An Interesting Probability Question

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1 Problem Statement

On a 5×5 chess board whose squares are coded by A, B, C, D, and E along the horizontal and by 1, 2, 3, 4, and 5 along the vertical direction, you have a remote-controlled robot on the square $C3$. The remote-controller has 4 buttons that can move the robot up, down, left, and right by 1 square; however, your michievous sibling has tampered with the placements of these buttons. Assuming that the buttons are fixed after having shuffled once, what is the expected value of the number of times you need to press the buttons in order to move the robot to the $E5$ square?

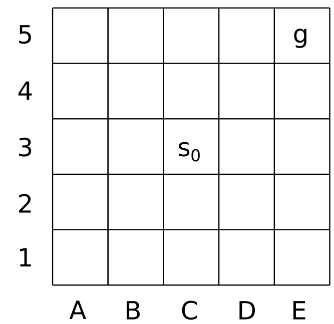

Figure 1: Schematic of the problem

2 Solution of the Problem

We invoke a celebrated theorem from probability theory that will help us solve the problem. This theorem is called the *law of total expectation* or *tower rule of probability theory* in the literaturer Bertsekas and Tsitsiklis (2002).

Theorem 2.1 (Tower rule) *Let X be a random variable and $\{A_i\}_i^m$ be a finite partition of the sample space. Then,*

$$\mathbb{E}[X] = \sum_i^m \mathbb{E}\left[X \mid A_i\right] \mathbb{P}(A_i).$$

We denote the square on which the robot resides at the k^{th} step by s_k and the goal square by g. Consider the following family of random variables.

$$C_n := \sum_{k=n+1}^{\infty} r_k, \qquad r_k = \begin{cases} 1 & \text{if } s_k \neq g \\ 0 & \text{if } s_k = g \end{cases}. \tag{1}$$

The sum in this definition is well-defined because once the robot reaches g, all the remaining r_k's take the value zero. We are being asked the value of $\mathbb{E}[C_0]$. To that end, we will use the following result of definition (1).

$$C_m = \sum_{i=m+1}^{n} r_i + C_n, \qquad m \leq n.$$

Using the properties of expectation, we deduce the following result.

$$\mathbb{E}[C_m] = \sum_{i=m+1}^{n} \mathbb{E}[r_i] + \mathbb{E}[C_n] = n - m + \mathbb{E}[C_n], \qquad m < i < n, \;\; s_i \neq g.$$

Similar identities to the above expression hold for conditional expecteations as well. Using the tower rule 2.1, we start to compute.

$$\mathbb{E}[C_0] = \mathbb{E}\left[C_0 \mid s_1 \in \{C4, D3\}\right] \underbrace{\mathbb{P}(s_1 \in \{C4, D3\})}_{=\frac{1}{2}} + \mathbb{E}\left[C_0 \mid s_1 \in \{B3, C2\}\right] \underbrace{\mathbb{P}(s_1 \in \{B3, C2\})}_{=\frac{1}{2}}. \tag{2}$$

From now on, we will assume $s_1 = C4$ in the first term on the right-hand side and $s_1 = C2$ in the second. Notice that, because of the inherent symmetry of the problem, while the state $s_1 = D3$ is equivalent to $s_1 = C4$; $s_1 = B3$ is equivalent to $s_1 = C2$.

Computation of the term $\mathbb{E}\left[C_0 \mid s_1 = C4\right]$. First of all, we consider the first expectation term on the right-hand side of equation (2).

$$\begin{aligned}\mathbb{E}\left[C_0 \mid s_1 = C4\right] = \overbrace{r_1 + r_2}^{=2} &+ \overbrace{\mathbb{E}[C_2 \mid s_1 = C4, s_3 = D5]}^{=2}\,\overbrace{\mathbb{P}(s_3 = D5 \mid s_1 = C4)}^{=\frac{1}{3}} \\ &+ \mathbb{E}[C_2 \mid s_1 = s_3 = C4]\,\underbrace{\mathbb{P}(s_3 = C4 \mid s_1 = C4)}_{=\frac{1}{3}} \\ &+ \mathbb{E}[C_2 \mid s_1 = C4, s_3 = B5]\,\underbrace{\mathbb{P}(s_3 = B5 \mid s_1 = C4)}_{=\frac{1}{3}}.\end{aligned} \tag{3}$$

We delve further into the computation of the two expected values in equation (3), whose values are not immediately apparent.

$$\begin{aligned}\mathbb{E}[C_2 \mid s_1 = s_3 = C4] = \underbrace{r_3 + r_4}_{=2} &+ \underbrace{\mathbb{E}[C_4 \mid s_1 = s_3 = C4, s_5 = D5]}_{=2}\,\underbrace{\mathbb{P}(s_5 = D5 \mid s_1 = s_3 = C4)}_{=\frac{1}{2}} \\ &+ \underbrace{\mathbb{E}[C_4 \mid s_1 = s_3 = C4, s_5 = B5]}_{=4}\,\underbrace{\mathbb{P}(s_5 = B5 \mid s_1 = s_3 = C4)}_{=\frac{1}{2}}.\end{aligned}$$

There is only one expected value that we have left to compute in equation (3):

$$\begin{aligned}\mathbb{E}[C_2 \mid \; s_1 = C4, s_3 = B5] = &\underbrace{\mathbb{E}[C_2 \mid s_1 = C4, s_3 = B5, s_4 = C5]}_{=4}\,\underbrace{\mathbb{P}(s_4 = C5 \mid s_1 = C4, s_3 = B5)}_{=\frac{1}{2}} \\ &+ \underbrace{\mathbb{E}[C_2 \mid s_1 = C4, s_3 = B5, s_4 = B4]}_{=6}\,\underbrace{\mathbb{P}(s_4 = B4 \mid s_1 = C4, s_3 = B5)}_{=\frac{1}{2}}.\end{aligned}$$

The computations above allows us to determine the first expected value in equation (2) using equation (3): $\boxed{\mathbb{E}[C_0 \mid s_1 = C4] = 6}$.

Computation of the term $\mathbb{E}\left[C_0 \mid s_1 = C2\right]$. We use similar techniques to compute the second expected value on the right-hand side of equation (2).

$$\begin{aligned}\mathbb{E}\left[C_0 \mid s_1 = C2\right] = &\underbrace{\mathbb{E}[C_0 \mid s_1 = C2, s_2 = B2]}_{=8}\underbrace{\mathbb{P}(s_2 = B2 \mid s_1 = C2)}_{=\frac{1}{3}} \\ &+ \mathbb{E}[C_0 \mid s_1 = C2, s_2 = C3]\underbrace{\mathbb{P}(s_2 = C3 \mid s_1 = C2)}_{=\frac{1}{3}} \\ &+ \mathbb{E}[C_0 \mid s_1 = C2, s_2 = D2]\underbrace{\mathbb{P}(s_2 = D2 \mid s_1 = C2)}_{=\frac{1}{3}}.\end{aligned} \tag{4}$$

Once more, we utilize the tower rule to compute the expected values in equation (4) whose values are not immediately apparent.

$$\begin{aligned}\mathbb{E}[C_0 \mid s_1 = C2, s_2 = C3] &= \overbrace{r_1 + r_2 + r_3 + r_4}^{=4} \\ &+ \underbrace{\mathbb{E}[C_4 \mid s_1 = C2, s_2 = C3, s_5 = D5]}_{=2}\underbrace{\mathbb{P}(s_5 = D5 \mid s_1 = C2, s_2 = C3)}_{=\frac{1}{2}} \\ &+ \underbrace{\mathbb{E}[C_4 \mid s_1 = C2, s_2 = C3, s_5 = B5]}_{=4}\underbrace{\mathbb{P}(s_5 = B5 \mid s_1 = C2, s_2 = C3)}_{=\frac{1}{2}}.\end{aligned}$$

There is only one expected value that we have left to compute in equation (4):

$$\begin{aligned}\mathbb{E}[C_0 \mid s_1 = C2, s_2 = D2] &= \overbrace{r_1 + r_2 + r_3}^{=3} \\ &+ \underbrace{\mathbb{E}[C_3 \mid s_1 = C2, s_2 = D2, s_4 = E3]}_{=3}\underbrace{\mathbb{P}(s_4 = E3 \mid s_1 = C2, s_2 = D2)}_{=\frac{1}{2}} \\ &+ \underbrace{\mathbb{E}[C_3 \mid s_1 = C2, s_2 = s_4 = D2]}_{=5}\underbrace{\mathbb{P}(s_4 = D2 \mid s_1 = C2, s_2 = D2)}_{=\frac{1}{2}}.\end{aligned}$$

We can now use equation (4) in order to compute the second expected value on the right-hand side of equation (2): $\boxed{\mathbb{E}[C_0 \mid s_1 = C2] = \frac{22}{3}}$. We have computed every quantity that we are interested in. Now, we go back to equation (2):

$$\boxed{\begin{aligned}\mathbb{E}[C_0] = &\underbrace{\mathbb{E}\left[C_0 \mid s_1 \in \{C4, D3\}\right]}_{=6}\underbrace{\mathbb{P}(s_1 \in \{C4, D3\})}_{=\frac{1}{2}} \\ &+ \underbrace{\mathbb{E}\left[C_0 \mid s_1 \in \{B3, C2\}\right]}_{=\frac{22}{3}}\underbrace{\mathbb{P}(s_1 \in \{B3, C2\})}_{=\frac{1}{2}} = \frac{20}{3} = 6.\bar{6}.\end{aligned}} \tag{5}$$

References

Bertsekas, D. P., & Tsitsiklis, J. N. (2002). *Introduction to probability* (Vol. 1). Athena Scientific Belmont, MA.