ECE 697 Modeling and High-Performance Control of Electric Machines
HW 7 Solutions
Spring 2022

Problem 1 *Solutions to Linear Time-Invariant Systems*

The Physics of the Induction Machine

Problem 2 *Torque Versus Slip for the induction motor*

Assuming the rotor loops to have an inductance L_R, the equations describing the current dynamics are $(\theta_S - \theta_R = (\omega_S - \omega_R)\, t\ )$

$$\begin{aligned} L_R\frac{di_{Ra}}{dt} + R_R i_{Ra} = \xi_{Ra}, \quad \xi_{Ra} &= +\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}(\omega_S - \omega_R)\sin\left((\omega_S - \omega_R)t\right) \\ L_R\frac{di_{Rb}}{dt} + R_R i_{Rb} = \xi_{Rb}, \quad \xi_{Rb} &= -\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}(\omega_S - \omega_R)\cos\left((\omega_S - \omega_R)t\right). \end{aligned}$$

The *stable linear time-invariant* system

$$L\frac{di}{dt} + Ri = A\cos(\omega t + \phi)$$

has the steady-state solution

$$i_{SS}(t) = |G(j\omega)|\, A\cos\left(\omega t + \phi + \angle G(j\omega)\right) \quad \text{where} \quad G(j\omega) = \frac{1}{R + j\omega L}.$$

The *steady-state* solution for the currents are then

$$\begin{aligned} i_{RaSS} &= \frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}}\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\sin\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right) \\ i_{RbSS} &= -\frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}}\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\cos\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right). \end{aligned}$$

The total torque on rotor phase a is given by

$$\begin{aligned} \tau_{Ra} &= \frac{\mu_0\ell_1\ell_2 N_S I_S}{2g} i_{Ra}\sin\Big((\omega_S - \omega_R)t\Big) \\ &= \frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}}\left(\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\right)^2 \sin\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right)\sin\Big((\omega_S - \omega_R)t\Big). \end{aligned}$$

Similarly, the total torque on rotor phase b is then

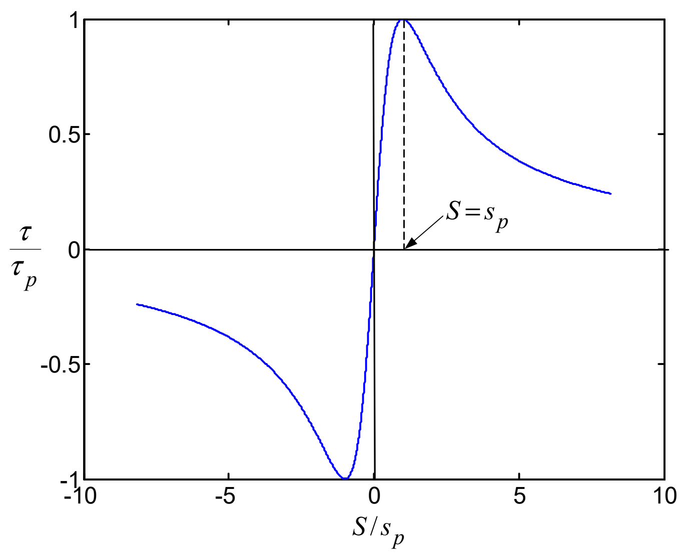
$$\begin{aligned} \tau_{Rb} &= -\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g} i_{Rb}\cos\Big(\ (\omega_S - \omega_R)t\Big) \\ &= \frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}}\left(\frac{\mu_0\ell_1\ell_2 N_S I_S}{2g}\right)^2 \cos\left((\omega_S - \omega_R)t - \tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right)\cos\Big(\ (\omega_S - \omega_R)t\Big). \end{aligned}$$

Combining the above results, the *total torque* is given by

$$
\begin{aligned}
\tau = \tau_{Ra} + \tau_{Rb} &= \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} \cos\left(-\tan^{-1}\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)\right) \\
&= \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{(\omega_S - \omega_R)}{\sqrt{R_R^2 + (\omega_S - \omega_R)^2 L_R^2}} \frac{1}{\sqrt{\left(\frac{(\omega_S - \omega_R)L_R}{R_R}\right)^2 + 1}} \\
&= \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{1}{L_R} \frac{(\omega_S - \omega_R)L_R/R_R}{((\omega_S - \omega_R)L_R/R_R)^2 + 1} \\
&= \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{1}{L_R} \frac{1}{2} \frac{2}{S/s_p + s_p/S}
\end{aligned}
$$

where

$$
\begin{aligned}
S &\triangleq \frac{\omega_S - \omega_R}{\omega_S} \\
s_p &\triangleq \frac{R_R}{\omega_S L_R}.
\end{aligned}
$$



Plot of the normalized torque $\dfrac{2}{S/s_p + s_p/S}$ versus S/s_p

The quantity $S \triangleq \dfrac{\omega_S - \omega_R}{\omega_S}$ is called the *normalized slip* and $S_p \triangleq \dfrac{R_R}{\sigma \omega_S L_R}$ is the *pull out slip* so that $s_p \triangleq \dfrac{R_R}{\omega_S L_R} = \sigma S_p$ where σ is the so-called leakage factor.

Problem 3 *Induction Motor Under Load*

In problem 2 it is shown that the torque produced by a two-phase induction motor with two rotor loops $\pi/2$ radians apart and non zero rotor inductance L_R is given by

$$
\tau = \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{1}{2L_R} \frac{2}{s_p/S + S/s_p} = \tau_p \frac{2}{s_p/S + S/s_p} \tag{1}
$$

where

$$S \triangleq \frac{\omega_S - \omega_R}{\omega_S}, \tau_p \triangleq \left(\frac{\mu_0 \ell_1 \ell_2 N_S I_S}{2g}\right)^2 \frac{1}{2L_R}, \text{ and } s_p \triangleq \frac{R_R}{\omega_S L_R}.$$

A plot of τ/τ_p versus S/s_p is given in Figure 1. Of course, an induction motor has more than two rotor loops. The squirrel cage rotor for an induction motor is shown in Figure 2. The cross sectional view of the rotor is shown in Figure 2(b) which shows that there are 6 rotor loops (12 sides) which can be viewed as three sets of 2 rotor loops. Each set consists of two rotor loops $\pi/2$ radians apart.

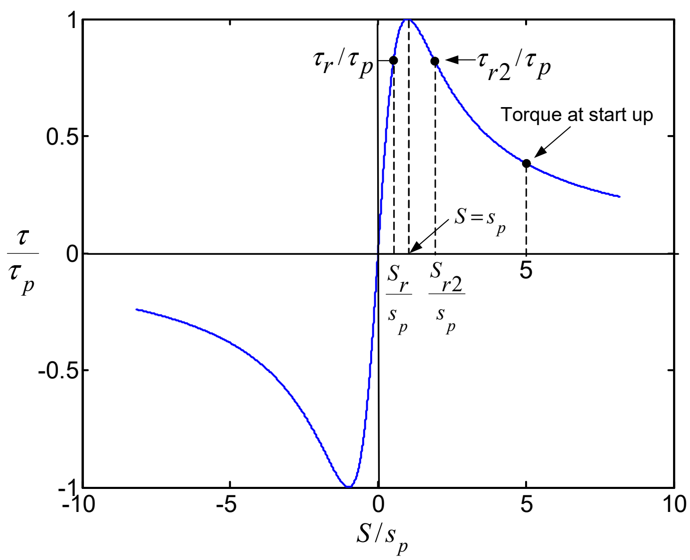


Figure 1: Torque versus slip.

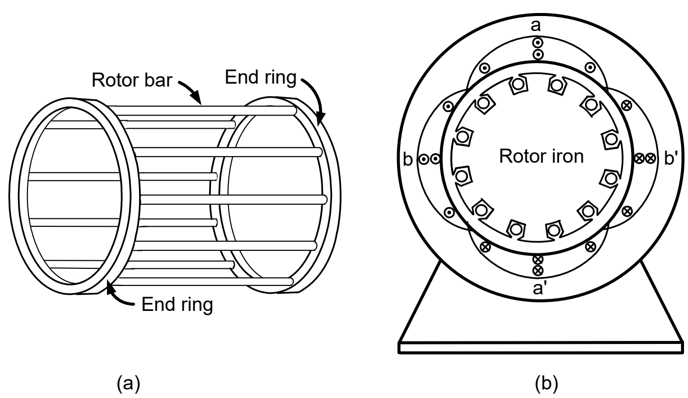


Figure 2: (a) Squirrel cage rotor for an induction motor. (b) Cross section.

(a) For the rotor with two loops $\pi/2$ radians apart, the torque is given by (1). The rotor of Figure 2(b) has three sets of such loops so the expression (1) must be multiplied by 3.

(b) Suppose the induction motor has a load on it and is producing the torque τ_r so that it is operating at the point $(S_r/s_p, \tau_r/\tau_p)$ shown in Figure 1. Further, suppose an additional load is put on the motor so that the total load torque τ_L now satisfies $\tau_r < \tau_L < \tau_p$. After the additional load is put on the motor:

Will the speed increase or decrease?

With more load torque on the motor, the motor will slow down, i.e., the speed decreases.

Will the normalized slip increase or decrease?

$S = (\omega_S - \omega_R)/\omega_S$ and, as ω_R decreases due to the increased load, the normalized slip S increases.

Will the motor torque increase to handle the increased load?

The motor was originally operating at $(S_r/s_p, \tau_r/\tau_p)$. S increases and thus the torque increases to handle the increased load (See Figure 1).

(c) Repeat part (b), but with the motor operating at $(S_{r2}/s_p, \tau_{r2}/\tau_p)$.

With more load torque on the motor, the motor will slow down, i.e., the speed decreases. As ω_R decreases due to the increased load, the normalized slip $S = (\omega_S - \omega_R)/\omega_S$ increases. As S increases the torque decreases showing that the motor cannot handle the increased load (See Figure 1).

(d) Suppose the induction motor is turned off (no currents applied to the stator phases) so that $\omega_R = 0$, but it has a load torque $\tau_L = \tau_r$ on it (the same τ_r as in Figure 1). Let $s_p = 0.2$. Now stator currents of frequency ω_S are applied to the stator phases.

What is the value of S/s_p?

$$\frac{S}{s_p} = \frac{\dfrac{\omega_S - \omega_R}{\omega_S}}{s_p} = \frac{\dfrac{\omega_S}{\omega_S}}{s_p} = \frac{1}{s_p} = \frac{1}{0.2} = 5.$$

Mark on Figure 1 the operating point of the motor.

Can the motor start with the load torque $\tau_L = \tau_r$ on it? Explain.

Just after applying the stator currents ω_R is still zero and so The startup torque is marked on Figure 1. As the startup torque is less than the load torque on the motor, the motor will not start.

Problem 4 *Torque*

On the two semicircular ends of the loop, $d\vec{\boldsymbol{\ell}} = r_R d\theta\hat{\boldsymbol{\theta}}$ for one side of the loop and $d\vec{\boldsymbol{\ell}} = -rd\theta\hat{\boldsymbol{\theta}}$ for the other side of the loop. As $\vec{\mathbf{B}}_S = B_S\hat{\mathbf{r}}$, $d\vec{\mathbf{F}} = i_{Ra}d\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}_S = \pm i_{Ra}r_R B_S\hat{\mathbf{z}}$ where the $+$ sign is for one end and the $-$ sign is for the other end. These two forces cancel each other and further neither one can produce a torque about the axis of rotation.

Problem 5 *Simple Induction Machine with Three Phases*

(a) Using a trigonometric identity,

$$\begin{aligned}
\vec{\mathbf{B}}_S &= \vec{\mathbf{B}}_{S1} + \vec{\mathbf{B}}_{S2} + \vec{\mathbf{B}}_{S3} \\
&= \frac{\mu_0 \ell_2 N_S I_S}{4gr}\left(\cos(\omega_S t)\cos(\theta) + \cos(\omega_S t - \frac{2\pi}{3})\cos(\theta - \frac{2\pi}{3}) + \cos(\omega_S t - \frac{4\pi}{3})\cos(\theta - \frac{4\pi}{3})\right)\hat{\mathbf{r}} \\
&= \frac{\mu_0 \ell_2 N_S I_S}{4gr}\frac{3}{2}\cos(\theta - \omega_S t)\hat{\mathbf{r}}.
\end{aligned}$$

(b) Yes.

(c) The only change is the factor 3/2. One need only replace I_S by $(3/2)I_S$ in the analysis of the two-phase machine in the text to obtain the expressions for this three-phase machine.

Problem 6 *A PM-Generator/Induction-Motor Machine.*

Synchronous Machine

Problem 7 *Magnetic Field of a Sinusoidally Wound Rotor*

Applying Ampère's law $\oint \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} = i_{\text{enclosed}}$ to the closed-path 1−2−3−4−1 shown in Figure 3.

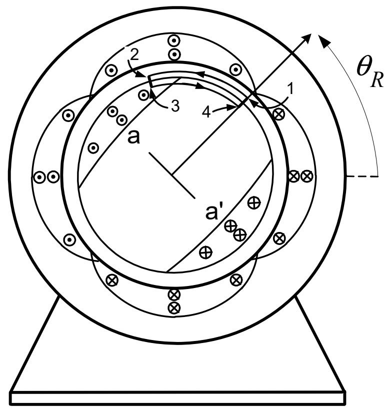


Figure 3: Sinusoidally Wound Rotor Phase

gives

$$
\begin{aligned}
\int_4^1 \vec{\mathbf{H}}_R \cdot d\vec{\boldsymbol{\ell}} + \int_1^2 \vec{\mathbf{H}}_R \cdot d\vec{\boldsymbol{\ell}} + \int_2^3 \vec{\mathbf{H}}_R \cdot d\vec{\boldsymbol{\ell}} + \int_3^4 \vec{\mathbf{H}}_R \cdot d\vec{\boldsymbol{\ell}} &= \int_{\theta_R}^{\theta} i_F (N_F/2) \sin(\theta' - \theta_R) d\theta' \\
\int_4^1 \vec{\mathbf{H}}_R \cdot d\vec{\boldsymbol{\ell}} + \int_2^3 \vec{\mathbf{H}}_R \cdot d\vec{\boldsymbol{\ell}} &= -i_F \frac{N_F}{2} \left(\cos(\theta - \theta_R) - \cos(0)\right) \\
\int_4^1 H_R(\theta_R)\hat{\mathbf{r}} \cdot d\ell\hat{\mathbf{r}} + \int_2^3 H_R(\theta)\hat{\mathbf{r}} \cdot (-d\ell\hat{\mathbf{r}}) &= -i_F \frac{N_F}{2} \cos(\theta - \theta_R) + i_F \frac{N_R}{2} \\
gH_R(\theta_R) - gH_R(\theta) &= -i_F \frac{N_F}{2} \cos(\theta - \theta_R) + i_F \frac{N_R}{2} \\
H_R(\theta) &= i_F \frac{N_F}{2g} \cos(\theta - \theta_R) + H_{Sa}(\theta_R) - i_F \frac{N_R}{2g}.
\end{aligned}
$$

Applying Gauss's law to a closed cylindrical surface in the air gap that encloses the rotor shows that $H_R(\theta_R) - i_F N_F/2g = 0$ so that

$$B_R(\theta) = \mu_0 i_F \frac{N_F}{2g} \cos(\theta - \theta_R).$$

Finally, the factor r_R/r is included in order to satisfy conservation of flux for *any* closed flux surface in the air gap to make the final expression for $B_R(r, \theta - \theta_R)$

$$B_R(r, \theta - \theta_R) = \mu_0 i_F \frac{N_F}{2g} \frac{r_R}{r} \cos(\theta - \theta_R).$$