SOLUTION OF PLANAR VECTOR EQUATIONS

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Questions. Consider the planar vector equation

(1) $$u + v + w = 0,$$

or, in terms of unit vectors ($\hat{u}$, etc.) and magnitudes ($|u|\,, etc.$),

(2) $$|u|\,\hat{u} + |v|\,\hat{v} + |w|\,\hat{w} = 0.$$

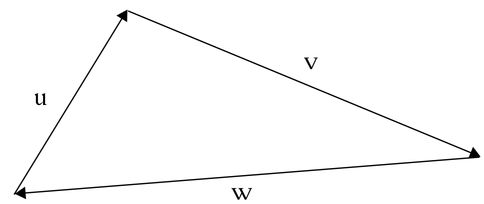

Fig. 1: `Graphical Representation`

Case 1. Find w, if the magnitude and direction of u and v are known.

Case 2. Suppose that w and the directions $\hat{u}$ and $\hat{v}$ of u and v are known. Determine the magnitudes $|u|$ and $|v|$ of u and v.

Case 3. Suppose that w and the magnitudes $|u|$ and $|v|$ of u and v are known. Determine the directions $\hat{u}$ and $\hat{v}$ of u and v.

Case 4. Suppose that w, the direction $\hat{u}$ of u, and the magnitude $|v|$ of v are known. Determine the magnitude $|u|$ of u and the direction $\hat{v}$ of v.

Solutions. Let $\hat{k}$ be a unit vector that points perpendicular to the plane.

Case 1. This is the simplest case. Just solve equation (1) for w:

$$w = -u - v$$

Case 2. In order to find $|u|$, we take the dot product of each term in equation (2) with $\hat{v} \times \hat{k}$. Noting that $\hat{v} \cdot \left(\hat{v} \times \hat{k}\right) = 0$, we obtain

$$|u|\,\hat{u} \cdot \left(\hat{v} \times \hat{k}\right) + w \cdot \left(\hat{v} \times \hat{k}\right) = 0,$$

from which the magnitude $|u|$ of vector u is given by

$$|u| = \frac{-w \cdot \left(\hat{v} \times \hat{k}\right)}{\hat{u} \cdot \left(\hat{v} \times \hat{k}\right)}.$$

Similarly, the magnitude $|v|$ of v is given by

$$|v| = \frac{-w \cdot \left(\hat{u} \times \hat{k}\right)}{\hat{v} \cdot \left(\hat{u} \times \hat{k}\right)}.$$

Case 3. Let

$$\Delta = \frac{|w|^2 + |v|^2 - |u|^2}{2\,|w|}.$$

In this case, we can find the vectors u and v by

$$
\begin{aligned}
u &= \mp\left(\sqrt{|v|^2 - \Delta^2}\right)\left(\hat{w} \times \hat{k}\right) + (\Delta - |w|)\,\hat{w}, \\
v &= \pm\left(\sqrt{|v|^2 - \Delta^2}\right)\left(\hat{w} \times \hat{k}\right) - \Delta\hat{w},
\end{aligned}
$$

where we use the upper set of signs if the vectors u, v, w are oriented in a counter clockwise manner and we use the lower set of signs if they are oriented in a clockwise manner.

Case 4. Let

$$
\Delta = w \cdot \left(\hat{u} \times \hat{k}\right).
$$

We can find the magnitude $|u|$ and the direction $\hat{v}$ by

$$
\begin{aligned}
u &= \left(-w \cdot \hat{u} \mp \sqrt{|v|^2 - \Delta^2}\right)\hat{u}, \\
v &= -\Delta\left(\hat{u} \times \hat{k}\right) \pm \left(\sqrt{|v|^2 - \Delta^2}\right)\hat{u},
\end{aligned}
$$

where we pick the correct signs by inspecting the mechanism in question.