Shortest Path on a Regular Pyramid

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***Abstract*— This problem, seen on Mind Your Decisions channel [1] on YouTube, concerns finding the shortest distance to a particular point on a regular pyramid, starting from a particular vertex.**

***Index Terms*— Geometry, Ti*k*Z**

I. Introduction

A regular square pyramid is not a smooth manifold. We want to find the shortest distance from a particular vertex of this pyramid to the midpoint of one of its edges. During the course of this quest, we will also identify the shortest path.

II. Problem Statement

In addition to the problem statement given in the caption of Figure 1, we will also seek to find the path that yields the shortest distance from the vertex P to the point T, which lies in the middle of the vertices O and R.

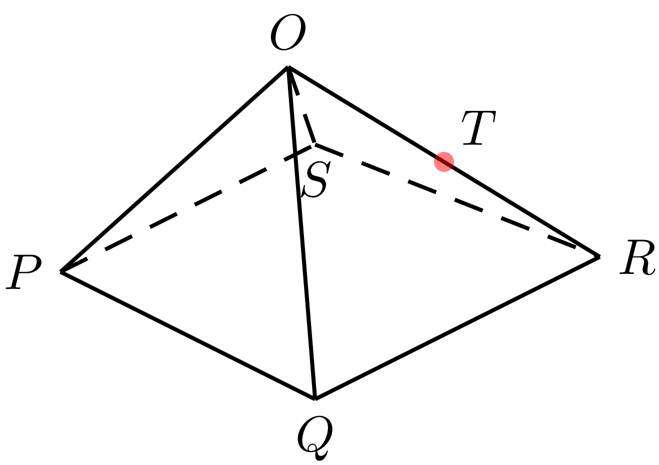

Fig. 1: Shown above is a regular square pyramid all of whose sides have length of 2ℓ. We seek to find the shortest distance from the vertex P to the midpoint T of points O and R.

III. Problem Solution

Shown in Figure 2, we show the regular pyramid of Figure 1 unfolded onto the plane in such a manner that the edge OQ remains glued, while the edges OR, OS, and OP are unglued. An arbitrary piecewise straight path γ from the vertex P to the midpoint T is then depicted as the green curve on this figure. The continuous portions, PU and UT, of this path have lengths ℓ_1 and ℓ_2, respectively.

The line PR that connects the vertices P and R is orthogonal to the edge OQ since the triangle $\triangle POQ$ is equilateral. Let the signed distance from the intersection M of PR with OQ to the intersection U of our path γ and OQ be given by x. In the following, we express the total length $\ell_1 + \ell_2$ of our curve γ as a function of this distance x and minimize it to arrive at the answer.

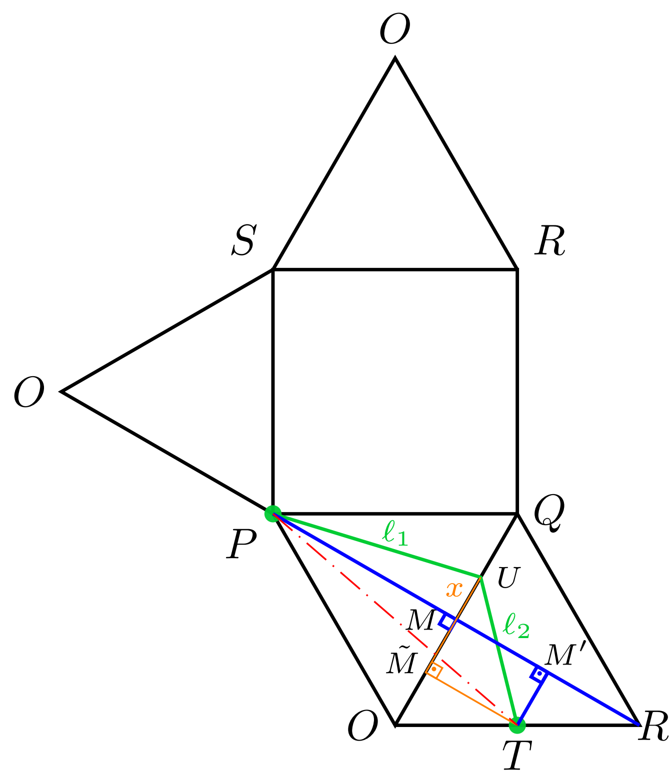

Fig. 2: The unfolded regular pyramid.

Since $PM \perp OM$, Pythagoras's theorem yields $|PM| = \sqrt{3}\ell$. By the same token, $\triangle PMU$ is a right triangle, yielding

$$\ell_1(x) = \sqrt{x^2 + 3\ell^2}. \tag{1}$$

The perpendicular TM' to PR is parallel to OQ so by the similarity of the triangles $\triangle OMR$ and $\triangle TM'R$, and the fact that $|OM| = \ell$, we deduce that $|TM'| = \ell/2$ and $|MM'| = \sqrt{3}\ell/2$. Using these lengths, along with x, as the side lenghts of the orange right triangle $\triangle T\tilde{M}M$ in Figure 2, we obtain

$$\ell_2(x) = \sqrt{\ell^2 + \ell x + x^2}. \tag{2}$$

The solution that we seek minimizes $\ell_1(x) + \ell_2(x)$. Therefore, we differentiate this function and set it equal to zero to

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obtain

$$\frac{2x}{\sqrt{3\ell^2 + x^2}} + \frac{\ell + 2x}{\sqrt{\ell^2 + \ell x + x^2}} = 0.$$

The solution to this equation is $x^\star = -\ell/3$ as a simple substitution will show. Plugging this solution into equations (1) and (2) yields

$$\ell_1^\star = \ell_1(x^\star) = \frac{2\sqrt{7}}{3}\ell, \qquad \ell_2^\star = \ell_2(x^\star) = \frac{\sqrt{7}}{3}\ell.$$

Summing these two optimal values gives the shortest distance between points P and T on the regular square pyramid as

$$\ell_1(x^\star) + \ell_2(x^\star) = \sqrt{7}\ell.$$

Lemma 1. The (green) path that yields the shortest distance $\sqrt{7}\ell$ is a straight line.

Proof: Perhaps the easiest way to show this is to use the law of cosines to show that the edge PT of the triangle $\triangle POT$ has length equal to $\sqrt{7}\ell$ so that our solution for the green path coincides with the dash-dotted red line on Figure 2.

$$|PT|^2_{\text{red}} = (2\ell)^2 + \ell^2 - 2 \cdot 2\ell \cdot \ell \cos 120^\circ = 7\ell^2$$

∎

IV. Conclusion

This is an interesting little brain teaser and a good excuse to practice some more Ti*k*Z.

References

[1] P. Talwalker, “Mind your decisions.” https://youtu.be/qo_wZ1zjh_Q. Accessed: 2022-09-17.