

# Estimation of the Parameters of a Second-Order Linear System

Aykut C. Satıcı, *Member, IEEE*

**Abstract**—We study the estimation of the parameters of a second-order linear ODE system using an adaptation law.

**Index Terms**—Adaptive control, estimation

## I. INTRODUCTION

We provide a Lyapunov analysis [1] to prove that our control and adaptation laws will stabilize the system to a reference trajectory while the estimates of the parameters tend to their correct values.

## II. ANALYSIS

Consider the scalar linear system with equations of motion given by the ODE

$$m\ddot{x} + b\dot{x} + kx = u(x, \dot{x}), \quad (1)$$

where  $u(x, \dot{x})$  is a control input that is to be determined for tracking. We are uncertain of the constant parameters  $\theta = [m \ b \ k]^\top$ , whose estimates are denoted by  $\hat{\theta} \in \mathbb{R}^3$ . Let us introduce the errors in the position  $x$ , and parameters  $\theta$  to be

$$\tilde{x} = x - x_r, \quad \tilde{\theta} = \theta - \hat{\theta}, \quad e = [\tilde{x} \ \dot{\tilde{x}}]^\top,$$

where  $x_r$  is a reference signal for the motion of the mechanical system. Inspired by Chapter 9 of [2], let us choose the control input according to

$$\begin{aligned} u(x, \dot{x}) &= Y(x, \dot{x}, a, v)\hat{\theta} - cr, \\ Y(x, \dot{x}, a, v) &= [a \ v \ x], \end{aligned} \quad (2)$$

where the quantities  $v$ ,  $a$ , and  $r$  are given as

$$\begin{aligned} v &= \dot{x}_r - \lambda\tilde{x}, \\ a &= \dot{v} = \ddot{x}_r - \lambda\dot{\tilde{x}}, \\ r &= \dot{x} - v = \dot{\tilde{x}} + \lambda\tilde{x}, \end{aligned}$$

where  $c, \lambda > 0$  are constant gains. Substituting the control law (2) into the system model (1) leads to

$$m\dot{r} + (b + c)r = -Y\tilde{\theta}. \quad (3)$$

The parameter estimate  $\hat{\theta}$  may be computed using standard methods of adaptive control such as gradients or least squares.

For example, for a positive definite matrix  $\Gamma$  of appropriate dimensions, we can use the gradient update law

$$\dot{\hat{\theta}} = -\Gamma^{-1}Y^\top(x, \dot{x}, a, v)r. \quad (4)$$

Consider the Lyapunov function candidate

$$V(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) = \frac{1}{2}mr^2 + c\lambda\tilde{x}^2 + \frac{1}{2}\tilde{\theta}^\top\Gamma\tilde{\theta}.$$

This is a positive definite function over the space of  $(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta})$ . We take the time derivative of the Lyapunov function candidate and substitute from the closed-loop system dynamics (3, 4). We suppress its functional dependence for brevity.

$$\begin{aligned} \dot{V} &= mr\dot{r} + 2c\lambda\tilde{x}\dot{\tilde{x}} + \tilde{\theta}^\top\Gamma\dot{\tilde{\theta}} \\ &= r\left(-(b+c)r - Y\tilde{\theta}\right) + 2c\lambda\tilde{x}\dot{\tilde{x}} + \tilde{\theta}^\top Y^\top r \\ &= -e^\top Qe \leq 0, \end{aligned} \quad (5)$$

where  $Q$  is a symmetric, positive-definite matrix given as

$$Q = \begin{bmatrix} (b+c)\lambda^2 & b\lambda \\ b\lambda & b+c \end{bmatrix}.$$

Integrating both sides of equation (5) gives

$$V(t) - V(0) = -\int_0^t e^\top(\sigma)Qe(\sigma) d\sigma < \infty.$$

We observe that  $\dot{\tilde{x}}$  is bounded because  $\dot{V} \leq 0$  implies that the terms  $r$ ,  $\tilde{x}$  and  $\tilde{\theta}$  are bounded functions of time. This allows us to invoke Barbalat's lemma [2] to deduce that  $\tilde{x} \rightarrow 0$  as  $t \rightarrow \infty$ . Furthermore, using equation (3), we can readily see that  $\dot{\tilde{x}}$  is bounded. Another application of Barbalat's lemma shows that the velocity error  $\dot{\tilde{x}} \rightarrow 0$  provided that the reference acceleration  $\ddot{x}_r(t)$  is bounded. Since  $x \rightarrow x_r$ , we also have that  $u(x, \dot{x}) \rightarrow \hat{m}\ddot{x}_r + \hat{b}\dot{x}_r + \hat{k}x_r$ .

For any  $\eta > 0$ , the set  $\Omega_\eta = \{(e, \tilde{\theta}) : V(\tilde{x}, \dot{\tilde{x}}, \tilde{\theta}) \leq \eta\}$  is positively invariant. The positive limit set of  $(e(t), \tilde{\theta}(t))$  is a subset of  $E = \{(e, \tilde{\theta}) : e = 0\}$ . Unfortunately, for a general nonautonomous system, the positive limit set is not necessarily a positively invariant set, precluding us to invoke LaSalle's theorem to conclude that  $\tilde{\theta} \rightarrow 0$  [1].

We can excite several frequencies by choosing

$$\begin{aligned} x_r(t) &= A \sin(\omega(t) + \varphi) \\ \omega(t) &= \pi \sum_{k=1}^n \left(1 - \frac{k-1}{n}\right) \sin kt \end{aligned}$$

for some constants  $\varphi$ ,  $A > 0$ , and a sufficiently large  $n \in \mathbb{N}$ . Simulation results in the next section suggests that this coaxes  $\tilde{\theta} \rightarrow 0$  even though we were only able to show that  $\tilde{\theta}$  is bounded in theory.

### III. RESULTS

In Figure 1, we plot the response of the system to the control and adaptation laws (2, 4), implemented in simulation. The constants that are used are as follows:  $(A, n, \varphi) = (3/10, 3, 0^\circ)$ ,  $\Gamma = \text{diag}(1, 1, 1/10)$  and  $(c, \lambda) = (1, 4)$ . The real mass, damping and stiffness of the system are  $(m, b, k) = (0.665, 1.819, 0)$  and their estimates start at  $(\hat{m}, \hat{b}) = (-1/2, -1/4, -1)$ .

### IV. CONCLUSION

We have shown that our control and adaptation laws for the linear second-order mechanical system guide the estimates of the parameters to tend to their correct values while tracking a judiciously chosen reference signal.

### REFERENCES

- [1] H. Khalil, *Nonlinear Control*. Always Learning, Pearson, 2015.
- [2] M. Spong, S. Hutchinson, and M. Vidyasagar, *Robot Modeling and Control*. Wiley, 2020.

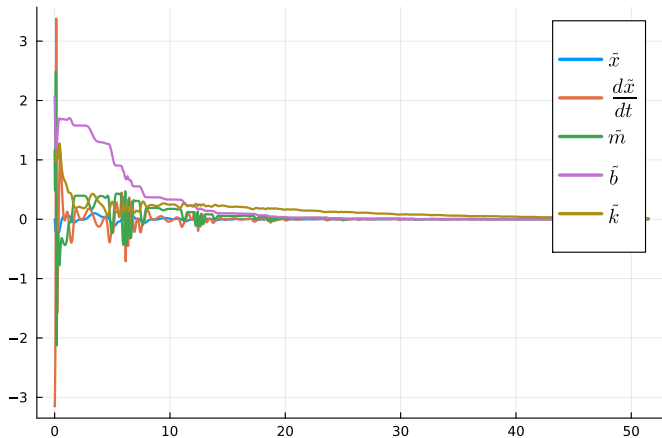


Fig. 1: Simulation showing the convergence of the system state and parameter estimates.