Estimation of the Mass and Damping of a Second-Order Linear System

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***Abstract*— We study the estimation of the mass and the damping of a second-order linear ODE system using an adaptation law.**

***Index Terms*— Adaptive control, estimation**

I. Introduction

We provide a Lyapunov analysis [1] to prove that our control and adaptation laws will stabilize the system to a reference trajectory while the estimates of the mass and the damping tend to their correct values.

II. Analysis

We have a linear translational mechanical system with viscous damping, whose motion is governed by the ODE

$$m\ddot{x} + b\dot{x} = u(x, \dot{x}), \tag{1}$$

where $u(x, \dot{x})$ is a control input that is to be determined for tracking. We are uncertain of the true values m and b so we denote our guesses for them by $\hat{m}$ and $\hat{b}$, respectively. Let us introduce the errors in x, m and b to be

$$\tilde{x} = x - x_r, \quad \tilde{m} = m - \hat{m}, \quad \tilde{b} = b - \hat{b},$$

where x_r is a reference signal for the motion of the mechanical system. We will postulate update rules for $\hat{m}$ and $\hat{b}$ and a control law for the mechanical system (1). The combined system has the dynamics

$$\begin{aligned} m\ddot{\tilde{x}} &= u(x, \dot{x}) - m\ddot{x}_r - b(\dot{x}_r + \dot{\tilde{x}}) \\ \dot{\hat{m}} &= f(x, \dot{x}) \Rightarrow \dot{\tilde{m}} = -f(x, \dot{x}) \\ \dot{\hat{b}} &= g(x, \dot{x}) \Rightarrow \dot{\tilde{b}} = -g(x, \dot{x}) \end{aligned} \tag{2}$$

Let us finally introduce the auxiliary variable $r := \dot{\tilde{x}} + \lambda\tilde{x}$, where $\lambda > 0$ is a constant whose value is to be determined. Its dynamics is therefore given by

$$m\dot{r} = m\left(\ddot{\tilde{x}} + \lambda\dot{\tilde{x}}\right) = u - m(\ddot{x}_r - \lambda\dot{\tilde{x}}) - b(\dot{x}_r + \dot{\tilde{x}}).$$

Consider the Lyapunov function candidate

$$V(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) = \frac{1}{2}mr^2 + k\lambda\tilde{x}^2 + \frac{1}{2}\tilde{m}^2 + \frac{1}{2}\tilde{b}^2,$$

where k is another constant to be determined. This is a positive definite function over the space of $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b})$. We take the time derivative of the Lyapunov function candidate and substitute from the system dynamics (2). We suppress its functional dependence for brevity.

$$\begin{aligned} \dot{V} &= r\dot{r} + 2k\lambda\tilde{x}\dot{\tilde{x}} + \tilde{m}\dot{\tilde{m}} + \tilde{b}\dot{\tilde{b}}, \\ &= r\left(u - m\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) - b\left(\dot{x}_r + \dot{\tilde{x}}\right)\right) + 2k\lambda\tilde{x}\dot{\tilde{x}} \\ &\quad - \tilde{m}f - \tilde{b}g. \end{aligned}$$

This expression informs the selection of the control law as

$$\boxed{u(x, \dot{x}) = \hat{m}\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) + \hat{b}\left(\dot{x}_r + \dot{\tilde{x}}\right) - kr}. \tag{3}$$

Substituting this controller into the expression for $\dot{V}$ gives

$$\begin{aligned} \dot{V} &= r\left(-\tilde{m}\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) - \tilde{b}\left(\dot{x}_r + \dot{\tilde{x}}\right)\right) - kr^2 + 2k\lambda\tilde{x}\dot{\tilde{x}} \\ &\quad - \tilde{m}f - \tilde{b}g, \\ &= -k\lambda^2\tilde{x}^2 - k\dot{\tilde{x}}^2 - \tilde{m}\left(r\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) + f\right) \\ &\quad - \tilde{b}\left(r\left(\dot{x}_r + \dot{\tilde{x}}\right) + g\right). \end{aligned}$$

Since we do not know the sign of $\tilde{m}$ or $\tilde{b}$, this expression informs the choices of the adaptation laws as

$$\boxed{f(x, \dot{x}) = -r\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right)}, \quad \boxed{g(x, \dot{x}) = -r\left(\dot{x}_r + \dot{\tilde{x}}\right)}. \tag{4}$$

yielding a negative semidefinite $\dot{V}$:

$$\dot{V} = -k\lambda^2\tilde{x}^2 - k\dot{\tilde{x}}^2 \le 0.$$

Hence the set $\Omega_c = \{(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) : V(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) \le c\}$ is positively invariant for any $c > 0$. Moreover, $\tilde{x}, \dot{\tilde{x}}, r \to 0$ as $t \to \infty$. This means $u(x, \dot{x}) \to \hat{m}\ddot{x}_r + \hat{b}\dot{x}_r$ and $f(x, \dot{x})$, $g(x, \dot{x}) \to 0$. We identify $S = \{(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) : (\tilde{x}, \dot{\tilde{x}}) = 0\}$ as the set of all points in Ω_c where $\dot{V} = 0$. We now show that for a specific choice of the reference signal $x_r(t)$, no solution can stay identically in S other than the trivial solution $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) = (0, 0, 0, 0)$ and invoke Corollary 4.1 of [1] proving that the errors converge to zero.

To this end, for any solution that belongs identically to S, the system dynamics yields

$$\tilde{m}\ddot{x}_r + \tilde{b}\dot{x}_r \equiv 0, \qquad \dot{\tilde{m}} \equiv \dot{\tilde{b}} \equiv 0.$$

Now, choose $\boxed{x_r(t) = A\sin\left(\omega t + \varphi\right)}$ for some constants $A, \omega > 0$, and φ (or any time function with a nontrivial second derivative). With $\theta(t) = \omega t + \varphi$, we have

$$\tilde{m}\omega\sin\theta(t) - \tilde{b}\cos\theta(t) = \begin{bmatrix}\sin\theta(t) & -\cos\theta(t)\end{bmatrix}\begin{bmatrix}\tilde{m}\omega \\ \tilde{b}\end{bmatrix} \equiv 0,$$

for all $t > 0$. This implies that $\tilde{m}, \tilde{b} \equiv 0$ because $\tilde{m}$ and $\tilde{b}$ must be constants within S. Thus the origin of $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b})$ is globally asymptotically stable. ∎

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III. Results

In Figure 1, we plot the response of the system to the control and adaptation laws (3, 4), implemented in simulation. The constants that are used are as follows: $(A, \omega, \varphi) = (3/10, 3\pi/5, 0^\circ)$ and $(k, \lambda) = (1, 4)$. The real mass and damping of the system is $(m, b) = (0.75, 1.22)$ and their estimates start at $(\hat{m}, \hat{b}) = (-1/2, -1/4)$.

In Figure 2, we plot the response of the system to the control and adaptation laws (3, 4), implemented on a real cart system. The constants used are as follows: $(A, \omega, \varphi) = (3/10, 3\pi/5, 0^\circ)$ and $(k, \lambda) = (1, 4)$. We did not know the real mass and damping of the system and set our initial guesses of them to be $(\hat{m}, \hat{b}) = (-1/2, -1/4)$.

IV. Conclusion

We have shown that our control and adaptation laws for the linear second-order mechanical system guide the estimates of the mass and the damping coefficient to tend to their correct values while tracking a judiciously chosen reference signal.

References

[1] H. Khalil, *Nonlinear Control*. Always Learning, Pearson, 2015.

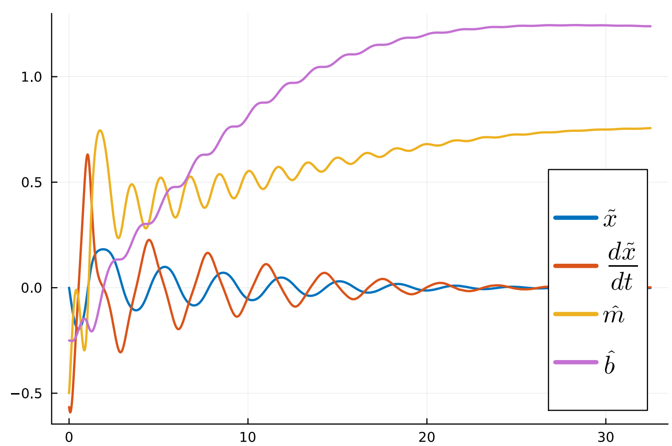

Fig. 1: Simulation showing the convergence of the system state and parameter estimates.

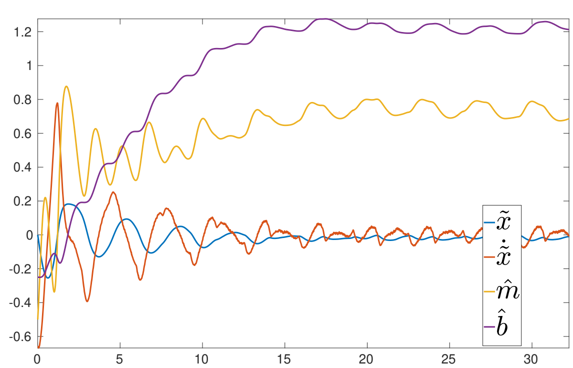

Fig. 2: Hardware implementation of the estimation algorithm.