Estimation of the Mass and Damping of a Second-Order Linear System

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***Abstract*—We study the estimation of the mass and the damping of a second-order linear ODE system using an adaptation law.**

***Index Terms*—Adaptive control, estimation**

I. Introduction

We provide a Lyapunov analysis [1] to prove that our control and adaptation laws will stabilize the system to a reference trajectory while the estimates of the mass and the damping tend to their correct values.

II. Analysis

We have a linear translational mechanical system with viscous damping, whose motion is governed by the ODE

$$m\ddot{x} + b\dot{x} = u(x, \dot{x}), \tag{1}$$

where $u(x, \dot{x})$ is a control input that is to be determined for tracking. We are uncertain of the true values m and b so we denote our guesses for them by $\hat{m}$ and $\hat{b}$, respectively. Let us introduce the errors in x, m and b to be

$$\tilde{x} = x - x_r, \quad \tilde{m} = m - \hat{m}, \quad \tilde{b} = b - \hat{b},$$

where x_r is a reference signal for the motion of the mechanical system. We will postulate update rules for $\hat{m}$ and $\hat{b}$ and a control law for the mechanical system (1). The combined system has the dynamics

$$\begin{aligned} m\ddot{\tilde{x}} &= u(x, \dot{x}) - m\ddot{x}_r - b(\dot{x}_r + \dot{\tilde{x}}) \\ \dot{\hat{m}} &= f(x, \dot{x}) \Rightarrow \dot{\tilde{m}} = -f(x, \dot{x}) \\ \dot{\hat{b}} &= g(x, \dot{x}) \Rightarrow \dot{\tilde{b}} = -g(x, \dot{x}) \end{aligned} \tag{2}$$

Let us finally introduce the auxiliary variable $r := \dot{\tilde{x}} + \lambda\tilde{x}$, where $\lambda > 0$ is a constant whose value is to be determined. Its dynamics is therefore given by

$$m\dot{r} = m\left(\ddot{\tilde{x}} + \lambda\dot{\tilde{x}}\right) = u - m(\ddot{x}_r - \lambda\dot{\tilde{x}}) - b(\dot{x}_r + \dot{\tilde{x}}).$$

Consider the Lyapunov function candidate

$$V(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) = \frac{1}{2}mr^2 + k\lambda\tilde{x}^2 + \frac{1}{2}\tilde{m}^2 + \frac{1}{2}\tilde{b}^2,$$

where k is another constant to be determined. This is a positive definite function over the space of $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b})$. We take the time derivative of the Lyapunov function candidate and substitute from the system dynamics (2). We suppress its functional dependence for brevity.

$$\begin{aligned} \dot{V} &= r\dot{r} + 2k\lambda\tilde{x}\dot{\tilde{x}} + \tilde{m}\dot{\tilde{m}} + \tilde{b}\dot{\tilde{b}}, \\ &= r\left(u - m\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) - b\left(\dot{x}_r + \dot{\tilde{x}}\right)\right) + 2k\lambda\tilde{x}\dot{\tilde{x}} \\ &\quad - \tilde{m}f - \tilde{b}g. \end{aligned}$$

This expression informs the selection of the control law as

$$\boxed{u(x, \dot{x}) = \hat{m}\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) + \hat{b}\left(\dot{x}_r + \dot{\tilde{x}}\right) - kr}. \tag{3}$$

Substituting this controller into the expression for $\dot{V}$ gives

$$\begin{aligned} \dot{V} &= r\left(-\tilde{m}\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) - \tilde{b}\left(\dot{x}_r + \dot{\tilde{x}}\right)\right) - kr^2 + 2k\lambda\tilde{x}\dot{\tilde{x}} \\ &\quad - \tilde{m}f - \tilde{b}g, \\ &= -k\lambda^2\tilde{x}^2 - k\dot{\tilde{x}}^2 - \tilde{m}\left(r\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right) + f\right) \\ &\quad - \tilde{b}\left(r\left(\dot{x}_r + \dot{\tilde{x}}\right) + g\right). \end{aligned}$$

Since we do not know the sign of $\tilde{m}$ or $\tilde{b}$, this expression informs the choices of the adaptation laws as

$$\boxed{f(x, \dot{x}) = -r\left(\ddot{x}_r - \lambda\dot{\tilde{x}}\right)}, \quad \boxed{g(x, \dot{x}) = -r\left(\dot{x}_r + \dot{\tilde{x}}\right)}. \tag{4}$$

yielding a negative semidefinite $\dot{V}$:

$$\dot{V} = -k\lambda^2\tilde{x}^2 - k\dot{\tilde{x}}^2 \leq 0.$$

Hence the set $\Omega_c = \{(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) : V(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) \leq c\}$ is positively invariant for any $c > 0$. Moreover, $\tilde{x}, \dot{\tilde{x}}, r \to 0$ as $t \to \infty$. This means $u(x, \dot{x}) \to \hat{m}\ddot{x}_r + \hat{b}\dot{x}_r$ and $f(x, \dot{x})$, $g(x, \dot{x}) \to 0$. We identify $S = \{(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) : (\tilde{x}, \dot{\tilde{x}}) = 0\}$ as the set of all points in Ω_c where $\dot{V} = 0$. We now show that for a specific choice of the reference signal $x_r(t)$, no solution can stay identically in S other than the trivial solution $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b}) = (0, 0, 0, 0)$ and invoke Corollary 4.1 of [1] proving that the errors converge to zero.

To this end, for any solution that belongs identically to S, the system dynamics yields

$$\tilde{m}\ddot{x}_r + \tilde{b}\dot{x}_r \equiv 0, \qquad \dot{\tilde{m}} \equiv \dot{\tilde{b}} \equiv 0.$$

Now, choose $\boxed{x_r(t) = A\sin(\omega t + \varphi)}$ for some constants $A, \omega > 0$, and φ (or any time function with a nontrivial second derivative). With $\theta(t) = \omega t + \varphi$, we have

$$\tilde{m}\omega\sin\theta(t) - \tilde{b}\cos\theta(t) = \begin{bmatrix}\sin\theta(t) & -\cos\theta(t)\end{bmatrix}\begin{bmatrix}\tilde{m}\omega \\ \tilde{b}\end{bmatrix} \equiv 0,$$

for all $t > 0$. This implies that $\tilde{m}, \tilde{b} \equiv 0$ because $\tilde{m}$ and $\tilde{b}$ must be constants within S. Thus the origin of $(\tilde{x}, \dot{\tilde{x}}, \tilde{m}, \tilde{b})$ is globally asymptotically stable. ■

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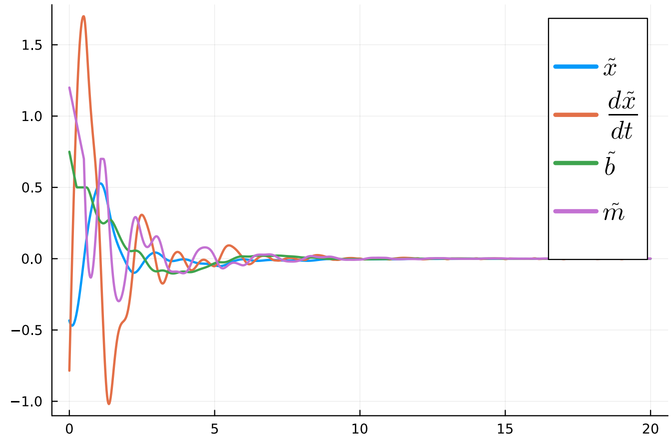

Fig. 1: Simulation showing that the system and estimation errors converge to zero.

III. Results

In Figure 1, we plot the response of the system to the control and adaptation laws (3, 4), implemented in simulation. The constants that are used are as follows: $(A, \omega, \varphi) = (1/2, \pi, 60°)$ and $(k, \lambda) = (1, 1)$. The real mass and damping of the system is $(m, b) = (0.7, 0.5)$ and their estimates start at $(\hat{m}, \hat{b}) = (-1/2, -1/4)$.

In Figure 2, we plot the response of the system to the control and adaptation laws (3, 4), implemented on a real cart system. The constants used are as follows: $(A, \omega, \varphi) = (3/10, 3\pi/5, 0°)$ and $(k, \lambda) = (1, 4)$. We did not know the real mass and damping of the system and set our initial guesses of them to be $(\hat{m}, \hat{b}) = (-1/2, -1/4)$.

IV. Conclusion

We have shown that our control and adaptation laws for the linear second-order mechanical system guide the estimates of the mass and the damping coefficient to tend to their correct values while tracking a judiciously chosen reference signal.

References

[1] H. Khalil, *Nonlinear Control*. Always Learning, Pearson, 2015.

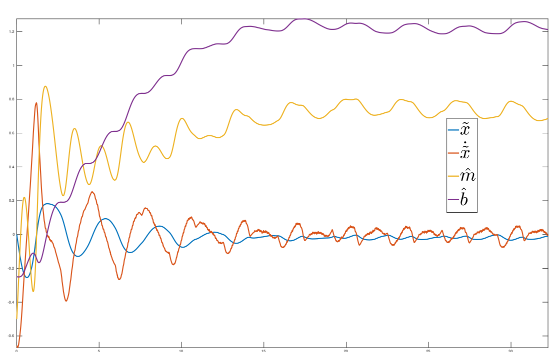

Fig. 2: Hardware implementation of the estimation algorithm.