Estimation of the Parameters of a Second-Order Linear System

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***Abstract*— We study the estimation of the parameters of a second-order linear ODE system using an adaptation law.**

***Index Terms*— Adaptive control, estimation**

I. Introduction

We provide a Lyapunov analysis [1] to prove that our control and adaptation laws will stabilize the system to a reference trajectory while the estimates of the parameters tend to their correct values.

II. Analysis

Consider the scalar linear system with equations of motion given by the ODE

$$m\ddot{x} + b\dot{x} + kx = u(x, \dot{x}), \tag{1}$$

where $u(x, \dot{x})$ is a control input that is to be determined for tracking. We are uncertain of the constant parameters $\theta = \begin{bmatrix} m & b & k \end{bmatrix}^\top$, whose estimates are denoted by $\hat{\theta} \in \mathbb{R}^3$. Let us introduce the errors in the position x, and parameters θ to be

$$\tilde{x} = x - x_r, \quad \tilde{\theta} = \theta - \hat{\theta}, \quad e = \begin{bmatrix} \tilde{x} & \dot{\tilde{x}} \end{bmatrix}^\top,$$

where x_r is a reference signal for the motion of the mechanical system. Inspired by Chapter 9 of [2], let us choose the control input according to

$$\begin{aligned} u(x, \dot{x}) &= Y(x, \dot{x}, a, v)\hat{\theta} - cr, \\ Y(x, \dot{x}, a, v) &= \begin{bmatrix} a & v & x \end{bmatrix}, \end{aligned} \tag{2}$$

where the quantities v, a, and r are given as

$$\begin{aligned} v &= \dot{x}_r - \lambda\tilde{x}, \\ a &= \dot{v} = \ddot{x}_r - \lambda\dot{\tilde{x}}, \\ r &= \dot{x} - v = \dot{\tilde{x}} + \lambda\tilde{x}, \end{aligned}$$

where $c, \lambda > 0$ are constant gains. Substituting the control law (2) into the system model (1) leads to

$$m\dot{r} + (b + c)r = -Y\tilde{\theta}. \tag{3}$$

The parameter estimate $\hat{\theta}$ may be computed using standard methods of adaptive control such as gradients or least squares.

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For example, for a positive definite matrix Γ of appropriate dimensions, we can use the gradient update law

$$\dot{\tilde{\theta}} = -\Gamma^{-1}Y^\top(x, \dot{x}, a, v)r. \tag{4}$$

Consider the Lyapunov function candidate

$$V(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) = \frac{1}{2}mr^2 + c\lambda\tilde{x}^2 + \frac{1}{2}\tilde{\theta}^\top\Gamma\tilde{\theta}.$$

This is a positive definite function over the space of $\left(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}\right)$. We take the time derivative of the Lyapunov function candidate and substitute from the closed-loop system dynamics (3, 4). We suppress its functional dependence for brevity.

$$\begin{aligned} \dot{V} &= mr\dot{r} + 2c\lambda\tilde{x}\dot{\tilde{x}} + \tilde{\theta}^\top\Gamma\dot{\tilde{\theta}} \\ &= r\left(-(b + c)r - Y\tilde{\theta}\right) + 2c\lambda\tilde{x}\dot{\tilde{x}} + \tilde{\theta}^\top Y^\top r \\ &= -e^\top Qe \le 0, \end{aligned} \tag{5}$$

where Q is a symmetric, positive-definite matrix given as

$$Q = \begin{bmatrix} (b + c)\lambda^2 & b\lambda \\ b\lambda & b + c \end{bmatrix}.$$

Integrating both sides of equation (5) gives

$$V(t) - V(0) = -\int_0^t e^\top(\sigma)Qe(\sigma)\,\mathrm{d}\sigma < \infty.$$

We observe that $\dot{\tilde{x}}$ is bounded because $\dot{V} \le 0$ implies that the terms r, $\tilde{x}$ and $\tilde{\theta}$ are bounded functions of time. This allows us to invoke Barbalat's lemma [2] to deduce that $\tilde{x} \to 0$ as $t \to \infty$. Furthermore, using equation (3), we can readily see that $\ddot{x}$ is bounded. Another application of Barbalat's lemma shows that the velocity error $\dot{\tilde{x}} \to 0$ provided that the reference acceleration $\ddot{x}_r(t)$ is bounded. Since $x \to x_r$, we also have that $u(x, \dot{x}) \to \hat{m}\ddot{x}_r + \hat{b}\dot{x}_r + \hat{k}x_r$.

For any $\eta > 0$, the set $\Omega_\eta = \{(e, \tilde{\theta}) : V(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) \le \eta\}$ is positively invariant. The positive limit set of $(e(t), \tilde{\theta}(t))$ is a subset of $E = \{(e, \tilde{\theta}) : e = 0\}$. Unfortunately, for a general nonautonomous system, the positive limit set is not necessarily a positively invariant set, precluding us to invoke LaSalle's theorem to conclude that $\tilde{\theta} \to 0$ [1].

We can excite several frequencies by choosing

$$\begin{aligned} x_r(t) &= A\sin\left(\omega(t) + \varphi\right) \\ \omega(t) &= \pi\sum_{k=1}^{n}\left(1 - \frac{k-1}{n}\right)\sin kt \end{aligned}$$

for some constants φ, $A > 0$, and a sufficiently large $n \in \mathbb{N}$. Simulation results in the next section suggests that this coaxes $\tilde{\theta} \to 0$ even though we were only able to show that $\tilde{\theta}$ is bounded in theory.

III. Results

In Figure 1, we plot the response of the system to the control and adaptation laws (2, 4), implemented in simulation. The constants that are used are as follows: $(A, n, \varphi) = ({}^3\!/_{10}, 3, 0°)$, $\Gamma = \mathrm{diag}(1, 1, {}^1\!/_{10})$ and $(c, \lambda) = (1, 4)$. The real mass, damping and stiffness of the system are $(m, b, k) = (0.665, 1.819, 0)$ and their estimates start at $(\hat{m}, \hat{b}) = (-{}^1\!/_2, -{}^1\!/_4, -1)$.

IV. Conclusion

We have shown that our control and adaptation laws for the linear second-order mechanical system guide the estimates of the parameters to tend to their correct values while tracking a judiciously chosen reference signal.

References

[1] H. Khalil, *Nonlinear Control*. Always Learning, Pearson, 2015.
[2] M. Spong, S. Hutchinson, and M. Vidyasagar, *Robot Modeling and Control*. Wiley, 2020.

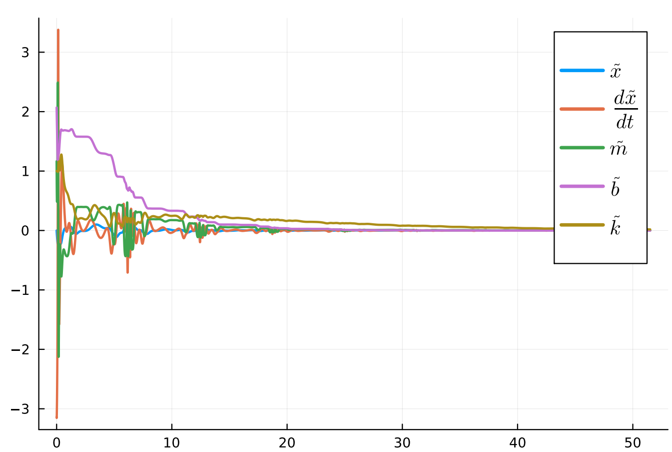

Fig. 1: Simulation showing the convergence of the system state and parameter estimates.