Modeling and High-Performance Control of Electric Machines

Chapter 9 PM Synchronous Motor Control

John Chiasson

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PM Synchronous Motor Control

- **Field-Oriented Control**
- **Optimal Field Weakening*** (no slides)
- **Identification of the PM Synchronous Motor Parameters**
- **PM Stepper Motors**
- **Appendix: Motor Parameters from Manufacturer's Data Sheet**

Field-Oriented Control

PM Model

$$
\begin{aligned}
L_S \frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m \sin(n_p\theta)\omega + u_{Sa} \\
L_S \frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m \cos(n_p\theta)\omega + u_{Sb} \\
J\frac{d\omega}{dt} &= K_m(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)) - \tau_L \\
\frac{d\theta}{dt} &= \omega.
\end{aligned}
$$

Direct Quadrature (DQ) Transformations

$$
\begin{bmatrix} u_d \\ u_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix}
$$

$$
\begin{bmatrix} i_d \\ i_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix}.
$$

- i_d corresponds to the component of $\vec{\mathbf{B}}_S$ **along the axis** of $\vec{\mathbf{B}}_R$ (same as rotor axis).
- i_q corresponds to the component of $\vec{\mathbf{B}}_S$ **orthogonal** to $\vec{\mathbf{B}}_R$.

Direct Quadrature (DQ) Model

$$
\begin{aligned}
L_S \frac{di_d}{dt} &= -R_S i_d + n_p \omega L_S i_q + u_d \\
L_S \frac{di_q}{dt} &= -R_S i_q - n_p \omega L_S i_d - K_m \omega + u_q \\
J \frac{d\omega}{dt} &= K_m i_q - \tau_L \\
\frac{d\theta}{dt} &= \omega
\end{aligned}
$$

- The *dq* currents i_d, i_q vary approximately at the **mechanical** frequency of the motor.
- These variables have bandwidths in the range of $0 - 100$ Hz.
- The original variables $u_{Sa}, u_{Sb}, i_{Sa}, i_{Sb}$ have bandwidths in the range $0 - 5$ kHz!

Direct Quadrature (DQ) Model

$$
\begin{aligned}
L_S \frac{di_d}{dt} &= -R_S i_d + n_p \omega L_S i_q + u_d \\
L_S \frac{di_q}{dt} &= -R_S i_q - n_p \omega L_S i_d - K_m \omega + u_q \\
J \frac{d\omega}{dt} &= K_m i_q - \tau_L \\
\frac{d\theta}{dt} &= \omega
\end{aligned}
$$

Energy Conversion

Multiply the 1^{st} eqn by i_d, the 2^{nd} eqn by i_q, add and rearange to get

$$i_d u_d + i_q u_q = \frac{d}{dt}\left(\frac{1}{2} L_S (i_d^2 + i_q^2)\right) + R_S (i_d^2 + i_q^2) + K_m \omega i_q.$$

Multiply the 3^{rd} eqn by ω and rearrange to get

$$\underbrace{K_m i_q}_{\tau} \omega = \frac{d}{dt}\left(\frac{1}{2} J \omega^2\right) + \tau_L \omega.$$

- The **back-emf** voltage $K_m \omega$ absorbs the electrical power electrical power $K_m \omega i_q$.
- This reappears as the mechanical power $\tau \omega = K_m i_q \omega$.

Brushless DC Motor (BLDC)

$$
\begin{aligned}
L_S \frac{di_d}{dt} &= -R_S i_d + n_p \omega L_S i_q + u_d \\
L_S \frac{di_q}{dt} &= -R_S i_q - n_p \omega L_S i_d - K_m \omega + u_q \\
J \frac{d\omega}{dt} &= K_m i_q - \tau_L \\
\frac{d\theta}{dt} &= \omega
\end{aligned}
$$

- Set $u_d = -K_{Pd}(i_d - 0) - K_{Id}\int_0^t (i_d - 0)dt'$ to force $i_d \to 0$. Then

$$
\begin{aligned}
L_S \frac{di_q}{dt} &= -R_S i_q - K_m \omega + u_q \\
J \frac{d\omega}{dt} &= K_m i_q - \tau_L \\
\frac{d\theta}{dt} &= \omega
\end{aligned}
$$

- These are the equations of a PM DC motor!
- We can use the same control methods as for a DC motor.
- Hence the terminology brushless DC motor or BLDC.

Direct Quadrature (DQ) Model

- Can do field weakening at high speeds by having i_d *not* be zero.
- Choosing u_d and u_q to be of the form

$$\begin{aligned} u_d &= R_S i_d - n_p \omega L_S i_q + w_d \\ u_q &= R_S i_q + n_p \omega L_S i_d + K_m \omega + w_q \end{aligned}$$

results in the **feedback linearized** system

$$\begin{aligned} \frac{di_d}{dt} &= w_d / L_S \\ \frac{di_q}{dt} &= w_q / L_q \\ \frac{d\omega}{dt} &= (K_m / J) i_q - \tau_L / J \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

- This is now two **linear decoupled** systems.
  - A **first-order** system for i_d with input w_d.
  - A **third-order** system for i_q, ω, θ with input w_q.
- **Linear** control techniques can now be used.

Field-Oriented Control

Voltage Constraints

With the motor running at constant speed the voltages have the form

$$\begin{aligned} u_{Sa} &= V\cos(n_p\omega t + \phi) \\ u_{Sb} &= V\sin(n_p\omega t + \phi). \end{aligned}$$

- The voltage (amplifier) constraint is $|u_{Sa}| \leq V_{\max}, |u_{Sb}| \leq V_{\max}$.
- At constant speed this is the same as

$$u_{Sa}^2 + u_{Sb}^2 \leq V_{\max}^2.$$

- We also have $u_d^2 + u_q^2 = u_{Sa}^2 + u_{Sb}^2$.
- In the *dq* system we will model the voltage constraint by

$$u_d^2 + u_q^2 = u_{Sa}^2 + u_{Sb}^2 \leq V_{\max}^2.$$

Field-Oriented Control

Feedback Cancellation and Voltage Constraints

- Recall the feedback cancellation:

$$\begin{aligned} u_d &= R_S i_d - n_p \omega L_S i_q + w_d \\ u_q &= R_S i_q + n_p \omega L_S i_d + K_m \omega + w_q \end{aligned}$$

- This uses **much more** voltage than required.
    - In u_q the **back-emf** voltage $K_m\omega$ is large at high speeds.
    - Force i_d **negative** so that in u_q we have $n_p \omega L_S i_d + K_m \omega < K_m \omega$
- Design a reference for i_d as a function of ω to partially cancel the back-emf.
- More generally design references for i_d, i_q, ω, θ and u_d, u_q.

Design of the Reference Trajectory and Inputs

Quadrature Current Reference

Let $\theta_{ref}, \omega_{ref} = d\theta_{ref}/dt, \alpha_{ref} = d\alpha_{ref}/dt$ be reference position, speed and acceleration. With $\tau_L = -f\omega$ the quadrature reference current is

$$i_{qref} = \frac{J}{K_m}\alpha_{ref} + \frac{f}{K_m}\omega_{ref}.$$

Direct Current Reference

- The voltage constraint $u_d^2 + u_q^2 = u_{Sa}^2 + u_{Sb}^2 \leq V_{\max}^2$.
- Determine i_{dref} by maximizing the torque $K_m i_q$ subject to $u_d^2 + u_q^2 \leq V_{\max}^2$.
- To obtain a tractable solution the optimization is carried out at **constant** speed.
- **Maximize** $K_m i_q$ subject to
$$u_d^2 + u_q^2 \leq V_{\max}^2.$$
Hint: To solve this get i_q in terms of u_d, u_q using
$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} = \begin{bmatrix} R_S & -n_p\omega L_S \\ n_p\omega L_S & R_S \end{bmatrix}\begin{bmatrix} i_d \\ i_q \end{bmatrix} + \begin{bmatrix} 0 \\ K_m\omega \end{bmatrix}.$$
The solution is
$$u_{qref}(\omega) = \frac{R_S}{\sqrt{R_S^2 + (n_p\omega L_S)^2}}V_{\max}, \quad \frac{u_{dref}(\omega)}{u_{qref}(\omega)} = -\frac{n_p\omega L_S}{R_S}, \quad i_{dref}(\omega) = -\frac{n_p L_S K_m \omega^2}{R_S^2 + (n_p\omega L_S)^2}.$$
- The inverse tangent of $-\dfrac{n_p\omega L_S}{R_S}$ is referred to as the optimal lead angle

Design of the Reference Trajectory and Inputs

Full Reference Trajectory

- **Specify** the mechanical reference trajectory $\theta_{ref}, \omega_{ref}, \alpha_{ref}$.
- $i_{qref} = \dfrac{J}{K_m}\alpha_{ref} + \dfrac{f}{K_m}\omega_{ref}$.
- **Redefine** i_{dref} as $i_{dref}(\omega)|_{\omega=\omega_{ref}} = -\dfrac{n_p L_S K_m^2 \omega_{ref}^2}{R_S^2 + (n_p \omega_{ref} L_S)^2}$.

  Then

$$
\begin{aligned}
\frac{di_{dref}}{dt} &= -\frac{d}{dt}\frac{n_p L_S K_m^2 \omega_{ref}^2}{R_S^2 + (n_p \omega_{ref} L_S)^2} = \frac{-2K_m n_p \omega_{ref} L_S R_S^2}{(R_S^2 + (n_p \omega_{ref} L_S)^2)^2}\alpha_{ref} \\
\frac{di_{qref}}{dt} &= \frac{d}{dt}\left(\frac{J}{K_m}\alpha_{ref} + \frac{f}{K_m}\omega_{ref}\right) = \frac{J}{K_m}\frac{d\alpha_{ref}}{dt} + \frac{f}{K_m}\alpha_{ref}.
\end{aligned}
$$

- **Choose** u_{dref}, u_{qref} to satisfy

$$
\begin{aligned}
L_S\frac{di_{dref}}{dt} &= -R_S i_{dref} + n_p \omega_{ref} L_S i_{qref} + u_{dref} \\
L_S\frac{di_{qref}}{dt} &= -R_S i_{qref} - n_p \omega_{ref} L_S i_{dref} - K_m \omega_{ref} + u_{qref} \\
J\frac{d\omega_{ref}}{dt} &= K_m i_{qref} - f\omega_{ref} \\
\frac{d\theta_{ref}}{dt} &= \omega_{ref}
\end{aligned}
$$

Error System

We subtract the system model

$$
\begin{aligned}
L_S \frac{di_d}{dt} &= -R_S i_d + n_p \omega L_S i_q + u_d \\
L_S \frac{di_q}{dt} &= -R_S i_q - n_p \omega L_S i_d - K_m \omega + u_q \\
J \frac{d\omega}{dt} &= K_m i_q - \tau_L \\
\frac{d\theta}{dt} &= \omega
\end{aligned}
$$

from the reference model

$$
\begin{aligned}
L_S \frac{di_{dref}}{dt} &= -R_S i_{dref} + n_p \omega_{ref} L_S i_{qref} + u_{dref} \\
L_S \frac{di_{qref}}{dt} &= -R_S i_{qref} - n_p \omega_{ref} L_S i_{dref} - K_m \omega_{ref} + u_{qref} \\
J \frac{d\omega_{ref}}{dt} &= K_m i_{qref} - f \omega_{ref} \\
\frac{d\theta_{ref}}{dt} &= \omega_{ref}.
\end{aligned}
$$

Error System

$$
\begin{aligned}
L_S \frac{d}{dt}(i_{dref} - i_d) &= -R_S(i_{dref} - i_d) + n_p\omega_{ref} L_S i_{qref} - n_p\omega L_S i_q + u_{dref} - u_d \\
L_S \frac{d}{dt}(i_{qref} - i_q) &= -R_S(i_{qref} - i_q) - n_p\omega_{ref} L_S i_{dref} + n_p\omega L_S i_d - K_m(\omega_{ref} - \omega) + u_{qref} - u_q \\
J\frac{d}{dt}(\omega_{ref} - \omega) &= K_m(i_{qref} - i_q) - f(\omega_{ref} - \omega) \\
\frac{d}{dt}(\theta_{ref} - \theta) &= \omega_{ref} - \omega.
\end{aligned}
$$

Use the **feedback linearizing** controller given by

$$
\begin{aligned}
u_d &= -n_p\omega L_S i_q + u_{dref} + n_p\omega_{ref} L_S i_{qref} - v_d \\
u_q &= +n_p\omega L_S i_d + u_{qref} - n_p\omega_{ref} L_S i_{dref} - v_q.
\end{aligned}
$$

The error system becomes the **linear** system

$$
\begin{aligned}
L_S\frac{d}{dt}(i_{dref} - i_d) &= -R_S(i_{dref} - i_d) + v_d \\
L_S\frac{d}{dt}(i_{qref} - i_q) &= -R_S(i_{qref} - i_q) - K_m(\omega_{ref} - \omega) + v_q \\
J\frac{d}{dt}(\omega_{ref} - \omega) &= K_m(i_{qref} - i_q) - f(\omega_{ref} - \omega) \\
\frac{d}{dt}(\theta_{ref} - \theta) &= \omega_{ref} - \omega.
\end{aligned}
$$

Error System

Define the **state tracking error** as

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} \triangleq \begin{bmatrix} i_{dref} - i_d \\ i_{qref} - i_q \\ \omega_{ref} - \omega \\ \theta_{ref} - \theta \\ \int_0^t (\theta_{ref}(\tau) - \theta(\tau)) d\tau \end{bmatrix}$$

- An integrator is added to the controller to reject constant disturbances.

The state error satisfies

$$\frac{de}{dt} = \begin{bmatrix} -R_S/L_S & 0 & 0 & 0 & 0 \\ 0 & -R_S/L_S & -K_m/L_S & 0 & 0 \\ 0 & K_m/J & -f/J & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} e + \begin{bmatrix} 1/L_S & 0 \\ 0 & 1/L_S \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} v, \quad v \triangleq \begin{bmatrix} v_d \\ v_q \end{bmatrix}$$

More compactly,

$$\frac{de}{dt} = Ae + Bv.$$

Error System

The input v is chosen as the **linear state feedback**

$$v = -Ke$$

where K is taken to be of the form

$$K = \begin{bmatrix} k_{11} & 0 & 0 & 0 & 0 \\ 0 & k_{22} & k_{23} & k_{24} & k_{25} \end{bmatrix}.$$

The **closed-loop error system** is then

$$\frac{de}{dt} = (A - BK)e.$$

Use K to place the closed-loop poles of $A - BK$ at any desired location (see Chapter 2).

Speed Observer

- Typically only the current and position measurements are available.
- To estimate ω define an **observer** by

$$\begin{aligned}\frac{d\hat{\theta}}{dt} &= \hat{\omega} + \ell_1(\theta - \hat{\theta}) \\ \frac{d\hat{\omega}}{dt} &= (K_m/J)i_q - (f/J)\hat{\omega} + \ell_2(\theta - \hat{\theta})\end{aligned}$$

for the system

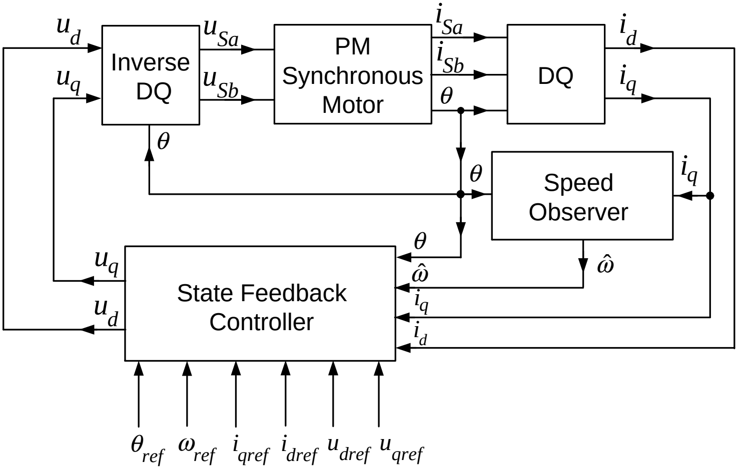
$$\begin{aligned}\frac{d\omega}{dt} &= (K_m/J)i_q - \tau_L/J \\ \frac{d\theta}{dt} &= \omega.\end{aligned}$$

- i_q is a **known input** to the observer as i_{Sa}, i_{Sb}, and θ are measured.
- With $\varepsilon_1 \overset{\Delta}{=} \theta - \hat{\theta}$, $\varepsilon_2 \overset{\Delta}{=} \omega - \hat{\omega}$ we have

$$\begin{aligned}\frac{d\varepsilon_1}{dt} &= \varepsilon_2 - \ell_1\varepsilon_1 \\ \frac{d\varepsilon_2}{dt} &= -(f/J)\varepsilon_2 - \ell_2\varepsilon_1\end{aligned}$$

ℓ_1, ℓ_2 can be chosen so that $\varepsilon_1(t) \to 0, \varepsilon_2(t) \to 0$ at **any** prescribed rate.

State Feedback Trajectory Tracking Controller



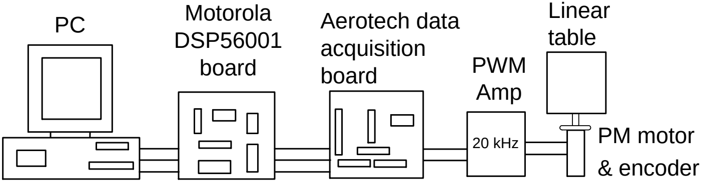
Experimental Results

Two Phase PM motor parameters

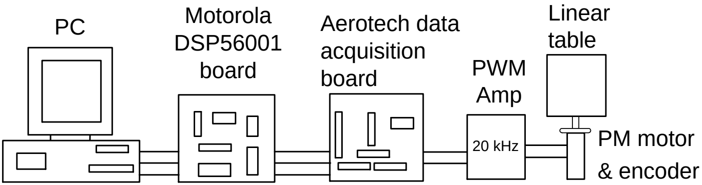
- $R_S = 0.55\ \Omega$, $L_S = 1.5$ mH, $K_m = 0.19$ N-m/A, $J = 4.5 \times 10^{-5}$ kg-m^2.
- $f = 0.0008$ N-m/rads/sec, $n_p = 50$ pole pairs, $I_{\max} = 6.0$ A, $V_{\max} = 40$ V.
- 2000-counts/rev optical encoder.
- Four 8-bit A/D converters for sampling the voltages and currents.
- Two 12-bit D/A converters to output the control voltages.
- The motor reaches speeds up to 3000 rpm which is an electrical frequency of

$$(2\pi/60) \times 3000 \times n_p = 1.57kHz.$$

- A sample rate of 10 kHz ($T = 100\ \mu$sec) was used.
- The observer gains were $\ell_1 = 5272$ and $\ell_2 = 7.0 \times 10^6$ (observer poles at -2646).



Control of a 1.8 millimeter Move

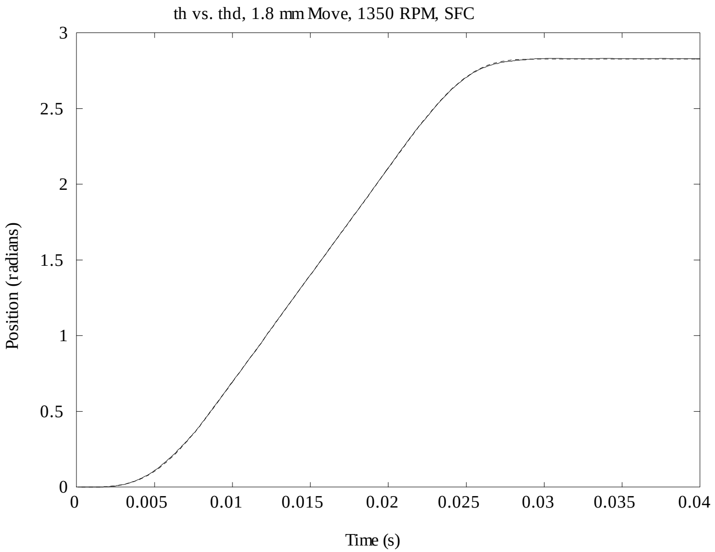


- Have the motor move a small linear positioning table 1.8 mm under 30 msec.
- The motor is attached to the linear stage through a ball screw.
- The motor must turn 0.9π radian for the motor to turn 1.8 millimeters.
- $\theta_{ref}, \omega_{ref} = d\theta_{ref}/dt$ and $\alpha_{ref} = d\omega_{ref}/dt$ are chosen to do this (see Chapter 2 for the trajectory design)
- The feedback gains are

$$K = \begin{bmatrix} 1.06 \times 10^4 & 0 & 0 & 0 & 0 \\ 0 & 1.78 \times 10^4 & 3.40 \times 10^2 & 3.80 \times 10^6 & 1.07 \times 10^7 \end{bmatrix}$$

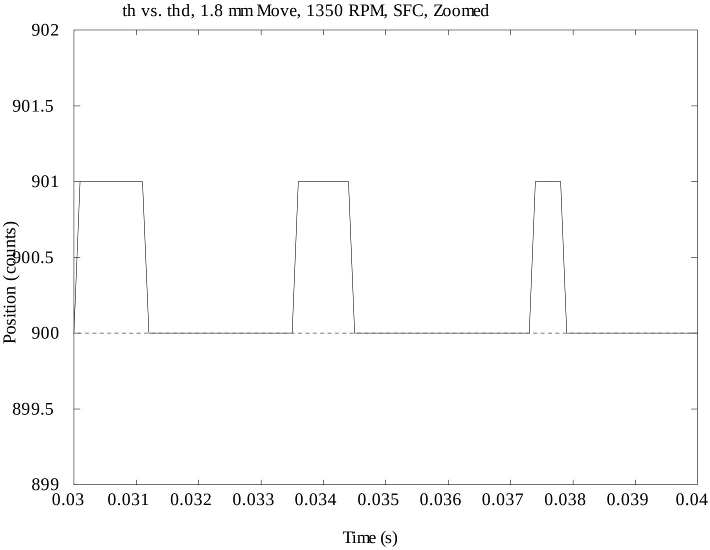
- This places the **closed-loop pole** of the first-order subsystem at -18178.
- It places the fourth-order subsystem **closed-loop poles** at $-11047, -28.3, -54.89 \pm j1190.7$

θ and θ_{ref}



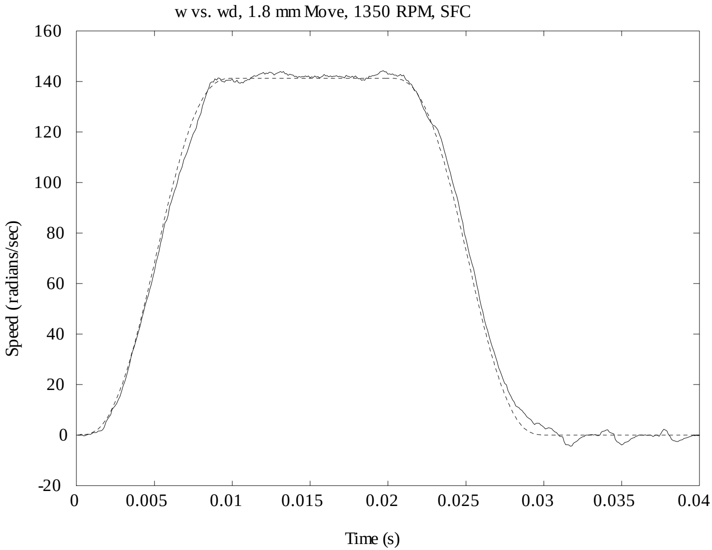
- $\theta_{ref}(t_f) = \theta_{ref}(0.03) = 0.9\pi$ radians.

θ and θ_{ref} in encoder counts for $t \geq 0.03$ sec



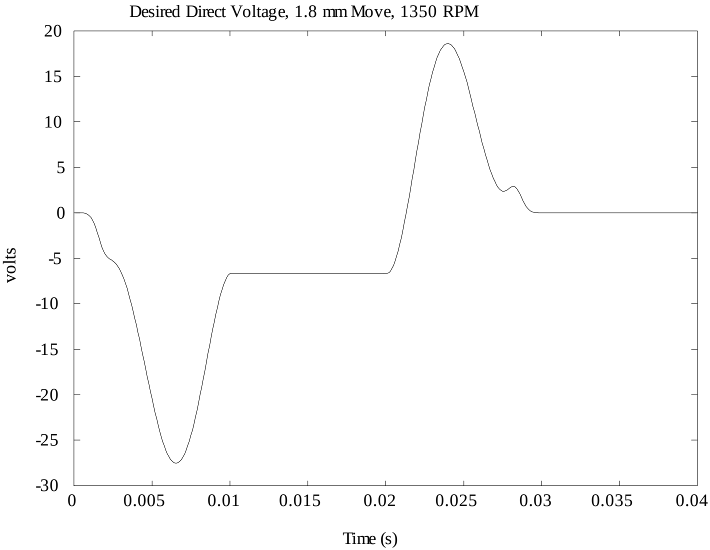
- 0.9π radians $= 0.9\pi \dfrac{2000}{2\pi}$ counts $= 900$ counts.

$\hat{\omega}$ and ω_{ref}



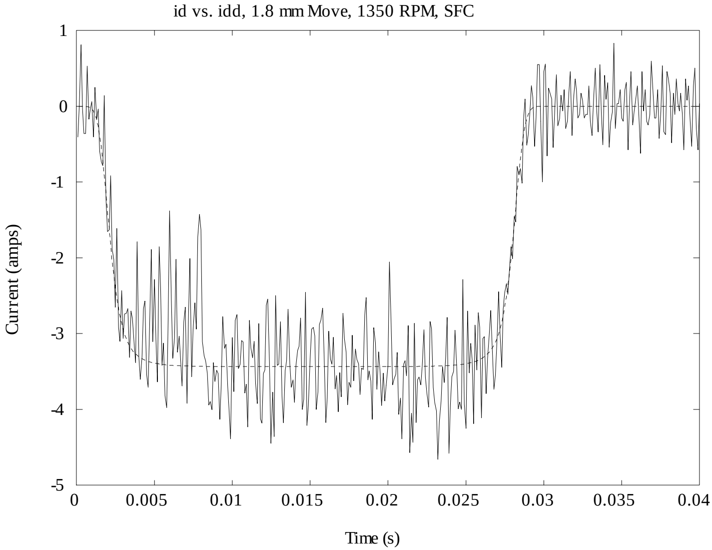
- $\hat{\omega}$ is the speed estimated by the observer.

u_{dref}



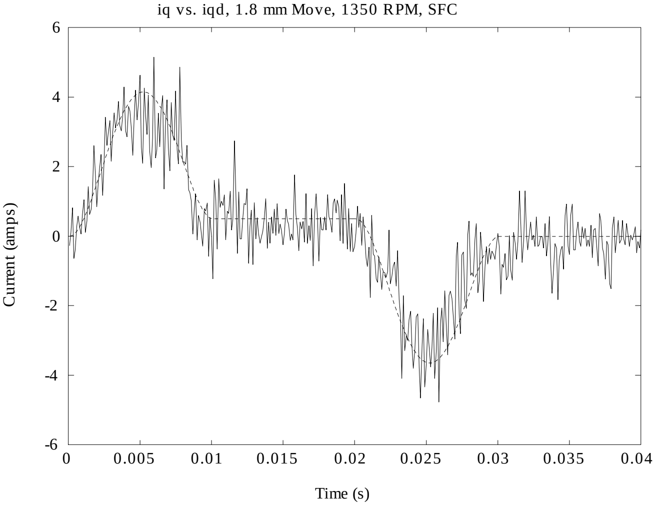
- u_{dref} goes below -25 V leaving less than 15 V for the feedback controller.

i_d and i_{dref}



- Note the **fluctuations/oscillations** in i_d.

i_q and i_{qref}



- Note the **fluctuations/oscillations** in i_q.
- i_{qref} is over 4 A (limit of 6 A) in order to achieve the required torque $K_m i_q$.

Remarks

Fluctuations in i_d and i_q are due to the **limited resolution** of the encoder.

- A simulation using a $2,000$-counts/rev encoder does have fluctuations.
- A simulation using a $50,000$-counts/rev encoder does **not** have fluctuations.

Oscillatory behavior of $\hat{\omega}$

- This is due to integrating the oscillating quadrature current i_q.
- The controller is maintaining the **correct** final position (within one encoder count).

Comparison with a brush PM DC motor

- $K_T = 0.2$ N-m/A ($K_m = 0.19$ N-m/A for the PM synchronous motor).
- $J = 3.6 \times 10^{-4}$ kg-m^2 (the inertia is much larger due to windings on the rotor).
- $V_{\max} = 80$ V, $I_{\max} = 22$ A.
- This DC brush motor is unable to make such a move under 30 msec.
  (Due to the fact that field weakening is not possible.)

Control of a 3000-rpm Run

This experiment brought the motor up to 3000 rpm.

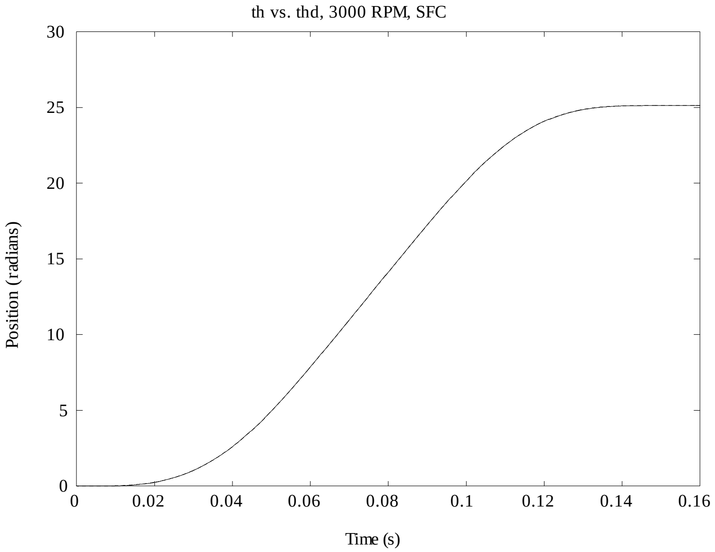
In the implementation, the gains for the 3000-rpm run were set as

$$K = \begin{bmatrix} 2.587\times10^4 & 0 & 0 & 0 & 0 \\ 0 & 2.446\times10^4 & 6.031\times10^2 & 2.935\times10^6 & 1.004\times10^7 \end{bmatrix}$$

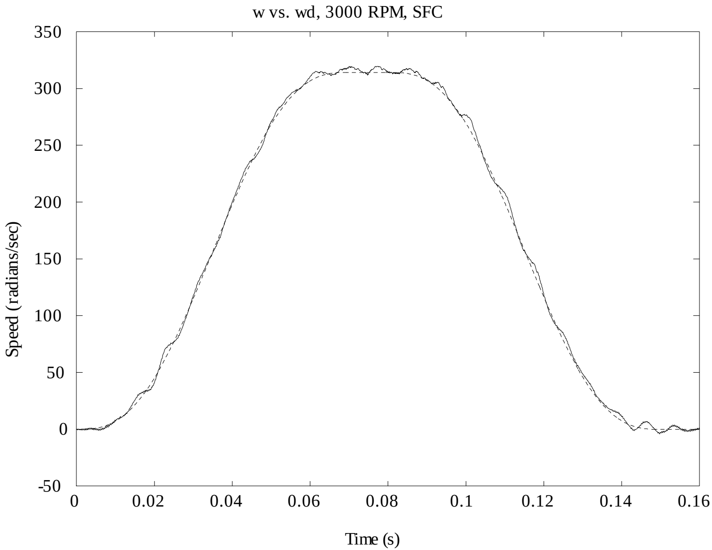
This places the **closed-loop pole** of the first-order subsystem at -26314.

It places the fourth-order subsystem **closed-loop poles** at $-24727, -3.4, -101 \pm j893$.

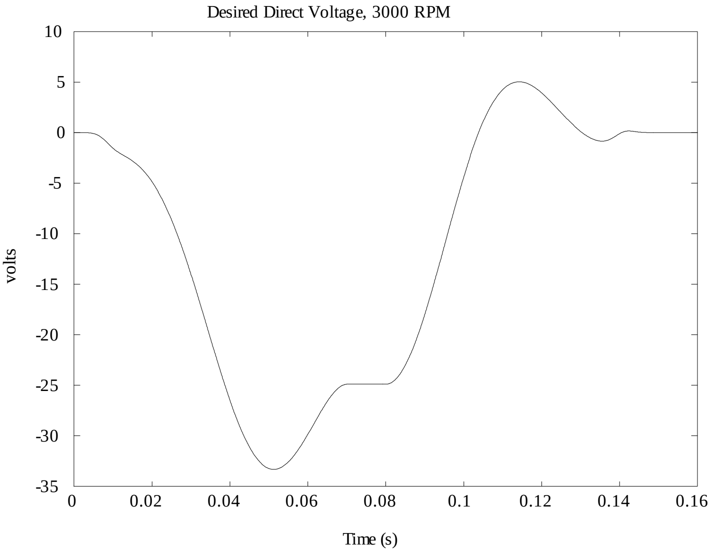
θ and θ_{ref}



$\hat{\omega}$ and ω_{ref}

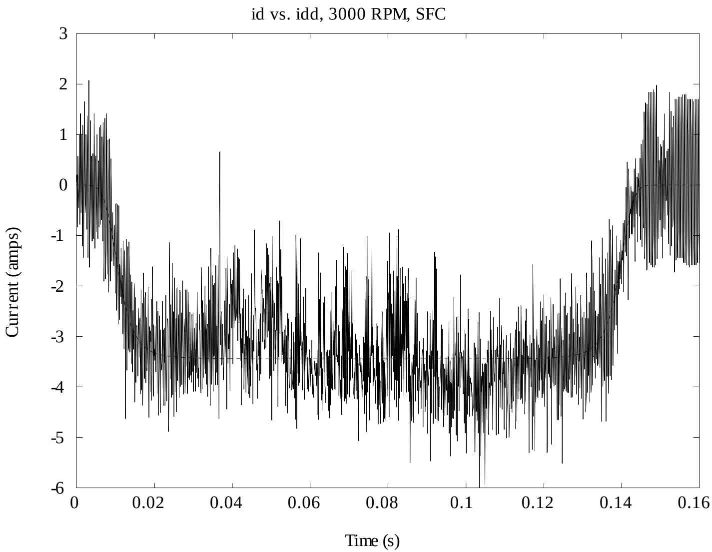


u_{dref}



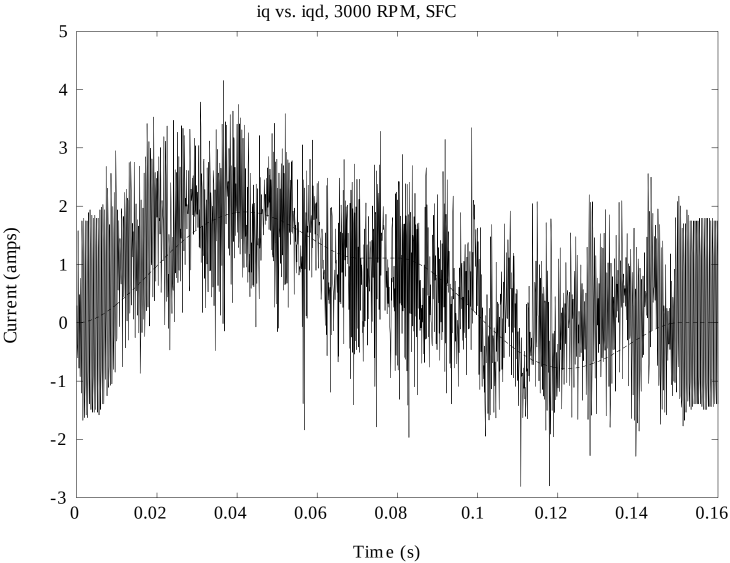
- Note that u_{dref} reaches -30 V leaving less than 10 V for the feedback controller.

i_d and i_{dref}



id vs. idd, 3000 RPM, SFC
Current (amps)
Time (s)
3
2
1
0
-1
-2
-3
-4
-5
-6
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16

i_q and i_{qref}


iq vs. iqd, 3000 RPM, SFC
Current (amps)
Time (s)
5
4
3
2
1
0
-1
-2
-3
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16

Current Command Control

- We have considered a **voltage-controlled** PM synchronous motor.
- As $n_p = 50$ is so large, voltage control was **essential** to obtain high performance.
  - To explain, recall our model as

$$\begin{aligned}
L_S\frac{di_d}{dt} &= -R_S i_d + n_p\omega L_S i_q + u_d \\
L_S\frac{di_q}{dt} &= -R_S i_q - n_p\omega L_S i_d - K_m\omega + u_q \\
J\frac{d\omega}{dt} &= K_m i_q - \tau_L \\
\frac{d\theta}{dt} &= \omega
\end{aligned}$$

  - $n_p\omega L_S$ is an **impedance** to changing i_d, i_q through the inputs u_d, u_q.
  - For a given $V_{\max}$ the larger n_p, the **more time** it takes to build up the current.

Current Command Control

- It is common to find PM synchronous motors with $n_p = 2$ rather than $n_p = 50$.
- In this case, **current control** can be used and high performance can still be obtained.
- A PI current controller has the form

$$u_q = K_p(i_{qref} - i_q) + K_I \int_0^t (i_{qref} - i_q)dt$$
$$u_d = K_p(i_{dref} - i_d) + K_I \int_0^t (i_{dref} - i_q)dt.$$

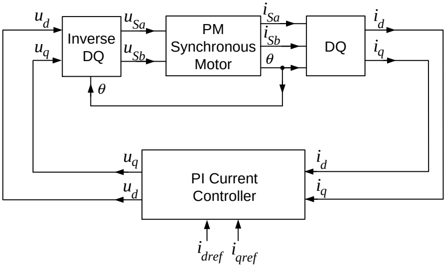
  - Adjusting the gains appropriately quickly forces $i_{qref} \to i_q, i_{dref} \to i_d$.
  - In this case we take $i_q \approx i_{qref}, i_d \approx i_{dref}$.

- We then have the reduced-order system model

$$\begin{aligned} \frac{d\omega}{dt} &= (K_m/J)\, i_{qref} - \tau_L/J \\ \frac{d\theta}{dt} &= \omega \end{aligned}$$

where i_{qref} is now considered as the input.

Current Command Control



- The reduced-order model is identically in form to a PM DC motor.

$$\begin{aligned}\frac{d\omega}{dt} &= (K_m/J)\, i_{qref} - \tau_L/J \\ \frac{d\theta}{dt} &= \omega\end{aligned}$$

- To reject constant disturbance and track $\theta_{ref}, \omega_{ref}$ choose the "input" i_{qref} as

$$i_{qref} = \frac{J}{K_m}\left(K_0\int_0^t(\theta_{ref}-\theta)dt + K_1(\theta_{ref}-\theta) + K_2(\omega_{ref}-\omega) + \alpha_{ref}\right).$$

- Experience shows that this works quite well if the number of pole pairs n_p is small.
- One often chooses $i_{dref} \equiv 0$ if the speeds are not too high.

Identification of the PM Synchronous Motor Parameters

- Model in the dq coordinate system

$$\begin{aligned} L_S \frac{di_d}{dt} &= -R_S i_d + n_p \omega L_S i_q + u_d \\ L_S \frac{di_q}{dt} &= -R_S i_q - n_p \omega L_S i_d - K_m \omega + u_q \\ J \frac{d\omega}{dt} &= K_m i_q - \tau_L . \end{aligned}$$

- For the identification of the motor parameters take

$$\tau_L = f\omega + f_c \operatorname{sgn}(\omega)$$

where

$$\operatorname{sgn}(\omega) = \begin{cases} 1, & \omega > 0 \\ 0, & \omega = 0 \\ -1, & \omega < 0. \end{cases}$$

Identification of the PM Synchronous Motor Parameters

The model

$$\begin{aligned} L_S\frac{di_d}{dt} &= -R_S i_d + n_p\omega L_S i_q + u_d \\ L_S\frac{di_q}{dt} &= -R_S i_q - n_p\omega L_S i_d - K_m\omega + u_q \\ J\frac{d\omega}{dt} &= K_m i_q - f\omega - f_c\mathrm{sgn}(\omega) \end{aligned}$$

is rewritten in regressor form as

$$\underbrace{\begin{bmatrix} u_d(nT) \\ u_q(nT) \\ 0 \end{bmatrix}}_{y(nT)} =$$

$$\underbrace{\begin{bmatrix} i_d(nT) & \dfrac{di_d(nT)}{dt} - n_p\omega(nT)i_q(nT) & 0 & 0 & 0 & 0 \\ i_q(nT) & \dfrac{di_q(nT)}{dt} + n_p\omega(nT)i_d(nT) & \omega(nT) & 0 & 0 & 0 \\ 0 & 0 & -i_q(nT) & \dfrac{d\omega(nT)}{dt} & \omega(nT) & -\mathrm{sgn}(\omega(nT)) \end{bmatrix}}_{W(nT)} \underbrace{\begin{bmatrix} R_S \\ L_S \\ K_m \\ J \\ f \\ f_c \end{bmatrix}}_{K}$$

or compactly as

$$y(nT) = W(nT)K$$

Identification of the PM Synchronous Motor Parameters

- The *error equation* is defined by

$$e(nT) \triangleq W(nT)K - y(nT) \in \mathbb{R}^2.$$

- If the model is exact, and the voltages, currents, position, & speed measured exactly, then there exists a K such that $e(nT) = 0$ for all n.
- In practice there are modeling errors and the measurements are not exact. There is no K such that $e(nT) = 0$ for all n.
- The *residual* or *squared error* is given by

$$E^2(K) = \sum_{n=n_0}^{n_1} e^T(nT)e(nT) = \sum_{n=n_0}^{n_1} \Big(W(nT)K - y(nT)\Big)^T \Big(W(nT)K - y(nT)\Big).$$

- The *least-squares estimate* K^* is the one that minimizes this squared error.

Identification of the PM Synchronous Motor Parameters

Expanding

$$E^2(K) = \sum_{n=n_0}^{n_1} e^T(nT)e(nT) = \sum_{n=n_0}^{n_1} \Big(W(nT)K - y(nT)\Big)^T\Big(W(nT)K - y(nT)\Big).$$

gives the residual error becomes

$$\begin{aligned}
E^2(K) &= \sum_{n=n_0}^{n_1} \Big(y^T(nT)y(nT) - y^T(nT)W(nT)K - K^T W^T(nT)y(nT) + K^T W^T(nT)W(nT)K\Big) \\
&= \sum_{n=n_0}^{n_1} y^T(nT)y(nT) - \Big(\sum_{n=n_0}^{n_1} y^T(nT)W(nT)\Big)K - K^T\Big(\sum_{n=n_0}^{n_1} W^T(nT)y(nT)\Big) + \\
&\quad K^T\Big(\sum_{n=n_0}^{n_1} W^T(nT)W(nT)\Big)K.
\end{aligned}$$

Define

$$R_W \triangleq \sum_{n=n_0}^{n_1} W^T(nT)W(nT) \in \mathbb{R}^{6\times 6}, \quad R_{Wy} \triangleq \sum_{n=n_0}^{n_1} W^T(nT)y(nT) \in \mathbb{R}^{6\times 1}$$

$$R_{yW} \triangleq \sum_{n=n_0}^{n_1} y^T(nT)W(nT) \in \mathbb{R}^{1\times 6}, \quad R_y \triangleq \sum_{n=n_0}^{n_1} y^T(nT)y(nT) \in \mathbb{R}$$

to rewrite the error as (assuming R_W is invertible)

$$\begin{aligned}
E^2(K) &= R_y - R_{yW}K - K^T R_{Wy} + K^T R_W K \\
&= R_y - R_{yW}R_W^{-1}R_{Wy} + \left(K - R_W^{-1}R_{Wy}\right)^T R_W\left(K - R_W^{-1}R_{Wy}\right)
\end{aligned}$$

Identification of the PM Synchronous Motor Parameters

From the previous slide we have

$$
\begin{aligned}
E^2(K) &= \sum_{n=n_0}^{n_1} \underbrace{\left(W(nT)K - y(nT)\right)^T}_{e^T(nT)} \underbrace{\left(W(nT)K - y(nT)\right)}_{e(nT)} \\
&= R_y - R_{yW}K - K^T R_{Wy} + K^T R_W K \\
&= R_y - R_{yW}R_W^{-1}R_{Wy} + \left(K - R_W^{-1}R_{Wy}\right)^T R_W \left(K - R_W^{-1}R_{Wy}\right)
\end{aligned}
$$

R_W is a *symmetric positive definite* matrix so this residual error is minimized by choosing

$$K = K^* \triangleq R_W^{-1}R_{Wy}.$$

Error Index

- The residual error indicates how well the output $y(nT)$ fits the linear model $W(nT)\hat{K}$.
- A residual error of zero would indicate a perfect match for every point n.
- Noise, quantization errors, and unmodeled dynamics prevent a residual error of zero.
- Compare the residual error using K^* to the residual error using $K = 0$.

$$\text{Error Index} \triangleq \sqrt{\frac{E^2(K^*)}{E^2(0)}} = \sqrt{\frac{R_y - R_{yW}R_W^{-1}R_{Wy}}{R_y}} \leq 1.$$

Identification of the PM Synchronous Motor Parameters

Parametric Error Index

- Find how well the parameters are determined from the data.
- The residual error due to a variation of $\delta K = K - K^* = K - R_W^{-1} R_{Wy}$ is given by

$$\begin{aligned} E^2(K)_{|K=K^*+\delta K} &= R_y - R_{yW} R_W^{-1} R_{Wy} + \left(K - R_W^{-1} R_{Wy}\right)^T R_W \left(K - R_W^{-1} R_{Wy}\right)_{|K^*=K+\delta K} \\ &= E^2(K^*) + \delta K^T R_W \delta K. \end{aligned}$$

  Setting $E^2(K)$ equal to $2E^2(K^*)$ yields $E^2(K^*) = \delta K^T R_W \delta K$.
- The solution is (see Chapter 2)

$$\delta K_i = \sqrt{E^2(K^*)(R_W^{-1})_{ii}} \ \text{ for } \ i = 1, \ldots, 6.$$

  This is the *parametric error* index.
- The percent parametric error index is then defined by

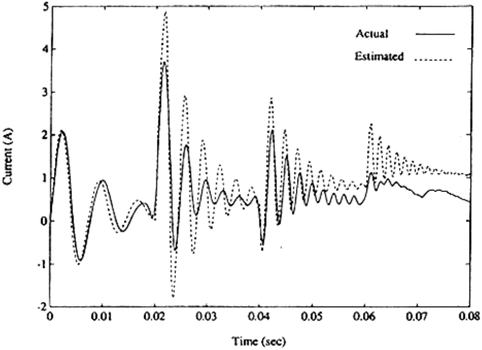
$$PE_i \triangleq \frac{\delta K_i}{K_i^*} \times 100.$$

- This is the percent change in K_i corresponding to a doubling of the residual error.
- Provides an indication of how sensitive the residual error is to each parameter estimate.
- If $PE_i = 10\%$ then the residual error $E^2(K)$ is doubled for only a 10% change in the ith parameter relative to the optimal estimate K_i^*.
  - In this case $E^2(K)$ is relatively sensitive to the estimated value of K_i^*.
  - K_i^* is probably a good estimate of the parameter.
- PE_i can be interpreted as a measure of the order of magnitude of the error.

Identification of the PM Synchronous Motor Parameters

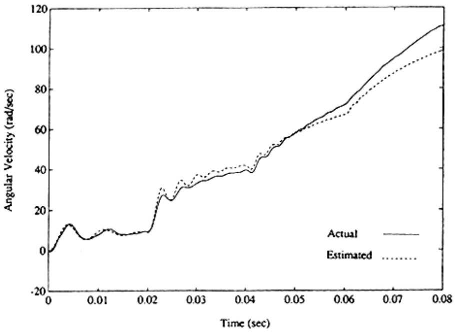
Experimental Results

A 50-pole-pair ($n_p = 50$) PM synchronous machine
2000-pulses/rev encoder.
Input voltages commanded to the motor so it would accelerate reasonably fast.
The sampling frequency is $T = 50$ kHz.
The figure shows i_q and $i_{q\text{sim}}$ (from a simulation using the identified parameters)



Identification of the PM Synchronous Motor Parameters

The figure below shows the corresponding speed response ω and the simulated speed ω_{sim}.



The data collected between 0.01 sec and 0.03 sec. The results are in the table below

Parameter	Estimate	Parametric Error
R_S ohms	0.269	399%
L_S H	0.0027	47.6%
K_m N-m/A	0.515	19.8%
J kg-m^2/ sec^2	0.000187	312%
f N-m/rad/sec	0.0032	5040%
f_c N-m	0.0693	4500%

The residual error was found to be 12%.

Identification of the PM Synchronous Motor Parameters

- The resistance R_S was also measured using an ohmmeter and found to be 0.6 ohm.
- The rotor was locked at $\theta = 0$ making phase a a simple RL circuit with time constant of L_S/R_S.
- A step voltages into phase a and a meauring the decay rate of the current gave
  - $L_S/R_S = 0.004$ so $L_S = (0.004)(0.6) = 0.0024$ H
- The value of the resistance computed using the static rotor tests is further from its least-squares value than the value of the inductance.
  - This is due to the fact that the value of R_S is not very significant compared to the value of $n_p\omega L_S$.
  - The speed is 10 rad/sec or higher after 0.01 sec so that $n_p\omega L_S \geq 50 \times 10 \times 0.0024 = 1.2$ while $R_S = 0.27$ ohm.
  - The value of R_S does not impact the value of the residual error compared to the value of L_S.
  - This is reflected in the relative values of their parametric errors.

Identification of the PM Synchronous Motor Parameters

- The parametric indices of f and f_c indicate they are much more uncertain than R_S.
- The moment of inertia J appears to be almost as uncertain as R_S.
- However, the parametric indices are not absolute indicators of error.

Consider a **Two-Stage Identification** procedure as follows.
First stage is to identify the electrical parameters R_S, L_S, and K_m.

$$\underbrace{\begin{bmatrix} u_d(nT) \\ u_q(nT) \end{bmatrix}}_{y_{elec}(nT)} = \underbrace{\begin{bmatrix} i_d(nT) & \dfrac{di_d(nT)}{dt} - n_p\omega(nT)i_q(nT) & 0 \\ i_q(nT) & \dfrac{di_q(nT)}{dt} + n_p\omega(nT)i_d(nT) & \omega(nT) \end{bmatrix}}_{W_{elec}(nT)} \underbrace{\begin{bmatrix} R_S \\ L_S \\ K_m \end{bmatrix}}_{K_{elec}}$$

or

$$y_{elec}(nT) = W_{elec}(nT)K_{elec}.$$

With

$$\begin{aligned} R_{W_{elec}} &\triangleq \sum_{n=n_0}^{n_1} W_{elec}^T(nT)W_{elec}(nT) \in \mathbb{R}^{2\times 2}, \quad R_{W_{elec}y} \triangleq \sum_{n=n_0}^{n_1} W_{elec}^T(nT)y_{elec}(nT) \in \mathbb{R}^{3\times 1} \\ R_{yW_{elec}} &\triangleq \sum_{n=n_0}^{n_1} y_{elec}^T(nT)W_{elec}(nT) \in \mathbb{R}^{1\times 3}, \quad R_{y_{elec}} \triangleq \sum_{n=n_0}^{n_1} y_{elec}^T(nT)y_{elec}(nT) \in \mathbb{R}, \end{aligned}$$

we have

$$K_{elec}^* = R_{W_{elec}}^{-1} R_{W_{elec}y}.$$

Identification of the PM Synchronous Motor Parameters

Two-Stage Identification

Using the same data as above the results are given in the table below.

Parameter	Estimate	Parametric Error
R_S ohms	0.268	399%
L_S H	0.0027	47.6%
K_m N-m/A	0.515	19.8%

Second stage is to use these values to identify J, f, f_c. We have

$$\underbrace{K_m i_q(nT)}_{y_{mech}(nT)} = \underbrace{\begin{bmatrix} \dfrac{d\omega(nT)}{dt} & \omega(nT) & \operatorname{sgn}(\omega(nT)) \end{bmatrix}}_{W_{mech}(nT)} \underbrace{\begin{bmatrix} J \\ f \\ f_c \end{bmatrix}}_{K_{mech}}$$

or

$$y_{mech}(nT) = W_{mech}(nT)K_{mech}.$$

With

$$\begin{aligned} R_{W_{mech}} &\triangleq \sum_{n=n_0}^{n_1} W_{mech}^T(nT)W_{mech}(nT) \in \mathbb{R}^{2\times 2}, \quad R_{W_{mech}y} \triangleq \sum_{n=n_0}^{n_1} W_{mech}^T(nT)y_{mech}(nT) \in \mathbb{R}^{3\times 1} \\ R_{yW_{mech}} &\triangleq \sum_{n=n_0}^{n_1} y_{mech}^T(nT)W_{mech}(nT) \in \mathbb{R}^{1\times 3}, \quad R_{y_{elec}} \triangleq \sum_{n=n_0}^{n_1} y_{mech}^T(nT)y_{mech}(nT) \in \mathbb{R}, \end{aligned}$$

we have

$$K_{mech}^* = R_{W_{mech}}^{-1} R_{W_{mech}y}.$$

Identification of the PM Synchronous Motor Parameters

Two-Stage Identification

The parameter estimates and their parametric errors are

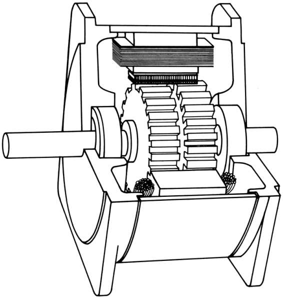
Parameter	Estimate	Parametric Error
J kg-m^2 / sec^2	0.000187	13.2%
f N-m/rad/sec	0.0032	214%
f_c N-m	0.0693	191%

The error index was 10% compared to 12% before

- The parameter estimates are the same as before, but the parametric errors are significantly reduced.
- However, the parametric errors still show the same *relative* uncertainty.
- For example, the estimate of J is much more certain than the values of f and f_c.

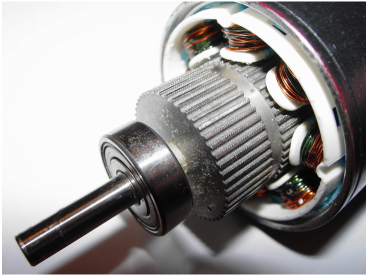
PM Stepper Motors

- Some motion (servo) control systems use a particular PM synchronous machine called the stepper motor.
- A PM stepper motor consists of a stator made of soft iron equipped with windings/coils and a PM rotor.
- The rotor has two sets of teeth (left and right in the figure) which are out of alignment by a tooth width.
- Typically there are 50 teeth in each set.
    - One set is magnetized as S poles and the other set is magnetized as N poles.
    - There are $n_p = 50$ pole pairs.

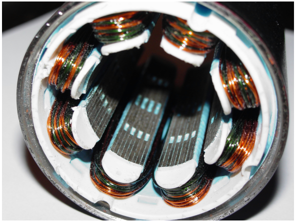


PM Stepper Motors

Photograph of a PM stepper motor

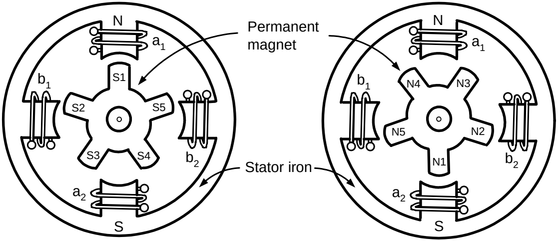


Rotor removed from the stepper motor.



PM Stepper Motors

- Simplified PM stepper motor with only 5 rotor teeth on each side.
- Two phase motor with 4 windings.
- Two windings make up phase *a* and two windings make up phase *b*.
- The rotor is made of a cylindrical PM core with teeth made of soft iron pressed onto either end.



- The PM magnetizes the soft iron of the rotor teeth, making one set a south pole and the other a north pole.
- The motor is operated as a two-phase machine by connecting windings a_1 & a_2 in series to make phase a and windings b_1 & b_2 in series to make phase b.

Open-Loop Operation of the Stepper Motor

- Stepper motors were originally designed to be used in open-loop.
- Their *stepping* ability allows for accurate positioning to a set of discrete positions.
- To explain consider Figure 1(a) with $\theta = 0$, $i_{Sa} = i_0 > 0$, $u_{Sa} = R_S i_0$ and $i_{Sb} = 0$.
- The direction for positive current i_{Sa} is as shown in the figure.
- With $i_{Sa} > 0$ winding a_1 of the stator is magnetized as a north pole and winding a_2 is magnetized as a south pole. Similar remarks hold for phase b.
- There are $360^\circ/5 = 72^\circ$ between each rotor tooth.
- $N2$ is 18° from being aligned with winding b_2.
- First step. Set $i_{Sa} = 0$ and $i_{Sb} = i_0$. $N2$ rotates to stator winding b_2.
- Similarly $S2$ is attracted to stator winding b_1.

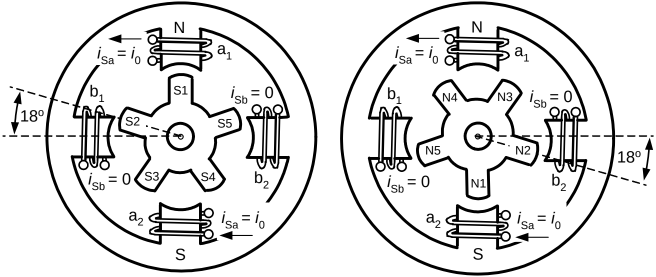
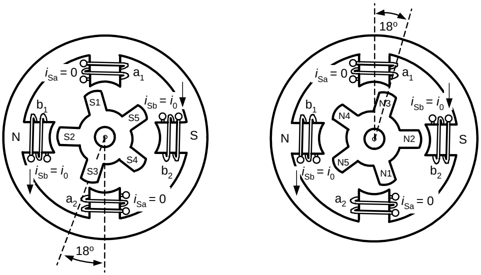


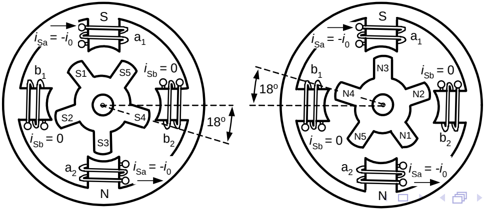
Figure: (a) Starting point with $\theta = 0$ where $u_{Sa} = u_0$, $u_{Sb} = 0$, $i_{Sa} = i_0$, and $i_{Sb} = 0$.

PM Stepper Motors

Rotor position at the end of step 1.

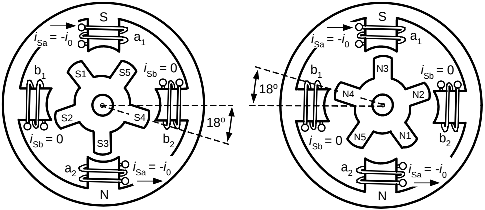


- **Second step.** Set $u_{Sa} = -u_0$ so $i_{Sa} = -i_0$ and $u_{Sb} = 0$ so $i_{Sb} = 0$
- Winding a_1 is now a S pole and winding a_2 is a N pole.
- $N3$ is attracted to stator winding a_1 and $S3$ is attracted to stator winding a_2.
- The motor rotates another 18° so that $\theta = 36^\circ$.

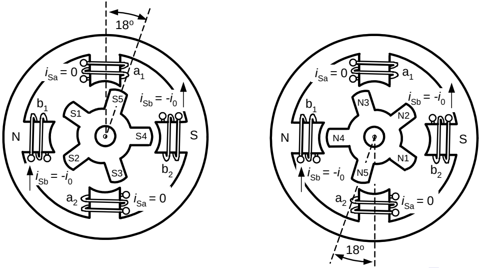


PM Stepper Motors

Rotor position at the end of step 2.

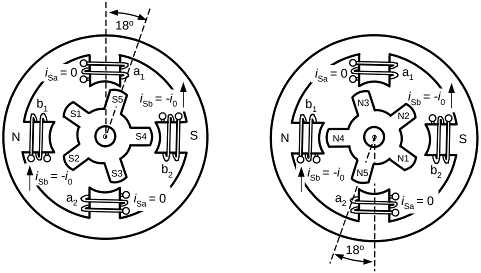


- **Third step.** Set $u_{Sa} = 0$, $u_{Sb} = -u_0$, $i_{Sa} = 0$, and $i_{Sb} = -i_0$.
- Winding b_1 is a S pole and winding b_2 is a N pole.
- $N4$ is attracted to stator winding b_1 and $S4$ is attracted to stator winding b_2.
- The motor rotates another 18° so that $\theta = 54°$.

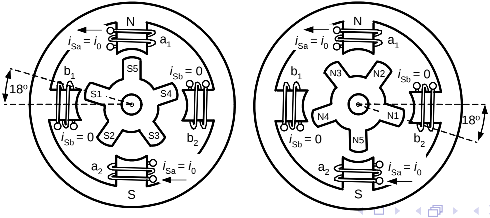


PM Stepper Motors

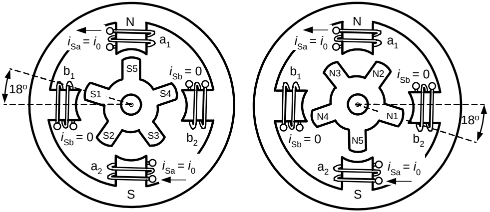
Rotor position at the end of step 3.



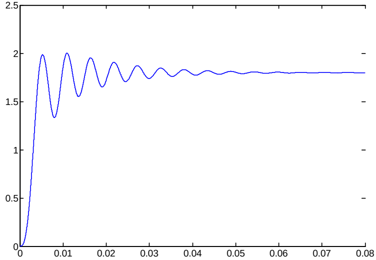
- **Fourth step**. Set $u_{Sa} = u_0$, $u_{Sb} = 0$, $i_{Sa} = i_0$, and $i_{Sb} = 0$.
- Winding a_1 is now a N pole and winding a_2 is a S pole.
- $S5$ is attracted to stator winding a_1 and $N5$ is attracted to stator winding a_2.
- The motor rotates another 18° so $\theta = 72°$.



PM Stepper Motors Rotor position at the end of step 4.



- Each step the motor rotates $360°/(4n_p) = 360°/(20) = 18°$.
- After four steps the motor has rotated $360°/n_p = 72°$.
- If $n_p = 50$ then the step size is $360°/200 = 1.8°$.
- A typical step response of a PM stepper motor with $n_p = 50$ is below.
- A step is not initiated before the transient from the previous step has died out.



Mathematical Model of a PM Stepper Motor

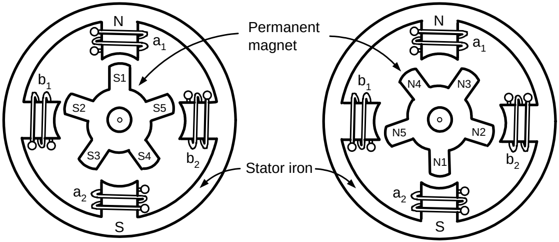
L_S is the self inductance of each phase winding.
The rotor's PM produces the flux linkages $\phi_a(\theta)$ and $\phi_b(\theta)$ in stator phases a and b.
The total flux linkage in the windings of the stator phases is then

$$\begin{aligned}\lambda_{Sa} &= L_S i_{Sa} + \phi_a(\theta)\\ \lambda_{Sb} &= L_S i_{Sb} + \phi_b(\theta).\end{aligned}$$

The surfaces of the rotor and stator teeth are shaped so that

$$\begin{aligned}\phi_a(\theta) &= \lambda_M \cos(n_p\theta)\\ \phi_b(\theta) &= \lambda_M \sin(n_p\theta)\end{aligned}$$

where λ_M is a constant and $\theta = 0$ corresponds to the rotor as shown in the figure below.



Mathematical Model of a PM Stepper Motor

Faraday's law, Ohm's law, and Kirchhoff's voltage law gives

$$\begin{aligned} u_{Sa} - R_S i_{Sa} - \frac{d}{dt}\lambda_{Sa} &= 0 \\ u_{Sb} - R_S i_{Sb} - \frac{d}{dt}\lambda_{Sb} &= 0 \end{aligned}$$

With

$$\begin{aligned} \lambda_{Sa} &= L_S i_{Sa} + \lambda_M \cos(n_p\theta) \\ \lambda_{Sb} &= L_S i_{Sb} + \lambda_M \sin(n_p\theta) \end{aligned}$$

we have

$$\begin{aligned} L_S \frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m\omega \sin(n_p\theta) + u_{Sa} \\ L_S \frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m\omega \cos(n_p\theta) + u_{Sb} \end{aligned}$$

where $K_m \triangleq n_p\lambda_M$ and $\omega = d\theta/dt$ is the rotor's angular speed.

Mathematical Model of a PM Stepper Motor

From the previous slide

$$
\begin{aligned}
L_S \frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m \omega \sin(n_p \theta) + u_{Sa} \\
L_S \frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m \omega \cos(n_p \theta) + u_{Sb}
\end{aligned}
$$

By conservation of energy (power)

$$
\begin{aligned}
i_{Sa} u_{Sa} + i_{Sb} u_{Sb} &= R_S i_{Sa}^2 + R_S i_{Sb}^2 + L_S i_{Sa} \frac{di_{Sa}}{dt} + L_S i_{Sb} \frac{di_{Sb}}{dt} - (i_{Sa} e_{Sa} + i_{Sb} e_{Sb}) \\
&= R_S (i_{Sa}^2 + i_{Sb}^2) + \frac{d}{dt} \frac{1}{2} L_S (i_{Sa}^2 + i_{Sb}^2) - (i_{Sa} e_{Sa} + i_{Sb} e_{Sb})
\end{aligned}
$$

where the back-emf voltages e_{Sa} and e_{Sb} are defined by

$$
\begin{aligned}
e_{Sa} &\triangleq -\frac{d}{dt} \phi_a(\theta) = K_m \omega \sin(n_p \theta) \\
e_{Sb} &\triangleq -\frac{d}{dt} \phi_b(\theta) = -K_m \omega \cos(n_p \theta).
\end{aligned}
$$

- The pwr absorbed by the back emf $-(i_{Sa} e_{Sa} + i_{Sb} e_{Sb})$ reappears as $\tau\omega$, i.e.,

  $$\tau\omega = -(i_{Sa} e_{Sa} + i_{Sb} e_{Sb})$$

  or

  $$\tau = -(i_{Sa} e_{Sa} + i_{Sb} e_{Sb})/\omega = -i_{Sa} K_m \sin(n_p \theta) + i_{Sb} K_m \cos(n_p \theta).$$

Mathematical Model of a PM Stepper Motor

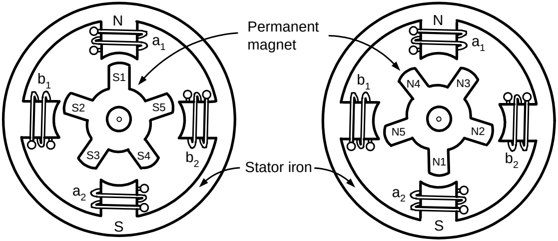
- J is the rotor's moment of inertia.
- τ_L is the load torque.

The equations of a PM stepper motor are given by

$$
\begin{aligned}
L_S \frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m \omega \sin(n_p\theta) + u_{Sa} \\
L_S \frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m \omega \cos(n_p\theta) + u_{Sb} \\
J \frac{d\omega}{dt} &= -i_{Sa} K_m \sin(n_p\theta) + i_{Sb} K_m \cos(n_p\theta) - \tau_L \\
\frac{d\theta}{dt} &= \omega.
\end{aligned}
$$

Oscillatory Behavior of the Open-Loop PM Stepper Motor

- Describe the oscillatory behavior of the PM stepper motor about its eq pt.
- The stepper motor is at $\theta = 0$ in the figure below.



- Look at oscillations about $\theta = 0$. With $i_{Sa} = i_0$, $i_{Sb} = 0$ and θ close to 0

$$\tau = -i_{Sa} K_m \sin(n_p\theta) + i_{Sb} K_m \cos(n_p\theta) = -i_{Sa} K_m \sin(n_p\theta) \approx -i_0 n_p K_m \theta.$$

- This restoring torque $-i_0 n_p K_m \theta$ is large if n_p is large.
- With $\tau_L = -f\omega$ the equation describing the rotor position for $|\theta|$ small is given by

$$J\frac{d^2\theta}{dt^2} = -(i_0 n_p K_m)\theta - f\frac{d\theta}{dt}.$$

- Set $\omega_n^2 \triangleq i_0 n_p K_m / J, 2\zeta\omega_n \triangleq f/J$ to get

$$\frac{d^2\theta}{dt^2} + 2\zeta\omega_n \frac{d\theta}{dt} + \omega_n^2\theta = 0.$$

- This is the equation of a damped pendulum which has an oscillatory response.

Motor Parameters from Manufacturer's data sheet

Manufacturer's parameters for a PM synchronous motor are as follows.

- Line-to-line inductance $L_{\ell-\ell} = 0.028$ H.
- Line-to-line resistance $R_{\ell-\ell} = 0.5$ ohms.
- Line-to-line back-emf constant $K_b^{\ell-\ell} = 23.6$ V(peak)/krpm.
- Torque constant $K_T = 0.28$ N-m/A(rms).

Motor Parameters from Manufacturer's data sheet

The electrical equations of a three-phase PM synchronous machine are ($n_p = 1$)

$$\begin{aligned}
L\frac{di_{S1}}{dt} - \frac{L}{2}\frac{di_{S2}}{dt} - \frac{L}{2}\frac{di_{S3}}{dt} &= K\omega\sin(\theta) - R_S i_{S1} + u_{S1} \\
-\frac{L}{2}\frac{di_{S1}}{dt} + L\frac{di_{S2}}{dt} - \frac{L}{2}\frac{di_{S3}}{dt} &= K\omega\sin(\theta - 2\pi/3) - R_S i_{S2} + u_{S2} \\
-\frac{L}{2}\frac{di_{S1}}{dt} - \frac{L}{2}\frac{di_{S2}}{dt} + L\frac{di_{S3}}{dt} &= K\omega\sin(\theta - 4\pi/3) - R_S i_{S3} + u_{S3}
\end{aligned}$$

- Line-to-line inductance and resistance measurements done with rotor held fixed.
- The voltage is applied between phases 1 and 2 with phase 3 open-circuited.
- Subtract second equation from the first equation ($i_{S2} = -i_{S1}, i_{S3} = 0, \omega = 0$)

$$3L\frac{di_{S1}}{dt} = u_{S1} - u_{S2} - 2R_S i_{S1}.$$

$3L = 0.028$ H. From Chapter 7 $L = (2/3)L_S$ so

$$L_S = \frac{3}{2}\frac{0.028}{3}\ \text{H} \ = 0.014\ \text{H}.$$

$2R_S = 0.5$ ohm so $R_S = 0.25$ ohm.

Motor Parameters from Manufacturer's data sheet

- Line-to-line back-emf voltage.
  - Externally drive the motor shaft at constant speed with the phases open circuited.
  - Measure the (back-emf) voltage between two-phases.
  - From $(\theta = \omega t)$

$$\begin{aligned} u_{S1} &= -K\omega \sin(\theta) \\ u_{S2} &= -K\omega \sin\left(\theta - 2\pi/3\right) = -K\omega(-0.5\sin(\theta) - \sqrt{3}/2\cos(\theta)) \\ u_{S3} &= -K\omega \sin(\theta - 4\pi/3) = -K\omega(-0.5\sin(\theta) + \sqrt{3}/2\cos(\theta)) \end{aligned}$$

  we have

$$\begin{aligned} u_{S2} - u_{S3} &= \sqrt{3}K\omega\cos(\theta) \\ K_b^{\ell-\ell} &= \frac{(u_{S2} - u_{S3})_{\text{peak}}}{\omega} = \sqrt{3}K \ \frac{\text{V}}{\text{rads/sec}} \end{aligned}$$

- $K_b^{\ell-\ell} = 23.6\ V(\text{peak})/\text{krpm}$ so in these units

$$K_b^{\ell-\ell} = \sqrt{3}K\frac{2\pi/60 \text{ rads/sec}}{\text{rpm}}\frac{1000 \text{ rpm}}{\text{krpm}} = 23.6\ V(\text{peak})/\text{krpm}.$$

Motor Parameters from Manufacturer's data sheet

- From previous slide

$$K_b^{\ell-\ell} = \sqrt{3}K\frac{2\pi/60 \text{ rads/sec}}{\text{rpm}}\frac{1000 \text{ rpm}}{\text{krpm}} = 23.6 \ V(\text{peak})/\text{krpm}.$$

- From Chapter 7 we haved $K = \sqrt{2/3}K_m$ so that

$$\begin{aligned} K_m = \sqrt{\frac{3}{2}}K = \frac{1}{\sqrt{2}}K_b^{\ell-\ell}\frac{\text{V(peak)}}{\text{krpm}}\frac{\text{krpm}}{1000 \text{ rpm}}\frac{\text{rpm}}{2\pi/60 \text{ rads/sec}} &= \frac{1}{\sqrt{2}}23.6\frac{1}{1000}\frac{60}{2\pi} \\ &= 0.159 \ \frac{\text{V(peak)}}{\text{rads/sec}}. \end{aligned}$$

- Compute the torque constant. With I_{3ph} the peak current, in steady-state we have

$$\begin{aligned} i_{S1} &= I_{3ph}\cos(\omega_S t) \\ i_{S2} &= I_{3ph}\cos(\omega_S t - 2\pi/3) \\ i_{S3} &= I_{3ph}\cos(\omega_S t - 4\pi/3). \end{aligned}$$

$$\tau_R = -K\Big(i_{S1}\sin(\theta_R) + i_{S2}\sin(\theta_R - 2\pi/3) + i_{S3}\sin(\theta_R - 4\pi/3)\Big).$$

Motor Parameters from Manufacturer's data sheet

The $3-2$ phase transformation results in

$$\begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix} = \begin{bmatrix} \sqrt{3/2} I_{3ph} \cos(\omega_S t) \\ \sqrt{3/2} I_{3ph} \sin(\omega_S t) \end{bmatrix}$$

and (I_{3ph_rms} is the rms current)

$$\begin{aligned} \tau = -\sqrt{\frac{3}{2}} K i_{Sa} \sin(\theta) + \sqrt{\frac{3}{2}} K i_{Sb} \cos(\theta) = \sqrt{\frac{3}{2}} K \sqrt{\frac{3}{2}} I_{3ph} \sin(\omega_S t - \theta) &= \sqrt{2}\frac{3}{2} K I_{3ph\_rms} \sin(\omega_S t - \theta) \\ &= K_T I_{3ph\_rms} \sin(\omega_S t - \theta). \end{aligned}$$

The manufacturer gives $K_T = 0.28$ N-m/A(rms). As

$$K = \frac{1}{\sqrt{2}} \frac{2}{3} K_T$$

we compute

$$K_m = \sqrt{\frac{3}{2}} K = \frac{K_T}{\sqrt{3}} = \frac{0.28}{\sqrt{3}} \frac{\text{N-m}}{\text{A(peak)}} = 0.162 \frac{\text{N-m}}{\text{A(peak)}}.$$

- The two-phase equivalent back-emf constant was computed to be $0.159 \dfrac{\text{V(peak)}}{\text{rads/sec}}$.
- The two-phase equivalent torque constant was computed to be $0.162 \dfrac{\text{N-m}}{\text{A(peak)}}$.