Probability on the Unit Square

Aykut C. Satici
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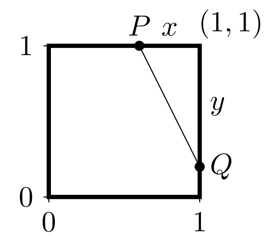
1 Problem Statement

Points P and Q are randomly sampled from a uniform distribution on two sides of a unit square. Let d be the random variable that denotes the length of the chord $|PQ|$ (see the figure). What is the probability that $d \geq 1$?



2 Solution of the Problem

We have the following cases:

C1. With $p_a = 1/2$ probability, P and Q are sampled on adjacent sides.

C2. With $p_o = 1/4$ probability, P and Q are sampled on opposite sides.

C3. With $p_s = 1/4$ probability, P and Q are sampled on the same side.

By the law of total probability, we have

$$\mathbb{P}(d \geq 1) = \mathbb{P}(d \geq 1|\text{adjacent sides})p_a + \mathbb{P}(d \geq 1|\text{opposite sides})p_o + \mathbb{P}(d \geq 1|\text{same sides})p_s \tag{1}$$

Let us analyze the case in which the points P and Q are sampled on adjacent sides of the unit square. Then $d \geq 1$ if and only if $d^2 = x^2 + y^2 \geq 1$ if and only if $y^2 \geq 1 - x^2$. The random variable x is uniformly distributed. Thus, for a given $0 \leq x \leq 1$, we have $p(y \geq \sqrt{1-x^2}) = 1 - \sqrt{1-x^2}$. As a result,

$$\mathbb{P}(d \geq 1|\text{adjacent sides}) = \mathbb{P}(y \geq \sqrt{1-x^2}) = \int_0^1 \left(1 - \sqrt{1-x^2}\right) \mathrm{d}x = 1 - \int_0^1 \sqrt{1-x^2}\, \mathrm{d}x = 1 - \frac{\pi}{4}. \tag{2}$$

Indeed, the integral in equation (2) can be evaluated by substituting $x = \sin(u)$ and $\mathrm{d}x = \cos(u)\,\mathrm{d}u$.

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\pi/2} \cos(u)^2\,\mathrm{d}u = \int_0^{\pi/2} \left(\frac{1}{2}\cos(2u) + \frac{1}{2}\right)\mathrm{d}u = \left[\frac{1}{4}\sin(2u) + \frac{u}{2}\right]_{u=0}^{u=\pi/2} = \frac{\pi}{4}.$$

The remaining terms in equation (1), namely, $\mathbb{P}(d \geq 1|\text{opposite sides})$ and $\mathbb{P}(d \geq 1|\text{same sides})$ are readily seen to equal 1 and 0, respectively. Plugging in the appropriate values in equation (1) thus yields

$$\begin{aligned}\mathbb{P}(d \geq 1) &= \mathbb{P}(d \geq 1|\text{adjacent sides})p_a + \mathbb{P}(d \geq 1|\text{opposite sides})p_o + \mathbb{P}(d \geq 1|\text{same sides})p_s \\ &= 1/2\,(1 - \pi/4) + 1/4 \times 1 + 1/4 \times 0 = \boxed{3/4 - \pi/8} \approx 35.73\%.\end{aligned}$$