Estimation of the Parameters of a Second-Order Linear System

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Abstract

We study the estimation of the parameters of a second-order linear ODE system using an adaptation law.

Keywords Adaptive control · Parameter estimation · Passivity

1 Introduction

We provide a Lyapunov analysis [1] to prove that our control and adaptation laws will stabilize the system to a reference trajectory while the estimates of the parameters tend to their correct values.

2 Analysis

Consider the scalar linear system with equations of motion given by the ODE

$$m\ddot{x} + b\dot{x} + kx = u(x, \dot{x}), \tag{1}$$

where $u(x, \dot{x})$ is a control input that is to be determined for tracking. We are uncertain of the constant parameters $\theta = \begin{bmatrix} m & b & k \end{bmatrix}^\top$, whose estimates are denoted by $\hat{\theta} \in \mathbb{R}^3$. Let us introduce the errors in the position x, and parameters θ to be

$$\tilde{x} = x - x_r, \quad \tilde{\theta} = \theta - \hat{\theta}, \quad e = \begin{bmatrix} \tilde{x} & \dot{\tilde{x}} \end{bmatrix}^\top,$$

where x_r is a reference signal for the motion of the mechanical system. Inspired by Chapter 9 of [2], let us choose the control input according to

$$\begin{aligned} u(x, \dot{x}) &= Y(x, \dot{x}, a, v)\hat{\theta} - cr, \\ Y(x, \dot{x}, a, v) &= \begin{bmatrix} a & v & x \end{bmatrix}, \end{aligned} \tag{2}$$

where the quantities v, a, and r are given as

$$\begin{aligned} v &= \dot{x}_r - \lambda\tilde{x}, \\ a &= \dot{v} = \ddot{x}_r - \lambda\dot{\tilde{x}}, \\ r &= \dot{x} - v = \dot{\tilde{x}} + \lambda\tilde{x}, \end{aligned}$$

where $c, \lambda > 0$ are constant gains. Substituting the control law (2) into the system model (1) leads to

$$m\dot{r} + (b + c)r = -Y\tilde{\theta}. \tag{3}$$

The parameter estimate $\hat{\theta}$ may be computed using standard methods of adaptive control such as gradients or least squares. For example, for a positive definite matrix Γ of appropriate dimensions, we can use the gradient update law

$$\dot{\hat{\theta}} = -\Gamma^{-1}Y^\top(x, \dot{x}, a, v)r. \tag{4}$$

Consider the Lyapunov function candidate

$$V(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) = \frac{1}{2} m r^2 + c\lambda \tilde{x}^2 + \frac{1}{2} \tilde{\theta}^\top \Gamma \tilde{\theta}.$$

This is a positive definite function over the space of $\left(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}\right)$. We take the time derivative of the Lyapunov function candidate and substitute from the closed-loop system dynamics (3, 4). We suppress its functional dependence for brevity.

$$\begin{aligned} \dot{V} &= m r \dot{r} + 2c\lambda \tilde{x}\dot{\tilde{x}} + \tilde{\theta}^\top \Gamma \dot{\tilde{\theta}} \\ &= r\left(-(b+c)r - Y\tilde{\theta}\right) + 2c\lambda \tilde{x}\dot{\tilde{x}} + \tilde{\theta}^\top Y^\top r \\ &= -e^\top Q e \leq 0, \end{aligned} \tag{5}$$

where Q is a symmetric, positive-definite matrix given as

$$Q = \begin{bmatrix} (b+c)\lambda^2 & b\lambda \\ b\lambda & b+c \end{bmatrix}.$$

Integrating both sides of equation (5) gives

$$V(t) - V(0) = -\int_0^t e^\top(\sigma) Q e(\sigma)\, \mathrm{d}\sigma < \infty.$$

We observe that $\dot{\tilde{x}}$ is bounded because $\dot{V} \leq 0$ implies that the terms r, $\tilde{x}$ and $\tilde{\theta}$ are bounded functions of time. This allows us to invoke Barbalat's lemma [2] to deduce that $\tilde{x} \to 0$ as $t \to \infty$. Furthermore, using equation (3), we can readily see that $\ddot{x}$ is bounded. Another application of Barbalat's lemma shows that the velocity error $\dot{\tilde{x}} \to 0$ provided that the reference acceleration $\ddot{x}_r(t)$ is bounded. Since $x \to x_r$, we also have that $u(x, \dot{x}) \to \hat{m}\ddot{x}_r + \hat{b}\dot{x}_r + \hat{k}x_r$.

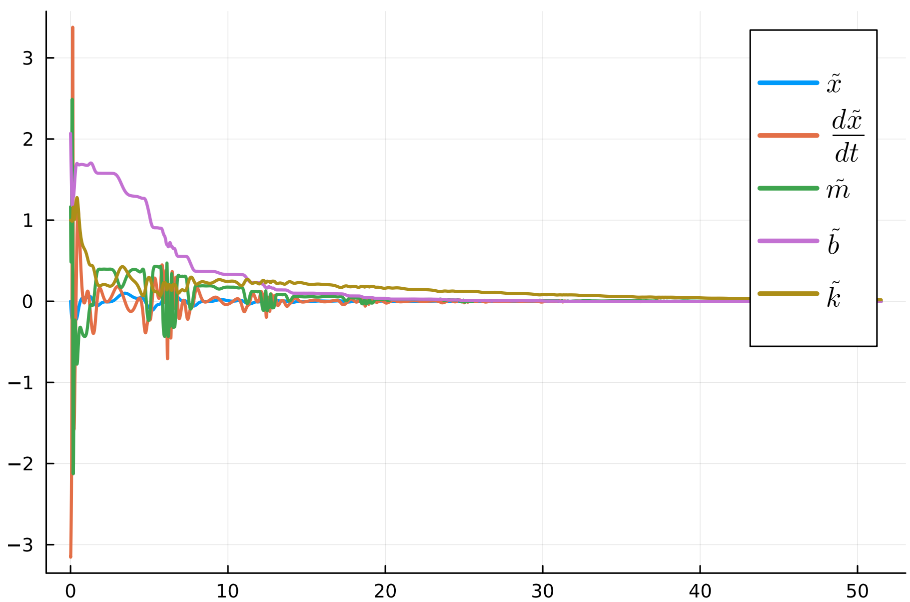

Figure 1: Simulation showing the convergence of the system state and parameter estimates.

For any $\eta > 0$, the set $\Omega_\eta = \{(e, \tilde{\theta}) : V(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) \leq \eta\}$ is positively invariant. The positive limit set of $(e(t), \tilde{\theta}(t))$ is a subset of $E = \{(e, \tilde{\theta}) : e = 0\}$. Unfortunately, for a general nonautonomous system, the positive limit set is not necessarily a positively invariant set, precluding us to invoke LaSalle's theorem to conclude that $\tilde{\theta} \to 0$ [1].

We can excite several frequencies by choosing

$$\begin{aligned} x_r(t) &= A \sin\left(\omega(t) + \varphi\right) \\ \omega(t) &= \pi \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) \sin kt \end{aligned}$$

for some constants φ, $A > 0$, and a sufficiently large $n \in \mathbb{N}$. Simulation results in the next section suggests that this coaxes $\tilde{\theta} \to 0$ even though we were only able to show that $\tilde{\theta}$ is bounded in theory.

3 Results

In Figure 1, we plot the response of the system to the control and adaptation laws (2, 4), implemented in simulation. The constants that are used are as follows: $(A, n, \varphi) = (3/10, 3, 0°)$, $\Gamma = \mathrm{diag}(1, 1, 1/10)$ and $(c, \lambda) = (1, 4)$. The real mass, damping and stiffness of the system are $(m, b, k) = (0.665, 1.819, 0)$ and their estimates start at $(\hat{m}, \hat{b}) = (-1/2, -1/4, -1)$.

4 Conclusion

We have shown that our control and adaptation laws for the linear second-order mechanical system guide the estimates of the parameters to tend to their correct values while tracking a judiciously chosen reference signal.

References

[1] H. Khalil, *Nonlinear Control*. Always Learning, Pearson, 2015.

[2] M. Spong, S. Hutchinson, and M. Vidyasagar, *Robot Modeling and Control*. Wiley, 2020.