\dfrac{j\omega_S L_R S}{R_R}}\right).$$

becomes

$$Z_S = R_S + j\omega_S L_S\left(\frac{1+\dfrac{jS}{S_p}}{1+\dfrac{jS}{\sigma S_p}}\right).$$

With $R_S = 0$ the **stator current phasor** $\underline{I}_S = \underline{U}_S/Z_S$ becomes

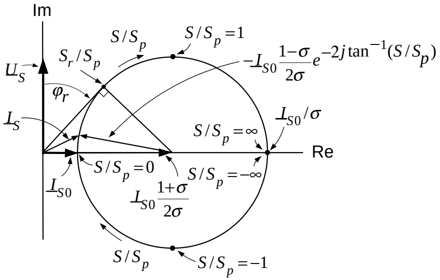
$$\begin{aligned}\underline{I}_S = \frac{\underline{U}_S}{j\omega_S L_S}\left(\frac{1+\dfrac{jS}{\sigma S_p}}{1+\dfrac{jS}{S_p}}\right) &= \underline{I}_{S0}\left(\frac{1+\sigma}{2\sigma} - \frac{1-\sigma}{2\sigma}\frac{1-\dfrac{jS}{S_p}}{1+\dfrac{jS}{S_p}}\right)\\ &= \underline{I}_{S0}\left(\frac{1+\sigma}{2\sigma} - \frac{1-\sigma}{2\sigma}e^{-j2\tan^{-1}(S/S_p)}\right).\end{aligned}$$

Stator Current Phasor Versus Slip

$$\underline{I}_S = \underline{I}_{S0}\left(\frac{1+\sigma}{2\sigma} - \frac{1-\sigma}{2\sigma}e^{-j2\tan^{-1}(S/S_p)}\right),\ \underline{I}_{S0} = \frac{\underline{U}_S}{j\omega_S L_S}$$

- Here $\underline{I}_{S0} \triangleq \underline{U}_S/(j\omega_S L_S)$ is the **no-load** stator current phasor. I.e., the stator current phasor when the slip is **zero**.
- Choose $\underline{U}_S = jU_S$ so that $\underline{I}_{S0} = I_{S0} = U_S/(\omega_S L_S)$ is **real**.

Circle Diagram: Plot of $\underline{I}_S$ versus S/S_p.



- As S/S_p varies from $-\infty$ to ∞, $\underline{I}_S$ traces out a **circle**.
- $\underline{I}_S$ lies on a circle with center $\underline{I}_{S0}\frac{1+\sigma}{2\sigma}$ and radius $-\underline{I}_{S0}\frac{1+\sigma}{2\sigma}e^{-j2\tan^{-1}(S/S_p)}$.

Steady-State Power

Average Power P_{stator}

$$P_{\text{stator}} \triangleq \frac{1}{2\pi/\omega_S}\int_0^{2\pi/\omega_S}\left(u_{S1}i_{S1}+u_{S2}i_{S2}+u_{S3}i_{S3}\right)dt.$$

Then

$$\begin{aligned}
\frac{1}{2\pi/\omega_S}\int_0^{2\pi/\omega_S} u_{S1}i_{S1}\,dt &= \frac{1}{2\pi/\omega_S}\int_0^{2\pi/\omega_S}\left(\frac{\sqrt{2}}{2}\left(\underline{U}_S e^{j\omega_S t}+\underline{U}_S^* e^{-j\omega_S t}\right)\right)\left(\frac{\sqrt{2}}{2}\left(\underline{I}_S e^{j\omega_S t}+\underline{I}_S^* e^{-j\omega_S t}\right)\right)\\
&= \frac{1}{2\pi/\omega_S}\int_0^{2\pi/\omega_S}\frac{1}{2}\left(\underline{U}_S\underline{I}_S^*+\underline{U}_S^*\underline{I}_S+\underline{U}_S\underline{I}_S e^{j2\omega_S t}+\underline{U}_S^*\underline{I}_S^* e^{-j2\omega_S t}\right)dt\\
&= \frac{1}{2}\left(\underline{U}_S\underline{I}_S^*+\underline{U}_S^*\underline{I}_S\right)\\
&= \mathrm{Re}\{\underline{U}_S\underline{I}_S^*\}\\
&= U_S I_S\cos(\angle\underline{U}_S-\angle\underline{I}_S)
\end{aligned}$$

as

$$\mathrm{Re}\{\underline{U}_S\underline{I}_S^*\}=\mathrm{Re}\left\{|\underline{U}_S|\,e^{j\angle\underline{U}_S}\,|\underline{I}_S|\,e^{-j\angle\underline{I}_S}\right\}=|\underline{U}_S|\,|\underline{I}_S|\cos(\angle\underline{U}_S-\angle\underline{I}_S).$$

- $\varphi \triangleq \angle\underline{U}_S-\angle\underline{I}_S$ is the **power factor angle**.
- Each phase contributes the same average power so $P_{\text{stator}}=3U_S I_S\cos(\varphi)$.
- $\cos(\varphi)$ is the **power factor.**

Stator Current Phasor

$$\underline{I}_S = \underbrace{\frac{\underline{U}_S}{j\omega_S L_S}}_{I_{S0}} \left(\frac{1 + \dfrac{jS}{\sigma S_p}}{1 + \dfrac{jS}{S_p}} \right), \quad S_p \triangleq \frac{R_R}{\sigma \omega_S L_R}$$

The **power factor angle** φ may be written

$$\varphi = \angle \underline{U}_S - \angle \underline{I}_S = \frac{\pi}{2} - \left(\tan^{-1}\left(\frac{S}{\sigma S_p}\right) - \tan^{-1}\left(\frac{S}{S_p}\right) \right).$$

The **rated slip** S_r is the value of S that **minimizes** φ (**maximizes** $\cos(\varphi)$). Solving $d\varphi/dS = 0$ gives

$$S_r = \sqrt{\sigma} S_p.$$

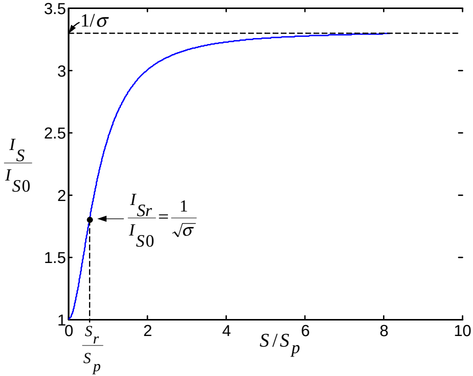
The ratio I_S/I_{S0} may be written as

$$\left(\frac{I_S}{I_{S0}}\right)^2 = \frac{1 + \left(\dfrac{S}{\sigma S_p}\right)^2}{1 + \left(\dfrac{S}{S_p}\right)^2} \Longrightarrow \left.\frac{I_S}{I_{S0}}\right|_{S=S_r} = \frac{1}{\sqrt{\sigma}} \quad \text{(rated stator current).}$$

Also

$$\lim_{S\to\infty} \frac{I_S}{I_{S0}} = \frac{1}{\sigma}.$$

Stator Current Phasor



$$\frac{I_S}{I_{S0}} = \left(\frac{1 + (S/\sigma S_p)^2}{1 + (S/S_p)^2}\right)^{1/2}.$$

At **rated slip** $S_r = \sqrt{\sigma} S_p$ the **power factor angle** is

$$\varphi_r = \pi/2 - \left(\tan^{-1}(S/\sigma S_p) - \tan^{-1}(S/S_p)\right)\Big|_{S=S_r} = \pi/2 - \left(\tan^{-1}(1/\sqrt{\sigma}) - \tan^{-1}(\sqrt{\sigma})\right).$$

The **rated power factor** is $\cos(\varphi_r) = \dfrac{1-\sigma}{1+\sigma}$.

Rated Conditions

Suppose the motor is operating at **rated slip** $S_r = \dfrac{\omega_S - \omega_R}{\omega_S}$.

Then the power **into** the motor is

$$3U_S I_{S_r}\cos(\varphi_r) = 3U_S \frac{I_{S0}}{\sqrt{\sigma}}\frac{1-\sigma}{1+\sigma} \quad \text{as} \quad I_{S_r} = \frac{I_{S0}}{\sqrt{\sigma}} \quad \text{and} \quad \cos(\varphi_r) = \frac{1-\sigma}{1+\sigma}$$

- At rated slip, the power factor $\cos(\varphi_r)$ is at its **maximum** value.
- I_S required to achieve the particular input power level is at a **minimum**.
- This power goes into mechanical work and losses given by

$$3U_S I_{S_r}\cos(\varphi_r) = \tau_r\omega_R + 3R_R I_R^2.$$

- At these operating conditions τ_r is the **rated** torque and ω_R is the **rated** speed.
- The losses $3R_S I_S^2$ are at **minimum** if motor is operating at **rated slip**.
- This last conclusion is a bit **bogus** as our analysis assumed $R_S = 0$!

Steady-State Torque

Recall the **space vector** model:

$$\begin{aligned}
R_S \underline{i}_S + L_S \frac{d}{dt}\underline{i}_S + M\frac{d}{dt}\left(\underline{i}_R e^{j\theta_R}\right) &= \underline{u}_S \\
R_R \underline{i}_R + L_R \frac{d}{dt}\underline{i}_R + M\frac{d}{dt}\left(\underline{i}_S e^{-j\theta_R}\right) &= 0 \\
M\,\mathrm{Im}\{\underline{i}_S(\underline{i}_R e^{j\theta_R})^*\} - \tau_L &= J\frac{d\omega_R}{dt} \\
\frac{d\theta_R}{dt} &= \omega_R.
\end{aligned}$$

With $\underline{i}_S = \sqrt{3}\underline{I}_S e^{j\omega_S t}$ and $\underline{i}_R(t)e^{j\theta_R(t)} = \sqrt{3}\underline{I}_R e^{j\omega_S t}$,

$$\tau = M\,\mathrm{Im}\{\underline{i}_S(\underline{i}_R e^{j\theta_R})^*\} = 3M\,\mathrm{Im}\{\underline{I}_S \underline{I}_R^*\}.$$

From the electrical equations

$$\begin{aligned}
(R_S + j\omega_S L_S)\underline{I}_S + j\omega_S M\underline{I}_R &= \underline{U}_S \\
(R_R/S + j\omega_S L_R)\underline{I}_R + j\omega_S M\underline{I}_S &= 0
\end{aligned}$$

we have

$$\underline{I}_R = -\frac{j\omega_S M}{R_R/S + j\omega_S L_R}\underline{I}_S.$$

Steady-State Torque

With

$$\underline{I}_R = -\frac{j\omega_S M}{R_R/S + j\omega_S L_R}\underline{I}_S$$

we have

$$\begin{aligned}
\tau = 3M\,\mathrm{Im}\{\underline{I}_S \underline{I}_R^*\} &= 3M\,\mathrm{Im}\left\{\underline{I}_S\left(-\frac{j\omega_S M}{R_R/S + j\omega S L_R}\underline{I}_S\right)^*\right\}\\
&= 3M\,\mathrm{Im}\left\{\underline{I}_S \frac{j\omega_S M}{R_R/S - j\omega_S L_R}\underline{I}_S^*\right\}\\
&= 3MI_S^2\,\mathrm{Im}\left\{\frac{\dfrac{j\omega_S M S}{R_R}}{1 - \dfrac{j\omega_S L_R S}{R_R}}\right\}.
\end{aligned}$$

Steady-State Torque

Recalling that $S_p = \dfrac{R_R}{\sigma\omega_S L_R}$ we have

$$\tau = 3MI_S^2\,\mathrm{Im}\left\{\frac{\dfrac{j\omega_S MS}{R_R}}{1-\dfrac{j\omega_S L_R S}{R_R}}\right\} = 3MI_S^2\,\mathrm{Im}\left\{\frac{\dfrac{j\omega_S MS}{R_R}}{1-\dfrac{jS}{\sigma S_p}}\,\frac{1+\dfrac{jS}{\sigma S_p}}{1+\dfrac{jS}{\sigma S_p}}\right\}$$

$$= 3MI_S^2\,\mathrm{Im}\left\{\frac{\dfrac{j\omega_S MS}{R_R}\left(1+\dfrac{jS}{\sigma S_p}\right)}{1+\left(\dfrac{S}{\sigma S_p}\right)^2}\right\}$$

$$= 3MI_S^2\,\frac{\omega_S MS/R_R}{1+\left(\dfrac{S}{\sigma S_p}\right)^2}$$

$$= 3M\left(\frac{I_S}{I_{S0}}\right)^2(I_{S0})^2\,\frac{\omega_S MS/R_R}{1+\left(\dfrac{S}{\sigma S_p}\right)^2}.$$

Steady-State Torque

$$\tau = 3M\frac{1+\left(\dfrac{S}{\sigma S_p}\right)^2}{1+\left(\dfrac{S}{S_p}\right)^2}\left(\frac{U_S}{\omega_S L_S}\right)^2\frac{\omega_S MS/R_R}{1+\left(\dfrac{S}{\sigma S_p}\right)^2} \quad = \quad 3\frac{U_S^2}{\omega_S^2 L_S^2}\frac{\omega_S M^2 S/R_R}{1+\left(\dfrac{S}{S_p}\right)^2}$$

$$= \quad 3\frac{U_S^2}{\omega_S^2 L_S^2}\frac{S_p/S}{S_p/S+S/S_p}\frac{\omega_S M^2 S}{R_R}.$$

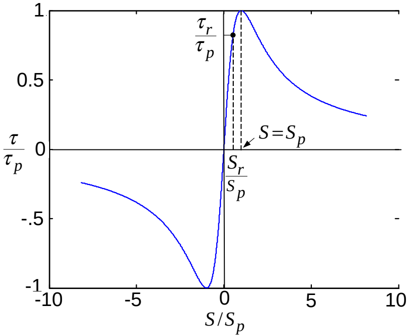
Substitute $M^2 = \dfrac{L_S}{1+\sigma_S}\dfrac{L_R}{1+\sigma_R} = L_S L_R(1-\sigma)$ to obtain

$$\tau = 3\frac{U_S^2}{\omega_S^2 L_S^2}\frac{S_p/S}{S_p/S+S/S_p}L_S(1-\sigma)\frac{\omega_S L_R}{R_R}S \quad = \quad 3\frac{U_S^2}{\omega_S^2 L_S}\frac{S_p/S}{S_p/S+S/S_p}\frac{1-\sigma}{\sigma}\frac{S}{S_p}$$

$$= \quad 3\frac{U_S^2}{\omega_S^2 L_S}\frac{1}{S_p/S+S/S_p}\frac{1-\sigma}{\sigma}.$$

Steady-State Torque-Slip Curve

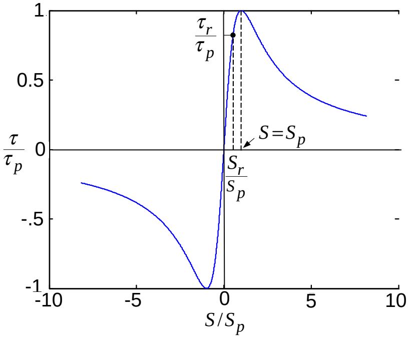
$$\tau = 3\frac{U_S^2}{\omega_S^2 L_S}\frac{1}{S_p/S + S/S_p}\frac{1-\sigma}{\sigma} = \underbrace{\frac{3}{2}\frac{1-\sigma}{\sigma}\frac{U_S^2}{\omega_S^2 L_S}}_{\tau_p}\frac{2}{S_p/S + S/S_p}.$$



The torque is a **maximum** for $S = S_p$.
Setting $S_r = \sqrt{\sigma}S_p$ we find the **rated torque** is

$$\tau_r \triangleq \tau_p \frac{2}{S/S_p + S_p/S}\Big|_{S=S_r} = \tau_p\frac{2}{\sqrt{\sigma} + 1/\sqrt{\sigma}} = \tau_p\frac{2\sqrt{\sigma}}{1+\sigma}.$$

Steady-State Torque

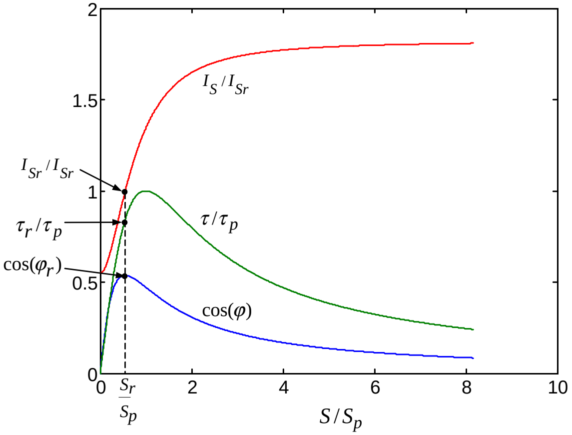


- S_p is called the **pull-out slip**.
- τ_p is called the **pull-out torque**.

Pull-Out Slip/Torque:

For $S > S_p$ the torque **decreases** as the slip **increases**.
If the motor operates at $S = S_p$ and more load torque is added, then ω_R **decreases**.
The slip $S = (\omega_S - \omega_R)/\omega_S$ then **increases** so even **less** torque is produced.
The machine slows down and stops, i.e., it **pulls out** from producing torque.
$S \geq S_p$ are not **stable** operating points.

Torque, Stator Current and Power Factor vs Slip



Steady-State Power Transfer in the Induction Motor

Steady-State Input Power

$$
\begin{aligned}
P_{\text{stator}} &= 3U_S I_S \cos(\varphi) \\
\frac{I_S}{I_{S0}} &= \left(\frac{1+(S/\sigma S_p)^2}{1+(S/S_p)^2}\right)^{1/2}, \qquad I_{S0} = \frac{U_S}{\omega_S L_S} \\
\varphi &= \angle \underline{U}_S - \angle \underline{I}_S = \frac{\pi}{2} - \left(\tan^{-1}\left(\frac{S}{\sigma S_p}\right) - \tan^{-1}\left(\frac{S}{S_p}\right)\right).
\end{aligned}
$$

The **power factor** may then be written as

$$
\begin{aligned}
\cos(\varphi) &= \cos\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{S}{\sigma S_p}\right) + \tan^{-1}\left(\frac{S}{S_p}\right)\right) \\
&= \sin\left(\tan^{-1}\left(\frac{S}{\sigma S_p}\right) - \tan^{-1}\left(\frac{S}{S_p}\right)\right) \\
&= \sin\left(\tan^{-1}\left(\frac{S}{\sigma S_p}\right)\right)\cos\left(\tan^{-1}\left(\frac{S}{S_p}\right)\right) - \sin\left(\tan^{-1}\left(\frac{S}{S_p}\right)\right)\cos\left(\tan^{-1}\left(\frac{S}{\sigma S_p}\right)\right) \\
&= \frac{\dfrac{S}{\sigma S_p}}{\sqrt{1+\left(\dfrac{S}{\sigma S_p}\right)^2}}\frac{1}{\sqrt{1+\left(\dfrac{S}{S_p}\right)^2}} - \frac{\dfrac{S}{S_p}}{\sqrt{1+\left(\dfrac{S}{S_p}\right)^2}}\frac{1}{\sqrt{1+\left(\dfrac{S}{\sigma S_p}\right)^2}}
\end{aligned}
$$

Steady-State Power Transfer in the Induction Motor

$$\cos(\varphi) = \frac{S/\sigma S_p - S/S_p}{\sqrt{1+(S/\sigma S_p)^2}\sqrt{1+(S/S_p)^2}}.$$

The power into the stator is then

$$\begin{aligned}
P_{\text{stator}} = 3U_S I_S \cos(\varphi) &= 3U_S\sqrt{\frac{1+(S/\sigma S_p)^2}{1+(S/S_p)^2}}\frac{U_S}{\omega_S L_S}\frac{S/\sigma S_p - S/S_p}{\sqrt{1+(S/\sigma S_p)^2}\sqrt{1+(S/S_p)^2}} \\
&= 3\frac{U_S^2}{\omega_S L_S}\frac{S/\sigma S_p - S/S_p}{1+(S/S_p)^2} \\
&= 3\frac{U_S^2}{\omega_S L_S}\frac{1/\sigma - 1}{S_p/S + S/S_p} \\
&= \frac{3}{2}\frac{U_S^2}{\omega_S L_S}\frac{1-\sigma}{\sigma}\frac{2}{S_p/S + S/S_p} \\
&= \omega_S \tau.
\end{aligned}$$

The **mechanical power** produced is $P_{\text{mech}} = \omega_R \tau$.

Steady-State Power Transfer in the Induction Motor

The difference between the **input stator power** and the **output mechanical power** is

$$P_{\text{stator}} - P_{\text{mech}} = (\omega_S - \omega_R)\tau = \omega_S S \tau.$$

Where does this power go? We now show $\omega_S S \tau = 3 I_R^2 R_R$. (See slide 55 for $\underline{I}_R$)

$$\begin{aligned}
I_R^2 = \frac{\omega_S^2 M^2}{(R_R/S)^2 + \omega_S^2 L_R^2} I_S^2 &= \frac{\left(\dfrac{\omega_S M S}{R_R}\right)^2}{1 + \left(\dfrac{\omega_S L_R S}{R_R}\right)^2} I_S^2 \\
&= \frac{\dfrac{1}{(1+\sigma_R)(1+\sigma_S)}\left(\dfrac{\omega_S L_R S}{R_R}\right)\left(\dfrac{\omega_S L_S S}{R_R}\right)}{1 + \left(\dfrac{S}{\sigma S_p}\right)^2} I_S^2 \\
&= \frac{(1-\sigma)\left(\dfrac{S}{\sigma S_p}\right)^2 \dfrac{L_S}{L_R}}{1 + \left(\dfrac{S}{\sigma S_p}\right)^2} I_S^2.
\end{aligned}$$

Eliminating I_S^2

$$
\begin{aligned}
I_R^2 &= \frac{(1-\sigma)\left(\dfrac{S}{\sigma S_p}\right)^2 \dfrac{L_S}{L_R}}{1+\left(\dfrac{S}{\sigma S_p}\right)^2} \frac{1+\left(\dfrac{S}{\sigma S_p}\right)^2}{1+\left(\dfrac{S}{S_p}\right)^2} \left(\frac{U_S}{\omega_S L_S}\right)^2 \\
&= \frac{(1-\sigma)\left(\dfrac{S}{\sigma S_p}\right)^2 \dfrac{L_S}{L_R}}{1+\left(\dfrac{S}{S_p}\right)^2} \left(\frac{U_S}{\omega_S L_S}\right)^2 \\
&= \frac{U_S^2}{\omega_S^2 L_S} \frac{1-\sigma}{\sigma} \frac{\dfrac{S}{\sigma S_p}}{\dfrac{S_p}{S}+\dfrac{S}{S_p}} \frac{1}{L_R} \\
&= \frac{1}{2} \frac{U_S^2}{\omega_S^2 L_S} \frac{1-\sigma}{\sigma} \frac{2}{S_p/S+S/S_p} S \frac{\omega_S}{R_R}
\end{aligned}
$$

where

$$
\frac{1}{\sigma S_p L_R} = \frac{1}{\sigma \dfrac{R_R}{\sigma \omega_S L_R} L_R} = \frac{\omega_S}{R_R}.
$$

Steady-State Power Transfer in the Induction Motor

Then

$$3I_R^2 R_R = \left(\underbrace{\frac{3}{2}\frac{U_S^2}{\omega_S^2 L_S}\frac{1-\sigma}{\sigma}}_{\tau_p} \frac{2}{S_p/S + S/S_p} \right) S\omega_S = \omega_S S\tau.$$

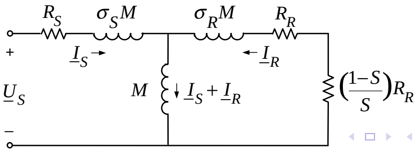
Summary

$$P_{\text{stator}} - P_{\text{mech}} = (\omega_S - \omega_R)\tau = \omega_S S\tau = 3I_R^2 R_R.$$

$$\begin{aligned} P_{\text{mech}} &= \omega_R\tau = \frac{\omega_R}{\omega_S S}\omega_S S\tau = \frac{(1-S)\,\omega_S}{\omega_S S}3I_R^2 R_R = 3I_R^2\frac{1-S}{S}R_R \\ P_{\text{stator}} &= P_{\text{stator}} - P_{\text{mech}} + P_{\text{mech}} = 3I_R^2 R_R + 3I_R^2\frac{1-S}{S}R_R = 3I_R^2\frac{R_R}{S}. \end{aligned}$$

View the energy into the rotor as being dissipated in an **"equivalent"** resistance of

$$\frac{R_R}{S} = R_R + \frac{1-S}{S}R_R.$$



Steady-State Power Transfer in the Induction Motor

Efficiency

The efficiency is ($R_S = 0$ is still assumed)

$$\text{Efficiency} \triangleq \frac{P_{\text{mech}}}{P_{\text{stator}}} = \frac{3I_R^2 \dfrac{1-S}{S} R_R}{3I_R^2 \dfrac{R_R}{S}} = 1 - S = \frac{\omega_R}{\omega_S} < 1.$$

- The input power not converted to mechanical power is lost as heat in the rotor.
- Consequently, the slip must not be too large.
- The smaller the slip the smaller the output torque.
- Trade off between torque and efficiency.

Theory Versus Experiment

Compare the theoretical values with the measured values of

$$S_r/S_p = \sqrt{\sigma}, \quad I_{S0}/I_{Sr} = \sqrt{\sigma}, \quad \cos(\varphi_r) = \frac{1-\sigma}{1+\sigma}, \quad \tau_r/\tau_p = \frac{2\sqrt{\sigma}}{1+\sigma}.$$

The first motor has two poles ($n_p = 1$) and $\sigma = 0.05$.

Quantity	Predicted Value $\sigma = 0.05$	Actual Value
S_r/S_p	$\sqrt{\sigma} = 0.22$	0.20
I_{S0}/I_{Sr}	$\sqrt{\sigma} = 0.22$	0.30
$\cos(\varphi_r)$	$\frac{1-\sigma}{1+\sigma} = 0.90$	0.90
τ_p/τ_r	$\frac{1+\sigma}{2\sqrt{\sigma}} = 2.35$	2.30

Figure: From Table 10.2 of *Control of Electrical Drives*, 3rd edition by W. Leonhard, Springer-Verlag, 2001. Reprinted with permission.

Theory Versus Experiment

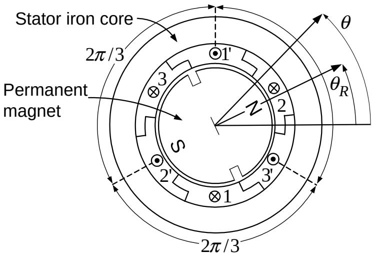
The second motor has eight poles ($n_p = 4$) and $\sigma = 0.10$.

Quantity	Predicted Value $\sigma = 0.1$	Actual Value
S_r / S_p	$\sqrt{\sigma} = 0.32$	0.30
I_{S0} / I_{Sr}	$\sqrt{\sigma} = 0.32$	0.40
$\cos(\varphi_r)$	$\frac{1-\sigma}{1+\sigma} = 0.82$	0.84
τ_p / τ_r	$\frac{1+\sigma}{2\sqrt{\sigma}} = 1.82$	2.0

Figure: From Table 10.2 of *Control of Electrical Drives,* 3rd edition by W. Leonhard, Springer-Verlag, 2001. Reprinted with permission.

- The predicted values and the measured values are in **very good** agreement.
- It is interesting to note that these open-loop characteristics depend **only** on the leakage parameter σ.
- The leakage parameter σ is introduced through the parameter κ ($\sigma = 1 - \kappa^2$) in a pretty much **ad hoc** manner!
- κ was used to account for fact that as the air gap is crossed the magnetic field **spreads out** in the axial and azimuthal directions.

Mathematical Model of a Three-Phase PM Synchronous Motor



Stator Magnetic Field

$$\vec{\mathbf{B}}_S(i_{S1}, i_{S2}, i_{S3}, r, \theta) = \frac{\mu_0 N_S}{2g}\frac{r_R}{r}\Big(i_{S1}\cos(\theta) + i_{S2}\cos(\theta - 2\pi/3) + i_{S3}\cos(\theta - 4\pi/3)\Big)\hat{\mathbf{r}}.$$

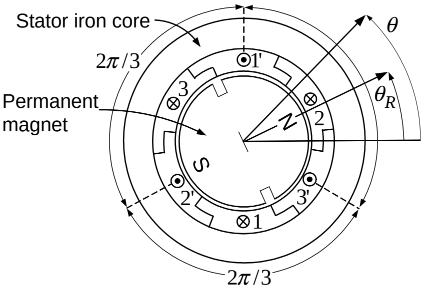
Rotor (PM) Magnetic Field

$$\vec{\mathbf{B}}_R(r, \theta - \theta_R) = B_m\frac{r_R}{r}\cos(\theta - \theta_R)\hat{\mathbf{r}}.$$

Rotor (PM) Magnetic Field at $r = r_S$

$$\vec{\mathbf{B}}_R(r_S, \theta - \theta_R) = \kappa B_m\frac{r_R}{r_S}\cos(\theta - \theta_R)\hat{\mathbf{r}}$$

Mathematical Model of a Three-Phase PM Synchronous Motor

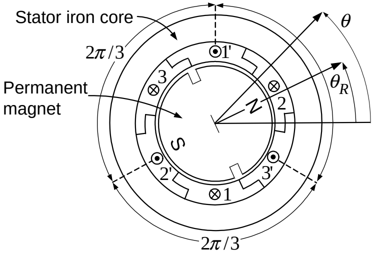


- Compute the stator flux linkages.
- Compute the torque $\vec{\boldsymbol{\tau}}_S$ exerted on the stator windings by $\vec{\mathbf{B}}_R$.
- $\vec{\boldsymbol{\tau}}_R = -\vec{\boldsymbol{\tau}}_S$.

Total Magnetic Field at $r = r_S$

$$\vec{\mathbf{B}}(i_{S1}, i_{S2}, i_{S3}, r_S, \theta, \theta_R) \triangleq \vec{\mathbf{B}}_S(i_{S1}, i_{S2}, i_{S3}, r_S, \theta) + \vec{\mathbf{B}}_R(r_S, \theta - \theta_R).$$

Stator Flux Linkages

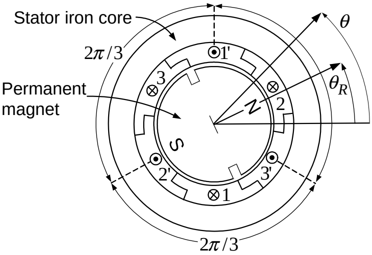


$$
\begin{aligned}
\psi_{S1}(t) &= \int_0^{\pi} \frac{N_S}{2}\sin(\theta)\left(\int_{\theta-\pi}^{\theta} \ell_1 r_S B(i_{S1}, i_{S2}, i_{S3}, r_S, \theta', \theta_R)\,d\theta'\right)d\theta \\
&= \int_0^{\pi} \frac{N_S}{2}\sin(\theta)\left(\int_{\theta-\pi}^{\theta} \ell_1 r_S B_S(i_{S1}, i_{S2}, i_{S3}, r_S, \theta')\,d\theta'\right)d\theta + \\
&\quad \int_0^{\pi} \frac{N_S}{2}\sin(\theta)\left(\int_{\theta-\pi}^{\theta} \ell_1 r_S B_R(r_S, \theta' - \theta_R)\,d\theta'\right)d\theta.
\end{aligned}
$$

The first integral is the **same** as in the case of the induction machine:

$$
\begin{aligned}
\int_0^{\pi} \frac{N_S}{2}\sin(\theta)\left(\int_{\theta-\pi}^{\theta} \ell_1 r_S B_S(i_{S1}, i_{S2}, i_{S3}, r_S, \theta')\,d\theta'\right)d\theta &= \frac{2}{3}L_S\Big(i_{S1} + i_{S2}\cos(2\pi/3) + i_{S3}\cos(4\pi/3)\Big) \\
L_S &= \frac{3}{2}\frac{\pi\mu_0\ell_1\ell_2 N_S^2}{8g}.
\end{aligned}
$$

Mathematical Model of a Three-Phase PM Synchronous Motor



The second integral evaluates as

$$\int_0^{\pi} \frac{N_S}{2}\sin(\theta)\left(\int_{\theta-\pi}^{\theta} \kappa B_m \frac{r_R}{r_S}\cos(\theta'-\theta_R)\ell_1 r_S\, d\theta'\right) d\theta = \int_0^{\pi} \frac{N_S}{2}\sin(\theta) 2\kappa B_m \frac{r_R}{r_S}\sin(\theta-\theta_R)\ell_1 r_S\, d\theta$$

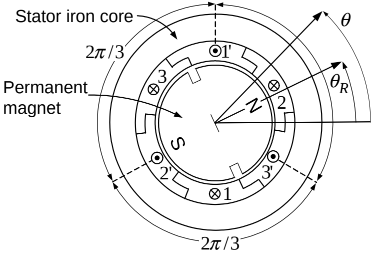
$$= \kappa \ell_1 r_R B_m N_S \cos(\theta_R)\int_0^{\pi}\sin^2(\theta)d\theta$$

$$= \kappa \ell_1 r_R B_m N_S \frac{\pi}{2}\cos(\theta_R)$$

$$= \sqrt{\frac{2}{3}} K_m \cos(\theta_R).$$

- $K_m \triangleq \sqrt{\dfrac{3}{2}}\dfrac{\kappa\pi\ell_1\ell_1 B_m N_S}{4}$ (K_m is the **two-phase equivalent** back-emf constant)

Mathematical Model of a Three-Phase PM Synchronous Motor



$$\psi_{S1}(t) = \frac{2}{3} L_S \Big(i_{S1} + i_{S2}\cos(2\pi/3) + i_{S3}\cos(4\pi/3) \Big) + \sqrt{\frac{2}{3}} K_m \cos(\theta_R).$$

Similarly,

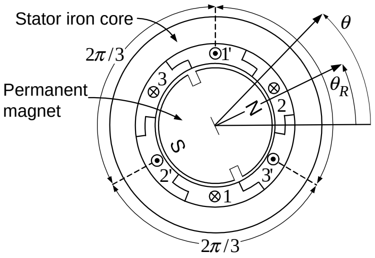
$$\psi_{S2}(t) = \int_{2\pi/3}^{2\pi/3+\pi} \frac{N_S}{2}\sin(\theta - 2\pi/3) \left(\int_{\theta-\pi}^{\theta} \ell_1 r_S B(i_{S1}, i_{S2}, i_{S3}, r_S, \theta', \theta_R) d\theta' \right) d\theta$$

$$= \frac{2}{3} L_S (i_{S1}\cos(2\pi/3) + i_{S2} + i_{S3}\cos(2\pi/3)) + \sqrt{\frac{2}{3}} K_m \cos(\theta_R - 2\pi/3)$$

$$\psi_{S3}(t) = \int_{4\pi/3}^{4\pi/3+\pi} \frac{N_S}{2}\sin(\theta - 4\pi/3) \left(\int_{\theta-\pi}^{\theta} \ell_1 r_S B(i_{S1}, i_{S2}, i_{S3}, r_S, \theta', \theta_R) d\theta' \right) d\theta$$

$$= \frac{2}{3} L_S (i_{S1}\cos(4\pi/3) + i_{S2}\cos(2\pi/3) + i_{S3}) + \sqrt{\frac{2}{3}} K_m \cos(\theta_R - 4\pi/3).$$

Mathematical Model of a Three-Phase PM Synchronous Motor



In **matrix form**

$$\begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix} = \frac{2}{3}L_S \begin{bmatrix} 1 & \cos(2\pi/3) & \cos(4\pi/3) \\ \cos(2\pi/3) & 1 & \cos(2\pi/3) \\ \cos(4\pi/3) & \cos(2\pi/3) & 1 \end{bmatrix} \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} + \sqrt{\frac{2}{3}}K_m \begin{bmatrix} \cos(\theta_R) \\ \cos(\theta_R - 2\pi/3) \\ \cos(\theta_R - 4\pi/3) \end{bmatrix}$$

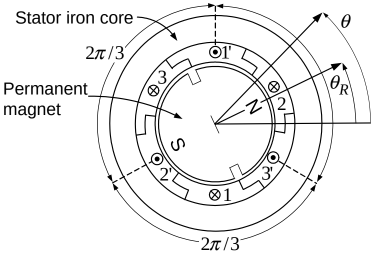
or

$$\begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix} = C_1 \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} + \sqrt{\frac{2}{3}}K_m \begin{bmatrix} \cos(\theta_R) \\ \cos(\theta_R - 2\pi/3) \\ \cos(\theta_R - 4\pi/3) \end{bmatrix}.$$

Note that

$$\lambda_{S0}(t) \triangleq \frac{1}{\sqrt{3}}\left(\psi_{S1}(t) + \psi_{S2}(t) + \psi_{S3}(t)\right) \equiv 0$$

Mathematical Model of a Three-Phase PM Synchronous Motor



- Let the stator voltages $u_{S1}(t), u_{S2}(t), u_{S3}(t)$ be **balanced**.
- Let R_S be the **resistance value** in each stator phase.
- Faraday's law and Ohm's law give

$$\begin{aligned} u_{S1}(t) &= R_S i_{S1} + \frac{d\psi_{S1}(t)}{dt} \\ u_{S2}(t) &= R_S i_{S2} + \frac{d\psi_{S2}(t)}{dt} \\ u_{S3}(t) &= R_S i_{S3} + \frac{d\psi_{S3}(t)}{dt}. \end{aligned}$$

Three-Phase to Two-Phase Transformation

Recall the **three-phase to two-phase** transformation

$$Q = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Define

$$\begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} \triangleq Q \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix}, \quad \begin{bmatrix} \lambda_{Sa}(t) \\ \lambda_{Sb}(t) \\ \lambda_{S0}(t) \end{bmatrix} \triangleq Q \begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix}, \quad \begin{bmatrix} u_{Sa}(t) \\ u_{Sb}(t) \\ u_{S0}(t) \end{bmatrix} \triangleq Q \begin{bmatrix} u_{S1}(t) \\ u_{S2}(t) \\ u_{S3}(t) \end{bmatrix}.$$

We assume a **wye connected** stator so that

$$i_{S0}(t) = \frac{1}{\sqrt{3}} \left(i_{S1}(t) + i_{S2}(t) + i_{S3}(t) \right) \equiv 0.$$

Three-Phase to Two-Phase Transformation

Recall

$$\begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix} = C_1 \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} + \sqrt{\frac{2}{3}} K_m \begin{bmatrix} \cos(\theta_R) \\ \cos(\theta_R - 2\pi/3) \\ \cos(\theta_R - 4\pi/3) \end{bmatrix}.$$

Then

$$\begin{aligned} \begin{bmatrix} \lambda_{Sa}(t) \\ \lambda_{Sb}(t) \\ \lambda_{S0}(t) \end{bmatrix} &= QC_1Q^{-1} \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} + Q\sqrt{\frac{2}{3}} K_m \begin{bmatrix} \cos(\theta_R) \\ \cos(\theta_R - 2\pi/3) \\ \cos(\theta_R - 4\pi/3) \end{bmatrix} \\ &= \begin{bmatrix} L_S & 0 & 0 \\ 0 & L_S & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} + K_m \begin{bmatrix} \cos\theta_R \\ \sin\theta_R \\ 0 \end{bmatrix} \end{aligned}$$

or

$$\begin{aligned} \lambda_{Sa}(t) &= L_S i_{Sa}(t) + K_m \cos(\theta_R) \\ \lambda_{Sb}(t) &= L_S i_{Sb}(t) + K_m \sin(\theta_R) \\ \lambda_{S0} &\equiv 0. \end{aligned}$$

Equivalent Two-Phase Model

$$\begin{bmatrix} u_{S1}(t) \\ u_{S2}(t) \\ u_{S3}(t) \end{bmatrix} = R_S \begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} \psi_{S1}(t) \\ \psi_{S2}(t) \\ \psi_{S3}(t) \end{bmatrix}$$

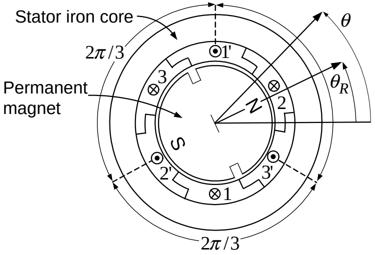
Multiply both sides on the left by Q to obtain

$$\begin{aligned}
\begin{bmatrix} u_{Sa}(t) \\ u_{Sb}(t) \\ u_{S0}(t) \end{bmatrix} &= R_S \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} \lambda_{Sa}(t) \\ \lambda_{Sb}(t) \\ \lambda_{S0}(t) \end{bmatrix} \\
&= R_S \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} L_S i_{Sa}(t) + K_m \cos(\theta_R) \\ L_S i_{Sb}(t) + K_m \sin(\theta_R) \\ \lambda_{S0}(t) \end{bmatrix}.
\end{aligned}$$

or

$$\begin{aligned}
u_{Sa} &= L_S \frac{d}{dt} i_{Sa} + K_m \frac{d}{dt}\cos(\theta_R) + R_S i_{Sa} \\
u_{Sb} &= L_S \frac{d}{dt} i_{Sb} + K_m \frac{d}{dt}\sin(\theta_R) + R_S i_{Sb}.
\end{aligned}$$

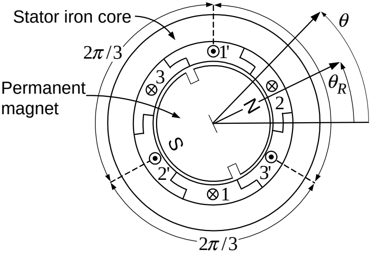
Torque



We compute $\vec{\tau}_S$ and then $\vec{\tau}_R = -\vec{\tau}_S$.

$$
\begin{aligned}
\vec{\mathbf{B}}_R(r_S, \theta - \theta_R) &= \kappa B_m \frac{r_R}{r_S}\cos(\theta - \theta_R)\hat{\mathbf{r}}. \\
\vec{\tau}_{S1} &= \int_{\theta=0}^{2\pi} r_S \hat{\mathbf{r}} \times \left(i_{S1}(t)\frac{N_S}{2}\sin(\theta)d\theta(+\ell_1\hat{\mathbf{z}}) \times \left(B_{R|_{r=r_S}}\hat{\mathbf{r}}\right)\right) \\
&= \int_{\theta=0}^{2\pi} r_S i_{S1}(t)\frac{\ell_1 N_S}{2}\sin(\theta)\left(\kappa B_m \frac{r_R}{r_S}\right)\cos(\theta-\theta_R)d\theta\hat{\mathbf{z}} \\
&= i_{S1}(t)\frac{\kappa B_m \ell_1 r_R N_S}{2}\int_{\theta=0}^{2\pi}\sin(\theta)\cos(\theta-\theta_R)d\theta\hat{\mathbf{z}} \\
&= \kappa\ell_1 r_R B_m N_S \frac{\pi}{2} i_{S1}(t)\sin(\theta_R)\hat{\mathbf{z}} \\
&= \sqrt{2/3}K_m i_{S1}(t)\sin(\theta_R)\hat{\mathbf{z}}.
\end{aligned}
$$

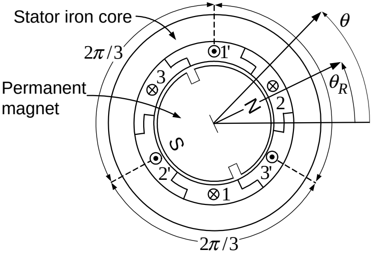
Torque



The torque on stator phase 2 is then

$$
\begin{aligned}
\vec{\boldsymbol{\tau}}_{S2} &= \int_{\theta=0}^{2\pi} r_S \hat{\mathbf{r}} \times \left(i_{S2}(t)\frac{N_S}{2}\sin(\theta - 2\pi/3)d\theta(+\ell_1\hat{\mathbf{z}}) \times \left(B_{R|_{r=r_S}}\hat{\mathbf{r}} \right) \right) \\
&= \int_{\theta=0}^{2\pi} r_S i_{S2}(t)\frac{\ell_1 N_S}{2}\sin(\theta - 2\pi/3)\left(\kappa B_m \frac{r_R}{r_S} \right)\cos(\theta - \theta_R)d\theta\hat{\mathbf{z}} \\
&= i_{S2}(t)\frac{\kappa B_m \ell_1 r_R N_S}{2}\int_{\theta=0}^{2\pi}\sin(\theta - 2\pi/3)\cos(\theta - \theta_R)d\theta\hat{\mathbf{z}} \\
&= i_{S2}(t)\frac{\kappa B_m \ell_1 r_R N_S}{2}\int_{\theta=0}^{2\pi}\frac{1}{2}\left(\sin(2\theta - \frac{2\pi}{3} - \theta_R) + \sin(\theta_R - \frac{2\pi}{3}) \right) d\theta\hat{\mathbf{z}} \\
&= \kappa\ell_1 r_R B_m N_S \frac{\pi}{2} i_{S2}(t)\sin(\theta_R - 2\pi/3)\hat{\mathbf{z}} \\
&= \sqrt{2/3}K_m i_{S2}(t)\sin(\theta_R - 2\pi/3)\hat{\mathbf{z}}.
\end{aligned}
$$

Torque



Finally, the torque on stator phase 3 is computed as

$$
\begin{aligned}
\vec{\boldsymbol{\tau}}_{S3} &= \int_{\theta=0}^{2\pi} r_S \hat{\mathbf{r}} \times \left(i_{S3}(t) \frac{N_S}{2} \sin(\theta - 4\pi/3) d\theta (+\ell_1 \hat{\mathbf{z}}) \times \left(B_{R|_{r=r_S}} \hat{\mathbf{r}} \right) \right) \\
&= \int_{\theta=0}^{2\pi} r_S i_{S3}(t) \frac{\ell_1 N_S}{2} \sin(\theta - 4\pi/3) \left(\kappa B_m \frac{r_R}{r_S} \right) \cos(\theta - \theta_R) d\theta \hat{\mathbf{z}} \\
&= i_{S3}(t) \frac{\kappa B_m \ell_1 r_R N_S}{2} \int_{\theta=0}^{2\pi} \sin(\theta - 4\pi/3) \cos(\theta - \theta_R) d\theta \hat{\mathbf{z}} \\
&= i_{S3}(t) \frac{\kappa B_m \ell_1 r_R N_S}{2} \int_{\theta=0}^{2\pi} \frac{1}{2} \left(\sin(2\theta - \frac{4\pi}{3} - \theta_R) + \sin(\theta_R - \frac{4\pi}{3}) \right) d\theta \hat{\mathbf{z}} \\
&= \kappa \ell_1 r_R B_m N_S \frac{\pi}{2} i_{S3}(t) \sin(\theta_R - 4\pi/3) \hat{\mathbf{z}} \\
&= \sqrt{2/3} K_m i_{S3}(t) \sin(\theta_R - 4\pi/3) \hat{\mathbf{z}}.
\end{aligned}
$$

Torque

The **total torque** is $\tau_S = \tau_{S1} + \tau_{S2} + \tau_{S3}$ or

$$\tau_S = \sqrt{\frac{2}{3}} K_m \Big(i_{S1} \sin(\theta_R) + i_{S2} \sin(\theta_R - 2\pi/3) + i_{S3} \sin(\theta_R - 4\pi/3) \Big).$$

Substitute

$$\begin{bmatrix} i_{S1}(t) \\ i_{S2}(t) \\ i_{S3}(t) \end{bmatrix} \triangleq Q^{-1} \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix} = \sqrt{\frac{3}{2}} \begin{bmatrix} 2/3 & 0 & \sqrt{2}/3 \\ -1/3 & 1/\sqrt{3} & \sqrt{2}/3 \\ -1/3 & -1/\sqrt{3} & \sqrt{2}/3 \end{bmatrix} \begin{bmatrix} i_{Sa}(t) \\ i_{Sb}(t) \\ i_{S0}(t) \end{bmatrix}$$

to obtain $(i_{S0}(t) \equiv 0)$

$$\tau_S = K_m \Big(i_{Sa} \sin(\theta_R) - i_{Sb} \cos(\theta_R) \Big).$$

The torque on the rotor is then

$$\tau_R = -K_m \Big(i_{Sa} \sin(\theta_R) - i_{Sb} \cos(\theta_R) \Big).$$

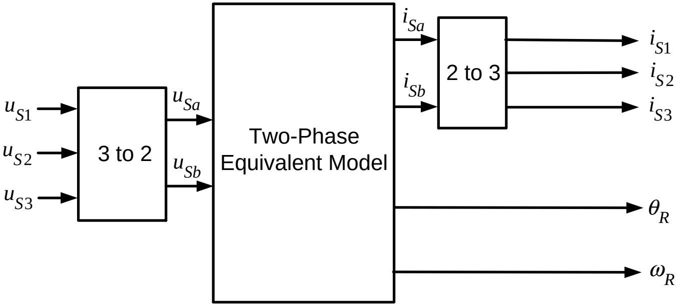
Two-Phase Equivalent Equations of a PM Synchronous Machine

$$
\begin{aligned}
u_{Sa} &= L_S \frac{di_{Sa}}{dt} + K_m \frac{d}{dt}\cos(\theta_R) + R_S i_{Sa} \\
u_{Sb} &= L_S \frac{di_{Sb}}{dt} + K_m \frac{d}{dt}\sin(\theta_R) + R_S i_{Sb} \\
J\frac{d\omega_R}{dt} &= K_m\Big(i_{Sb}\cos(\theta_R) - i_{Sa}\sin(\theta_R)\Big) - \tau_L \\
\frac{d\theta_R}{dt} &= \omega_R
\end{aligned}
$$

or in **statespace** form

$$
\begin{aligned}
L_S\frac{di_{Sa}}{dt} &= -R_S i_{Sa} - K_m\omega_R\sin(\theta_R) + u_{Sa} \\
L_S\frac{di_{Sb}}{dt} &= -R_S i_{Sb} + K_m\omega_R\cos(\theta_R) + u_{Sb} \\
J\frac{d\omega_R}{dt} &= K_m\Big(i_{Sb}\cos(\theta_R) - i_{Sa}\sin(\theta_R)\Big) - \tau_L \\
\frac{d\theta_R}{dt} &= \omega_R.
\end{aligned}
$$

Simulation of a Three-Phase of a PM Synchronous Machine



$$
\begin{aligned}
L_S\frac{di_{Sa}}{dt} &= -R_S i_{Sa} - K_m\omega_R\sin(\theta_R) + u_{Sa}\\
L_S\frac{di_{Sb}}{dt} &= -R_S i_{Sb} + K_m\omega_R\cos(\theta_R) + u_{Sb}\\
J\frac{d\omega_R}{dt} &= K_m\Big(i_{Sb}\cos(\theta_R) - i_{Sa}\sin(\theta_R)\Big) - \tau_L\\
\frac{d\theta_R}{dt} &= \omega_R.
\end{aligned}
$$

Equations in Three-Phase Form

$L = (2/3)L_S$, $K = \sqrt{2/3}K_m$ (see slide 75)

$$\begin{bmatrix} \psi_{S1} \\ \psi_{S2} \\ \psi_{S3} \end{bmatrix} = \begin{bmatrix} L & -L/2 & -L/2 \\ -L/2 & L & -L/2 \\ -L/2 & -L/2 & L \end{bmatrix}\begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} + K\begin{bmatrix} \cos(\theta_R) \\ \cos(\theta_R - 2\pi/3) \\ \cos(\theta_R - 4\pi/3) \end{bmatrix}$$

so that

$$\begin{bmatrix} L & -L/2 & -L/2 \\ -L/2 & L & -L/2 \\ -L/2 & -L/2 & L \end{bmatrix}\begin{bmatrix} di_{S1}/dt \\ di_{S2}/dt \\ di_{S3}/dt \end{bmatrix} = K\omega_R\begin{bmatrix} \sin(\theta_R) \\ \sin(\theta_R - 2\pi/3) \\ \sin(\theta_R - 4\pi/3) \end{bmatrix} - R_S\begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} + \begin{bmatrix} u_{S1} \\ u_{S2} \\ u_{S3} \end{bmatrix}.$$

The torque is (see slide 83)

$$\tau_R = -K\Big(i_{S1}\sin(\theta_R) + i_{S2}\sin(\theta_R - 2\pi/3) + i_{S3}\sin(\theta_R - 4\pi/3)\Big).$$

Using $i_{S1} + i_{S2} + i_{S3} \equiv 0$

$$\begin{bmatrix} \psi_{S1} \\ \psi_{S2} \\ \psi_{S3} \end{bmatrix} = \begin{bmatrix} 3L/2 & 0 & 0 \\ 0 & 3L/2 & 0 \\ 0 & 0 & 3L/2 \end{bmatrix}\begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} + K\begin{bmatrix} \cos(\theta_R) \\ \cos(\theta_R - 2\pi/3) \\ \cos(\theta_R - 4\pi/3) \end{bmatrix}.$$

Then

$$\begin{bmatrix} di_{S1}/dt \\ di_{S2}/dt \\ di_{S3}/dt \end{bmatrix} = -\frac{2}{3}\frac{R_S}{L}\begin{bmatrix} i_{S1} \\ i_{S2} \\ i_{S3} \end{bmatrix} + \frac{2}{3}\frac{K}{L}\omega_R\begin{bmatrix} \sin(\theta_R) \\ \sin(\theta_R - 2\pi/3) \\ \sin(\theta_R - 4\pi/3) \end{bmatrix} + \frac{2}{3}\frac{1}{L}\begin{bmatrix} u_{S1} \\ u_{S2} \\ u_{S3} \end{bmatrix}.$$