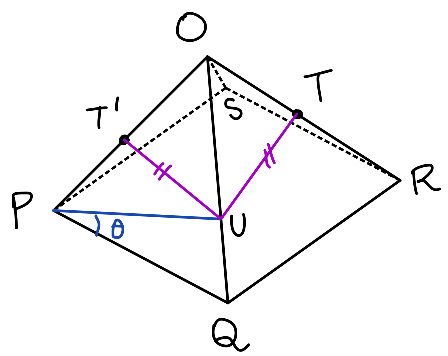
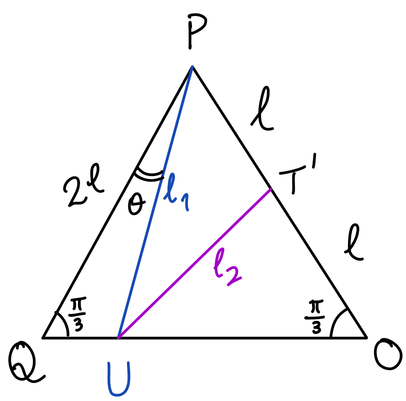


A square pyramid (pentahedron) has base PQRS and vertex O. Each edge has length 20.

Calculate the shortest distance along the surface of the pentahedron from point P to point T, where T is the midpoint of OR.

In order to get to T, a point starting at P must cross one of the lines OQ or OS since a simple check reveals that staying on the square is suboptimal. By symmetry we can choose to cross the line OQ, the question becomes where should one cross it? To determine this, let U be the point on OQ to be visited while crossing it. It can be parametrized by the angle between PU and PQ. Let's call it θ. The remainder of the path is determined as the straight line UT. Hence all paths are parametrized by the starting heading direction θ. We can summarize the whole path on one single equilateral triangle by symmetry: $|UT| = |UT'|$, where T' is the reflection of point T through the plane containing Q, O, S.



By the law of sines, applied to ΔOUP, we obtain

$$\frac{\sin(\pi/3)}{\ell_1} = \frac{\sin\left(\frac{2\pi}{3} - \theta\right)}{2\ell} \Rightarrow \ell_1(\theta) = \frac{\sqrt{3}\,\ell}{\sin\left(\frac{2\pi}{3} - \theta\right)}$$

$$\frac{\sin(\theta)}{|PU|} = \frac{\sin\left(\frac{2\pi}{3} - \theta\right)}{2\ell} \Rightarrow |PU| = 2\ell \frac{\sin(\theta)}{\sin\left(\frac{2\pi}{3} - \theta\right)}$$

$$|UQ| = 2\ell\left(1 - \frac{\sin(\theta)}{\sin\left(\frac{2\pi}{3} - \theta\right)}\right) = 2\ell \frac{\sin\left(\frac{2\pi}{3} + \theta\right)}{\sin\left(\frac{2\pi}{3} - \theta\right)} .$$

Law of cosines applied to $\Delta UQT'$ yields

$$\ell_2(\theta)^2 = \ell^2 + |UQ|^2 - 2\ell|UQ|\cos(\pi/6)$$

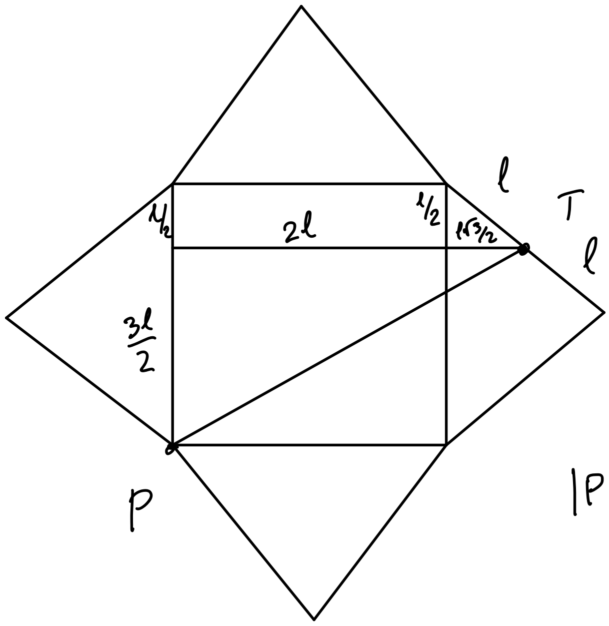
$$= \ell^2\left[1 - 2\frac{\sin\left(\frac{2\pi}{3} + \theta\right)}{\sin\left(\frac{2\pi}{3} - \theta\right)} + 4\left(\frac{\sin\left(2\pi/3 + \theta\right)}{\sin\left(\frac{2\pi}{3} - \theta\right)}\right)^2\right]$$

Hence we optimize (minimize)

$$f(\theta) = \ell_1(\theta) + \ell_2(\theta)$$

subject to $0 \leq \theta \leq \pi/3$. For $\ell = 10$, this yields

$$f^*(\theta^*) \approx 26.4575 \quad \text{and} \quad \theta^* \approx 0.713719 \approx 40.893^\circ$$



Along the base, the shortest distance is

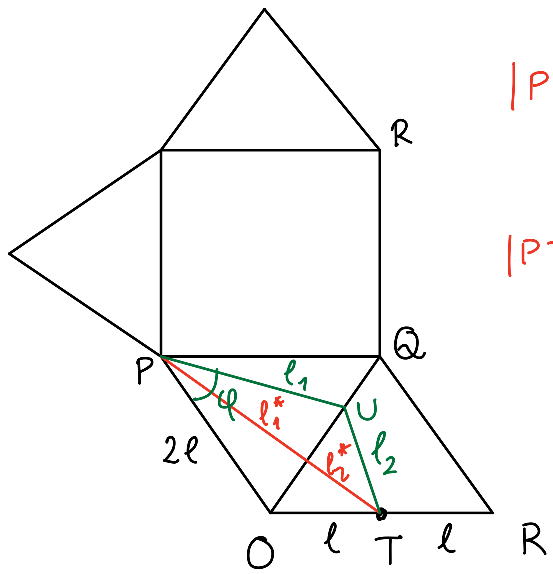
$$|PT|^2 = \left(\frac{3l}{2}\right)^2 + \left(2l + \frac{l\sqrt{3}}{2}\right)^2$$

$$= \frac{9l^2}{4} + 4l^2 + \frac{3l^2}{4} + 2\sqrt{3}\,l^2$$

$$= \left(7 + 2\sqrt{3}\right) l^2$$

$$|PT| = l\sqrt{7+2\sqrt{3}}$$

For $l=10$, this gives $|PT| = 10\sqrt{7+2\sqrt{3}} \approx 32.348$; larger than 26.4575 one can achieve over by traversing other faces.



$$|PT|^2 = (2\ell)^2 + \ell^2 - 2\ell\cdot\ell\cdot\cos(2\pi/3)$$
$$= 7\ell^2$$
$$|PT| = \ell\sqrt{7}$$

$$\frac{S_\varphi}{|OU|} = \frac{\sqrt{3}/2}{\ell_1} = \frac{\sin\left(\frac{2\pi}{3}-\varphi\right)}{2\ell}$$

$$\ell_1 = \frac{\sqrt{3}\,\ell}{\sin\left(\frac{2\pi}{3}-\varphi\right)}$$

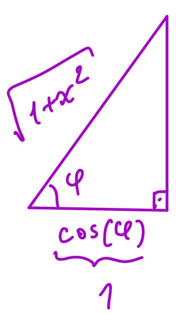
$$|OU| = 2\ell\,\frac{\sin(\varphi)}{\sin\left(\frac{2\pi}{3}-\varphi\right)}$$

$$\ell\sqrt{7} < \ell\sqrt{7+2\sqrt{3}}\,.$$

$$\ell_2^2 = \ell^2 + 4\ell^2\left(\frac{\sin(\varphi)}{\sin\left(\frac{2\pi}{3}-\varphi\right)}\right)^2 - 2\ell^2\,\frac{\sin(\varphi)}{\sin\left(\frac{2\pi}{3}-\varphi\right)}$$

$$\frac{\sin(\varphi^*)}{\ell} = \frac{\sin\left(\frac{\pi}{3}-\varphi^*\right)}{2\ell} \Rightarrow 2\sin(\varphi^*) = \frac{\sqrt{3}}{2}\cos(\varphi^*) - \frac{1}{2}\sin(\varphi^*)$$

$$\tan(\varphi^*) = \frac{\sqrt{3}}{5} \qquad \varphi^* = \operatorname{atan}\left(\frac{\sqrt{3}}{5}\right)$$



$\sin(\varphi) = x$

$$\varphi = \operatorname{atan}(x)$$

$$\cos\left(\operatorname{atan}(x)\right) = \cos(\varphi) = \frac{1}{\sqrt{1+x^2}}$$

$$\sin\left(\operatorname{atan}(x)\right) = \sin(\varphi) = \frac{x}{\sqrt{1+x^2}}$$

$$\ell_1^* = \frac{\sqrt{3}\,\ell}{\sin\left(\frac{2\pi}{3}-\varphi^*\right)} = \frac{\sqrt{3}\,\ell}{\frac{\sqrt{3}}{2}\cos(\varphi^*) + \frac{1}{2}\sin(\varphi^*)} = \frac{2\sqrt{3}\,\ell}{\sqrt{3}\,\frac{1}{\sqrt{1+3/25}} + \frac{\sqrt{3}/5}{\sqrt{1+3/25}}} = \frac{5\ell}{3\cdot\frac{1}{\sqrt{28/25}}}$$

$$= \frac{5\ell}{3\cdot\frac{5}{2\sqrt{7}}} = \frac{2\sqrt{7}}{3}\,\ell$$

$$\ell_2^* = \ell\sqrt{7} - \ell_1^* = \sqrt{7}\left(\ell - \frac{2}{3}\ell\right) = \frac{\sqrt{7}}{3}\ell .$$