Modeling and High-Performance Control of Electric Machines

Chapter 3 Magnetic Fields and Materials

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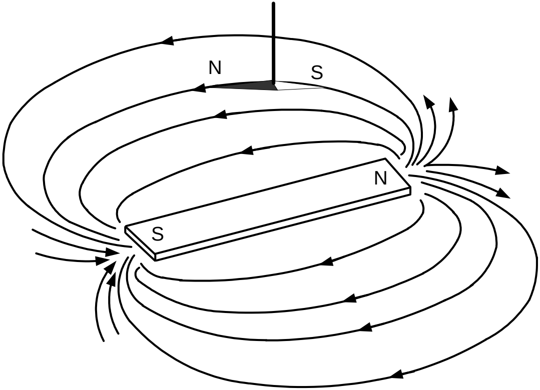
Magnetic Fields and Materials

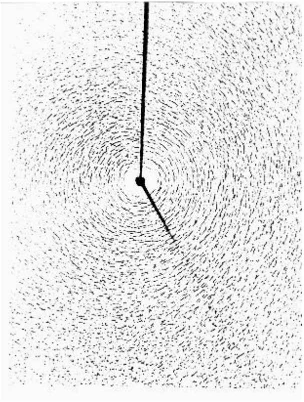
- **Magnetic Field $\vec{\mathbf{B}}$ and Gauss's Law**
- **Modeling Magnetic Materials**
- **Magnetic Intensity Field Vector $\vec{\mathbf{H}}$**
- **$\mathbf{B} - \mathbf{H}$ Hysteresis Curve**
- **Permanent Magnets***
- **Chapter 3, Problems 7, 8, and 9 on Transformers**

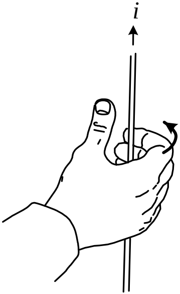
Direction of the Magnetic Field



- The curves drawn show the **direction** of the $\vec{\mathbf{B}}$ field.
- The $\vec{\mathbf{B}}$ field is **defined** to have the direction that a small compass needle points.

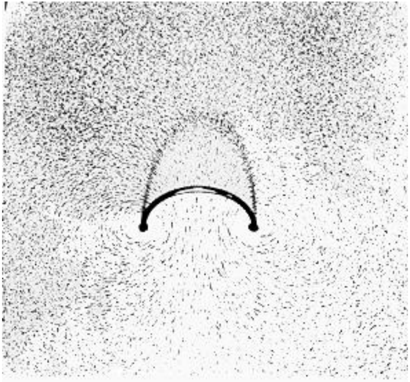
Iron Filings Experiments - Long Straight Wire

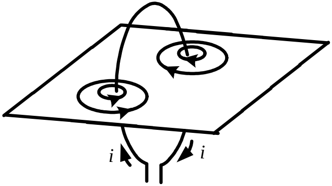




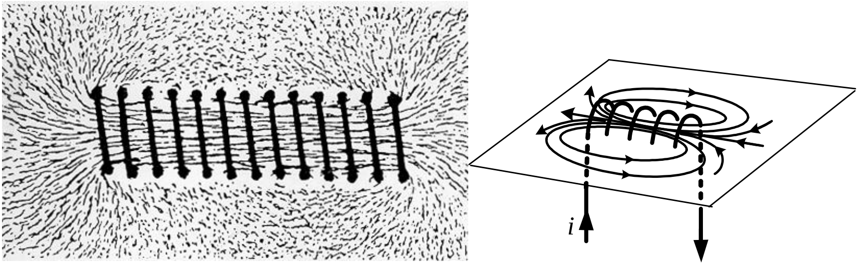
- Small iron filings, i.e., **compass needles**, are spread over a piece of paper.
- The iron filings **align** with the direction of the magnetic field.
- Close to the wire the iron filings are aligned in **concentric circles**.
- The further from the wire the **weaker** the magnetic field.
  Eventually it cannot align the iron filings against the friction forces of the paper.

Iron Filings Experiments - Circular Loop





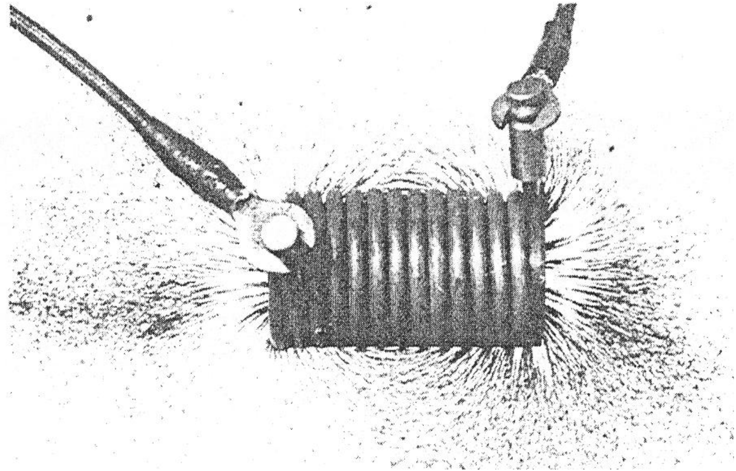
Iron Filings Experiments - Solenoid Coil with an Air Core



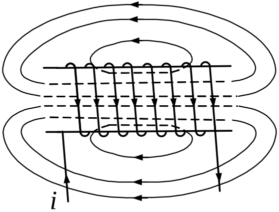
(a) (b)

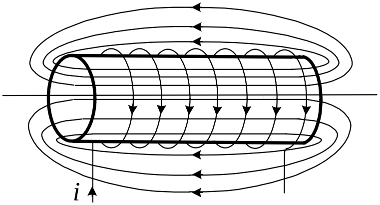
- Normally the coil is **tightly wound**, i.e., no space between successive turns.
- Loosely wound to show the orientation of the iron filings **inside** the coil.

Short Solenoid Coil



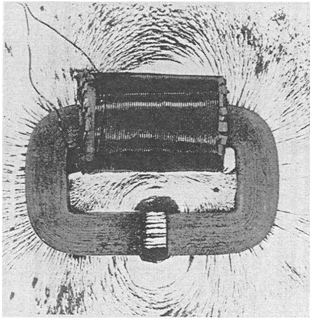
Air versus Iron Core Coils (Solenoids)
Left: Coil with **air** core.
Right: Coil with **iron** core.





- Both coils have the **same** dimensions, number of windings, and current.
- The $\vec{\mathbf{B}}$ field of the iron core coil is **much stronger** than the air filled coil.
- The $\vec{\mathbf{B}}$ field lines of the coil with the iron core are drawn **closer together**. Indicates its magnetic field is much stronger than that of the air filled coil.

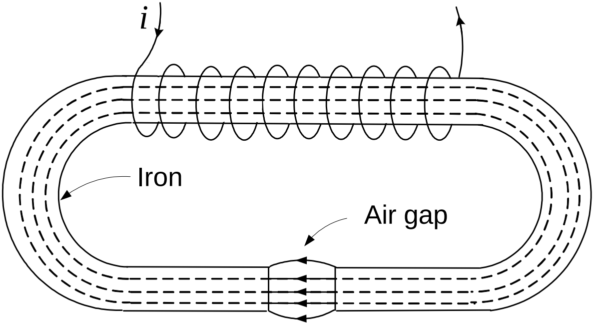
Iron Filings Experiments - Solenoid Coil with an Iron Core and Air Gap



Important properties of iron:

- The $\vec{\mathbf{B}}$ field inside the iron core is typically **1000 times** stronger compared to an air core.
- For small air gaps, the $\vec{\mathbf{B}}$ field in the air gap is the **same** as in the iron.
- The $\vec{\mathbf{B}}$ field **follows the path** of the iron.

Solenoid Coil with an Iron Core and Air Gap



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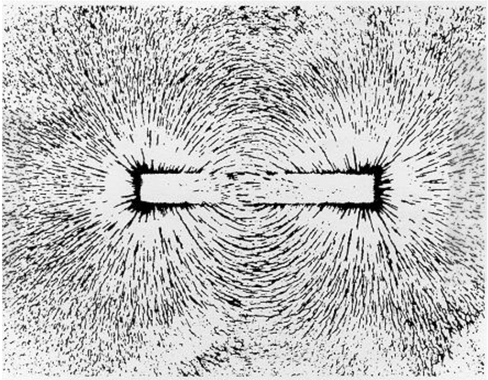
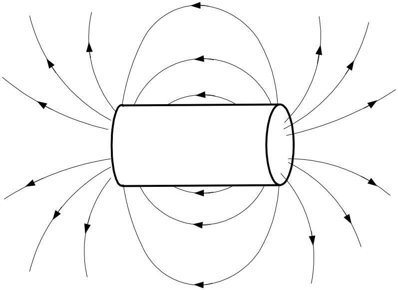
- The $\vec{\mathbf{B}}$ field **follows the path** of the iron

Iron Filings Experiments - Permanent Magnet



(a)

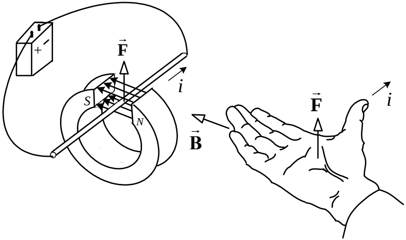


(b)

The magnetic field lines look similar to that of a solenoid carrying a current.

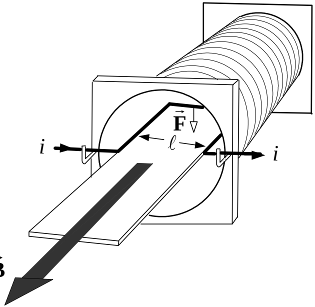
The Magnetic Field $\vec{B}$ and Gauss's Law

- Magnetism was discovered because naturally occurring magnets (e.g., the earth and a compass needle) exerted forces on each other.
- In 1819 Hans Ørsted discovered that magnets can also **exert forces** on currents in wires.
- He also discovered that currents in wires **produce** magnetic fields.



The **magnitude** of the $\vec{B}$ field is defined in terms of the **force** it produces.

Apparatus to Measure $\vec{B}$



- The $\vec{\mathbf{B}}$ field has the **direction** that a small compass needle points.
- The **magnitude** $B \triangleq \left|\vec{\mathbf{B}}\right|$ of $\vec{\mathbf{B}}$ is defined by

$$B \triangleq \frac{F}{\ell i}$$

  where F is the **force** on the conductor of length ℓ carrying the current i.
- The wire ℓ carrying the current i is assumed to be **perpendicular** to $\vec{\mathbf{B}}$.
- The **units** of the magnetic field are tesla $\triangleq \dfrac{\text{N}}{\text{m-A}} = \dfrac{\text{weber}}{\text{m}^2} = 10^4$ gauss.

Magnetic Force Law

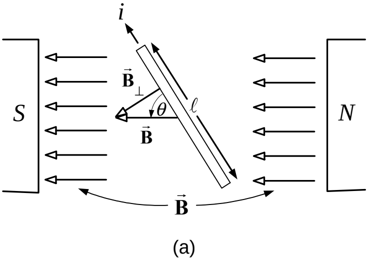
- The force depends only on the component of $\vec{\mathbf{B}}$ **perpendicular** to the current.
- $\vec{\ell}$ has **magnitude** given by the length ℓ of the wire in the magnetic field.
- $\vec{\ell}$ has **direction** given by the direction of positive current flow in the wire.
- The **magnetic force law** is

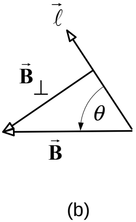
$$\vec{\mathbf{F}} = i\vec{\ell} \times \vec{\mathbf{B}}.$$

Let θ denote the angle between $\vec{\ell}$ and $\vec{\mathbf{B}}$ as shown below.

$$F \triangleq |\ \vec{\mathbf{F}}\ | = i\ell B\sin(\theta) = i\ell B_{\perp}$$

where $B_{\perp} = B\sin(\theta)$ is just the component of $\vec{\mathbf{B}}$ **perpendicular** to the wire.





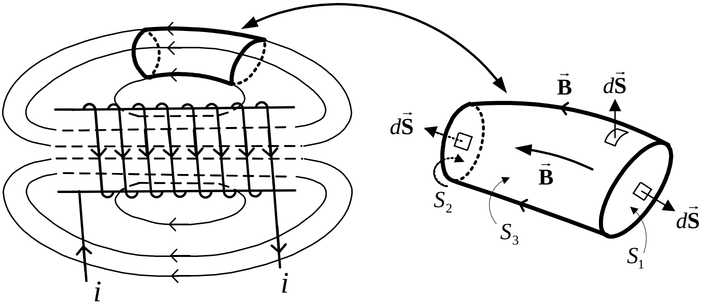
Conservation of Flux

Gauss's Law: The integral of the flux over a **closed** surface is zero, that is,

$$\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \equiv 0.$$

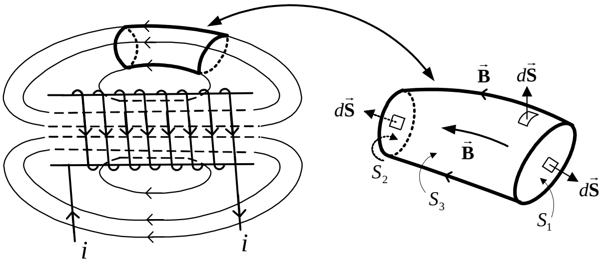
Consider the **closed** surface S shown in the figure below.
The closed is made of two disk-shaped surfaces S_1, S_2 and a tube-shaped surface S_3.
The surface S_3 was chosen so that $\vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$ on it.
By Gauss's law

$$\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \int_{S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \int_{S_3} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \equiv 0$$



Conservation of Flux

$$\int_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} + \int_{S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \equiv 0.$$



$$\phi_{S_2} = \int_{S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = -\int_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}' = \phi_{S_1}$$

where $d\vec{\mathbf{S}}' \triangleq -d\vec{\mathbf{S}}$ is the **inward** normal to the closed surface S on S_1.

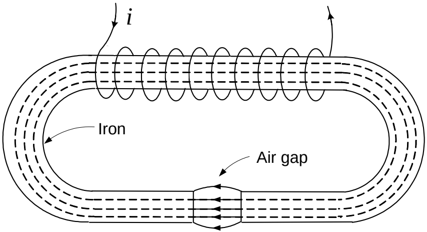
$$\phi_{S_1} = \phi_{S_2} \quad \textbf{conservation of flux}$$

The volume enclosed by S_1, S_2, S_3 is called a **flux tube.**

By "flux tube" is meant $\vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \equiv 0$ on the tube shaped part of the surface, i.e., on S_3.

Shaping the Magnetic Field

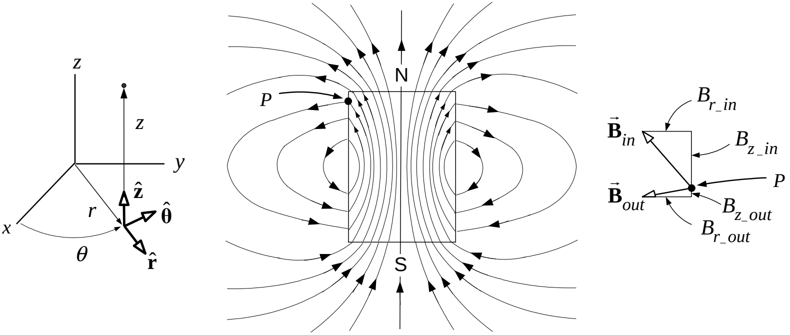
Magnetic materials provide the capability of **constructing** flux tubes.



- The $\vec{\mathbf{B}}$ field inside the coil produced by the **current** and **magnetized iron** follows along the path of the iron to the air gap.
- To a reasonable approximation, the magnetic material keeps the magnetic field **inside** the magnetic material.
- The tube-shaped surface of the magnetic material forms a surface for which $\vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \mathbf{0}$.
- The magnetic flux through any cross section of the iron tube is approximately the **same** throughout the tube and in particular at the air gap.

Continuity of the Normal Component of $\vec{\mathbf{B}}$

Example Cylindrical Permanent Magnet



At point P we have

$$\begin{aligned}\vec{\mathbf{B}}_{in} &= B_{r\_in}\,\hat{\mathbf{r}} + B_{z\_in}\,\hat{\mathbf{z}}\\ \vec{\mathbf{B}}_{out} &= B_{r\_out}\,\hat{\mathbf{r}} + B_{z\_out}\,\hat{\mathbf{z}}\end{aligned}$$

The **normal** component ($\hat{\mathbf{r}}$ direction at P) being **continuous** means

$$B_{r\_in} = B_{r\_out}.$$

At P, $B_{z\_in} > 0$ while $B_{z\_out} < 0$ so $\vec{\mathbf{B}}$ is **not** continuous at P.

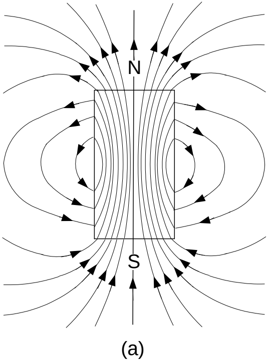
Continuity of the Normal Component of $\vec{\mathbf{B}}$

Gauss's law $\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \equiv 0$ is used to show that the **normal component** of $\vec{\mathbf{B}}$ is always continuous even at a surface boundary.

See the book for the proof.

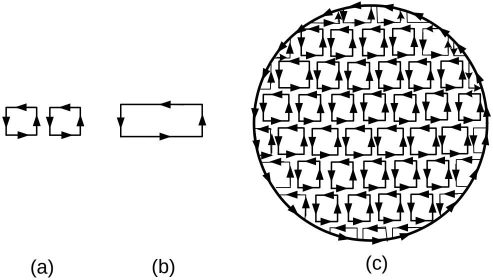
Modeling Magnetic Materials

Left: Cylindrical permanent magnet made of iron and some alloys.
Right:Tightly wound cylindrical (solenoid) coil.



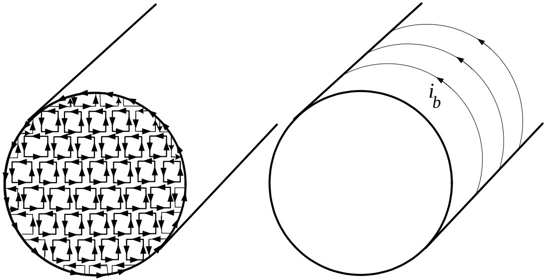
Observation: Their $\vec{\mathbf{B}}$ fields in the air look the same.
Conjecture: The $\vec{\mathbf{B}}$ field lines inside the PM is the **same** as inside the coil.
Picture the PM as having a "bound" current which goes around its periphery.
This bound current produces a $\vec{\mathbf{B}}$ field analogously to i in the air-filled coil.

Modeling Magnetic Materials



- Consider an **iron atom** in the bar magnet as a small **current loop**.
- The motion of the electrons in the individual iron atoms are considered to be **equivalent** to a current loop.
- Iron has the property that the loops can **align** to have the same orientation.
- Inside the magnet the current in adjacent sides of neighboring loops **cancel**.
- Loops on the **surface** of the material have no neighboring loop to cancel.
- Net effect is a **bound current** around the periphery of the magnetic material.

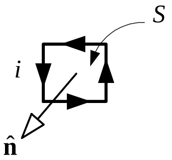
Modeling Magnetic Materials



(a) (b)

- **Bound** current means that an individual electron does not go around the surface of the magnet, but stays with the particular iron atom.
- **Free** current refers to current flow in e.g., copper conductors where individual electrons actually move down the wire through the crystal lattice of the copper.
- **Magnetic materials** have the property that the motion (spin) of the electrons in their atoms are able to **align** themselves to the same orientation.
- In **nonmagnetic materials** the current loops are at **random** orientations.
- We **model** the magnetic material by an equivalent **bound current** i_b going around the periphery of the cylinder.

Magnetic Dipole Moments



$$\vec{\mathbf{m}} = iS\hat{\mathbf{n}}$$

- Consider a small planar loop of area S with a current i going around the loop.
- The **scalar magnetic moment** is defined to be $m \triangleq iS$.
- The **direction** of the magnetic moment is the normal $\hat{\mathbf{n}}$ to the loop specified by right-hand rule.
- The **vector magnetic moment** is

$$\vec{\mathbf{m}} \triangleq iS\hat{\mathbf{n}}.$$

- Magnetic material (such as iron) is now **pictured** by modeling the atoms as magnetic moments which are able to **align** themselves in the same direction.
- The net effect is then a **bound surface current** that produces the external magnetic field analogous to a current-carrying coil with air inside.

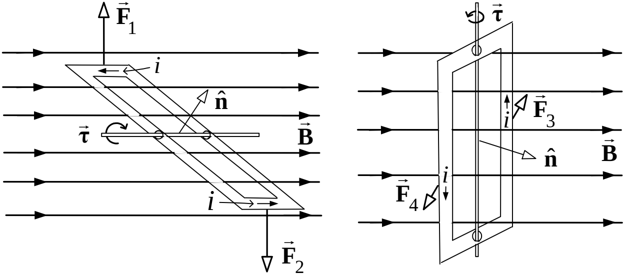
Iron Magnetism

The actual explanation of the magnetic property of iron (ferromagnetism) is through the theory of Quantum Mechanics. Quoting from Melvin Schwartz:

> *Iron has two remarkable atomic properties that have far-reaching consequences with respect to its macroscopic magnetic behavior. First, 4 of the 26 electrons in an isolated atom of iron have their intrinsic angular momenta lined up. Second, within solid iron, there are very strong quantum-mechanical forces tending to make the intrinsic angular momenta of neighboring atoms line up. This results in domains of macroscopic size having net magnetizations corresponding to about two aligned electron moments per atom on the average. Applying a magnetic field causes those domains, which are aligned in the same direction as the field, to grow until the iron finally reaches a saturated state of magnetization. Typically, saturation occurs at fields in the neighborhood of 10,000 to 20,000 gauss. Needless to say, removal of the applied field does not lead to complete randomization of the domains. The residual magnetization can be quite large, as in the case of special permanent magnetic alloys, or it can be quite small, as in the case of soft iron. These phenomena are characteristic of ferromagnetism.*

Aligning Magnetic Dipoles by an External Magnetic Field

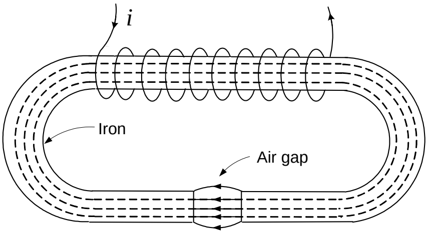
- Picture any material made up of **magnetic dipole moments** (small current loops).
- The loops can **align** themselves when an **external** magnetic field is applied.



(a) (b)

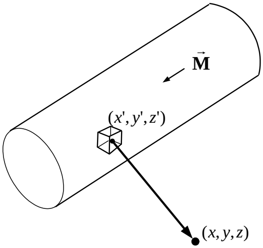
- Left: The forces $\vec{F}_1, \vec{F}_2$ provide a torque $\vec{\tau}$ to align the dipole moment with $\vec{B}$.
- Right: The forces $\vec{F}_3, \vec{F}_4$ provide a torque $\vec{\tau}$ to align the dipole moment with $\vec{B}$.

Aligning Magnetic Dipoles by an External Magnetic Field



- The current in the coil produces a magnetic field.
- This magnetic field causes dipole moments of the iron atoms to align with it.
- We say that the current in the coil **magnetizes** the iron.
- This results in a much larger (1000 times) magnetic field in the iron than if we had no iron inside the coil.

The Magnetization Vector $\vec{\mathbf{M}}$



$$\vec{\mathbf{m}}(x', y', z') \triangleq \text{Average magnetic dipole moment of the atoms in a neighborhood of the point } (x', y', z').$$

$$N_m \triangleq \text{Number of atoms (e.g., iron) per unit volume.}$$

$$\vec{\mathbf{M}}(x', y', z') \triangleq N_m \vec{\mathbf{m}}(x', y', z').$$

- $M = |\vec{\mathbf{M}}|$ is the magnetic dipole moment per unit volume.
- Need to determine the magnetic field $\vec{\mathbf{B}}(x, y, z)$ **outside** the material due to the magnetization $\vec{\mathbf{M}}(x', y', z')$ **inside** the material.
- This will require **Ampère's law**.

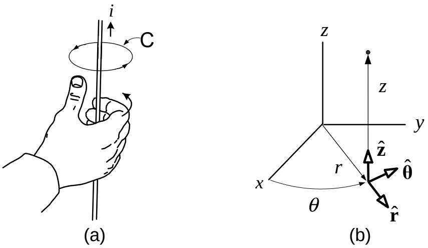
Ampère's Law

When magnetic materials are **not** present, **Ampère's law** is

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 i_{\text{enclosed}}$$

where i_{enclosed} is the current **enclosed** by the closed curve (**contour**) C.

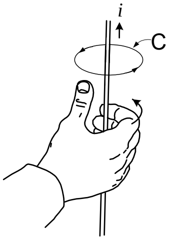
Right-Hand Rule:



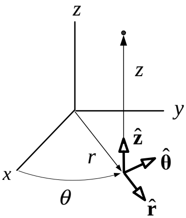
- Let C be a **closed curve**.
- The thumb points in the direction of **positive current** in the wire.
- The fingers point in the **positive direction of travel** along the curve C.

Example Long Straight Wire

Use Ampère's law to the $\vec{\mathbf{B}}$ field due to the current in a long straight wire.



(a)



(b)

- Due to symmetry $\vec{\mathbf{B}}$ does **not** depend on z or θ.

$$\vec{\mathbf{B}} = B_r(r)\hat{\mathbf{r}} + B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}}.$$

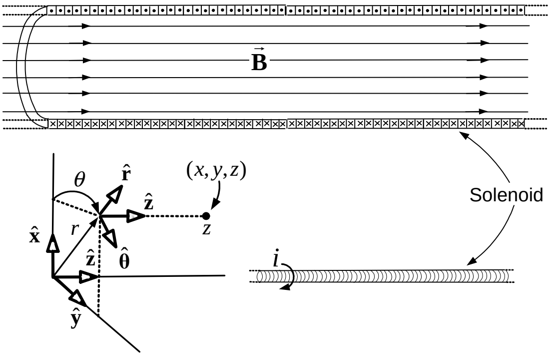
- Ampere's law can be used to show $B_r(r) \equiv 0, B_z(r) \equiv 0$ (see book).
- To find $B_\theta(r)$, apply Ampère's law to the closed curve C.

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \int_0^{2\pi} \left(B_\theta\hat{\boldsymbol{\theta}}\right) \cdot \left(rd\theta\hat{\boldsymbol{\theta}}\right) = \int_0^{2\pi} B_\theta r d\theta = 2\pi r B_\theta = \mu_0 i$$

or,

$$B_\theta(r) = \frac{\mu_0 i}{2\pi r}.$$

Example Ideal Infinitely Long Solenoid

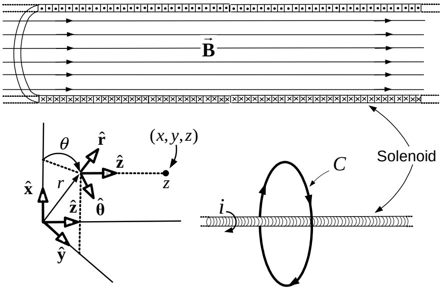


- A current is put through the coil (solenoid) wire.
- Compute the magnetic field **inside** and **outside** the coil.
- Due to symmetry, $\vec{\mathbf{B}}$ does **not** depend on z or θ, that is,

$$\vec{\mathbf{B}} = B_r(r)\hat{\mathbf{r}} + B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}}.$$

Example Ideal Infinitely Long Solenoid

Magnetic field **outside** the coil: $\vec{\mathbf{B}} = B_r(r)\hat{\mathbf{r}} + B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}}$.



- Gauss's law can be used (see book) to show $B_r(r) \equiv 0$ outside the solenoid.
- Ampere's law can be used (see book) to show that $B_z(r) \equiv 0$ outside the solenoid.
- To find $B_\theta(r)$ apply Ampère's law to the closed curve C.

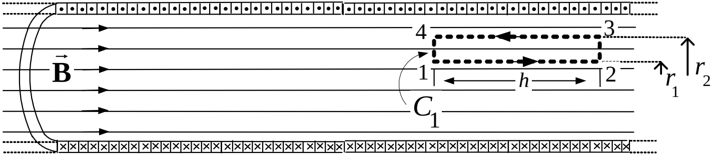
$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \int_0^{2\pi} \left(B_\theta \hat{\boldsymbol{\theta}}\right) \cdot \left(r d\theta \hat{\boldsymbol{\theta}}\right) = \int_0^{2\pi} B_\theta r d\theta = 2\pi r B_\theta = \mu_0 i$$

or,

$$\vec{\mathbf{B}}_{outside} = B_{\theta\_outside}(r)\hat{\boldsymbol{\theta}} = \frac{\mu_0 i}{2\pi r}\hat{\boldsymbol{\theta}}.$$

Example Ideal Infinitely Long Solenoid

Magnetic field **inside** the coil: $\vec{\mathbf{B}} = B_r(r)\hat{\mathbf{r}} + B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}}$.



$$\oint_{C_1} \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 i_{\text{enclosed}}$$

$$\underbrace{\int_1^2 \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}}}_{B_z(r_1)h} + \underbrace{\int_2^3 \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}}}_{\int_{r_1}^{r_2} B_r(r)dr} + \underbrace{\int_3^4 \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}}}_{-B_z(r_2)h} + \underbrace{\int_4^1 \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}}}_{\int_{r_2}^{r_1} B_r(r)dr} = \mu_0 i_{\text{enclosed}}$$

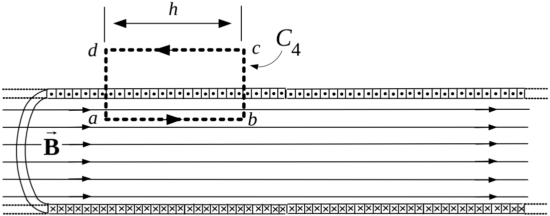
$$B_z(r_1)h - B_z(r_2)h = 0$$

$$B_z(r_1) = B_z(r_2)$$

- **Inside** the coil, $B_z(r)$ does **not** depend on r, i.e., $B_z(r)$ is **constant**.
- Similarly, **outside** the coil, $B_z(r)$ does **not** depend on r, i.e., it is **constant**.

Example Ideal Infinitely Long Solenoid

Magnetic field **inside** the coil: $\vec{\mathbf{B}} = B_r(r)\hat{\mathbf{r}} + B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}}$.
n is number of coil turns per unit length.



$$\oint_{C_4} \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 i_{\text{enclosed}}$$

$$\underbrace{\int_a^b \vec{\mathbf{B}} \cdot d\vec{\ell}}_{\int_a^b B_{z\_inside}\,dz} + \underbrace{\int_b^c \vec{\mathbf{B}} \cdot d\vec{\ell}}_{\int_b^c B_r(r)dr} + \underbrace{\int_c^d \vec{\mathbf{B}} \cdot d\vec{\ell}}_{\int_c^d B_{z\_outside}\,dz=0} + \underbrace{\int_d^a \vec{\mathbf{B}} \cdot d\vec{\ell}}_{\int_d^a B_r(r)dr} = \mu_0 nhi$$

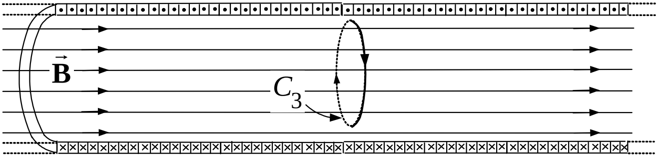
$$B_{z\_inside}\,h = \mu_0 nhi$$

$$B_{z\_inside} = \mu_0 ni$$

- $B_{z\_outside} = 0$ (see book).

Example Ideal Infinitely Long Solenoid

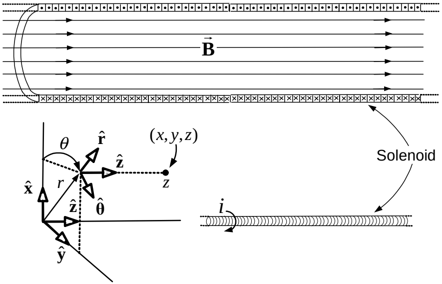
Magnetic field **inside** the coil: $\vec{\mathbf{B}} = B_r(r)\hat{\mathbf{r}} + B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}}$.



$$\begin{aligned}
\oint_{C_3} \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} &= \mu_0 i_{\text{enclosed}} \\
\int_0^{2\pi} (B_\theta(r)\hat{\boldsymbol{\theta}}) \cdot (r d\theta \hat{\boldsymbol{\theta}}) &= 0 \\
2\pi B_\theta(r) &= 0 \\
B_\theta(r) &= 0.
\end{aligned}$$

- **Inside** the coil $B_\theta(r) = 0$.
- Gauss's law also implies $B_r(r) = 0$ **inside** the coil (see book).

Example Ideal Infinitely Long Solenoid



$$\vec{\mathbf{B}}(r,\theta,z) = \begin{cases} \mu_0 n i \hat{\mathbf{z}} & \text{inside the solenoid} \\ \dfrac{\mu_0 i}{2\pi r}\hat{\boldsymbol{\theta}} & \text{outside the solenoid.} \end{cases}$$

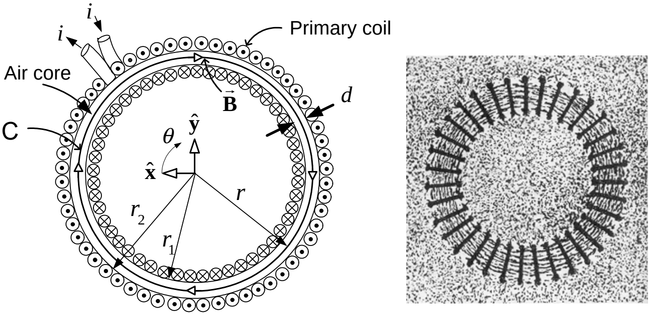
This expression is consistent with the **iron filings** experiment of slide 6.

The quantity $\lambda = ni$ is the **surface current** per unit length of the coil.

$\vec{\boldsymbol{\lambda}} = ni\hat{\boldsymbol{\theta}}$ is the **surface current vector**.

$\vec{\mathbf{B}} = \mu_0 \lambda \hat{\mathbf{z}}$ **inside** the coil.

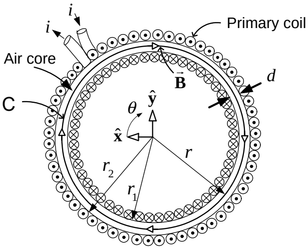
Example Toroidal Solenoid



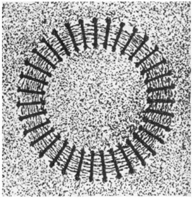
(a) (b)

- Iron filing experiment - $\vec{\mathbf{B}}$ goes in circles inside an air filled toroidal solenoid.
- Outside of the toroid the magnetic field is quite **weak**.
- Inside the toroid, the magnetic field is only **nonzero** in the **azimuthal direction** (see Problem 6 of Chapter 3).

Example Toroidal Solenoid



(a)



(b)

n denotes the number of loops (turns) **per unit length** along the radius r_1 of the toroid. $2\pi r_1 n$ is the **total** number of loops wound around the toroid.

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 i_{\text{enclosed}} \Rightarrow B_\theta 2\pi r = \mu_0 2\pi r_1 n i \Rightarrow B_\theta(r) = \frac{r_1}{r}\mu_0 n i.$$

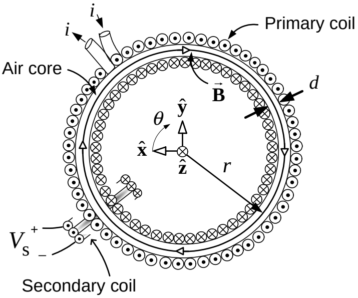
This is valid for $r_1 < r < r_2$.
With $d = r_2 - r_1 << r_1$ so $r_1/r \approx 1$ we have **inside** the toroid

$$\vec{\mathbf{B}} \approx \mu_0 n i \hat{\boldsymbol{\theta}}.$$

We will always assume that $d = r_2 - r_1 << r_1$ so that $r_1/r \approx 1$ is valid.

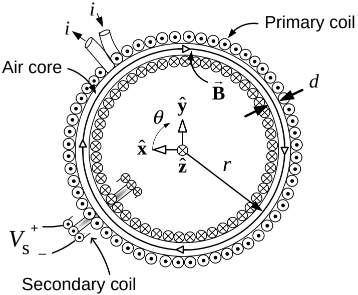
Relating $\vec{\mathbf{B}}$ to $\vec{\mathbf{M}}$

We just showed that $B \approx \mu_0 ni$ in an **air core** toroidal coil.
We now verify this **experimentally**.



- A **secondary** coil is wrapped around the toroid as shown.
- We use the secondary coil to **measure** $\vec{\mathbf{B}} = B\hat{\boldsymbol{\theta}}$ inside the toroid.
- N_s is the number of turns in the secondary coil.
- $S = \pi(d/2)^2$ the cross-sectional area of the toroidal coil.

Relating $\vec{B}$ to $\vec{M}$



The **flux linkage** λ_s through the **secondary** coil is $\lambda_s = N_s B(t) S$.
By Faraday's law,

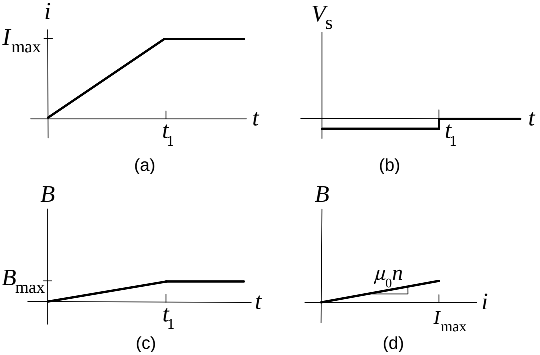
$$V_s(t) = -\frac{d\lambda_s}{dt} = -\frac{d}{dt}\left(N_s B(t) S\right) = -N_s S \frac{dB(t)}{dt}$$

so that

$$B(t) = -\frac{1}{N_s S}\int_0^t V_s(\tau)d\tau.$$

- Measure $i(t)$ in the **primary** windings and $V_s(t)$ of **secondary** coil.
- $B(t)$ inside the toroid can then be computed as a function of the current $i(t)$.

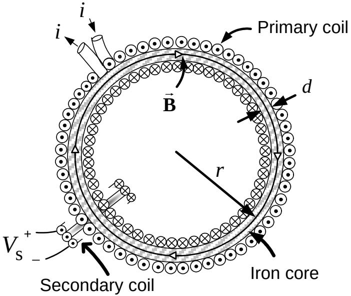
Relating $\vec{B}$ to $\vec{M}$



- Ramp up the **primary** current $i(t)$ from 0 to some I_{max}.
- At the same time measure the **secondary** voltage $V_s(t)$.
- Calculate $B(t)$.
- Plot B versus i.
- These measurements verify that $B \approx \mu_0 ni$.

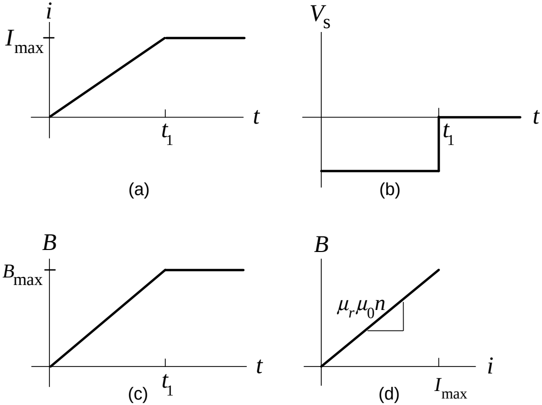
Relating $\vec{B}$ to $\vec{M}$

This **same** experimental setup can also be used to measure the magnetic field inside the toroidal coil when it has an **iron** core.



- The experiment is now repeated, but with the toroid filled with an **iron** core.
- There is now a **non zero** magnetization $\vec{M}$ in the iron core.
- $B(t)$ inside the toroid is computed as a function of the current $i(t)$.

Relating $\vec{B}$ to $\vec{M}$



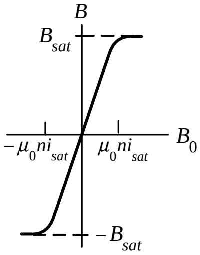
- Ramp up the **primary** current $i(t)$ from 0 to some I_{max}.
- At the same time measure the **secondary** voltage $V_s(t)$.
- Calculate $B(t)$.
- Plot B versus i.
- The slope of B vs. i is **greater** by a factor μ_r **compared** to the air filled toroid.
- μ_r is the **relative permeability** and is typically on the order of $1000 - 2000$.

Relating $\vec{B}$ to $\vec{M}$

Rewrite B in the **iron core** of the toroid as

$$B = B_0 + B_M.$$

- $B_0 = \mu_0 ni$ denotes the magnetic field when **no magnetic** material is present.
- B_M is the additional strength in the magnetic field **due** to the magnetic material.
- Now plot B versus $B_0 = \mu_0 ni$ (rather than versus i).
- The **slope** of the straight line part of the curve **is** the relative permeability μ_r.
- $B = \mu_r B_0 = \mu_r \mu_0 ni$ for $i \leq i_{\text{sat}}$.
- Increase the current **above** i_{sat} and B only increases at the rate $\mu_0 n$ and **not** $\mu_r \mu_0 n$. This is because the magnetic dipole moments are **all aligned** for $i > i_{\text{sat}}$.

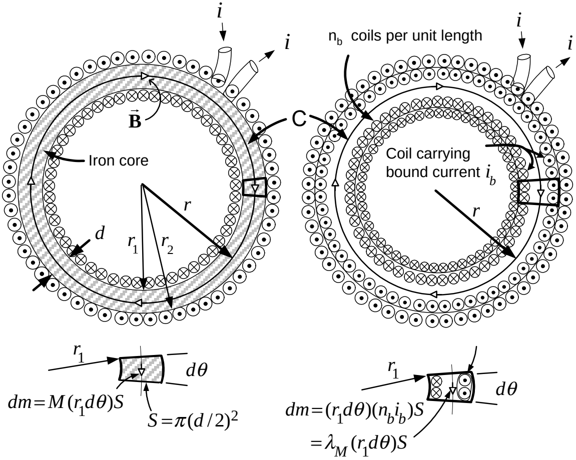


Relating $\vec{B}$ to $\vec{M}$

- **Important:** B_M is typically on the order of a 1000 times **greater** than B_0.
- I.e., for the **same current** in the toroidal coil, the one filled with **iron** will have a $\vec{\mathbf{B}}$ field that is on the order of a **1000 times** that of the **air filled** toroid.
- As will be described shortly, the **magnetic field intensity** H is $H \triangleq B_0/\mu_0 = ni$.
- It turns out that the **magnetization** M is given by $M = B_M/\mu_0$.
- Thus
$$B = B_0 + B_M = \mu_0 H + \mu_0 M$$
where
$$H << M \text{ as } B_0 << B_M.$$
- In soft magnetic materials one can take $H = 0$ as a reasonable approximation.
- In soft magnetic materials the B field is **primarily** due to the magnetization (alignment of the magnetic dipoles) of the magnetic material **rather** than the current that magnetizes the magnetic material.

The Magnetic Intensity Field Vector $\vec{\mathbf{H}}$

We now define the **magnetic intensity** vector $\vec{\mathbf{H}}$ in terms of the **magnetization** vector $\vec{\mathbf{M}}$ and the **magnetic induction** vector $\vec{\mathbf{B}}$.



A toroid with an iron core is replaced with an "equivalent" air-filled coil carrying the **bound** current i_b.

The Magnetic Intensity Field Vector $\vec{\mathbf{H}}$

- Ampère's law must be **modified** to account for **magnetic materials**.
- Ampère's law showed that $B \approx \mu_0 n i$ in the **air-filled** toroid.
- With an **iron core**, experiments showed B was on the order of a 1000 times **greater**.
- The **magnetization (dipole moment per unit volume)** can be pictured as an equivalent bound current going around the surface of the toroidal core.
- Modify Ampère's law to be

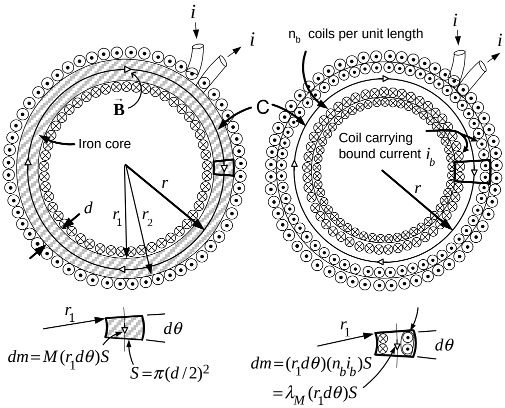
$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 \left(i_{\text{enclosed}} + i_{M\text{enclosed}} \right)$$

where

$$i_{M\text{enclosed}} = 2\pi r_1 n_b i_b.$$

is the equivalent **bound** current enclosed by the curve C.
n_b denotes the **number** of loops/coils **per** unit length of the **equivalent** coil.

The Magnetic Intensity Field Vector $\vec{\mathbf{H}}$



Define the **surface current density** $\lambda_M \triangleq \dfrac{i_{M\text{enclosed}}}{2\pi r_1} = n_b i_b$.

The **bound** current going around the toroid between θ and $\theta + d\theta$ is $\lambda_M(r_1 d\theta)$.
Magnetic dipole moment dm **due** to λ_M between θ and $\theta + d\theta$ is $dm = \lambda_M(r_1 d\theta)S$.
By **definition** of M we also have $dm = M(r_1 d\theta)S$.
That is, in this example, $\lambda_M = M$.
In words, the equivalent surface current λ_M due to i_b **equals** the magnetization M.

The Magnetic Intensity Field Vector $\vec{\mathbf{H}}$

For the toroid coil, the **surface current density** λ_M equals the **magnetization** M.

$$i_{M\text{enclosed}} = \lambda_M 2\pi r = M 2\pi r = \oint_C \left(M\hat{\boldsymbol{\theta}}\right) \cdot \left(r d\theta \hat{\boldsymbol{\theta}}\right) = \oint_C \vec{\mathbf{M}} \cdot d\vec{\boldsymbol{\ell}}.$$

Ampère's law is then

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 \left(i_{\text{enclosed}} + \oint_C \vec{\mathbf{M}} \cdot d\vec{\boldsymbol{\ell}} \right).$$

Rearranging

$$\oint_C \left(\frac{\vec{\mathbf{B}} - \mu_0 \vec{\mathbf{M}}}{\mu_0} \right) \cdot d\vec{\boldsymbol{\ell}} = i_{\text{enclosed}}.$$

Define the **magnetic intensity** field $\vec{\mathbf{H}}$ as

$$\vec{\mathbf{H}} \triangleq \frac{\vec{\mathbf{B}}}{\mu_0} - \vec{\mathbf{M}}.$$

Ampère's law is then

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} = i_{\text{enclosed}} = i_{\text{free}}$$

where i_{free} is the **free** current enclosed by C.
Both $\vec{\mathbf{B}}$ and $\vec{\mathbf{M}}$ were defined in terms of **physical** phenomena.
$\vec{\mathbf{H}}$ is **defined** as the difference between $\vec{\mathbf{B}}/\mu_0$ and $\vec{\mathbf{M}}$.

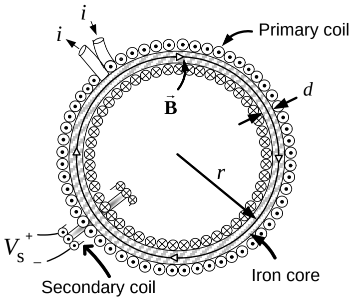
The B − H Curve

With the definition $\vec{\mathbf{H}} \triangleq \vec{\mathbf{B}}/\mu_0 - \vec{\mathbf{M}}$, Ampère's law was modified to

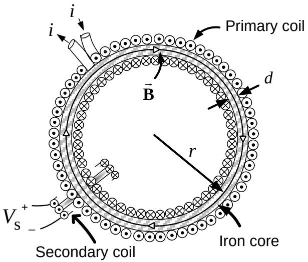
$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = i_{\text{free}}.$$

- Ampère's law can be used to calculate $\vec{\mathbf{H}}$, but $\vec{\mathbf{B}}$ is the desired quantity!
- $\vec{\mathbf{B}}$ produces forces on currents and magnetic materials.
- $\vec{\mathbf{M}}$ is typically **not** known - thus can't compute $\vec{\mathbf{B}}$ from the definition of $\vec{\mathbf{H}}$.
- Experimental methods are used to relate $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$.

Experimental Setup



The B – H Curve



The initial magnetization of the iron is zero (just heat up the iron core). Ampère's law gives[1]

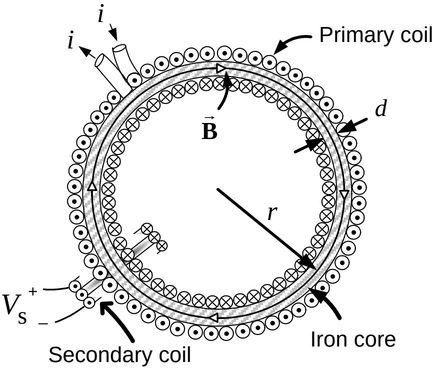
$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = i_{\text{free}} \Longrightarrow H2\pi r = 2\pi r_1 ni \Rightarrow H = \frac{r_1}{r} ni \approx ni$$

$$H \approx ni = B_0/\mu_0.$$

B_0 is the magnetic field **without** the iron core.

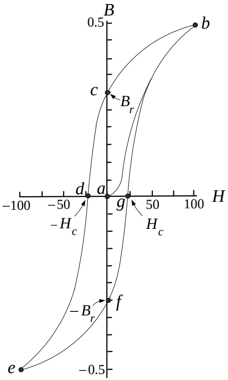
[1]By definition $\vec{\mathbf{H}} \triangleq \vec{\mathbf{B}}/\mu_0 - \vec{\mathbf{M}}$. On physical grounds $\vec{\mathbf{B}}$ and $\vec{\mathbf{M}}$ are aligned pointing in the $\hat{\boldsymbol{\theta}}$ direction. Thus $\vec{\mathbf{H}}$ will also be in the $\hat{\boldsymbol{\theta}}$ direction, i.e., $\vec{\mathbf{H}} = H\hat{\boldsymbol{\theta}}$.

The B − H Curve



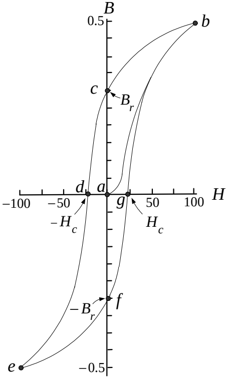
- $H(t) = ni(t)$
- $B(t) = -\frac{1}{N_s S}\int_0^t V_s(\tau)d\tau$
- Plot B versus H (rather than versus i)

The B − H Curve (continued)



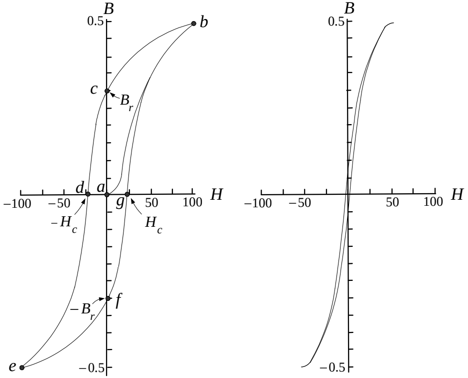
- **Curve a − b** : i is increased from 0 to $I_{\max}$ (H goes from 0 to $H_{\max} = nI_{\max} = 100\,\frac{\text{A-t}}{\text{m}}$)
- **Curve b − c** : i is decreased from $I_{\max}$ to 0 (H decreases from $H_{\max}$ to 0).
- **Point c** : $B = B_r > 0,\ H = 0, i = 0$. The iron permanently magnetized.
- **Curve c − d** : $i < 0$, i.e., the current is **reversed** with H negative and B decreasing.
  $H < 0, B > 0$ and $M > 0$.
- **Point d** : $B = 0$ where the magnetic field produced by the (negative) current in the coil is canceling the magnetic field produced by the aligned dipoles of the iron.

The B − H Curve (continued)



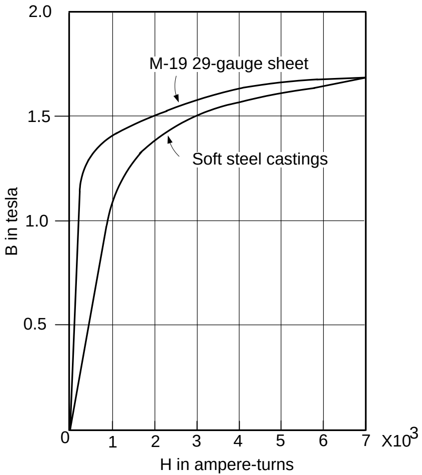
- **Point d** : $H = -H_c$ where $H_c > 0$ is the **coercive force** or **coercivity**.
- **Curve d − e** : i is decreased until H goes down to $-H_{\text{max}} = -100\frac{\text{A-t}}{\text{m}}$.
- **Curve d − e** : $H < 0$ and $B < 0$.
- **Point e** : B is the **negative** of the value at point b.
- **Curve e − f − g − b** : Similarly, H is brought from $-H_{\text{max}}$ to 0 to H_{max}.

The B − H Curve (continued)



- H varies from $H_{\max}$ to 0 to $-H_{\max}$ and then from $-H_{\max}$ to 0 back to $H_{\max}$.
- The plot of B versus H follows the path $b - c - d - e - f - g - b$.
- This is called the **hysteresis loop**.
- If the hysteresis loop is **thick**, then it is **hard** iron.
- If the hysteresis loop of the iron is **thin**, it is **soft** iron.

The B − H Curve (continued)



- **Soft iron** can be characterized by a **single** curve lying in the 1^{st} and 3^{rd} quadrants.
- Above are the $B - H$ curves for two **soft** magnetic materials.
- At smaller values of H, the $B - H$ curves are approximately **linear**.

Example

- Consider the $B - H$ for soft steel casting.
- $\vec{\mathbf{B}}$ and $\vec{\mathbf{M}}$ (and therefore $\vec{\mathbf{H}}$) tend to point in the **same** direction in soft iron.
- Thus
$$\vec{\mathbf{B}} = \mu_0 \vec{\mathbf{H}} + \mu_0 \vec{\mathbf{M}}$$
becomes
$$\vec{\mathbf{B}} = \mu_r \mu_0 \vec{\mathbf{H}} \ \text{ or } \ B = \mu_r \mu_0 H.$$
- μ_r is determined from the **experimentally** measured $B - H$ curve.
- E.g., choose i in the primary coil so that $H = 1000$ A-t.
- The $B - H$ data curve gives $B = 1.2$ tesla. Thus,
$$\mu_r = \frac{B}{\mu_0 H} = \frac{1.2}{(4\pi \times 10^{-7}) \times 1000} = 955.$$

Example (continued)

- In an **air core** toroid

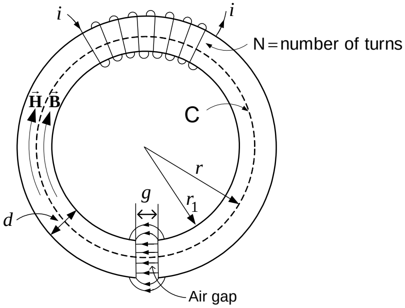
$$B_{\text{inside}} = \mu_0 H_{\text{inside}} = \mu_0 ni.$$

- In an identical toroid coil carrying the **same** current i, but with an **iron core**

$$B_{\text{inside}} = \mu_r \mu_0 H_{\text{inside}} = \mu_r \mu_0 ni.$$

- The H field is the **same** in both coils as the current is the **same** in each coil.
- As $\mu_r \sim 1000$ the magnetic field B in the iron core coil is **much larger**.
- The terminology **ideal** soft magnetic material is one in which $\mu_r = \infty$.

Computing $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$ in Magnetic Circuits

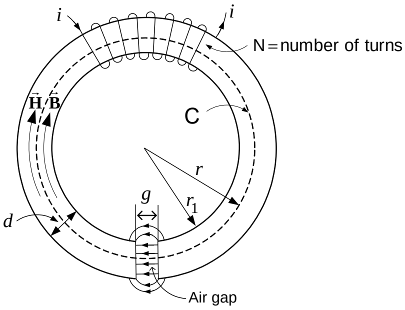


Ampère's law applied to the circular curve C of radius r is

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} = i_{\text{free}}.$$

- In the magnetic material, $\vec{\mathbf{B}} = B_m \hat{\boldsymbol{\theta}}$ while in the air gap $\vec{\mathbf{B}} = B_a \hat{\boldsymbol{\theta}}$.
- The air gap is small and the core diameter $d << r_1 =$ radius of the toroid.
  $\Longrightarrow B_m \approx$ **constant** in the iron core.
  $\Longrightarrow B_a \approx$ **constant** in the air gap.
  $\Longrightarrow \vec{\mathbf{M}} = M\hat{\boldsymbol{\theta}}$ with $M \approx$ **constant** in the iron core and **zero** in the **air gap**.

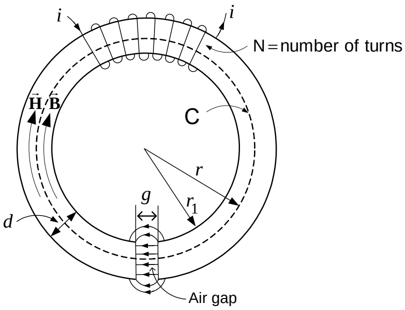
Computing $\vec{B}$ and $\vec{H}$ in Magnetic Circuits



- $H_m = B_m/\mu_0 - M \approx$ constant in the iron core.
- $H_a = B_a/\mu_0 \approx$ constant in the air gap.
- Ampère's law gives $H_m(2\pi r - g) + H_a g = Ni$.
- Also

$$\begin{aligned} B_m &= \mu_0 H_m + \mu_0 M = \mu_r \mu_0 H_m \ (\mu_r \text{ is from the } B-H \text{ curve}) \\ B_a &= \mu_0 H_a. \end{aligned}$$

Computing $\vec{B}$ and $\vec{H}$ in Magnetic Circuits



- By Gauss's law, the **normal component** of $\vec{\mathbf{B}}$ is continuous.
- Thus at the **air-iron interface** $B_m = B_a$ which implies

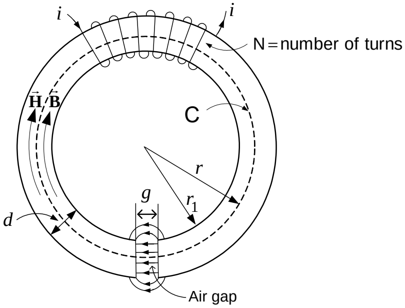
$$\mu_0 H_a = B_a = B_m = \mu_r \mu_0 H_m$$

or

$$H_m = \frac{H_a}{\mu_r}.$$

- In good magnetic materials, $\mu_r \sim 1000$ so that $H_m << H_a$.
- We will usually just take $H_m = 0$ in good magnetic materials.

Computing $\vec{B}$ and $\vec{H}$ in Magnetic Circuits



Taking $H_m = 0$ in

$$H_m(2\pi r - g) + H_a g = Ni$$

leads to

$$H_a g = Ni.$$

Finally

$$B_a = \mu_0 H_a = \frac{\mu_0 Ni}{g} \qquad (H_m = 0).$$

- This gives us the magnetic field B_a in the **air gap** in terms of **known** quantities.

Is the approximation $H_m = 0$ valid?

One must check that iron path length $2\pi r - g$ is not **too large**.

E.g., with $\mu_r = 1000$ suppose $2\pi r - g \approx 1000g$. Then

$$H_m(2\pi r - g) + H_a g = \frac{H_a}{1000}1000g + H_a g = 2H_a g = Ni$$

or

$$B_a = \mu_0 H_a = \frac{\mu_0 Ni}{2g}.$$

- The magnetic field is reduced by 1/2 because the iron path is **too long.**

Now suppose $2\pi r - g = 100g$. Then

$$H_m(2\pi r - g) + H_a g = \frac{H_a}{1000}100g + H_a g = 1.1H_a g$$

resulting in

$$B_a = \mu_0 H_a = \frac{\mu_0 Ni}{1.1g}.$$

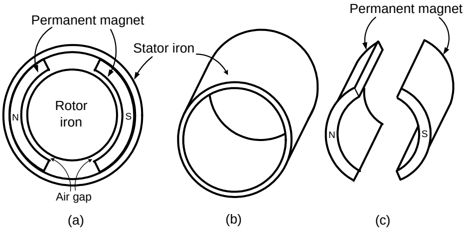
- **Not** much different from when $H_m = 0$ was assumed!

$\vec{\mathbf{B}}$ is Normal to the Surface of SOFT Magnetic Material

$\vec{\mathbf{B}}$ is **normal** to the surface at an air/magnetic-material interface under two assumptions:
(1) It is **ideal** ($\mu_r = \infty$) **soft** magnetic material. As $B = \mu_r \mu_0 H \Longrightarrow H = 0$.
(2) There are **no** current carrying wires on the surface.

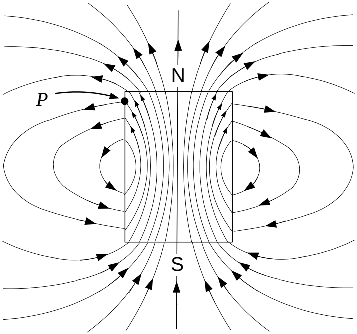
(Assumption (2) is a result of Ampère's law $\oint_C \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} = i_{\text{free}}$ - see book).

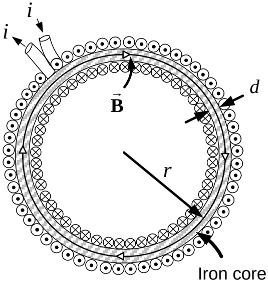
Example Consider the magnetic system of a DC machine.



- The magnetic field will be **perpendicular** on the **rotor iron surface**.
- The magnetic field will also be **perpendicular** on the two **disk-shaped ends** of the rotor iron, but it will be **weak** there.
- It turns out that the magnetic field is also **perpendicular** on the **cylindrical surfaces** of the **permanent magnet** because the magnet was **made** that way.

$\vec{\mathbf{B}}$ is Normal to the Surface of Soft Magnetic Material

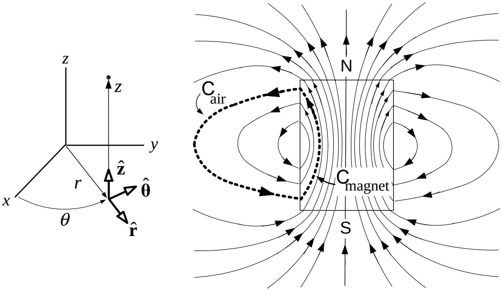




Example $\vec{\mathbf{B}}$ is **not necessarily** normal to the surface of a **permanent magnet**.

Example On the surface of the **soft iron core** of a toroidal-coil carrying current, $\vec{\mathbf{B}}$ is **not** normal to the surface. In fact, $\vec{\mathbf{B}} = B_\theta \hat{\boldsymbol{\theta}}$ in the iron core **even** at its surface.

Permanent Magnets



(a) (b)

Choose the path C to follow the closed curve of one of the magnetic lines of $\vec{\mathbf{B}}$.

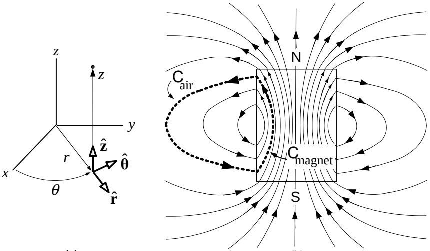
$$\begin{aligned}\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} &= \int_{C_{\text{air}}} \vec{\mathbf{H}} \cdot d\vec{\ell} + \int_{C_{\text{magnet}}} \vec{\mathbf{H}} \cdot d\vec{\ell} = 0 \\ d\vec{\ell} &= dr\hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + dz\hat{\mathbf{z}} = dr\hat{\mathbf{r}} + dz\hat{\mathbf{z}} \ \text{ as } d\theta = 0 \text{ on } C.\end{aligned}$$

As $\vec{\mathbf{H}} = \vec{\mathbf{B}}/\mu_0$ in the air, $\int_{C_{\text{air}}} \vec{\mathbf{H}} \cdot d\vec{\ell} > 0$.

Consequently, in order that $\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = 0$ it must be that

$$\int_{C_{\text{magnet}}} \vec{\mathbf{H}} \cdot d\vec{\ell} < 0.$$

Permanent Magnets

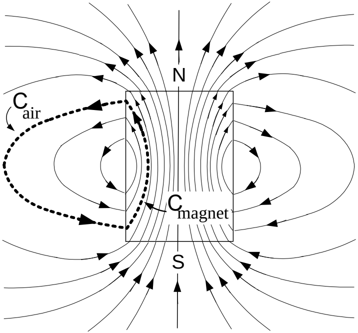


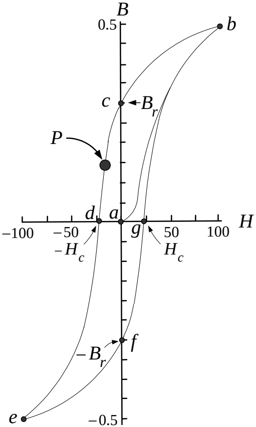
$H_\theta = 0$ and on C we have $d\theta = 0$. So

$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \underbrace{\int_{C_{\text{air}}} (H_r \hat{\mathbf{r}} + H_z \hat{\mathbf{z}}) \cdot d\vec{\ell}}_{>0} + \underbrace{\int_{C_{\text{magnet}}} (H_r \hat{\mathbf{r}} + H_z \hat{\mathbf{z}}) \cdot d\vec{\ell}}_{<0} = 0.$$

- On C_{air}, $d\vec{\ell} = dr\hat{\mathbf{r}} + dz\hat{\mathbf{z}}$ and $\vec{\mathbf{H}} = \vec{\mathbf{B}}/\mu_0$ point in the **same** direction.
- On C_{magnet} the component of $\vec{\mathbf{H}}$ tangent to the curve C_{magnet} points **opposite** to $d\vec{\ell}$.
- $\vec{\mathbf{H}}$ must be pointing **downwards** inside the magnet, but $\vec{\mathbf{B}}$ is pointing **upwards** there.
- The **z** components of $\vec{\mathbf{B}}$ and $\vec{\mathbf{M}}$ are in the $+\hat{\mathbf{z}}$ direction **inside** the magnet.
- As $H_z = B_z/\mu_0 - M_z < 0$ it follows that $M_z > B_z/\mu_0$ $(B_z > 0)$.

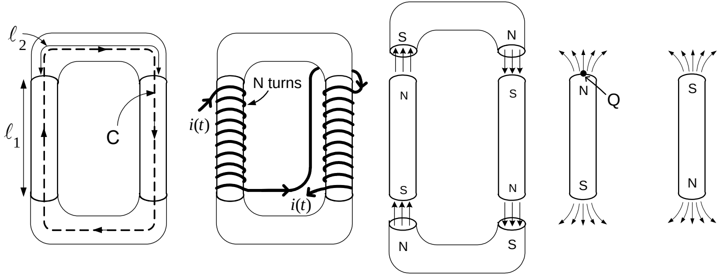
Permanent Magnets





- **Inside** the magnet we have $B_z > 0$ and $H_z = B_z/\mu_0 - M_z < 0$.
- A PM operates in the **second** quadrant of the $B - H$ curve. See point P above.
- How can this be understood?

Permanent Magnets

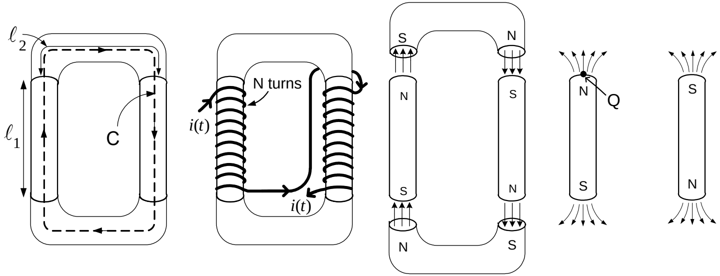


- Make a PM out of carbon steel by forming it into a tubular oblong shape.
- The straight segments are already "cut out" to become the cylindrical magnets.
- Wrap wire around the carbon steel and apply current to magnetize the iron.
- $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$ are taken as constant (in magnitude) throughout the core.

$$H(2\ell_1 + 2\ell_2) = ni$$

- H is the magnitude of $\vec{\mathbf{H}}$ in the core.

Permanent Magnets



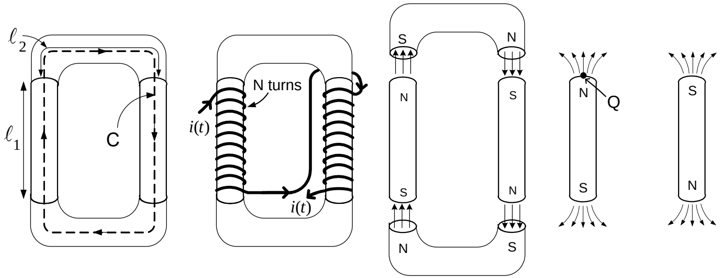
- $H(2\ell_1 + 2\ell_2) = ni$
- The current is increased up to the saturation point of the core.
- The current is then brought back down to **zero** (See point c in slide 67).
- The magnet is operating at the point c of the $B - H$ curve ($H = 0$ as $i = 0$).
- The winding is now removed and the magnet is still operating at point c.
- At each point of the core,

$$\vec{\mathbf{H}} = \frac{\vec{\mathbf{B}}}{\mu_0} - \vec{\mathbf{M}} = 0$$

that is,

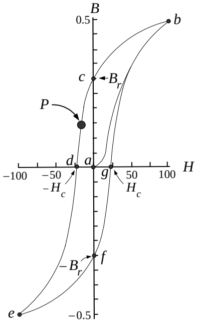
$$\vec{\mathbf{B}} = \mu_0 \vec{\mathbf{M}} \quad \text{or} \quad B = \mu_0 M.$$

Permanent Magnets



- The two end pieces of the steel core are pulled away.
- The **magnetization** M in the steel stays pretty much the **same**.
  I.e., the magnetic dipoles of the steel tend to stay **aligned**.
- At the point Q there is only **air** above it instead of **steel** as before.
- The B field at Q is primarily due to the magnetized **steel** atoms **close** to Q.
- At Q there is now only **half** as many steel atoms around it as before.
- B will be reduced at Q even assuming M stays the same.
- $H = \dfrac{B}{\mu_0} - M < 0$ at Q as B is reduced.

Permanent Magnets

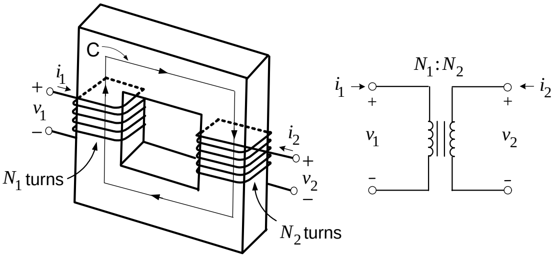


- M in the steel core remains constant after the two end pieces are removed.
- B at the two **end surfaces** of the cylindrical magnets is reduced.
- $H = \dfrac{B}{\mu_0} - M < 0$ at the magnet ends because B is reduced in strength.
- While the end pieces were removed, the **operating point** for the steel **moved** down the $B - H$ curve from c to some new point (say) P.
- In the middle of the cylinder (away from the ends) we expect H to be less negative, i.e., closer to 0.
- If the cylindrical magnet was long then H would be essentially zero in the middle.

Transformers

The next slides work out problems 7, 8, 9 of Chapter 3 of the book.

Ideal Transformer Model via Ampère's Law



- Assume **ideal soft iron** so $H = 0$ in the iron core.

$$0 = \oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = N_1 i_1 + N_2 i_2$$

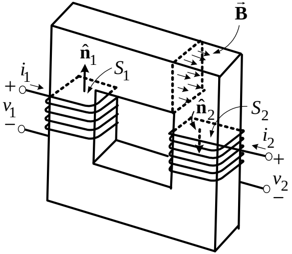
or

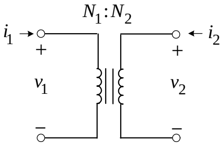
$$i_1 = -\frac{N_2}{N_1} i_2.$$

- In this **ideal** transformer if $i_2 = 0$ (winding 2 open) then $i_1 = 0$.
- By **conservation of energy** $v_1 i_1 + v_2 i_2 = 0$ so

$$v_1 = \frac{N_1}{N_2} v_2.$$

Ideal Transformer via Faraday's Law





- The iron core is a **flux tube** so (conservation of flux)

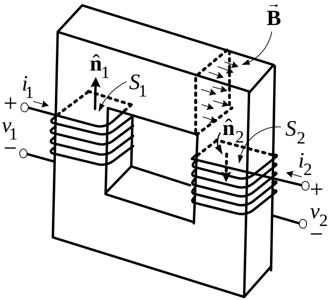
$$\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_{S_1} \vec{\mathbf{B}} \cdot (-dS\hat{\mathbf{n}}_1) + \int_{S_2} \vec{\mathbf{B}} \cdot dS\hat{\mathbf{n}}_2 = 0$$

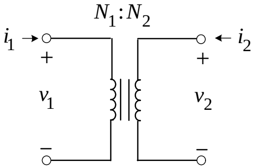
- The flux ϕ in **any** cross section of the iron core is

$$\phi = \int_{S_1} \vec{\mathbf{B}} \cdot (dS\hat{\mathbf{n}}_1) = \int_{S_2} \vec{\mathbf{B}} \cdot (dS\hat{\mathbf{n}}_2).$$

- **Flux linkage** $\lambda_1 \triangleq N_1\phi$ and the **induced voltage** in winding 1 is $-\dfrac{d\lambda_1}{dt}$.

Transformer via Faraday's Law (continued)





- With R_1 the resistance of winding 1 we have

$$v_1 + \left(-\frac{d\lambda_1}{dt}\right) = R_1 i_1.$$

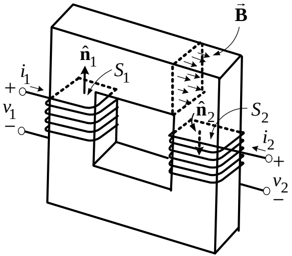
- Taking $R_1 = 0$

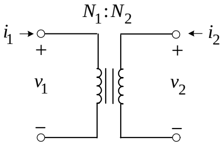
$$v_1 = \frac{d\lambda_1}{dt} = N_1 \frac{d\phi}{dt}.$$

- Similarly,

$$v_2 = \frac{d\lambda_2}{dt} = N_2 \frac{d\phi}{dt}.$$

Ideal Transformer via Faraday's Law (continued)





- Eliminate $d\phi/dt$ from (assume $d\phi/dt \neq 0$)

$$v_1 = N_1 \frac{d\phi}{dt} \text{ and } v_2 = N_2 \frac{d\phi}{dt}$$

  to obtain

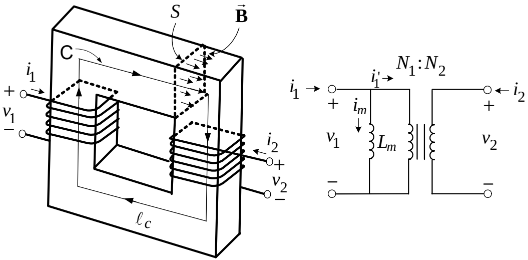
$$v_1 = \frac{N_1}{N_2} v_2.$$

- Substitute into $v_1 i_1 + v_2 i_2 = 0$ to obtain

$$i_1 = -\frac{N_2}{N_1} i_2.$$

- In the **ideal** transformer if $i_2 = 0$ (winding 2 open) then $i_1 = 0$.

Magnetizing Inductance



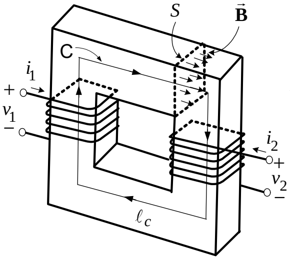
- **No** longer assume H is zero in the iron (μ_r is large but finite).
- Still assume that the magnetic fields B, H are confined to the core (no leakage).
- With winding 2 **open** and a voltage applied to winding 1

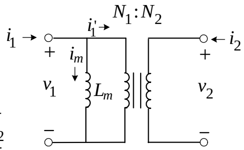
$$\oint_C \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} = H\ell_c = N_1 i_m$$

- i_m is the (magnetizing) current in **winding 1** with the secondary winding 2 **open**.
- The (average) value of H in the core is $H = \dfrac{N_1 i_m}{\ell_c}$.
- With winding 2 **open**, the magnetic field B in the core is

$$B = \mu_r \mu_0 H = \frac{\mu_r \mu_0 N_m}{\ell_c} i_m.$$

Magnetizing Inductance (continued)





- The flux ϕ in the core is given by $\phi = BS = \dfrac{\mu_r \mu_0 N_1 S}{\ell_c} i_m$.
- The **flux linkage** in winding 1 is

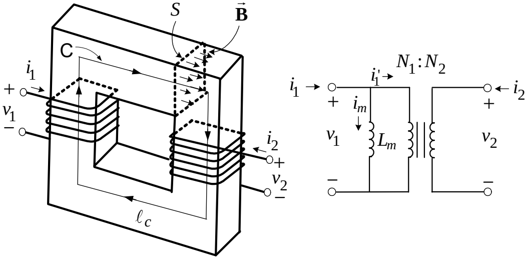
$$\lambda_1 = N_1 \phi = \frac{\mu_r \mu_0 N_1^2 S}{\ell_c} i_m = L_m i_m \quad \text{(winding 2 open).}$$

- In this **non ideal** transformer, with winding 2 **open**, then $i_1 = i_m \neq 0$ and

$$v_1 = L_m \frac{di_m}{dt} \text{ (winding 2 open).}$$

- The current i_m is called the **magnetizing** current.

Non Ideal Transformer Model



- i_m is the current required to produce the B field and thus ϕ in the core.
- The effect of finite μ_r in a transformer is accounted for by the inductance L_m.
- As $\mu_r \to \infty$, $L_m \to \infty$ and we have the ideal transformer.
- By Ampère's law, $\oint_C \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} = H\ell_c = N_1 i_1 + N_2 i_2$.
- Then

$$B = \mu_r \mu_0 H = \mu_r \mu_0 \frac{N_1 i_1 + N_2 i_2}{\ell_c}$$

and

$$\phi = BS = \frac{\mu_r \mu_0 S}{\ell_c} \left(N_1 i_1 + N_2 i_2 \right).$$

- Whether or not winding 2 is open, $\phi = \dfrac{1}{N_1} \int_0^t v_1(\tau) d\tau$ as $v_1 = N_1 d\phi/dt$.

Non Ideal Transformer Model (continued)

- Apply a voltage v_1 to winding 1 so $\phi = \dfrac{1}{N_1}\int_0^t v_1(\tau)d\tau$.
- If winding 2 is **open**,

$$\phi = \frac{\mu_r\mu_0 S}{\ell_c} N_1 i_m.$$

- Write $i_1 = i_m + i_1'$ where i_m is the (magnetizing) current with phase 2 **open**. i_1' is the additional current in winding 1 when winding 2 is **not** open, i.e.,

$$i_1' \triangleq i_1 - i_m.$$

- The flux can then be written as

$$\phi = \frac{\mu_r\mu_0 S}{\ell_c}\left(N_1 i_m + N_1 i_1' + N_2 i_2\right).$$

- The flux is determined by v_1 so these two flux expressions are equal, i.e.,

$$\frac{\mu_r\mu_0 S}{\ell_c} N_1 i_m = \frac{\mu_r\mu_0 S}{\ell_c}\left(N_1 i_m + N_1 i_1' + N_2 i_2\right)$$

  which gives $N_1 i_1' + N_2 i_2 \equiv 0$ or

$$i_1' = -\frac{N_2}{N_1} i_2.$$

Non Ideal Transformer Model (continued)

- As $i_1' = -\dfrac{N_2}{N_1} i_2$, conservation of energy (power) gives

$$v_1 i_1 + v_2 i_2 = v_1 i_m + v_1 i_1' + v_2 i_2 = \frac{d}{dt}\left(\frac{1}{2} L_m i_m^2\right) + v_1 i_1' + v_2 i_2$$

  using

$$v_1 = L \frac{di_m}{dt}.$$

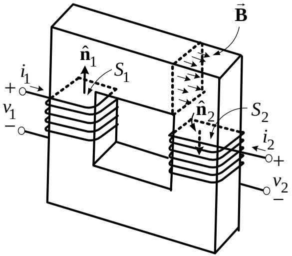
- $v_1 i_1'$ is the power into transformer from winding 1 that is **not** stored in the magnetic field of the core.
- $v_2 i_2$ is the power into the transformer from winding 2.
- There is **no** other mechanism for energy storage in the transformer so

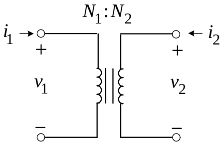
$$v_1 i_1' + v_2 i_2 = 0$$

  or

$$v_2 = -\frac{i_1'}{i_2} v_1 = \frac{N_2}{N_1} v_1.$$

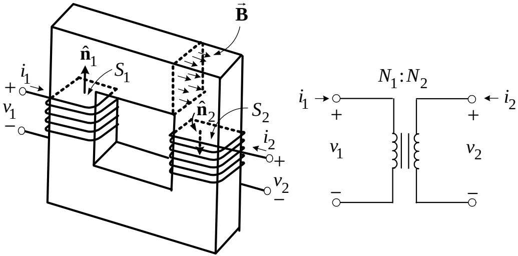
The Flux is Zero in an Ideal Transformer Model





- The flux ϕ in an ideal transformer must be **zero!**
- $i_1 = -\dfrac{N_2}{N_1} i_2 \Longrightarrow$ If winding 2 is **open**, then $i_1 = 0$ and the flux is **zero**.
- Let $N_1 = N_2 = 1$.
- A current $i_1 > 0$ in the winding 1 will produce a magnetic field B_1.
- The current $i_2 = -i_1 < 0$ in winding 2 will produce a magnetic field $B_2 = -B_1$.
- The **total** magnetic field $B = B_1 + B_2 = 0$ so the flux is **zero**.

The Flux is Zero in an Ideal Transformer Model



- Consider arbitrary values for N_1 and N_2.
- Each turn in winding 1 will produce B_1 for a total of $N_1 B_1$ due i_1.
- Each turn in winding 2 has the current $i_2 = -(N_1/N_2)i_1$ in it.
- Each turn in winding 2 will produce a magnetic field $B_2 = -(N_1/N_2)B_1$ in the core.
- Winding 2 will produce a magnetic field $N_2 B_2 = -N_2(N_1/N_2)B_1 = -N_1 B_1$.
- The total magnetic field in the core is zero and thus the flux is **zero**.
- Consider the ideal transformer model as a limiting case as $\mu_r \to \infty$.