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Vector Analysis of Linkages

Expression of vectors in spherical coordinates within an appropriately defined reference frame facilitates explicit, general solutions to a wide range of linkage problems. To demonstrate this, a position, velocity, and acceleration analysis is performed on an example of each of three linkage types: simple, complex, and spatial. Also, the mechanical advantage and the transmission criterion of a four-bar spatial linkage are determined. Solutions are obtained in a form which simplifies digital computer programming.

Introduction

In essence, linkage analysis is equivalent to the solution of vector equations for position, motion, and force. Simultaneous solution is often not required, except for synthesis or the analysis of complex linkages. Thus, the usual difficulty lies in obtaining solutions to single vector equations, most of which cannot be solved by simple rearrangement of terms. It is not widely appreciated that manageable, explicit, general solutions can be derived for practically all such equations in two or three dimensions. This leads to a direct, completely mathematical method of linkage analysis. However, physical interpretation is not obscured, because the solutions obtained are in the form of ordinary vector equations.

Certainly there exist many more or less related means for linkage analysis, some of which make direct use of vector methods. A reference list to several approaches is included here [1–12],[1] though no overall review is intended. Of particular relevance, a recent paper by R. Beyer [2] states the solution to the vector equation in which the only unknowns are vector magnitudes. This is of frequent use in finding magnitudes of velocity and acceleration but is seldom sufficient to solve position vector equations, because these usually involve unknown directions. Important suggestions for these solutions are provided by J. E. Shigley's unit vector approach [12] in which emphasis is placed on stating vectors in spherical coordinates as products of magnitude and unit vector. This focuses attention on the possibility of solving for magnitude or direction individually, and makes the solution itself more convenient. The only remaining technique required is appropriate orientation of the reference frame.

To demonstrate the scope of the approach analysis is performed on an example of each of three linkage types: the simple four-bar linkage, the complex Wanzer needle-bar mechanism [15], and the spatial four-bar linkage with one rotating pair and three cylinder pairs. For each, explicit vector expressions are obtained for position, velocity, and acceleration. Additionally, a summary is made of all solutions to the plane vector polygon (Table 1) and expressions are derived for the mechanical advantage and transmission criterion of the spatial four-bar linkage. A discussion of the general procedure of the analysis concludes the paper and a brief review of essential vector operations is included in the Appendix.

[1] Numbers in brackets designate References at end of paper.

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The Plane Four-Bar Linkage

The Vector Polygon. The link position vectors of the plane four-bar linkage form a special case of the n-vector polygon shown in Fig. 1. Analysis of this polygon yields results of general applicability, from which the equations for the link position vectors of the four-bar linkage are obtained.

It is assumed that the only equation governing the polygon is

$$\mathbf{C} + s\hat{\mathbf{s}} + t\hat{\mathbf{t}} = 0 \qquad (1)$$

If the polygon lies in a single plane, equation (1) represents two independent scalar equations. These can only be solved if there are no more than two scalar unknowns. However, each vector is defined by two scalars: the magnitude and the angle specifying the direction, or unit vector. Thus, no more than two vectors ($s\hat{\mathbf{s}}$ and $t\hat{\mathbf{t}}$) can have either magnitude or direction unknown. All other vectors are summed into a single known vector.

Nomenclature

General

$\mathbf{u}$ = vector quantity
u = magnitude of $\mathbf{u}$
$\hat{\mathbf{u}}$ = unit vector of $\mathbf{u}$

Specific

$\mathbf{a}_{m_i n_j}$ = linear acceleration. Subscripts as for $\mathbf{v}_{m_i n_j}$
$\mathbf{C}_i$ = a known vector which is side i of the n-vector polygon (Fig. 1)
c_i = scalar constant determined by link design
$\mathbf{f}$ = force exerted by link i on link $i+1$ (Fig. 6)
$\mathbf{f}_o$ = output force
f_L = number of degrees of freedom of a linkage
f_{P_i} = number of degrees of freedom of pair i
h = number of higher pairs
$\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ = unit vectors of the ground reference frame
l = number of lower pairs
n = number of links
p = number of pairs
$\mathbf{r}$ = position vector
$\mathbf{r}_i$ = position vector of link i from joint with $i-1$ link to joint with $i+1$ link
$\mathbf{r}_{mn}$ = position vector to point m from point n
$\mathbf{s}, \mathbf{t}$ = vectors which may be unknown in magnitude and/or direction in the n-vector polygon (Fig. 1)
$\mathbf{v}_{m_i n_j}$ = linear velocity of point m fixed in link i relative to point n fixed in link j. Omit subscript n for velocity relative to ground. Omit i or j for points not fixed in a link.
$\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_{ij}$ = angular acceleration (see p. 290, item 6). Subscripts as for $\boldsymbol{\omega}_i, \boldsymbol{\omega}_{ij}$.
θ, ϕ = angles specifying direction of a unit vector
$\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\nu}}$ = unit vectors of a reference frame to be oriented for convenience relative to the $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ frame
$\boldsymbol{\tau}_{ij}$ = torque exerted by link i on link j
$\boldsymbol{\tau}_i, \boldsymbol{\tau}_o$ = input and output torques
$\boldsymbol{\omega}_i$ = angular velocity of link i relative to ground
$\boldsymbol{\omega}_{ij}$ = angular velocity of link i relative to link j

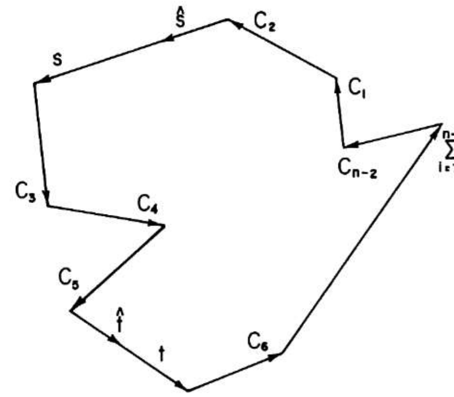

Fig. 1 The plane *n*-vector polygon. If determinate, this will always reduce to a vector triangle in which one side is the vector sum of all constant vectors C_i. Any two of the four quantities s, $\hat{s}$, t, and $\hat{t}$ may be unknown.

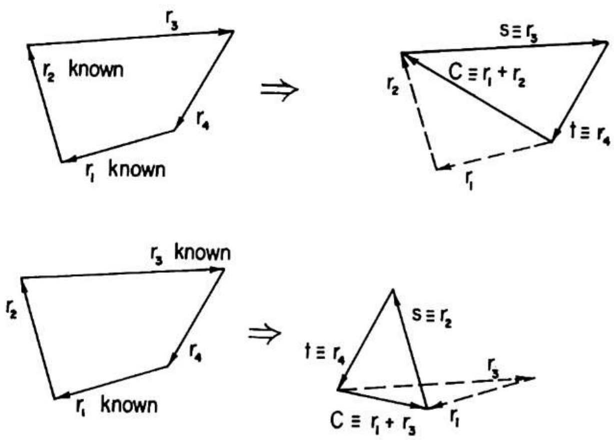

Fig. 2 Reduction of the plane four-bar linkage to a vector triangle

$$\mathbf{C} = \sum_{i=1}^{n-2} \mathbf{C}_i \tag{2}$$

Thus, the n-vector polygon reduces to a vector triangle of sides $\mathbf{C}$, $s\hat{\mathbf{s}}$, and $t\hat{\mathbf{t}}$ in which there are only six possible combinations of unknowns: s, $\hat{\mathbf{s}}$; t, $\hat{\mathbf{t}}$; s, t; $\hat{\mathbf{s}}$, t; s, $\hat{\mathbf{t}}$; and $\hat{\mathbf{s}}$, $\hat{\mathbf{t}}$. Mathematical solutions for all these combinations are derived in the Appendix and summarized in Table 1.

By their generality, the Table 1 equations apply to any two-dimensional situation governed only by equation (1), no matter what vector quantity the symbols represent. Of interest here, equations (111a, b) apply to a plane four-bar linkage when the links are replaced by position vectors, as shown in Fig. 2. In the usual four-bar analysis $\mathbf{r}_1$, $\mathbf{r}_2$, r_3, and r_4 are known and $\hat{\mathbf{r}}_3$ and $\hat{\mathbf{r}}_4$ are unknown. $\mathbf{r}_3$ and $\mathbf{r}_4$ are then given by equations (111a) and (111b), with $\mathbf{C}$, $\mathbf{s}$, and $\mathbf{t}$ replaced by $(\mathbf{r}_1 + \mathbf{r}_2)$, $\mathbf{r}_3$, and $\mathbf{r}_4$, respectively. The ambiguity in sign in equations (109a) through (111b) means that the position vectors may have either of two consistent configurations, depending on how the linkage was first assembled. This situation also occurs in many spatial linkages (see equation (50)).

Velocity and Acceleration. In principle, expressions for motion can be obtained directly by differentiating previously obtained expressions for position. However, in practice it is usually easier to substitute the appropriate derivative of $\mathbf{r}$ into a general equation of relative motion (e.g., equation (7)), then solve for the unknown motions by the same techniques used to solve position equations. Expressions for $\frac{d\mathbf{r}}{dt}$ and $\frac{d^2\mathbf{r}}{dt^2}$ are stated here, as derived [13] under the following assumptions:

1 Vector $\mathbf{r}$ is directed to point p from point q_i. The velocity and acceleration of p relative to q_i are equivalent to $d\mathbf{r}/dt$ and $d^2\mathbf{r}/dt^2$, respectively.

2 Point q_i and unit vector $\hat{\mathbf{r}}$ are fixed relative to link i, but length r is time-dependent.

3 Link i is part of a linkage having n links, of which link 1 is ground.

4 Link i undergoes linear and angular motion. In general, the motion vectors themselves are time-dependent in both magnitude and direction (e.g., ω_i and $\hat{\boldsymbol{\omega}}_i$ time-dependent).

5 Each link has an associated direction of rotation relative to an adjoining link (e.g., $\hat{\boldsymbol{\omega}}_{i,i-1}$).

6 $\boldsymbol{\alpha}_i$ is defined as $\sum_{j=2}^{i} \boldsymbol{\alpha}_{j,j-1}$, not $\frac{d\boldsymbol{\omega}_i}{dt}$.

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_{pq_i} = \left(\frac{dr}{dt}\right)\hat{\mathbf{r}} + (\boldsymbol{\omega}_i \times \mathbf{r}) \tag{3}$$

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}_{pq_i} = \left(\frac{d^2r}{dt^2}\right)\hat{\mathbf{r}} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}) + (\boldsymbol{\alpha}_i \times \mathbf{r}) + 2\left[\boldsymbol{\omega}_i \times \left(\frac{dr}{dt}\right)\hat{\mathbf{r}}\right] + \sum_{j=3}^{i} [(\boldsymbol{\omega}_{j-1} \times \boldsymbol{\omega}_{j,j-1}) \times \mathbf{r}] \tag{4}$$

Equations (3) and (4) are used directly when three-dimensional motion is involved, but for two-dimensional motion they can be simplified to equations (5) and (6). Here $\hat{\mathbf{k}}$ is defined as the unit vector perpendicular to the plane of motion.

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_{pq_i} = \left(\frac{dr}{dt}\right)\hat{\mathbf{r}} + \omega_i(\hat{\mathbf{k}} \times \mathbf{r}) \tag{5}$$

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}_{pq} = \left[\left(\frac{d^2r}{dt^2}\right) - \omega_i^2 r\right]\hat{\mathbf{r}} + \left[\alpha_i r + 2\left(\frac{dr}{dt}\right)\omega_i\right](\hat{\mathbf{k}} \times \hat{\mathbf{r}}) \tag{6}$$

For the four-bar linkage of Fig. 3,

$$\mathbf{v}_b + \mathbf{v}_{cb} - \mathbf{v}_c = 0 \tag{7}$$

Substitute equation (5) for each of the terms in equation (7). Here, ω_3 and ω_4 are unknown, ω_2 and $\mathbf{r}_2$ are known, and $\omega_1 = 0$. $\mathbf{r}_3$ and $\mathbf{r}_4$ have already been determined by equations (111a, b).

$$\mathbf{V}_1 + \omega_3(\hat{\mathbf{k}} \times \mathbf{r}_3) + \omega_4(\hat{\mathbf{k}} \times \mathbf{r}_4) = 0 \tag{8}$$

$$\mathbf{V}_1 \equiv \omega_2(\hat{\mathbf{k}} \times \mathbf{r}_2)$$

To solve equation (8) for ω_3 take the dot product throughout with $\mathbf{r}_4$ to eliminate ω_4. Thus

$$\omega_3 = -\frac{\mathbf{V}_1 \cdot \mathbf{r}_4}{(\hat{\mathbf{k}} \times \mathbf{r}_3) \cdot \mathbf{r}_4} = -\omega_2 \frac{(\hat{\mathbf{k}} \times \mathbf{r}_2) \cdot \mathbf{r}_4}{(\hat{\mathbf{k}} \times \mathbf{r}_3) \cdot \mathbf{r}_4} \tag{9}$$

Similarly,

$$\omega_4 = -\frac{\mathbf{V}_1 \cdot \mathbf{r}_3}{(\hat{\mathbf{k}} \times \mathbf{r}_4) \cdot \mathbf{r}_3} = +\omega_2 \frac{(\hat{\mathbf{k}} \times \mathbf{r}_2) \cdot \mathbf{r}_3}{(\hat{\mathbf{k}} \times \mathbf{r}_3) \cdot \mathbf{r}_4} \tag{10}$$

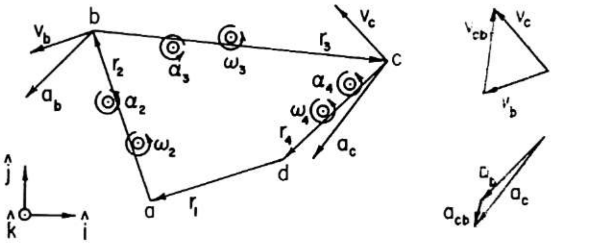

Fig. 3 Velocity and acceleration of the plane four-bar linkage. Link 1 is stationary with respect to $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$.

Linkage accelerations can be obtained by the same procedure. From Fig. 3,

$$\mathbf{a}_b + \mathbf{a}_{cb} - \mathbf{a}_c = 0 \quad (11)$$

Substitute equation (6) for each of the terms in equation (11). Here, α_3 and α_4 are unknown; α_2, ω_2, and $\mathbf{r}_2$ are known, and $\alpha_1 = \omega_1 = 0$. Also, ω_3, ω_4, $\mathbf{r}_3$, and $\mathbf{r}_4$ have already been determined.

$$\mathbf{V}_2 + \alpha_3(\hat{\mathbf{k}} \times \mathbf{r}_3) + \alpha_4(\hat{\mathbf{k}} \times \mathbf{r}_4) = 0 \quad (12)$$

$$\mathbf{V}_2 \equiv \alpha_2(\hat{\mathbf{k}} \times \mathbf{r}_2) - \omega_2^2\mathbf{r}_2 - \omega_3^2\mathbf{r}_3 - \omega_4^2\mathbf{r}_4$$

The solution of equation (12) proceeds in the same way as that of equation (8) because the two have identical forms. Thus,

$$\alpha_3 = -\frac{[(\boldsymbol{\alpha}_2 \times \mathbf{r}_2) - \omega_2^2\mathbf{r}_2 - \omega_3^2\mathbf{r}_3 - \omega_4^2\mathbf{r}_4]\cdot\mathbf{r}_4}{(\hat{\mathbf{k}} \times \mathbf{r}_3)\cdot\mathbf{r}_4} \quad (13)$$

$$\alpha_4 = +\frac{[(\boldsymbol{\alpha}_2 \times \mathbf{r}_2) - \omega_2^2\mathbf{r}_2 - \omega_3^2\mathbf{r}_3 - \omega_4^2\mathbf{r}_4]\cdot\mathbf{r}_3}{(\hat{\mathbf{k}} \times \mathbf{r}_3)\cdot\mathbf{r}_4} \quad (14)$$

Most of the essential equations for the plane four-bar linkage have now been derived. Beyond this, the following remarks are made:

1 Expressions for higher order motions can be derived by the same procedure as for velocity and acceleration, using the proper order derivative of $r\hat{\mathbf{r}}$.

2 Vector expressions for all linear motions can be written in terms of position vectors and angular motions.

3 In many plane linkages one or more of the link lengths can be considered to vary with time (e.g., the slider-crank mechanism). For such linkages it is necessary to substitute more terms from equations (3) and (4) than were used to develop equations (8) and (12). However, the solution can usually still be obtained by taking dot products to eliminate unknown magnitudes, in a way similar to the derivation of equation (9).

The Complex Wanzer Needle-Bar Mechanism

Analysis of complex linkages typically requires simultaneous solutions of sets of vector equations, a set each for the position, velocity, and acceleration vectors. The velocity and acceleration solutions usually present little difficulty. However, the position vector solution may be more involved, depending on how many unit vectors are unknown.

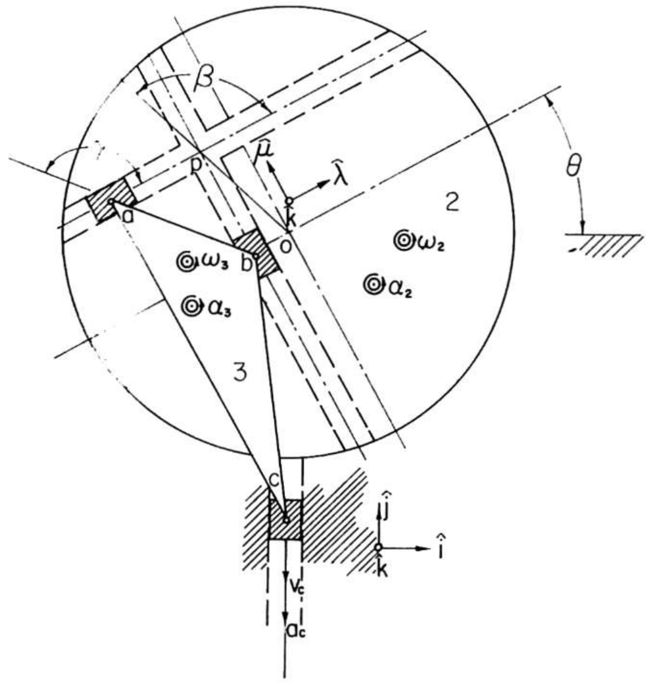

Fig. 4 The Wanzer needle-bar mechanism. Link 2 rotates about O and link 3 moves according to the constraints of sliders a, b, and c. Reference frame $\hat{\lambda}$, $\hat{\mu}$ facilitates the solution.

Position Vectors. Refer to Fig. 4. The following set of equations is sufficient to determine the possible positions of the linkage, although there are other sets which would also serve the purpose.

$$\mathbf{r}_{ab} + \mathbf{r}_{pa} + \mathbf{r}_{bp} = 0 \quad (15)$$

$$\mathbf{r}_{bp} + \mathbf{r}_{cb} + \mathbf{r}_{pc} = 0 \quad (16)$$

A reference frame is now defined so as to simplify the eventual form of equations (15) and (16) as much as possible. A natural choice is that shown, where the unit vectors can be defined in terms of the ground frame and the instantaneous value of the wheel angle, θ. At this point $\mathbf{r}_{po}$ can also be defined in terms of θ and the constant angle β.

$$\hat{\boldsymbol{\lambda}} \equiv (\cos\theta)\hat{\mathbf{i}} + (\sin\theta)\hat{\mathbf{j}} \quad (17)$$

$$\hat{\boldsymbol{\mu}} \equiv -(\sin\theta)\hat{\mathbf{i}} + (\cos\theta)\hat{\mathbf{j}} \quad (18)$$

$$\mathbf{r}_{po} \equiv r_{po}\{[\cos(\theta+\beta)]\hat{\mathbf{i}} + [\sin(\theta+\beta)]\hat{\mathbf{j}}\} \quad (19)$$

Substitutions are now made in equations (15) and (16) to reduce the number of unknowns to $\hat{\mathbf{r}}_{ab}$, r_{pa}, r_{bp}, and r_{oc}. These can be counted as only four scalar unknowns because $\hat{\mathbf{r}}_{ab}$ is completely determined by a single scalar quantity, the angle γ. c_1 and c_2 are constants of the triangle.

$$r_{ab}\hat{\mathbf{r}}_{ab} + r_{pa}\hat{\boldsymbol{\lambda}} + r_{bp}\hat{\boldsymbol{\mu}} = 0 \quad (20)$$

$$r_{bp}\hat{\boldsymbol{\mu}} + [c_1\hat{\mathbf{r}}_{ab} + c_2(\hat{\mathbf{k}} \times \hat{\mathbf{r}}_{ab})] + [\mathbf{r}_{po} + r_{oc}\hat{\mathbf{j}}] = 0 \quad (21)$$

Equations (20) and (21) are now expressed in terms of their components on the $\hat{\boldsymbol{\lambda}}$, $\hat{\boldsymbol{\mu}}$ directions.

$$r_{ab}\cos\gamma + r_{pa} = 0 \quad (22)$$

$$r_{ab}\sin\gamma + r_{bp} = 0 \quad (23)$$

$$(c_1\cos\gamma - c_2\sin\gamma) + \mathbf{r}_{po}\cdot\hat{\boldsymbol{\lambda}} + r_{oc}(\hat{\mathbf{j}}\cdot\hat{\boldsymbol{\lambda}}) = 0 \quad (24)$$

$$r_{bp} + (c_1\sin\gamma + c_2\cos\gamma) + \mathbf{r}_{po}\cdot\hat{\boldsymbol{\lambda}} + r_{oc}(\hat{\mathbf{j}}\cdot\hat{\boldsymbol{\mu}}) = 0 \quad (25)$$

Equations (22) through (25) can now be solved as follows:

1 Note that r_{pa} does not occur in equations (23), (24), or (25).

2 Use equation (23) to replace r_{bp} in equation (25).

3 Eliminate r_{oc} by subtraction, using equations (24) and (25).

4 State the resulting equation entirely in terms of either $\cos\gamma$ or $\sin\gamma$ by means of the identity $\cos^2\gamma + \sin^2\gamma = 1$, then solve for $\cos\gamma$ and $\sin\gamma$.

5 Express $\hat{\mathbf{r}}_{ab}$ as $(\cos\gamma)\hat{\boldsymbol{\lambda}} + (\sin\gamma)\hat{\boldsymbol{\mu}}$. Equation (26) results.

$$\hat{\mathbf{r}}_{ab} = \frac{1}{(A_3^2 + A_4^2)}\{\pm[(A_3^2 + A_4^2) - A_2^2]^{1/2}(A_4\hat{\boldsymbol{\lambda}} - A_3\hat{\boldsymbol{\mu}}) - A_2(A_3\hat{\boldsymbol{\lambda}} + A_4\hat{\boldsymbol{\mu}})\} \quad (26)$$

$$A_1 \equiv (\hat{\mathbf{j}}\cdot\hat{\boldsymbol{\mu}})/(\hat{\mathbf{j}}\cdot\hat{\boldsymbol{\lambda}})$$

$$A_2 \equiv \mathbf{r}_{po}\cdot[\hat{\boldsymbol{\mu}} - A_1\hat{\boldsymbol{\lambda}}]$$

$$A_3 \equiv (c_2 - A_1c_1)$$

$$A_4 \equiv (c_1 + A_1c_2 - r_{ab})$$

All other position vectors can now be determined directly.

Velocity. With the position vectors known the following two general equations are sufficient to determine the linkage velocities:

$$\mathbf{v}_{c_3} = \mathbf{v}_{c_3a_3} + \mathbf{v}_{a_3a_2} + \mathbf{v}_{a_2} \quad (27)$$

$$\mathbf{v}_{c_3} = \mathbf{v}_{c_3b_3} + \mathbf{v}_{b_3b_2} + \mathbf{v}_{b_2} \quad (28)$$

Substitute equation (5) for the first and third terms in equations (27) and (28). Also, substitute $\hat{\mathbf{v}}_{c_3} = \hat{\mathbf{j}}$, $\hat{\mathbf{v}}_{a_3a_2} = \hat{\boldsymbol{\lambda}}$, and $\hat{\mathbf{v}}_{b_3b_2} = \hat{\boldsymbol{\mu}}$.

$$v_{c_3}\hat{\mathbf{j}} = \omega(\mathbf{k} \times \mathbf{r}_{ca}) + v_{a_3a_2}\hat{\boldsymbol{\lambda}} + \mathbf{V}_3 \qquad (29)$$

$$v_{c_3}\hat{\mathbf{j}} = \omega_3(\hat{\mathbf{k}} \times \mathbf{r}_{cb}) + v_{b_3b_2}\hat{\boldsymbol{\mu}} + \mathbf{V}_4 \qquad (30)$$

$$\mathbf{V}_3 \equiv \boldsymbol{\omega}_2 \times \mathbf{r}_{ao}$$

$$\mathbf{V}_4 \equiv \boldsymbol{\omega}_2 \times \mathbf{r}_{bo}$$

Equations (29) and (30) contain only four scalar unknowns: v_{c_3}, ω_3, $v_{a_3a_2}$, and $v_{b_3b_2}$. Of these, $v_{a_3a_2}$ and $v_{b_3b_2}$ can be eliminated by taking dot products throughout with $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\lambda}}$, respectively. Then ω_3 can be eliminated by subtraction and the resulting equation solved for v_{c_3}.

$$v_{c_3} = \frac{\mathbf{V}_3 \cdot \hat{\boldsymbol{\mu}} - A_5 \mathbf{V}_4 \cdot \hat{\boldsymbol{\lambda}}}{\hat{\mathbf{j}} \cdot (\hat{\boldsymbol{\mu}} - A_5 \hat{\boldsymbol{\lambda}})} \qquad (31)$$

$$A_5 \equiv \frac{(\hat{\mathbf{k}} \times \mathbf{r}_{ca}) \cdot \hat{\boldsymbol{\mu}}}{(\hat{\mathbf{k}} \times \mathbf{r}_{cb}) \cdot \hat{\boldsymbol{\lambda}}}$$

Alternately, ω_3 can be found by eliminating v_{c_3} by subtraction. With v_{c_3} and ω_3 known, $v_{a_3a_2}$ and $v_{b_3b_2}$ can be obtained directly from equations (29) and (30).

Acceleration. Accelerations are found similarly except that care must be exercised in dealing with Coriolis terms. The general equations are

$$\mathbf{a}_{c_3} = \mathbf{a}_{c_3a_3} + \mathbf{a}_{a_3a_2} + \mathbf{a}_{a_2} \qquad (32)$$

$$\mathbf{a}_{c_3} = \mathbf{a}_{c_3b_3} + \mathbf{a}_{b_3b_2} + \mathbf{a}_{b_2} \qquad (33)$$

Equation (4) may be substituted for the first and third terms in equations (32) and (33), just as equation (3) was substituted in the velocity analysis. Also, $\hat{\mathbf{a}}_{c_3} = \hat{\mathbf{j}}$. However, $\hat{\mathbf{a}}_{a_3a_2}$ and $\hat{\mathbf{a}}_{b_3b_2}$ are not known, so that $\mathbf{a}_{a_3a_2}$ and $\mathbf{a}_{b_3b_2}$ must be entered in equations (32) and (33) in the following form:

$$\mathbf{a}_{a_3a_2} = (\mathbf{a}_{a_3a_2} \cdot \hat{\boldsymbol{\lambda}})\hat{\boldsymbol{\lambda}} + 2(\boldsymbol{\omega}_2 \times \mathbf{v}_{a_3a_2}) \qquad (34)$$

$$\mathbf{a}_{b_3b_2} = (\mathbf{a}_{b_3b_2} \cdot \hat{\boldsymbol{\mu}})\hat{\boldsymbol{\mu}} + 2(\boldsymbol{\omega}_2 \times \mathbf{v}_{b_3b_2}) \qquad (35)$$

When the indicated substitutions are made,

$$a_{c_3}\hat{\mathbf{j}} = \alpha_3(\hat{\mathbf{k}} \times \mathbf{r}_{ca}) + (\mathbf{a}_{a_3a_2} \cdot \hat{\boldsymbol{\lambda}})\hat{\boldsymbol{\lambda}} + \mathbf{V}_5 \qquad (36)$$

$$a_{c_3}\hat{\mathbf{j}} = \alpha_3(\hat{\mathbf{k}} \times \mathbf{r}_{cb}) + (\mathbf{a}_{b_3b_2} \cdot \hat{\boldsymbol{\mu}})\hat{\boldsymbol{\mu}} + \mathbf{V}_6 \qquad (37)$$

$$\mathbf{V}_5 \equiv (\boldsymbol{\alpha}_2 \times \mathbf{r}_{ao}) + 2(\boldsymbol{\omega}_2 \times \mathbf{v}_{a_3a_2}) - \omega_2^2\mathbf{r}_{ao} - \omega_3^2\mathbf{r}_{ca}$$

$$\mathbf{V}_6 \equiv (\boldsymbol{\alpha}_2 \times \mathbf{r}_{bo}) + 2(\boldsymbol{\omega}_2 \times \mathbf{v}_{b_3b_2}) - \omega_2^2\mathbf{r}_{bo} - \omega_3^2\mathbf{r}_{cb}$$

Equations (36) and (37) have the same form and the same number of scalar unknowns as equations (29) and (30). By analogy, the solution for a_{c_3} is,

$$a_{c_3} = \frac{\mathbf{V}_5 \cdot \hat{\boldsymbol{\mu}} - A_5 \mathbf{V}_6 \cdot \hat{\boldsymbol{\lambda}}}{\hat{\mathbf{j}} \cdot (\hat{\boldsymbol{\mu}} - A_5 \hat{\boldsymbol{\lambda}})} \qquad (38)$$

The Spatial Four-Bar Linkage

Use of the present vector method seems particularly appropriate for spatial problems. To illustrate, the linkage shown in Fig. 5[2] is analyzed for position, velocity, acceleration, mechanical advantage, and transmission criterion by essentially the same procedure that would be used for a plane linkage. In general, there is a very wide range of other spatial linkages which can be analyzed in this way, despite the correspondingly wide variety of position vector solutions required.

Position Vectors. The following three conditions are sufficient to determine the possible positions of the linkage shown in Fig. 5:

$$\mathbf{r}_1 + r_2\hat{\mathbf{r}}_2 + r_3\hat{\mathbf{r}}_3 + r_4\hat{\mathbf{r}}_4 = 0 \qquad (39)$$

$$(\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{r}}_3) \text{ known} \qquad (40)$$

$$(\hat{\mathbf{r}}_3 \cdot \hat{\mathbf{r}}_4) \text{ known} \qquad (41)$$

[2] The most general such linkage has bent links. However, the analysis proceeds similarly.

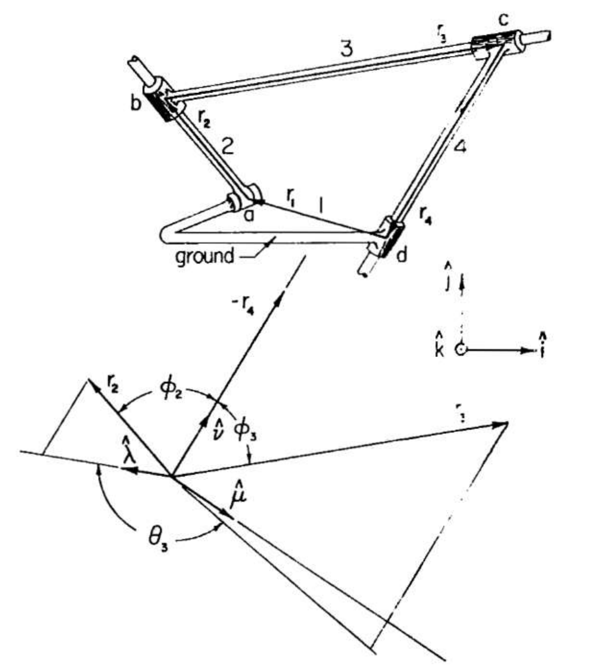

Fig. 5 A spatial four-bar linkage with one turning pair and three cylinder pairs. Link 4 is driven by link 2, and screw motion occurs at b, c, and d. The position vectors of links 2, 3, and 4 are shown in spherical coordinates in the $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$ reference frame, defined to facilitate the solution.

Here, $\mathbf{r}_1$ and $\hat{\mathbf{r}}_4$ are constants, $\hat{\mathbf{r}}_2$ is given, and r_2, r_3, $\hat{\mathbf{r}}_3$, and r_4 are unknown. First, conditions (40) and (41) are used to determine $\hat{\mathbf{r}}_3$. Magnitudes r_2, r_3, and r_4 can then readily be determined from equation (39). Though conditions (40) and (41) are simple in form, they involve an unknown unit vector and hence must be solved using an appropriately oriented reference frame; $\hat{\boldsymbol{\lambda}}$, $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\nu}}$. To deduce this orientation it is helpful to express $\hat{\mathbf{r}}_2$, $\hat{\mathbf{r}}_3$, and $\hat{\mathbf{r}}_4$ in terms of the $\hat{\boldsymbol{\lambda}}$, $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\nu}}$ frame even though it has not yet been defined. In spherical coordinates,

$$\hat{\mathbf{r}}_2 = (\sin\phi_2 \cos\theta_2)\hat{\boldsymbol{\lambda}} + (\sin\phi_2 \sin\theta_2)\hat{\boldsymbol{\mu}} + (\cos\phi_2)\hat{\boldsymbol{\nu}} \quad (42)$$

$$\hat{\mathbf{r}}_3 = (\sin\phi_3 \cos\theta_3)\hat{\boldsymbol{\lambda}} + (\sin\phi_3 \sin\theta_3)\hat{\boldsymbol{\mu}} + (\cos\phi_3)\hat{\boldsymbol{\nu}} \quad (43)$$

$$\hat{\mathbf{r}}_4 = (\sin\phi_4 \cos\theta_4)\hat{\boldsymbol{\lambda}} + (\sin\phi_4 \sin\theta_4)\hat{\boldsymbol{\mu}} + (\cos\phi_4)\hat{\boldsymbol{\nu}} \quad (44)$$

When equations (43) and (44) are substituted into equation (41) the resulting equation is much simpler if ϕ_4 equals 0 or π. As shown in Fig. 5, if $\hat{\boldsymbol{\nu}}$ is defined as $-\hat{\mathbf{r}}_4$, then ϕ_4 equals π. It is now impossible to make the corresponding simplification in equations (42), (43), and (40) (i.e., $\phi_2 = 0$ or π) unless $\hat{\mathbf{r}}_2$ and $\hat{\mathbf{r}}_4$ are parallel. However, some simplification can be achieved by defining $\hat{\boldsymbol{\mu}}$ so that θ_2 equals 0. The definition of $\hat{\boldsymbol{\lambda}}$ follows from that of $\hat{\boldsymbol{\nu}}$ and $\hat{\boldsymbol{\mu}}$. Thus,

$$\hat{\boldsymbol{\nu}} \equiv -\hat{\mathbf{r}}_4 \qquad (45)$$

$$\hat{\boldsymbol{\mu}} \equiv \frac{(\hat{\boldsymbol{\nu}} \times \hat{\mathbf{r}}_2)}{|\hat{\boldsymbol{\nu}} \times \hat{\mathbf{r}}_2|} = \frac{(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_4)}{|\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_4|} \qquad (46)$$

$$\hat{\boldsymbol{\lambda}} \equiv \hat{\boldsymbol{\mu}} \times \hat{\boldsymbol{\nu}} = \frac{\hat{\mathbf{r}}_4 \times (\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_4)}{|\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_4|} \qquad (47)$$

With this definition of $\hat{\boldsymbol{\lambda}}$, $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\nu}}$, conditions (41) and (40) become,

$$(\hat{\mathbf{r}}_3 \cdot \hat{\mathbf{r}}_4) = -\cos\phi_3 \qquad (48)$$

$$\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{r}}_3 = \sin\phi_2 \sin\phi_3 \cos\theta_3 + \cos\phi_2 \cos\phi_3 \qquad (49)$$

Solve equations (48) and (49) for ϕ_3 and θ_3 and substitute the resulting expressions into equation (43). When $\sin\phi_2$ and $\cos\phi_2$ are expressed in terms of vectors the following equation is obtained:

$$\hat{\mathbf{r}}_3 = \frac{1}{[1 - (\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{r}}_4)^2]^{1/2}} \{ [(\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{r}}_3) - (\hat{\mathbf{r}}_3 \cdot \hat{\mathbf{r}}_4)(\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{r}}_4)]\hat{\boldsymbol{\lambda}}$$

$$\pm\ [2(\hat{t}_2\cdot\hat{t}_3)(\hat{t}_3\cdot\hat{t}_4)(\hat{t}_2\cdot\hat{t}_4) - (\hat{t}_2\cdot\hat{t}_3)^2 - (\hat{t}_3\cdot\hat{t}_4)^2 - (\hat{t}_2\cdot\hat{t}_4)^2 + 1]^{1/2}\hat{\mathbf{u}}\} - (\hat{t}_3\cdot\hat{t}_4)\hat{\mathbf{v}} \tag{50}$$

With $\hat{t}_3$ known, r_2, r_3, and r_4 are determined by taking dot products throughout equation (39) with $(\hat{t}_3 \times \hat{t}_4)$, $(\hat{t}_4 \times \hat{t}_2)$, and $(\hat{t}_2 \times \hat{t}_3)$, respectively.

$$r_2 = -\frac{\mathbf{r}_1\cdot(\hat{t}_3\times\hat{t}_4)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{51}$$

$$r_3 = -\frac{\mathbf{r}_1\cdot(\hat{t}_4\times\hat{t}_2)}{\hat{t}_3\cdot(\hat{t}_4\times\hat{t}_2)} = -\frac{\mathbf{r}_1\cdot(\hat{t}_4\times\hat{t}_2)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{52}$$

$$r_4 = -\frac{\mathbf{r}_1\cdot(\hat{t}_2\times\hat{t}_3)}{\hat{t}_4\cdot(\hat{t}_2\times\hat{t}_3)} = -\frac{\mathbf{r}_1\cdot(\hat{t}_2\times\hat{t}_3)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{53}$$

Velocity and Acceleration. With the position vectors known, the following three general equations are sufficient to determine the unknown angular and linear velocities.

$$\omega_{32}\hat{\omega}_{32} + \omega_2 - \omega_3 = 0 \tag{54}$$

$$\omega_{34}\hat{\omega}_{34} + \omega_4\hat{\omega}_4 - \omega_3 = 0 \tag{55}$$

$$\mathbf{v}_{b_3} + \mathbf{v}_{c_4b_3} - \mathbf{v}_{c_4} = 0 \tag{56}$$

First, equations (54) and (55) are solved. ω_2 is known and in Fig. 5 it can be seen that $\hat{\omega}_{32}$, $\hat{\omega}_{34}$, and $\hat{\omega}_4$ equal $\hat{t}_2$, $\hat{t}_3$, and $\hat{t}_4$, respectively. Vector ω_3 can be eliminated by subtraction. Thus,

$$\omega_{32}\hat{t}_2 - \omega_{34}\hat{t}_3 - \omega_4\hat{t}_4 + \omega_2 = 0 \tag{57}$$

Expressions for ω_{32}, ω_{34}, and ω_4 are now obtained by taking dot products throughout equation (57) with $(\hat{t}_3 \times \hat{t}_4)$, $(\hat{t}_4 \times \hat{t}_2)$, and $(\hat{t}_2 \times \hat{t}_3)$, respectively.

$$\omega_{32} = -\frac{\omega_2\cdot(\hat{t}_3\times\hat{t}_4)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{58}$$

$$\omega_{34} = -\frac{\omega_2\cdot(\hat{t}_2\times\hat{t}_4)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{59}$$

$$\omega_4 = +\frac{\omega_2\cdot(\hat{t}_2\times\hat{t}_3)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{60}$$

These results can now be substituted into equations (54) or (55) to obtain ω_3.

$$\omega_3 = -\omega_2\left[\frac{(\hat{\omega}_2\times\hat{t}_2)\times(\hat{t}_3\times\hat{t}_4)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)}\right] \tag{61}$$

Now, equation (3) is substituted for each of the terms in equation (56).

$$v_{b_3b_2}\hat{t}_2 + v_{c_4c_3}\hat{t}_3 - v_{c_4}\hat{t}_4 + \mathbf{V}_7 = 0 \tag{62}$$

$$\mathbf{V}_7 \equiv (\omega_2 \times \mathbf{r}_2) + (\omega_3 \times \mathbf{r}_3)$$

Equation (62) is solved in the same way as equation (57).

$$v_{b_3b_2} = -\frac{\mathbf{V}_7\cdot(\hat{t}_3\times\hat{t}_4)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{63}$$

$$v_{c_4c_3} = -\frac{\mathbf{V}_7\cdot(\hat{t}_4\times\hat{t}_2)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{64}$$

$$v_{c_4} = +\frac{\mathbf{V}_7\cdot(\hat{t}_2\times\hat{t}_3)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{65}$$

The accelerations are found similarly. Take the time-derivative of equations (54) and (55) and state the general equation for relative linear acceleration.

$$\alpha_{32}\hat{t}_2 + (\omega_2 \times \omega_{32}) + \alpha_2 - \frac{d\omega_3}{dt} = 0 \tag{66}$$

$$\alpha_{34}\hat{t}_3 + (\omega_4 \times \omega_{34}) + \alpha_4\hat{t}_4 - \frac{d\omega_3}{dt} = 0 \tag{67}$$

$$\mathbf{a}_{b_3} + \mathbf{a}_{c_4b_3} - \mathbf{a}_{c_4} = 0 \tag{68}$$

In equations (66) and (67) α_2, $(\omega_2 \times \omega_{32})$, and $(\omega_4 \times \omega_{34})$ are known. By analogy to the solution of equations (54) and (55),

$$\alpha_{32} = \frac{-[\alpha_2 + (\omega_2\times\omega_{32}) - (\omega_4\times\omega_{34})]\cdot(\hat{t}_3\times\hat{t}_4)}{[\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)]} \tag{69}$$

$$\alpha_{34} = \frac{-[\alpha_2 + (\omega_2\times\omega_{32}) - (\omega_4\times\omega_{34})]\cdot(\hat{t}_2\times\hat{t}_4)}{[\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)]} \tag{70}$$

$$\alpha_4 = \frac{+[\alpha_2 + (\omega_2\times\omega_{32}) - (\omega_4\times\omega_{34})]\cdot(\hat{t}_2\times\hat{t}_3)}{[\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)]} \tag{71}$$

$$\alpha \equiv \alpha_{32} + \alpha_2 \tag{72}$$

Equation (4) is now substituted for each of the terms in equation (68).

$$a_{b_3b_2}\hat{t}_2 + a_{c_4c_3}\hat{t}_3 - a_{c_4}\hat{t}_4 + \mathbf{V}_8 = 0 \tag{73}$$

$$\mathbf{V}_8 \equiv \omega_2 \times (\omega_2 \times \mathbf{r}_2) + \omega_3 \times (\omega_3 \times \mathbf{r}_3) + \alpha_2(\hat{\omega}_2 \times \mathbf{r}_2) + \alpha_3(\hat{\omega}_3 \times \mathbf{r}_3) + 2(\omega_2 \times v_{b_3b_2}\hat{t}_2) + 2(\omega_3 \times v_{c_4c_3}\hat{t}_3) + \omega_{32}(\omega_2 \times \hat{t}_2) \times \mathbf{r}_2$$

By analogy to the solution of equation (62),

$$a_{b_3b_2} = -\frac{\mathbf{V}_8\cdot(\hat{t}_3\times\hat{t}_4)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{74}$$

$$a_{c_4c_3} = -\frac{\mathbf{V}_8\cdot(\hat{t}_4\times\hat{t}_2)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{75}$$

$$a_{c_4} = +\frac{\mathbf{V}_8\cdot(\hat{t}_2\times\hat{t}_3)}{\hat{t}_2\cdot(\hat{t}_3\times\hat{t}_4)} \tag{76}$$

Force and Torque. Referring to Fig. 5, consider that link 1 exerts an input torque component $\tau_1\hat{\omega}_2$ on link 2. This will result in a torque and force $\boldsymbol{\tau}_{41}$ and $\mathbf{f}$ applied to link 1 at joint d, providing the output motion of link 4 is adequately resisted. Fig. 6 shows the free body diagrams for the links, assuming infinite stiffness, no mass, and no frictional forces. From the conditions of force equilibrium the forces exerted on successive links are equal and can be represented by the single symbol, $\mathbf{f}$. The conditions of torque equilibrium are,

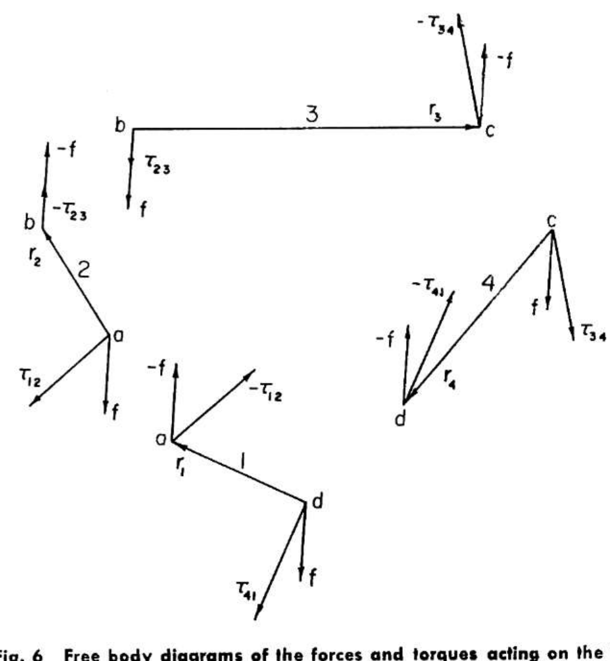

Fig. 6 Free body diagrams of the forces and torques acting on the individual links of the spatial four-bar linkage (Fig. 5)

$$\tau_{12}\hat{\boldsymbol{\tau}}_{12} + f(\hat{\mathbf{f}} \times \mathbf{r}_2) - \tau_{23}\hat{\boldsymbol{\tau}}_{23} = 0 \tag{77}$$

$$\tau_{23}\hat{\boldsymbol{\tau}}_{23} + f(\hat{\mathbf{f}} \times \mathbf{r}_3) - \tau_{34}\hat{\boldsymbol{\tau}}_{34} = 0 \tag{78}$$

$$\tau_{34}\hat{\boldsymbol{\tau}}_{34} + f(\hat{\mathbf{f}} \times \mathbf{r}_4) - \tau_{41}\hat{\boldsymbol{\tau}}_{41} = 0 \tag{79}$$

Torque $\hat{\boldsymbol{\tau}}_{12}$ exerted at joint a (Fig. 5) equals the sum of two torques: the input torque $\tau_i\hat{\boldsymbol{\omega}}_2$ and a reaction torque $\tau_h\hat{\boldsymbol{\tau}}_h$ which must be perpendicular to $\hat{\boldsymbol{\omega}}_2$. To eliminate $\hat{\boldsymbol{\tau}}_h$ take the dot product throughout this relation with $\hat{\boldsymbol{\omega}}_2$. Rearranging,

$$\tau_{12} = \frac{\tau_i}{(\hat{\boldsymbol{\tau}}_{12}\cdot\hat{\boldsymbol{\omega}}_2)} \tag{80}$$

Joint b (Fig. 5) does not permit $\mathbf{f}$ or $\boldsymbol{\tau}_{23}$ to have $\hat{\mathbf{r}}_2$ components. Thus, $\mathbf{f}\cdot\hat{\mathbf{r}}_2$ and $\boldsymbol{\tau}_{23}\cdot\hat{\mathbf{r}}_2$ are zero. Likewise, joint c requires that $\mathbf{f}\cdot\hat{\mathbf{r}}_3$ and $\boldsymbol{\tau}_{34}\cdot\hat{\mathbf{r}}_3$ be zero. If the dot products with $\hat{\mathbf{r}}_2$ and $\hat{\mathbf{r}}_3$ are taken throughout equations (77) and (78), respectively, it is seen that $\boldsymbol{\tau}_{12}\cdot\hat{\mathbf{r}}_2$ and $\boldsymbol{\tau}_{23}\cdot\hat{\mathbf{r}}_3$ are zero. Finally, from the sum of equations (78) and (79), $\hat{\boldsymbol{\tau}}_{41}\cdot\{\hat{\mathbf{f}} \times [\hat{\mathbf{f}} \times (\mathbf{r}_3 + \mathbf{r}_4)]\}$ must be zero. From the conditions above on $\mathbf{f}$ and $\boldsymbol{\tau}_{23}$,

$$\hat{\mathbf{f}} = \hat{\boldsymbol{\tau}}_{23} = \frac{(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)}{|\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3|} \tag{81}$$

Further solution of equations (77)–(79) requires specification of the output resistance. Two statically determinate cases are considered here.

1 Resistance by force. Assume that $\mathbf{f}$ has a component $f_0\hat{\mathbf{r}}_4$, but $\boldsymbol{\tau}_{41}\cdot\hat{\mathbf{r}}_4$ is zero. If the dot product with $\hat{\mathbf{r}}_4$ is taken throughout equation (79) it is seen that $\boldsymbol{\tau}_{34}\cdot\hat{\mathbf{r}}_4$ is zero. Since $\boldsymbol{\tau}_{34}\cdot\hat{\mathbf{r}}_3$ is also zero,

$$\hat{\boldsymbol{\tau}}_{34} = \frac{(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)}{|\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4|} \tag{82}$$

From the sum of equations (77) and (78), the quantity $\hat{\boldsymbol{\tau}}_{12}\cdot\{\hat{\boldsymbol{\tau}}_{34} \times [\hat{\mathbf{f}} \times (\mathbf{r}_2 + \mathbf{r}_3)]\}$ must be zero. Since $\boldsymbol{\tau}_{12}\cdot\hat{\mathbf{r}}_2$ is also zero,

$$\begin{aligned}\hat{\boldsymbol{\tau}}_{12} &= (\text{const})_1[\hat{\mathbf{r}}_2 \times \{\hat{\boldsymbol{\tau}}_{34} \times [\hat{\mathbf{f}} \times (\mathbf{r}_2 + \mathbf{r}_3)]\}]\\ &= (\text{const})_2\{r_2[\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)][\hat{\mathbf{r}}_2 \times (\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)]\\ &\qquad - r_3[(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)](\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)\}\end{aligned} \tag{83}$$

Now an expression for f_0 can be obtained. First express f by taking the dot product with $\hat{\mathbf{r}}_3$ throughout equation (77), then substitute equations (80) and (83),

$$f = \frac{\tau_i}{r_2}\,\frac{[\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)]\,|\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3|}{[\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)]\{[\hat{\mathbf{r}}_2 \times (\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)]\cdot\hat{\boldsymbol{\omega}}_2\} - r_3\{[(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)][\hat{\boldsymbol{\omega}}_2\cdot(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)]\}} \tag{84}$$

Using equations (81) and (84),

$$f_0 = f(\hat{\mathbf{f}}\cdot\hat{\mathbf{r}}_4) = (\text{M.A.})_f\,\frac{\tau_i}{r_2} \tag{85}$$

$$(\text{M.A.})_f \equiv \frac{[\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)]^2}{[\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)]\{[\hat{\mathbf{r}}_2 \times (\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)]\cdot\hat{\boldsymbol{\omega}}_2\} - \frac{r_3}{r_2}\{[(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)][\hat{\boldsymbol{\omega}}_2\cdot(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)]\}}$$

2 Resistance by torque. Assume that torque $\boldsymbol{\tau}_{41}$ has a component $\tau_0\hat{\mathbf{r}}_4$ but $\mathbf{f}\cdot\hat{\mathbf{r}}_4$ is zero. Thus, either f, $(\hat{\mathbf{f}}\cdot\hat{\mathbf{r}}_4)$, or both are zero. If f is zero, equations (77)–(79) show that $\boldsymbol{\tau}_{12}$, $\boldsymbol{\tau}_{23}$, $\boldsymbol{\tau}_{34}$, and $\boldsymbol{\tau}_{41}$ are equal and can be represented by the single symbol $\boldsymbol{\tau}$. By equation (81),

$$\hat{\boldsymbol{\tau}} = \frac{(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)}{|\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3|} \tag{86}$$

An expression for τ_0 can be obtained from equations (80) and (86).

$$\tau_0 = \tau(\hat{\boldsymbol{\tau}}\cdot\hat{\mathbf{r}}_4) = (\text{M.A.})_\tau\tau_i \tag{87}$$

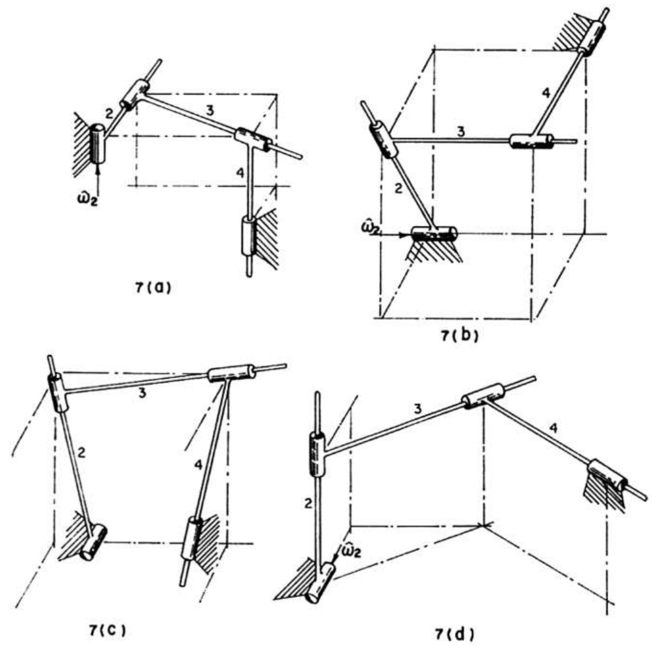

Fig. 7 Special positions of the spatial four-bar linkage:

7(a) Rotational output

7(b) Translational output

7(c) $|\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)|$ is zero. All links lie in the same plane and the linkage is locked against torque exerted on link 2.

7(d) $|\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)|$ is unity. Links 2, 3, and 4 are mutually perpendicular and transmission is optimum.

$$(\text{M.A.})_\tau \equiv \frac{[\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)]}{[\hat{\boldsymbol{\omega}}_2\cdot(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)]}$$

If $\mathbf{f}\cdot\hat{\mathbf{r}}_4$ is zero, by equation (81) the quantity $\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)$ must be zero (planar position). Equation (60) then shows that τ_0 must approach zero unless $\boldsymbol{\omega}_2\cdot(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)$ is zero, because the power output $\tau_0\omega_4$ must remain finite. Thus, $\boldsymbol{\tau}_{41}\cdot\hat{\mathbf{r}}_4$ is zero and equation (84) applies, showing that f is zero, unless the denominator of equation (84) is zero. Therefore, equation (87) holds even if $\hat{\mathbf{f}}\cdot\hat{\mathbf{r}}_4$ is zero, except for special cases.

The performance of the linkage can now be discussed:

1 The term $\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)$ is a factor in the numerators of equations (85) and (87). Moreover, it constitutes the denominator of every expression for dependent velocity and acceleration. Thus, if $|\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)|$ becomes small, less force and torque are transmitted and links 3 and 4 must move further to accommodate a given input movement of link 2. At best $|\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)|$ is unity; $\hat{\mathbf{r}}_2$, $\hat{\mathbf{r}}_3$, and $\hat{\mathbf{r}}_4$ are mutually perpendicular and transmission is optimum (Fig. 7(d)). Therefore the *Transmission Criterion* is that $|\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)|$ be near unity.[3]

2 When $\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)$ is zero all links lie in the same plane and f is zero (unless the denominator of equation (84) is zero). Thus, equation (86) usually applies and all torques, including $\boldsymbol{\tau}_{12}$, are perpendicular to the linkage plane. As suggested by Fig. 7(c), the linkage is locked against this particular input since $\hat{\mathbf{r}}_2$ cannot rotate in the $\hat{\boldsymbol{\tau}}_{12}$ direction. However, the linkage is movable by other inputs. For example link 3 can be moved up and down on 2 and 4. Also, if $|\hat{\boldsymbol{\omega}}_2\cdot\hat{\mathbf{f}}|$ is not unity the linkage can be moved into a spatial position. It can be shown that when both $\hat{\mathbf{r}}_2\cdot(\hat{\mathbf{r}}_3 \times \hat{\mathbf{r}}_4)$ and $\hat{\boldsymbol{\omega}}_2\cdot(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)$ approach zero the situation is the same except that all torques become very large.

3 When $\hat{\boldsymbol{\omega}}_2\cdot(\hat{\mathbf{r}}_2 \times \hat{\mathbf{r}}_3)$ approaches zero τ_0 becomes very large (equation (87)) but ω_4 approaches zero (equation (60)). The quantities f_0 and v_{c4} behave similarly when the denominator of equation (85) approaches zero. Thus, realistically, the linkages

[3] Analogously, optimum transmission occurs in the plane four-bar linkage when the coupler is perpendicular to the follower.

shown in Figs. 7(b) and 7(a) can transmit power only by force, $f_0\hat{r}_4$ and only by torque, $\tau_0\hat{r}_4$.

General Procedure

Appraisal.

1 Determine if the linkage has a single degree of freedom. This can be done by means of equations (88) and (89), if there are no redundant constraints [14].

$$\text{Two dimensions:}\quad f_L = 3(n-1) - h - 2l \tag{88}$$

$$\text{Three dimensions:}\quad f_L = 6(n-p-1) + \sum_{i=1}^{p} f_{P_i} \tag{89}$$

2 Write the general vector equation or equations for position and check that the product of the number of equations and the number of dimensions equals the number of scalar unknowns. In two dimensions each vector is specified by two scalars: r and θ. In three dimensions each vector is specified by three scalars: r, θ, and ϕ.

3 If item 1 is satisfied but item 2 is not, try to determine enough scalars to satisfy item 2 independently of the general vector equation. In many instances it is possible to determine the direction of unit position vectors just by reasoning about the constraints. Also, there may be auxiliary conditions (e.g., conditions (40) and (41)). If all conditions on the position can be expressed in a single general vector equation such that item 2 is satisfied, expressions for the dependent position vectors can probably be found without serious difficulty.

Outline of Approach.

1 Find explicit expressions for the unknown position vectors in terms of those which are known. With this done, regard all position vectors as known.

2 Write the general vector equations which completely relate the linkage velocities (e.g., equations (54), (55), and (56)). Substitute equation (3) or (5) for all unknown linear velocities and simplify the resulting equation by summing all completely known terms into a single vector constant. Then, obtain explicit expressions for the unknown velocities. With this done, regard all velocities as known.

3 Continue the step 2 procedure for higher order motions, substituting the corresponding order derivative of $r\hat{r}$. Such a stepwise procedure will generally be more convenient than successive differentiation of the position vector expressions found in step 1.

4 Write the equations for the force and moment equilibrium of the individual links. Determine as many quantities as possible from inspection of the constraints, then solve the equilibrium equations regarding all position, velocity, and acceleration vectors as known.

Solution of Vector Equations.

1 Adapt the manner of expression of vectors to the situation.

a. Equations can be simplified by expressing groups of known vectors by a single vector symbol.

b. The product form ($u\hat{\mathbf{u}}$) may be useful when either magnitude or direction is unknown.

c. Expression in spherical coordinates (e.g., equations (102) or (42)) is helpful in derivations that involve unknown unit vectors.

d. Once solutions have been derived they are most conveniently evaluated using vectors expressed in terms of rectangular coordinates in the ground reference frame ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$).

2 If the only unknowns in an equation are magnitudes, solutions can be obtained by taking dot products throughout so as to eliminate all terms except knowns and the terms containing the unknown sought.

3 When an equation involves unknown unit vectors it is generally necessary to define a special reference frame ($\hat{\boldsymbol{\lambda}}$, $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\nu}}$) to obtain a solution. Though only a few examples have been shown here, this technique makes possible a wide range of solutions to complex and spatial mechanisms.

4 Early insertion of numerical values usually achieves no advantage in derivation and destroys the generality of any result that is obtained.

Computation. As shown in the preceding examples, all operations required for complete solution to complicated kinematic problems can be combined in a few vector equations. Thus, the procedure required for computation is clearly defined, whether the computation itself is to be done by hand or programmed for a digital computer. Computation is best performed by a computer when many linkage positions are analyzed, especially when higher-order motions of spatial linkages are involved.

APPENDIX

Vector Operations and Identities

A few essential vector operations and identities are listed here. Several texts, including that of W. Kaplan [16], afford a thorough review.

1 A vector, $\mathbf{a}$, is the product of a magnitude, a, and a unit vector, $\hat{\mathbf{a}}$, which specifies its direction.

2 Two vectors are equal if their magnitudes and directions are the same, regardless of their location.

3 Let γ be the angle between two vectors, $\mathbf{a}$ and $\mathbf{b}$. Then,

$$\mathbf{a}\cdot\mathbf{b} \equiv ab\cos\gamma \tag{90}$$

$$\mathbf{a}\times\mathbf{b} \equiv (ab\sin\gamma)\hat{\mathbf{d}} \tag{91}$$

In equation (91), $\hat{\mathbf{d}}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$, and completes a positive triple: $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{d}}$ (right-hand rule).

4 Let $\hat{\boldsymbol{\lambda}}$, $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\nu}}$ be a positive triple of mutually perpendicular unit vectors. Then,

$$\mathbf{a}\cdot\mathbf{b} = a_\lambda b_\lambda + a_\mu b_\mu + a_\nu b_\nu \tag{92}$$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\boldsymbol{\lambda}} & \hat{\boldsymbol{\mu}} & \hat{\boldsymbol{\nu}} \\ a_\lambda & a_\mu & a_\nu \\ b_\lambda & b_\mu & b_\nu \end{vmatrix} \tag{93}$$

5

$$|\mathbf{a}| = (\mathbf{a}\cdot\mathbf{a})^{1/2} \tag{94}$$

$$\mathbf{a}\cdot\mathbf{a} = a^2 \tag{95}$$

$$\mathbf{a}\times\mathbf{a} = 0 \tag{96}$$

$$(\mathbf{a}\times\mathbf{b}) = -(\mathbf{b}\times\mathbf{a}) \tag{97}$$

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0 \tag{98}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{99}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a} \tag{100}$$

6 The quantity $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ changes sign if the cyclic order of the vectors is changed. Otherwise, its value is unaffected by interchange of vectors or by exchange of cross and dot.

Derivation of Equations (107a) Through (111b)

1 Equations (107*a*, *b*)

Rearrange terms.

2 Equations (108*a*, *b*)

Take the dot product with $(\hat{\mathbf{i}}\times\hat{\mathbf{k}})$ throughout equation (1).

$$\mathbf{C}\cdot(\hat{\mathbf{i}}\times\hat{\mathbf{k}}) + s\hat{\mathbf{s}}\cdot(\hat{\mathbf{i}}\times\hat{\mathbf{k}}) = 0 \tag{101}$$

Magnitude s is obtained by dividing equation (101) through by $\hat{\mathbf{s}}\cdot(\hat{\mathbf{i}}\times\hat{\mathbf{k}})$. This determines $\mathbf{s}$, because $\hat{\mathbf{s}}$ is already known. Equation (108b) is derived similarly.

3 Equations (109*a*, *b*)

Divide equation (101) through by s to obtain an expression for the cosine of the angle, θ_s, between $\hat{\mathbf{s}}$ and $(\hat{\mathbf{i}}\times\hat{\mathbf{k}})$. If $(\hat{\mathbf{i}}\times\hat{\mathbf{k}})$ and

Table 1 Solutions of the vector polygon equation

$$\mathbf{C} + s\hat{\mathbf{s}} + t\hat{\mathbf{t}} = 0 \tag{1}$$

$$\mathbf{C} \equiv \sum_{i=1}^{n-2} \mathbf{C}_i \tag{2}$$

	Unknown	Known	Solution	
1	$s,\ \hat{\mathbf{s}}$	$\mathbf{C}, t, \hat{\mathbf{t}}$	$\mathbf{s} = -(\mathbf{C} + \mathbf{t})$	(107*a*)
2	$t,\ \hat{\mathbf{t}}$	$\mathbf{C}, s, \hat{\mathbf{s}}$	$\mathbf{t} = -(\mathbf{C} + \mathbf{s})$	(107*b*)
3	$s,\ t$	$\mathbf{C}, \hat{\mathbf{s}}, \hat{\mathbf{t}}$	$\mathbf{s} = -\left[\dfrac{\mathbf{C}\cdot(\hat{\mathbf{t}} \times \hat{\mathbf{k}})}{\hat{\mathbf{s}}\cdot(\hat{\mathbf{t}} \times \hat{\mathbf{k}})}\right]\hat{\mathbf{s}}$	(108*a*)
			$\mathbf{t} = -\left[\dfrac{\mathbf{C}\cdot(\hat{\mathbf{s}} \times \hat{\mathbf{k}})}{\hat{\mathbf{t}}\cdot(\hat{\mathbf{s}} \times \hat{\mathbf{k}})}\right]\hat{\mathbf{t}}$	(108*b*)
4	$\hat{\mathbf{s}},\ t$	$\mathbf{C}, s, \mathbf{t}$	$\mathbf{s} = -[\mathbf{C}\cdot(\hat{\mathbf{t}} \times \hat{\mathbf{k}})](\hat{\mathbf{t}} \times \hat{\mathbf{k}}) \pm \{s^2 - [\mathbf{C}\cdot(\hat{\mathbf{t}} \times \hat{\mathbf{k}})]^2\}^{1/2}\hat{\mathbf{t}}$	(109*a*)
			$\mathbf{t} = [-(\mathbf{C}\cdot\hat{\mathbf{t}}) \mp \{s^2 - [\mathbf{C}\cdot(\hat{\mathbf{t}} \times \hat{\mathbf{k}})]^2\}^{1/2}]\hat{\mathbf{t}}$	(109*b*)
5	$s,\ \hat{\mathbf{t}}$	$\mathbf{C}, \hat{\mathbf{s}}, t$	$\mathbf{s} = [-(\mathbf{C}\cdot\hat{\mathbf{s}}) \mp \{t^2 - [\mathbf{C}\cdot(\hat{\mathbf{s}} \times \hat{\mathbf{k}})]^2\}^{1/2}]\hat{\mathbf{s}}$	(110*a*)
			$\mathbf{t} = -[\mathbf{C}\cdot(\hat{\mathbf{s}} \times \hat{\mathbf{k}})](\hat{\mathbf{s}} \times \hat{\mathbf{k}}) \pm \{t^2 - [\mathbf{C}\cdot(\hat{\mathbf{s}} \times \hat{\mathbf{k}})]^2\}^{1/2}\hat{\mathbf{s}}$	(110*b*)
6	$\hat{\mathbf{s}},\ \hat{\mathbf{t}}$	$\mathbf{C}, s, t$	$\mathbf{s} = \mp\left[t^2 - \left(\dfrac{C^2 + t^2 - s^2}{2C}\right)^2\right]^{1/2}(\hat{\mathbf{C}} \times \hat{\mathbf{k}}) + \left(\dfrac{C^2 + t^2 - s^2}{2C} - C\right)\hat{\mathbf{C}}$	(111*a*)
			$\mathbf{t} = \pm\left[t^2 - \left(\dfrac{C^2 + t^2 - s^2}{2C}\right)^2\right]^{1/2}(\hat{\mathbf{C}} \times \hat{\mathbf{k}}) - \left(\dfrac{C^2 + t^2 - s^2}{2C}\right)\hat{\mathbf{C}}$	(111*b*)

$\hat{\mathbf{t}}$ are defined as the reference frame unit vectors $\hat{\mathbf{s}}$ can be stated as follows:

$$\hat{\mathbf{s}} = (\cos\theta_s)(\hat{\mathbf{t}} \times \hat{\mathbf{k}}) + (\sin\theta_s)\hat{\mathbf{t}} \tag{102}$$

$$\cos\theta_s = -\frac{\mathbf{C}\cdot(\hat{\mathbf{t}} \times \hat{\mathbf{k}})}{s}$$

$$\sin\theta_s = \pm(1 - \cos^2\theta_s)^{1/2} = \pm\frac{1}{s}\{s^2 - [\mathbf{C}\cdot(\hat{\mathbf{t}} \times \hat{\mathbf{k}})]^2\}^{1/2}$$

Equation (109*a*) is obtained directly from equation (102). Then equation (109*b*) is obtained by substituting equation (109*a*) into equation (107*b*).

4 Equations (110*a*, *b*)

Equation (1) has the same form when s and $\hat{\mathbf{t}}$ are unknown as when $\hat{\mathbf{s}}$ and t are unknown. Thus, equations (110*a*, *b*) can be obtained by interchanging $\mathbf{s}$, s, and $\hat{\mathbf{s}}$ with $\mathbf{t}$, t, and $\hat{\mathbf{t}}$ in equations (109*a*, *b*).

5 Equations (111*a*, *b*)

Define $\hat{\mathbf{C}} \times \hat{\mathbf{k}}$ and $\hat{\mathbf{C}}$ as the reference frame unit vectors. Then rearrange equation (1) and express it in spherical coordinates as follows:

$$s[(\cos\theta_s)(\hat{\mathbf{C}} \times \hat{\mathbf{k}}) + (\sin\theta_s)\hat{\mathbf{C}}] = -\mathbf{C} - t[(\cos\theta_t)(\hat{\mathbf{C}} \times \hat{\mathbf{k}}) + (\sin\theta_t)\hat{\mathbf{C}}] \tag{103}$$

Take the dot product throughout equation (103), first with $\hat{\mathbf{C}} \times \hat{\mathbf{k}}$, then with $\hat{\mathbf{C}}$, to form two scalar equations:

$$s(\cos\theta_s) = -t(\cos\theta_t) \tag{104}$$

$$s(\sin\theta_s) = -[C + t\sin\theta_t] \tag{105}$$

Square both sides of equations (104) and (105), then add. The resulting equation is easily solved for $\sin\theta_t$. Thus,

$$\mathbf{t} = t[(\cos\theta_t)(\hat{\mathbf{C}} \times \hat{\mathbf{k}}) + (\sin\theta_t)\hat{\mathbf{C}}] \tag{106}$$

$$\sin\theta_t = -\left\{\frac{C^2 + t^2 - s^2}{2Ct}\right\}$$

$$\cos\theta_t = \pm\left[1 - \left\{\frac{C^2 + t^2 - s^2}{2Ct}\right\}^2\right]^{1/2}$$

Equation (106) becomes equation (111*b*), and equation (111*a*) is obtained by substituting equation (111*b*) into equation (107*a*).

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