Modeling and High-Performance Control of Electric Machines

Chapter 1 Physics of DC Motors

John Chiasson

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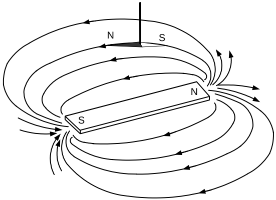
The Physics of the DC Motor

- **Magnetic Force**
- **Faraday's Law**
- **Dynamic Equations of the DC motor**
- **Tachometer for a DC Machine**
- **Multiloop Motor**
- **Microscopic Viewpoint*** [1]

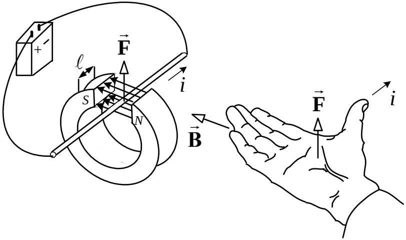
[1] An asterisk "*" denotes an optional section that can be skipped without loss of continuity.

Magnetic Field

- Motor principle: Magnetic fields produce **forces** on wires carrying a current.
- Direction of $\vec{\mathbf{B}}$ at any point: The direction a compass needle points.



- The magnetic force is **proportional** to ℓ and i, i.e., $F_{\text{magnetic}} \propto \ell i$



- The **magnitude** $\vec{\mathbf{B}}$: $B = |\,\vec{\mathbf{B}}\,| \triangleq \dfrac{F_{\text{magnetic}}}{\ell i}$.

Magnetic Force Law

In general the magnetic force is proportional to

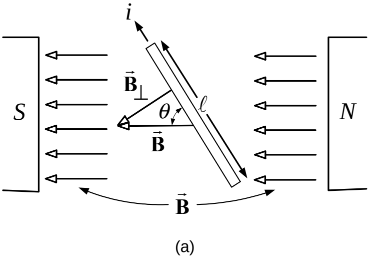
- The amount of current i in the wire
- The length ℓ of wire
- The strength of the magnetic field $B = |\vec{\mathbf{B}}|$
- The sine of angle between $\vec{\mathbf{B}}$ and the wire.

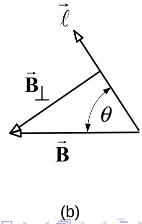
Define a vector $\vec{\ell}$

- $\ell = |\vec{\ell}|$ is the **length** of the wire in the magnetic field.
- The **direction** of $\vec{\ell}$ is defined as the direction of positive current in the wire.

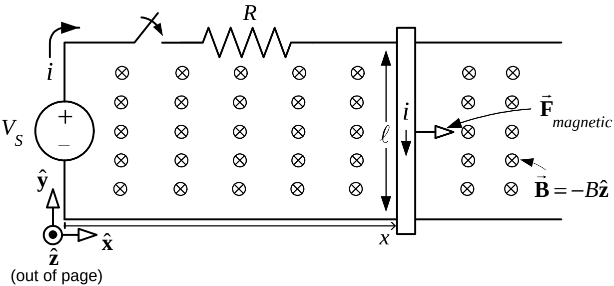
The magnetic force on a wire of length ℓ carrying the current i is given by

$$\begin{aligned} \vec{\mathbf{F}}_{\text{magnetic}} &= i\vec{\ell} \times \vec{\mathbf{B}} \\ F_{\text{magnetic}} &= i\ell B \sin(\theta) = i\ell B_{\perp} \end{aligned}$$





Example Linear DC Machine



$$\vec{\mathbf{B}} = -B\hat{\mathbf{z}} \quad B > 0$$
$$\vec{\ell} = -\ell\hat{\mathbf{y}}.$$

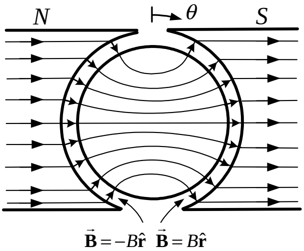
$$\vec{\mathbf{F}}_{\text{magnetic}} = i\vec{\ell} \times \vec{\mathbf{B}} = i(-\ell\hat{\mathbf{y}}) \times (-B\hat{\mathbf{z}}) = i\ell B\hat{\mathbf{x}}.$$

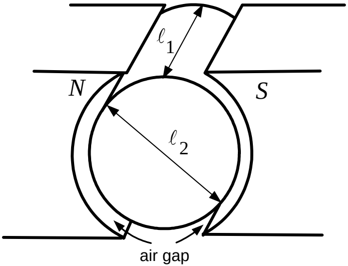
- f is the coefficient of viscous (sliding) friction.
- m_ℓ is the mass of the bar.
- **Equation of motion** of the bar:

$$i\ell B - f dx/dt = m_\ell d^2x/dt^2.$$

Example - Single Loop Motor

- Soft iron cylindrical core.
- Placed inside a hollowed out **Permanent Magnet**.
- Magnetic field tends to be **perpendicular** to the surface of magnetic materials.
- The cylindrical shape results in $\vec{\mathbf{B}}$ being **radially directed** in the air gap.

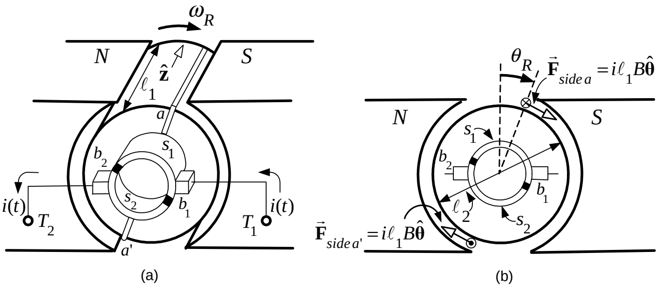




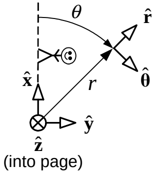
The magnetic field in the **air gap** is given by $(B > 0)$

$$\vec{\mathbf{B}} = \begin{cases} +B\hat{\mathbf{r}} & \text{for } 0 < \theta < \pi \\ -B\hat{\mathbf{r}} & \text{for } \pi < \theta < 2\pi. \end{cases}$$

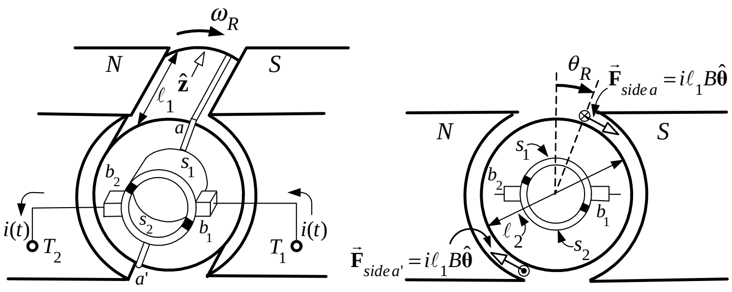
Rotor Loop and Slip Rings



- $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$ denote unit cylindrical coordinate vectors.
- The unit vector $\hat{\mathbf{z}}$ points along the rotor axis into the page.
- $\hat{\boldsymbol{\theta}}$ is in the direction of increasing θ and $\hat{\mathbf{r}}$ is in the direction of increasing r.



Single Loop Motor - Torque



(a) (b)

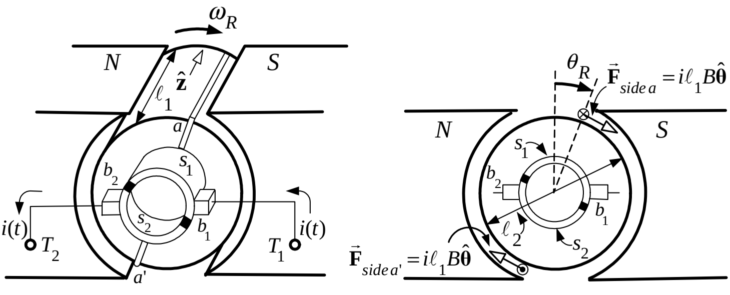
- For $i > 0$, the current in side a of the loop is going into the page (denoted by $\otimes$).
- On side a, $\vec{\ell} = \ell_1 \hat{\mathbf{z}}$.
- The magnetic force $\vec{\mathbf{F}}_{\text{side } a}$ on side a is then

$$\vec{\mathbf{F}}_{\text{side } a} = i\vec{\ell} \times \vec{\mathbf{B}} = i(\ell_1 \hat{\mathbf{z}}) \times (B\hat{\mathbf{r}}) = i\ell_1 B\hat{\boldsymbol{\theta}}.$$

- The resulting torque is

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{\text{side } a} = (\ell_2/2)\hat{\mathbf{r}} \times \vec{\mathbf{F}}_{\text{side } a} &= (\ell_2/2) i\ell_1 B\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \\ &= (\ell_2/2) i\ell_1 B\hat{\mathbf{z}}.\end{aligned}$$

Single Loop Motor - Torque



(a) (b)

- The magnetic force $\vec{\mathbf{F}}_{\text{side } a'}$ on side a' is then

$$\vec{\mathbf{F}}_{\text{side } a'} = i\vec{\ell} \times \vec{\mathbf{B}} = i(-\ell_1 \hat{\mathbf{z}}) \times (-B\hat{\mathbf{r}}) = i\ell_1 B\hat{\boldsymbol{\theta}}$$

- The resulting torque is

$$\vec{\boldsymbol{\tau}}_{\text{side } a'} = (\ell_2/2)\hat{\mathbf{r}} \times \vec{\mathbf{F}}_{\text{side } a'} = (\ell_2/2)i\ell_1 B\hat{\mathbf{z}}$$

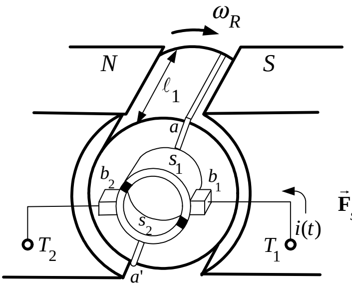
The **total** torque on the rotor loop is then

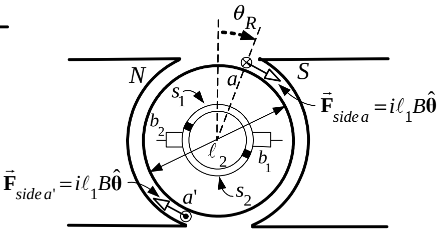
$$\begin{aligned} \vec{\boldsymbol{\tau}}_m &= \vec{\boldsymbol{\tau}}_{\text{side } a} + \vec{\boldsymbol{\tau}}_{\text{side } a'} = 2(\ell_2/2)i\ell_1 B\hat{\mathbf{z}} = \ell_1\ell_2 Bi\hat{\mathbf{z}} \\ \text{or} \quad \tau_m &= K_T i \quad \text{where } K_T \triangleq \ell_1\ell_2 B. \end{aligned}$$

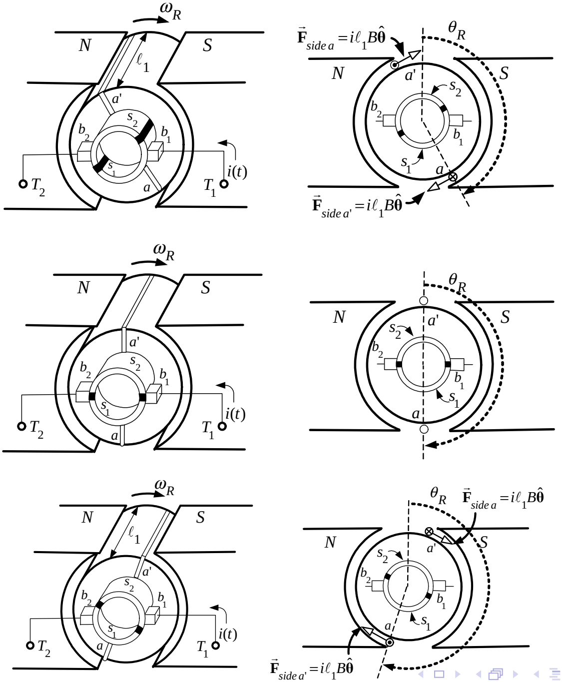
Current Commutation

To obtain positive torque $\tau_m = K_T i > 0$:

- The current in the loop side under the **South Pole** must be **into** the page.
- The current in the loop side under the **North Pole** must be **out** of the page.
- That is, every **half turn**, the direction of current in the loop must be **reversed**.






ω_R
N
S
ℓ_1
a'
s_2
b_2
b_1
s_1
a
T_2
T_1
$i(t)$
$\vec{\mathbf{F}}_{side\,a} = i\ell_1 B\hat{\boldsymbol{\theta}}$
$\vec{\mathbf{F}}_{side\,a'} = i\ell_1 B\hat{\boldsymbol{\theta}}$
θ_R

Faraday's Law

A **changing flux** in a loop produces an induced voltage in the loop.

This induced voltage is also called an **electromotive force (emf)** and denoted by ξ.

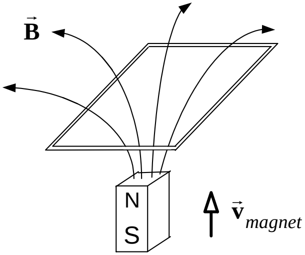
Mathematically,

$$\xi = -\frac{d\phi}{dt}.$$

Here

$$\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}$$

is the flux in the loop and S is any surface with the loop as its boundary.

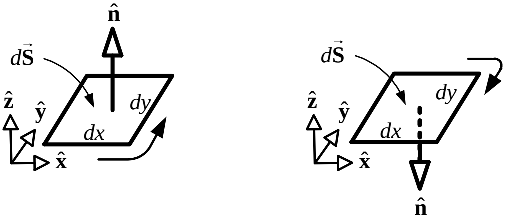


The Surface Element Vector $d\vec{\mathbf{S}}$

The **surface element vector** is defined by

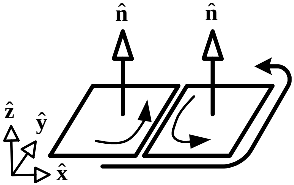
$$d\vec{\mathbf{S}} \triangleq dxdy\hat{\mathbf{z}} \quad \text{or} \quad d\vec{\mathbf{S}} \triangleq -dxdy\hat{\mathbf{z}}$$

Positive direction of travel around the surface given by the **right-hand rule**.

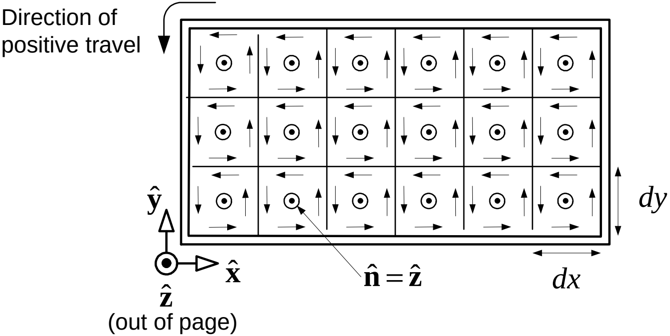


(a) (b)

Connecting Two Surface Elements



Net Direction of Travel Around a Surface Boundary



- The normal to each surface element is $\hat{\mathbf{n}} = \hat{\mathbf{z}}$.
- Surface element: $d\vec{\mathbf{S}} = dxdy\hat{\mathbf{n}} = dxdy\hat{\mathbf{z}}$.
- $\odot$ denotes the direction of the normal $\hat{\mathbf{n}}$ (out of the page).
- Positive direction of travel around the surface boundary is *counterclockwise*.

Interpreting the Sign of ξ

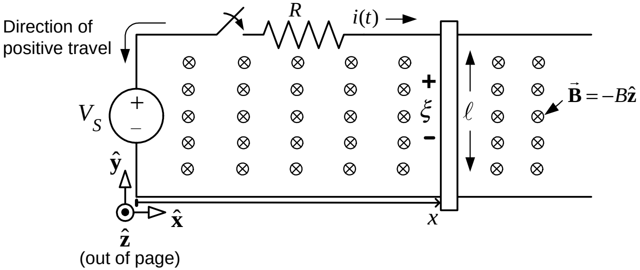
Faraday's law

$$\begin{aligned}\xi &= -\frac{d\phi}{dt} \\ \phi &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}.\end{aligned}$$

- If $\xi > 0$, the induced emf will force current in the positive direction of travel.
- If $\xi < 0$, the induced emf will force current in the opposite direction.

This is all better understood by some simple examples which we now do.

Example Linear DC Machine



- $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$, where $B > 0$.
- Let $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ be the normal to the surface.
- Let $d\vec{\mathbf{S}} = dxdy\hat{\mathbf{z}}$ where $dS = dxdy$.

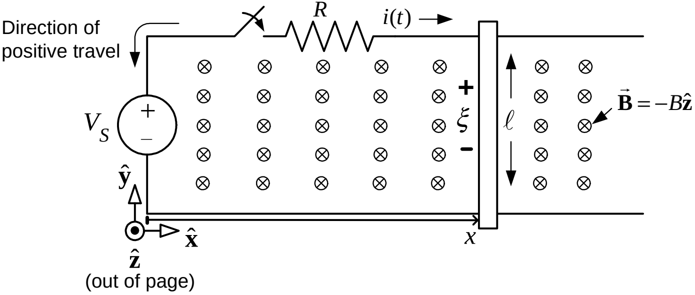
Then

$$\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_0^\ell \int_0^x (-B\hat{\mathbf{z}}) \cdot (dxdy\hat{\mathbf{z}}) = \int_0^\ell \int_0^x -Bdxdy = -B\ell x.$$

The induced (back) emf is therefore given by

$$\xi = -\frac{d\phi}{dt} = -\frac{d(-B\ell x)}{dt} = B\ell v.$$

Example Linear DC Machine (continued)

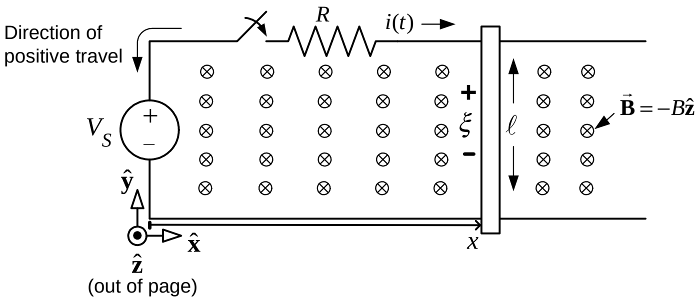


We just computed

$$\xi = -\frac{d\phi}{dt} = -\frac{d(-B\ell x)}{dt} = B\ell v.$$

- The magnetic force on the bar is $\vec{\mathbf{F}}_{\text{magnetic}} = i\ell B\hat{\mathbf{x}}$ so $v = dx/dt > 0$.
- The induced voltage $\xi > 0$ and **opposes** the source voltage V_S.

Energy Conversion



- $F_{\text{magnetic}} = i\ell B$.
- Mechanical power is $F_{\text{magnetic}} v = i\ell B v$.
- The back emf $\xi = B\ell v$ **opposes** the current i.
- Electrical power **absorbed** is $-i\xi = -iB\ell v$.

The electrical power **absorbed** by the back emf **reappears** as mechanical power.

Conservation of energy:

$$-i\xi + F_{\text{magnetic}} v = -iB\ell v + i\ell B v = 0$$

Linear DC Machines - Equations of Motion

- The self-inductance L of the circuit loop is taken to be zero.
- m_ℓ the mass of the bar.
- f the coefficient of viscous-friction.

$$\begin{aligned} V_S - \xi &= Ri \\ m_\ell \frac{dv}{dt} &= F_{\text{magnetic}} - fv. \end{aligned}$$

Or

$$\begin{aligned} V_S - B\ell v &= Ri \\ m_\ell \frac{dv}{dt} &= i\ell B - fv. \end{aligned}$$

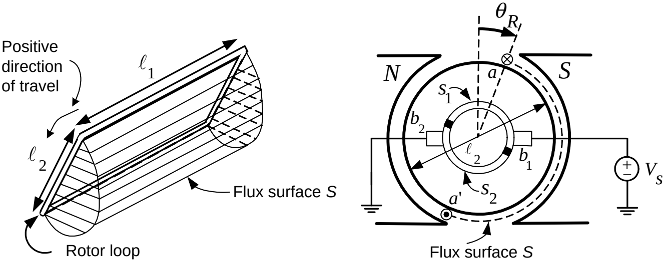
Eliminating the current i, one obtains

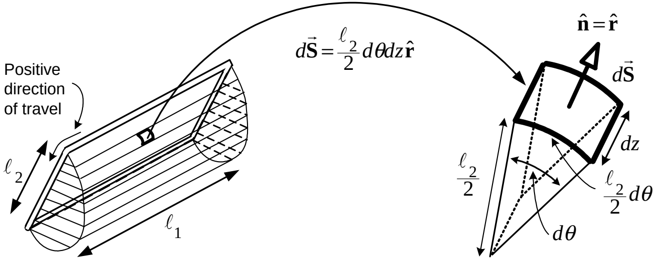
$$m_\ell \frac{d^2x}{dt^2} = \frac{\ell B(V_S - B\ell v)}{R} - fv = -\left(\frac{B^2\ell^2}{R} + f\right)\frac{dx}{dt} + \frac{\ell B}{R}V_S$$

or

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\left(\frac{B^2\ell^2}{m_\ell R} + \frac{f}{m_\ell}\right)v + \frac{\ell B}{m_\ell R}V_S. \end{aligned}$$

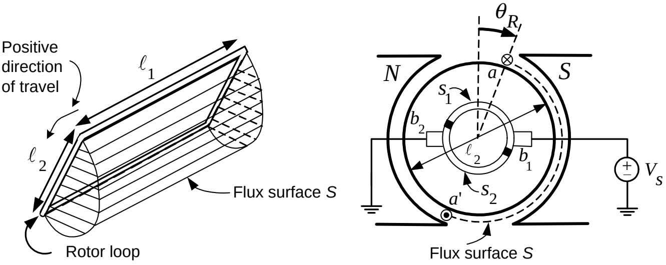
Example - EMF in the Single Loop Motor





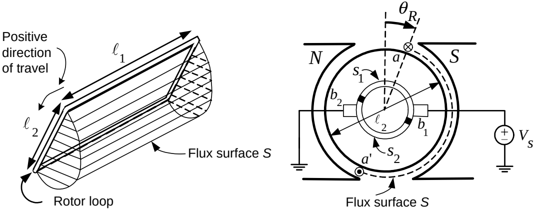
- On the cylindrical part of the surface $d\vec{\mathbf{S}} = (\ell_2/2)d\theta dz\hat{\mathbf{r}}$.
- On the two ends (half-disks) of the cylindrical surface $\vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$.

Example - EMF in the Single Loop Motor

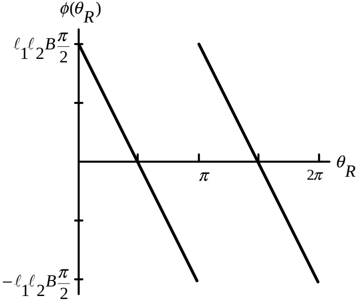


$$
\begin{aligned}
\phi(\theta_R) &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \\
&= \int_0^{\ell_1} \int_{\theta=\theta_R}^{\theta=\pi} (B\hat{\mathbf{r}}) \cdot (\frac{\ell_2}{2} d\theta dz \hat{\mathbf{r}}) + \int_0^{\ell_1} \int_{\theta=\pi}^{\theta=\pi+\theta_R} (-B\hat{\mathbf{r}}) \cdot (\frac{\ell_2}{2} d\theta dz \hat{\mathbf{r}}) \\
&= \frac{\ell_1 \ell_2 B}{2}(\pi - \theta_R) - \frac{\ell_1 \ell_2 B}{2}\theta_R \quad \text{for } 0 < \theta_R < \pi \\
&= -\ell_1 \ell_2 B\,(\theta_R - \pi/2) \qquad \text{for } 0 < \theta_R < \pi.
\end{aligned}
$$

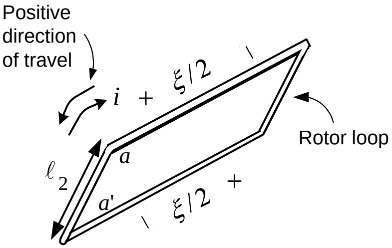
Example - EMF in the Single Loop Motor (continued)

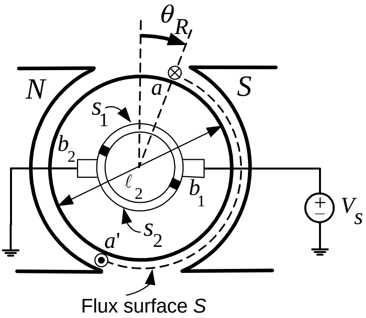


$$\phi(\theta_R) = -\ell_1\ell_2 B(\theta_R - \pi/2) \text{ for } 0 < \theta_R < \pi.$$



Example - EMF in the Single Loop Motor (continued)





$$\phi(\theta_R) = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = -\ell_1 \ell_2 B(\theta_R - \pi/2) \quad \text{for } 0 < \theta_R < \pi.$$

By Faraday's law

$$\xi = -\frac{d\phi}{dt} = (\ell_1 \ell_2 B)\frac{d\theta_R}{dt} = K_b \omega_R.$$

- $K_b \triangleq \ell_1 \ell_2 B$ is called the **back-emf** constant.
- If $\omega_R > 0$ then $\xi > 0$ showing the back-emf **opposes** V_S.

Example - EMF in the Single Loop Motor (continued)

- The mechanical power **produced** is

$$\tau_m \omega_R = K_T i \omega_R = \ell_1 \ell_2 B i \omega_R.$$

- The electrical power **absorbed** by the back emf is

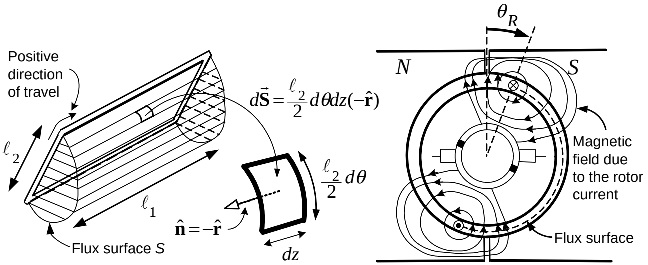
$$-\xi i = -K_b \omega_R i = -\ell_1 \ell_2 B \omega_R i.$$

- The electrical power absorbed **reappears** as mechanical power, i.e.,

$$-\xi i + \tau_m \omega_R = -\ell_1 \ell_2 B \omega_R i + \ell_1 \ell_2 B i \omega_R = 0.$$

- Note that $K_T = K_b$ **required** for energy conservation to hold.

Self Inductance L - Rotor current also produces a flux



Let $r_R \triangleq \ell_2/2$ denote the radius of the rotor.
The magnetic field on the flux surface due to the **armature current** has the form

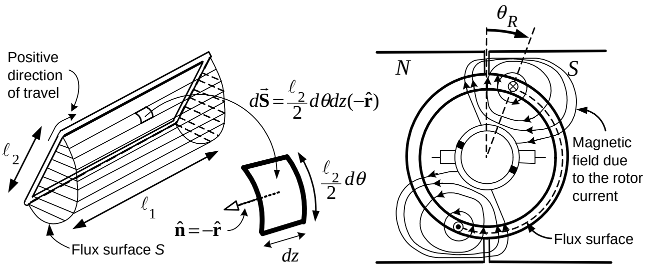
$$\vec{\mathbf{B}}(r_R, \theta - \theta_R, i) = iK(r_R, \theta - \theta_R)(-\hat{\mathbf{r}})$$

where

$$K(r_R, \theta - \theta_R) > 0 \quad \text{for} \quad 0 < \theta - \theta_R < \pi$$
$$K(r_R, \theta - \theta_R) < 0 \quad \text{for} \quad \pi < \theta - \theta_R < 2\pi$$

- The exact expression for $K(r_R, \theta - \theta_R)$ is not needed.
- The surface element is chosen to be $d\vec{\mathbf{S}} = r_R d\theta dz(-\hat{\mathbf{r}})$.
- The positive direction of travel around the surface coincides with the positive direction of the current i in the loop.

Self Inductance L - Rotor current also produces flux



$$
\begin{aligned}
\psi(i) = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} &= \int_0^{\ell_1} \int_{\theta_R}^{\theta_R+\pi} iK(r_R, \theta - \theta_R)(-\hat{\mathbf{r}}) \cdot (-r_R d\theta dz \hat{\mathbf{r}}) \\
&= \underbrace{i \int_0^{\ell_1} \int_{\theta_R}^{\theta_R+\pi} K(r_R, \theta - \theta_R) r_R d\theta dz}_{L>0} \\
&= Li.
\end{aligned}
$$

- If $-d\psi/dt = -Ldi/dt > 0$, the induced emf will force current into the page $\otimes$ on side a and out of the page $\odot$ on side a'.
- $-d\psi/dt$ has the same sign convention as V_S.

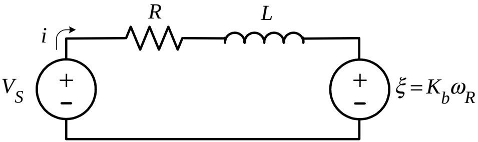
Equations of the DC Motor

$$V_S - K_b\omega_R - L\frac{di}{dt} = Ri$$

or

$$L\frac{di}{dt} = -Ri - K_b\omega_R + V_S.$$

Equivalent circuit



The mechanical equation is then

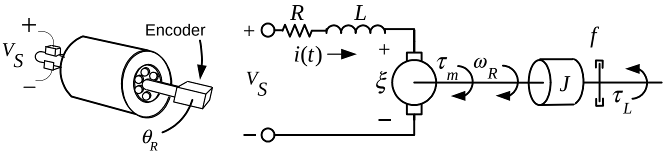
$$\tau_m - \tau_L - f\omega_R = J\frac{d\omega_R}{dt}$$

- J is the moment of inertia of rotor assembly (armature, etc.).
- τ_L is the load torque.
- f is the coefficient of viscous friction.

Equations of the DC Motor

The system of equations characterizing the DC motor is then

$$
\begin{aligned}
L\frac{di}{dt} &= -Ri - K_b\omega_R + V_S \\
J\frac{d\omega_R}{dt} &= K_T i - f\omega_R - \tau_L \\
\frac{d\theta_R}{dt} &= \omega_R.
\end{aligned}
$$



Energy Conversion

- The mechanical power **produced** by the DC motor is

$$\tau_m \omega_R = K_T i \omega_R = i \ell_1 \ell_2 B \omega_R.$$

- The electrical power **absorbed** by the back emf is

$$-i\xi = -iK_b \omega_R = -i \ell_1 \ell_2 B \omega_R.$$

- Thus the electrical power absorbed is **converted** to mechanical power, that is,

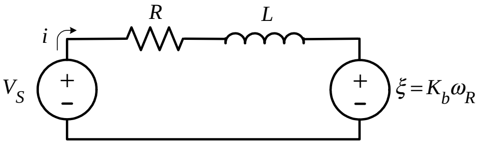
$$(\tau_m \omega_R) + (-i\xi) = 0.$$

- $K_T = K_b = \ell_1 \ell_2 B$ must hold by **conservation of energy**.

Energy Conversion

Another way to view this energy conversion is to write the electrical equation as

$$V_S = Ri + L\frac{di}{dt} + \xi.$$



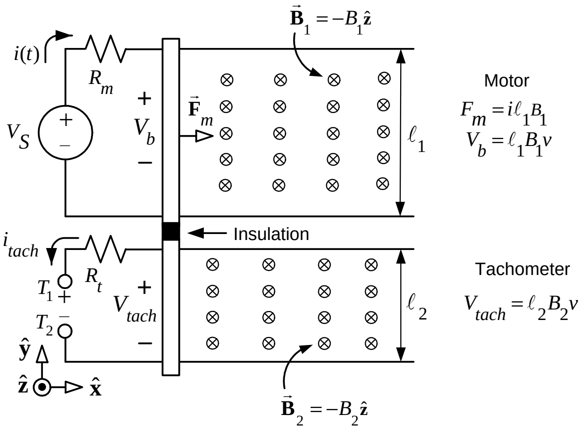
The power out of the voltage source $V_S(t)$ is given by

$$\begin{aligned} V_S(t)i(t) &= Ri^2 + Li\frac{di}{dt} + iK_b\omega_R \\ &= Ri^2 + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + K_T i\omega_R \\ &= Ri^2 + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + \tau_m\omega_R. \end{aligned}$$

- Ri^2 power lost to heat.
- $\frac{d}{dt}\left(\frac{1}{2}Li^2\right)$ power stored/recovered from the magnetic field of the armature current.
- $\tau_m\omega_R$ mechanical power.

Tachometer for a DC Machine

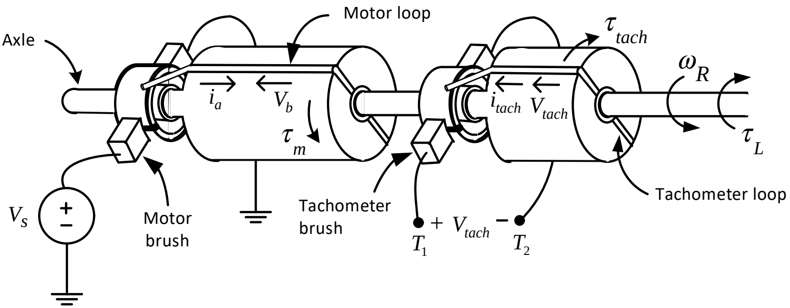
A tachometer is a device for measuring speed of a motor.



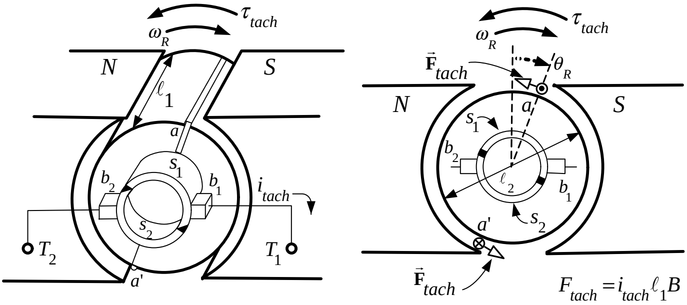
- The motor force is $F_m = i\ell_1 B_1$.
- The induced (back) emf in the motor is $V_b = B_1\ell_1 v$.
- The induced (back) emf in the tachometer is $V_{tach} = B_2\ell_2 v$.
- Measure the voltage between T_1 and T_2 to compute v.

Tachometer for the Single-Loop DC Motor

- Attach a second loop to the shaft of the single loop DC motor.
- The motor and tachometer loop rotate together in an external magnetic field.
- The external magnetic field is not drawn in the figure below.
- The voltage V_{tach} between T_1 and T_2 is proportional to the motor speed ω_R.



Tachometer for the Single-Loop DC Motor



$$\begin{aligned}\phi_{tach} = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} &= \int_0^{\ell_1}\int_{\theta_R}^{\pi} (B\hat{\mathbf{r}})\cdot(\frac{\ell_2}{2}d\theta dz\hat{\mathbf{r}}) + \int_0^{\ell_1}\int_{\pi}^{\pi+\theta_R} (-B\hat{\mathbf{r}})\cdot(\frac{\ell_2}{2}d\theta dz\hat{\mathbf{r}}) \\ &= -\ell_1\ell_2 B\theta_R + (\ell_1\ell_2 B/2)\pi.\end{aligned}$$

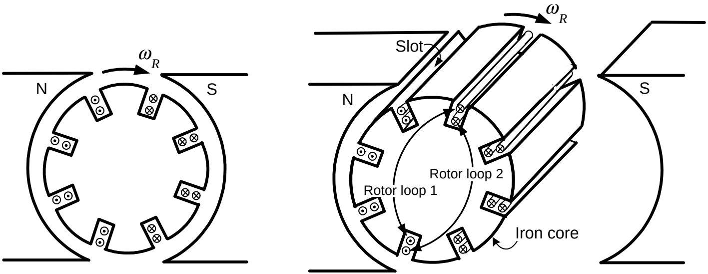
The induced emf is then

$$V_{\text{tach}} = -d\phi_{tach}/dt = (\ell_1\ell_2 B)d\theta_R/dt = K_{b\_\text{tach}}\omega_R.$$

Measure V_{tach} to compute ω_R.

The Multiloop DC Motor

In the figure there are 8 slots in the rotor.
There are two loops in each pair of slots which are 180° apart.
$n = 8$ is the number of rotor loops.

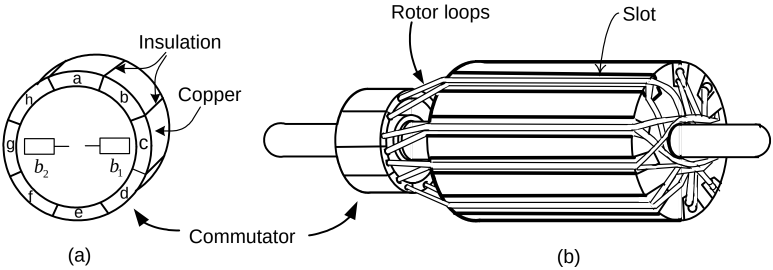


The torque on the rotor is now $\tau_m = n\ell_1\ell_2 Bi$.

For $\tau_m > 0$ we must have:

- The current going **into** the page $\otimes$ under the **south** pole face
- The current coming **out** of the page $\odot$ under **north** pole face.

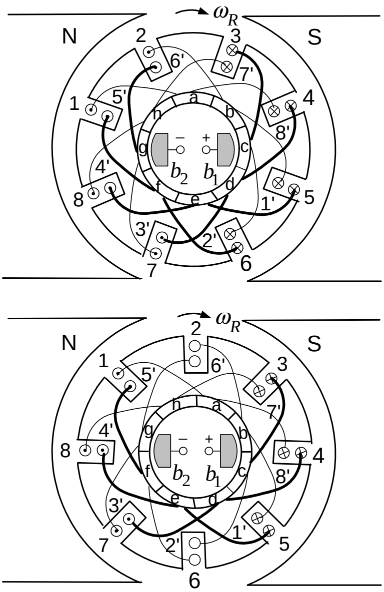
Commutation of the Armature Current



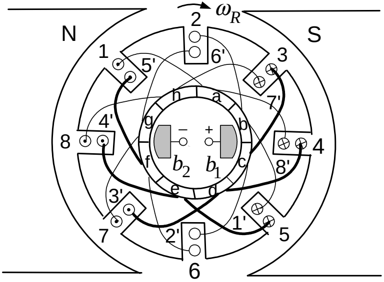
- The **commutator** consists of 8 copper segments labeled *a-h*.
- The commutator and rotor loops rotate **together**.
- The two brushes (labeled b_1 and b_2) remain **stationary**.

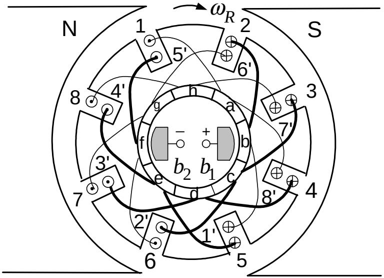
Commutation of the Armature Current

The eight rotor loops are labeled as $1-1'$,..., $8-8'$.

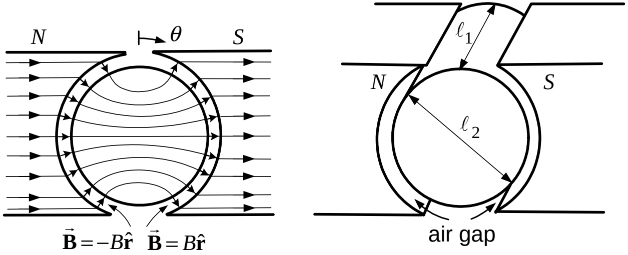


Commutation of the Armature Current

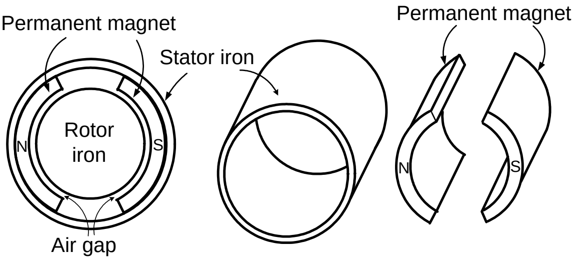




Stator Iron Construction

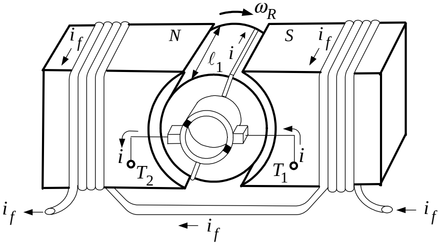


Realistic depiction of a (single pole-pair) **permanent magnet** DC motor.

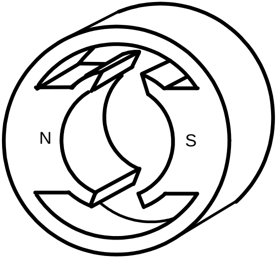


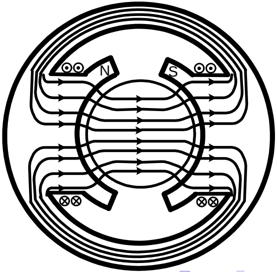
Wound Field DC Machine

- Difficult to make a large DC machine using a PM.
- An electromagnet is used to produce the magnetic field B in the air gap.



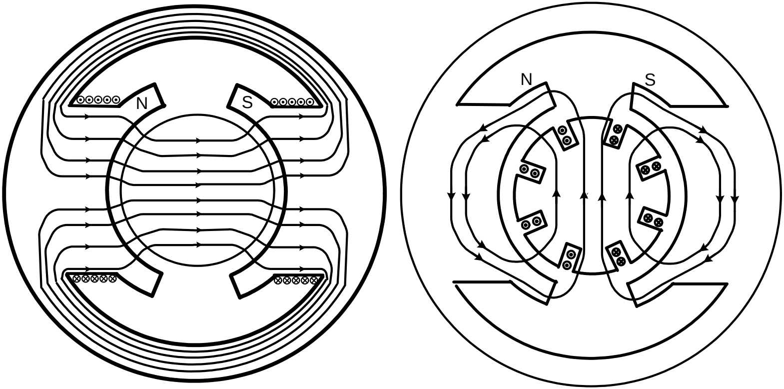
Realistic depiction of a single pole-pair **wound field** machine.





Armature Reaction

Left: Magnetic field due to the **field** current i_f.
Right: Magnetic field due to the **armature** current i.



- The net flux produced by the armature in the field windings is **zero**.
- The changing armature current does **not** induce voltages in the field windings.
- i_f remains **constant**.

Microscopic Viewpoint*

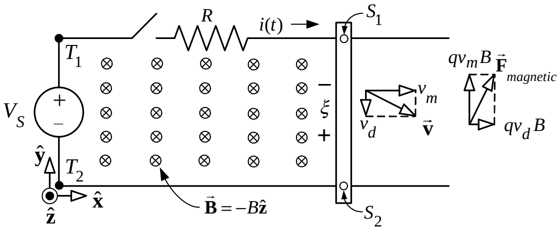
*This optional section presents the previous results from a different point of view.

Microscopic Viewpoint

If a charged particle with charge q is moving with velocity $\vec{\mathbf{v}}$ in a magnetic field $\vec{\mathbf{B}}$ then experiments show that the magnetic force on q is

$$\vec{\mathbf{F}}_{\text{magnetic}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}.$$

Example Linear DC Machine



$\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$ where $B > 0$.

Let the motor (bar) move to the right with speed $v_m > 0$.

Each charge q in the sliding bar has total velocity $\vec{\mathbf{v}} = v_m\hat{\mathbf{x}} - v_d\hat{\mathbf{y}}$.

v_d is the **drift** speed of the charges down the wire.

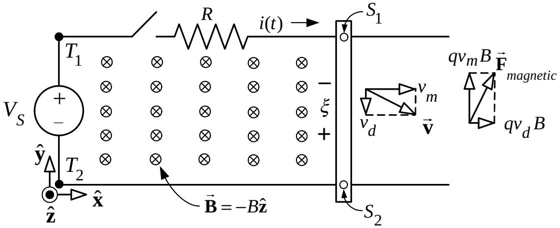
$$\vec{\mathbf{F}}_{\text{magnetic}} = q\vec{\mathbf{v}}\times\vec{\mathbf{B}} = q\left(v_m\hat{\mathbf{x}} - v_d\hat{\mathbf{y}}\right) \times \left(-B\hat{\mathbf{z}}\right) = qv_mB\hat{\mathbf{y}} + qv_dB\hat{\mathbf{x}}.$$

- The component of force $qv_dB\hat{\mathbf{x}}$ causes the bar to move to the right.
- The component of force $qv_mB\hat{\mathbf{y}}$ along the bar opposes the current flow.

Microscopic Viewpoint

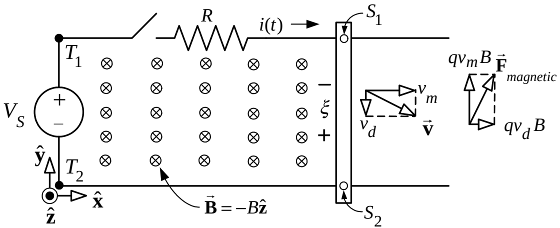
The source voltage V_S sets up an electric field $\vec{\mathbf{E}}_S$ in the bar to overcome the magnetic force $qv_m B\hat{\mathbf{y}}$ and the resistance. We have

$$V_S = \int_{\mathrm{T}_1-\mathrm{S}_1-\mathrm{S}_2-\mathrm{T}_2} \vec{\mathbf{E}}_S \cdot d\vec{\ell}.$$



- $q\vec{\mathbf{E}}_S$ is the force on each charge carrier.
- qV_S is the energy given to the charge carrier by the source voltage as the charge goes around the loop.
- The magnetic force $qv_m B\hat{\mathbf{y}}$ is acting against $q\vec{\mathbf{E}}_S$.

Microscopic Viewpoint - Induced emf



The energy per unit charge ξ that the magnetic force takes from the charge carrier as it goes down the bar is

$$\xi \triangleq \frac{1}{q}\int_{S_1}^{S_2} \vec{\mathbf{F}}_{\text{magnetic}} \cdot d\vec{\boldsymbol{\ell}} = \frac{1}{q}\int_{S_1}^{S_2} q(\vec{\mathbf{v}}\times\vec{\mathbf{B}}) \cdot d\vec{\boldsymbol{\ell}} = \int_0^{\ell} (v_d B\hat{\mathbf{x}} + v_m B\hat{\mathbf{y}}) \cdot (-d\ell\hat{\mathbf{y}}) = -v_m B\ell.$$

- ξ being **negative** indicates that the magnetic force is taking energy out of the charge carrier as it goes down the bar.
- V_S was computed by integrating $\vec{\mathbf{E}}_S$ in the **CW** direction around the loop.
- ξ was computed by integrating $\vec{\mathbf{v}}\times\vec{\mathbf{B}}$ also in the **CW** direction around the loop.
- V_S and ξ have the same sign convention.

Connecting the Microscopic to the Macroscopic

The total emf in a loop is the integral of the force per unit charge around the loop.

The total emf is the sum of the **source voltage** and the **induced emf**.

This total emf goes into producing the current, that is,

$$V_S + \xi = V_S - v_m \ell B = Ri.$$

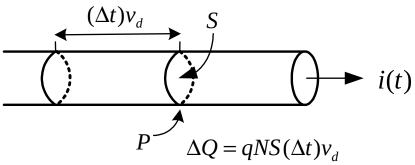
This is **identical** to that found in the macroscopic case using Faraday's law.

The **total magnetic force** on all the charge carriers in the bar in the $\hat{\mathbf{x}}$ direction is

$$q(NS\ell)v_d B\hat{\mathbf{x}}.$$

- N is the number of charge carriers/volume.
- S is the cross sectional area of the sliding bar.
- $NS\ell$ is the total number of charge carriers in the sliding bar.

Connecting the Microscopic to the Macroscopic (continued)



- In a time Δt, the charges in the volume $NS(v_d \Delta t)$ have moved past the point P.
- I.e., the charge $\Delta Q = qNS(v_d \Delta t)$ has moved past the point P in the time Δt so

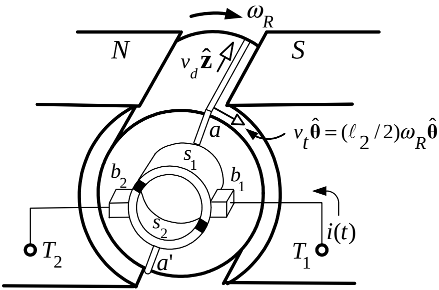
$$i = \Delta Q / \Delta t = qNSv_d.$$

Consequently, the **total magnetic force** on the bar is

$$q(NS\ell)v_d B\hat{\mathbf{x}} = (qNSv_d)\ell B\hat{\mathbf{x}} = i\ell B\hat{\mathbf{x}}.$$

This is **identical** to the expression derived from the macroscopic point of view.

Microscopic Viewpoint of the Single-Loop DC Motor

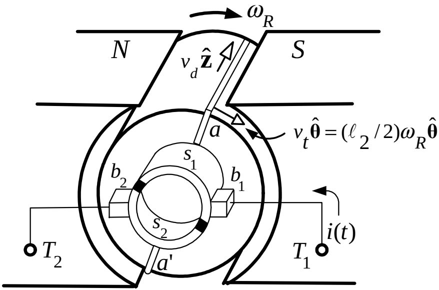


- Let the loop rotate at angular speed $\omega_R > 0$.
- The velocity $\vec{\mathbf{v}}$ of the **charge carriers** that make up the current is given by

$$\vec{\mathbf{v}} = \begin{cases} v_t\hat{\boldsymbol{\theta}} + v_d\hat{\mathbf{z}} & \text{for side } a \\ v_t\hat{\boldsymbol{\theta}} - v_d\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$$

- v_d is the **drift speed** of the charge carriers along the wire.
- $v_t = (\ell_2/2)\omega_R$ is the **tangential velocity** due to the rotating loop.

Microscopic Viewpoint of the Single-Loop DC Motor



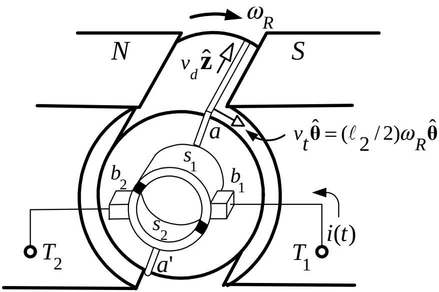
- The drift speed has the **same** sign as the current, that is, $v_d > 0$ for $i > 0$.
- The rotor has **angular velocity** $\vec{\omega}_R = \omega_R \hat{\mathbf{z}}$ where $\hat{\mathbf{z}}$ is the axis of rotation.
- The **magnetic force per unit charge** is $\vec{\mathbf{F}}_{\text{magnetic}}/q = \vec{\mathbf{v}} \times \vec{\mathbf{B}}$ where

$$\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \begin{cases} (v_t\hat{\boldsymbol{\theta}} + v_d\hat{\mathbf{z}}) \times (+B)\hat{\mathbf{r}} = -v_t B\hat{\mathbf{z}} + v_d B\hat{\boldsymbol{\theta}} & \text{for side } a \\ (v_t\hat{\boldsymbol{\theta}} - v_d\hat{\mathbf{z}}) \times (-B)\hat{\mathbf{r}} = +v_t B\hat{\mathbf{z}} + v_d B\hat{\boldsymbol{\theta}} & \text{for side } a' \end{cases}$$

or

$$\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \begin{cases} v_d B\hat{\boldsymbol{\theta}} - \omega_R(\ell_2/2)B\hat{\mathbf{z}} & \text{for side } a \\ v_d B\hat{\boldsymbol{\theta}} + \omega_R(\ell_2/2)B\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$$

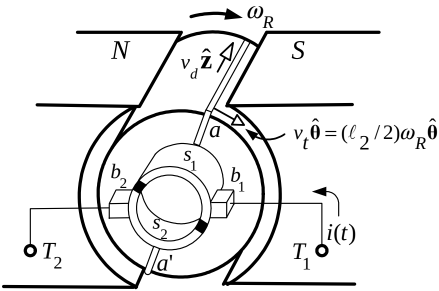
Microscopic Viewpoint of the Single-Loop DC Motor - Torque



$$\frac{\vec{\mathbf{F}}_{\text{magnetic}}}{q} = \vec{\mathbf{v}} \times \vec{\mathbf{B}} = \begin{cases} v_d B\hat{\boldsymbol{\theta}} - \omega_R(\ell_2/2)B\hat{\mathbf{z}} & \text{for side } a \\ v_d B\hat{\boldsymbol{\theta}} + \omega_R(\ell_2/2)B\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$$

- The component $v_d B\hat{\boldsymbol{\theta}}$ produces the torque.
- Let N be the number of charge carriers/unit volume
- Let S be the cross-sectional area of the wire loop.
- $NS\ell_1$ is the total number of charge carriers on each side of the loop.
- The current due to the movement of these charges is $i = qNSv_d$.

Microscopic Viewpoint of the Single-Loop DC Motor - Torque



The total tangential forces on the axial sides of the rotor loop are given by

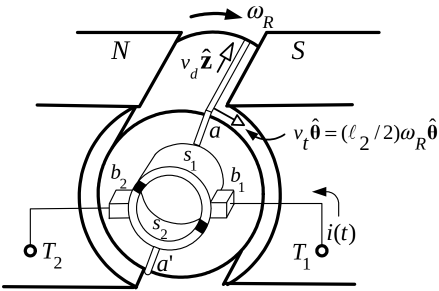
$$
\begin{aligned}
\vec{\mathbf{F}}_{\text{side } a} = (qNS\ell_1)v_d B\hat{\boldsymbol{\theta}} &= i(t)\ell_1 B\hat{\boldsymbol{\theta}} \\
\vec{\mathbf{F}}_{\text{side } a'} = (qNS\ell_1)v_d B\hat{\boldsymbol{\theta}} &= i(t)\ell_1 B\hat{\boldsymbol{\theta}}.
\end{aligned}
$$

The torque is then

$$\vec{\boldsymbol{\tau}} = \frac{\ell_2}{2}\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\text{side } a} + \frac{\ell_2}{2}\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\text{side } a'} = 2\left(\frac{\ell_2}{2}\hat{\mathbf{r}}\right)\times\left(i\ell_1 B\hat{\boldsymbol{\theta}}\right) = i\ell_1\ell_2 B\hat{\mathbf{z}}$$

which is the **same** result as in the macroscopic case.

Microscopic Viewpoint of the Single-Loop DC Motor - Back EMF



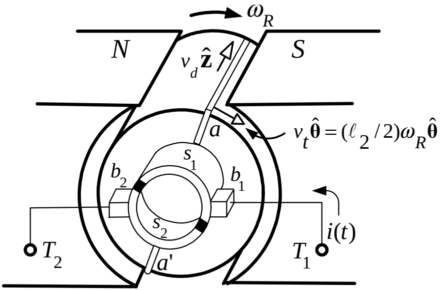
The $\hat{\mathbf{z}}$ component of $\vec{\mathbf{F}}_{\text{magnetic}}$ is

$$(\vec{\mathbf{F}}_{\text{magnetic}}/q)_z\hat{\mathbf{z}} = \begin{cases} -\omega_R(\ell_2/2)B\hat{\mathbf{z}} & \text{for side } a \\ +\omega_R(\ell_2/2)B\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$$

$(\vec{\mathbf{F}}_{\text{magnetic}}/q)_z$ opposes the electric field $\vec{\mathbf{E}}_S$ set up in the loop by V_S.
The relationship between V_S and $\vec{\mathbf{E}}_S$ is

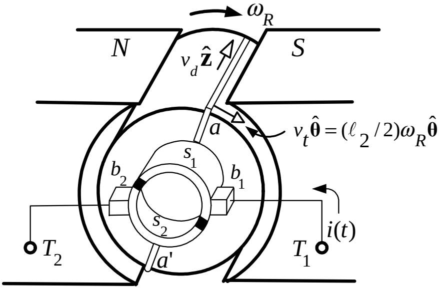
$$V_S = \int_{T_1}^{T_2} \vec{\mathbf{E}}_S \cdot d\vec{\boldsymbol{\ell}}, \quad d\vec{\boldsymbol{\ell}} = \begin{cases} +d\ell\hat{\mathbf{z}} & \text{for side } a \\ -d\ell\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$$

Microscopic Viewpoint of the Single-Loop DC Motor - Back EMF



$$
\begin{aligned}
\xi &\triangleq \int_{T_1}^{T_2} (\vec{\mathbf{F}}_{\text{magnetic}}/q)\cdot d\vec{\boldsymbol{\ell}} \\
&= \int_{\text{side } a} (-\omega_R(\ell_2/2)B\hat{\mathbf{z}})\cdot(d\ell\hat{\mathbf{z}}) + \int_{\text{side } a'} (\omega_R(\ell_2/2)B\hat{\mathbf{z}})\cdot(-d\ell\hat{\mathbf{z}}) \\
&= \int_{\ell=0}^{\ell=\ell_1} -\omega_R(\ell_2/2)Bd\ell + \int_{\ell=0}^{\ell=\ell_1} -\omega_R(\ell_2/2)Bd\ell \\
&= -\omega_R(\ell_2/2)B\ell_1 - \omega_R(\ell_2/2)B\ell_1 \\
&= -\ell_1\ell_2 B\omega_R.
\end{aligned}
$$

Microscopic Viewpoint of the Single-Loop DC Motor - Back EMF



$$\xi \triangleq \int_{T_1}^{T_2} (\vec{\mathbf{F}}_{\text{magnetic}}/q) \cdot d\vec{\boldsymbol{\ell}} \; = -\ell_1 \ell_2 B \omega_R.$$

- In the single-loop motor, the (back) emf ξ is due to the **magnetic force**.
- V_S is due to the **electric field** set up by the voltage source.
- The induced emf ξ and V_S have the **same** sign convention.
- The minus sign in the expression for ξ shows that it opposes V_S.
- The equation governing the current in the rotor loop is

$$V_S - \ell_1 \ell_2 B \omega_R - L\frac{di}{dt} = Ri \quad \text{or} \quad L\frac{di}{dt} = -Ri - \ell_1 \ell_2 B \omega_R + V_S$$