Position-Level Kinematic Analysis

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ME380: Kinematics and Machine Dynamics
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Overview

Introduction

Forward Kinematics

Inverse Kinematics

Vector Cross Product Method

Introduction

Most modern manipulators consist of a set of rigid links connected together by a set of joints. Motors are attached to some of the joints so that the overall motion of the mechanism can be controlled to perform a given task. A tool, typically a gripper of some sort, is attached to the end of the robot to interact with the environment.

The kinematics of a mechanism or robot manipulator describes the relationship between the motion of the joints of the manipulator and the resulting motion of the rigid bodies which form the manipulator.

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Forward Kinematics

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Vector Cross Product Method

Definition

The *forward kinematics* of a robot determines the configuration of the end-effector (the gripper or tool mounted on the end of the robot) given the relative configurations of some (or all in the case of open-chain mechanisms) of the links of the mechanism.

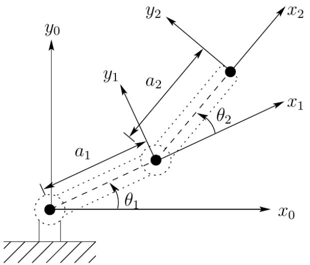
For a parallel manipulator, this procedure might also involve computing the configuration of the remaining links of the mechanism.

Computation of forward kinematics map

1. Designate a point fixed to each link (world or base link included) and attach a coordinate frame at this point.
2. Figure out the relationship between adjacent coordinate frames.
3. Find the relationship from the base frame to the end-effector frame, by composing adjacent transformations.
4. If the mechanism is closed-loop (parallel), work out the loop equations.

Two-Link Manipulator

- The link lenghts a_1 and a_2 are given.
- Given θ_1 and θ_2, figure out the pose (position and orientation) of frame 2, $\Sigma_2 = o_2 x_2 y_2 z_2$.

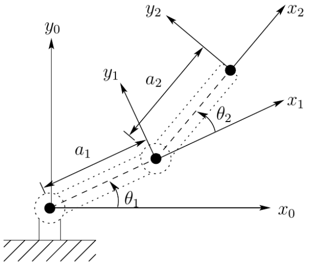


Two-link planar manipulator. The z-axes all point out of the page, and are not shown in the figure.

Two-Link Manipulator

$\Sigma_0 \to \Sigma_1$

- The origin o_1 is displaced from the origin o_0 by a_1 units in the x_1-direction.
- x_1 expressed in the Σ_1 coordinates is $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$.
- We must find the coordinates of x_1 in Σ_0. Notice that x_1 is a vector!
- $x_1^0 = R_{01} x_1^1$. What is R_{01}?



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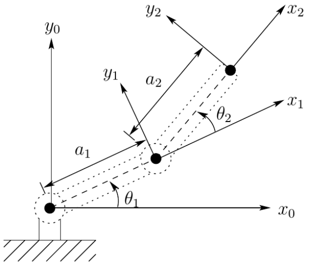
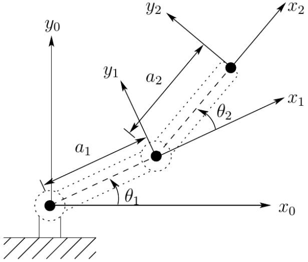


$$R_{01} = R_{z,\theta_1} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Two-Link Manipulator

$\Sigma_0 \rightarrow \Sigma_1$

- The position vector p_{01} from o_0 to o_1 is thus given by



$$p_{01} = R_{01} \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix}.$$

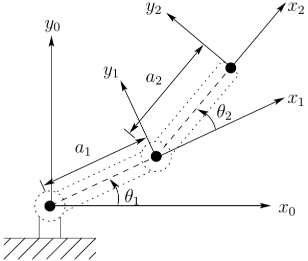
$$g_{01} = \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Two-Link Manipulator

$\Sigma_1 \to \Sigma_2$

- $p_{12} = a_2 R_{12} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$
- $R_{12} = R_{z,\theta_2} = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $p_{12} = \begin{bmatrix} a_2 c_2 & a_2 s_2 & 0 \end{bmatrix}^\top.$



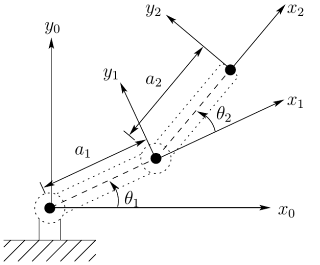
$$
\begin{aligned}
g_{12} &= \begin{bmatrix} R_{12} & p_{12} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Two-Link Manipulator

Recall

$$g_{01} = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$g_{12} = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$



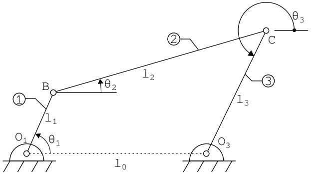
$\Sigma_0 \to \Sigma_2$

- Compose the two homogeneous transformations to find $g_{02} = g_{01} g_{12}$.

$$\begin{aligned} g_{02} &= g_{01} g_{12} \\ &= \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_{02} & p_{02} \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Four-bar linkage

- The link lengths l_0, l_1, l_2, and l_3 are given.
- Given θ_1, figure out the location of point C.
- Place coordinate frame $\Sigma_i,\ i \in 0, 1, 2, 3$ at O_1, B, C, and O_3, respectively.
- Directions:
    - x_1: O_1 to B,
    - x_2: B to C,
    - x_3: C to O_3.



Four-bar mechanism. The z-axes of all coordinate frames point out of the page.

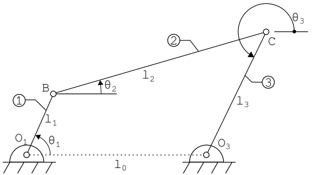
Four-bar linkage

Rotation matrices between adjacent links

$$R_{01} = R_{z,\theta_1} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_{02} = R_{z,\theta_2} = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_{03} = R_{z,\theta_3} = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



$$p_{01} = R_{01} \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}.$$

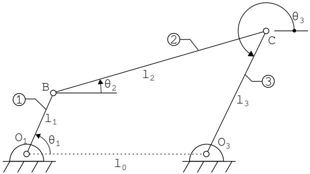
$$p_{12}^0 = R_{02} \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} l_2 c_2 \\ l_2 s_2 \\ 0 \end{bmatrix}.$$

Four-bar linkage

Loop equations

Let's write two position vectors from O_1 to C.

1. $p_{02} = p_{01} + p_{12}^0 = \begin{bmatrix} l_1 c_1 + l_2 c_2 \\ l_1 s_1 + l_2 s_2 \end{bmatrix}.$



Objective

We solve the loop equations for θ_2 and θ_3 and use the solution in the expression for g_{02} to find the location of point C (or the origin of Σ_2).

Four-bar linkage

Loop equations

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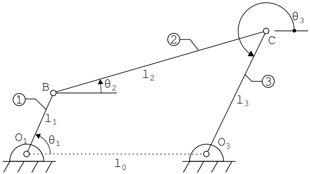
1. $p_{02} = p_{01} + p_{12}^0 = \begin{bmatrix} l_1 c_1 + l_2 c_2 \\ l_1 s_1 + l_2 s_2 \end{bmatrix}$.
2. $p_{02} = p_{O_1 O_3} + p_{O_3 C} = \begin{bmatrix} l_0 - l_3 c_3 \\ -l_3 s_3 \end{bmatrix}$ (why?).

Equating these two, we obtain the loop equations.

$$\begin{bmatrix} l_1 c_1 + l_2 c_2 + l_3 c_3 - l_0 \\ l_1 s_1 + l_2 s_2 + l_3 s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$



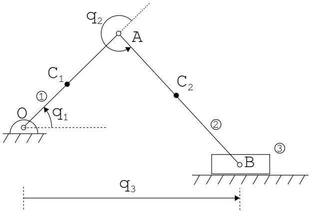
Objective

We solve the loop equations for θ_2 and θ_3 and use the solution in the expression for g_{02} to find the location of point C (or the origin of Σ_2).

Slider-crank linkage

Your turn!

- The link lengths a_1, a_2, and a_0 are given, where a_0 is the vertical distance between O and B.
- Figure out the position of the slider, given the driver crank angle q_1.



Slider-crank mechanism. The z-axes of all coordinate frames point out of the page. Lengths of links 1 and 2 are given by a_1 and a_2, while the vertical distance between points O and B is given by a_0.

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Definition

The general problem of *inverse kinematics* can be stated as follows. Given a 4×4 homogeneous transformation

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3),$$

with $R \in SO(3)$, find (one or all) solutions of the equation

$$g_{0n}(q_1, \ldots, q_n) = H.$$

Remark

For closed-loop (parallel) mechanisms, we will still have to deal with the additional loop constraint equation when solving for the inverse kinematics problem.

Two-link manipulator

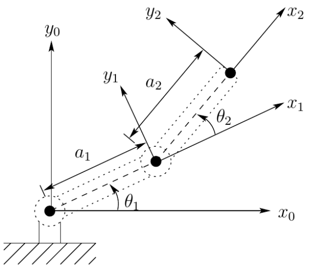
Given a pose for frame 2, find the values of θ_1 and θ_2.

Method 1

Solve the equations

$$x = a_1 c_1 + a_2 c_{12}$$
$$y = a_1 s_1 + a_2 s_{12}$$

for θ_1 and θ_2.

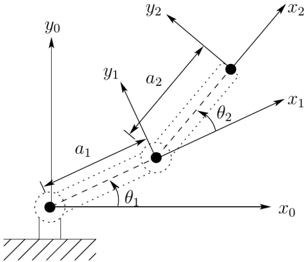


Two-link manipulator

Method 2

We invoke trigonometry. Use law of cosines to get

$$\cos\theta_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2} =: D.$$



We could now determine θ_2 as

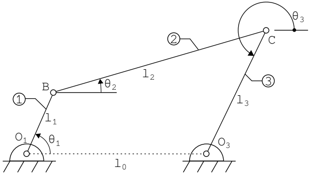
$$\theta_2 = \mathrm{atan}_2\left(\pm\sqrt{1-D^2}, D\right).$$

Exercise: Show that θ_1 is given by

$$\theta_1 = \mathrm{atan}_2(y,x) - \mathrm{atan}_2\left(a_2\sin\theta_2, a_1 + a_2\cos\theta_2.\right)$$

Four-bar linkage

Given the location of point C, find the joint angles θ_1, θ_2, and θ_3.

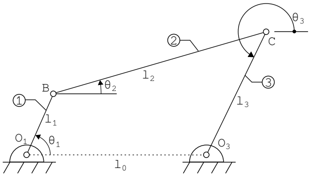


Four-bar linkage

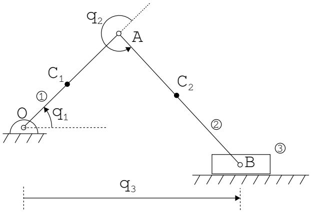
Given the location of point C, find the joint angles θ_1, θ_2, and θ_3.

The left chain: $O_1 - B - C$ is a two-link manipulator that we have already discussed.
The right-chain: is a simple pendulum and so if (x, y) denotes the position vector from O_1 to C, then we have

$$\theta_3 = -\pi + \operatorname{atan}_2(y, x - l_0).$$



Slider-crank linkage



Slider-crank mechanism. The z-axes of all coordinate frames point out of the page. Lengths of links 1 and 2 are given by a_1 and a_2, while the vertical distance between points O and B is given by a_0.

Your turn!

Given the location of point B, determine the joint variables q_1, q_2, and q_3.

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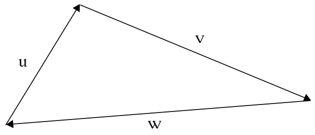
Solving Planar Vector Equations

Consider the following planar vector equation

$$u + v + w = 0, \qquad (1)$$

Let $\hat{u}$, $\hat{v}$, $\hat{w}$ be unit vectors along u, v, w, resp. Then (1) becomes

$$|u|\,\hat{u} + |v|\,\hat{v} + |w|\,\hat{w} = 0. \qquad (2)$$



Graphical Representation

Questions.

Case 1. Given u and v, find w (exercise).
Case 2. Given w, $\hat{u}$, and $\hat{v}$, find $|u|$ and $|v|$.
Case 3. Given w, $|u|$, and $|v|$, find $\hat{u}$ and $\hat{v}$.
Case 4. Given w, $\hat{u}$, and $|v|$, find $|u|$ and $\hat{v}$.

Case 2

Take the dot product of each term in equation (2) with $\hat{v} \times \hat{k}$:

$$|u|\,\hat{u} \cdot \left(\hat{v} \times \hat{k}\right) + w \cdot \left(\hat{v} \times \hat{k}\right) = 0,$$

and solve for $|u|$.

$$|u| = \frac{-w \cdot \left(\hat{v} \times \hat{k}\right)}{\hat{u} \cdot \left(\hat{v} \times \hat{k}\right)}.$$

Similarly, take the dot product of each term in equation (2) with $\hat{u} \times \hat{k}$ and solve for $|v|$:

$$|v| = \frac{-w \cdot \left(\hat{u} \times \hat{k}\right)}{\hat{v} \cdot \left(\hat{u} \times \hat{k}\right)}.$$

Case 3

Define

$$\Delta = \frac{|w|^2 + |v|^2 - |u|^2}{2\,|w|}.$$

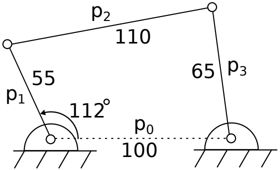
Now, we can find the vectors u and v (`NOT` the unit vectors along them!) by

$$u = \mp\left(\sqrt{|v|^2 - \Delta^2}\right)\left(\hat{w}\times\hat{k}\right) + (\Delta - |w|)\,\hat{w},$$

$$v = \pm\left(\sqrt{|v|^2 - \Delta^2}\right)\left(\hat{w}\times\hat{k}\right) - \Delta\hat{w},$$

where the upper set of signs are used if the vectors u, v, and w are oriented in a counter-clockwise manner.

Example: The four-bar linkage



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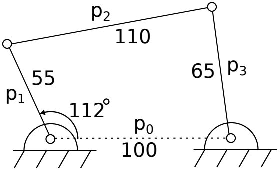


$$p_0 = \begin{bmatrix} -100 \\ 0 \end{bmatrix}, p_1 = 55 \begin{bmatrix} \cos 112 \\ \sin 112 \end{bmatrix},$$
$$|p_2| = 110, \quad |p_3| = 65.$$
$$p_d := p_0 + p_1 = \begin{bmatrix} -120.603 \\ 50.995 \end{bmatrix}.$$
$$\hat{p}_d = \begin{bmatrix} -0.92105 \\ 0.38945 \end{bmatrix}, \;\; |p_d| = 130.941,$$
$$\Delta := \frac{|p_d|^2 + |p_3|^2 - |p_2|^2}{2|p_d|} = 35.4.$$

Example: The four-bar linkage

$$p_2 + p_3 + p_d = 0,$$

$$p_2 = \mp\left(\sqrt{|p_3|^2 - \Delta^2}\right)\begin{bmatrix}-0.92105\\0.38945\\0\end{bmatrix} \times \begin{bmatrix}0\\0\\1\end{bmatrix}$$

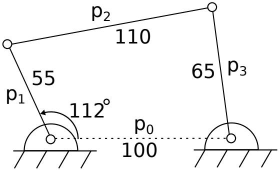
$$+(\Delta + |p_d|)\begin{bmatrix}-0.92105\\0.38945\\0\end{bmatrix}$$

$$\in \left\{\begin{bmatrix}66.7675\\-87.4191\end{bmatrix}, \begin{bmatrix}109.2289\\13.0019\end{bmatrix}\right\}.$$

$$p_3 = \pm\left(\sqrt{|p_3|^2 - \Delta^2}\right)\begin{bmatrix}-0.92105\\0.38945\\0\end{bmatrix} \times \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$-\Delta\begin{bmatrix}-0.92105\\0.38945\\0\end{bmatrix}$$

$$\in \left\{\begin{bmatrix}53.8358\\36.4240\end{bmatrix}, \begin{bmatrix}11.3745\\-63.9970\end{bmatrix}\right\}.$$



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Case 4

Define

$$\Delta = w \cdot \left(\hat{u} \times \hat{k}\right).$$

We can find u and v by

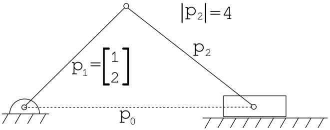
$$u = \left(-w \cdot \hat{u} \mp \sqrt{|v|^2 - \Delta^2}\right)\hat{u},$$

$$v = -\Delta\left(\hat{u} \times \hat{k}\right) \pm \left(\sqrt{|v|^2 - \Delta^2}\right)\hat{u},$$

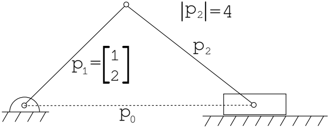
where we pick the correct signs by inspecting the mechanism in question.

Notice: These equations yield the vectors u and v. $|u|$ and $\hat{v}$ need to be computed afterwards if that is what is desired.

Example: Slider-Crank



Example: Slider-Crank



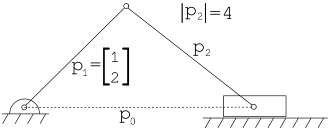
$$\hat{p}_0 \times \hat{k} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$
$$\Delta := p_1 \cdot (\hat{p}_0 \times \hat{k}) = 2,$$
$$p_1 \cdot \hat{p}_0 = -1.$$

Example: Slider-Crank

$$p_0 + p_2 + p_1 = 0,$$

$$\begin{aligned} p_0 &= (1 \mp \sqrt{16 - \Delta^2})\hat{p}_0 \\ &\in \left\{ -2.464 \begin{bmatrix} -1 \\ 0 \end{bmatrix}, 4.466 \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}, \end{aligned}$$

$$\begin{aligned} p_2 &= -2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \pm \sqrt{16 - \Delta^2} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \\ &\in \left\{ -2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \pm 2\sqrt{3} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}. \end{aligned}$$



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$$p_1 \cdot \hat{p}_0 = -1.$$