Rectangle in a Triangle

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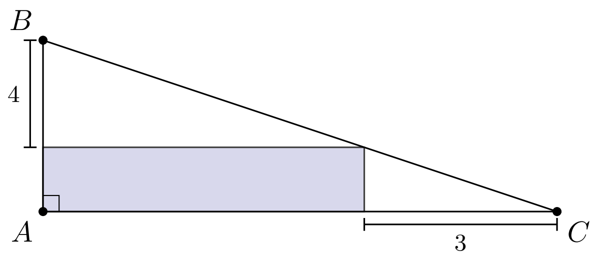
1 Problem Statement

Consider the blue rectangle that shares its bottom left corner with that of the triangle $\triangle ABC$ as seen in the figure to the right. The length of the line segment between points B and the top left corner of the rectangle is 4 units while the length of the line segment between the bottom right corner of the rectangle and point C is 3 units.

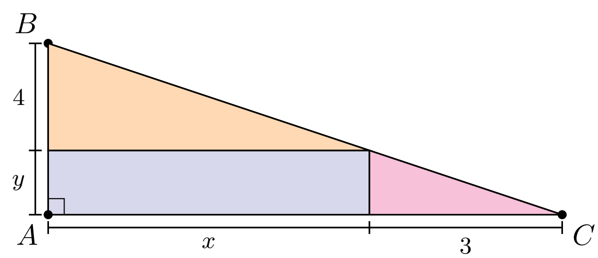


1. Find the area of the rectangle.
2. What is the minimum area of the triangle $\triangle ABC$?

2 Solution of the Problem

Let us define the positive real numbers x and y as in the figure to the right. Note that the area of the rectangle is given by xy and twice the area of the triangle is given by one of the following:

$$
\begin{aligned}
2A(\triangle ABC) &= (3+x)(4+y) = 12 + 4x + 3y + xy \\
&= 4x + 3y + 2xy.
\end{aligned}
$$



1. There are multiple methods we can invoke the find the area of the rectangle. Let us illustrate two methods here.

   (i) Equating the two equivalent expressions for twice the area of the triangle (see above) immediately implies that $xy = 12$.

   (ii) The triangles shaded in orange and magenta are similar yielding the relationship $4/x = y/3$, implying that $xy = 12$.

2. Set up the following optimization problem

$$
\begin{aligned}
\underset{x,\,y}{\text{minimize}} \quad & 24 + 4x + 3y, \\
\text{subject to} \quad & xy = 12, \\
& x, y \geq 0.
\end{aligned}
\tag{1}
$$

We will simply ignore the nonnegativity constraints on the decision variables for the moment and verify that the solution we end up with satisfies these conditions. To solve the optimization problem (1), we introduce the Lagrangian

$$\mathcal{L} = 24 + 4x + 3y + \lambda(12 - xy),$$

and perform unconstrained optimization on it. To that end, consider the first-order conditions

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 4 - \lambda y = 0 \quad &\Longrightarrow \quad &\lambda y = 4, \\
\frac{\partial \mathcal{L}}{\partial y} &= 3 - \lambda x = 0 \quad &\Longrightarrow \quad &\lambda x = 3, \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 12 - xy = 0 \quad &\Longrightarrow \quad &xy = 12.
\end{aligned}$$

Substituting the ratio of the first two equations, which yields $y = {}^{4x}/_{3}$, into the last equation gives $x^2 = 9$, implying that $x^\star = 3$ (it needs to be nonnegative). Now, we can use this solution in the last equation above to get $y^\star = 4$. At this solution, we also have $\lambda^\star = 1$. Note that the Hessian of the Lagrangian given by

$$H(x^\star, y^\star, \lambda^\star) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

is positive definite on the tangent space of the constraint manifold $T(x^\star, y^\star) := \{z \in \mathbb{R}^2 : \begin{bmatrix} y & x \end{bmatrix} z = 0\} = \{z : z = t \begin{bmatrix} x & -y \end{bmatrix}^\top, t \in \mathbb{R}\}$, verifying the sufficient conditions for minimality. Indeed, for $z \in T(x^\star, y^\star)$, we have

$$z \cdot Hz = t^2 xy = 12t^2 > 0 \quad \text{for all} \quad t \neq 0.$$

Hence the triangle with the minimum area satisfies $A(\triangle ABC) = 24$ and looks like the one below.