

# Estimation of the Mass and Damping of a Second-Order Linear System

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**Abstract**—We study the estimation of the mass and the damping of a second-order linear ODE system using an adaptation law.

**Index Terms**—Adaptive control, estimation

## I. INTRODUCTION

We provide a Lyapunov analysis [1] to prove that our control and adaptation laws will stabilize the system to a reference trajectory while the estimates of the mass and the damping tend to their correct values.

## II. ANALYSIS

We have a linear translational mechanical system with viscous damping, whose motion is governed by the ODE

$$m\ddot{x} + b\dot{x} = u(x, \dot{x}), \quad (1)$$

where  $u(x, \dot{x})$  is a control input that is to be determined for tracking. We are uncertain of the true values  $m$  and  $b$  so we denote our guesses for them by  $\hat{m}$  and  $\hat{b}$ , respectively. Let us introduce the errors in  $x$ ,  $m$  and  $b$  to be

$$\tilde{x} = x - x_r, \quad \tilde{m} = m - \hat{m}, \quad \tilde{b} = b - \hat{b},$$

where  $x_r$  is a reference signal for the motion of the mechanical system. We will postulate update rules for  $\hat{m}$  and  $\hat{b}$  and a control law for the mechanical system (1). The combined system has the dynamics

$$\begin{aligned} m\ddot{\tilde{x}} &= u(x, \dot{x}) - m\ddot{x}_r - b(\dot{x}_r + \dot{\tilde{x}}) \\ \dot{\tilde{m}} &= f(x, \dot{x}) \Rightarrow \dot{\tilde{m}} = -f(x, \dot{x}) \\ \dot{\tilde{b}} &= g(x, \dot{x}) \Rightarrow \dot{\tilde{b}} = -g(x, \dot{x}) \end{aligned} \quad (2)$$

Let us finally introduce the auxiliary variable  $r := \dot{\tilde{x}} + \lambda\tilde{x}$ , where  $\lambda > 0$  is a constant whose value is to be determined. Its dynamics is therefore given by

$$m\dot{r} = m(\ddot{\tilde{x}} + \lambda\dot{\tilde{x}}) = u - m(\ddot{x}_r - \lambda\dot{\tilde{x}}) - b(\dot{x}_r + \dot{\tilde{x}}).$$

Consider the Lyapunov function candidate

$$V(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) = \frac{1}{2}mr^2 + k\lambda\tilde{x}^2 + \frac{1}{2}\tilde{m}^2 + \frac{1}{2}\tilde{b}^2, \quad (3)$$

where  $k$  is another constant to be determined. This is a positive definite function over the space of  $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b})$ . We

take the time derivative of the Lyapunov function candidate and substitute from the system dynamics. We suppress the functional dependence of brevity.

$$\begin{aligned} \dot{V} &= r\dot{r} + 2k\lambda\tilde{x}\dot{\tilde{x}} - \dot{\tilde{m}}f - \dot{\tilde{b}}g, \\ &= r(u - m(\ddot{x}_r - \lambda\dot{\tilde{x}}) - b(\dot{x}_r + \dot{\tilde{x}})) + 2k\lambda\tilde{x}\dot{\tilde{x}} - \dot{\tilde{m}}f - \dot{\tilde{b}}g. \end{aligned}$$

This expression informs the selection of the control law as

$$u(x, \dot{x}) = \hat{m}(\ddot{x}_r - \lambda\dot{\tilde{x}}) + \hat{b}(\dot{x}_r + \dot{\tilde{x}}) - kr.$$

Substituting this controller into the expression for  $\dot{V}$  gives

$$\begin{aligned} \dot{V} &= r(-\tilde{m}(\ddot{x}_r - \lambda\dot{\tilde{x}}) - \tilde{b}(\dot{x}_r + \dot{\tilde{x}})) - kr^2 + 2k\lambda\tilde{x}\dot{\tilde{x}} - \dot{\tilde{m}}f \\ &\quad - \dot{\tilde{b}}g, \\ &= -k\lambda^2\tilde{x}^2 - k\dot{\tilde{x}}^2 - \tilde{m}(r(\ddot{x}_r - \lambda\dot{\tilde{x}}) + f) \\ &\quad - \tilde{b}(r(\dot{x}_r + \dot{\tilde{x}}) + g). \end{aligned}$$

Since we do not know the sign of  $\tilde{m}$  or  $\tilde{b}$ , this expression informs the choices of the adaptation laws as

$$f(x, \dot{x}) = -r(\ddot{x}_r - \lambda\dot{\tilde{x}}), \quad g(x, \dot{x}) = -r(\dot{x}_r + \dot{\tilde{x}}).$$

yielding a negative semidefinite  $\dot{V}$ :

$$\dot{V} = -k\lambda^2\tilde{x}^2 - k\dot{\tilde{x}}^2 \leq 0.$$

Hence the set  $\Omega_c = \{(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) : V(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) \leq c\}$  is positively invariant for any  $c > 0$ . Moreover,  $\tilde{x}, \dot{\tilde{x}}, r \rightarrow 0$  as  $t \rightarrow \infty$ . This means  $u(x, \dot{x}) \rightarrow \hat{m}\ddot{x}_r + \hat{b}\dot{x}_r$  and  $f(x, \dot{x}), g(x, \dot{x}) \rightarrow 0$ . We identify  $S = \{(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) : (\tilde{x}, \dot{\tilde{x}}) = 0\}$  as the set of all points in  $\Omega_c$  where  $\dot{V} = 0$ . We now show that for specific choice of reference signal  $x_r(t)$ , no solution can stay identically in  $S$  other than the trivial solution  $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) = (0, 0, 0, 0)$  and invoke Corollary 4.1 of [1] proving that the errors converge to zero.

To this end, for any solution that belongs identically to  $S$ , the system dynamics yields

$$\tilde{m}\ddot{x}_r + \tilde{b}\dot{x}_r \equiv 0, \quad \dot{\tilde{m}} \equiv \dot{\tilde{b}} \equiv 0.$$

Now, choose  $x_r(t) = A \sin(\omega t + \varphi)$  for some constants  $A, \omega > 0$ , and  $\varphi$ . With  $\theta(t) = \omega t + \varphi$ , we have

$$\tilde{m}\omega \sin \theta(t) - \tilde{b} \cos \theta(t) = \begin{bmatrix} \sin \theta(t) & -\cos \theta(t) \end{bmatrix} \begin{bmatrix} \tilde{m}\omega \\ \tilde{b} \end{bmatrix} \equiv 0,$$

for all  $t > 0$ . This implies that  $\tilde{m}, \tilde{b} \equiv 0$  because  $\tilde{m}$  and  $\tilde{b}$  must be constants within  $S$ . Thus the origin of  $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b})$  is globally asymptotically stable. ■

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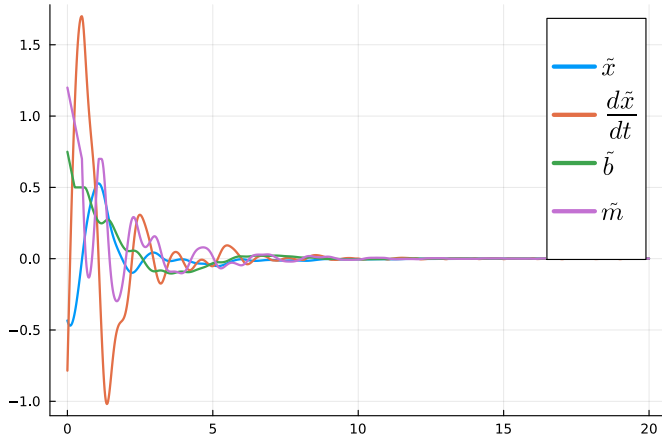


Fig. 1: Simulation showing that the system and estimation errors converge to zero.

### III. RESULTS

In Figure 1, we plot the response of the system to the control and adaptation laws laid out in Section II with  $(A, \omega, \varphi) = (1/2, \pi, 60^\circ)$  and  $(k, \lambda) = (1, 1)$ . The real mass and damping of the system is  $(m, b) = (0.7, 0.5)$  and their estimates start at  $(\hat{m}, \hat{b}) = (-1/2, -1/4)$ .

### IV. CONCLUSION

We have shown that our control and adaptation laws for the linear second-order mechanical system guide the estimates of the mass and the damping coefficient to tend to their correct values while tracking a judiciously chosen reference signal.

### REFERENCES

- [1] H. Khalil, *Nonlinear Control*. Always Learning, Pearson, 2015.