Estimation of the Parameters of a Second-Order Linear System

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***Abstract*— We study the estimation of the parameters of a second-order linear ODE system using an adaptation law.**

***Index Terms*— Adaptive control, estimation**

I. Introduction

We provide a Lyapunov analysis [1] to prove that our control and adaptation laws will stabilize the system to a reference trajectory while the estimates of the parameters tend to their correct values.

II. Analysis

Consider the scalar linear system with equations of motion given by the ODE

$$m\ddot{x} + b\dot{x} + kx = u(x, \dot{x}), \tag{1}$$

where $u(x, \dot{x})$ is a control input that is to be determined for tracking. We are uncertain of the constant parameters $\theta = \begin{bmatrix} m & b & k \end{bmatrix}^\top$, whose estimates are denoted by $\hat{\theta} \in \mathbb{R}^3$. Let us introduce the errors in the position x, and parameters θ to be

$$\tilde{x} = x - x_r, \quad \tilde{\theta} = \theta - \hat{\theta},$$

where x_r is a reference signal for the motion of the mechanical system. Inspired by Chapter 9 of [2], let us choose the control input according to

$$\begin{aligned} u(x, \dot{x}) &= Y(x, \dot{x}, a, v)\hat{\theta} - cr, \\ Y(x, \dot{x}, a, v) &= \begin{bmatrix} a & v & x \end{bmatrix}, \end{aligned} \tag{2}$$

where the quantities v, a, and r are given as

$$\begin{aligned} v &= \dot{x}_r - \lambda\tilde{x}, \\ a &= \dot{v} = \ddot{x}_r - \lambda\dot{\tilde{x}}, \\ r &= \dot{x} - v = \dot{\tilde{x}} + \lambda\tilde{x}, \end{aligned}$$

where $c, \lambda > 0$ are constant gains. Substituting the control law (2) into the system model (1) leads to

$$m\dot{r} + (b + c)r = -Y\tilde{\theta}. \tag{3}$$

The parameter estimate $\hat{\theta}$ may be computed using standard methods of adaptive control such as gradients or least squares.

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For example, for a positive definite matrix Γ of appropriate dimensions, we can use the gradient update law

$$\dot{\hat{\theta}} = -\Gamma^{-1}Y^\top(x, \dot{x}, a, v)r. \tag{4}$$

Consider the Lyapunov function candidate

$$V(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) = \frac{1}{2}mr^2 + c\lambda\tilde{x}^2 + \frac{1}{2}\tilde{\theta}^\top\Gamma\tilde{\theta}.$$

This is a positive definite function over the space of $(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta})$. We take the time derivative of the Lyapunov function candidate and substitute from the closed-loop system dynamics (3, 4). We suppress its functional dependence for brevity.

$$\begin{aligned} \dot{V} &= mr\dot{r} + 2c\lambda\tilde{x}\dot{\tilde{x}} + \tilde{\theta}^\top\Gamma\dot{\tilde{\theta}} \\ &= r\left(-(b + c)r - Y\tilde{\theta}\right) + 2c\lambda\tilde{x}\dot{\tilde{x}} + \tilde{\theta}^\top Y^\top r \\ &= -c\lambda^2\tilde{x}^2 - c\dot{\tilde{x}}^2 - br^2 \leq 0. \end{aligned}$$

Hence the set $\Omega_c = \{(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) : V(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) \leq c\}$ is positively invariant for any $c > 0$. Moreover, $\tilde{x}, \dot{\tilde{x}}, r \to 0$ as $t \to \infty$. This means $u(x, \dot{x}) \to \hat{m}\ddot{x}_r + \hat{b}\dot{x}_r + \hat{k}x_r$ (since $x \to x_r$). We identify $S = \{(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) : (\tilde{x}, \dot{\tilde{x}}) = 0\}$ as the set of all points in Ω_c where $\dot{V} = 0$. Under some mild persistence of excitations consitions on $x_r(t)$, no solution can stay identically in S other than the trivial solution $(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) = (0, 0, 0)$. Therefore, invoking Corollary 4.1 of [1] proves that all the errors converge to zero. Indeed, for any solution that belongs identically to S, the system dynamics yields

$$\tilde{m}\ddot{x}_r + \tilde{b}\dot{x}_r + \tilde{k}x_r \equiv 0, \qquad \dot{\tilde{m}} \equiv \dot{\tilde{b}} \equiv \dot{\tilde{k}} \equiv 0.$$

We can excite several frequencies by choosing

$$\begin{aligned} x_r(t) &= A\sin\left(\omega(t) + \varphi\right) \\ \omega(t) &= \pi\sum_{k=1}^{n}\left(1 - \frac{k-1}{n}\right)\sin kt \end{aligned}$$

for some constants φ, $A > 0$, and a sufficiently large $n \in \mathbb{N}$ This coaxes $\tilde{\theta} \to 0$, making the origin of $(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta})$ globally asymptotically stable.

III. Results

In Figure 1, we plot the response of the system to the control and adaptation laws (2, 4), implemented in simulation. The constants that are used are as follows: $(A, n, \varphi) = (3/10, 3, 0^\circ)$, $\Gamma = \text{diag}(1, 1, 1/10)$ and $(c, \lambda) = (1, 4)$. The real mass, damping and stiffness of the system are $(m, b, k) = (0.665, 1.819, 0)$ and their estimates start at $(\hat{m}, \hat{b}) = (-1/2, -1/4, -1)$.

IV. Conclusion

We have shown that our control and adaptation laws for the linear second-order mechanical system guide the estimates of the parameters to tend to their correct values while tracking a judiciously chosen reference signal.

References

[1] H. Khalil, *Nonlinear Control*. Always Learning, Pearson, 2015.
[2] M. Spong, S. Hutchinson, and M. Vidyasagar, *Robot Modeling and Control*. Wiley, 2020.

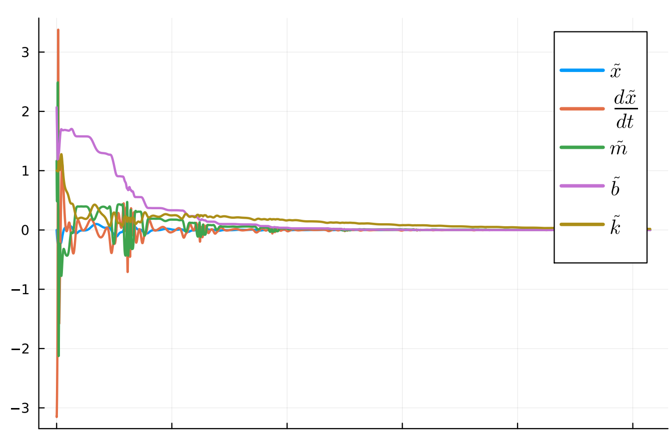

Fig. 1: Simulation showing the convergence of the system state and parameter estimates.