Estimation of the Mass and Damping of a Second-Order Linear System

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***Abstract*— We study the estimation of the mass and the damping of a second-order linear ODE system using an adaptation law.**

***Index Terms*— Adaptive control, estimation**

I. Introduction

We provide a Lyapunov analysis [1] to prove that our adaptation law will stabilize the system to a reference trajectory while the estimates of the mass and the damping tend to their correct values.

II. Analysis

We have a linear translational mechanical system with viscous damping, whose motion is governed by the ODE

$$m\ddot{x} + b\dot{x} = u(x, \dot{x}), \tag{1}$$

where $u(x, \dot{x})$ is a control input that is to be determined for tracking. We are uncertain of the true values m and b so we denote our starting guess of them as $\hat{m}$ and $\hat{b}$, respectively. Let us introduce the errors in x, m and b to be

$$\tilde{x} = x - x_r, \quad \tilde{m} = m - \hat{m}, \quad \tilde{b} = b - \hat{b},$$

where x_r is a reference signal for the motion of the mechanical system. We will postulate update rules for $\hat{m}$ and $\hat{b}$ and a control law for the mechanical system (1). The combined system has the dynamics

$$\begin{aligned} m\ddot{\tilde{x}} &= u(x, \dot{x}) - m\ddot{x}_r - b(\dot{x}_r + \dot{\tilde{x}}) \\ \dot{\hat{m}} &= f(x, \dot{x}) \Rightarrow \dot{\tilde{m}} = -f(x, \dot{x}) \\ \dot{\hat{b}} &= g(x, \dot{x}) \Rightarrow \dot{\tilde{b}} = -g(x, \dot{x}) \end{aligned} \tag{2}$$

Let us finally introduce the auxiliary variable $r := \dot{\tilde{x}} + \lambda\tilde{x}$, where $\lambda > 0$ is a constant whose value is to be determined. Its dynamics is therefore given by

$$m\dot{r} = m\left(\ddot{\tilde{x}} + \lambda\dot{\tilde{x}}\right) = u - m(\ddot{x}_r - \lambda\dot{\tilde{x}}) - b(\dot{x}_r + \dot{\tilde{x}}).$$

Consider the Lyapunov function candidate

$$V(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) = \frac{1}{2}mr^2 + k\lambda\tilde{x}^2 + \frac{1}{2}\tilde{m}^2 + \frac{1}{2}\tilde{b}^2, \tag{3}$$

where k is another constant to be determined. This is positive definite function over the space of $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b})$. We take the time derivative of the Lyapunov function candidate and substitute from the system dynamics. We suppress the functional dependence of brevity.

$$\begin{aligned} \dot{V} &= r\dot{r} + 2k\lambda\tilde{x}\dot{\tilde{x}} - \tilde{m}f - \tilde{b}g, \\ &= r\left(u - m\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) - b\left(\dot{x}_r + \dot{\tilde{x}}\right)\right) + 2k\lambda\tilde{x}\dot{\tilde{x}} - \tilde{m}f - \tilde{b}g. \end{aligned}$$

This expression informs the selection of the control law as

$$\boxed{u(x, \dot{x}) = \hat{m}\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) + \hat{b}\left(\dot{x}_r + \dot{\tilde{x}}\right) - kr}.$$

Substituting this controller into the expression for $\dot{V}$ gives

$$\begin{aligned} \dot{V} &= r\left(-\tilde{m}\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) - \tilde{b}\left(\dot{x}_r + \dot{\tilde{x}}\right)\right) - kr^2 + 2k\lambda\tilde{x}\dot{\tilde{x}} - \tilde{m}f \\ &\quad - \tilde{b}g, \\ &= -k\lambda^2\tilde{x}^2 - k\dot{\tilde{x}}^2 - \tilde{m}\left(r\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) + f\right) \\ &\quad - \tilde{b}\left(r\left(\dot{x}_r + \dot{\tilde{x}}\right) + g\right). \end{aligned}$$

Since we do not know the sign of $\tilde{m}$ or $\tilde{b}$, this expression informs the choices of the adaptation laws as

$$\boxed{f(x, \dot{x}) = -r\left(\ddot{x}_r - \lambda\tilde{x}\right)}, \qquad \boxed{g(x, \dot{x}) = -r\left(\dot{x}_r + \dot{\tilde{x}}\right)}.$$

yielding a negative semidefinite $\dot{V}$:

$$\dot{V} = -k\lambda^2\tilde{x}^2 - k\dot{\tilde{x}}^2 \leq 0.$$

It follows that $\tilde{x}, \dot{\tilde{x}}, r \to 0$ as $t \to \infty$. This means $u(x, \dot{x}) \to \hat{m}\ddot{x}_r + \hat{b}\dot{x}_r$ and $f(x, \dot{x})$, $g(x, \dot{x}) \to 0$. Hence the the set $\Omega_c = \{V(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) \leq c\}$ is positively invariant for any $c > 0$. Invoking LaSalle's theorem [1], we identify $S = \{(\tilde{x}, \dot{\tilde{x}}) = 0\}$ as the set of all points in Ω_c where $\dot{V} = 0$. We now show that for specific choice of reference signal $x_r(t)$, no solution can stay identically in S other than the trivial solution $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) = (0, 0, 0, 0)$ and invoke Corollary 4.1 of [1] proving that the errors converge to zero.

To this end, for any solution that belongs identitcally to S, the system dynamics yields

$$\tilde{m}\ddot{x}_r + \tilde{b}\dot{x}_r \equiv 0.$$

Now, choose $\boxed{x_r(t) = A\sin(\omega t + \varphi)}$ for some constants $A, \omega > 0$, and φ. Then we have

$$-\tilde{m}\omega\sin(\omega t + \varphi) + \tilde{b}\cos(\omega t + \varphi) \equiv 0.$$

which implies that $\tilde{m}, \tilde{b} \equiv 0$. Thus the origin of $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b})$ is asymptotically stable.

Not submitted for review in February 2023.

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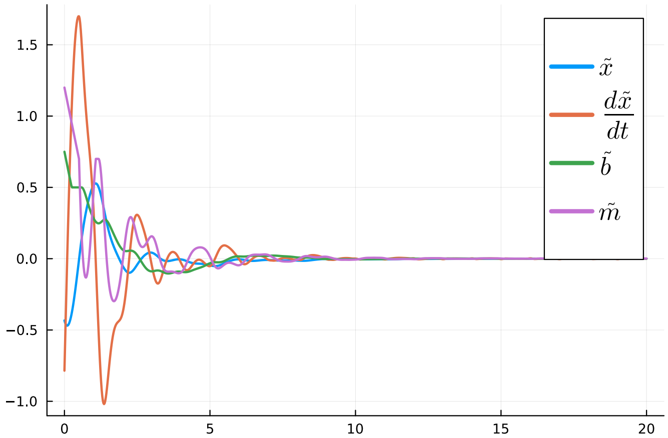

Fig. 1: Simulation showing that the system and estimation errors converge to zero.

III. Results

In Figure 1, we plot the response of the system to the control and adaptation laws laid out in Section II with $(A, \omega, \varphi) = (1/2, \pi, 60°)$ and $(k, \lambda) = (1, 1)$. The real mass and damping of the system is $(m, b) = (0.7, 0.5)$ and their estimates start at $(\hat{m}, \hat{b}) = (-1/2, -1/4)$.

IV. Conclusion

We have shown that our estimation algorithm for the linear second-order mechanical system guides the estimates of the mass and the damping coefficient to tend to their correct values.

References

[1] H. Khalil, *Nonlinear Control*. Always Learning, Pearson, 2015.