ECE 697 Modeling and High-Performance Control of Electric Machines
HW 3 Solutions
Spring 2022

Problem 16 *A Two-Phase Four-Pole Generator*
(a) Using an outward normal for the flux surface ($\hat{\mathbf{n}} = \hat{\mathbf{r}}$), the flux linkage in stator loop $a - a'$ due to the rotor's magnetic field is

$$\begin{aligned}
\lambda_a &= \int_0^{\ell_1}\int_{-\pi/2}^{0} B_{R\max}\cos\left(n_p(\theta-\theta_R)\right)\hat{\mathbf{r}}\cdot\left(r_S d\theta d\ell\hat{\mathbf{r}}\right) + \int_0^{\ell_1}\int_{\pi/2}^{\pi} B_{R\max}\cos\left(n_p(\theta-\theta_R)\right)\hat{\mathbf{r}}\cdot\left(r_S d\theta d\ell\hat{\mathbf{r}}\right)\\
&= \frac{r_S\ell_1 B_{R\max}}{n_p}\sin\left(n_p(\theta-\theta_R)\right)\Big|_{-\pi/2}^{0} + \frac{r_S\ell_1 B_{R\max}}{n_p}\sin\left(n_p(\theta-\theta_R)\right)\Big|_{\pi/2}^{\pi}\\
&= \frac{r_S\ell_1 B_{R\max}}{n_p}\Big(-\sin\left(n_p\theta_R\right)+\sin\left(n_p\left(\pi/2+\theta_R\right)\right)+\sin\left(n_p\left(\pi-\theta_R\right)\right)-\sin\left(n_p\left(\pi/2-\theta_R\right)\right)\Big)\\
&= \frac{r_S\ell_1 B_{R\max}}{2}\Big(-\sin\left(2\theta_R\right)+\sin\left(2\left(\pi/2+\theta_R\right)\right)+\sin\left(2\left(\pi-\theta_R\right)\right)-\sin\left(2\left(\pi/2-\theta_R\right)\right)\Big)\\
&= \frac{r_S\ell_1 B_{R\max}}{2}\left(-4\sin(2\theta_R)\right)\\
&= -2r_S\ell_1 B_{R\max}\sin(2\theta_R).
\end{aligned}$$

(b) Does the positive direction of travel around each of the two flux surfaces of phase a coincide with the positive direction of current? **Yes**
(c)

$$\xi_{a-a'} = -\frac{d\lambda_a}{dt} = 4r_S\ell_1 B_{R\max}\omega_R\cos(2\theta_R) = 4r_S\ell_1 B_{R\max}\omega_R\cos(2\omega_R t)$$

(d) Similarly ($n_p = 2$)

$$\begin{aligned}
\lambda_b(\theta_R) &= \int_0^{\ell_1}\int_{-\pi/4}^{+\pi/4} B_{R\max}\cos\left(n_p(\theta-\theta_R)\right)\hat{\mathbf{r}}\cdot\left(r_S d\theta d\ell\hat{\mathbf{r}}\right) + \int_0^{\ell_1}\int_{3\pi/4}^{5\pi/4} B_{R\max}\cos\left(n_p(\theta-\theta_R)\right)\hat{\mathbf{r}}\cdot\left(r_S d\theta d\ell\hat{\mathbf{r}}\right)\\
&= \frac{r_S\ell_1 B_{R\max}}{n_p}\sin\left(n_p(\theta-\theta_R)\right)\Big|_{-\pi/4}^{+\pi/4} + \frac{r_S\ell_1 B_{R\max}}{n_p}\sin\left(n_p(\theta-\theta_R)\right)\Big|_{3\pi/4}^{5\pi/4}\\
&= \frac{r_S\ell_1 B_{R\max}}{n_p}\Big(\sin\left(2(\pi/4)-2\theta_R\right)\big)-\sin(2\left(-\pi/4\right)-2\theta_R))+\sin(2\left(5\pi/4\right)-2\theta_R)-\sin(2\left(3\pi/4\right)-2\theta_R)\Big)\\
&= \frac{r_S\ell_1 B_{R\max}}{2}\Big(\cos\left(2\theta_R\right)+\cos(2\theta_R)+\cos\left(2\theta_R\right)+\cos(2\theta_R)\Big)\\
&= \frac{r_S\ell_1 B_{R\max}}{2}\left(4\cos(2\theta_R)\right)\\
&= 2r_S\ell_1 B_{R\max}\cos(2\theta_R).
\end{aligned}$$

Note that

$$\lambda_b(\theta_R) = \lambda_a(\theta_R-\pi/4) = -2r_S\ell_1 B_{R\max}\sin(2\left(\theta_R-\pi/4\right)) = 2r_S\ell_1 B_{R\max}\cos(2\theta_R).$$

The induced voltage in phase $b - b'$ is

$$\xi_{b-b'} = -d\lambda_b/dt = 4r_S\ell_1 B_{R\max}\omega_R\sin(2\theta_R) = 4r_S\ell_1 B_{R\max}\omega_R\sin(2\omega_R t).$$

(e) With the electric field given by

$$\vec{\mathbf{E}}(\theta-\theta_R) = \omega_R B_{R\max} r_S\cos\left(n_p(\theta-\theta_R)\right)\hat{\mathbf{z}}$$

the voltage in phase $a-a'$ is computed as

$$
\begin{aligned}
\xi_{a-a'} &= \int_{a'}^{a} \vec{\mathbf{E}}(\theta-\theta_R)\cdot d\vec{\boldsymbol{\ell}} \\
&= \int_{side\ a_1'} \left(\omega_R B_{R\,\max} r_S \cos\left(n_p\left(-\pi/2-\theta_R\right)\right)\hat{\mathbf{z}}\right)\cdot\left(-d\ell\hat{\mathbf{z}}\right) + \int_{side\ a_1} \left(\omega_R B_{R\,\max} r_S \cos\left(n_p\left(0-\theta_R\right)\right)\hat{\mathbf{z}}\right)\cdot\left(d\ell\hat{\mathbf{z}}\right) \\
&\quad + \int_{side\ a_2'} \left(\omega_R B_{R\,\max} r_S \cos\left(n_p\left(\pi/2-\theta_R\right)\right)\hat{\mathbf{z}}\right)\cdot\left(-d\ell\hat{\mathbf{z}}\right) + \int_{side\ a_2} \left(\omega_R B_{R\,\max} r_S \cos\left(n_p\left(\pi-\theta_R\right)\right)\hat{\mathbf{z}}\right)\cdot\left(d\ell\hat{\mathbf{z}}\right) \\
&= \omega_R B_{R\,\max} r_S \ell_1 \Big(-\cos\left(2\left(-\pi/2-\theta_R\right)\right)+\cos\left(2\theta_R\right)-\cos\left(2\left(\pi/2-\theta_R\right)\right)+\cos\left(2\left(\pi-\theta_R\right)\right)\Big) \\
&= \omega_R B_{R\,\max} r_S \ell_1 \Big(-\cos\left(\pi+2\theta_R\right)+\cos\left(2\theta_R\right)-\cos\left(\pi-2\theta_R\right)+\cos\left(2\theta_R\right)\Big) \\
&= 4 r_S \ell_1 B_{R\,\max}\omega_R \cos\left(2\theta_R\right) \\
&= 4 r_S \ell_1 B_{R\,\max}\omega_R \cos\left(2\omega_R t\right)
\end{aligned}
$$

which is the same result as using Faraday's law in part (c).