A Trigonometry Question

Aykut C. Satici *Member, IEEE*

I. Question

Solve the following equation for $0 \leq x \leq 2\pi$.

$$2 \cos x \left(\cos 2x - 2 \sin^2 x\right) = 1.$$

II. Answer

Using the fact that $2 \sin^2 x = 1 - \cos 2x$, we have

$$1 = 2 \cos x \left(\cos 2x - 2 \sin^2 x\right) = 2 \cos x \left(2 \cos 2x - 1\right) \quad (1)$$

Next, we use the identity

$$\cos \theta_1 \cos \theta_2 = \frac{1}{2} \cos\left(\theta_1 + \theta_2\right) + \frac{1}{2} \cos\left(\theta_1 - \theta_2\right)$$

in equation (1) to obtain

$$\cos 3x = \frac{1}{2}. \quad (2)$$

This means that $3x + 2\pi k = \frac{\pi}{3}$ or $3x + 2\pi k = \frac{5\pi}{3}$, for $k \in \mathbb{Z}$.

a) Case 1:

$$3x + 2\pi k = \frac{\pi}{3} \quad \Rightarrow \quad x = \frac{\pi}{9}(1 - 6k).$$

Since $0 \leq x \leq 2\pi$, we must have that $k \in \{-2, -1, 0\}$; i.e.,

$$x \in \left\{ \frac{\pi}{9}, \frac{7\pi}{9}, \frac{13\pi}{9} \right\}.$$

b) Case 2:

$$3x + 2\pi k = \frac{5\pi}{3} \quad \Rightarrow \quad x = \frac{\pi}{9}(5 - 6k).$$

Since $0 \leq x \leq 2\pi$, we must have that $k \in \{-2, -1, 0\}$; i.e.,

$$x \in \left\{ \frac{5\pi}{9}, \frac{11\pi}{9}, \frac{17\pi}{9} \right\}.$$

Putting together the two cases, we have the final solution as

$$\boxed{x \in \frac{\pi}{9} \times \{1, 5, 7, 11, 13, 17\}.}$$