Probability on the Unit Square

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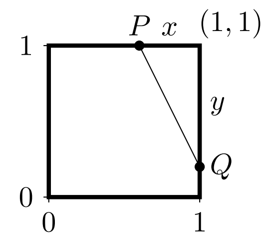
1 Problem Statement

Points P and Q are randomly sampled from a uniform distribution on two sides of a unit square. Let d be the random variable that denotes the length of the chord $|PQ|$ (see the figure). What is the probability that $d \geq 1$?



2 Solution of the Problem

$d \geq 1$ if and only if $d^2 = x^2 + y^2 \geq 1$ if and only if $y^2 \geq 1 - x^2$. The random variable x is uniformly distributed. Thus, for a given $0 \leq x \leq 1$, we have $p(y \geq \sqrt{1-x^2}) = 1 - \sqrt{1-x^2}$. As a result,

$$\mathbb{P}(y \geq \sqrt{1-x^2}) = \int_0^1 \left(1 - \sqrt{1-x^2}\right) \mathrm{d}x \tag{1}$$

Let us take the indefinite integral $\int \sqrt{1-x^2}\,\mathrm{d}x$ by substituting $x = \sin(u)$ and $\mathrm{d}x = \cos(u)\,\mathrm{d}u$.

$$\begin{aligned}\int \sqrt{1-x^2}\,\mathrm{d}x &= \int \cos(u)^2\,\mathrm{d}u = \int \left(\frac{1}{2}\cos(2u) + \frac{1}{2}\right)\mathrm{d}u = \frac{1}{4}\sin(2u) + \frac{u}{2} + c = \frac{1}{2}\sin(u)\cos(u) + \frac{u}{2} + c \\ &= \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin(x)\right) + c\end{aligned}$$

Using the above computation to evaluate the definite integral in equation (1), we obtain

$$\mathbb{P}(y \geq \sqrt{1-x^2}) = 1 - \frac{\pi}{4}.$$

Now, we have the following cases:

1. With $1/2$ probability, the points P and Q are sampled on adjacent sides, in which case, the above computation shows that $\mathbb{P}(y \geq \sqrt{1-x^2}) = 1 - \frac{\pi}{4}$.

2. With $1/4$ probability, the points P and Q are sampled on opposite sides, in which case, $\mathbb{P}(y \geq \sqrt{1-x^2}) = 1$.

3. With $1/4$ probability the points P and Q are sampled on the same side, in which case, $\mathbb{P}(y \geq \sqrt{1-x^2}) = 0$.

By the law of total probability, we have

$$\begin{aligned}\mathbb{P}(d \geq 1) = &\ \mathbb{P}(d \geq 1|\text{adjacent sides})\mathbb{P}(\text{adjacent sides}) + \mathbb{P}(d \geq 1|\text{same sides})\mathbb{P}(\text{same sides}) \\ &+ \mathbb{P}(d \geq 1|\text{opposite sides})\mathbb{P}(\text{opposite sides}) = 1/2\,(1 - \pi/4) + 1/4 \cdot 0 + 1/4 \cdot 1 = \boxed{3/4 - \pi/8} \\ &\approx 0.3573.\end{aligned}$$