Differential-Geometric Approach to Nonlinear Control

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Preface

Differential Geometry is the study of mathematical objects that are invariant under changes of notation.

Acknowledgments

Any comments, criticisms, and corrections are most welcome and may be sent to the author at *chiasson@ieee.org*.

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Linear and Nonlinear Control Systems

In this first chapter we review the use of state feedback and state estimation for linear systems. Specifically the transformation of a controllable linear system into control canonical form and the transformation of an observable linear system are considered. The procedure of linearizing a nonlinear system about an equilibrium point is also reviewed.

Next examples of nonlinear systems are presented. These examples are used to show how the linear methods of state feedback control and state estimation can be generalized without having to linearize the system about an equilibrium points. These examples are used to motivate the differential geometric approach to the control of nonlinear systems.

1.1 The Control Canonical Form for Linear Time Invariant Systems

Recall that a single-input linear time invariant (LTI) statespace model is given as

$$\frac{dx}{dt} = Ax + bu,\ x \in \mathbb{R}^n,\ u \in \mathbb{R},\ A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n. \tag{1.1}$$

With $n = 3$ suppose A and b have special form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.2}$$

Then a simple calculation shows that the characteristic equation is

$$\det(sI - A) = s^3 + a_2 s^2 + a_1 s + a_0.$$

With the state feedback

$$u = -kx + r = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + r \tag{1.3}$$

the closed-loop system is

$$\frac{dx}{dt} = (A - bk)x + br.$$

More explicitly we have

$$\begin{aligned} \frac{dx}{dt} &= \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r. \end{aligned}$$

Choosing the control gains as $k_1 = \alpha_0 - a_1, k_2 = \alpha_2 - k_2, k_3 = \alpha_3 - a_3$ results in

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

with $\det\Big(sI - \ (A - bk)\Big) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$.

The special form of the pair (A, b) above is called the control canonical form and its usefulness is clear: If (A, b) is in control canonical form then the feedback (row)vector may be straightforwardly chosen (by inspection!) to assign the coefficients of the closed-loop characteristic equation to any desired values. That is, arbitrary pole placement is possible.

We now consider the above in the context of a physical system: the armature controlled DC motor. The figure below is a physical representation of a single-loop DC motor from [1].

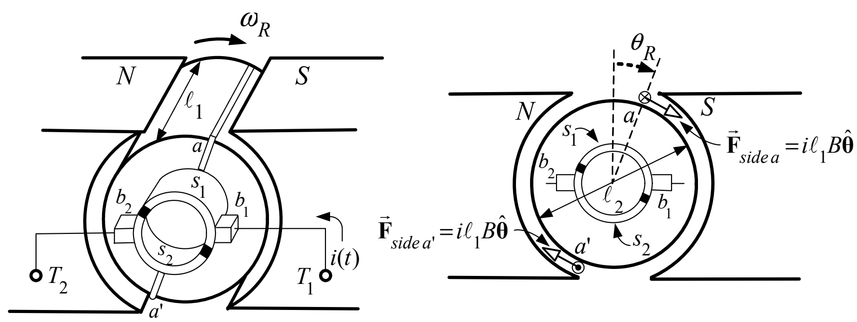


FIGURE 1.1. Physical structure of a single-loop DC motor.

The standard schematic diagram for the DC motor is shown on the right side of Figure 1.2.

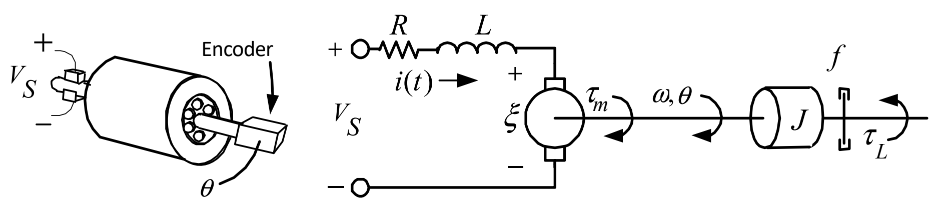


FIGURE 1.2. DC motor drawing and schematic.

Here R is the resistance of the rotor loop, L is the inductance of the rotor loop, i is the current in the rotor loop (armature current), ω is the rotor speed, θ is the rotor position, $\tau_m = K_T i$ is the torque produced by the motor, $\xi = K_b\omega$ is the induced voltage (back emf) in the rotor loop, J is the moment of inertia of the rotor, f is the viscous friction of the rotor (small and due to the ball bearings), and τ_l represents any load torque on the rotor. The differential equation model for the DC motor is then [1]

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega + V_S \\ J\frac{d\omega}{dt} &= K_T i - f\omega - \tau_L \\ \frac{d\theta}{dt} &= \omega. \end{aligned} \tag{1.4}$$

In matrix form this is

$$\frac{d}{dt}\begin{bmatrix} i \\ \omega \\ \theta \end{bmatrix} = \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 \\ K_T/J & -f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} i \\ \omega \\ \theta \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{b} V_S - \begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix}\tau_L. \tag{1.5}$$

For now let $\tau_l = 0$. Note that the pair (A, b) is not in control canonical form and

$$\det(sI - A) = s\left((s + R/L)(s + f/J) + K_T K_b\right) = s^3 + (R/L + f/J)s^2 + \left(K_T K_b + Rf/(JL)\right)s \quad (1.6)$$

showing the system is unstable as it has a pole at $s = 0$.

Next we look at transforming the system into control canonical form. Set $x_1 = i, x_2 = \omega, x_3 = \theta$, and $u = V_S$ and consider the statespace transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \underbrace{\frac{JL}{K_T}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ K_T/J & -f/J & 0 \end{bmatrix}}_{T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Tx. \quad (1.7)$$

To get into the "spirit" of how we will work with nonlinear transformations we take what may seem to be a rather (unnecessarily) involved approach. Start with rewriting this transformation as

$$\begin{aligned} x_1^* &= T_1(x) \triangleq \frac{JL}{K_T}x_3 \\ x_2^* &= T_2(x) \triangleq \frac{JL}{K_T}x_2 \\ x_3^* &= T_3(x) \triangleq \frac{JL}{K_T}\left(\frac{K_T}{J}x_1 - \frac{f}{J}x_2\right). \end{aligned} \quad (1.8)$$

Then

$$\begin{aligned} \frac{dx_1^*}{dt} = \frac{dT_1}{dt} &= \frac{\partial T_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial T_1}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial T_1}{\partial x_3}\frac{dx_3}{dt} \\ &= \underbrace{\begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \end{bmatrix}}_{dT_1} \underbrace{\begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{bmatrix}}_{Ax+bu}. \end{aligned}$$

Digression on Gradients and Dual Product

The gradient of the scalar function $T_1(x) : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$dT_1 \triangleq \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \end{bmatrix} \in \mathbb{R}^{1\times 3}.$$

We can also write $\dfrac{\partial T_1}{\partial x}$ to denote the gradient, i.e.,

$$\frac{\partial T_1}{\partial x} = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \end{bmatrix}.$$

The gradient will always be take to be a row vector.

Definition 1 *Dual Product*

With the gradient (row) vector dT_1 and the column vector $\dfrac{dx}{dt}$, their *dual product* $\left\langle dT_1, \dfrac{dx}{dt}\right\rangle$ is defined by

$$\left\langle dT_1, \frac{dx}{dt}\right\rangle \triangleq \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{bmatrix} = \frac{\partial T_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial T_1}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial T_1}{\partial x_3}\frac{dx_3}{dt}.$$

Remark 1 An *inner product* is defined similarly except both must be row vectors or column vectors.

Remark 2 We can also write $\left\langle dT_1, \dfrac{dx}{dt}\right\rangle = dT_1\dfrac{dx}{dt}$. The gradient (row) vector is a *covariant* vector and the (column) vector $\dfrac{dx}{dt}$ is a *contravariant* vector. This will be explained in the next chapter.

Let's return to the task of finding the system equations in the x^* coordinates. We have

$$\frac{dx_1^*}{dt} = \left\langle dT_1, \frac{dx}{dt}\right\rangle = \langle dT_1, Ax + bu\rangle = \langle dT_1, Ax\rangle + \langle dT_1, b\rangle\, u.$$

Then

$$\begin{aligned}
\langle dT_1, Ax\rangle &= \underbrace{\frac{JL}{K_T}\begin{bmatrix} 0 & 0 & 1\end{bmatrix}}_{dT_1}\underbrace{\begin{bmatrix} -(R/L)x_1 - (K_b/L)x_2 \\ (K_T/J)x_1 - (f/J)x_2 \\ x_2 \end{bmatrix}}_{Ax} = \frac{JL}{K_T}x_2 = x_2^* \\
\langle dT_1, b\rangle\, u &= \underbrace{\frac{JL}{K_T}\begin{bmatrix} 0 & 0 & 1\end{bmatrix}}_{dT_1}\underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{b} u = 0.
\end{aligned}$$

Next

$$\frac{dx_2^*}{dt} = \left\langle dT_2, \frac{dx}{dt}\right\rangle = \langle dT_2, Ax + bu\rangle = \langle dT_2, Ax\rangle + \langle dT_2, b\rangle\, u$$

and

$$\begin{aligned}
\langle dT_2, Ax\rangle &= \underbrace{\frac{JL}{K_T}\begin{bmatrix} 0 & 1 & 0\end{bmatrix}}_{dT2}\underbrace{\begin{bmatrix} -(R/L)x_1 - (K_b/L)x_2 \\ (K_T/J)x_1 - (f/J)x_2 \\ x_2 \end{bmatrix}}_{Ax} = \frac{JL}{K_T}\left(\frac{K_T}{J}x_1 - \frac{f}{J}x_2\right)x_2 = x_3^* \\
\langle dT_2, b\rangle\, u &= \underbrace{\frac{JL}{K_T}\begin{bmatrix} 0 & 1 & 0\end{bmatrix}}_{dT_2}\underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{b} u = 0.
\end{aligned}$$

Finally

$$\frac{dx_3^*}{dt} = \left\langle dT_3, \frac{dx}{dt}\right\rangle = \langle dT_3, Ax + bu\rangle = \langle dT_3, Ax\rangle + \langle dT_3, b\rangle\, u$$

and

$$\begin{aligned}
\langle dT_3, Ax\rangle &= \underbrace{\frac{JL}{K_T}\begin{bmatrix} \dfrac{K_T}{J} & -\dfrac{f}{J} & 0\end{bmatrix}}_{dT_3}\underbrace{\begin{bmatrix} -(R/L)x_1 - (K_b/L)x_2 \\ (K_T/J)x_1 - (f/J)x_2 \\ x_2 \end{bmatrix}}_{Ax} \\
&= \frac{JL}{K_T}\left(-\frac{fK_T}{J^2} - \frac{RK_T}{JL}\right)x_1 + \frac{JL}{K_T}\left(\frac{f^2}{J^2} - \frac{K_T K_b}{JL}\right)x_2 \\
\langle dT_3, b\rangle\, u &= \underbrace{\frac{JL}{K_T}\begin{bmatrix} \dfrac{K_T}{J} & -\dfrac{f}{J} & 0\end{bmatrix}}_{dT_3}\underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{b} u = u.
\end{aligned}$$

Summarizing we have shown

$$
\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^* \\
\frac{dx_2^*}{dt} &= x_3^* \\
\frac{dx_3^*}{dt} &= \underbrace{\frac{JL}{K_T}\left(-\frac{fK_T}{J^2} - \frac{RK_T}{JL}\right)x_1 + \frac{JL}{K_T}\left(\frac{f^2}{J^2} - \frac{K_T K_b}{JL}\right)x_2}_{\langle dT_3, Ax\rangle} + u.
\end{aligned}
\tag{1.9}
$$

However, $\langle dT_3, Ax\rangle = \frac{JL}{K_T}\left(-\frac{fK_T}{J^2} - \frac{RK_T}{JL}\right)x_1 + \frac{JL}{K_T}\left(\frac{f^2}{J^2} - \frac{K_T K_b}{JL}\right)x_2$ needs to be rewritten in terms of the x^* coordinates. The inverse of (1.8) is

$$
\begin{aligned}
x_1 &= \frac{f}{JL}x_2^* + \frac{1}{L}x_3^* \\
x_2 &= \frac{K_T}{JL}x_2^* \\
x_3 &= \frac{K_T}{JL}x_1^*.
\end{aligned}
$$

Substituting for x_1 and x_2 into the third equation of (1.9) gives

$$
\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^* \\
\frac{dx_2^*}{dt} &= x_3^* \\
\frac{dx_3^*}{dt} &= -\frac{Rf + K_T K_b}{JL}x_2^* - \frac{fL + RJ}{JL}x_3^* + u.
\end{aligned}
\tag{1.10}
$$

With $a_0 = 0, a_1 = \dfrac{Rf + K_T K_b}{JL}, a_2 = \dfrac{fL + RJ}{JL}$ this is written in matrix form as

$$
\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b_c} u \tag{1.11}
$$

which is in control canonical form.

Note from (1.8) that, except for the factor $\dfrac{JL}{K_T}$, the transformed coordinates x_1^*, x_2^*, and x_3^* are the position, speed, and acceleration of the motor, respectively. It is usually more convenient to specify the trajectory of a mechanical system by its desired position, speed, and acceleration rather than by position, speed, and current. Let

$$
x_d^* = \frac{JL}{K_T}\begin{bmatrix} \theta_{Rd} \\ \omega_{Rd} \\ \alpha_{Rd} \end{bmatrix}
$$

be the reference trajectory where $\omega_{Rd} = d\theta_{Rd}/dt, \alpha_{Rd} = d\omega_{Rd}/dt$. With

$$
r_d(t) \triangleq \frac{JL}{K_T}\frac{d\alpha_{Rd}}{dt} + a_0 x_{d1}^* + a_1 x_{d2}^* + a_2 x_{d3}^*
$$

we have

$$
\frac{d}{dt}\begin{bmatrix} x_{d1}^* \\ x_{d2}^* \\ x_{d3}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_{d1}^* \\ x_{d2}^* \\ x_{d3}^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r_d. \tag{1.12}
$$

Subtracting (1.12) from (1.11) gives the error system

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x_1^* - x_{d1}^* \\ x_2^* - x_{d2}^* \\ x_3^* - x_{d3}^* \end{bmatrix}}_{\epsilon^*} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c}\underbrace{\begin{bmatrix} x_1^* - x_{d1}^* \\ x_2^* - x_{d2}^* \\ x_3^* - x_{d3}^* \end{bmatrix}}_{\epsilon^*} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b_c}\underbrace{(u - r_d)}_{w} \tag{1.13}$$

or in more compact form

$$\epsilon^* = A_c^*\epsilon + b_c w.$$

With

$$k = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \triangleq \begin{bmatrix} \alpha_0 - a_0 & \alpha_1 - a_1 & \alpha_2 - a_2 \end{bmatrix}$$

the feedback

$$w = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\begin{bmatrix} x_1^* - x_{d1}^* \\ x_2^* - x_{d2}^* \\ x_3^* - x_{d3}^* \end{bmatrix}$$

results in

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x_1^* - x_{d1}^* \\ x_2^* - x_{d2}^* \\ x_3^* - x_{d3}^* \end{bmatrix}}_{\epsilon^*} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix}}_{A_{CL}}\underbrace{\begin{bmatrix} x_1^* - x_{d1}^* \\ x_2^* - x_{d2}^* \\ x_3^* - x_{d3}^* \end{bmatrix}}_{\epsilon^*} \tag{1.14}$$

With the α_i chosen so that A_{CL} is stable the error $\epsilon^*(t) \triangleq x^*(t) - x_d^*(t)$ is given by

$$\epsilon^*(t) = e^{A_{CL}t}\epsilon^*(0) \to 0_{3\times 1}.$$

The feedback results in the motor tracking the trajectory. The actual input to the motor is

$$\begin{aligned} u = w + r_d &= -k_1(x_1^* - x_{d1}^*) - k_2(x_2^* - x_{d2}^*) - k_3(x_3^* - x_{d3}^*) + r_d \\ &= -k_1(T_1(x) - x_{d1}^*) - k_2(T_2(x) - x_{d2}^*) - k_3(T_3(x) - x_{d3}^*) + r_d. \end{aligned}$$

So if the original coordinates are the signals that are measured, then they must be transformed to the x^* coordinate system to apply this feedback. In this particular example

$$T_3(x) = \frac{JL}{K_T}\left(\frac{K_T}{J}x_1 - \frac{f}{J}x_2\right) = \frac{JL}{K_T}\left(\frac{K_T}{J}i - \frac{f}{J}\omega\right) = \frac{JL}{K_T}\alpha.$$

If the current and speed are measured then the acceleration can be computed. However this requires the motor parameters being known accurately to get an accurate value of $\dfrac{JL}{K_T}\alpha$.

General Procedure To Transform a SISO LTI System to Control Canonical Form

For single-input single-output (SISO) linear time invariant (LTI) there is a simple procedure for transforming it to control canonical form if it is *controllable*. Recall that the LTI system

$$\frac{dx}{dt} = Ax + bu, \quad A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n$$

is controllable if and only if

$$\mathcal{C} \triangleq \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$$

is nonsingular. To continue with the procedure of transforming this LTI system to control canonical form, let $n = 3$ to simplify the presentation. Assume $\mathcal{C} \triangleq \begin{bmatrix} b & Ab & A^2b \end{bmatrix}$ is nonsingular so it has an inverse, i.e.,

$$\mathcal{C}^{-1}\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define $q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ be the last row of $\mathcal{C}^{-1}$ so that

$$q\mathcal{C} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{1.15}$$

That is, $qb = 0, qAb = 0, qA^2b = 1$. Then the change of coordinates to transform this system into control canonical form is given by

$$\begin{aligned} T_1(x) &\triangleq qx = q_1x_1 + q_2x_2 + q_3x_3, & dT_1 = q \in \mathbb{R}^{1\times 3} \\ T_2(x) &\triangleq \langle dT_1, Ax\rangle = qAx, & qA \in \mathbb{R}^{1\times 3} \\ T_3(x) &\triangleq \langle dT_2, Ax\rangle = qA^2x, & qA^2 \in \mathbb{R}^{1\times 3}. \end{aligned}$$

That is, with

$$\begin{aligned} x_1^* &= T_1(x) = qx \\ x_2^* &= T_2(x) = qAx \\ x_3^* &= T_3(x) = qA^2x. \end{aligned} \tag{1.16}$$

and using (1.15) we have

$$\begin{aligned} \frac{dx_1^*}{dt} &= \left\langle dT_1, \frac{dx}{dt} \right\rangle = \langle q, Ax + bu\rangle = qAx + qbu = qAx = T_2(x) \\ \frac{dx_2^*}{dt} &= \left\langle dT_2, \frac{dx}{dt} \right\rangle = \langle qA, Ax + bu\rangle = qA^2x + qAbu = qA^2x = T_3(x) \\ \frac{dx_3^*}{dt} &= \left\langle dT_3, \frac{dx}{dt} \right\rangle = \left\langle qA^2, Ax + bu\right\rangle = qA^3x + qA^2bu = qA^3x + u \end{aligned}$$

or

$$\begin{aligned} \frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= x_3^* \\ \frac{dx_3^*}{dt} &= qA^3x + u. \end{aligned} \tag{1.17}$$

By (1.16) we have

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \underbrace{\begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix}}_{T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

so $x = T^{-1}x^*$ and (1.17) becomes

$$\begin{aligned} \frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= x_3^* \\ \frac{dx_3^*}{dt} &= (qA^3T^{-1})x^* + u = -a_0x_1^* - a_1x_2^* - a_2x_3^* + u \end{aligned}$$

where $\begin{bmatrix} -a_0 & -a_1 & -a_2 \end{bmatrix} \triangleq qA^3T^{-1}$.

Exercise 1 Given $A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3$ with the pair (A, b) controllable and q the last row of $\mathcal{C}^{-1} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix}^{-1}$, show that

$$T = \begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix}$$

is nonsingular.

Exercise 2 With $A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3$ and $\det(sI - A) = s^3 + a_2 s^2 + a_1 s + a_0$ show, by direct computation, that

$$\frac{dx^*}{dt} = TAT^{-1}x^* + Tbu = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c} x^* + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b_c} u.$$

Hint: Show $TA = A_cT$ by using the Cayley-Hamilton theorem.

1.2 Lie Derivatives

We now introduce the Lie derivative which will be used extensively in our dealings with nonlinear systems. For simplicity of exposition we stay with $n = 3$. Let

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} \in \mathbb{R}^3$$

be a (in general nonlinear) function from $\mathbb{R}^3 \to \mathbb{R}^3$ and $h(x) : \mathbb{R}^3 \to \mathbb{R}$ be a scalar function. Then the Lie derivative $\mathcal{L}_f(h)$ of h with respect to f is defined as

$$\mathcal{L}_f(h) \triangleq \langle dh, f\rangle = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}.$$

More precisely, let $\mathcal{C}^\infty$ denote all infinitely differentiable functions on some open subset $U \subset \mathbb{R}^3$. By a function h being infinitely differentiable on U is meant that partial derivatives of h of all orders exist on U.[1] Let $h(x) : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function of x defined on U and the components of $f(x) : \mathbb{R}^3 \to \mathbb{R}^3$ also $\mathcal{C}^\infty$ functions of x defined on U. Then the operator $\mathcal{L}_f(h) : \mathcal{C}^\infty \to \mathcal{C}^\infty$ is defined by

$$\mathcal{L}_f(h) \triangleq \langle dh, f\rangle$$

is the *Lie derivative* of h with respect f (Lie is pronounced as "Lee").

Repeated Lie derivatives are defined recursively as follows.

$$\mathcal{L}_f^2(h) \triangleq \mathcal{L}_f(\mathcal{L}_f(h)), \mathcal{L}_f^3(h) \triangleq \mathcal{L}_f(\mathcal{L}_f^2(h)), \text{ etc.}$$

Now consider the LTI system

$$\frac{dx}{dt} = Ax + bu,\ x \in \mathbb{R}^3,\ u \in \mathbb{R},\ A \in \mathbb{R}^{3\times 3},\ b \in \mathbb{R}^3.$$

Define

$$f(x) \triangleq Ax \in \mathbb{R}^3$$

[1] So $\frac{\partial h}{\partial x_1}, \frac{\partial^2 h}{\partial x_1 \partial x_3}, \frac{\partial^3 h}{\partial x_2^3}$, etc. all exist.

so we may write this system as

$$\frac{dx}{dt} = f(x) + bu.$$

With $h(x)$ a differentiable function we define the derivative of h along the trajectory $x(t)$ to be $\frac{dh}{dt}$ where $x(t)$ is the solution to the above system. In the notation of Lie derivatives we write

$$\begin{aligned}
\frac{dh(x(t))}{dt} = \mathcal{L}_{f+bu}(h) = \langle dh, f+bu \rangle &= \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1(x) + b_1 u \\ f_2(x) + b_2 u \\ f_3(x) + b_3 u \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} + \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} b_1 u \\ b_2 u \\ b_3 u \end{bmatrix} \\
&= \mathcal{L}_f(h) + u\mathcal{L}_b(h)
\end{aligned}$$

For example, with

$$h(x) = qx = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad f(x) = Ax$$

we have

$$\frac{dh}{dt} = \frac{dh(x(t))}{dt} = \mathcal{L}_f(h) + u\mathcal{L}_b(h) = qAx + uqb.$$

More generally, with $h(x) = qx$ and $\frac{dx}{dt} = f(x) + bu = Ax + bu$ we have

$$\begin{aligned}
\mathcal{L}_{f+bu}(h) &= \mathcal{L}_f(h) + u\mathcal{L}_b(h) = qAx + uqb \in \mathbb{R} \\
\mathcal{L}^2_{f+bu}(h) &= \mathcal{L}_{f+bu}(\mathcal{L}_{f+bu}) = \mathcal{L}_{f+bu}(qAx + uqb) = \langle qA, Ax + bu \rangle = qA^2x + qAbu \in \mathbb{R} \\
\mathcal{L}^2_{f+bu}(h) &= \mathcal{L}_{f+bu}(\mathcal{L}^2_{f+bu}) = \mathcal{L}_{f+bu}(qA^2x + qAbu) = \left\langle qA^2, Ax + bu \right\rangle = qA^3x + qA^2bu \in \mathbb{R}
\end{aligned}$$

Remark Though $\frac{dh}{dt} = \mathcal{L}_{f+bu}(h)$, $\frac{d^2h}{dt^2} \neq \mathcal{L}^2_{f+bu}(h)$ (why?).

1.3 Linearization about an Equilibrium Point

We now look at the standard approach of dealing with nonlinear control systems which is to find a linear approximate model for it. To explain, consider the nonlinear control system

$$\frac{dx_1}{dt} = f_1(x_1, x_2) + g_1(x_1, x_2)u \tag{1.18}$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2) + g_2(x_1, x_2)u \tag{1.19}$$

written compactly as

$$\frac{dx}{dt} = f(x) + g(x)u.$$

Let $x_0 = (x_{01}, x_{02})$ be a constant equilibrium point with reference input u_0 such that x_0, u_0 satisfy

$$\begin{aligned}
\frac{dx_{01}}{dt} &= 0 = f_1(x_{01}, x_{02}) + g_1(x_{01}, x_{02})u_0 \\
\frac{dx_{02}}{dt} &= 0 = f_2(x_{01}, x_{02}) + g_2(x_{01}, x_{02})u_0
\end{aligned}$$

or

$$0_{2\times 1} = f(x_0) + g(x_0)u_0.$$

Next do a Taylor series expansion of $f_1(x_1,x_2), f_2(x_1,x_2), g_1(x_1,x_2)$, and $g_2(x_1,x_2)$ about $x_0 = (x_{01}, x_{02})$. Specifically, we have

$$\begin{aligned} f_1(x_1,x_2) &= f_1(x_{01},x_{02}) + \frac{\partial f_1(x_{01},x_{02})}{\partial x_1}(x_1 - x_{01}) + \frac{\partial f_1(x_{01},x_{02})}{\partial x_2}(x_2 - x_{02}) \\ &\quad + \frac{1}{2!}\left(\frac{\partial^2 f_1(x_{01},x_{02})}{\partial x_1^2}(x_1 - x_{01})^2 + \frac{\partial^2 f_1(x_{01},x_{02})}{\partial x_1 \partial x_2}(x_1 - x_{01})(x_2 - x_{02}) + \frac{\partial^2 f_1(x_{01},x_{02})}{\partial x_2^2}(x_2 - x_{02})^2\right) \\ &\quad + \cdots \end{aligned}$$

$$\begin{aligned} g_1(x_1,x_2) &= g_1(x_{01},x_{02}) + \frac{\partial g_1(x_{01},x_{02})}{\partial x_1}(x_1 - x_{01}) + \frac{\partial g_1(x_{01},x_{02})}{\partial x_2}(x_2 - x_{02}) \\ &\quad + \frac{1}{2!}\left(\frac{\partial^2 g_1(x_{01},x_{02})}{\partial x_1^2}(x_1 - x_{01})^2 + \frac{\partial^2 g_1(x_{01},x_{02})}{\partial x_1 \partial x_2}(x_1 - x_{01})(x_2 - x_{02}) + \frac{\partial^2 g_1(x_{01},x_{02})}{\partial x_2^2}(x_2 - x_{02})^2\right) \\ &\quad + \cdots \end{aligned}$$

and similarly for $f_2(x_1,x_2)$ and $g_2(x_1,x_2)$. With $u = w + u_0$ we may rewrite (1.18) and (1.19) as

$$\begin{aligned} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} &= \begin{bmatrix} f_1(x_{01},x_{02}) \\ f_2(x_{01},x_{02}) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1(x_{01},x_{02})}{\partial x_1} & \frac{\partial f_1(x_{01},x_{02})}{\partial x_2} \\ \frac{\partial f_2(x_{01},x_{02})}{\partial x_1} & \frac{\partial f_2(x_{01},x_{02})}{\partial x_2} \end{bmatrix} \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix} + \cdots \\ &\quad + \begin{bmatrix} g_1(x_{01},x_{02}) \\ g_2(x_{01},x_{02}) \end{bmatrix}(w + u_0) + \begin{bmatrix} \frac{\partial g_1(x_{01},x_{02})}{\partial x_1}(x_1 - x_{01}) + \frac{\partial g_1(x_{01},x_{02})}{\partial x_2}(x_2 - x_{02}) \\ \frac{\partial g_2(x_{01},x_{02})}{\partial x_1}(x_1 - x_{01}) + \frac{\partial g_2(x_{01},x_{02})}{\partial x_2}(x_2 - x_{02}) \end{bmatrix}(w + u_0) + \cdots \end{aligned}$$

or

$$\begin{aligned} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} &= \underbrace{\begin{bmatrix} f_1(x_{01},x_{02}) \\ f_2(x_{01},x_{02}) \end{bmatrix} + \begin{bmatrix} g_1(x_{01},x_{02}) \\ g_2(x_{01},x_{02}) \end{bmatrix} u_0}_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]} \\ &\quad + \begin{bmatrix} \frac{\partial f_1(x_{01},x_{02})}{\partial x_1} + \frac{\partial g_1(x_{01},x_{02})}{\partial x_1}u_0 & \frac{\partial f_1(x_{01},x_{02})}{\partial x_2} + \frac{\partial g_1(x_{01},x_{02})}{\partial x_2}u_0 \\ \frac{\partial f_2(x_{01},x_{02})}{\partial x_1} + \frac{\partial g_2(x_{01},x_{02})}{\partial x_1}u_0 & \frac{\partial f_2(x_{01},x_{02})}{\partial x_2} + \frac{\partial g_2(x_{01},x_{02})}{\partial x_2}u_0 \end{bmatrix} \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix} \\ &\quad + \begin{bmatrix} g_1(x_{01},x_{02}) \\ g_2(x_{01},x_{02}) \end{bmatrix} w + \cdots \end{aligned}$$

The terms left out all have factors of the form $w(x_1 - x_{01}), w(x_2 - x_{02}), (x_1 - x_{01})^2, (x_1 - x_{01})(x_2 - x_{02}), (x_2 - x_{02})^2$ or higher and are referred to as higher order terms. The idea is that the state $[x_1, x_2]^T$ starts off close to the equilibrium point $[x_{01}, x_{02}]^T$ and that a feedback controller is designed to keep the state close to the equilibrium point for all time with $w = u - u_0$ also small. In this case these higher order terms are small and we take them to be zero to end up with the linear system

$$\frac{d}{dt}\begin{bmatrix}(x_1-x_{01})\\ \\(x_2-x_{02})\end{bmatrix} = \underbrace{\begin{bmatrix}\dfrac{\partial f_1(x_{01},x_{02})}{\partial x_1}+\dfrac{\partial g_1(x_{01},x_{02})}{\partial x_1}u_0 & \dfrac{\partial f_1(x_{01},x_{02})}{\partial x_2}+\dfrac{\partial g_1(x_{01},x_{02})}{\partial x_2}u_0\\ \dfrac{\partial f_2(x_{01},x_{02})}{\partial x_1}+\dfrac{\partial g_2(x_{01},x_{02})}{\partial x_1}u_0 & \dfrac{\partial f_2(x_{01},x_{02})}{\partial x_2}+\dfrac{\partial g_2(x_{01},x_{02})}{\partial x_2}u_0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1-x_{01}\\ \\x_2-x_{02}\end{bmatrix}}_{z}$$
$$+\underbrace{\begin{bmatrix}g_1(x_{01},x_{02})\\ \\g_2(x_{01},x_{02})\end{bmatrix}}_{b}w. \tag{1.20}$$

This system can then be controlled by linear methods. For example, with $z = x - x_0$ we rewrite (1.20) as

$$\frac{dz}{dt} = Az + bw.$$

If the pair (A, b) is controllable then, as shown previously, we can find $k = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ such that the feedback $w = -kz$ results in the closed-loop system

$$\frac{dz}{dt} = Az - bkz = (A - bk)z$$

being stable. More generally, consider a nonlinear system described by

$$\frac{dx}{dt} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m.$$

Let x_0 be an equilibrium point with corresponding input reference u_0 so that

$$\frac{dx_0}{dt} = 0_{n\times 1} = f(x_0, u_0).$$

With $z \triangleq x - x_0 \in \mathbb{R}^n, w \triangleq u - u_0 \in \mathbb{R}^m$ and $||z|| = \sqrt{z_1^2 + \cdots + z_n^2}, ||w|| = \sqrt{w_1^2 + \cdots + w_m^2}$ small we have

$$\begin{aligned}\frac{d}{dt}(x - x_0) &= f(x, u) - f(x_0, u_0)\\ &\approx \frac{\partial f(x_0, u_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, u_0)}{\partial u}(u - u_0)\end{aligned}$$

or

$$\frac{dz}{dt} = Az + Bw$$

where $A = \dfrac{\partial f(x_0, u_0)}{\partial x} \in \mathbb{R}^{n\times n}, B = \dfrac{\partial f(x_0, u_0)}{\partial u} \in \mathbb{R}^{n\times m}$. This is a LTI system which is (hopefully) valid for x close to x_0 and u close to u_0.

Linear Statespace Model of a Magnetically Levitated Steel Ball

Figure 1.3 shows a representation of using an electromagnet to provide a magnetic force F_{mag} to keep a steel ball levitated against the force of gravity. A current command amplifier is used, and, with the gains K_p, K_I chosen appropriately, the current can be considered as the input, that is, $i_r = i$. That is, a PI controller forces $i(t) \to i_r(t)$ so fast (compared to the motion of the steel ball) that the current $i(t)$ can be considered as the input.

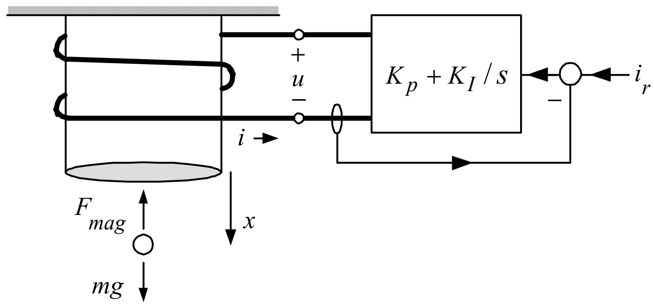


FIGURE 1.3. Current command amplifier for the magnetic levitation system.

With $u = i$ the input and $x_1 = x, x_2 = dx/dt$, the model is given by [2]

$$\frac{dx_1}{dt} = x_2 \tag{1.21}$$

$$\frac{dx_2}{dt} = g - \frac{C}{m}\frac{u^2}{x_1^2}. \tag{1.22}$$

Here C is an empirically determined force constant, m is the mass of the steel ball, and g is the acceleration due to gravity. An equilibrium point for the system is a *constant* solution to the system equations (1.21) and (1.22). We choose an equilibrium point so that the steel ball is at a distance x_{eq} below the electromagnet. Specifically, choose the equilibrium point to be $(x_{01}, x_{02}) = (x_{eq}, 0)$ for some u_0. At the equilibrium point (1.21) and (1.22) are simply

$$\frac{dx_{01}}{dt} = 0 = x_{02}$$

$$\frac{dx_{02}}{dt} = 0 = g - \frac{C}{m}\frac{u_0^2}{x_{01}^2}.$$

For the right side of the second equation to hold it must be that $(x_{01} = x_{eq})$

$$g = \frac{C}{m}\frac{u_0^2}{x_{eq}^2} \quad \text{or} \quad u_0 = x_{eq}\sqrt{mg/C}. \tag{1.23}$$

The model (1.21) and (1.22) is nonlinear due to the $\dfrac{u^2}{x_1^2}$ term. To find a linear approximate model a Taylor series expansion of

$$f(x_1, u) = g - \frac{C}{m}\frac{u^2}{x_1^2}$$

is done about the equilibrium point $(x_{01}, x_{02}) = (x_{eq}, 0)$ with reference input $u_0 = x_0\sqrt{g/C}$. First note that

$$f(x_0, u_0) = g - \frac{C}{m}\frac{u_0^2}{x_{eq}^2} = 0$$

as u_0 was chosen to make this true. The Taylor series expansion of $f(x, u)$ about (x_{eq}, u_0) is then (dropping higher-order terms)

$$
\begin{aligned}
f(x_1,u) &\approx f(x_{eq},u_0)+\frac{\partial f(x_{eq},u_0)}{\partial x}(x_1-x_{eq})+\frac{\partial f(x_{eq},u_0)}{\partial u}(u-u_0)\\
&= \underbrace{g-\frac{C}{m}\frac{u_0^2}{x_{eq}^2}}_{0}+2\frac{C}{m}\frac{u_0^2}{x^3}(x_1-x_{eq})-2\frac{C}{m}\frac{u_0}{x^2}(u-u_0)\\
&= 2\frac{C}{m}\frac{u_0^2}{x^3}(x_1-x_{eq})-2\frac{C}{m}\frac{u_0}{x^2}(u-u_0)\\
&= \frac{2g}{x_0}(x_1-x_{eq})-\frac{2g}{u_0}(u-u_0)
\end{aligned}
$$

where in the third line we used (1.23) to obtain

$$
\frac{2g}{x_{eq}}=2\frac{C}{m}\frac{u_0^2}{x_{eq}^3},\quad \frac{2g}{u_0}=2\frac{C}{m}\frac{u_0}{x_{eq}^2}.
$$

The linear statespace model is then

$$
\begin{aligned}
\frac{d}{dt}(x_1-x_{eq}) &= x_2-0\\
\frac{d}{dt}(x_2-0) &= \frac{2g}{x_0}(x_1-x_{eq})-\frac{2g}{u_0}(u-u_0).
\end{aligned}
$$

With $z_1=x_1-x_{eq}, z_2=x_2-0, w=u-u_0$ this becomes

$$
\frac{d}{dt}\begin{bmatrix} z_1\\ z_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 2g/x_{eq} & 0\end{bmatrix}\begin{bmatrix} z_1\\ z_2\end{bmatrix}+\begin{bmatrix}0\\ -2g/u_0\end{bmatrix}w
$$

1.4 Feedback Linearization of Nonlinear Control Systems

The fundamental reason we want to approximate a nonlinear system by a linear system is that we know how to stabilize a linear system.[2] However, for the linear system to be a "good" approximation to the nonlinear system, the state must start off "close" to the equilibrium point and the controller must keep the state "close" to the equilibrium point. We use the quotes "good" and "close" indicate that these are vague terms. In reality one only knows if the controller will work by implementing it. The concept of feedback linearization is to use a nonlinear change of coordinates so that in the new coordinate system the nonlinear system can be made linear by using feedback to cancel out the nonlinearities. We use this section to present several examples of this approach. The remaining chapters develop the theory of how one finds these nonlinear transformations.

Example 1 *Feedback Linearization*

Consider the nonlinear control system given by

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2\\
\frac{dx_2}{dt} &= x_3\\
\frac{dx_3}{dt} &= f(x_1,x_2,x_3)+u.
\end{aligned}
$$

[2] Assuming it is a *controllable* linear system.

With the feedback

$$u = -f(x_1, x_2, x_3) + v$$

the nonlinear control system becomes the *linear* control system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v.$$

Setting

$$v = -\underbrace{\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{bmatrix}}_{k}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

gives the closed-loop system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

with closed-loop characteristic polynomial

$$\det\left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix}\right) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0.$$

Note that the feedback to the system is

$$u = -f(x_1, x_2, x_3) - kx.$$

Feedback linearization is a generalization of the previous example. To explain, consider the nonlinear second-order control system given by

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2) + g_1(x_1, x_2)u \\ \frac{dx_2}{dt} &= f_2(x_1, x_2) + g_2(x_1, x_2)u. \end{aligned}$$

Suppose we can find a nonlinear change of coordinates

$$\begin{aligned} x_1^* &= T_1(x_1, x_2) \\ x_2^* &= T_2(x_1, x_2) \end{aligned}$$

such that in the x^* coordinates

$$\begin{aligned} \frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= f^*(x_1, x_2) + g^*(x_1, x_2)u. \end{aligned}$$

With the feedback

$$u = \frac{-f^*(x_1, x_2) + v}{g^*(x_1, x_2)}$$

this nonlinear system becomes the linear system (double integrator)

$$\begin{aligned} \frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= v. \end{aligned}$$

By means of a nonlinear transformation and state feedback the system is now linear from the input v to the state (x_1^*, x_2^*).

Example 2 *Direct Current to Direct Current (DC-DC) Converter* [3]
A DC-to-DC converter is an switching electronic circuit which converts a source of direct current (DC) from one voltage level to another. A circuit model of a DC-DC converter is shown in Figure 1.4. The variable $d(t)$ indicates if the switch is attached to ground ($d = 1$) or attached to the output ($d = 0$).

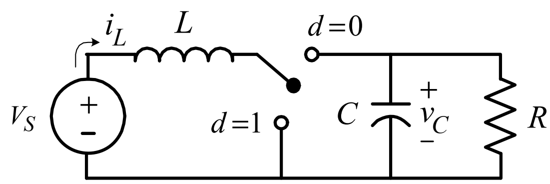


FIGURE 1.4. DC-DC converter.

Given the value of $d \in \{0, 1\}$ the equations of the circuit are

$$\frac{di_L}{dt} = -(1-d)\frac{v_C}{L} + \frac{V_S}{L} \tag{1.24}$$
$$\frac{dv_C}{dt} = (1-d)\frac{i_L}{C} - \frac{v_C}{RC}. \tag{1.25}$$

Let T_s denote the switching period so that $f_s \triangleq 1/T_s$ is the switching rate which is typically $10 - 20$ kHz. The duty ratio D for each period is the fraction of time that $d = 1$. In more detail, for the n^{th} time period starting at nT_s, $d(t)$ is given as

$$d(t) = \begin{cases} 1, & nT_s \le t \le nT_s + DT_s \\ 0, & nT_s + DT_s \le t \le nT_s + T_s. \end{cases}$$

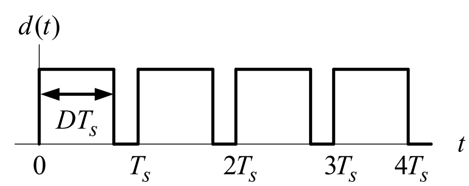


FIGURE 1.5. Illustration of the duty ratio.

The duty ratio is defined as

$$D = < d > \triangleq \frac{1}{T_s}\int_{t-T_s}^{t} d(\tau)d\tau.$$

Define

$$\begin{aligned} V_0 &\triangleq < V_S > = \frac{1}{T_s}\int_{t-T_s}^{t} V_S(\tau)d\tau \\ x_1(t) &\triangleq < i_L > = \frac{1}{T_s}\int_{t-T_s}^{t} i_L(\tau)d\tau \\ x_2(t) &\triangleq < v_C > = \frac{1}{T_s}\int_{t-T_s}^{t} v_C(\tau)d\tau. \end{aligned}$$

Typically V_S is constant so $V_0 \triangleq < V_S > = V_S$. Over a single switching period the duty ratio is constant. Averaging the set of equations (1.24) and (1.24) over one period gives

$$\begin{aligned}
\frac{d}{dt} < i_L > &= -(1-D)\frac{< v_C >}{L} + \frac{< V_S >}{L} \\
\frac{d}{dt} < v_C > &= (1-D)\frac{< i_L >}{C} - \frac{< v_C >}{RC}
\end{aligned}$$

where approximations

$$\begin{aligned}
\langle (1-d) v_C \rangle &\approx \langle 1-d \rangle \langle v_C \rangle = (1-D) \langle v_C \rangle \\
\langle (1-d) i_L \rangle &\approx \langle 1-d \rangle \langle i_L \rangle = (1-D) \langle i_L \rangle .
\end{aligned}$$

were used. Summarizing, a mathematical model of a DC-DC converter is given by

$$\begin{aligned}
\frac{dx_1}{dt} &= -(1-u)\frac{x_2}{L} + \frac{V_0}{L} \\
\frac{dx_2}{dt} &= (1-u)\frac{x_1}{C} - \frac{x_2}{RC}
\end{aligned}$$

where $u = D$ is the input (duty ratio) with $0 \le u \le 1$, x_1 is the (average) current in the inductor, and x_2 is the (average) voltage across the capacitor. The nonlinear terms are ux_2 and ux_1 in the two equations. This model of the DC-DC is given in the standard nonlinear form by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -x_2/L + V_0/L \\ x_1/C - x_2/(RC) \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} x_2/L \\ -x_1/C \end{bmatrix}}_{g(x)} u.$$

This way of setting up the nonlinear statespace equations going from the discrete input d to the continuous input $u = D$ is called *statespace averaging* [4]. Consider the nonlinear transformation

$$\begin{aligned}
x_1^* &= T_1(x) \triangleq \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2 \\
x_2^* &= T_2(x) \triangleq x_1 V_0 - x_2^2/R.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d}{dt}x_1^* &\triangleq Lx_1\frac{d}{dt}x_1 + Cx_2\frac{d}{dt}x_2 \\
&= Lx_1\left(-(1-u)\frac{x_2}{L} + \frac{V_0}{L}\right) + Cx_2\Big((1-u)\frac{x_1}{C} - \frac{x_2}{RC}\Big) \\
&= x_1 V_0 - x_2^2/R \\
&= x_2^*
\end{aligned}$$

and

$$\begin{aligned}
\frac{dx_2^*}{dt} &= V_0\frac{dx_1}{dt} - \frac{2}{R}x_2\frac{dx_2}{dt} \\
&= V_0\left(-(1-u)\frac{x_2}{L} + \frac{V_0}{L}\right) - \frac{2}{R}x_2\left((1-u)\frac{x_1}{C} - \frac{x_2}{RC}\right) \\
&= \underbrace{V_0\left(-\frac{x_2}{L} + \frac{V_0}{L}\right) - \frac{2}{R}x_2\left(-\frac{x_1}{C} + \frac{x_2}{RC}\right)}_{f_2^*(x_1,x_2)} + u\underbrace{\left(\frac{V_0}{L}x_2 + \frac{2}{RC}x_1x_2\right)}_{g_2^*(x_1,x_2)}.
\end{aligned}$$

We now have

$$\begin{aligned}\frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= f^*(x_1,x_2) + ug^*(x_1,x_2).\end{aligned}$$

If we set the input as

$$u = \frac{-f^*(x_1,x_2)+v}{g^*(x_1,x_2)}$$

we obtain

$$\begin{aligned}\frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= v\end{aligned}$$

which is a *linear* system.

Example 3 *Series Connected DC Motor* [5][6]
A series connected DC motor is one in which the field circuit is connected in series with the armature (rotor) loop. This is indicated in Figure 1.6 which shows the terminal T_2' of the field winding connected to the terminal T_1 of the armature loop. R_f is the resistance of the field windings and R_a is the resistance of the rotor loops. The total flux (flux linkage) in the field windings due to the current i_f in the field windings is $L_f i_f$ with L_f the self inductance of the field windings. The total flux in the rotor loops is $L_a i_a$ due to the current i_a in the rotor loops with L_a the self inductance of the rotor loop. It turns out (see [1]) that the torque produced by the motor is

$$\tau_m = K_T L_f i_f i_a$$

and the voltage induced in rotor loops as they rotate in the magnetic field of the air gap produced by the field current is

$$\xi = K_b L_f i_f \omega.$$

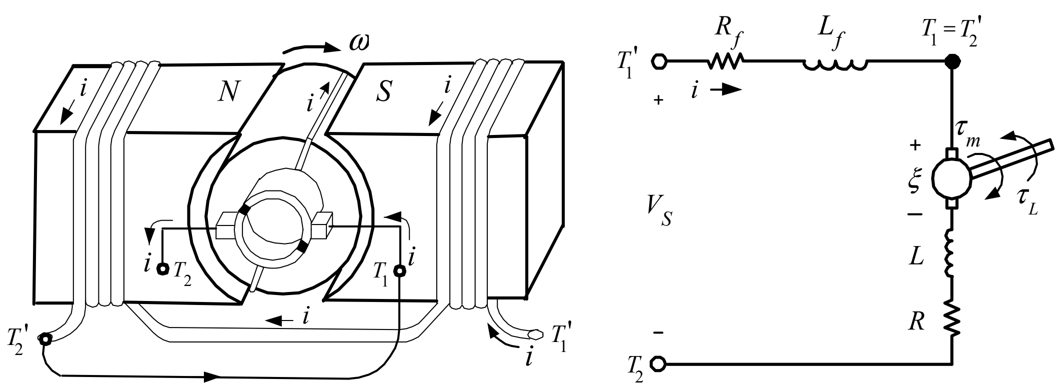


FIGURE 1.6. Series connected DC motor.

As this is a series connected DC motor we have $i = i_f = i_a$. Conservation of energy requires $\tau_m\omega = \xi i$ which implies $K_T = K_b$. Using the equivalent circuit on the right side of Figure 1.6 it is seen that the equations describing this motor are

$$\begin{aligned}
V_S &= (R_f + R_a)i + (L_f + L_a)\frac{di}{dt} + \xi = Ri + L\frac{di}{dt} + K_b L_f i\omega \\
J\frac{d\omega}{dt} &= \tau_m - \tau_L = K_T L_f i^2 - \tau_L \\
\frac{d\theta}{dt} &= \omega.
\end{aligned}$$

Here $R \triangleq R_f + R_a, L \triangleq L_f + L_a$, and J is the moment of inertia of the motor shaft. Rearranging these equations into statespace form we have

$$\begin{aligned}
\frac{di}{dt} &= -\frac{R}{L}i - \frac{K_b L_f}{L}i\omega + \frac{V_S}{L} \\
\frac{d\omega}{dt} &= \frac{K_T L_f}{J}i^2 - \frac{\tau_L}{J} \\
\frac{d\theta}{dt} &= \omega
\end{aligned}$$

Set $x_1 = \theta, x_2 = \omega, x_3 = i$, and $u = V_S/L$ to have

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= \frac{K_T L_f}{J}x_3^2 - \frac{\tau_L}{J} \\
\frac{dx_3}{dt} &= -\frac{R}{L}x_3 - \frac{K_b L_f}{L}x_3 x_2 + u
\end{aligned}$$

or

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}}_{p} \tau_L$$

with $c_1 \triangleq K_T L_f/J, c_2 \triangleq R/L$, and $c_3 \triangleq K_b L_f/L$.

In this case of the series connected DC motor the nonlinearity in the dx_3/dt equation can be canceled out by feedback, but we would still have a nonlinearity in the dx_2/dt equation. To get around this consider the nonlinear statespace transformation

$$\begin{aligned}
x_1^* &= T_1(x) = x_1 \\
x_2^* &= T_2(x) = \mathcal{L}_f(T_1) = \langle dT_1, f\rangle = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix} = x_2 \\
x_3^* &= T_3(x) = \mathcal{L}_f(T_2) = \langle dT_2, f\rangle = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix} = c_1 x_3^2.
\end{aligned}$$

We now show that this nonlinear statespace transformation, i.e.,

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_2(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ c_1 x_3^2 \end{bmatrix}$$

will transform the equations of the series connected DC motor into a form where the nonlinearities can be canceled out by feedback. We proceed as follows.

$$\begin{aligned}
\frac{dx_1^*}{dt} = \frac{d}{dt}T_1(x) &= \langle dT_1, f(x) + g(x)u + p\tau_L\rangle \\
&= \mathcal{L}_f(T_1) + u\mathcal{L}_g(T_1) + \tau_L\mathcal{L}_p(T_1) \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ c_1x_3^2 \\ -c_2x_3 - c_3x_3x_2 \end{bmatrix} + u\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \tau_L\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix} \\
&= x_2
\end{aligned}$$

$$\begin{aligned}
\frac{dx_2^*}{dt} = \frac{d}{dt}T_2(x) &= \langle dT_2, f(x) + g(x)u + p\tau_L\rangle \\
&= \mathcal{L}_f(T_2) + u\mathcal{L}_g(T_2) + \tau_L\mathcal{L}_p(T_2) \\
&= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ c_1x_3^2 \\ -c_2x_3 - c_3x_3x_2 \end{bmatrix} + u\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \tau_L\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix} \\
&= c_1x_3^2 - \tau_L/J
\end{aligned}$$

$$\begin{aligned}
\frac{dx_3^*}{dt} = \frac{d}{dt}T_3(x) &= \langle dT_3, f(x) + g(x)u + p\tau_L\rangle \\
&= \mathcal{L}_f(T_3) + u\mathcal{L}_g(T_3) + \tau_L\mathcal{L}_p(T_3) \\
&= \begin{bmatrix} 0 & 0 & 2c_1x_3 \end{bmatrix}\begin{bmatrix} x_2 \\ c_1x_3^2 \\ -c_2x_3 - c_3x_3x_2 \end{bmatrix} + u\begin{bmatrix} 0 & 0 & 2c_1x_3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \tau_L\begin{bmatrix} 0 & 0 & 2c_1x_3 \end{bmatrix}\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix} \\
&= -2c_1c_2x_3^2 - 2c_1c_3x_2x_3^2 + 2c_1x_3u
\end{aligned}$$

Summarizing we have

$$\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^* \\
\frac{dx_2^*}{dt} &= x_3^* - \tau_L/J \\
\frac{dx_3^*}{dt} &= \underbrace{-2c_1c_2x_3^2 - 2c_1c_3x_2x_3^2}_{a(x)} + \underbrace{2c_1x_3}_{b(x)}u.
\end{aligned}$$

For $x_3 = i \neq 0$ we can apply the state feedback

$$u = \frac{-a(x) + w}{b(x)}$$

to obtain the linear statespace system

$$\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^* \\
\frac{dx_2^*}{dt} &= x_3^* - \tau_L/J \\
\frac{dx_3^*}{dt} &= w.
\end{aligned}$$

As $\tau_m = K_TL_fi^2$ a series connected DC motor can only produce torque in one direction so it is used for applications (e.g., locomotives) where speed needs to be controlled, but it is not required to reverse it. By decreasing the current the load torque decreases the speed.

With a constant load torque τ_{L0} consider speed control with ω_0 the desired constant speed. Define $(x_2^* = \omega)$

$$\begin{aligned}
z_0 &\triangleq \int_0^t (x_2^*(t') - \omega_0)dt' = \int_0^t (\omega(t') - \omega_0)dt' \\
z_1 &\triangleq x_2^* - \omega_0 = \omega - \omega_0 \\
z_2 &\triangleq x_3^*
\end{aligned}$$

with corresponding linear model

$$\begin{aligned}
\frac{dz_0}{dt} &= z_1 \\
\frac{dz_1}{dt} &= z_2 - \tau_{L0}/J \\
\frac{dz_2}{dt} &= w.
\end{aligned}$$

In matrix form this is

$$\frac{d}{dt}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}\tau_{L0}.$$

The feedback

$$w = -k_0 z_0 - k_1 z_1 - k_2 z_2$$

results in

$$\frac{d}{dt}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_0 & -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}\tau_{L0}.$$

Taking Laplace transform we have

$$\begin{aligned}
\begin{bmatrix} Z_0(s) \\ Z_1(s) \\ Z_2(s) \end{bmatrix} &= \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k_0 & k_1 & s + k_2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}\frac{\tau_{L0}}{s} \\
&= \frac{1}{s^3 + k_2 s^2 + k_1 s + k_0}\begin{bmatrix} s^2 + k_2 s + k_1 & s + k_2 & 1 \\ -k_0 & s^2 + k_2 s & s \\ -s k_0 & -k_0 - s k_1 & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}\frac{\tau_{L0}}{s}.
\end{aligned}$$

By the final value theorem $(z_1(t) = \omega(t) - \omega_0)$ it follows that

$$\begin{bmatrix} z_0(\infty) \\ z_1(\infty) \\ z_2(\infty) \end{bmatrix} = \lim_{s \to 0} s \begin{bmatrix} Z_0(s) \\ Z_1(s) \\ Z_2(s) \end{bmatrix} = \begin{bmatrix} -k_2 \tau_{L0}/J \\ 0 \\ \tau_{L0}/J \end{bmatrix}$$

Though the controller is designed without knowledge of the value of τ_{L0}, it forces $\omega(t) \to \omega_0$.
Remark The series connected DC motor puts out more torque per Ampere than any other DC motor. Historically they were used in applications that require high torque such as diesel electric locomotives. In the case of the locomotive the diesel engine runs a generator with the output of the generator connected to the input of the series connected DC motor. The motor is controlled to bring the locomotive up to some speed $\omega > 0$. As the train moves across the landscape the speed can be increased or decreased, but keeping $\omega > 0$ and $i > 0$. However, in the last thirty years or so series connected DC motors have been replaced by AC induction drives made possible by the technological advancements in power electronics.)
Remark Note that if an AC voltage is applied to the series connected DC motor then the current will also change direction due to this alternating voltage input. However, the torque $\tau_m = K_T i^2$ will remain positive

as it depends on the square of the current. For this reason the series connected DC motor is also referred to as a *universal* motor as it can run either of an AC or a DC voltage source yet keeping the torque always positive. For this reason vacuum cleaners, blenders, hair dyers, drills, sanders, jig saws and starter motors in cars typically use series connected DC motors.

1.5 Permanent Magnet Synchronous Motor [7][8]

Figure 1.7 is an illustration of the structure of a two-phase permanent magnet synchronous motor. The rotor is a permanent magnet and the stator has two sets of windings (phases) denoted as $a - a'$ and $b - b'$. The current in phase $a - a'$ is i_{Sa} and the current in phase $b - b'$ is i_{Sb}. L_S denotes the self-inductance of each phase, R_S denote the resistance of each phase, J is the moment of inertia of the rotor, K_m is the torque constant, and τ_L denotes the load torque. u_{Sa} is the voltage applied to phase $a - a'$ and u_{Sb} is the voltage applied to phase $b - b'$. Details on the physical operation of this motor based on elementary electricity and magnetism are given in [1].

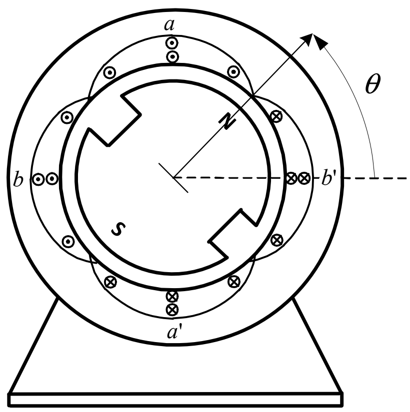


FIGURE 1.7. Two-phase permanent magnet synchronous machine.

The differential equation model of this motor is derived in [1] and given by

$$\begin{aligned}
L_S\frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m\sin(n_p\theta)\omega + u_{Sa}\\
L_S\frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m\cos(n_p\theta)\omega + u_{Sb}\\
J\frac{d\omega}{dt} &= K_m(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)) - \tau_L\\
\frac{d\theta}{dt} &= \omega.
\end{aligned} \tag{1.26}$$

More common in use are three-phase PM synchronous machines. However these machines have an equivalent two-phase model also given by (1.26) so the development below holds for them as well.

The currents and voltages are transformed by the *direct-quadrature* (dq) transformation defined by

$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix} \tag{1.27}$$

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix} \tag{1.28}$$

where u_d is the direct voltage, u_q is the quadrature voltage, i_d is the direct current, and i_q is the quadrature current. The direct current i_d corresponds to the component of the stator magnetic field along the axis of the rotor magnetic field, while the quadrature current i_q corresponds to the orthogonal component of the stator current that produces torque. The application of the dq transformation to the original system (1.26) yields the system of equations

$$
\begin{aligned}
L_S\frac{di_d}{dt} &= -R_S i_d + n_p\omega L_S i_q + u_d && (1.29)\\
L_S\frac{di_q}{dt} &= -R_S i_q - n_p\omega L_S i_d - K_m\omega + u_q && (1.30)\\
J\frac{d\omega}{dt} &= K_m i_q - \tau_L && (1.31)\\
\frac{d\theta}{dt} &= \omega && (1.32)
\end{aligned}
$$

The resulting dq system model (1.29)−(1.32) is still nonlinear, however the nonlinear terms can now be canceled by state feedback. Specifically, choosing u_d and u_q to be

$$
\begin{aligned}
u_d &= R_S i_d - n_p\omega L_S i_q + L_S w_d && (1.33)\\
u_q &= R_S i_q + n_p\omega L_S i_d + K_m\omega + L_S w_q && (1.34)
\end{aligned}
$$

results in the *feedback linearized* system

$$
\begin{aligned}
\frac{di_d}{dt} &= w_d && (1.35)\\
\frac{di_q}{dt} &= w_q && (1.36)\\
\frac{d\omega}{dt} &= (K_m/J)i_q - \tau_L/J && (1.37)\\
\frac{d\theta}{dt} &= \omega. && (1.38)
\end{aligned}
$$

Note that the original fourth-order system has been transformed into the first-order linear system (1.35) and the third-order linear system given by Equations (1.36)−(1.38) which are decoupled from each other.

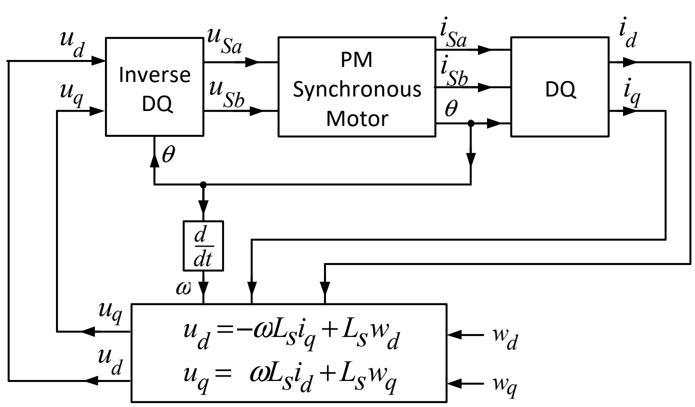


FIGURE 1.8. Block diagram for controlling a PM synchronous machine using the dq coordinates.

However, rather than proceeding with the feedback linearized system of equations (1.35)−(1.38), it is much better to carry out the feedback linearization on the *error* system as this makes it easier to avoid actuator saturation. To explain, suppose a desired trajectory has been designed for the motor which satisfies the system equations (1.29)−(1.32), that is,

$$L_S \frac{di_{dref}}{dt} = -R_S i_{dref} + n_p \omega_{ref} L_S i_{qref} + u_{dref} \tag{1.39}$$

$$L_S \frac{di_{qref}}{dt} = -R_S i_{qref} - n_p \omega_{ref} L_S i_{dref} - K_m \omega_{ref} + u_{qref} \tag{1.40}$$

$$J \frac{d\omega_{ref}}{dt} = K_m i_{qref} \tag{1.41}$$

$$\frac{d\theta_{ref}}{dt} = \omega_{ref}. \tag{1.42}$$

Let τ_L be constant and subtract the system model (1.29)−(1.32) from the reference model (1.39)− (1.42) to obtain the error system

$$\begin{aligned} L_S \frac{d}{dt}(i_{dref} - i_d) &= -R_S(i_{dref} - i_d) + n_p \omega_{ref} L_S i_{qref} - n_p \omega L_S i_q \\ &\quad + u_{dref} - u_d \end{aligned} \tag{1.43}$$

$$\begin{aligned} L_S \frac{d}{dt}(i_{qref} - i_q) &= -R_S(i_{qref} - i_q) - n_p \omega_{ref} L_S i_{dref} + n_p \omega L_S i_d \\ &\quad - K_m(\omega_{ref} - \omega) + u_{qref} - u_q \end{aligned} \tag{1.44}$$

$$J \frac{d}{dt}(\omega_{ref} - \omega) = K_m(i_{qref} - i_q) + \tau_L / J \tag{1.45}$$

$$\frac{d}{dt}(\theta_{ref} - \theta) = \omega_{ref} - \omega. \tag{1.46}$$

Next apply the *feedback linearizing* controller given by

$$u_d = -n_p \omega L_S i_q + u_{dref} + n_p \omega_{ref} L_S i_{qref} - L_S v_d \tag{1.47}$$

$$u_q = +n_p \omega L_S i_d + u_{qref} - n_p \omega_{ref} L_S i_{dref} - L_S v_q \tag{1.48}$$

where v_d and v_q are new inputs yet to be defined. If the controller keeps i_q close to i_{qref} then the two terms $-n_p \omega L_S i_q$ and $\omega_{ref} L_S i_{qref}$ in (1.47) approximately cancel each other so that a lot of the direct voltage u_d is not used to cancel them. Similarly, if the controller keeps i_d close to i_{dref} then the two terms $n_p \omega L_S i_d$ and $-n_p \omega_{ref} L_S i_{dref}$ in (1.48) approximately cancel each other so that a lot of the quadrature voltage u_q is not used to cancel them.

Substituting equations (1.47) and (1.48) for u_d and u_q into (1.43)−(1.44) results in the *linear* error system of equations

$$L_S \frac{d}{dt}(i_{dref} - i_d) = -R_S(i_{dref} - i_d) + v_d \tag{1.49}$$

$$L_S \frac{d}{dt}(i_{qref} - i_q) = -R_S(i_{qref} - i_q) - K_m(\omega_{ref} - \omega) + v_q \tag{1.50}$$

$$J \frac{d}{dt}(\omega_{ref} - \omega) = K_m(i_{qref} - i_q) + \tau_L \tag{1.51}$$

$$\frac{d}{dt}(\theta_{ref} - \theta) = \omega_{ref} - \omega. \tag{1.52}$$

Define the tracking error as

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} \triangleq \begin{bmatrix} i_{dref} - i_d \\ i_{qref} - i_q \\ \omega_{ref} - \omega \\ \theta_{ref} - \theta \\ \int_0^t (\theta_{ref}(\tau) - \theta(\tau)) d\tau \end{bmatrix} \tag{1.53}$$

where the integrator is added to the controller to eliminate any steady-state error due to τ_L. Using (1.49)−(1.52), the error (1.53) satisfies

$$\frac{de}{dt} = \begin{bmatrix} -R_S/L_S & 0 & 0 & 0 & 0 \\ 0 & -R_S/L_S & -K_m/L_S & 0 & 0 \\ 0 & K_m/J & -f/J & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} e + \begin{bmatrix} 1/L_S & 0 \\ 0 & 1/L_S \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1/J \\ 0 \\ 0 \end{bmatrix} \tau_L. \tag{1.54}$$

where $v \triangleq \begin{bmatrix} v_d & v_q \end{bmatrix}^T$. More compactly,

$$\frac{de}{dt} = Ae + Bv + p\tau_L$$

with the obvious definitions for A, B, and p. Through the use of a nonlinear state transformation, input transformation, and nonlinear feedback, a *linear time-invariant* error system has been obtained. The input v is chosen as the linear state feedback

$$v = -Ke \tag{1.55}$$

where K is taken to be of the form

$$K = \begin{bmatrix} k_{11} & 0 & 0 & 0 & 0 \\ 0 & k_{22} & k_{23} & k_{24} & k_{25} \end{bmatrix}. \tag{1.56}$$

The closed-loop error system is then

$$\frac{de}{dt} = (A - BK)\, e + p\tau_L.$$

Using linear statespace techniques one can choose K to place the closed-loop poles of $A - BK$ in the open left-half plane at any desired location there. It is straightforward to show that $e_4(t) = \theta_{ref}(t) - \theta(t) \to 0$ and $e_5(t) = \omega_{ref}(t) - \omega(t) \to 0$ despite any constant load torque τ_L.

Remarks The details of using this approach for controlling actual PM synchronous machines are given in [9][1]. The dq transformation is the feedback linearizing transformation for the model (1.26). If one identifies the direct current i_d with the field current i_f of a wound field DC motor and the quadrature current i_q with the armature current i_a of a wound field DC motor then the model (1.29)−(1.32) is similar to the model of a wound field DC motor and the control scheme just presented for the PM synchronous machine is similar to that used for wound field DC motor. For these reasons a PM synchronous machine with this type of control is referred to as a *brushless* DC motor.

1.6 Magnetic Levitation - Again

Let's go back to the levitation of a steel ball using an electromagnet shown in Figure 1.9. Now we take the voltage applied to coil of the electromagnet as the input and the state variables to be the current i in the coil, the position x of the steel ball below the electromagnet, and the velocity $v = dx/dt$ of the steel ball.

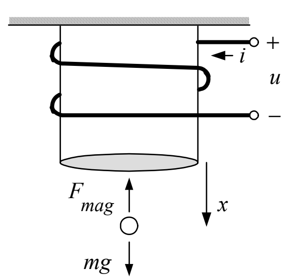


FIGURE 1.9. Magnetic levitation system.

With $L(x) = L_0 + rL_1/x$ denoting the inductance of the coil with the ball at distance x below the magnet, the nonlinear statespace model of this magnetic levitation system is [2]

$$\frac{di}{dt} = rL_1 \frac{i}{x^2 L(x)} v - \frac{R}{L(x)} i + \frac{1}{L(x)} u \tag{1.57}$$

$$\frac{dx}{dt} = v \tag{1.58}$$

$$\frac{dv}{dt} = g - \frac{rL_1}{2m} \frac{i^2}{x^2}. \tag{1.59}$$

Define a nonlinear statespace transformation by

$$x_1 \triangleq x \tag{1.60}$$

$$x_2 \triangleq v \tag{1.61}$$

$$x_3 \triangleq g - \frac{rL_1}{2m} \frac{i^2}{x^2}. \tag{1.62}$$

We then have

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
\frac{dx_3}{dt} &= \frac{d}{dt}\left(g - \frac{rL_1}{2m}\frac{i^2}{x^2}\right).
\end{aligned}$$

x_1 is the position of the steel ball, x_2 is its velocity, x_3 is its acceleration. We calculate $\frac{dx_3}{dt}$ as follows.

$$\begin{aligned}
\frac{d}{dt}\left(g - \frac{rL_1}{2m}\frac{i^2}{x^2}\right) &= -\frac{rL_1}{m}\frac{i}{x^2}\frac{di}{dt} + \frac{rL_1}{m}\frac{i^2}{x^3}\frac{dx}{dt} \\
&= -\frac{rL_1}{m}\frac{i}{x^2}\left(rL_1 \frac{i}{x^2 L(x)} v - \frac{R}{L(x)} i + \frac{1}{L(x)} u\right) + \frac{rL_1}{m}\frac{i^2}{x^3} v \\
&= -\frac{rL_1}{m}\frac{i^2}{x^4}\frac{rL_1}{L(x)} v + \frac{rL_1}{m}\frac{i^2}{x^2}\frac{R}{L(x)} - \frac{rL_1}{m}\frac{i}{x^2}\frac{u}{L(x)} + \frac{rL_1}{m}\frac{i^2}{x^3} v \\
&= \underbrace{-\frac{rL_1}{m}\frac{i^2}{x^4}\frac{rL_1}{L(x)} v + \frac{rL_1}{m}\frac{i^2}{x^2}\frac{R}{L(x)} + \frac{rL_1}{m}\frac{i^2}{x^3} v}_{f(i,x,v)} + \underbrace{\left(-\frac{rL_1}{m}\frac{i}{x^2 L(x)}\right)}_{g(i,x)} u.
\end{aligned}$$

In terms of the state variables x_1, x_2, and x_3, the nonlinear statespace model is

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
\frac{dx_3}{dt} &= f(i,x,v) + g(i,x)u.
\end{aligned}$$

The point of this transformation is that the nonlinear terms are in the *same* equation as the input. Setting

$$u = \frac{-f(i,x,v) + w}{g(i,x)}$$

we obtain the *linear* system

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= x_3\\ \frac{dx_3}{dt} &= w.\end{aligned}$$

Let's first design a trajectory for the system. Let x_{d1} be the position reference, $x_{d2} \triangleq dx_{d1}/dt$ be the speed reference, $x_{d3} \triangleq dx_{d2}/dt$ be the acceleration reference, and $j_d \triangleq dx_{d3}/dt$ be the jerk reference so that

$$\begin{aligned}\frac{dx_{d1}}{dt} &= x_{d2}\\ \frac{dx_{d2}}{dt} &= x_{d3}\\ \frac{dx_{d3}}{dt} &= j_d.\end{aligned}$$

Using (1.60), (1.61), and (1.62) we have

$$x_d \triangleq x_{d1}, v_d \triangleq x_{d2}, \text{ and } i_d \triangleq \sqrt{\frac{2m}{rL_1}\left(g - x_{d3}\right) x_{d1}^2}.$$

We want

$$\frac{dx_{d3}}{dt} = f(i_d, x_d, v_d) + g(i_d, x_d)u_d = j_d,$$

so define the reference input voltage u_d to be

$$u_d \triangleq \frac{j_d - f(i_d, x_d, v_d)}{g(i_d, x_d)}.$$

With $\epsilon_1 = x_{d1} - x_1, \epsilon_2 = x_{d2} - x_2, \epsilon_3 = x_{d3} - x_3$ the error system is

$$\begin{aligned}\frac{d\epsilon_1}{dt} &= \epsilon_2\\ \frac{d\epsilon_2}{dt} &= \epsilon_3\\ \frac{d\epsilon_3}{dt} &= f(i_d, x_d, v_d) + g(i_d, x_d)u_d - f(i, x, v) - g(i, x)u\\ &= f(i_d, x_d, v_d) - f(i, x, v) + g(i_d, x_d)u_d - g(i, x)u.\end{aligned}$$

Then choose the input u to satisfy

$$f(i_d, x_d, v_d) - f(i, x, v) + g(i_d, x_d)u_d - g(i, x)u = -\underbrace{\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}}_{k}\underbrace{\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}}_{\epsilon}.$$

That is, let the input voltage be given by (see Figure 1.10)

$$u = \frac{f(i_d, x_d, v_d) - f(i, x, v) + g(i_d, x_d)u_d + k\epsilon}{g(i, x)} = \frac{-f(i, x, v) + j_d + k\epsilon}{g(i, x)}. \tag{1.63}$$

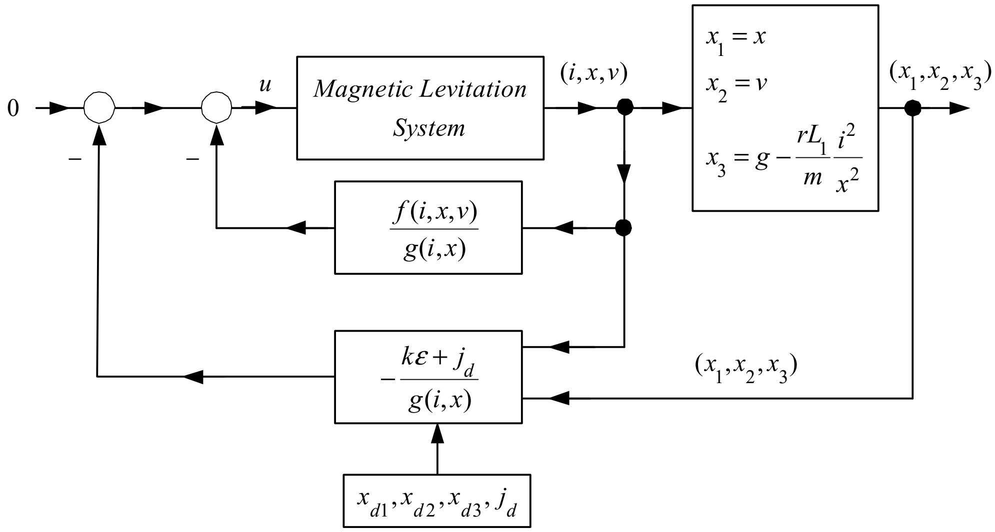


FIGURE 1.10. Block diagram for the magnetic levitation system.

The error system is then

$$\begin{aligned}\frac{d\epsilon_1}{dt} &= \epsilon_2\\ \frac{d\epsilon_2}{dt} &= \epsilon_3\\ \frac{d\epsilon_3}{dt} &= -k_1\epsilon_1 - k_2\epsilon_2 - k_3\epsilon_3.\end{aligned}$$

In matrix form we have

$$\frac{d}{dt}\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -k_1 & -k_2 & -k_3\end{bmatrix}\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\end{bmatrix}$$

with

$$\det(sI - (A_c - b_ck)) = s^3 + k_3s^2 + k_2s + k_1.$$

The feedback gains k_1, k_2, k_3 can be chosen to place the poles of $A_c - b_ck$ at any desired location in the open left half-plane. The feedback voltage u to the amplifier is then

$$u = \frac{-f(i,x,v) + j_d + k\epsilon}{g(i,x)}.$$

The state variables i, x, v are sampled in real-time and, along with the stored variables $i_d, x_d, v_d, x_{d1}, x_{d2}, x_{d3}, u_d$, the input u computed in real-time according to (1.63) as the voltage commanded to the amplifier. This nonlinear controller allows for trajectory tracking and does *not* require (i,x,v) to be close to an equilibrium state. However, it must be ensured that the controller does not violate the voltage limit of the amplifier, that is, the feedback gains k_1, k_2, k_3 must be chosen so that

$$|u| = \left|\frac{-f(i,x,v) + j_d + k\epsilon}{g(i,x)}\right| \le V_{\max}.$$

1.7 State Observers for Linear Systems

We first review the concept of observers for state estimation in LTI systems. Consider the single-input single-output (SISO) linear time invariant system given by

$$\begin{aligned} \frac{dx}{dt} &= Ax + bu, \quad A \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^n \\ y &= cx, \quad c \in \mathbb{R}^{1 \times n}. \end{aligned}$$

Define an observer for this system by

$$\begin{aligned} \frac{d\hat{x}}{dt} &= A\hat{x} + bu + \ell(y - \hat{y}), \quad \ell \in \mathbb{R}^n \\ \hat{y} &= c\hat{x}. \end{aligned}$$

Note that with $\ell = 0_{n\times 1}$ the observer is simply a simulation of the original system. The addition of the error term $\ell(y - \hat{y})$ provides a way to force $\hat{x}(t) \to x(t)$ despite $\hat{x}(0)$ not being known. The idea here is that the observer puts out the value $\hat{x}(t)$ and uses it to predict the value of $\hat{y}(t) = c\hat{x}(t)$. Then, as $y(t) = cx(t)$ is measured, the observer uses the difference (error) $y(t) - \hat{y}(t) = cx(t) - c\hat{x}(t)$ to adjust the state estimate $\hat{x}(t)$ through $\ell(y(t) - \hat{y}(t))$. To show that this actually works, let

$$\epsilon(t) \triangleq x(t) - \hat{x}(t)$$

which has the dynamics

$$\frac{d\epsilon}{dt} = Ax + bu - (A\hat{x} + bu + \ell(y - \hat{y})) = A(x - \hat{x}) + \ell c(x - \hat{x}) = (A - \ell c)\epsilon(t).$$

If ℓ can be chosen so that $A - \ell c$ is stable, then $\epsilon(t) \to 0$. That is, the state estimate $\hat{x}(t)$ goes to $x(t)$ for any initial condition $x(0)$. To see how ℓ can be chosen to make $A - \ell c$ stable let's consider an example where A and c have a special form.

Example 4 *The Pair* (c, A) *in Observer Canonical Form*
Let the matrices c and A have the special form

$$c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}.$$

A straightforward calculation shows $\det(sI - A) = s^3 + a_2 s^2 + a_1 s + a_0$. With $\ell = \begin{bmatrix} \ell_0 & \ell_1 & \ell_2 \end{bmatrix}^T$ we have

$$A - \ell c = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} - \begin{bmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\ell_0 - a_0 \\ 1 & 0 & -\ell_1 - a_1 \\ 0 & 1 & -\ell_2 - a_2 \end{bmatrix}.$$

Setting $\ell = \begin{bmatrix} a_0 - \alpha_0 & a_1 - \alpha_1 & a_2 - \alpha_2 \end{bmatrix}^T$ we then have

$$A - \ell c = \begin{bmatrix} 0 & 0 & -\alpha_0 \\ 1 & 0 & -\alpha_1 \\ 0 & 1 & -\alpha_2 \end{bmatrix}$$

with $\det(sI - (A - \ell c)) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$. That is, the coefficients of the characteristic polynomial of $A - \ell c$ can be arbitrarily assigned. This special form of the pair (c, A) is called the *observer canonical* form.

For arbitrary $c \in \mathbb{R}^{1\times 3}$ and $A \in \mathbb{R}^{3\times 3}$ the pair (c, A) is called observable if matrix $\mathcal{O}$ defined by

$$\mathcal{O} \triangleq \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix}$$

is nonsingular. If this is true then the eigenvalues of $A - \ell c$ can be assigned arbitrarily. Problem 5 shows how to transform an observable linear system into observer form.

Example 5 *An Observer for Speed and Current in a DC motor*
Recall the model of the DC motor given as

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega + V_S \\ J\frac{d\omega}{dt} &= K_T i - f\omega - \tau_L \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

With $x_1 = i, x_2 = \omega, x_3 = \theta, u = V_S$, and $\tau_L = 0$ we may write

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 \\ K_T/J & -f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{b} u \\ y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

where the rotor angle is taken as the output, i.e., it is measured. The observability matrix is

$$\mathcal{O} \triangleq \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ K_T/J & -f/J & 0 \end{bmatrix}$$

which is nonsingular. Define a coordinate transformation

$$x^* = Tx = \begin{bmatrix} K_T/J & R/L & (Rf + K_T K_b)/(JL) \\ 0 & 1 & R/L + f/J \\ 0 & 0 & 1 \end{bmatrix} x$$

which has inverse

$$x = T^{-1}x^* = \begin{bmatrix} J/K_T & -RJ/(LK_T) & R^2J/(L^2K_T) - K_b/L \\ 0 & 1 & -\left(RJ + Lf\right)/(JL) \\ 0 & 0 & 1 \end{bmatrix} x^*.$$

In the new coordinate system we have

$$\begin{aligned} \frac{dx^*}{dt} &= TAT^{-1}x^* + Tbu \\ y &= cT^{-1}x^* \end{aligned}$$

where

$$\begin{aligned} A_o &\triangleq TAT^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & (Rf + K_T K_b)/(JL) \\ 0 & 1 & R/L + f/J \end{bmatrix}, b_o \triangleq Tb = \begin{bmatrix} K_T/(JL) \\ 0 \\ 0 \end{bmatrix} \\ c_o &\triangleq cT^{-1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

With $a_0 = 0, a_1 = (Rf + K_T K_b)/(JL), a_2 = R/L + f/J$ we see that the pair

$$c_o = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \ A_o \begin{bmatrix} 0 & 0 & a_0 \\ 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{bmatrix}$$

is in observer canonical form. Thus the procedure given in the previous example can be used to estimate x^* and thus the estimate of $\hat{x}$ is given by

$$\hat{x} = T^{-1}\hat{x}^*$$

Problem 9 shows how to estimate the load torque τ_L in addition to the speed ω and the current i.

1.8 State Observers for Nonlinear Systems

Consider a nonlinear system in the following special form.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}}_{A_o} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \varphi_1(y) \\ \varphi_2(y) \\ \varphi_3(y) \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b_o} u \tag{1.64}$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{c_o} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{1.65}$$

where the $\varphi_i(y) = \varphi_i(x_3)$ are arbitrary functions of the output $y = x_3$. Define an observer by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} \varphi_1(y) \\ \varphi_2(y) \\ \varphi_3(y) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u + \begin{bmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \end{bmatrix} (y - \hat{y}) \\ \hat{y} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}. \end{aligned}$$

Then the error $\epsilon \triangleq x - \hat{x}$ satisfies

$$\frac{d}{dt}\epsilon = (A_o - \ell c_o)\epsilon.$$

This is a linear system with the pair (c_o, A_o) in observer canonical form so $\ell \in \mathbb{R}^3$ can used to place the poles of $A_o - \ell c_o$ in the open left-half plane so that $\epsilon(t) \triangleq x(t) - \hat{x}(t) \to 0_{3\times 1}$.

The differential-geometric approach to the design of observers for nonlinear systems is to find a statespace transformation so that in the new coordinates the system has the form of (1.64) and (1.65).

Example 6 *Series Connected DC Motor*

Recall the nonlinear equations of the series connected DC motor.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ c_2 x_3 - c_3 x_3 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}}_{p} \tau_L$$

where $x_1 = \theta, x_2 = \omega, x_3 = i = i_a = i_f, u = V_S/L, c_1 = K_T L_f/J, c_2 = -R/L$, and $c_3 = K_b L_f/L$. As discussed above, the series connected DC motor is used in speed control applications so let's remove $x_1 = \theta$

from the model. The load torque is taken to be constant, but is unknown so it will need to be estimated in order to estimate the motor speed ω. To do this τ_L/J is added to the model as a *state variable* with $\frac{d}{dt}(\tau_L/J) = 0$. With $z_1 = x_2 = \omega, z_2 = x_3 = i$, $z_3 = \tau_L/J$, and assuming only the current is measured, the system equations are now

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 z_2^2 - z_3 \\ -c_2 z_2 - c_3 z_1 z_2 \\ 0 \end{bmatrix}}_{f(z)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{g(z)} u \tag{1.66}$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{c} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \tag{1.67}$$

More compactly this is written as

$$\begin{aligned} \frac{dz}{dt} &= f(z) + g(z)u \\ y &= cz. \end{aligned}$$

$f(z)$ in (1.66) is *not* linear in the unmeasured state variable $z_1 = \omega$. We transform the equations into a new set of coordinates for which the statespace model is linear in i and ω. Specifically, consider the nonlinear transformation given by

$$\begin{aligned} z_1^* &= T_1(z) = c_3 z_3 \\ z_2^* &= T_2(z) = -c_3 z_1 \\ z_3^* &= T_3(z) = \ln(z_2) \end{aligned}$$

with inverse

$$\begin{aligned} z_1 &= -z_2^*/c_3 \\ z_2 &= e^{z_3^*} \\ z_3 &= z_1^*/c_3. \end{aligned}$$

The system equations in the z^* coordinate system are then

$$\begin{aligned} \frac{dz_1^*}{dt} = \mathcal{L}_{f+gu}(T_1(z)) = \langle dT_1, f+gu\rangle &= \begin{bmatrix} 0 & 0 & c_3 \end{bmatrix}\left(\begin{bmatrix} c_1 z_2^2 - z_3 \\ -c_2 z_2 - c_3 z_1 z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u\right) \\ &= 0 \\ \frac{dz_2^*}{dt} = \mathcal{L}_{f+gu}(T_2(z)) = \langle dT_2, f+gu\rangle &= \begin{bmatrix} -c_3 & 0 & 0 \end{bmatrix}\left(\begin{bmatrix} c_1 z_2^2 - z_3 \\ -c_2 z_2 - c_3 z_1 z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u\right) \\ &= -c_1 c_3 z_2^2 + c_3 z_3 \\ &= -c_1 c_3 y^2 + c_3 z_3 \\ \frac{dz_3^*}{dt} = \mathcal{L}_{f+gu}(T_3(z)) = \langle dT_3, f+gu\rangle &= \begin{bmatrix} 0 & 1/z_2 & 0 \end{bmatrix}\left(\begin{bmatrix} c_1 z_2^2 - z_3 \\ -c_2 z_2 - c_3 z_1 z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u\right) \\ &= -c_2 - c_3 z_1 + u/z_2. \end{aligned}$$

In summary the system in the z^* coordinates is given by

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A_o}\begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ -c_1c_3y^2 \\ -c_2 + u/z_2 \end{bmatrix} \\ y^* &= \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{c_o}\begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix}.\end{aligned}$$

Note that the pair (c_o, A_o) is in observer canonical form. The output is taken to be $y^* \triangleq z_3^* = \ln(z_2) = \ln(i)$. As the current is measured the output y^* is known.

The observer for $z_1^* = c_3z_3 = \dfrac{K_bL_f}{L}\dfrac{\tau_L}{J}$ and $z_2^* = -c_3z_1 = -\dfrac{K_bL_f}{L}\omega$ is

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} \hat{z}_1^* \\ \hat{z}_2^* \\ \hat{z}_3^* \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{z}_1^* \\ \hat{z}_2^* \\ \hat{z}_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ -c_1c_3y^2 \\ -c_2 + u/z_2 \end{bmatrix} + \begin{bmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \end{bmatrix}(y^* - \hat{y}^*) \\ \hat{y}^* &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{z}_1^* \\ \hat{z}_2^* \\ \hat{z}_3^* \end{bmatrix}.\end{aligned}$$

The error system is

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} z_1^* - \hat{z}_1^* \\ z_2^* - \hat{z}_2^* \\ z_3^* - \hat{z}_3^* \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} z_1^* - \hat{z}_1^* \\ z_2^* - \hat{z}_2^* \\ z_3^* - \hat{z}_3^* \end{bmatrix} - \begin{bmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} z_1^* - \hat{z}_1^* \\ z_2^* - \hat{z}_2^* \\ z_3^* - \hat{z}_3^* \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -\ell_0 \\ 1 & 0 & -\ell_1 \\ 0 & 1 & -\ell_2 \end{bmatrix}\begin{bmatrix} z_1^* - \hat{z}_1^* \\ z_2^* - \hat{z}_2^* \\ z_3^* - \hat{z}_3^* \end{bmatrix}\end{aligned}$$

The gains ℓ_0, ℓ_1, ℓ_2 can be chosen to place the eigenvalues of this error system arbitrarily so $\hat{z}^*(t) \to z^*(t)$. As $z_1^* = \dfrac{K_bL_f}{L}\dfrac{\tau_L}{J}$ and $z_2^* = -\dfrac{K_bL_f}{L}\omega$ the parameters $K_b, L_f, L = L_f + L_a$ must be known accurately to obtain accurate estimates for τ_L/J and ω.

Remark An implementation of this observer is given in [10].

1.9 Lie Brackets, Lie Derivatives, and Differential Equations

Let's look at a general nonlinear control system given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{bmatrix}}_{g(x)}u. \tag{1.68}$$

Examples of nonlinear control systems have been presented showing that it is possible to find a nonlinear change of coordinates such that the system can be made linear using feedback. The resulting linear system can then be straightforwardly controlled. The first goal is determine the conditions on a given nonlinear system that determine if such a transformation exists. If such a transformation exists, the next goal is to find the transformation. This will done in the following chapters. Here we develop some tools that are needed to achieve these goals.

Consider an invertible nonlinear change of coordinates $x^* = T(x) : \mathbb{R}^3 \to \mathbb{R}^3$ for (1.68) given by

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} \triangleq \begin{bmatrix} T_1(x_1, x_2, x_3) \\ T_2(x_1, x_2, x_3) \\ T_3(x_1, x_2, x_3) \end{bmatrix}.$$

In the x^* coordinate system the system (1.68) has the form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \underbrace{\begin{bmatrix} f_1^*(x^*) \\ f_2^*(x^*) \\ f_3^*(x^*) \end{bmatrix}}_{f^*(x^*)} + \underbrace{\begin{bmatrix} g_1^*(x^*) \\ g_2^*(x^*) \\ g_3^*(x^*) \end{bmatrix}}_{g^*(x^*)} u$$

where

$$f^*(x^*) = \frac{\partial T}{\partial x} f(x)\bigg|_{x=T^{-1}(x^*)}, \quad g^*(x^*) = \frac{\partial T}{\partial x} g(x)\bigg|_{x=T^{-1}(x^*)} \tag{1.69}$$

and

$$\frac{\partial T}{\partial x} \triangleq \begin{bmatrix} \partial T_1/\partial x_1 & \partial T_1/\partial x_2 & \partial T_1/\partial x_3 \\ \partial T_2/\partial x_1 & \partial T_2/\partial x_2 & \partial T_2/\partial x_3 \\ \partial T_3/\partial x_1 & \partial T_3/\partial x_2 & \partial T_3/\partial x_3 \end{bmatrix}. \tag{1.70}$$

The matrix $\dfrac{\partial T}{\partial x} \in \mathbb{R}^{3\times 3}$ is referred to as the *Jacobian* of the transformation $T \in \mathbb{R}^3$.

Example 7 *Linear Systems*

Let $f(x) = Ax, g(x) = b$ so that $\dfrac{dx}{dt} = f(x) + g(x)u = Ax + bu$ is a linear system. Further, let $T(x) \triangleq \mathbf{T}x$ be a linear transformation with $\mathbf{T} \in \mathbb{R}^{3\times 3}$ and $\det(\mathbf{T}) \neq 0$.

Then

$$\begin{aligned} f^*(x) &\triangleq \frac{\partial T}{\partial x} f(x) = \mathbf{T}Ax \\ g^*(x) &\triangleq \mathbf{T}b \end{aligned}$$

and

$$\begin{aligned} f^*(x^*) &= \frac{\partial T}{\partial x} f(x)\bigg|_{x=T^{-1}(x^*)} = \mathbf{T}Ax_{x=T^{-1}(x^*)} = \mathbf{T}A\mathbf{T}^{-1}x^* \\ g^*(x^*) &= \frac{\partial T}{\partial x} g(x)\bigg|_{x=T^{-1}(x^*)} = \mathbf{T}b\big|_{x=T^{-1}(x^*)} = \mathbf{T}b. \end{aligned}$$

We have the following definition of the *Lie bracket* of $f(x)$ and $g(x)$.

Definition 2 *Lie Bracket*

Let $f(x)$ and $g(x)$ be as in (1.68). Define the *Lie bracket* of the pair $f(x), g(x)$ as

$$[f,g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \triangleq \begin{bmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \partial g_1/\partial x_3 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 & \partial g_2/\partial x_3 \\ \partial g_3/\partial x_1 & \partial g_3/\partial x_2 & \partial g_3/\partial x_3 \end{bmatrix}\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} - \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \end{bmatrix}\begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{bmatrix}.$$

We will find out in Chapter 3 that the Lie bracket is a tool used in determining whether or not a feedback linearizing transformation exists for a system of the form $dx/dt = f(x) + g(x)u$.

Lie Brackets Under a Change of Coordinates $[f^*, g^*] = \dfrac{\partial T}{\partial x}[f, g]$

An important fact of the Lie bracket is that under a change of coordinates the Lie bracket $[f, g]$ transforms the *same* way as f, g transform as given in (1.69). That is, with $f^*(x^*), g^*(x^*)$ defined as in (1.69) it is now shown that

$$[f^*, g^*] = \frac{\partial g^*(x^*)}{\partial x^*} f^*(x^*) - \frac{\partial f^*(x^*)}{\partial x^*} g^*(x^*) = \frac{\partial T}{\partial x}\left(\frac{\partial g(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x)\right)\Bigg|_{x=T^{-1}(x^*)} = \frac{\partial T}{\partial x}[f, g]\Bigg|_{|x=T^{-1}(x^*)}. \tag{1.71}$$

To begin let $S = T^{-1}$ so that $x = T^{-1}(x^*) = S(x^*)$. By the chain rule we have

$$\begin{aligned}
\frac{\partial f^*(x^*)}{\partial x^*} = \frac{\partial}{\partial x^*}\left(\frac{\partial T}{\partial x} f(x)\Bigg|_{x=T^{-1}(x^*)}\right) &= \frac{\partial}{\partial x^*}\left(\begin{bmatrix} \partial T_1/\partial x_1 & \partial T_1/\partial x_2 & \partial T_1/\partial x_3 \\ \partial T_2/\partial x_1 & \partial T_2/\partial x_2 & \partial T_2/\partial x_3 \\ \partial T_3/\partial x_1 & \partial T_3/\partial x_2 & \partial T_3/\partial x_3 \end{bmatrix}\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}\Bigg|_{x=T^{-1}(x^*)}\right) \\
&= \underbrace{\frac{\partial}{\partial x}\begin{bmatrix} d_1(x) \\ d_2(x) \\ d_3(x) \end{bmatrix}}_{\left(\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} f(x)\right)\right)} \underbrace{\begin{bmatrix} \partial x_1/\partial x_1^* & \partial x_1/\partial x_2^* & \partial x_1/\partial x_3^* \\ \partial x_2/\partial x_1^* & \partial x_2/\partial x_2^* & \partial x_2/\partial x_3^* \\ \partial x_3/\partial x_1^* & \partial x_3/\partial x_2^* & \partial x_3/\partial x_3^* \end{bmatrix}}_{\frac{\partial x}{\partial x^*} = \frac{\partial S(x^*)}{\partial x^*}} \\
&= \underbrace{\begin{bmatrix} \partial d_1/\partial x_1 & \partial d_1/\partial x_2 & \partial d_1/\partial x_3 \\ \partial d_2/\partial x_1 & \partial d_2/\partial x_2 & \partial d_2/\partial x_3 \\ \partial d_3/\partial x_1 & \partial d_3/\partial x_2 & \partial d_3/\partial x_3 \end{bmatrix}}_{\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} f(x)\right)} \underbrace{\begin{bmatrix} \partial x_1/\partial x_1^* & \partial x_1/\partial x_2^* & \partial x_1/\partial x_3^* \\ \partial x_2/\partial x_1^* & \partial x_2/\partial x_2^* & \partial x_2/\partial x_3^* \\ \partial x_3/\partial x_1^* & \partial x_3/\partial x_2^* & \partial x_3/\partial x_3^* \end{bmatrix}}_{\frac{\partial x}{\partial x^*} = \frac{\partial S(x^*)}{\partial x^*}} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} f(x)\right)\Bigg|_{x=T^{-1}(x^*)} \frac{\partial S(x^*)}{\partial x^*}.
\end{aligned} \tag{1.72}$$

Then

$$\begin{aligned}
\frac{\partial f^*(x^*)}{\partial x^*} g^*(x^*) &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} f(x)\right)\Bigg|_{x=T^{-1}(x^*)} \frac{\partial S(x^*)}{\partial x^*}\, \frac{\partial T}{\partial x} g(x)\Bigg|_{x=T^{-1}(x^*)} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} f(x)\right)\Bigg|_{x=T^{-1}(x^*)} \underbrace{\frac{\partial S(x^*)}{\partial x^*} \frac{\partial T}{\partial x}_{|x=T^{-1}(x^*)}}_{I_{3\times 3}} g(x)_{|x=T^{-1}(x^*)} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} f(x)\right)\Bigg|_{x=T^{-1}(x^*)} g(x)_{|x=T^{-1}(x^*)}.
\end{aligned}$$

The last step used the fact that $x = S(x^*) = T^{-1}(x^*)$ so $S(x^*) = S(T(x)) = x$ and therefore

$$\frac{\partial}{\partial x} S(T(x)) = \frac{\partial S(x^*)}{\partial x^*}\frac{\partial T(x)}{\partial x} = \frac{\partial x}{\partial x} = I_{3\times 3}.$$

Next we find a more convenient expression for $\dfrac{\partial}{\partial x}\left(\dfrac{\partial T}{\partial x}f(x)\right)$. To do this note that $\dfrac{\partial}{\partial x}\left(\dfrac{\partial T}{\partial x}f(x)\right)$ is more explicitly written as

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}f(x)\right) = \begin{bmatrix} \dfrac{\partial}{\partial x}\left(\dfrac{\partial T_1}{\partial x}f(x)\right) \\ \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial T_2}{\partial x}f(x)\right) \\ \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial T_3}{\partial x}f(x)\right) \end{bmatrix}.$$

Expanding $\dfrac{\partial}{\partial x}\left(\dfrac{\partial T_1}{\partial x}f(x)\right)$ we have

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right) \\
&= \begin{bmatrix} \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial T_1}{\partial x}f(x)\right) & \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial T_1}{\partial x}f(x)\right) & \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial T_1}{\partial x}f(x)\right) \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial}{\partial x_1}\left(\sum_{i=1}^{3}\dfrac{\partial T_1}{\partial x_i}f_i(x)\right) & \dfrac{\partial}{\partial x_2}\left(\sum_{i=1}^{3}\dfrac{\partial T_1}{\partial x_i}f_i(x)\right) & \dfrac{\partial}{\partial x_3}\left(\sum_{i=1}^{3}\dfrac{\partial T_1}{\partial x_i}f_i(x)\right) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{3}\dfrac{\partial^2 T_1}{\partial x_1 \partial x_i}f_i(x) + \dfrac{\partial T_1}{\partial x_i}\dfrac{\partial f_i(x)}{\partial x_1} & \sum_{i=1}^{3}\dfrac{\partial^2 T_1}{\partial x_2 \partial x_i}f_i(x) + \dfrac{\partial T_1}{\partial x_i}\dfrac{\partial f_i(x)}{\partial x_2} & \sum_{i=1}^{3}\dfrac{\partial^2 T_1}{\partial x_3 \partial x_i}f_i(x) + \dfrac{\partial T_1}{\partial x_i}\dfrac{\partial f_i(x)}{\partial x_3} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{3}\left(\dfrac{\partial^2 T_1}{\partial x_1 \partial x_i}f_i(x)\right) & \sum_{i=1}^{3}\left(\dfrac{\partial^2 T_1}{\partial x_2 \partial x_i}f_i(x)\right) & \sum_{i=1}^{3}\left(\dfrac{\partial^2 T_1}{\partial x_3 \partial x_i}f_i(x)\right) \end{bmatrix} + \\
&\quad \begin{bmatrix} \sum_{i=1}^{3}\left(\dfrac{\partial T_1}{\partial x_i}\dfrac{\partial f_i(x)}{\partial x_1}\right) & \sum_{i=1}^{3}\left(\dfrac{\partial T_1}{\partial x_i}\dfrac{\partial f_i(x)}{\partial x_2}\right) & \sum_{i=1}^{3}\left(\dfrac{\partial T_1}{\partial x_i}\dfrac{\partial f_i(x)}{\partial x_3}\right) \end{bmatrix}.
\end{aligned}$$

Continuing the expansion of $\dfrac{\partial}{\partial x}\left(\dfrac{\partial T_1}{\partial x}f(x)\right)$ gives

$$\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right) =$$

$$\left[\begin{bmatrix} \dfrac{\partial^2 T_1}{\partial x_1 \partial x_1} & \dfrac{\partial^2 T_1}{\partial x_1 \partial x_2} & \dfrac{\partial^2 T_1}{\partial x_1 \partial x_3} \end{bmatrix}\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} \; \begin{bmatrix} \dfrac{\partial^2 T_1}{\partial x_2 \partial x_1} & \dfrac{\partial^2 T_1}{\partial x_2 \partial x_2} & \dfrac{\partial^2 T_1}{\partial x_2 \partial x_3} \end{bmatrix}\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} \; \begin{bmatrix} \dfrac{\partial^2 T_1}{\partial x_3 \partial x_1} & \dfrac{\partial^2 T_1}{\partial x_3 \partial x_2} & \dfrac{\partial^2 T_1}{\partial x_3 \partial x_3} \end{bmatrix}\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}\right]$$

$$+\left[\begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \end{bmatrix}\begin{bmatrix} \dfrac{\partial f_1(x)}{\partial x_1} \\ \dfrac{\partial f_2(x)}{\partial x_1} \\ \dfrac{\partial f_3(x)}{\partial x_1} \end{bmatrix} \; \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \end{bmatrix}\begin{bmatrix} \dfrac{\partial f_1(x)}{\partial x_2} \\ \dfrac{\partial f_2(x)}{\partial x_2} \\ \dfrac{\partial f_3(x)}{\partial x_2} \end{bmatrix} \; \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \end{bmatrix}\begin{bmatrix} \dfrac{\partial f_1(x)}{\partial x_2} \\ \dfrac{\partial f_2(x)}{\partial x_2} \\ \dfrac{\partial f_3(x)}{\partial x_2} \end{bmatrix}\right]$$

Using $\dfrac{\partial^2 T_1}{\partial x_i \partial x_j} = \dfrac{\partial^2 T_1}{\partial x_j \partial x_i}$ we rewrite this last expression as

$$\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f(x)\right) =$$

$$\left[\begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 T_1}{\partial x_1 \partial x_1} \\ \frac{\partial^2 T_1}{\partial x_2 \partial x_1} \\ \frac{\partial^2 T_1}{\partial x_3 \partial x_1} \end{bmatrix} \quad \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 T_1}{\partial x_1 \partial x_2} \\ \frac{\partial^2 T_1}{\partial x_2 \partial x_2} \\ \frac{\partial^2 T_1}{\partial x_3 \partial x_2} \end{bmatrix} \quad \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 T_1}{\partial x_1 \partial x_3} \\ \frac{\partial^2 T_1}{\partial x_2 \partial x_3} \\ \frac{\partial^2 T_1}{\partial x_3 \partial x_3} \end{bmatrix} \right]$$

$$+ \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \end{bmatrix} \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \end{bmatrix}$$

Finally we have the expression we were looking for given by

$$\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f(x)\right) =$$

$$\underbrace{\begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \end{bmatrix}}_{f^T} \underbrace{\begin{bmatrix} \frac{\partial^2 T_1}{\partial x_1 \partial x_1} & \frac{\partial^2 T_1}{\partial x_1 \partial x_2} & \frac{\partial^2 T_1}{\partial x_1 \partial x_3} \\ \frac{\partial^2 T_1}{\partial x_2 \partial x_1} & \frac{\partial^2 T_1}{\partial x_2 \partial x_2} & \frac{\partial^2 T_1}{\partial x_2 \partial x_3} \\ \frac{\partial^2 T_1}{\partial x_3 \partial x_1} & \frac{\partial^2 T_1}{\partial x_3 \partial x_2} & \frac{\partial^2 T_1}{\partial x_3 \partial x_3} \end{bmatrix}}_{\frac{\partial^2 T_1}{\partial x^2}} + \underbrace{\begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \end{bmatrix}}_{\frac{\partial T_1}{\partial x}} \underbrace{\begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \end{bmatrix}}_{\frac{\partial f}{\partial x}}$$

or

$$\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f(x)\right) = f^T \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial T_1}{\partial x}\frac{\partial f}{\partial x}. \tag{1.73}$$

The *symmetric* matrix $\dfrac{\partial^2 T_1}{\partial x^2}$ is referred to as the *Hessian* of T_1. Now we can write

$$\begin{aligned} \frac{\partial f^*(x^*)}{\partial x^*} g^*(x^*) = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} f(x)\right) g(x)_{|x=T^{-1}(x^*)} &= \left. \begin{bmatrix} f^T \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial T_1}{\partial x}\frac{\partial f}{\partial x} \\ f^T \frac{\partial^2 T_2}{\partial x^2} + \frac{\partial T_2}{\partial x}\frac{\partial f}{\partial x} \\ f^T \frac{\partial^2 T_3}{\partial x^2} + \frac{\partial T_3}{\partial x}\frac{\partial f}{\partial x} \end{bmatrix} \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{bmatrix} \right|_{x=T^{-1}(x^*)} \\ &= \begin{bmatrix} f^T \frac{\partial^2 T_1}{\partial x^2} g(x) + \frac{\partial T_1}{\partial x}\frac{\partial f}{\partial x} g(x) \\ f^T \frac{\partial^2 T_2}{\partial x^2} g(x) + \frac{\partial T_2}{\partial x}\frac{\partial f}{\partial x} g(x) \\ f^T \frac{\partial^2 T_3}{\partial x^2} g(x) + \frac{\partial T_3}{\partial x}\frac{\partial f}{\partial x} g(x) \end{bmatrix}_{|x=T^{-1}(x^*)} \end{aligned} \tag{1.74}$$

Thus

$$\begin{aligned}
\frac{\partial g^*(x^*)}{\partial x^*}f^*(x^*) - \frac{\partial f^*(x^*)}{\partial x^*}g^*(x^*) &= \begin{bmatrix} g^T\dfrac{\partial^2 T_1}{\partial x^2}f(x) + \dfrac{\partial T_1}{\partial x}\dfrac{\partial g}{\partial x}f(x) \\ g^T\dfrac{\partial^2 T_2}{\partial x^2}f(x) + \dfrac{\partial T_2}{\partial x}\dfrac{\partial g}{\partial x}f(x) \\ g^T\dfrac{\partial^2 T_3}{\partial x^2}f(x) + \dfrac{\partial T_3}{\partial x}\dfrac{\partial g}{\partial x}f(x) \end{bmatrix} - \begin{bmatrix} f^T\dfrac{\partial^2 T_1}{\partial x^2}g(x) + \dfrac{\partial T_1}{\partial x}\dfrac{\partial f}{\partial x}g(x) \\ f^T\dfrac{\partial^2 T_2}{\partial x^2}g(x) + \dfrac{\partial T_2}{\partial x}\dfrac{\partial f}{\partial x}g(x) \\ f^T\dfrac{\partial^2 T_3}{\partial x^2}g(x) + \dfrac{\partial T_3}{\partial x}\dfrac{\partial f}{\partial x}g(x) \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial T_1}{\partial x}\dfrac{\partial g}{\partial x}f(x) \\ \dfrac{\partial T_2}{\partial x}\dfrac{\partial g}{\partial x}f(x) \\ \dfrac{\partial T_3}{\partial x}\dfrac{\partial g}{\partial x}f(x) \end{bmatrix} - \begin{bmatrix} \dfrac{\partial T_1}{\partial x}\dfrac{\partial f}{\partial x}g(x) \\ \dfrac{\partial T_2}{\partial x}\dfrac{\partial f}{\partial x}g(x) \\ \dfrac{\partial T_3}{\partial x}\dfrac{\partial f}{\partial x}g(x) \end{bmatrix} \\
&= \frac{\partial T}{\partial x}\left(\frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g\right)_{|x=T^{-1}(x^*)}
\end{aligned}$$

We have shown

$$[f^*, g^*] = \frac{\partial T}{\partial x}\,[f,g]\Big|_{|x=T^{-1}(x^*)}$$

Theorem 1 *Lie brackets and Lie derivatives*
It is now shown that

$$\mathcal{L}_{[f,g]}(T_1) = \mathcal{L}_f(\mathcal{L}_g(T_1)) - \mathcal{L}_g(\mathcal{L}_f(T_1)).$$

Proof. There is a connection between Lie brackets and Lie derivatives. To show this connection recall by definition that

$$\mathcal{L}_f(T_1) = \frac{\partial T_1}{\partial x}f(x) \in \mathbb{R}$$

so

$$\mathcal{L}_g(\mathcal{L}_f(T_1)) = \frac{\partial(\mathcal{L}_f(T_1))}{\partial x}g(x) = \frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right)g(x).$$

By (1.74) we have

$$\mathcal{L}_g(\mathcal{L}_f(T_1)) = \frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right)g(x) = f^T\frac{\partial^2 T_1}{\partial x^2}g(x) + \frac{\partial T_1}{\partial x}\frac{\partial f}{\partial x}g(x). \tag{1.75}$$

It then follows that

$$\begin{aligned}
\mathcal{L}_f(\mathcal{L}_g(T_1)) - \mathcal{L}_g(\mathcal{L}_f(T_1)) &= \left(g^T\frac{\partial^2 T_1}{\partial x^2}f(x) + \frac{\partial T_1}{\partial x}\frac{\partial g}{\partial x}f(x)\right) - \left(f^T\frac{\partial^2 T_1}{\partial x^2}g(x) + \frac{\partial T_1}{\partial x}\frac{\partial f}{\partial x}g(x)\right) \\
&= \frac{\partial T_1}{\partial x}\left(\frac{\partial g}{\partial x}f(x) - \frac{\partial f}{\partial x}g(x)\right) \\
&= \mathcal{L}_{[f,g]}(T_1).
\end{aligned}$$

That is

$$\mathcal{L}_{[f,g]}(T_1) = \mathcal{L}_f(\mathcal{L}_g(T_1)) - \mathcal{L}_g(\mathcal{L}_f(T_1)). \tag{1.76}$$

∎

Repeated Lie Brackets

Definition 3 *Repeated Lie Brackets*
Define

$$\begin{aligned} ad_f^0 g &\triangleq g \\ ad_f^1 g &\triangleq [f, g] \\ ad_f^2 g &= [f, [f, g]] \\ &\vdots \\ ad_f^k g &= [f, ad_f^{k-1} g]. \end{aligned}$$

Example 8 *Linear Systems*
Let $f(x) = Ax, g(x) = b$ so that $\dfrac{dx}{dt} = f(x) + g(x)u = Ax + bu$ is a linear system. Then

$$\begin{aligned} ad_f^0 g &= b, \\ ad_f^1 g &\triangleq [Ax, b] = \frac{\partial b}{\partial x} Ax - \frac{\partial (Ax)}{\partial x} b = -Ab \\ ad_f^2 g &\triangleq [Ax, ad_f^1 g] = \frac{\partial ad_f^1 g}{\partial x} Ax - \frac{\partial (Ax)}{\partial x} ad_f^1 g = A^2 b \end{aligned}$$

or, in general,

$$ad_f^k g = (-1)^k A^k b.$$

Example 9 *Lie Brackets and Steering an Automobile* [11]
Figure 1.11 represents a three wheeled vehicle with front steering. The back wheel is powered and the axle of the front two wheels can be turned. We take the model to be

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \sin(x_3) \\ \cos(x_3) \\ 0 \end{bmatrix}}_{g_1(x)} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g_2} u_2$$

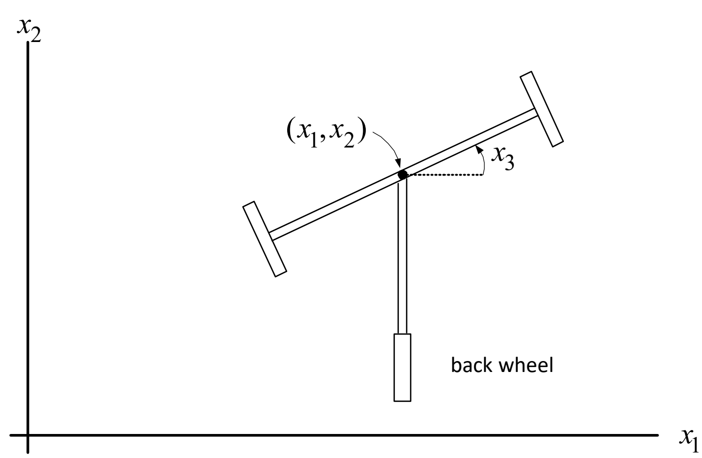


FIGURE 1.11. Three wheeled vehicle with front steering.

Suppose the system starts at (x_{01}, x_{02}, x_{03}) and consider the sequence of inputs u_1, u_2 given by

t	u_1	u_2	
$[0, h]$	1	0	rolling
$(h, 2h]$	0	1	rotation
$(2h, 3h]$	-1	0	rolling
$(3h, 4h]$	0	-1	rotation

We have

$$\begin{aligned}
\phi_h^{g_1}(x_0) &= \begin{bmatrix} x_{01} + h\sin(x_{03}) \\ x_{02} + h\cos(x_{03}) \\ x_{03} \end{bmatrix} \\
\phi_h^{g_2}(x_0') &= \begin{bmatrix} x_{01}' \\ x_{02}' \\ x_{03}' + h \end{bmatrix} \\
\phi_{-h}^{g_1}(x_0'') &= \begin{bmatrix} x_{01}'' - h\sin(x_{03}'') \\ x_{02}'' - h\cos(x_{03}'') \\ x_{03}'' \end{bmatrix} \\
\phi_{-h}^{g_2}(x_0''') &= \begin{bmatrix} x_{01}''' \\ x_{02}''' \\ x_{03}''' - h \end{bmatrix}
\end{aligned}$$

At time $t = 4h$ we have

$$\begin{aligned}
x(4h) = \phi_{-h}^{g_2}(\phi_{-h}^{g_1}(\phi_h^{g_2}(\phi_h^{g_1}(x_0)))) &= \begin{bmatrix} x_{01}''' \\ x_{02}''' \\ x_{03}''' - h \end{bmatrix}_{|x_0'''=\phi_{-h}^{g_1}(x_0''))} \\
&= \begin{bmatrix} x_{01}'' - h\sin(x_{03}'') \\ x_{02}'' - h\cos(x_{03}'') \\ x_{03}'' - h \end{bmatrix}_{|x_0''=\phi_h^{g_2}(x_0'))} \\
&= \begin{bmatrix} x_{01}' - h\sin(x_{03}' + h) \\ x_{02}' - h\cos(x_{03}' + h) \\ x_{03}' \end{bmatrix}_{|x_0'=\phi_h^{g_1}(x_0))} \\
&= \begin{bmatrix} x_{01} + h\sin(x_{03}) - h\sin(x_{03} + h) \\ x_{02} + h\cos(x_{03}) - h\cos(x_{03} + h) \\ x_{03} \end{bmatrix}.
\end{aligned}$$

For h small we have

$$\begin{aligned}
\sin(x_{03} + h) &\approx \sin(x_{03}) + \cos(x_{03})h \\
\cos(x_{03} + h) &\approx \cos(x_{03}) - \sin(x_{03})h
\end{aligned}$$

so we approximate $x(4h)$ by

$$\begin{aligned}
x(4h) = \begin{bmatrix} x_{01} + h\sin(x_{03}) - h\sin(x_{03} + h) \\ x_{02} + h\cos(x_{03}) - h\cos(x_{03} + h) \\ x_{03} \end{bmatrix} &\approx \begin{bmatrix} x_{01} - h^2\cos(x_{03}) \\ x_{02} + h^2\sin(x_{03}) \\ x_{03} \end{bmatrix} \\
&= \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} - h^2 \underbrace{\begin{bmatrix} -\cos(x_{03}) \\ \sin(x_{03}) \\ 0 \end{bmatrix}}_{[g_1, g_2]}.
\end{aligned}$$

Note that the vector

$$\begin{bmatrix} -\cos(x_{03}) \\ \sin(x_{03}) \\ 0 \end{bmatrix} = [g_1, g_2]$$

and is orthogonal to

$$\begin{bmatrix} \sin(x_{03}) \\ \cos(x_{03}) \\ 0 \end{bmatrix} = g_1(x_0).$$

At x_0 the vehicle is going in the direction $g_1(x_0)$ and by the control scheme just shown with h small we can move in a direction perpendicular to the vehicle (parallel parking).

Now consider a different approach to this calculation. For $0 \leq t \leq h$ we have

$$\begin{aligned} \frac{dx}{dt} &= g_1(x(t)) \\ \frac{d^2x}{dt^2} &= \frac{\partial g_1}{\partial x}\frac{dx}{dt} = \frac{\partial g_1}{\partial x} g_1. \end{aligned}$$

For $h < t \leq 2h$ we have

$$\begin{aligned} \frac{dx}{dt} &= g_2(x(t)) \\ \frac{d^2x}{dt^2} &= \frac{\partial g_2}{\partial x}\frac{dx}{dt} = \frac{\partial g_2}{\partial x} g_2. \end{aligned}$$

For $2h < t \leq 3h$ we have

$$\begin{aligned} \frac{dx}{dt} &= -g_1(x(t)) \\ \frac{d^2x}{dt^2} &= -\frac{\partial g_1}{\partial x}\frac{dx}{dt} = \frac{\partial g_1}{\partial x} g_1. \end{aligned}$$

For $3h < t \leq 4h$ we have

$$\begin{aligned} \frac{dx}{dt} &= -g_2(x(t)) \\ \frac{d^2x}{dt^2} &= -\frac{\partial g_2}{\partial x}\frac{dx}{dt} = \frac{\partial g_2}{\partial x} g_2. \end{aligned}$$

Then, only keeping terms in h^2 or lower we have.

$$x(h) \approx x(0) + h\frac{dx}{dt}_{|t=0} + \frac{1}{2}h^2\frac{d^2x}{dt^2}_{|t=0} = x_0 + hg_1(x_0) + \frac{1}{2}h^2\frac{\partial g_1}{\partial x}_{|x_0} g_1(x_0).$$

Again, keeping only terms in h^2 or lower we write

$$\begin{aligned}
x(2h) &\approx x(h) + h\frac{dx}{dt}_{|t=h} + \frac{1}{2}h^2\frac{d^2x}{dt^2}_{|t=h} \\
&= x(h) + hg_2(x(h)) + \frac{1}{2}h^2\frac{\partial g_2}{\partial x}_{|x(h)} g_2(x(h)) \\
&\approx x_0 + hg_1(x_0) + \frac{1}{2}h^2\frac{\partial g_1}{\partial x}_{|x_0} g_1(x_0) + hg_2(x_0 + hg_1(x_0)) + \\
&\quad \frac{1}{2}h^2\frac{\partial g_2}{\partial x}_{|(x_0+hg_1(x_0))}\left(g_2(x_0) + h\frac{\partial g_2}{\partial x}_{|x_0} g_2(x_0)\right) \\
&\approx x_0 + hg_1(x_0) + \frac{1}{2}h^2\frac{\partial g_1}{\partial x}_{|x_0} g_1(x_0) + hg_2(x_0) + h^2\frac{\partial g_2}{\partial x}g_1(x_0)) + \frac{1}{2}h^2\frac{\partial g_2}{\partial x}_{|x_0} g_2(x_0) \\
&\approx x_0 + h\Big(g_1(x_0) + g_2(x_0)\Big) + \frac{1}{2}h^2\frac{\partial g_1(x_0)}{\partial x}g_1(x_0) + h^2\frac{\partial g_2}{\partial x}g_1(x_0)) + \frac{1}{2}h^2\frac{\partial g_2(x_0)}{\partial x}g_2(x_0) \\
&= x_0 + h\Big(g_1(x_0) + g_2(x_0)\Big) + h^2\left(\frac{1}{2}\frac{\partial g_1(x_0)}{\partial x}g_1(x_0) + \frac{\partial g_2}{\partial x}g_1(x_0)) + \frac{1}{2}\frac{\partial g_2(x_0)}{\partial x}g_2(x_0)\right)
\end{aligned}$$

Next we compute

$$\begin{aligned}
x(3h) &\approx x(2h) - hg_1(x(2h)) + \frac{1}{2}h^2\frac{\partial g_1}{\partial x}_{|x(2h)} g_1(x(2h)) \\
&\approx x_0 + h\Big(g_1(x_0) + g_2(x_0)\Big) + h^2\left(\frac{1}{2}\frac{\partial g_1(x_0)}{\partial x}g_1(x_0) + \frac{\partial g_2}{\partial x}g_1(x_0)) + \frac{1}{2}\frac{\partial g_2(x_0)}{\partial x}g_2(x_0)\right) \\
&\quad -hg_1\left(x_0 + h\Big(g_1(x_0) + g_2(x_0)\Big)\right) + \frac{1}{2}h^2\frac{\partial g_1}{\partial x}_{|x_0} g_1(x_0) \\
&\approx x_0 + h\Big(g_1(x_0) + g_2(x_0)\Big) + h^2\left(\frac{1}{2}\frac{\partial g_1(x_0)}{\partial x}g_1(x_0) + \frac{\partial g_2}{\partial x}g_1(x_0)) + \frac{1}{2}\frac{\partial g_2(x_0)}{\partial x}g_2(x_0)\right) \\
&\quad -h\left(g_1(x_0) + h\frac{\partial g_1(x_0)}{\partial x}\Big(g_1(x_0) + g_2(x_0)\Big)\right) + \frac{1}{2}h^2\frac{\partial g_1}{\partial x}_{|x_0} g_1(x_0) \\
&= x_0 + hg_2(x_0) + h^2\left(\frac{\partial g_2}{\partial x}g_1(x_0) - \frac{\partial g_1}{\partial x}g_2(x_0) + \frac{1}{2}\frac{\partial g_2(x_0)}{\partial x}g_2(x_0)\right).
\end{aligned}$$

Finally

$$\begin{aligned}
x(4h) &\approx x(3h) - hg_2(x(3h)) + \frac{1}{2}h^2\frac{\partial g_2}{\partial x}_{|x(3h)} g_2(3h) \\
&\approx x_0 + hg_2(x_0) + h^2\left(\frac{\partial g_2}{\partial x}g_1(x_0) - \frac{\partial g_1}{\partial x}g_2(x_0) + \frac{1}{2}\frac{\partial g_2(x_0)}{\partial x}g_2(x_0)\right) \\
&\quad -hg_2(x_0 + hg_2(x_0)) + \frac{1}{2}h^2\frac{\partial g_2}{\partial x}_{|x_0} g_2(x_0) \\
&= x_0 + hg_2(x_0) + h^2\left(\frac{\partial g_2}{\partial x}g_1(x_0) - \frac{\partial g_1}{\partial x}g_2(x_0) + \frac{1}{2}\frac{\partial g_2(x_0)}{\partial x}g_2(x_0)\right) \\
&\quad -hg_2(x_0) - h^2\frac{\partial g_2}{\partial x}g_2(x_0)) + \frac{1}{2}h^2\frac{\partial g_2}{\partial x}_{|x_0} g_2(x_0) \\
&= x_0 + h^2\left(\frac{\partial g_2}{\partial x}g_1(x_0) - \frac{\partial g_1}{\partial x}g_2(x_0)\right) \\
&= x_0 + h^2[g_1, g_2].
\end{aligned}$$

This calculation shows that we can steer the car in the direction $[g_1, g_2]$ by successively going in the directions $g_1, g_2, -g_1, -g_2$. In this example it turns out that $[g_1, g_2]$ is orthogonal to the direction the wheels are pointing.

Remark 3 If the system was given by

$$\frac{dx}{dt} = f + ug$$

then we can look at this

$$\frac{dx}{dt} = fu_1 + gu_2$$

where $u_1 \equiv 1$. Although this cannot go in the negative direction of f, this system can be "steered" in the directions determined by the Lie bracket of f and g. In fact, with $f, g \in \mathbb{R}^n$ smooth vector fields let

$$\mathcal{C} \triangleq \left[\begin{array}{ccccc} g & ad_f g & ad_f^2 g & \cdots & ad_f^{n-1} g \end{array} \right].$$

If $rank[C] = n$ at any $x_0 \in \mathbb{R}^n$ then it is possible to steer the state to any point in a neighborhood of x_0.

Exercise 3 For notation clarity denote g_2 by $g^{(2)}$. Show that

$$\begin{aligned} \frac{\partial g^{(2)}}{\partial x}_{|(x_0+hg_1(x_0))} &= \begin{bmatrix} \frac{\partial g_1^{(2)}}{\partial x_1} & \frac{\partial g_1^{(2)}}{\partial x_2} & \frac{\partial g_1^{(2)}}{\partial x_3} \\ \frac{\partial g_2^{(2)}}{\partial x_1} & \frac{\partial g_2^{(2)}}{\partial x_2} & \frac{\partial g_2^{(2)}}{\partial x_3} \\ \frac{\partial g_3^{(2)}}{\partial x_1} & \frac{\partial g_3^{(2)}}{\partial x_2} & \frac{\partial g_3^{(2)}}{\partial x_3} \end{bmatrix}_{|(x_0+hg_1(x_0))} \\ &\approx \begin{bmatrix} \frac{\partial g_1^{(2)}}{\partial x_1} & \frac{\partial g_1^{(2)}}{\partial x_2} & \frac{\partial g_1^{(2)}}{\partial x_3} \\ \frac{\partial g_2^{(2)}}{\partial x_1} & \frac{\partial g_2^{(2)}}{\partial x_2} & \frac{\partial g_2^{(2)}}{\partial x_3} \\ \frac{\partial g_3^{(2)}}{\partial x_1} & \frac{\partial g_3^{(2)}}{\partial x_2} & \frac{\partial g_3^{(2)}}{\partial x_3} \end{bmatrix}_{|x_0} + \begin{bmatrix} \left(\nabla \frac{\partial g_1^{(2)}}{\partial x_1}\right) hg_1(x_0) & \left(\nabla \frac{\partial g_1^{(2)}}{\partial x_2}\right) hg_1(x_0) & \left(\nabla \frac{\partial g_1^{(2)}}{\partial x_3}\right) hg_1(x_0) \\ \left(\nabla \frac{\partial g_2^{(2)}}{\partial x_1}\right) hg_1(x_0) & \left(\nabla \frac{\partial g_2^{(2)}}{\partial x_2}\right) hg_1(x_0) & \left(\nabla \frac{\partial g_2^{(2)}}{\partial x_3}\right) hg_1(x_0) \\ \left(\nabla \frac{\partial g_3^{(2)}}{\partial x_1}\right) hg_1(x_0) & \left(\nabla \frac{\partial g_3^{(2)}}{\partial x_2}\right) hg_1(x_0) & \left(\nabla \frac{\partial g_3^{(2)}}{\partial x_3}\right) hg_1(x_0) \end{bmatrix} \end{aligned}$$

1.10 Problems

Problem 1 *Nonlinear Transformations*

(a) Let a nonlinear system be given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ g_2(x_1,x_2) \end{bmatrix} u.$$

Using the statespace transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} T_1(x_1,x_2) \\ T_2(x_1,x_2) \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ \mathcal{L}_f(T_1) \end{bmatrix} = \begin{bmatrix} x_1 \\ \mathcal{L}_f(x_1) \end{bmatrix}$$

and appropriate feedback the system can be put in the linear form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(b) Let a nonlinear system be given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_3(x_1,x_2,x_3) \end{bmatrix} u.$$

Using the statespace transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} T_1(x_1,x_2,x_3) \\ T_2(x_1,x_2,x_3) \\ T_3(x_1,x_2,x_3) \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ \mathcal{L}_f(T_1) \\ \mathcal{L}_f^2(T_1) \end{bmatrix} = \begin{bmatrix} x_1 \\ \mathcal{L}_f(x_1) \\ \mathcal{L}_f^2(x_1) \end{bmatrix}$$

and appropriate feedback the system can be put in the linear form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

(c) Let a nonlinear system be given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3,x_4) \\ f_3(x_1,x_2,x_3,x_4) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_4(x_1,x_2,x_3,x_4) \end{bmatrix} u.$$

Using the statespace transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} T_1(x_1,x_2,x_3,x_4) \\ T_2(x_1,x_2,x_3,x_4) \\ T_3(x_1,x_2,x_3,x_4) \\ T_4(x_1,x_2,x_3,x_4) \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ \mathcal{L}_f(T_1) \\ \mathcal{L}_f^2(T_1) \\ \mathcal{L}_f^3(T_1) \end{bmatrix} = \begin{bmatrix} x_1 \\ \mathcal{L}_f(x_1) \\ \mathcal{L}_f^2(x_1) \\ \mathcal{L}_f^3(x_1) \end{bmatrix}$$

and appropriate feedback the system can be put in the linear form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Problem 2 *Tank Reactor* [12]

A dimensionless pair of equations for the model of a constant volume, non-adiabatic (occurring with loss or gain of heat), stirred tank reactor with first irreversible kinetics is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + c_1(1-x_1)e^{\frac{x_2}{1+x_2/\gamma}} \\ -x_2 + c_1c_2(1-x_1)e^{\frac{x_2}{1+x_2/\gamma}} \end{bmatrix} + \begin{bmatrix} 0 \\ -(x_2 - x_c) \end{bmatrix} u$$

with $c_1 = 0.05, c_2 = 8$, $x_c = 0$, and γ is a constant.

(a) Show that the equilibrium point

$$x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 3 \end{bmatrix}$$

implies $e^{\frac{x_2}{1+x_2/\gamma}} = 20$ and $u_0 = 1/3$.

(b) Rewrite the model equations in terms of

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix} \quad \text{and} \quad w = -u_0.$$

(c) Use part (a) of Problem 1 to find a transformation

$$z^* = T(z)$$

so that in the new coordinates along with feedback of the form

$$u = f^*(z) + g^*(z)w$$

the model in part (b) becomes

$$\frac{d}{dt}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Explicitly given $a(z)$ and $\beta(z)$. Also, with the state feedback

$$w = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix}$$

find the values of the gains k_1, k_2 such that the closed-loop poles of the z^* system are $-3, -3$.

Problem 3 *Nonlinear Regulator for a Synchronous Generator* [13]

A nonlinear statespace model of a synchronous generator connected to an infinite bus is given in [13] as

$$\frac{d}{dt}\begin{bmatrix} \delta \\ \omega \\ \psi_f \\ \psi_A \\ \psi_B \end{bmatrix} = \underbrace{\begin{bmatrix} \omega - \omega_s \\ c_{mo} - K_2\omega\psi_f\sin(\delta) - K_3\omega\psi_A\sin(\delta) - K_4\omega\psi_B\sin(\delta) + K_5\sin(\delta)\cos(\delta) \\ \nu_{f0} - K_8\omega\psi_f + K_9\omega\psi_A + K_{10}\cos(\delta) \\ K_{11}\omega\psi_f - K_{12}\omega\psi_A + K_{13}\cos(\delta) \\ -K_{14}\omega\psi_B - K_{15}\sin(\delta) \end{bmatrix}}_{f(\delta,\omega,\psi_f,\psi_A,\psi_B)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{g_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{g_2} u_2$$

where δ is the rotor angle referred to the infinite bus, ω is the rotor angular speed, ψ_f is the field flux linkage, ψ_A is the direct axis flux linkage, ψ_B is the quadrature axis flux linkage, ω_s is the constant synchronous angular velocity, c_m is the rotor angular acceleration produced by the input torque, c_{mo} is the reference input angular acceleration, ν_f is the field excitation voltage, ν_{f0} is the constant reference field excitation voltage, $u_1 = c_m - c_{mo}$, and $u_2 = \nu_f - \nu_{f0}$.

Let $\begin{bmatrix} \delta_0 & \omega_0 & \psi_{f0} & \psi_{A0} & \psi_{B0} \end{bmatrix}^T$ denote a constant stable operating point with $u_1 = 0, u_2 = 0$. All the parameters $K_1, ..., K_{15}$ are positive constants. With

$$\begin{aligned} x &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T \triangleq \begin{bmatrix} \delta & \omega & \psi_f & \psi_A & \psi_B \end{bmatrix}^T \\ u &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} c_m - c_{mo} & \nu_f - \nu_{f0} \end{bmatrix} \end{aligned}$$

the model of the synchronous generator has the compact form

$$\frac{d}{dt}x = f(x) + g(x)u.$$

(a) Define a nonlinear transformation given by

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \\ T_4(x) \\ T_5(x) \end{bmatrix} = \begin{bmatrix} \ln(x_5)/K_{14} + x_1 + c_1 \\ -\omega_s - (K_{15}/K_{14})\sin(x_1)/x_5 \\ -(K_{15}/K_{14})(x_2 - \omega_s)\cos(x_1)/x_5 - K_{15}x_2\sin(x_1)/x_5 - (K_{15}^2/K_{14})\sin^2(x_1)/x_5^2 \\ x_4 + c_2 \\ K_{11}x_2x_3 - K_{12}x_2x_4 + K_{13}\cos(x_1) \end{bmatrix}.$$

The system equations in this new coordinate system have the form

$$\frac{d}{dt}x^* = f^*(x)_{|x=T^{-t}(x^*)} + g_1^*(x)_{|x=T^{-t}(x^*)}u_1 + g_2^*(x)_{|x=T^{-t}(x^*)}u_2.$$

Compute $f^*(x), g_1^*(x)$, and $g_2^*(x)$.

(b) Show that the nonlinear feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{g_1^*}(T_3) & \mathcal{L}_{g_2^*}(T_3) \\ \mathcal{L}_{g_1^*}(T_5) & \mathcal{L}_{g_2^*}(T_5) \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathcal{L}_{f^*}(T_3) \\ \mathcal{L}_{f^*}(T_5) \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right)$$

results in the linear model

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w_2.$$

(c) What condition is needed for the feedback in part (b) to be valid. When does the condition hold?

Problem 4 *Observer for a Predator-Prey Model* [14]
A nonlinear differential equation model for a predator-prey system is given by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} \alpha x_1 - \beta x_1 x_2 \\ \gamma x_1 x_2 - \lambda x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ -x_2 \end{bmatrix}}_{g(x)} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

$x_1 \geq 0$ is the prey population and $x_2 \geq 0$ is the predator population. The constants, $\alpha > 0$ and $\gamma > 0$ are the birth rates of prey and predator populations, respectively while the constants $\beta > 0$ and $\lambda > 0$ are the death rates of the prey and predator populations, respectively. The input $u \geq 0$ represents the rate at which humans can decimate the predator population (e.g., by hunting). The output y is the predator population while the prey population is considered too big to measure. Consider the nonlinear transformation

$$\begin{aligned}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} &= \begin{bmatrix} T_1(x_1,x_2) \\ T_2(x_1,x_2) \end{bmatrix} \triangleq \begin{bmatrix} \gamma x_1 + \beta x_2 - \alpha \ln(x_2) + c_1 \\ \ln(x_2) + c_2 \end{bmatrix} \\ y^* &\triangleq \ln(y) = \ln(x_2)\end{aligned}$$

where c_1, c_2 can be any arbitrary constants.

(a) Find the system equation in the x^* coordinates with $c_1 = c_2 = 0$.

(b) Design an observer for x_1 with linear error dynamics that places the poles of the error system at $-2, -2$.

(c) For any given reference input u_0 find all equilibrium points x_0, that is, the solutions to

$$0 = f(x_0) + g(x_0)u_0.$$

Explain why the only physically interesting equilibrium point is

$$\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} (\lambda + u_0)/\gamma \\ \alpha/\beta \end{bmatrix}.$$

(d) With

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \triangleq \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix}, \quad w \triangleq u - u_0, \quad \text{and} \quad \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} (\lambda + u_0)/\gamma \\ \alpha/\beta \end{bmatrix}$$

find the system equations in terms of z_1, z_2, and w. That is, show the equations can be written in the form

$$\frac{dz}{dt} = f'(z) + g'(z)w.$$

Explicitly give $f'(z)$ and $g'(z)$.

(e) With

$$\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} \triangleq \begin{bmatrix} z_1 \\ \mathcal{L}_{f'}(z_1) \end{bmatrix}$$

find the statespace representation in the z^* coordinates. Choose feedback of the form $w = f^*(z) + g^*(z)u$ so that the system dynamics in z^* are linear given by

$$\frac{d}{dt}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w.$$

With the state feedback

$$w = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix}$$

find the values of the gains k_1, k_2 such that the closed-loop poles of the z^* system are $-1, -1$.

(f) Draw a block diagram illustrating the interconnection of the observer, controller, and predator-prey model.

Problem 5 *Linear System Not in Observer Canonical Form*
Let

$$\begin{aligned}\frac{dx}{dt} &= Ax + bu, \quad A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3 \\ y &= cx, \quad c \in \mathbb{R}^{1\times 3}\end{aligned}$$

with $\det(sI - A) = s^3 + a_2 s^2 + a_1 s + a_0$ the characteristic polynomial of A and the observability matrix given by

$$\mathcal{O} \triangleq \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix}.$$

Show that if this system is observable, i.e., $\mathcal{O}$ has full rank, then it can be transformed to observer canonical form.

Hint. Define q as the vector that satisfies

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathcal{O}q.$$

That is, q is the last column of $\mathcal{O}^{-1}$. Show that the transformation $x^* = Tx$ with $T \triangleq \begin{bmatrix} q & Aq & A^2q \end{bmatrix}^{-1}$ takes the system to observer canonical form. You need to show

$$TAT^{-1} = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}, cT^{-1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

or

$$\begin{aligned} A\begin{bmatrix} q & Aq & A^2q \end{bmatrix} &= \begin{bmatrix} q & Aq & A^2q \end{bmatrix}\begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \\ c\begin{bmatrix} q & Aq & A^2q \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Problem 6 *PM Synchronous Motor with a Salient Rotor* [3]

Some PM synchronous machines have rotors constructed so that the air gap is not uniform and are referred to as a *salient* rotors. In this case the flux linkages (total flux) in the stator phases due to the stator currents depends on the rotor position. Specifically, the stator flux linkages are given by

$$\begin{aligned} \lambda_{Sa} &= L_S i_{Sa} + K_m \cos(n_p\theta) + L_g\left(i_{Sa}\cos(2n_p\theta) + i_{Sb}\sin(2n_p\theta)\right) \\ \lambda_{Sb} &= L_S i_{Sb} + K_m \sin(n_p\theta) + L_g\left(i_{Sa}\sin(2n_p\theta) - i_{Sb}\cos(2n_p\theta)\right) \end{aligned}$$

where the third term in each expression is due to the nonuniform air gap (salient rotor). Notice that these terms have a period of π/n_p while the flux linkage due to the permanent magnet rotor has a period of $2\pi/n_p$.

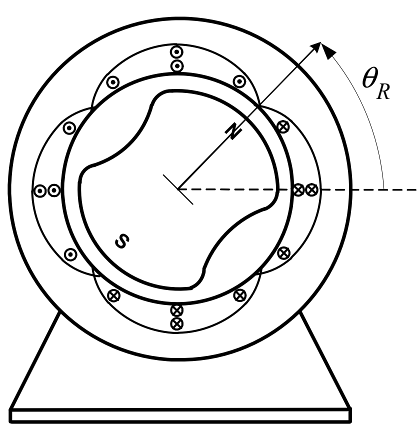


FIGURE 1.12. PM sychronous motor with a salient rotor ($n_p = 1$).

By Faraday's law we have

$$\begin{aligned}\frac{d\lambda_{Sa}}{dt} + Ri_{Sa} &= u_{Sa}\\ \frac{d\lambda_{Sb}}{dt} + Ri_{Sb} &= u_{Sb}.\end{aligned}$$

It follows from these equations that the electrical power $i_{Sa}u_{Sa} + i_{Sb}u_{Sb}$ into the motor is equal to

$$i_{Sa}u_{Sa} + i_{Sb}u_{Sb} = i_{Sa}\frac{d\lambda_{Sa}}{dt} + i_{Sb}\frac{d\lambda_{Sb}}{dt} + R_S(i_{Sa}^2 + i_{Sb}^2).$$

First compute

$$\begin{aligned}\lambda_{Sa} &= L_S i_{Sa} + K_m\cos(n_p\theta) + L_g\left(i_{Sa}\cos(2n_p\theta) + i_{Sb}\sin(2n_p\theta)\right)\\ \lambda_{Sb} &= L_S i_{Sb} + K_m\sin(n_p\theta) + L_g\left(i_{Sa}\sin(2n_p\theta) - i_{Sb}\cos(2n_p\theta)\right)\end{aligned}$$

$$\begin{aligned}&i_{Sa}\frac{d\lambda_{Sa}}{dt} + i_{Sb}\frac{d\lambda_{Sb}}{dt}\\ &= \frac{d}{dt}\frac{1}{2}L_S(i_{Sa}^2 + i_{Sb}^2) + i_{Sa}K_m\frac{d}{dt}\cos(n_p\theta) + i_{Sb}K_m\frac{d}{dt}\sin(n_p\theta) +\\ &\quad i_{Sa}L_g\left(\frac{di_{Sa}}{dt}\cos(2n_p\theta) + \frac{di_{Sb}}{dt}\sin(2n_p\theta)\right) + i_{Sb}L_g\left(\frac{di_{Sa}}{dt}\sin(2n_p\theta) - \frac{di_{Sb}}{dt}\cos(2n_p\theta)\right) +\\ &\quad i_{Sa}\left(L_g\left(i_{Sa}\frac{d}{dt}\cos(2n_p\theta) + i_{Sb}\frac{d}{dt}\sin(2n_p\theta)\right)\right) + i_{Sb}\left(L_g\left(i_{Sa}\frac{d}{dt}\sin(2n_p\theta) - i_{Sb}\frac{d}{dt}\cos(2n_p\theta)\right)\right)\end{aligned}$$

Continuing this becomes

$$\begin{aligned}&i_{Sa}\frac{d\lambda_{Sa}}{dt} + i_{Sb}\frac{d\lambda_{Sb}}{dt}\\ &= \frac{d}{dt}\frac{1}{2}L_S(i_{Sa}^2 + i_{Sb}^2) - i_{Sa}K_m n_p\omega\sin(n_p\theta) + i_{Sb}K_m n_p\omega\cos(n_p\theta)\\ &\quad i_{Sa}L_g\left(\frac{di_{Sa}}{dt}\cos(2n_p\theta) + \frac{di_{Sb}}{dt}\sin(2n_p\theta)\right) + i_{Sb}L_g\left(\frac{di_{Sa}}{dt}\sin(2n_p\theta) - \frac{di_{Sb}}{dt}\cos(2n_p\theta)\right)\\ &\quad -L_g i_{Sa}^2 2n_p\omega\sin(2n_p\theta) + L_g i_{Sa}i_{Sb}2n_p\omega\cos(2n_p\theta) + L_g i_{Sb}i_{Sa}2n_p\omega\cos(2n_p\theta) + L_g i_{Sb}^2 2n_p\omega\sin(2n_p\theta)\end{aligned}$$

Combining the above expression the power into the motor may now be written as

$$\begin{aligned}i_{Sa}u_{Sa} + i_{Sb}u_{Sb} &= R_S(i_{Sa}^2 + i_{Sb}^2) + i_{Sa}\frac{d\lambda_{Sa}}{dt} + i_{Sb}\frac{d\lambda_{Sb}}{dt}\\ &= R_S(i_{Sa}^2 + i_{Sb}^2) + \frac{d}{dt}\left(\frac{1}{2}L_S(i_{Sa}^2 + i_{Sb}^2)\right) + \omega\left(n_pK_m(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta))\right) +\\ &\quad i_{Sa}L_g\left(\frac{di_{Sa}}{dt}\cos(2n_p\theta) + \frac{di_{Sb}}{dt}\sin(2n_p\theta)\right) + i_{Sb}L_g\left(\frac{di_{Sa}}{dt}\sin(2n_p\theta) - \frac{di_{Sb}}{dt}\cos(2n_p\theta)\right)\\ &\quad \omega 2n_pL_g\left((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) + 2i_{Sa}i_{Sb}\cos(2n_p\theta)\right).\end{aligned}$$

Again, $i_{Sa}u_{Sa} + i_{Sb}u_{Sb}$ is the electrical power supplied to the motor. Part of this power changes the stored magnetic energy according to

$$\frac{d}{dt}\left(\frac{1}{2}L_S(i_{Sa}^2 + i_{Sb}^2)\right) + i_{Sa}L_g\left(\frac{di_{Sa}}{dt}\cos(2n_p\theta) + \frac{di_{Sb}}{dt}\sin(2n_p\theta)\right) + i_{Sb}L_g\left(\frac{di_{Sa}}{dt}\sin(2n_p\theta) - \frac{di_{Sb}}{dt}\cos(2n_p\theta)\right)$$

and part of this power goes into Ohmic losses according to

$$R_S(i_{Sa}^2 + i_{Sb}^2).$$

The rest of the power given by

$$\omega\left(n_p K_m(-i_{Sa}\sin(n_p\theta)\ + i_{Sb}\cos(n_p\theta))\right) + \omega 2n_p L_g\left((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) +\ 2i_{Sa}i_{Sb}\cos(2n_p\theta)\right)$$

is the power absorbed by the windings. By conservation of energy, this power reappears as the mechanical power $\tau\omega$. That is, conservation of energy requires

$$\omega\left(n_p K_m(-i_{Sa}\sin(n_p\theta)\ + i_{Sb}\cos(n_p\theta))\right) + \omega 2n_p L_g\left((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) +\ 2i_{Sa}i_{Sb}\cos(2n_p\theta)\right) = \tau\omega$$

or

$$\tau = n_p K_m\Big(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)\Big) + 2n_p L_g\Big((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) + 2i_{Sa}i_{Sb}\cos(2n_p\theta)\Big).$$

The complete set of equations for this PM salient machine are then

$$\begin{aligned}
\frac{d\lambda_{Sa}}{dt} + Ri_{Sa} &= u_{Sa}\\
\frac{d\lambda_{Sb}}{dt} + Ri_{Sb} &= u_{Sb}\\
J\frac{d\omega}{dt} &= n_p K_m\Big(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)\Big) +\\
&\quad 2n_p L_g\Big((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) + 2i_{Sa}i_{Sb}\cos(2n_p\theta)\Big) - \tau_L
\end{aligned}$$

with

$$\begin{aligned}
\lambda_{Sa} &= L_S i_{Sa} + K_m\cos(n_p\theta) + L_g\left(i_{Sa}\cos(2n_p\theta) + i_{Sb}\sin(2n_p\theta)\right)\\
\lambda_{Sb} &= L_S i_{Sb} + K_m\sin(n_p\theta) + L_g\left(i_{Sa}\sin(2n_p\theta) - i_{Sb}\cos(2n_p\theta)\right).
\end{aligned}$$

Transforming these equations in the dq frame greatly simplifies them from which a feedback controller can be designed.

(a) *Fluxes in the dq coordinate system.*

With

$$\begin{bmatrix}\lambda_d\\ \lambda_q\end{bmatrix} \triangleq \begin{bmatrix}\cos(n_p\theta) & \sin(n_p\theta)\\ -\sin(n_p\theta) & \cos(n_p\theta)\end{bmatrix}\begin{bmatrix}\lambda_{Sa}\\ \lambda_{Sb}\end{bmatrix}, \begin{bmatrix}i_d\\ i_q\end{bmatrix} \triangleq \begin{bmatrix}\cos(n_p\theta) & \sin(n_p\theta)\\ -\sin(n_p\theta) & \cos(n_p\theta)\end{bmatrix}\begin{bmatrix}i_{Sa}\\ i_{Sb}\end{bmatrix}$$

show that

$$\begin{bmatrix}\lambda_{Sa}\\ \lambda_{Sb}\end{bmatrix} = \begin{bmatrix}L_S & 0\\ 0 & L_S\end{bmatrix}\begin{bmatrix}i_{Sa}\\ i_{Sb}\end{bmatrix} + K_m\begin{bmatrix}\cos(n_p\theta)\\ \sin(n_p\theta)\end{bmatrix} + L_g\begin{bmatrix}\cos(2n_p\theta) & \sin(2n_p\theta)\\ \sin(2n_p\theta) & -\cos(2n_p\theta)\end{bmatrix}\begin{bmatrix}i_{Sa}\\ i_{Sb}\end{bmatrix}$$

are represented in the dq frame by

$$\begin{bmatrix}\lambda_d\\ \lambda_q\end{bmatrix} = \begin{bmatrix}L_S & 0\\ 0 & L_S\end{bmatrix}\begin{bmatrix}i_d\\ i_q\end{bmatrix} + \begin{bmatrix}K_m\\ 0\end{bmatrix} + L_g\begin{bmatrix}i_d\\ -i_q\end{bmatrix} = \begin{bmatrix}L_d i_d + K_m\\ L_q i_q\end{bmatrix}$$

where $L_d \triangleq L_S + L_g, L_q \triangleq L_S - L_g$.

(b) *Flux equations in the dq coordinate system*

Show that the flux equations

$$\begin{aligned} \frac{d\lambda_{Sa}}{dt} + Ri_{Sa} &= u_{Sa} \\ \frac{d\lambda_{Sb}}{dt} + Ri_{Sb} &= u_{Sb} \end{aligned}$$

in the dq system are given by

$$\frac{d}{dt}\begin{bmatrix} L_d i_d \\ L_q i_q \end{bmatrix} - n_p\omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} L_d i_d + K_m \\ L_q i_q \end{bmatrix} = -R\begin{bmatrix} i_d \\ i_q \end{bmatrix} + \begin{bmatrix} u_d \\ u_q \end{bmatrix}$$

where

$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix}\begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix}.$$

(c) *Torque Equation*

Show that the motor torque

$$\tau = n_p K_m\Big(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)\Big) + 2n_p L_g\Big((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) + 2i_{Sa}i_{Sb}\cos(2n_p\theta)\Big)$$

is given in the dq system by

$$\tau = 2n_p(L_d - L_q)i_d i_q + n_p K_m i_q.$$

(d) Equations of the salient PM synchronous machine in the dq system.

Show that the equations of the salient PM synchronous motor are given by

$$\begin{aligned} L_d\frac{di_d}{dt} &= -Ri_d + n_p\omega L_q i_q + u_d \\ L_q\frac{di_q}{dt} &= -Ri_q - n_p\omega L_d i_d - n_p\omega K_m + u_q \\ J\frac{d\omega}{dt} &= n_p\left(K_m i_q + 2(L_d - L_q)i_q i_d\right) \\ \frac{d\theta}{dt} &= \omega \end{aligned}$$

where

$$\begin{aligned} L_d &\triangleq L_S + L_g \\ L_q &\triangleq L_S - L_g. \end{aligned}$$

(e) Consider the change of coordinates

$$\begin{aligned} \theta &= \theta \\ \omega &= \omega \\ \alpha &= \frac{n_p}{J}\left(K_m i_q + 2(L_d - L_q)i_q i_d\right) \\ i_d &= i_d. \end{aligned}$$

Show that the motor model is then given by

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= \alpha \\ \frac{d\alpha}{dt} &= f_1(\omega, i_d, i_q) + 2L_g n_p u_d/L_d + n_p\left(K_m + 2L_g i_d\right)u_q/L_q \\ \frac{di_d}{dt} &= f_2(\omega, i_d, i_q) + u_d/L_d. \end{aligned}$$

Give the explicit expressions for f_1 and f_2. Show that the nonlinear feedback

$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} = \begin{bmatrix} \dfrac{2L_g n_p}{L_d} & \dfrac{n_p\,(K_m + 2L_g i_d)}{L_q} \\ \dfrac{n_p\,(K_m + 2L_g i_d)}{L_q} & 0 \end{bmatrix}^{-1} \begin{bmatrix} -f_1 + v_d \\ -f_2 + v_q \end{bmatrix}$$

results in a *linear* system. Under what conditions is the nonlinear feedback of part (e) singular? Is it a practical problem? Explain.

Problem 7 *Series Connected DC Motor*
With $x_1 = \theta, x_2 = \omega$, $x_3 = i$, and $u = V_S/L$ the equations describing the series connected DC motor are

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ c_2 x_3 - c_3 x_3 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}}_{p} \tau_L.$$

Using the transformation

$$\begin{aligned} x_1^* &= T_1(x) = x_1 \\ x_2^* &= T_2(x) = 2c_2 x_2 + c_3 x_2^2 + c_1 x_3^2 \\ x_3^* &= T_3(x) = x_2 \end{aligned}$$

give the equations of this system in the x^* coordinate system.
If $x_1 = \theta$ and $x_2 = \omega$ are measured, can an observer for $x_3 = i$ be constructed using the equations in the x^* coordinate system? If so, do so.

Problem 8 *Magnetic Levitation*
Recall the equations of the current command magnetic levitation system given by

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= g - \frac{C}{m}\frac{u^2}{x_1^2}. \end{aligned}$$

Note that this nonlinear system is not of the form $dx/dt = f(x) + g(x)u$ as it is not linear in the input u.

(a) Show that this system can be made linear by the appropriate choice of u. Design a state feedback controller to keep the steel ball at $x_1 = x_0$.

(b) Given that the position $x = x_1$ and coil current $u = i$ are measured, design an observer to estimate the velocity $v = x_2$.

Problem 9 *An Observer for Speed, Current, and Load Torque of a DC Motor*
Recall the model of the DC motor given as

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega + V_S \\ J\frac{d\omega}{dt} &= K_T i - f\omega - \tau_L \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

With $x_1 = i, x_2 = \omega, x_3 = \theta$, and $u = V_S$ we may write

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 \\ K_T/J & -f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{b} u + \begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix} \tau_L$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where the rotor angle is taken as the output, i.e., it is measured. The load torque affects the speed and so it must be included in the observer. With the load torque taken to be constant and setting $x_4 = \tau_L/J$ the system is now modeled by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 & 0 \\ K_T/J & -f/J & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_a}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_a} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}}_{c_a}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Design an observer that estimates $x_1 = i, x_2 = \omega$, and $x_4 = \tau/J$ based on the measurement $y = \theta$.

Problem 10 *Shunt Connected DC Motor*

A shunt connected DC motor has the field circuit and the armature circuit connected in parallel as illustrated in Figure 1.13. By connected in parallel is meant that the T_1 terminal of the armature is connected to the T_1' terminal of the field circuit and similarly for the T_2 and T_2'.

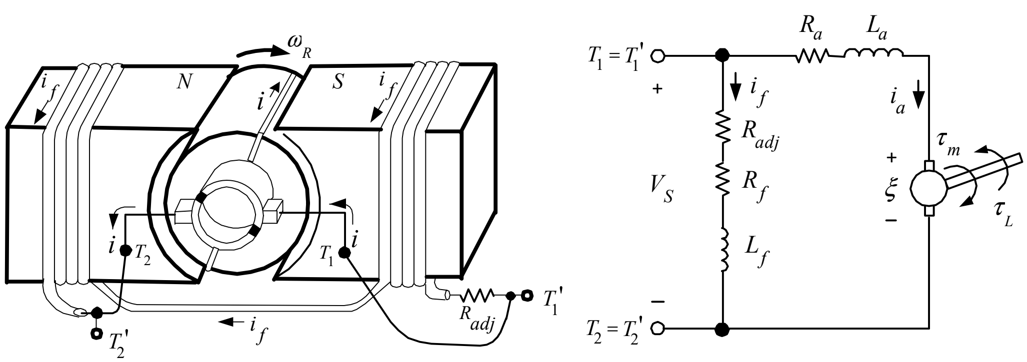


FIGURE 1.13. Shunt connected DC motor.

The equations describing the shunt motor are [1]

$$\begin{aligned} J\frac{d\omega}{dt} &= K_T L_f i_f i_a - \tau_L \\ L_a\frac{di_a}{dt} &= -R_a i_a - K_b L_f i_f \omega + V_S \\ L_f\frac{di_f}{dt} &= -(R_{adj} + R_f) i_f + V_S. \end{aligned}$$

Here ω is the rotor angular speed, V_S is the terminal (source) voltage, i_a is the armature current, i_f is the field current, τ_L is the load torque, K_T is the torque constant, and K_b is the back-emf constant. The armature resistance and armature inductance are denoted by R_a and L_a,respectively, and the field resistance and field inductance are R_f and L_f, respectively. R_{adj} is an adjustable resistor so that the total field resistance $R_{adj} + R_f$ can be varied.

Let $x_1 = \omega, x_2 = i_a, x_3 = i_f, u = V_S$, and define the constants $c_1 \triangleq \dfrac{K_T L_f}{J}, c_2 = \dfrac{R_a}{L_a}, c_3 = \dfrac{K_b L_f}{L_a}, c_4 = \dfrac{1}{L_a}, c_5 = \dfrac{R_{adj} + R_f}{L_f}, c_6 = \dfrac{1}{L_f}$ so that the statespace model becomes

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 x_2 x_3 \\ -c_2 x_2 - c_3 x_1 x_3 \\ -c_f x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ c_4 \\ c_6 \end{bmatrix} u + \begin{bmatrix} -1/J \\ 0 \\ 0 \end{bmatrix} \tau_L.$$

The constant load torque is not known. Assuming that $x_1 = \omega, x_2 = i_a$, and $x_3 = i_f$ are measured this problem shows how to design an observer to estimate τ_L/J.

(a) Let $x_4 \triangleq \tau_L/J$ with $dx_4/dt = 0$ and suppose $x_1 = \omega, x_2 = i_a$, and $x_3 = i_f$ are all measured. Show the model of the shunt connected DC motor is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} c_1 y_2 y_3 \\ -c_2 y_2 - c_3 y_1 y_3 \\ -c_f y_3 \\ 0 \end{bmatrix}}_{\varphi(y)} + \underbrace{\begin{bmatrix} 0 \\ c_4 \\ c_6 \\ 0 \end{bmatrix}}_{b} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(b) Is the pair (C, A) observable? Explain.

(c) Let

$$T_o \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, T_o^{-1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and define the linear transformation $x^* = T_o x$. Transform the model given in part (a) into the x^* coordinate system.

(d) With $A_o \triangleq T_o A T_o^{-1}$ and $C_o = C T_o^{-1}$ let

$$L_o = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \\ \ell_{41} & \ell_{42} & \ell_{43} \end{bmatrix}$$

give the equations for an observer to estimate the state x^*.

(e) Give the equations for the estimate error $x^* - \hat{x}^*$ and show that the components of L_o can be chosen so that poles of the estimation error system can be put at $-r_1, -r_2, -r_3, -r_4$.

Problem 11 *Speed and Load Torque Estimation for the PM Synchronous Motor*
The differential equation model of PM synchronous motor is

$$\begin{aligned}
L_S \frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m \sin(n_p\theta)\omega + u_{Sa} \\
L_S \frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m \cos(n_p\theta)\omega + u_{Sb} \\
J\frac{d\omega}{dt} &= K_m(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)) - \tau_L \\
\frac{d\theta}{dt} &= \omega.
\end{aligned}$$

With currents and voltages transformed by *direct-quadrature* (dq) transformation

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix}, \quad \begin{bmatrix} u_d \\ u_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix}$$

the model is given by

$$\begin{aligned}
L_S \frac{di_d}{dt} &= -R_S i_d + n_p\omega L_S i_q + u_d \\
L_S \frac{di_q}{dt} &= -R_S i_q - n_p\omega L_S i_d - K_m\omega + u_q \\
J\frac{d\omega}{dt} &= K_m i_q - \tau_L \\
\frac{d\theta}{dt} &= \omega.
\end{aligned}$$

With the currents i_{Sa}, i_{Sb} measured along with θ design an observer for ω and τ_L/J.

Problem 12 *Parallel Parking* [15]
Figure 1.14 is schematic to model the steering of a car. R_1 denotes the midpoint of the real axle while F_1 denotes the midpoint of the front axle and has coordinates (x_1, x_2). φ denotes the angle from the x_1 axis to the body axis $R_1 - F_1$ which has length ℓ. θ denotes the angle of the front wheels are pointing with respect to the body axis. The angle θ is an input whose value is set by the steering wheel.

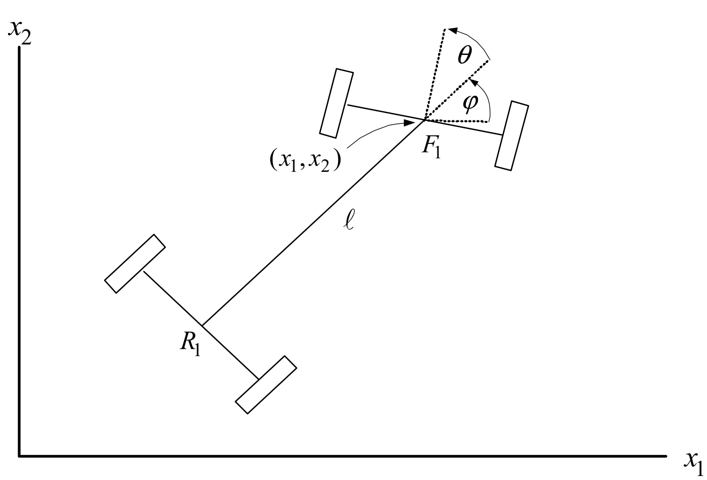


FIGURE 1.14. Schematic for steering a car.

Figure 1.15 is a schematic indicating the motion of car of a small distance h in the direction the front wheels are pointing. The midpoint of the front axle goes from F_1 to F_2. The coordinates of F_2 are then

$$(x_1 + h\cos(\theta+\varphi), x_2 + h\sin(\theta+\varphi)).$$

The distance $F_2 - E$ is a $h\sin(\theta)$ and with h small we have

$$\Delta\varphi = \frac{h\sin(\theta)}{\ell}.$$

The *drive* vector field g_D is

$$g_D \triangleq \cos(\theta+\varphi)\hat{x}_1 + \sin(\theta+\varphi))\hat{x}_2 + h\sin(\theta)\hat{\varphi}$$

where $\hat{x}_1$ and $\hat{x}_2$ are unit vectors in the direction of increasing x_1 and x_2, respectively and $\hat{\varphi}$ is a unit vector in the direction of increasing φ. This is the motion we get by applying power to the rear wheels.

The *steer* vector field g_{Steer} is

$$g_{Steer} \triangleq \hat{\theta}$$

where $\hat{\theta}$ is a unit vector in the direction of increasing θ.

Define the *slide* vector field g_{Slide} to be

$$g_{Slide} \triangleq -\sin(\varphi)\hat{x}_1 + \cos(\varphi)\hat{x}_2$$

and the *rotate* vector field g_R to be

$$g_R \triangleq \hat{\varphi}$$

where $\hat{\varphi}$ is a unit vector in the direction of increasing φ.

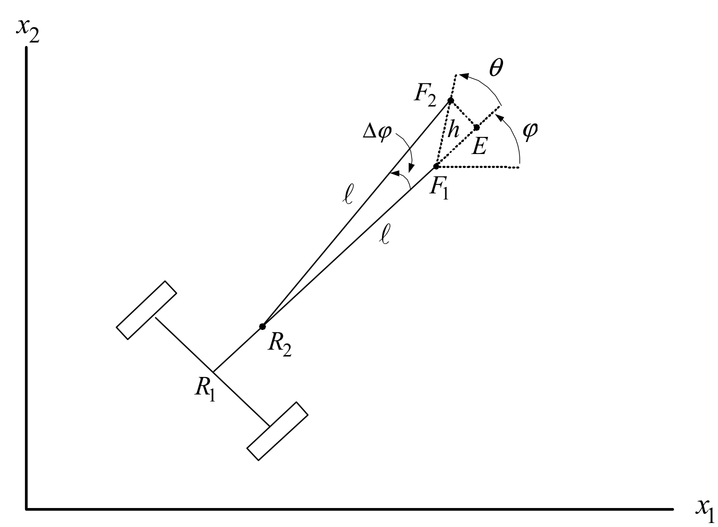


FIGURE 1.15. Computing drive.

We then have the model

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \theta \\ \varphi \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta+\varphi) \\ \sin(\theta+\varphi) \\ 0 \\ h\sin(\theta) \end{bmatrix}}_{g_D} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{g_{Steer}} u_2$$

where u_1 represents applying power to the rear wheels and u_1 represents turning the steering wheel.

(a) Let g_D, g_{Steer} denote the drive and steer vector fields, respectively. Define the *wriggle* vector field g_W to be

$$g_W \triangleq [g_D, g_{Steer}].$$

Compute g_W. Show that with $\theta = 0$ we have

$$g_W = g_{Slide} + g_R.$$

Remark Note that with $\theta = 0$ the vector field g_{Slide} is orthogonal to g_D. So trying to get your parked car out of a tight spot between two cars requires motion orthogonal to drive and a rotation which is given by wriggle. So to get wriggle one must: steer, drive, reverse steer, reverse drive and repeat.

(b) Show

$$\begin{aligned}[g_{Steer}, g_W] &= -g_D \\ [g_W, g_D] &= g_{Slide}\end{aligned}$$

2

Manifolds and Tangent Vectors

This chapter introduces manifolds and their tangent vectors through the use of examples. This chapter is not intended as a formal treatment of differential geometry; rather the intent is that the presentation here will provide the motivation for the abstract mathematical definitions of manifolds and tangent vectors given in formal treatments of the subject. For example, what is the difference between points in a manifold and their coordinates? Why is a tangent vector defined to be a derivative operator? What is the distinction between a vector tangent and its components?

This and the next chapter are written to give the reader enough background to tackle the more sophisticated treatments such as given in [16][11][17][18]. In particular, see Appendix A of [16], Chapter 2 of [11], Appendix A of [17] and Chapter 3 of [18]. A very nice introduction to abstract manifold theory is given in Chapter 7 of [19]. Further, a very readable introduction to Differential Geometry and General Relativity theory is the book [20].

Before getting into manifolds, we first look at the distinction between $\mathbb{R}^n$ and the Euclidean Space $\mathbf{E}^n$. $\mathbb{R}^n \triangleq \{(x_1, x_2, ..., x_n) | \; x_i \in \mathbb{R}\}$ is just the set of all ordered n-tuples of real numbers. On the other hand, $\mathbf{E}^n$ represents Euclidean n-space. That is, with $\mathbf{e}_1 \triangleq \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \mathbf{e}_2 \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T, ..., \mathbf{e}_n \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T$ the vector

$$p = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n \in \mathbf{E}^n$$

represents a point in space. This distinction is seemly quite fuzzy as the coordinates $(x_1, x_2, ..., x_n)$ of the point p and the point p itself given by

$$p = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

seem to be the same thing. In fact, because of this similarity the point $p = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_3\mathbf{e}_n \in \mathbf{E}^n$ is identified its coordinates $(x_1, x_2, ..., x_n)$. This identification of the coordinates in $\mathbb{R}^n$ with a point in Euclidean space $\mathbf{E}^n$ is called the *Cartesian* coordinate system. That is, n-tuples in $\mathbb{R}^n$ directly correspond to points in $\mathbf{E}^n$. This is not true in other coordinate systems. For example the cylindrical coordinates (ρ, θ, z) of a point in $\mathbf{E}^n$ are a 3-tuple $\mathbb{R}^3$; however, they do *not* represent the point $\rho\mathbf{e}_1 + \theta_2\mathbf{e}_2 + z\mathbf{e}_1$ with $\mathbf{e}_1 \triangleq \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \mathbf{e}_2 \triangleq \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, and $\mathbf{e}_3 \triangleq \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

As pointed out by Boothby [21] (p. 4), Euclidean space should be thought of as studied in high school geometry where definitions are made, theorems are proved, and so forth without the use of coordinates. This is what Euclid did. It was not until the invention of analytic geometry by Fermat and Descartes that coordinate systems were used. The point here is that $\mathbf{E}^n$ represents physical points (locations) in space while an element of $\mathbb{R}^n$ is just an n-tuple of real numbers. They only correspond (seem the same) when the Cartesian coordinate system is used.

$\mathbf{E}^n$ is a vector space with an inner product (e.g., see [22]). For example, with $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$ and $\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + \cdots + y_n\mathbf{e}_n$ their inner product is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \sum_{i=1}^{n} x_i y_i.$$

Using the inner product the norm of a vector is defined by

$$||\mathbf{x}|| \triangleq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

The distance between two points $d(\mathbf{x}, \mathbf{y})$ is defined by

$$d(\mathbf{x}, \mathbf{y}) \triangleq ||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Exercise 4 *Norm*

Show that $||\mathbf{x}|| = \sqrt{\sum_{i=1}^{n} x_i^2}$ has the properties of a norm. That is, (1) $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = 0$, (2) $||c\mathbf{x}|| = |c| \cdot ||\mathbf{x}||$, and (3) $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

2.1 Linear Manifolds

The basic idea of a manifold is that we have a subset of points (often a subset of $\mathbf{E}^n$) with various coordinate systems attached to it. Let's start with a concrete example.

Let $\mathcal{M} \subset \mathbf{E}^3$ be given by

$$\mathcal{M} \triangleq \left\{ \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T \in \mathbf{E}^3 | \ \mathbf{z} = \mathbf{a} + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \right\}$$

where $\mathbf{e}_1 \triangleq \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T, \mathbf{e}_2 \triangleq \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

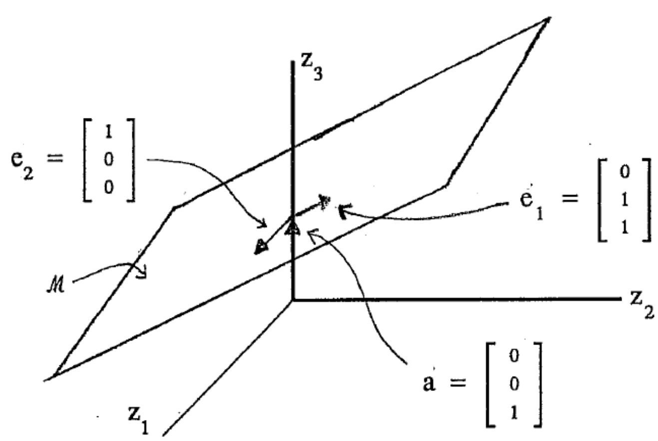


FIGURE 2.1. Linear manifold.

Exercise 5 *Implicit Definition of* $\mathcal{M}$

Show that $\mathcal{M}$ may also be defined by

$$\begin{aligned} \mathcal{M} &\triangleq \left\{ \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T \in \mathbf{E}^3 | \ \mathbf{n} \cdot \mathbf{z} = 1 \text{ where } \mathbf{n} = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T \right\} \\ &= \left\{ \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T \in \mathbf{E}^3 | \ -z_2 + z_3 = 0 \right\} \end{aligned}$$

Note that $\mathbf{n} \cdot \mathbf{e}_1 = 0, \mathbf{n} \cdot \mathbf{e}_2 = 0$ showing that $\mathbf{n}$ is orthogonal to $\mathbf{e}_1$ and $\mathbf{e}_2$.

The set $\mathcal{M}$ is an example of a *linear manifold.*

Definition 1 *Linear Manifold in* $\mathbf{E}^3$
A *linear manifold* in $\mathbf{E}^3$ is the set of vectors in $\mathbf{a}+\mathcal{S}$ where $\mathcal{S}$ is a subspace (sub vector space) of $\mathbf{E}^3$.

In the above example $\mathcal{S} \triangleq \left\{\mathbf{z} \in \mathbf{E}^3 | \ \mathbf{z} = x_1\mathbf{e}_2 + x_2\mathbf{e}_2 \text{ with } (x_1, x_2) \in \mathbb{R}^2\right\}$. For each $p \in \mathcal{M}$ there is a unique pair $(x_1, x_2) \in \mathbb{R}^2$ corresponding to p. The pair $(x_1, x_2) \in \mathbb{R}^2$ are the coordinates of the point $p \in \mathcal{M}$. The coordinates are in $\mathbb{R}^2$ while the points are in $\mathbf{E}^3$. We can explicitly write down the relationship between the coordinates of p and the point p itself. This is $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathcal{M} \subset \mathbf{E}^3$ given by

$$\mathbf{z}(x_1, x_2) = \mathbf{a} + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = \begin{bmatrix} x_2 \\ x_1 \\ x_1+1 \end{bmatrix}.$$

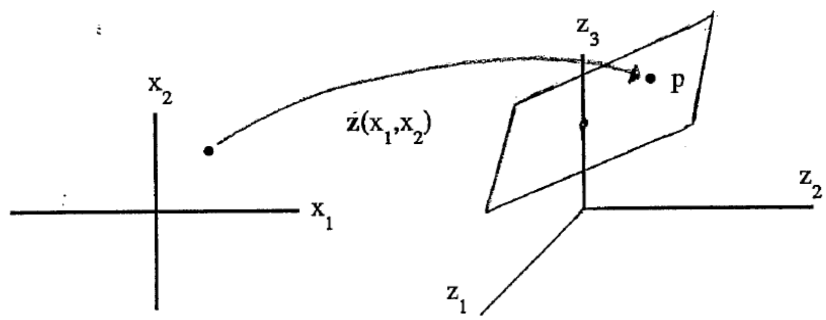


FIGURE 2.2. Coordinate map for a linear manifold.

Of course, (x_1, x_2) is not the only set of coordinates. Consider a different set of basis vectors for the subspace $\mathcal{S}$ given by

$$\mathbf{e}_1^* = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{e}_2^* = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}.$$

Then the above linear manifold is also given by

$$\mathcal{M} \triangleq \{ \ \mathbf{z} \in \mathbf{E}^3 | \ \mathbf{z} = \mathbf{a} + x_1^*\mathbf{e}_1^* + x_2^*\mathbf{e}_2^* \text{ with } (x_1^*, x_2^*) \in \mathbb{R}^2$$

This is clear because

$$\begin{aligned} \mathbf{e}_1^* &= \mathbf{e}_1 + 2\mathbf{e}_2 \\ \mathbf{e}_2^* &= 3\mathbf{e}_1 + 4\mathbf{e}_2 \end{aligned}$$

so the span$\{\mathbf{e}_1^*, \mathbf{e}_2^*\}$ =span$\{\mathbf{e}_1, \mathbf{e}_2\}$.

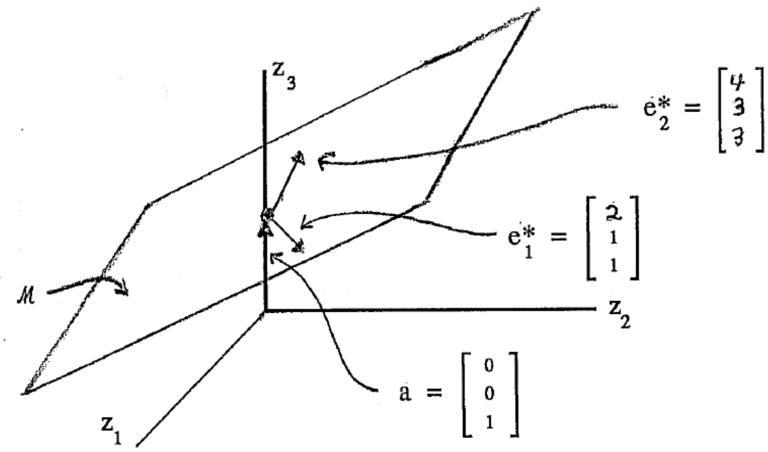


FIGURE 2.3. A second basis for the subspace $\mathcal{S}$.

For each point $p \in \mathcal{M}$ there is a unique pair of coordinates $(x_1^*, x_2^*) \in \mathbb{R}^2$ corresponding to this point. The function relating the coordinates (x_1^*, x_2^*) to the point $p \in \mathcal{M}$ is given by $\mathbf{z}(x_1^*, x_2^*) \mid \mathbb{R}^2 \to \mathcal{M} \subset \mathbf{E}^3$

$$\mathbf{z}(x_1^*, x_2^*) = \mathbf{a} + x_1^* \mathbf{e}_1^* + x_2^* \mathbf{e}_2^* = \begin{bmatrix} 2x_1^* + 4x_2^* \\ x_1^* + 2x_2^* \\ x_1^* + 3x_2^* + 1 \end{bmatrix}.$$

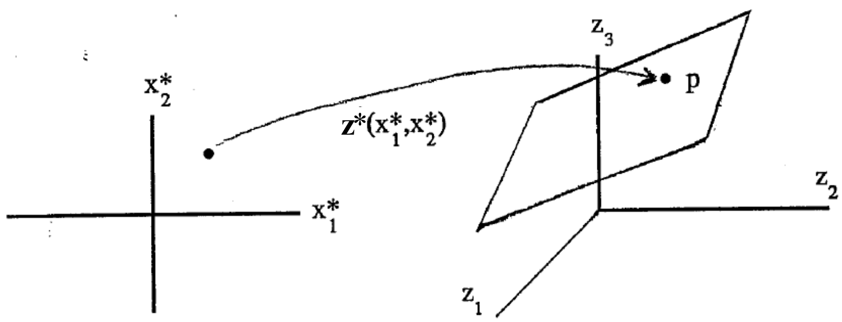


FIGURE 2.4. A second coordinates system for the linear manifold $\mathcal{M}$.

Change of Coordinates

What is the relationship between the two sets of coordinates for the above linear manifold? We have

$$\begin{aligned} \mathbf{e}_1^* &= \mathbf{e}_1 + 2\mathbf{e}_2 \\ \mathbf{e}_2^* &= 3\mathbf{e}_1 + 4\mathbf{e}_2 \end{aligned}$$

with inverse

$$\begin{aligned} \mathbf{e}_1 &= -2\mathbf{e}_1^* + \mathbf{e}_2^* \\ \mathbf{e}_2 &= \frac{3}{2}\mathbf{e}_1^* - \frac{1}{2}\mathbf{e}_2^*. \end{aligned}$$

Then

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = x_1(-2\mathbf{e}_1^* + \mathbf{e}_2^*) + x_2(\frac{3}{2}\mathbf{e}_1^* - \frac{1}{2}\mathbf{e}_2^*) = (-2x_1 + \frac{3}{2}x_2)\mathbf{e}_1^* + (x_1 - \frac{1}{2}x_2)\mathbf{e}_2^*$$

or

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

More generally let two bases for the subspace $\mathcal{S}$ be related by

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^* \\ \mathbf{e}_2^* \end{bmatrix}. \tag{2.1}$$

The matrix in (2.1) is the *change of basis* matrix. Writing this vertically we have

$$\begin{array}{cc} \mathbf{e}_1 & \mathbf{e}_2 \\ \| & \| \\ a_{11}\mathbf{e}_1^* & a_{21}\mathbf{e}_1^* \\ + & + \\ a_{12}\mathbf{e}_2^* & a_{22}\mathbf{e}_2^* \end{array}$$

so that

$$\begin{array}{cc} x_1\mathbf{e}_1 & x_2\mathbf{e}_2 \\ \| & \| \\ a_{11}x_1\mathbf{e}_1^* & a_{21}x_2\mathbf{e}_1^* \\ + & + \\ a_{12}x_1\mathbf{e}_2^* & a_{22}x_2\mathbf{e}_2^* \end{array}$$

By inspection of this last diagram it follows that

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.2}$$

The matrix in (2.2) is the *change of coordinates* matrix. Notice that the change of coordinates matrix is the transpose of the change of basis matrix.

$\mathbf{E}^2$ as a manifold

$\mathbf{E}^2$ is itself a linear manifold. Let

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

be an orthogonal basis for $\mathbf{E}^2$. Any point $p \in \mathbf{E}^2$ may be represented by coordinates (x_1, x_2) corresponding the point $p = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. The Cartesian coordinate mapping[1] $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathbf{E}^2$ is given by

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{E}^2.$$

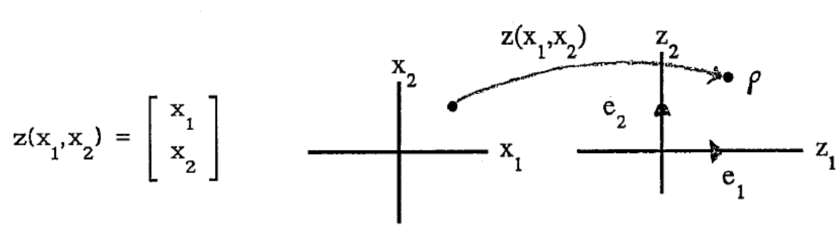


FIGURE 2.5. The manifold $\mathbf{E}^2$.

[1] The terminology *mapping* and *function* are used interchangably throughout this text.

$\mathbf{E}^2$ is a linear manifold because $\mathbf{E}^2 = \mathbf{0} + \mathcal{S}$ where

$$\mathcal{S} \triangleq \{\mathbf{z} \in \mathbf{E}^2 | \ \mathbf{z} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \text{ with } (x_1, x_2) \in \mathbb{R}^2\}.$$

Consider another set of basis vectors

$$\mathbf{e}_1^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{e}_2^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where any point $p \in \mathcal{M}$ is represented by the coordinates (x_1^*, x_2^*) corresponding to $x_1^*\mathbf{e}_1^* + x_2^*\mathbf{e}_2^*$. The coordinate mapping $\mathbf{z}^*(x_1^*, x_2^*) : \mathbb{R}^2 \to \mathbf{E}^2$ is then

$$\mathbf{z}^*(x_1^*, x_2^*) = \begin{bmatrix} x_1^* \\ x_1^* + x_2^* \end{bmatrix}.$$

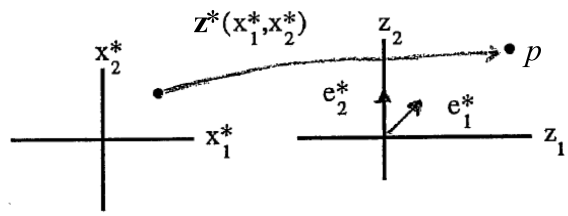


FIGURE 2.6. Another coordinate system for the manifold $\mathbf{E}^3$.

The two bases are related by

$$\begin{bmatrix} \mathbf{e}_1^* \\ \mathbf{e}_2^* \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} \text{ or } \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^* \\ \mathbf{e}_2^* \end{bmatrix}.$$

Next we find the relationship between the (x_1^*, x_2^*) and (x_1, x_2) coordinates. We compute

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = x_1(\mathbf{e}_1^* - \mathbf{e}_2^*) + x_2\mathbf{e}_2^* = x_1\mathbf{e}_1^* + (x_2 - x_1)\mathbf{e}_2^* = x_1^*\mathbf{e}_1^* + x_2^*\mathbf{e}_2^*$$

or in matrix form

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Exercise 6 *Linear Manifold*

Consider the linear manifold defined by

$$\mathcal{M} \triangleq \left\{ \mathbf{z} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T \in \mathbf{E}^3 | \ \mathbf{n} \cdot \mathbf{z} = -1 \text{ with } \mathbf{n} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T \right\}.$$

(a) Sketch this manifold in $\mathbf{E}^3$ to show it is a planar subset in $\mathbf{E}^3$.

(b) Show that $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathcal{M}$ given by

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + 1 \end{bmatrix}$$

is a coordinate mapping, i.e., $\mathbf{z}(x_1, x_2)$ is a $1 - 1$ mapping from $\mathbb{R}^2 \to \mathcal{M}$.

(c) Show that $\mathbf{z}'(x_1', x_2') : \mathbb{R}^2 \to \mathcal{M}$ given by

$$\mathbf{z}'(x_1', x_2') = \begin{bmatrix} x_1' + x_2' \\ x_1' \\ x_1' + 1 \end{bmatrix}$$

is a coordinate mapping, i.e., $\mathbf{z}'(x_1', x_2')$ is a $1-1$ mapping from $\mathbb{R}^2 \to \mathcal{M}$.

(d) Find the change of coordinates from (x_1, x_2) to (x_1', x_2') and its inverse.

Exercise 7 *Linear Manifold*
Consider the linear manifold defined by

$$\mathcal{M} \triangleq \mathbf{E}^3.$$

(a) Show that $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathbf{E}^3$ given by

$$\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_3 \end{bmatrix} = x_2\mathbf{e}_1 + x_1\mathbf{e}_2 + (x_1 + x_3)\vec{e}_3 = x_2\mathbf{e}_1 + x_1(\mathbf{e}_2 + \vec{e}_3) + e_3.$$

is a coordinate mapping, i.e., $\mathbf{z}(x_1, x_2)$ is a $1-1$ mapping from $\mathbb{R}^2 \to \mathbf{E}^3$.

(b) Show that $\mathbf{z}'(x_1', x_2', x_2') : \mathbb{R}^2 \to \mathbf{E}^3$ given by

$$\mathbf{z}'(x_1', x_2', x_3') = \begin{bmatrix} x_1' + x_2' \\ x_1' \\ x_1' + x_3' \end{bmatrix}$$

is a coordinate mapping, i.e., $\mathbf{z}'(x_1', x_2', x_3')$ is a $1-1$ mapping from $\mathbb{R}^3 \to \mathbf{E}^3$.

(d) Find the change of coordinates from (x_1, x_2, x_3) to (x_1', x_2', x_3') and its inverse.

2.2 Functions and Their Inverses

Before going onto general manifolds (not necessarily linear) we need to review some basic notions about functions.

Definition 2 *One-to-One (1-1) and Onto Functions*
Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^{n\mathbb{R}\mathbb{R}\mathbb{R}}$ be two sets and $f : X \to Y$ a function with domain X and range Y. Then f is $1-1$ if for $x_1 \in X$ and $x_2 \in X$ with $f(x_1) = f(x_2)$ then $x_1 = x_2$.
f is onto if for any $y \in Y$ there is at least one $x \in X$ such that $f(x) = y$.

Example 1 *Cartesian and Polar Coordinates*
Let

$$\begin{aligned} X &\triangleq \left\{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 > 0, 0 < x_2 < 2\pi\right\} \\ Y &\triangleq \left\{(y_1, y_2) \in \mathbb{R}^2 |\ \text{If } y_1 > 0 \text{ then } y_2 \neq 0\right\} \end{aligned}$$

and define the mapping from X to Y given by

$$f(x_1, x_2) \triangleq (x_1 \cos(x_2), x_1 \sin(x_2)) = (y_1, y_2).$$

This is just the coordinate transformation from polar coordinates to Cartesian coordinates.

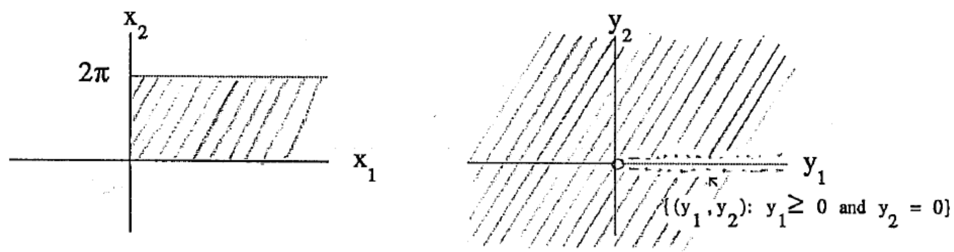


FIGURE 2.7. $X \triangleq \left\{(x_1,x_2) \in \mathbb{R}^2 |\ x_1 > 0, 0 < x_2 < 2\pi\right\}$, $Y \triangleq \left\{(y_1,y_2) \in \mathbb{R}^2 |\ \text{If } y_1 > 0 \text{ the } y_2 \neq 0\right\}$.

The inverse transformation is

$$f^{-1}(y_1,y_2) \triangleq \left(\sqrt{y_1^2+y_2^2}, \tan^{-1}(y_1,y_2)\right) = (x_1,x_2)$$

where $\tan^{-1}(y_1,y_2) \triangleq \mathtt{atan2}(y_2,y_1)$. With this particular domain X and range Y the function f is $1-1$ and onto. Note that both f and f^{-1} are continuous.

Definition 3 *Homeomorphism*
Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ be two sets and $f : X \to Y$ a function with domain X and range Y. If f is $1-1$ on X and onto Y with both f and f^{-1} continuous, then f and f^{-1} are *homeomorphisms.*

Definition 4 *Diffeomorphism*
Let $f : X \subset \mathbb{R}^n \to Y \subset \mathbb{R}^n$ be a homemorphic function. If both f and f^{-1} have continuous derivatives then f and f^{-1} are *diffeomorphisms.*

Example 2 *Cartesian and Polar Coordinates*
Let

$$\begin{aligned} X^* &\triangleq \left\{(x_1,x_2) \in \mathbb{R}^2 |\ x_1 > 0, 0 \leq x_2 < 2\pi\right\} \\ Y^* &\triangleq \left\{(y_1,y_2) \in \mathbb{R}^2 |\ (y_1,y_2) \neq (0,0)\right\} \end{aligned}$$

and define the mapping from X^* to Y^* given by

$$f(x_1,x_2) \triangleq (x_1\cos(x_2), x_1\sin(x_2)) = (y_1,y_2).$$

This is the coordinate transformation from polar coordinates to Cartesian coordinates.
f is $1-1$ and onto from X^* to Y^* so its inverse exists, i.e., f^{-1} is $1-1$ and onto from Y^* to X^*.
f is also continuous from X^* to Y^*. (What does continuity mean on the particular set of points (x_1,x_2) with $x_1 > 0$ and $x_2 = 0$?)
The inverse f^{-1} is *not* continuous from Y^* to X^*. Specifically, f^{-1} is *not* continuous on the set of points (y_1,y_2) with $y_1 > 0, y_2 = 0$. To see this consider the two sets of points in Y^* given by

$$\begin{aligned} y_n &= (y_{n1},y_{n2}) \triangleq (1,1/n) \text{ for } n = 1,2,3,... \\ \bar{y}_n &= (\bar{y}_{n1},\bar{y}_{n2}) \triangleq (1,-1/n) \text{ for } n = 1,2,3,... \end{aligned}$$

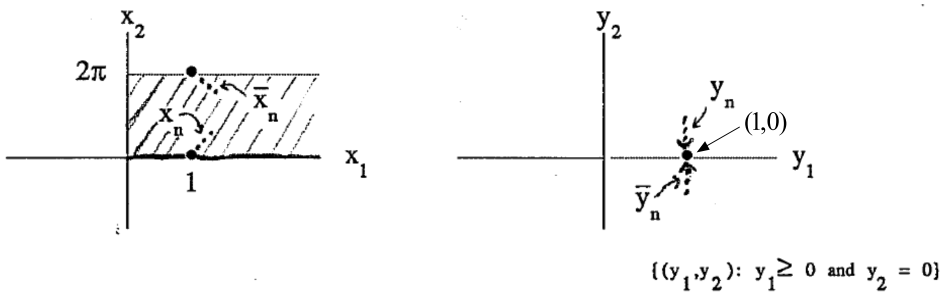


FIGURE 2.8. f^{-1} is not continous on the half-line $(y_1, 0)$, $y_1 \geq 0$.

Both of the sequences y_n and $\bar{y}_n$ converge to $(1, 0)$. However

$$\begin{aligned} x_n &= (x_{n1}, x_{n2}) = f^{-1}(y_n) = \left(\sqrt{1+(1/n)^2}, \tan^{-1}(1, 1/n)\right) \to (1, 0) \\ \bar{x}_n &= (\bar{x}_{n1}, \bar{x}_{n2}) = f^{-1}(\bar{y}_n) = \left(\sqrt{1+(-1/n)^2}, \tan^{-1}(1, -1/n)\right) \to (1, 2\pi). \end{aligned}$$

This shows that f^{-1} is not continuous at $(y_1, y_2) = (1, 0)$. A similar argument shows that f^{-1} is not continuous at the any point in the set $\left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_1 > 0, y_2 = 0\right\}$.

Exercise 8 *Cartesian and Polar Coordinates*
Let

$$\begin{aligned} X^{**} &\triangleq \left\{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 > 0, 0 < x_2 < 4\pi\right\} \\ Y^{**} &\triangleq \left\{(y_1, y_2) \in \mathbb{R}^2 |\ (y_1, y_2) \neq (0, 0)\right\} \end{aligned}$$

and define the mapping from X^{**} to Y^{**} given by

$$f((x_1, x_2)) \triangleq (x_1 \cos(x_2), x_1 \sin(x_2)) = (y_1, y_2).$$

Is this function $1 - 1$ from X^{**} to Y^{**}?
Is this function onto Y^{**}?
Does the function f have an inverse?

Single Variable Inverse Function Theorem

Given a function form $f(x) : \mathcal{D} \subset \mathbb{R}$ onto $\mathcal{U} \subset \mathbb{R}$ we would like to know when it has an inverse. I.e., is there a function $g(y) : \mathcal{U} \to \mathcal{D}$ such that

$$\begin{aligned} f(g(y)) &= y \\ g(f(x)) &= x? \end{aligned}$$

Let's consider an example to see what issues are involved.

Example 3 $y = x^2$
Let $\mathcal{D} \triangleq \mathbb{R}$ and $\mathcal{U} \triangleq \{y |\ y \geq 0\}$ with $f : \mathcal{D} \to \mathcal{U}$ given by

$$y = f(x) \triangleq x^2.$$

The function $f(x)$ is graphed on the left side of the figure below.

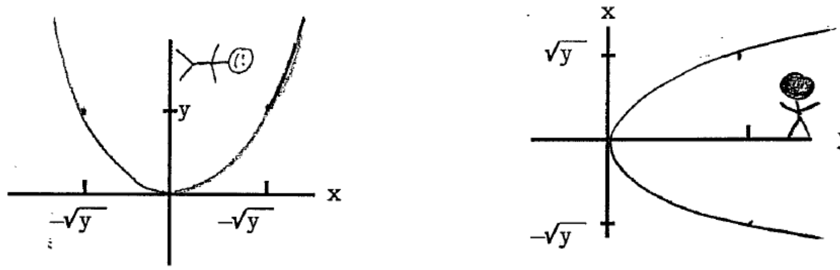


FIGURE 2.9. Left: $y = x^2$. Right: $x = \pm\sqrt{y}$.

The stick man on the left sees a relation (not function!) from $\mathcal{U} \triangleq \{y|\ y \geq 0\}$ to $\mathcal{D} \triangleq \mathbb{R}$ given by $g(y) = \pm\sqrt{y}$. That is, from the point of view of the stick man, the number y is taken is taken to both $\sqrt{y}$ and $-\sqrt{y}$. Thus g is not a function and so f does not have an inverse.

Example 4 $y = x^2$
Let $\mathcal{D} \triangleq \{x|\ x \geq 0\}$ and $\mathcal{U} \triangleq \{y|\ y \geq 0\}$ with $f : \mathcal{D} \to \mathcal{U}$ given by

$$y = f(x) \triangleq x^2.$$

The function $f(x)$ is graphed on the left side of the figure below.

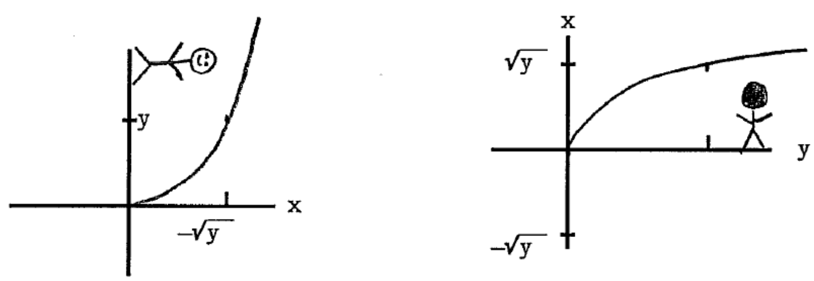


FIGURE 2.10. Left: $f : \mathcal{D} \to \mathcal{U}$ given by $f(x) = x^2$. $g : \mathcal{U} \to \mathcal{D}$ given by $g(y) = \sqrt{y}$.

The stick man on the left sees a function from $\mathcal{U} \triangleq \{y|\ y \geq 0\}$ to $\mathcal{D} \triangleq \{x|\ x \geq 0\}$ given by $g(y) = \sqrt{y}$. That is, from the point of view of the stick man, the number y is taken is to the unique point $\sqrt{y}$. Thus g is a function.

Theorem 1 *Inverse Function Theorem*
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Suppose that the derivative $f'(x) \neq 0$ on the interval $[a, b]$. Then $f(x)$ maps the interval $[a, b]$ in a one-to-one fashion to some interval $[\alpha, \beta]$ and there is a function g defined on $[\alpha, \beta]$ such that $g(f(x)) = x$ for all $x \in [a, b]$ and $f(g(y)) = y$ for all $y \in [\alpha, \beta]$.

Example 5 $y = \sin(x)$
Let $f(x) = \sin(x)$. Restrict f to the interval $[\pi/2, 3\pi/2]$ indicated by the solid line on the left hand side of the figure below. Let $g(y)$ defined on $[-1, 1]$ be the function graphed (solid line) on the right hand side of the figure below. Then $g(f(x)) = x$ for all $x \in [\pi/2, 3\pi/2]$ and $f(g(y)) = y$ for all $y \in [-1, 1]$

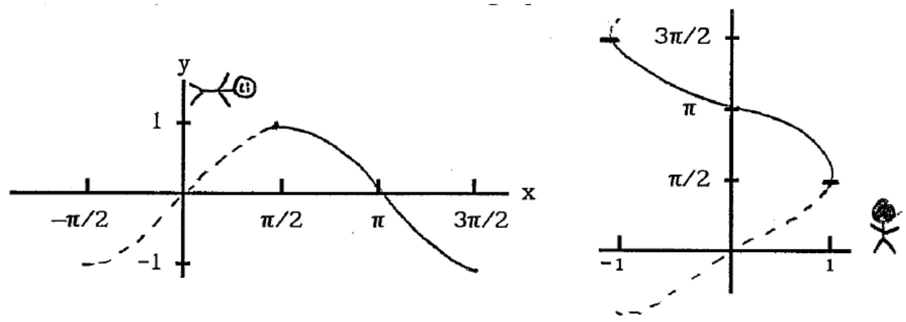


FIGURE 2.11. Left. $\sin(x) : [\pi/2, 3\pi/2] \to [-1, 1]$. Right. Inverse

Remark The conditions of the theorem are sufficient, but not necessary. For example, let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$ which is graphed in the figure below. Though $f'(0) = 0$ this function has an inverse $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) = \sqrt[3]{y}$.

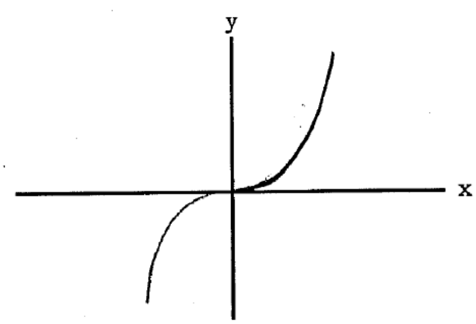


FIGURE 2.12. $f(x) = x^3$.

Multivariable Inverse Function Theorem

Let's start with a simple example.

Example 6 *Polar and Cartesian Coordinates*
Let

$$\begin{aligned} \mathcal{D} &\triangleq \left\{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 > 0, 0 < x_2 < 2\pi\right\} \\ \mathcal{U} &\triangleq \left\{(y_1, y_2) \in \mathbb{R}^2 |\ \text{If } y_1 \geq 0 \text{ then } y_2 \neq 0\right\} \end{aligned}$$

and define the function mapping $f : \mathcal{D} \to \mathcal{U}$ by

$$\begin{aligned} y_1 &= f_1(x_1, x_2) = x_1 \cos(x_2) \\ y_2 &= f_2(x_1, x_2) = x_1 \sin(x_2). \end{aligned}$$

This function has an inverse $g : \mathcal{U} \to \mathcal{D}$ given by

$$\begin{aligned} x_1 &= g_1(y_1, y_2) = \sqrt{y_1^2 + y_2^2} \\ x_2 &= g_2(y_1, y_2) = \tan^{-1}(y_1, y_2). \end{aligned}$$

We would like to know in general when does a function $f : \mathcal{D} \subset \mathbb{R}^2 \rightarrow \mathcal{U} \subset \mathbb{R}^2$ have an inverse? The next theorem provides conditions for such a function to have an inverse.

Theorem 2 *Inverse Function Theorem*
Let $f : X \subset \mathbb{R}^2 \rightarrow Y \subset \mathbb{R}^2$ given by

$$\begin{aligned} y_1 &= f_1(x_1, x_2) \\ y_2 &= f_2(x_1, x_2) \end{aligned}$$

or, more compactly,

$$y = f(x).$$

Let $x_0 = (x_{01}, x_{02}) \in \mathbb{R}^2$ and suppose

$$J_f(x_0) \triangleq \left[\begin{array}{cc} \partial f_1(x_1, x_2)/\partial x_1 & \partial f_1(x_1, x_2)/\partial x_2 \\ \partial f_2(x_1, x_2)/\partial x_1 & \partial f_2(x_1, x_2)/\partial x_2 \end{array} \right]_{|x_0=(x_{01},x_{02})}$$

is nonsingular, that is,

$$\det J_f(x_0) = \det \left[\begin{array}{cc} \partial f_1(x_1, x_2)/\partial x_1 & \partial f_1(x_1, x_2)/\partial x_2 \\ \partial f_2(x_1, x_2)/\partial x_1 & \partial f_2(x_1, x_2)/\partial x_2 \end{array} \right]_{|x_0=(x_{01},x_{02})} \neq 0.$$

Let $y_0 = (y_{01}, y_{02})$ be given by

$$\begin{aligned} y_{01} &= f_1(x_{01}, x_{02}) \\ y_{02} &= f_2(x_{01}, x_{02}). \end{aligned}$$

Then in a neighborhood of y_0 the function $y = f(x)$ has an inverse. Specifically, there exists an open neighborhood of $\mathcal{D}$ of x_0 and an open neighborhood $\mathcal{U}$ of y_0 such that f is an 1-1 and onto function from $\mathcal{D}$ to $\mathcal{U}$. Further, there exists functions $g_1(y_1, y_2), g_2(y_1, y_2)$ such that

$$\begin{aligned} x_{01} &= g_1(y_{01}, y_{02}) \\ x_{02} &= g_2(y_{01}, y_{02}) \end{aligned}$$

and for $x \in \mathcal{D}$

$$x_1 = g_1(f_1(x_1, x_2), f_2(x_1, x_2)) \tag{2.3}$$
$$x_2 = g_2(f_1(x_1, x_2), f_2(x_1, x_2)) \tag{2.4}$$

is a 1-1 and onto function from $\mathcal{D}$ to $\mathcal{U}$.
It also follows for all $y \in \mathcal{U}$ that

$$y_1 = f_1(g_1(y_1, y_2), g_2(y_1, y_2)) \tag{2.5}$$
$$y_2 = f_2(g_1(y_1, y_2), g_2(y_1, y_2)). \tag{2.6}$$

Exercise 9 The conditions of this theorem are sufficient, but not necessary. To see this consider the example

$$\begin{aligned} y_1 &= x_1^3 \\ y_2 &= x_2^3. \end{aligned}$$

Show that this has a global inverse, but its Jacobian is singular at $(0, 0)$.

Exercise 10 Show using (2.3) and (2.3) that for all $x \in \mathcal{D}$

$$\underbrace{\begin{bmatrix} \partial g_1(y_1,y_2)/\partial y_1 & \partial g_1(y_1,y_2)/\partial y_2 \\ \partial g_2(y_1,y_2)/\partial y_1 & \partial g_2(y_1,y_2)/\partial y_2 \end{bmatrix}_{y=f(x)}}_{J_g(y)_{y=f(x)}} \underbrace{\begin{bmatrix} \partial f_1(x_1,x_2)/\partial x_1 & \partial f_1(x_1,x_2)/\partial x_2 \\ \partial f_2(x_1,x_2)/\partial x_1 & \partial f_2(x_1,x_2)/\partial x_2 \end{bmatrix}}_{J_f(x)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Similarly, using (2.5) and (2.6) that for all $y \in \mathcal{U}$

$$\begin{bmatrix} \partial f_1(x_1,x_2)/\partial x_1 & \partial f_1(x_1,x_2)/\partial x_2 \\ \partial f_2(x_1,x_2)/\partial x_1 & \partial f_2(x_1,x_2)/\partial x_2 \end{bmatrix}_{x=g(y)} \begin{bmatrix} \partial g_1(y_1,y_2)/\partial y_1 & \partial g_1(y_1,y_2)/\partial y_2 \\ \partial g_2(y_1,y_2)/\partial y_1 & \partial g_2(y_1,y_2)/\partial y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 7 *Polar and Cartesian Coordinates*
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\begin{aligned} y_1 &= f_1(x_1,x_2) = x_1\cos(x_2) \\ y_2 &= f_2(x_1,x_2) = x_1\sin(x_2). \end{aligned}$$

The Jacobian matrix of f is

$$J_f(x_1,x_2) = \begin{bmatrix} \partial f_1(x_1,x_2)/\partial x_1 & \partial f_1(x_1,x_2)/\partial x_2 \\ \partial f_2(x_1,x_2)/\partial x_1 & \partial f_2(x_1,x_2)/\partial x_2 \end{bmatrix} = \begin{bmatrix} \cos(x_2) & -x_1\sin(x_2) \\ \sin(x_2) & x_1\cos(x_2) \end{bmatrix}$$

and

$$\det J_f(x_1,x_2) = x_1\cos^2(x_2) + x_1\sin^2(x_2) = x_1.$$

Around any point $x_0 = (x_{01}, x_{02})$ with $x_{01} \neq 0$ this function has an inverse. For example, take $x_0 = (1, 3\pi)$ which is mapped to $(-1, 0)$ where in some neighborhood of $(-1, 0)$ the inverse is given by[2]

$$\begin{aligned} x_1 &= g_1(y_1,y_2) = \sqrt{y_1^2 + y_2^2} \\ x_2 &= g_2(y_1,y_2) = \tan^{-1}(y_1,y_2) + 2\pi. \end{aligned}$$

In this *particular* example it turns out that f is invertible for all $x \in \mathcal{D}$ where

$$\mathcal{D} \triangleq \left\{(x_1,x_2) \in \mathbb{R}^2 |\ x_1 > 0, 2\pi < x_2 < 4\pi\right\}.$$

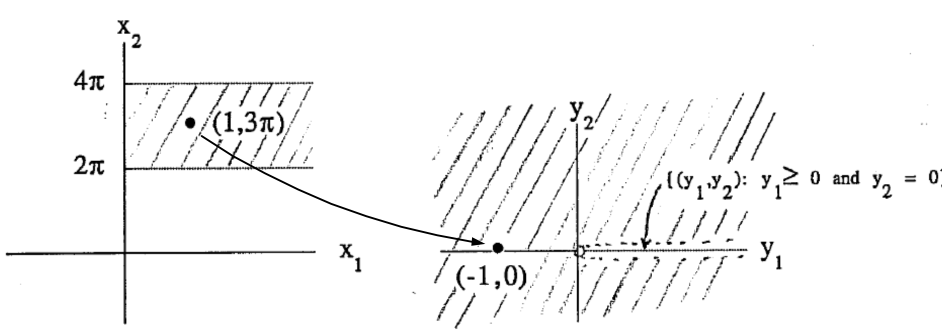


FIGURE 2.13. $\mathcal{D} \triangleq \left\{(x_1,x_2) \in \mathbb{R}^2 |\ x_1 > 0, 2\pi < x_2 < 3\pi\right\}$, $\mathcal{U} \triangleq \left\{(y_1,y_2) \in \mathbb{R}^2 |\ \text{If } y_1 > 0 \text{ the } y_2 \neq 0\right\}$

[2]Recall that we are taking $\theta = \tan^{-1}(y_1,y_2)$ to be in the interval $0 \leq \theta < 2\pi$.

2.3 Manifolds and Coordinate Systems

As far as the examples in this book, a manifold is an n dimensional subset of $\mathbf{E}^N$ $(n \leq N)$ with a collection of coordinate systems attache to (defined on) it. Hopefully the following examples clarify what this means.

Example 8 *Spherical Coordinates for* $\mathbf{E}^3$
The standard spherical coordinate system is shown in Figure 2.14.

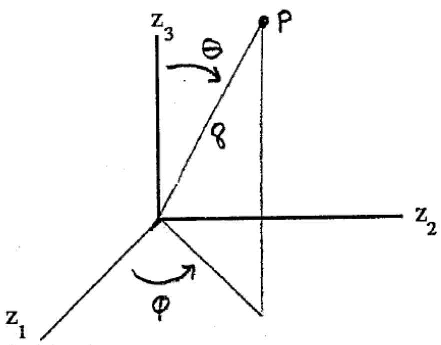


FIGURE 2.14. Spherical coordinate system.

With $x_1 = \rho, x_2 = \theta, x_3 = \varphi$, define the sets

$$\begin{aligned} \mathcal{D} &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\} \\ \mathcal{U} &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\}. \end{aligned}$$

The spherical coordinate system $\mathbf{z}(x_1, x_2, x_3) : \mathcal{D} \to \mathcal{U}$ is

$$\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3.$$

and is illustrated in Figure 2.15. $\mathbf{z}(x_1, x_2, x_3)$ is a one-to-one and onto function from $\mathcal{D}$ to $\mathcal{U}$. Further it is continuously differentiable on $\mathcal{D}$ and it has a continuously differentiable inverse $\mathbf{z}^{-1}(x_1, x_2, x_3) : \mathcal{U} \to \mathcal{D}$.

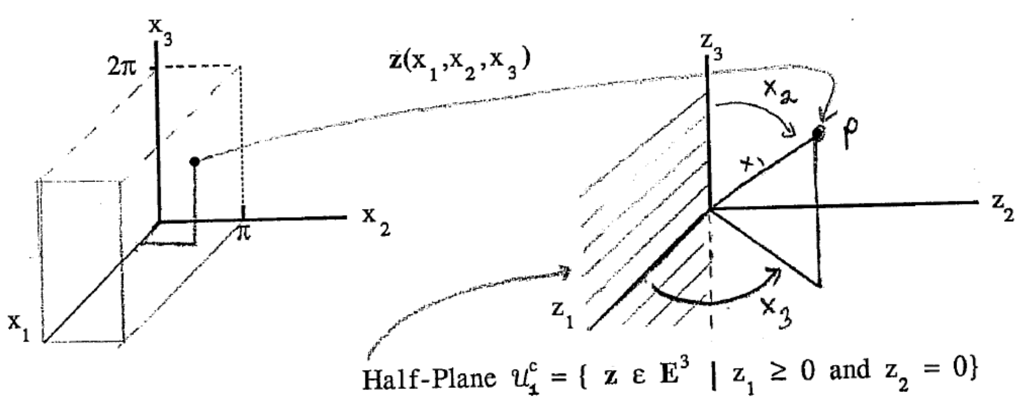


FIGURE 2.15. Spheical coordinate system as mapping from $\mathcal{D}$ to $\mathcal{U}$.

Note that this coordinate system mapping as defined in this example does not cover all of $\mathbf{E}^3$. In particular the points of $\mathbf{E}^3$ in $\mathcal{U}^c \triangleq \{\mathbf{z} \in \mathbf{E}^3 | \text{ If } z_1 \geq 0 \text{ the } z_2 = 0\}$ do not have spherical coordinates.[3] The reason for leaving this part of $\mathbf{E}^3$ out is so that $\mathbf{z}^{-1}(x_1, x_2, x_3)$ will be a continuously differentiable function from $\mathcal{U} \to \mathcal{D}$. If the points $\mathcal{U}^c$ are included then in any neighborhood[4] of $\mathcal{U}^c \triangleq \{\mathbf{z} \in \mathbf{E}^3 | \text{ If } z_1 \geq 0 \text{ the } z_2 = 0\}$ there are points whose $x_3 = \varphi$ coordinate is close to 0 and points whose $x_3 = \varphi$ coordinate are close 2π and thus $\mathbf{z}^{-1}(x_1, x_2, x_3)$ would not be continuous (let alone differentiable).

Example 9 *Cylindrical Coordinates*
The standard cylindrical coordinates (ρ, φ, z) are illustrated in Figure 2.16.

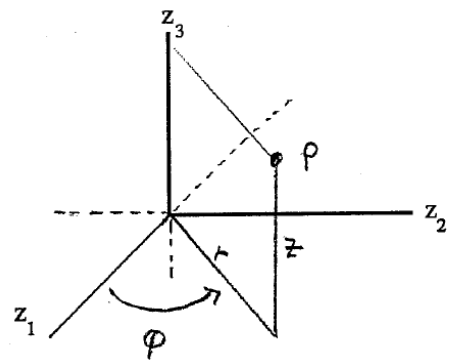


FIGURE 2.16. Cylindrical coordinates.

With $\bar{x}_1 = r, \bar{x}_2 = \varphi, \bar{x}_3 = z$, define the sets

$$\begin{aligned} \mathcal{D} &= \{(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 | \ \bar{x}_1 > 0, 0 < \bar{x}_2 < 2\pi, -\infty < \bar{x}_3 < \infty\} \\ \mathcal{U} &\triangleq \{\mathbf{z} \in \mathbf{E}^3 | \text{ If } z_1 > 0 \text{ the } z_2 \neq 0\}. \end{aligned}$$

The cylindrical coordinate system $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) : \mathcal{D} \to \mathcal{U}$ is

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \begin{bmatrix} \bar{x}_1 \cos(\bar{x}_2) \\ \bar{x}_1 \sin(\bar{x}_2) \\ \bar{x}_3 \end{bmatrix} \in \mathbf{E}^3.$$

and is illustrated in Figure 2.17. $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is a one-to-one and onto function from $\mathcal{D}$ to $\mathcal{U}$. Further it is continuously differentiable on $\mathcal{D}$ and it has a continuously differentiable inverse $\bar{\mathbf{z}}^{-1}(\bar{x}_1, \bar{x}_2, \bar{x}_3) : \mathcal{U} \to \mathcal{D}$.

[3] $\mathcal{U}^c$ denotes the complement of the set $\mathcal{U}$.
[4] Just take neighborhood of p to mean an open set that contains p.

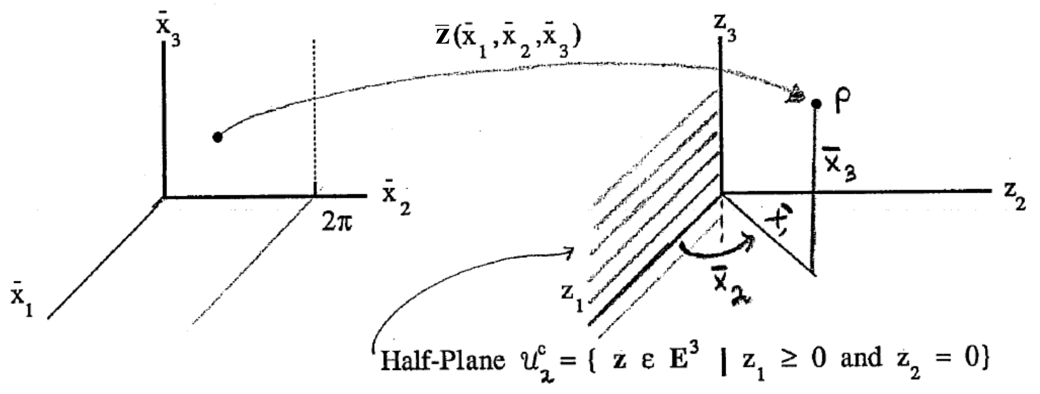


FIGURE 2.17. Cylindrical coordinate system as a mapping from $\mathcal{D}$ to $\mathcal{U}$.

Example 10 *Cartesian Coordinates*
The Cartesian coordinate system is shown in Figure 2.18.

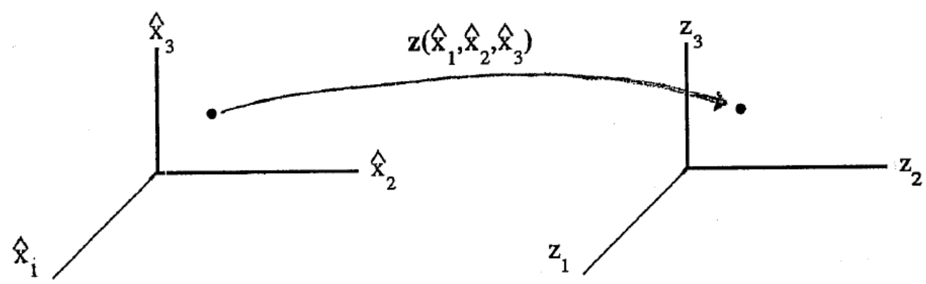


FIGURE 2.18. Cartesian coordinate system as mapping from $\mathcal{D}$ to $\mathcal{U}$.

The Cartesian coordinate system $\hat{\mathbf{z}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) : \mathcal{D} = \mathbb{R}^3 \rightarrow \mathcal{U} = \mathbb{R}^3$ is

$$\hat{\mathbf{z}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} \in \mathbf{E}^3.$$

These three examples above are meant to illustrate that a coordinate system for the manifold $\mathbf{E}^3$ consists of a set of coordinates $(x_1, x_2, x_3), (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, or $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ which lie in an open set of $\mathbb{R}^3$ with a mapping (function) that takes each 3-tuple of coordinates to a unique point p of $\mathbf{E}^3$.

The Manifold $\mathbf{S}^2$

The manifold $\mathbf{S}^2$ is a two dimensional subset of $\mathbf{E}^3$ defined by

$$\mathbf{S}^2 \triangleq \left\{ \mathbf{z} \in \mathbf{E}^3 \middle| \; z_1^2 + z_2^2 + z_3^2 = 1 \right\}.$$

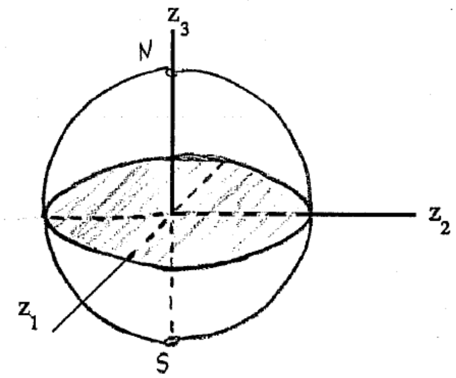


FIGURE 2.19.

Let's now look at some coordinate systems for $\mathbf{S}^2$.

Example 11 *Spherical Coordinate System for* $\mathbf{S}^2$
The spherical coordinates for $\mathbf{S}^2$ are illustrated in Figure 2.20.

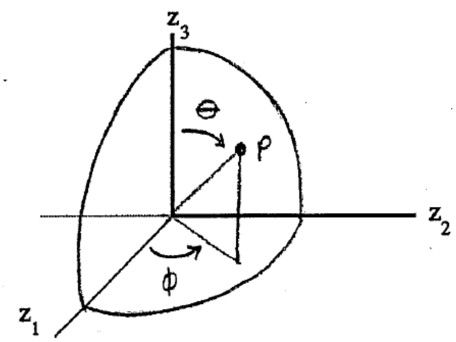


FIGURE 2.20. Spherical coordinates for $\mathbf{S}^2$.

With $x_1 = \theta$ and $x_2 = \varphi$, define the sets

$$\begin{aligned} \mathcal{R}_1 &= \left\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 < x_1 < \pi, 0 < x_2 < 2\pi\right\} \\ \mathcal{V}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ \text{If } z_1 > 0 \text{ then } z_2 \neq 0\right\}. \end{aligned}$$

The spherical coordinate system $\mathbf{z}(x_1, x_2) : \mathcal{R}_1 \to \mathcal{V}_1$ for $\mathbf{S}^2$ is

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} \sin(x_1)\cos(x_2) \\ \sin(x_1)\sin(x_2) \\ \cos(x_1) \end{bmatrix} \in \mathbf{S}^2.$$

and is illustrated in Figure 2.21. $\mathbf{z}(x_1, x_2)$ is a one-to-one and onto function from $\mathcal{R}_1$ to $\mathcal{V}_1$. Further it is continuously differentiable on $\mathcal{V}_1$ and it has a continuously differentiable inverse $\mathbf{z}^{-1}(z_1, z_2, z_3) : \mathcal{V}_1 \to \mathcal{R}_1$.

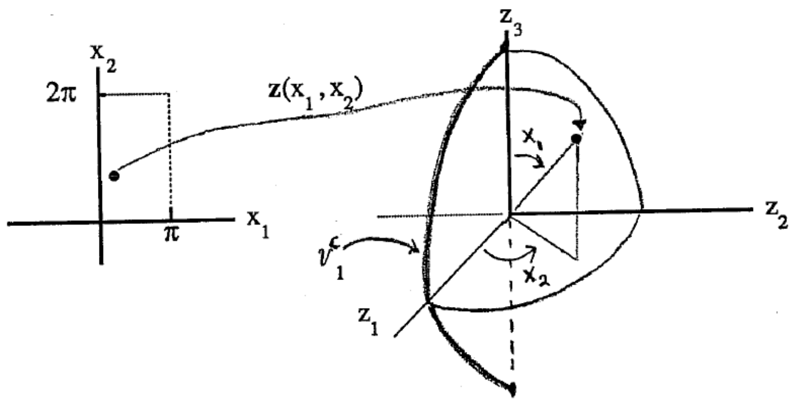


FIGURE 2.21. Spherical coordinate system for S^2.

This particular coordinate system leaves out the points $\mathcal{V}_1^c \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | \text{ If } z_1 > 0 \text{ then } z_2 = 0\right\}$. If this set is included then $\mathbf{z}^{-1}(x_1, x_2)$ would not be continuous.

Example 12 *A Second Spherical Coordinate System for* $\mathbf{S}^2$
Another spherical coordinate system for $\mathbf{S}^2$ is illustrated in Figure 2.22

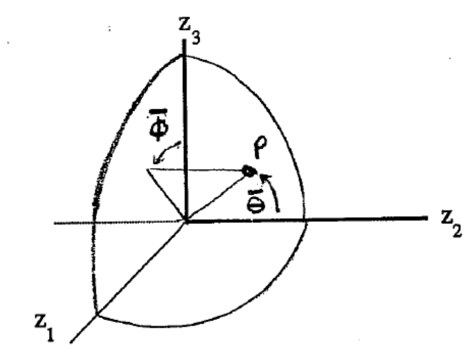


FIGURE 2.22. A second spherical coordinate system for $\mathbf{S}^2$.

With $\bar{x}_1 = \bar{\theta}$ and $\bar{x}_2 = \bar{\varphi}$ define the sets

$$\begin{aligned} \mathcal{R}_2 &= \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 | \ 0 < \bar{x}_1 < \pi, 0 < \bar{x} < 2\pi\right\} \\ \mathcal{V}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | \text{ If } z_1 > 0 \text{ the } z_2 \neq 0\right\}. \end{aligned}$$

This spherical coordinate system $\mathbf{z}(\bar{x}_1, \bar{x}_2) : \mathcal{R}_2 \to \mathcal{V}_2$ for $\mathbf{S}^2$ is

$$\mathbf{z}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \sin(\bar{x}_1)\sin(\bar{x}_2) \\ \cos(\bar{x}_1) \\ \sin(\bar{x}_1)\cos(\bar{x}_2) \end{bmatrix} \in \mathbf{S}^2.$$

and is illustrated in Figure 2.23. $\mathbf{z}(\bar{x}_1, \bar{x}_2)$ is a one-to-one and onto function from $\mathcal{R}_2$ to $\mathcal{V}_2$.

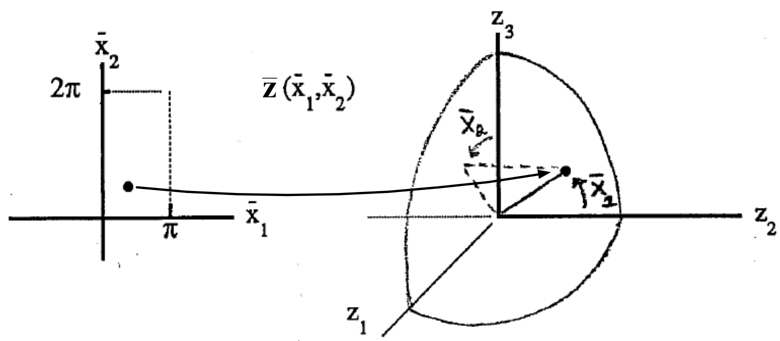


FIGURE 2.23. $\mathcal{R}_2 = \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 \middle|\ 0 < \bar{x}_1 < \pi, 0 < \bar{x} < 2\pi\right\}$, $\mathcal{V}_2 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 \middle|\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\}$.

Except for the north pole of $\mathbf{S}^2$, i.e., $\mathbf{z} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ all other points of $\mathbf{S}^2$ are either in $\mathcal{V}_2$ or in $\mathcal{V}_1$ from the previous example.

We now give some Cartesian coordinate systems for $\mathbf{S}^2$.

Example 13 *Northern Hemisphere Coordinate Patch*
Let

$$\begin{aligned} \mathcal{D}_1 &= \left\{(x_1, x_2) \in \mathbb{R}^2 \middle|\ 0 \leq x_1^2 + x_2^2 < 1\right\} \\ \mathcal{U}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}. \end{aligned}$$

Define the northern hemisphere coordinate map $\mathbf{z}(x_1, x_2) : \mathcal{D}_1 \to \mathcal{U}_1$ to be

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} \in \mathbf{S}^2. \tag{2.7}$$

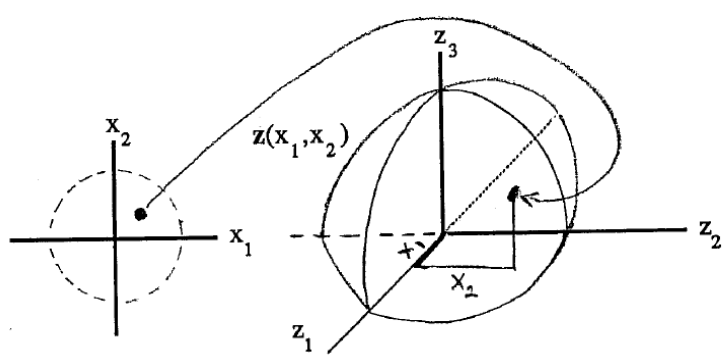


FIGURE 2.24. Northern hemisphere coordinates for $\mathbf{S}^2$.

This is a $1-1$ and onto mapping from $\mathcal{D}_1 \to \mathcal{U}_1$. The sets $\mathcal{D}_1$ and $\mathcal{U}_1$ along with the mapping (2.7) is called a *coordinate patch* or *coordinate chart*. All points of $\mathbf{S}^2$ with $z_3 > 0$ have a unique pair of northern hemisphere coordinates associated with them.

Example 14 *Southern Hemisphere Coordinate Patch*
Let

$$\begin{aligned}\mathcal{D}_2 &= \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 |\ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1\right\}\\ \mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}.\end{aligned}$$

Define the southern hemisphere coordinate map $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) : \mathcal{D}_2 \to \mathcal{U}_2$ to be

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ -\sqrt{1 - (\bar{x}_1^2 + \bar{x}_2^2)} \end{bmatrix} \in \mathbf{S}^2. \tag{2.8}$$

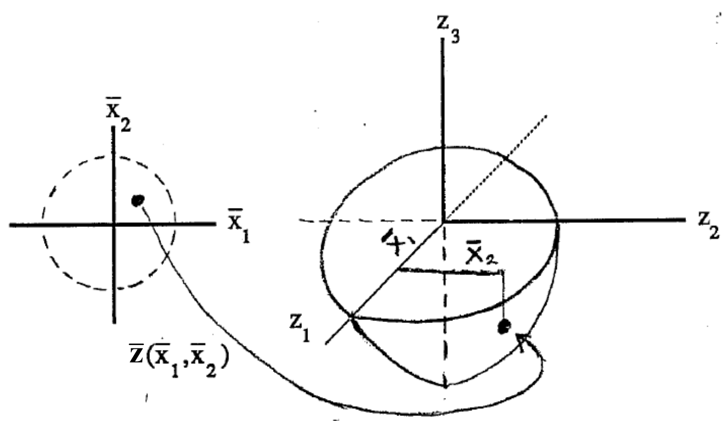


FIGURE 2.25. Southern hemisphere coordinates for $\mathbf{S}^2$.

This is a $1-1$ and onto mapping from $\mathcal{D}_2 \to \mathcal{U}_2$. The sets $\mathcal{D}_2$ and $\mathcal{U}_2$ along with the mapping (2.8) is called a *coordinate patch* or *coordinate chart*. All points of $\mathbf{S}^2$ with $z_3 < 0$ have a unique pair of southern hemisphere coordinates associated with them.

Example 15 *Eastern Hemisphere Coordinate Patch*
Let

$$\begin{aligned}\mathcal{D}_3 &= \left\{(x_1^*, x_2^*) \in \mathbb{R}^2 |\ 0 \le (x_1^*)^2 + (x_2^*)^2 < 1\right\}\\ \mathcal{U}_3 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}.\end{aligned}$$

Define the eastern hemisphere coordinate map $\mathbf{z}^*(x_1^*, x_2^*) : \mathcal{D}_3 \to \mathcal{U}_3$ to be

$$\mathbf{z}^*(x_1^*, x_2^*) = \begin{bmatrix} x_2^* \\ \sqrt{1 - ((x_1^*)^2 + (x_2^*)^2)} \\ x_1^* \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \tag{2.9}$$

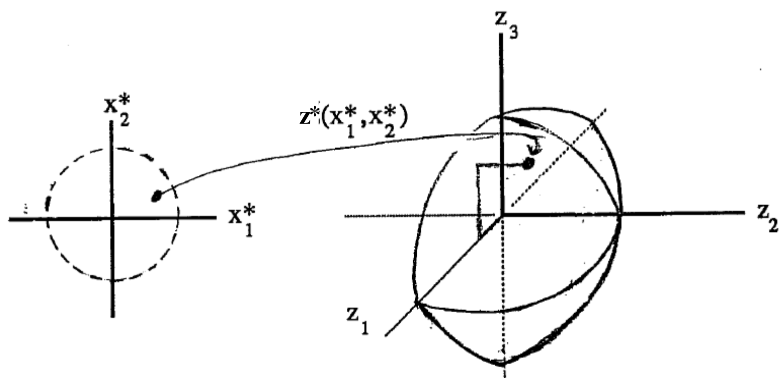


FIGURE 2.26. Eastern hemisphere coordinates for $\mathbf{S}^2$.

This is a $1-1$ and onto mapping from $\mathcal{D}_3 \to \mathcal{U}_3$. The sets $\mathcal{D}_3$ and $\mathcal{U}_3$ along with the mapping (2.9) is called a *coordinate patch* or *coordinate chart*. All points of $\mathbf{S}^2$ with $z_2 > 0$ have a unique pair of eastern hemisphere coordinates associated with them.

We can make similar coordinate patches (charts) for the remaining three hemispheres: $\mathcal{U}_4 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_2 < 0\right\}, \mathcal{U}_5 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_1 > 0\right\}$, and $\mathcal{U}_5 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_1 < 0\right\}$. Note that the six patches $\mathcal{U}_1$ through $\mathcal{U}_6$ cover the whole manifold $\mathbf{S}^2$. That is, each point of $\mathbf{S}^2$ is in at least one of the six patches (charts) and therefore each point has at least one set of coordinates associated with it.

How Mathematicians Describe Coordinate Systems

Mathematicians take a different perspective on defining coordinate systems. This is best explained by examples.

Example 16 *Spherical Coordinates for* $\mathbf{S}^2$
Recall with $x_1 = \theta, x_2 = \varphi$ and

$$\begin{aligned} \mathcal{R}_1 &= \left\{(x_1, x_2) \in \mathbb{R}^3 |\ 0 < x_1 < \pi, 0 < x_2 < 2\pi\right\} \\ \mathcal{V}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\}, \end{aligned}$$

the spherical coordinate map $\mathbf{z}(x_1, x_2) : \mathcal{R}_1 \to \mathcal{V}_1$ for $\mathbf{S}^2$ given by

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} \sin(x_1)\cos(x_2) \\ \sin(x_1)\sin(x_2) \\ \cos(x_1) \end{bmatrix} \in \mathbf{S}^2.$$

This is illustrated in Figure 2.27. The inverse of $\mathbf{z}(x_1, x_2)$ is

$$\mathbf{z}^{-1}(z_1, z_2, z_3) = \left(\tan^{-1}(\sqrt{z_1^2 + z_2^2}/z_3),\ \tan^{-1}(z_1, z_2)\right) = (x_1, x_2).$$

This is one-to-one and onto function from $\mathcal{V}_1$ to $\mathcal{R}_1$.

However, mathematicians view the coordinate patch as a mapping from the manifold to the coordinate system and write it as

$$\varphi_1(p) = \varphi_1\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \left(\tan^{-1}(z_3, \sqrt{z_1^2 + z_2^2}),\ \tan^{-1}(z_1, z_2)\right) = (x_1, x_2).$$

This is just $\mathbf{z}^{-1}(z_1, z_2, z_3)$.

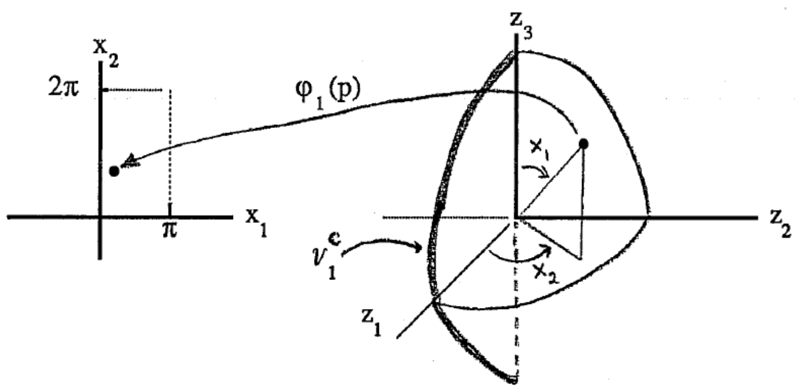


FIGURE 2.27. Spherical Coordinates for $\mathbf{S}^2$.

This makes a lot of sense because the points of the manifold are the fundamental objects of interest while the coordinates are used as a means to access the points conveniently. The mapping $\varphi_1{:}\mathcal{V}_1 \to \varphi_1(\mathcal{V}_1)$ takes any point $p \in \mathcal{V}_1 \subset S^2$ to its coordinates $\varphi_1(p)$ where $\mathcal{R}_1$ is denoted as $\varphi_1(\mathcal{V}_1)$.

Example 17 *Northern Hemisphere Coordinates for* $\mathbf{S}^2$
Recall with

$$\begin{aligned} \mathcal{D}_2 &= \left\{(x_1, x_2) \in \mathbb{R}^3 |\ 0 \le x_1^2 + x_2^2 < 1\right\} \\ \mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}, \end{aligned}$$

the northern hemisphere coordinate map $\mathbf{z}(x_1, x_2) : \mathcal{D}_2 \to \mathcal{U}_2$ is

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} \in \mathbf{S}^2.$$

The inverse of this map $\varphi_2 : \mathcal{U}_2 \to \varphi_2(\mathcal{U}_1)$ is

$$\varphi_2(p) = \varphi_2\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (z_1, z_2) = (x_1, x_2)$$

where $\varphi_2(\mathcal{U}_2) = \mathcal{D}_2$.

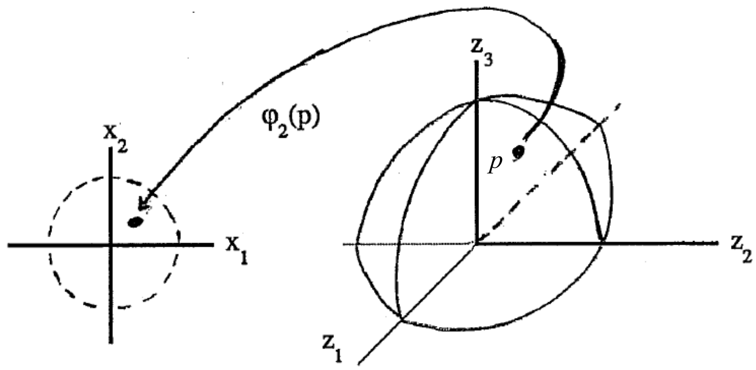


FIGURE 2.28. Northern hemisphere coordinates for $\mathbf{S}^2$.

Example 18 *The Manifold* $\mathbf{E}^3$ *with Spherical Coordinates*
As a last illustration we again look at $\mathbf{E}^3$ with the spherical coordinate system. With $x_1 = \rho, x_2 = \theta, x_3 = \varphi$, and

$$\begin{aligned} \mathcal{R} &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\} \\ \mathcal{U} &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\}, \end{aligned}$$

the spherical coordinate system $\mathbf{z}(x_1, x_2, x_3) : \mathcal{R} \to \mathcal{U}$ is

$$\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3.$$

The inverse of this is $\varphi(z_1, z_2, z_3) : \mathcal{U} \to \varphi(\mathcal{U})$ is

$$\varphi(p) = \varphi\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \left(\sqrt{z_1^2 + z_2^2 + z_3^2}, \tan^{-1}\left(\sqrt{z_1^2 + z_2^2}/z_3\right), \tan^{-1}(z_1, z_2)\right) = (x_1, x_2, x_3)$$

where $\varphi(\mathcal{U}) = \mathcal{R}$.

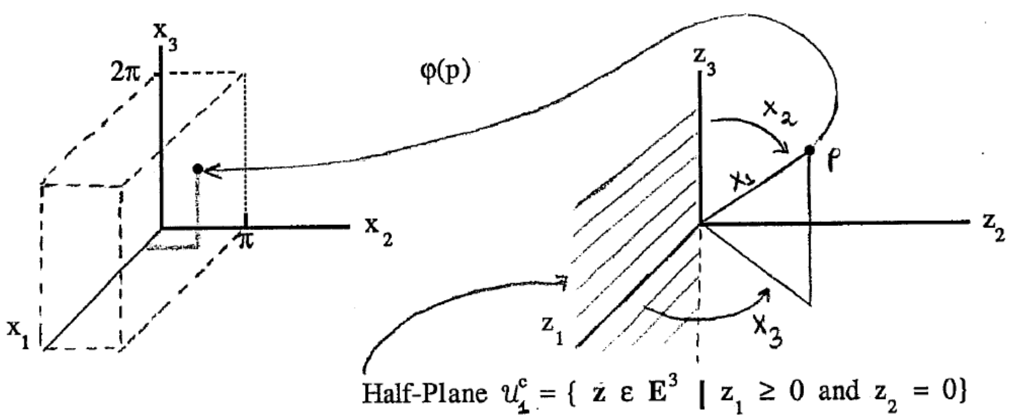


FIGURE 2.29. The manifold $\mathbf{E}^3$ with spherical coordinates.

Coordinate Transformations on Manifolds

We have seen any point p of a given manifold can have many different sets of coordinates attached to it. We now look at the relationship between different coordinates for the same point p of a manifold. Again, this is easiest to understand using examplse. A word about terminology: the particular coordinate patch (chart) being used is also referred to as the *local coordinates.*

Example 19 *Northern and Eastern Coordinate Patches on* $\mathbf{S}^2$
Patch 1 The northern hemisphere coordinate patch for $\mathbf{S}^2$ is $\varphi_1 : \mathcal{U}_1 \to \varphi_1(\mathcal{U}_1) \subset \mathbb{R}^2$ is

$$\varphi_1(p) = \varphi_1\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (z_1, z_2) = (x_1, x_2)$$

where

$$\begin{aligned} \mathcal{U}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\} \\ \varphi_1(\mathcal{U}_1) &= \left\{(x_1, x_2) \in \mathbb{R}^3 |\ 0 \le x_1^2 + x_2^2 < 1\right\}. \end{aligned}$$

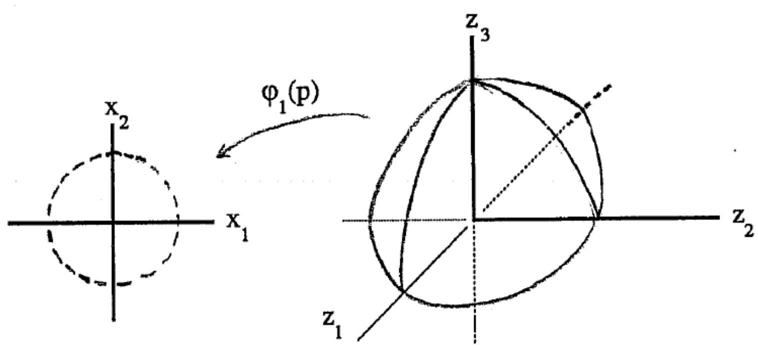


FIGURE 2.30. Northern coordinate patches on $\mathbf{S}^2$.

The inverse $\varphi_1^{-1} : \varphi_1(\mathcal{U}_1) \to \mathcal{U}_1$ is

$$\varphi_1^{-1}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

Patch 2 The eastern hemisphere coordinate patch for $\mathbf{S}^2$ is $\varphi_2 : \mathcal{U}_2 \to \varphi_2(\mathcal{U}_2) \subset \mathbb{R}^2$ is

$$\varphi_2(p) = \varphi_2\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (z_1, z_3) = (\bar{x}_1, \bar{x}_2)$$

where

$$\begin{aligned} \mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ z_2 > 0\right\} \\ \varphi_2(\mathcal{U}_2) &= \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 |\ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1\right\}. \end{aligned}$$

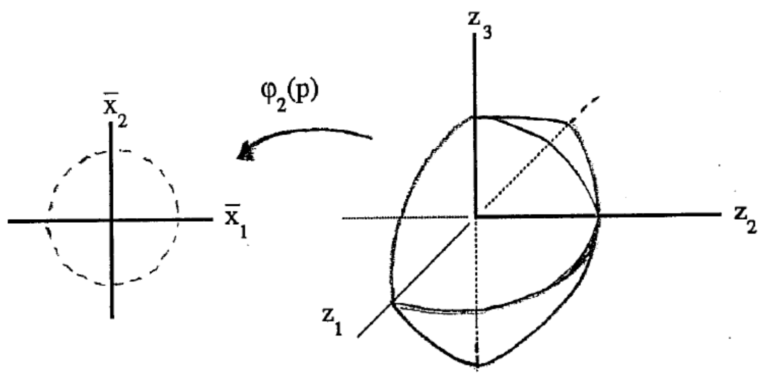


FIGURE 2.31. Eastern coordinate patches on $\mathbf{S}^2$.

The inverse $\varphi_2^{-1} : \varphi_2(\mathcal{U}_2) \to \mathcal{U}_2$ is

$$\varphi_2^{-1}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_1 \\ \sqrt{1-(\bar{x}_1^2+\bar{x}_2^2)} \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

The coordinate patch 1 covers the northern hemisphere $\mathcal{U}_1$ while coordinated patch 2 covers the eastern hemisphere $\mathcal{U}_2$. The intersection of these two sets is

$$\mathcal{U}_1 \cap \mathcal{U}_2 = \left\{ \mathbf{z} \in \mathbf{S}^2 \middle|\ z_2 > 0, z_3 > 0 \right\}$$

and is illustrated in Figure 2.32.

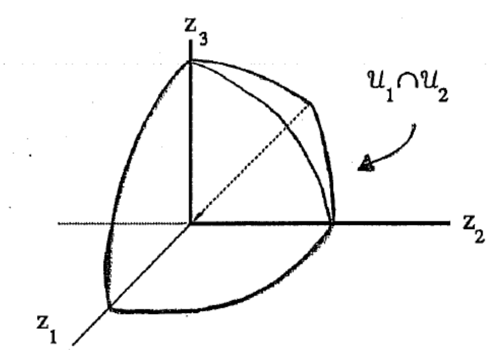


FIGURE 2.32. Intersection of the northern and eastern hemispheres.

Let $p \in \mathcal{U}_1 \cap \mathcal{U}_2$ and suppose we know the coordinates (x_1, x_2) of p in $\mathcal{U}_1$ (northern hemisphere) are known. What are the corresponding coordinates $(\bar{x}_1, \bar{x}_2)$ of p in $\mathcal{U}_2$ (eastern hemisphere)? Well the northern hemisphere coordinates (x_1, x_2) of p in $\mathcal{U}_1$ correspond to the point

$$\varphi_1^{-1}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1-(x_1^2+x_2^2)} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

This point has eastern hemisphere coordinates

$$\varphi_2(\varphi_1^{-1}(x_1, x_2)) = \varphi_2\left(\begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1-(x_1^2+x_2^2)} \end{bmatrix}\right) = (x_1, \sqrt{1-(x_1^2+x_2^2)}) = (\bar{x}_1, \bar{x}_2).$$

That is, the change of coordinates from (x_1, x_2) to $(\bar{x}_1, \bar{x}_2)$ is

$$\bar{x}_1 = x_1 \tag{2.10}$$

$$\bar{x}_2 = \sqrt{1-(x_1^2+x_2^2)}. \tag{2.11}$$

Exercise 11 *Coordinate Transformation from the Eastern Hemisphere to the Northern Hemisphere*
For $p \in \mathcal{U}_1 \cap \mathcal{U}_2$ show that the coordinate transformation from $\mathcal{U}_2$ to $\mathcal{U}_1$ is the inverse of the transformation given by (2.10) and (2.11).

How Mathematicians View Coordinate Transformations

As already pointed out, mathematicians view the coordinate system mapping φ to be from the manifold $\mathcal{M}$ to the coordinates in $\mathbb{R}^n$. We illustrate this point of view using the previous example of the change of coordinates between the northern and eastern coordinate systems.

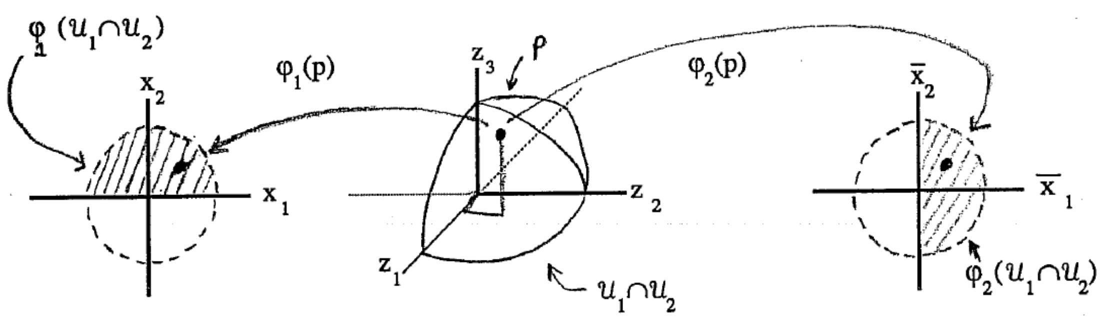


FIGURE 2.33. Change of coordinates between the northern hemisphere patch and the eastern hemisphere patch.

Now $\varphi_1(\mathcal{U}_1) = \left\{(x_1, x_2) \in \mathbb{R}^2 | \ 0 \le x_1^2 + x_2^2 < 1\right\}$ and

$$\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) = \left\{(x_1, x_2) | \ 0 \le x_1^2 + x_2^2 < 1, x_2 > 0\right\}$$

are all the (x_1, x_2) northern hemisphere coordinates of points $p \in \mathcal{U}_1 \cap \mathcal{U}_2$.
Similarly, $\varphi_2(\mathcal{U}_2) = \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 | \ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1\right\}$ and

$$\varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2) = \left\{(\bar{x}_1, \bar{x}_2) | \ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1, \bar{x}_1 > 0\right\}$$

are all the $(\bar{x}_1, \bar{x}_2)$ eastern hemisphere coordinates of points $p \in \mathcal{U}_1 \cap \mathcal{U}_2$. Then $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$ is given by

$$\varphi_2 \circ \varphi_1^{-1}(x_1, x_2) = \varphi_2(\varphi_1^{-1}(x_1, x_2)) = \varphi_2\left(\begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1-(x_1^2+x_2^2)} \end{bmatrix}\right) = (x_1, \sqrt{1-(x_1^2+x_2^2)}) = (\bar{x}_1, \bar{x}_2).$$

This is of course the same expression we got before.
Keep in mind that $\varphi_2 \circ \varphi_1^{-1}(x_1, x_2)$ is a one-to-one map from $\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \subset \mathbb{R}^2$ onto $\varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2) \subset \mathbb{R}^2$ and is nothing more than fancy notation for the change of coordinates between the northern hemisphere and the eastern hemisphere.

Example 20 *Spherical to Cylindrical Coordinates on* $\mathbf{E}^3$

Patch 1 Spherical Coordinates on $\mathbf{E}^3$

With

$$\begin{aligned}\mathcal{U}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 \middle| \text{ If } z_1 > 0 \text{ then } z_2 \neq 0\right\} \\ \varphi_1(\mathcal{U}_1) &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \middle| \; x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\}\end{aligned}$$

the spherical coordinate patch $\varphi_1 : \mathcal{U}_1 \to \varphi_1(\mathcal{U}_1)$ is given by

$$\varphi_1(p) = \varphi_1\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \left(\sqrt{z_1^2 + z_2^2 + z_3^2}, \tan^{-1}\left(z_3, \sqrt{z_1^2 + z_2^2}\right), \tan^{-1}(z_1, z_2)\right) \in \mathbb{R}^3.$$

The inverse $\varphi_1^{-1} : \varphi_1(\mathcal{U}_1) \to \mathcal{U}_1$ is

$$\varphi_1^{-1}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3.$$

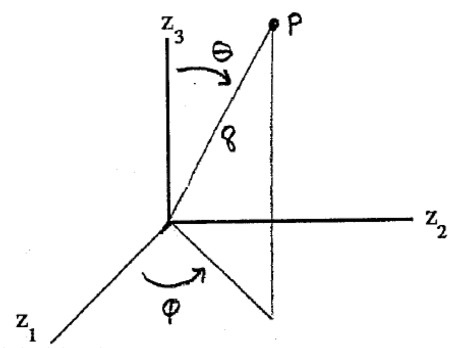


FIGURE 2.34. Spherical coordinates.

Patch 2 Cylindrical Coordinates on $\mathbf{E}^3$

With

$$\begin{aligned}\mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 \middle| \text{ If } z_1 > 0 \text{ the } z_2 \neq 0\right\} \\ \varphi_2(\mathcal{U}_2) &= \left\{(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 \middle| \; \bar{x}_1 > 0, 0 < \bar{x}_2 < 2\pi, -\infty < \bar{x}_3 < \infty\right\},\end{aligned}$$

the cylindrical coordinate patch $\varphi_2 : \mathcal{U}_2 \to \varphi_1(\mathcal{U}_2)$ is given by

$$\varphi_2(p) = \varphi_2\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (\sqrt{z_1^2 + z_2^2}, \tan^{-1}(z_1, z_2), z_3) \in \mathbb{R}^3.$$

The inverse $\varphi_2^{-1} : \varphi_2(\mathcal{U}_2) \to \mathcal{U}_2$ is

$$\varphi_2^{-1}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \begin{bmatrix} \bar{x}_1 \sin(\bar{x}_2)\cos(\bar{x}_3) \\ \bar{x}_1 \sin(\bar{x}_2)\sin(\bar{x}_3) \\ \bar{x}_1 \cos(\bar{x}_2) \end{bmatrix} \in \mathbf{E}^3.$$

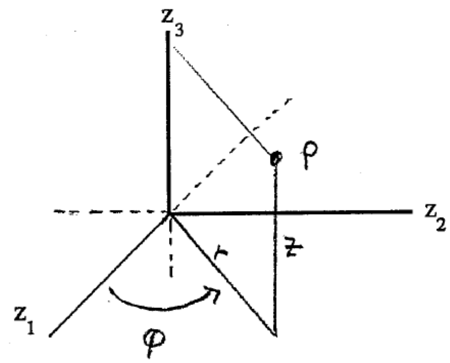


FIGURE 2.35. Cylindrical coordinates.

The change of coordinates from spherical to cylindrical are

$$\begin{aligned}
&\varphi_2 \circ \varphi_1^{-1}((x_1, x_2, x_3)) \\
&= \varphi_2 \left(\begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1\cos(x_2) \end{bmatrix} \right) \\
&= \left(\sqrt{(x_1\sin(x_2)\cos(x_3))^2 + (x_1 \sin(x_2)\sin(x_3))^2}, \tan^{-1}\left(x_1\sin(x_2)\cos(x_3),\ x_1\sin(x_2)\sin(x_3)\right), x_1\cos(x_2)\right) \\
&= (x_1\sin(x_2), x_3, x_1\cos(x_2)) \\
&= (\bar{x}_1, \bar{x}_2, \bar{x}_3).
\end{aligned}$$

That is, the change of coordinates from spherical to cylindrical are

$$\begin{aligned}
\bar{x}_1 &= x_1 \sin(x_2) \\
\bar{x}_2 &= x_3 \\
\bar{x}_3 &= x_1\cos(x_2).
\end{aligned}$$

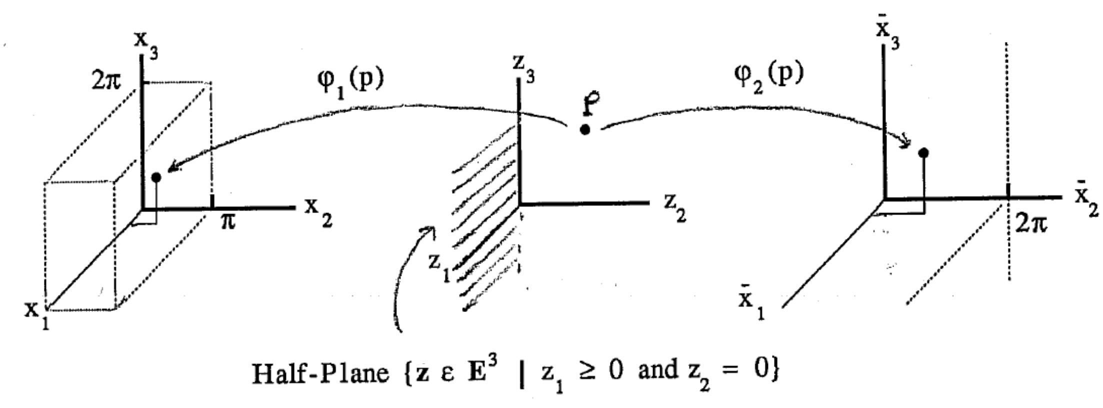


FIGURE 2.36. Change of coordinates from spherical to cylindrical on $\mathbf{E}^3$.

2.4 Tangent Vectors

We now look at the notion of a tangent vector to a manifold. Intuitively, the tangent vectors at a point p of a manifold give the possible directions one can move on the manifold at that point. We introduce tangent vectors using examples.

Tangent vectors on the manifold $\mathbf{S}^2$

Let the manifold be $\mathcal{M} = \mathbf{S}^2$, i.e., the unit sphere given by

$$\mathbf{S}^2 \triangleq \left\{ \mathbf{z} \in \mathbf{E^3} \middle| \; z_1^2 + z_2^2 + z_3^2 = 1 \right\}.$$

The next three examples show how three different coordinate charts can be used to formulate a tangent vector.

Example 21 *Tangent Vectors on* $\mathbf{S}^2$ *using the Northern Hemisphere Coordinate Patch*
As previously shown with

$$\begin{aligned} \mathcal{D}_1 &= \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \; 0 \leq x_1^2 + x_2^2 < 1 \right\} \\ \mathcal{U}_1 &\triangleq \left\{ \mathbf{z} \in \mathbf{S}^2 | z_3 > 0 \right\}, \end{aligned}$$

the northern hemisphere coordinate map $\mathbf{z}(x_1, x_2) : \mathcal{D}_1 \to \mathcal{U}_1$ is

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

Let $(x_1(t), x_2(t))$ be a curve in $\mathcal{D}_1$. Then

$$\mathbf{z}(x_1(t), x_2(t)) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \sqrt{1 - (x_1^2(t) + x_2^2(t))} \end{bmatrix}$$

is a curve on $\mathcal{U}_1 \subset \mathbf{S}^2$.

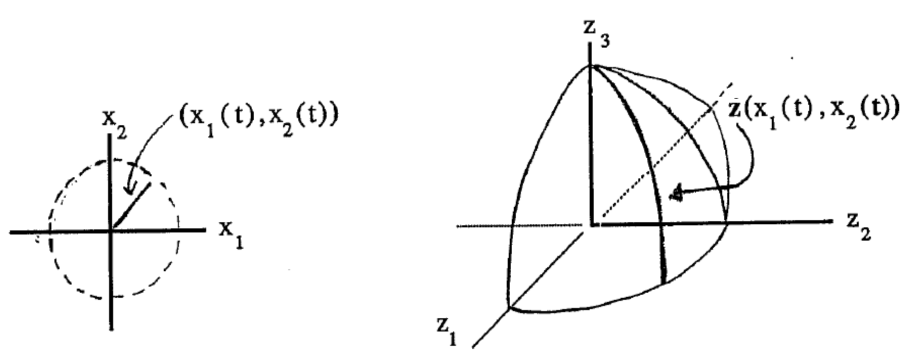


FIGURE 2.37. A curve on $\mathbf{S}^2$ usimg the northern hemisphere coordinate patch.

The tangent to the curve $(x_1(t), x_2(t))$ in $\mathcal{D}_1 \subset \mathbb{R}^2$ is $(dx_1(t)/dt, dx_2(t)/dt)$. The tangent to the curve

$\mathbf{z}(x_1(t), x_2(t))$ on $\mathcal{U}_1 \subset \mathbf{S}^2$ is

$$\begin{aligned}
\frac{d}{dt}\mathbf{z}(x_1(t), x_2(t)) &= \frac{\partial \mathbf{z}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{z}}{\partial x_2}\frac{dx_2}{dt} \\
&= \begin{bmatrix} 1 \\ 0 \\ \dfrac{-x_1(t)}{\sqrt{1-(x_1^2(t)+x_2^2(t))}} \end{bmatrix}\frac{dx_1}{dt} + \begin{bmatrix} 0 \\ 1 \\ \dfrac{-x_2(t)}{\sqrt{1-(x_1^2(t)+x_2^2(t))}} \end{bmatrix}\frac{dx_2}{dt} \\
&= \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x2}\frac{dx_2}{dt}.
\end{aligned}$$

The tangent vectors $\mathbf{z}_{x_1}$ and $\mathbf{z}_{x2}$ are defined on the northern hemisphere $\mathcal{U}_1$ or, in terms of local coordinates (x_1, x_2), $\mathbf{z}_{x_1}$ and $\mathbf{z}_{x2}$ are defined on $\mathcal{D}_1$. In particular, let

$$(x_1(t), x_2(t)) = (x_{01} + t, x_{02})$$

so that

$$\mathbf{z}_{x_1}(x_{01}, x_{02})$$

is the tangent vector to the curve $\mathbf{z}_{x_1}(x_{01} + t, x_{02})$ at $t = 0$. This is illustrated in Figure 2.38.

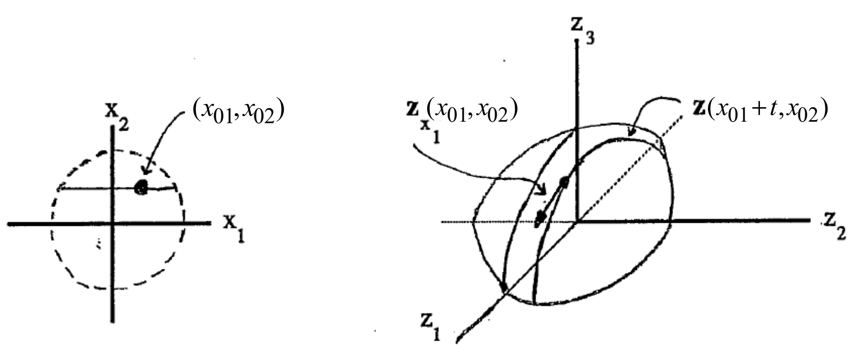


FIGURE 2.38. Tangent vector to the curve $\vec{z}(x_{01} + t, x_{02})$ at $t = 0$.

As similar interpretation holds for $\mathbf{z}_{x_2}(x_{01}, x_{02})$. Let $(x_1(t), x_2(t))$ in $\mathcal{D}_1$ be any curve goes through the coordinates (x_{01}, x_{02}) at $t = 0$, i.e., $(x_1(0), x_2(0)) = (x_{01}, x_{02})$. Then the tangent vector at $\mathbf{z}(x_{01}, x_{02})$ on $\mathbf{S}^2$ is

$$\left.\frac{d}{dt}\mathbf{z}(x_1(t), x_2(t))\right|_{t=0} = \mathbf{z}_{x_1}(x_{01}, x_{02}) \left.\frac{dx_1}{dt}\right|_{t=0} + \mathbf{z}_{x2}(x_{01}, x_{02}) \left.\frac{dx_2}{dt}\right|_{t=0}.$$

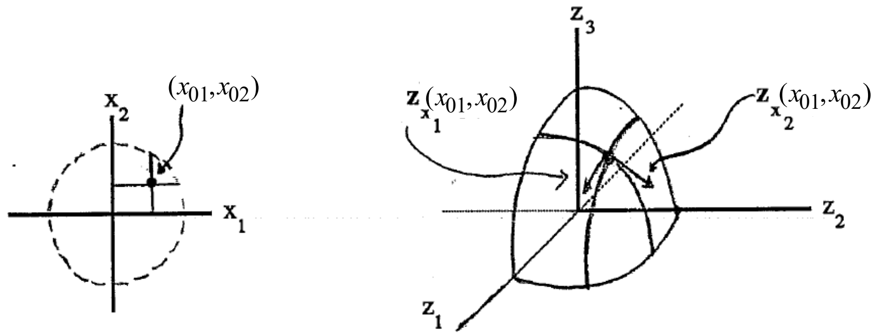


FIGURE 2.39. Basis vectors of the tangent space to $\mathbf{S}^2$ at $\vec{z}(x_{01}, x_{02})$

Any tangent vector to $\mathbf{S}^2$ at $\mathbf{z}(x_{01}, x_{02})$ is a linear combination of $\mathbf{z}_{x_1}(x_{01}, x_{02})$ and $\mathbf{z}_{x2}(x_{01}, x_{02})$. The *tangent space* at a point $p \in \mathbf{S}^2$ is the set of all tangent vectors at that point. In this example the tangent space at $\mathbf{z}(x_{01}, x_{02})$ is

$$\mathbf{T}_p(\mathbf{S}^2) = \left\{ a_1 \mathbf{z}_{x_1}(x_{01}, x_{02}) + a_2 \mathbf{z}_{x_2}(x_{01}, x_{02}) | \ (a_1, a_2) \in \mathbb{R}^2 \right\}.$$

$\mathbf{T}_p(\mathbf{S}^2)$ is a two dimensional *vector space* with basis vectors $\mathbf{z}_{x_1}(x_{01}, x_{02}), \mathbf{z}_{x_2}(x_{01}, x_{02})$. The pair (a_1, a_2) are referred to as the *components* of the tangent vector $a_1 \mathbf{z}_{x_1}(x_{01}, x_{02}) + a_2 \mathbf{z}_{x_2}(x_{01}, x_{02})$.

Example 22 *Tangent Vectors on* $\mathbf{S}^2$ *using the Eastern Hemisphere Coordinate Patch*
As shown previously, with

$$\begin{aligned} \mathcal{D}_2 &= \left\{ (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 | \ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1 \right\} \\ \mathcal{U}_2 &\triangleq \left\{ \mathbf{z} \in \mathbf{S}^2 | z_3 > 0 \right\}, \end{aligned}$$

the eastern hemisphere coordinate map $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) : \mathcal{D}_2 \to \mathcal{U}_2$ is

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_2 \\ \sqrt{1 - (\bar{x}_1^2 + \bar{x}_2^2)} \\ \bar{x}_1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

The left side of Figure 2.40 shows a curve in $\mathcal{D}_2$ which maps to a curve on $\mathbf{S}^2$ shown on the right side of Figure 2.40.

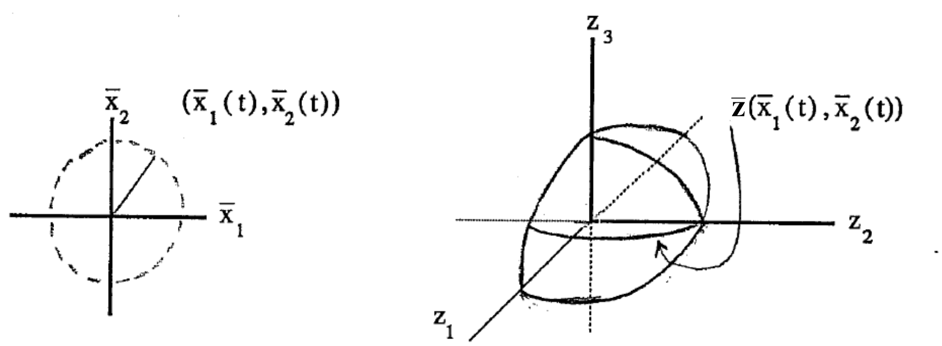


FIGURE 2.40. A curve on $\mathbf{S}^2$ using the eastern hemisphere coordinate patch.

More generally, for any curve $(\bar{x}_1(t), \bar{x}_2(t))$ in $\mathcal{D}_2$ the curve $\mathbf{z}(\bar{x}_1(t), \bar{x}_2(t))$ lies on the manifold $\mathbf{S}^2$ as illustrated in Figure 2.40. The tangent to this curve at $\bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t))$ is given by

$$\begin{aligned}
\frac{d}{dt}\bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t)) &= \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_1}\frac{d\bar{x}_1}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_2}\frac{d\bar{x}_2}{dt} \\
&= \begin{bmatrix} 0 \\ \dfrac{-\bar{x}_1(t)}{\sqrt{1-(\bar{x}_1^2(t)+\bar{x}_2^2(t))}} \\ 1 \end{bmatrix}\frac{d\bar{x}_1}{dt} + \begin{bmatrix} 1 \\ \dfrac{-\bar{x}_2(t)}{\sqrt{1-(\bar{x}_1^2(t)+\bar{x}_2^2(t))}} \\ 0 \end{bmatrix}\frac{d\bar{x}_2}{dt} \\
&= \bar{\mathbf{z}}_{\bar{x}_1}\frac{d\bar{x}_1}{dt} + \bar{\mathbf{z}}_{\bar{x}_2}\frac{d\bar{x}_2}{dt}.
\end{aligned}$$

The left side of Figure 2.41 shows two straight lines going through $(\bar{x}_{01}, \bar{x}_{02})$ and their corresponding curves on $\mathbf{S}^2$.

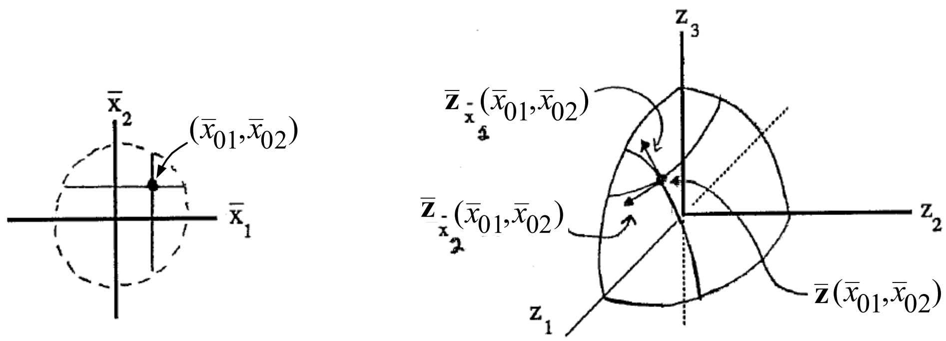


FIGURE 2.41. Tangent space using the eastern hemisphere coordinate chart.

Note that $\bar{\mathbf{z}}_{\bar{x}_1}, \bar{\mathbf{z}}_{\bar{x}_2}$ are defined for all points $p \in \mathcal{U}_2$ (eastern hemisphere) or, in terms of the local coordinates, $\bar{\mathbf{z}}_{\bar{x}_1}$ and $\bar{\mathbf{z}}_{\bar{x}_2}$ are defined for all $(\bar{x}_1, \bar{x}_2) \in \mathcal{D}_2$. Any curve going through the point p of $\mathbf{S}^2$ which has local coordinates $(\bar{x}_{01}, \bar{x}_{02})$ has a tangent vector that is a linear combination of $\bar{\mathbf{z}}_{\bar{x}_1}$ and $\bar{\mathbf{z}}_{\bar{x}_2}$. As in the previous example, the set of all possible tangent vectors at $p \in \mathcal{U}_2 \subset \mathbf{S}^2$ is the tangent space given by

$$\mathbf{T}_p(\mathbf{S}^2) = \left\{\bar{a}_1\bar{\mathbf{z}}_{\bar{x}_1}(\bar{x}_{01}, \bar{x}_{02}) + \bar{a}_2\bar{\mathbf{z}}_{\bar{x}_2}(\bar{x}_{01}, \bar{x}_{02}) |\ (\bar{a}_1, \bar{a}_2) \in \mathbb{R}^2\right\}.$$

$\bar{\mathbf{z}}_{\bar{x}_1}$ and $\bar{\mathbf{z}}_{\bar{x}_2}$ are the basis vectors of the two dimensional vector space $\mathbf{T}_p(\mathbf{S}^2)$ and $(\bar{a}_1, \bar{a}_2)$ are the components of the tangent vector $\bar{a}_1\bar{\mathbf{z}}_{\bar{x}_1}(\bar{x}_{01}, \bar{x}_{02}) + \bar{a}_2\bar{\mathbf{z}}_{\bar{x}_2}(\bar{x}_{01}, \bar{x}_{02})$.

Example 23 *Tangent Vectors on* $\mathbf{S}^2$ *using the Spherical Coordinate Patch*
With

$$\begin{aligned}
\mathcal{D}_3 &= \left\{(x_1^*, x_2^*) \in \mathbb{R}^2 |\ 0 < x_1^* < \pi, 0 < x_2^* < 2\pi\right\} \\
\mathcal{U}_3 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ \text{If } z_1 > 0 \text{ then } z_2 \neq 0\right\}.
\end{aligned}$$

The spherical coordinate patch $\mathbf{z}^*(x_1^*, x_2) : \mathcal{D}_3 \to \mathcal{U}_3$ for $\mathbf{S}^2$ is

$$\mathbf{z}^*(x_1^*, x_2^*) = \begin{bmatrix} \sin(x_1^*)\cos(x_2^*) \\ \sin(x_1^*)\sin(x_2^*) \\ \cos(x_1^*) \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

The left side of Figure 2.42 shows a curve in $\mathcal{D}_3$ which maps to a curve on $\mathbf{S}^2$ shown on the right side of Figure 2.42.

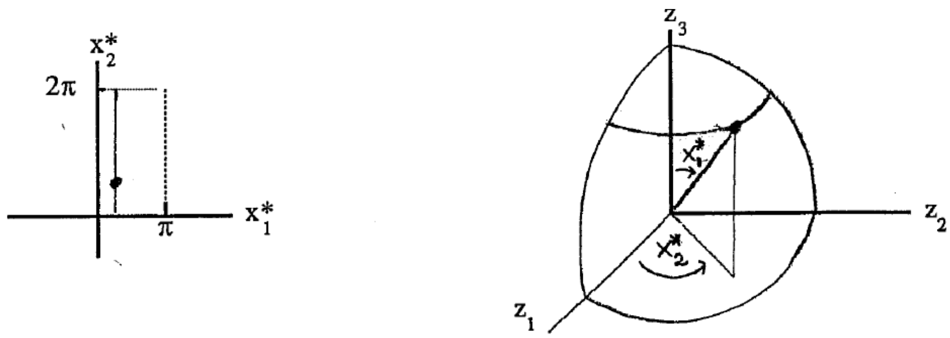


FIGURE 2.42. Spherical coordinate patch.

In general, for any curve $(x_1^*(t), x_2^*(t))$ in $\mathcal{D}_2$ the curve $\mathbf{z}^*(x_1^*(t), x_2^*(t))$ lies on the manifold $\mathbf{S}^2$. The tangent to this curve at $\mathbf{z}^*(x_1^*(t), x_2^*(t))$ is given by

$$\begin{aligned}
\frac{d}{dt}\mathbf{z}^*(x_1^*(t), x_2^*(t)) &= \frac{\partial \mathbf{z}^*}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_2^*}\frac{dx_2^*}{dt} \\
&= \begin{bmatrix} \cos(x_1^*)\cos(x_2^*) \\ \cos(x_1^*)\sin(x_2^*) \\ -\sin(x_1^*) \end{bmatrix}\frac{dx_1^*}{dt} + \begin{bmatrix} -\sin(x_1^*)\sin(x_2^*) \\ \sin(x_1^*)\cos(x_2^*) \\ \cos(x_1^*) \end{bmatrix}\frac{dx_2^*}{dt} \\
&= \mathbf{z}^*_{x_1^*}\frac{dx_1^*}{dt} + \mathbf{z}^*_{x_2^*}\frac{dx_2^*}{dt}.
\end{aligned}$$

Figure 2.43 illustrates the tangents to the curve $\mathbf{z}^*(x_{01}^*, t)$ for $0 < t < 2\pi$ and the curve $\mathbf{z}^*(t, x_{02}^*)$ for $0 < t < \pi$.

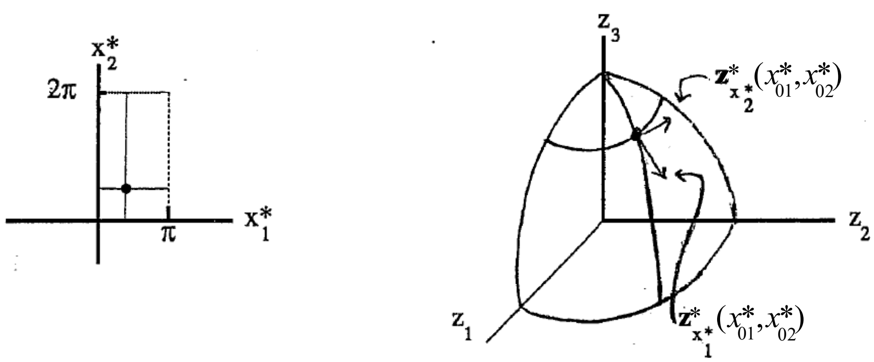


FIGURE 2.43. Basis vectors of the tangent space to $\mathbf{S}^2$ at $\mathbf{z}^*(x_{01}^*, x_{02}^*)$.

Again the set of all possible tangent vectors at $p = \mathbf{z}^*(x_1^*, x_2^*) \in \mathcal{U}_1 \subset \mathbf{S}^2$ is the tangent space given by

$$\mathbf{T}_p(\mathbf{S}^2) = \left\{ a_1^*\mathbf{z}^*_{x_1^*}(x_1^*, x_2^*) + a_2^*\mathbf{z}^*_{x_2^*}(x_1^*, x_2^*) |\ (a_1^*, a_2^*) \in \mathbb{R}^2 \right\}$$

where $\mathbf{z}^*_{x_1^*}(x_1^*, x_2^*)$ and $\mathbf{z}^*_{x_2^*}(x_1^*, x_2^*)$ are the basis vectors of the two dimensional vector space $\mathbf{T}_p(\mathbf{S}^2)$. For any vector $a_1^*\mathbf{z}^*_{x_1^*}(x_1^*, x_2^*) + a_2^*\mathbf{z}^*_{x_2^*}(x_1^*, x_2^*)$ in the tangent space, the pair $(a_1^*, a_2^*) \in \mathbb{R}^2$ are the components of this vector.

The three previous examples illustrated the idea of a manifold and its tangent vectors. We considered three different sets of local coordinates (patches/charts) and there are many points of $\mathbf{S}^2$ that are in all three coordinate charts. Specifically (see Figure 2.44)

$$\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 = \left\{ \mathbf{z} = \begin{bmatrix} z_1 & z_2 & z_2 \end{bmatrix}^T \in \mathbf{S}^2 \middle| \, z_2 > 0, z_3 > 0 \right\}.$$

For $0 < t < 1/2$ consider the curve $c(t)$ in $\mathbf{S}^2$ given by

$$c(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} \sin(\pi t) \\ \frac{1}{\sqrt{2}} \sin(\pi t) \\ \cos(\pi t) \end{bmatrix} \in \mathbf{S}^2.$$

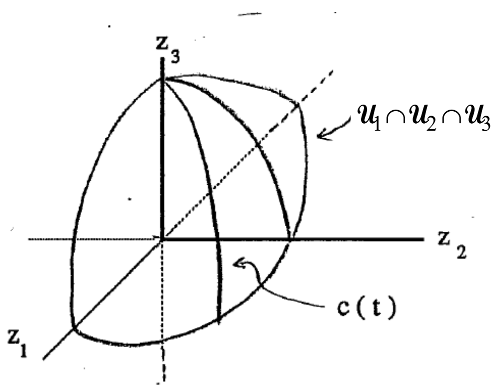


FIGURE 2.44. $c(t)$ on $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 = \left\{ \vec{z} \in \mathbf{S}^2 \middle| \, z_2 > 0, z_3 > 0 \right\}$.

The tangent vector along the curve c is

$$\frac{d}{dt} c(t) = \begin{bmatrix} \frac{\pi}{\sqrt{2}} \cos(\pi t) \\ \frac{\pi}{\sqrt{2}} \cos(\pi t) \\ -\pi \sin(\pi t) \end{bmatrix} \in \mathbf{S}^2$$

and is illustrated in Figure 2.45.

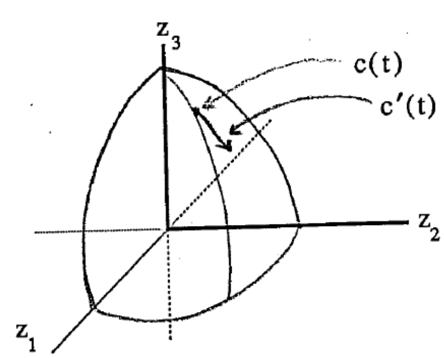


FIGURE 2.45. $c'(t)$ on $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 = \left\{ \vec{z} \in \mathbf{S}^2 \middle| \, z_2 > 0, z_3 > 0 \right\}$.

In the three coordinate systems the curve $c(t)$ is represented by

$$c(t) = \mathbf{z}(x_1(t), x_2(t)) = \bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t)) = \mathbf{z}^*(x_1^*(t), x_2^*(t))$$

where $(x_1(t), x_2(t)), (\bar{x}_1(t), \bar{x}_2(t))$, and $(x_1^*(t), x_2^*(t))$ are, respectively, the local coordinates for the curve $c(t)$ in the northern, eastern, and spherical coordinate systems. By the chain rule for partial differentiation the tangent vector represented in these three sets of local coordinates are

$$\begin{aligned}
\frac{dc}{dt} &= \frac{\partial \mathbf{z}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{z}}{\partial x_2}\frac{dx_2}{dt} = \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x_2}\frac{dx_2}{dt} \\
&= \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_1}\frac{d\bar{x}_1}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_2}\frac{d\bar{x}_2}{dt} = \bar{\mathbf{z}}_{\bar{x}_1}\frac{d\bar{x}_1}{dt} + \bar{\mathbf{z}}_{\bar{x}_2}\frac{d\bar{x}_2}{dt} \\
&= \frac{\partial \mathbf{z}^*}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_2^*}\frac{dx_2^*}{dt} = \mathbf{z}^*_{x_1^*}\frac{dx_1^*}{dt} + \mathbf{z}^*_{x_2^*}\frac{dx_2^*}{dt}.
\end{aligned}$$

Transformation of the Components of a Tangent Vector in $\mathbf{S}^2$

We just saw how the same curve on a manifold can represented in three different coordinate charts. Let's focus on the northern hemisphere and spherical coordinate charts where

$$c(t) = \mathbf{z}(x_1(t), x_2(t)) = \mathbf{z}^*(x_1^*(t), x_2^*(t))$$

with tangent vector

$$\frac{dc}{dt} = \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x_2}\frac{dx_2}{dt} = \mathbf{z}^*_{x_1^*}\frac{dx_1^*}{dt} + \mathbf{z}^*_{x_2^*}\frac{dx_2^*}{dt}.$$

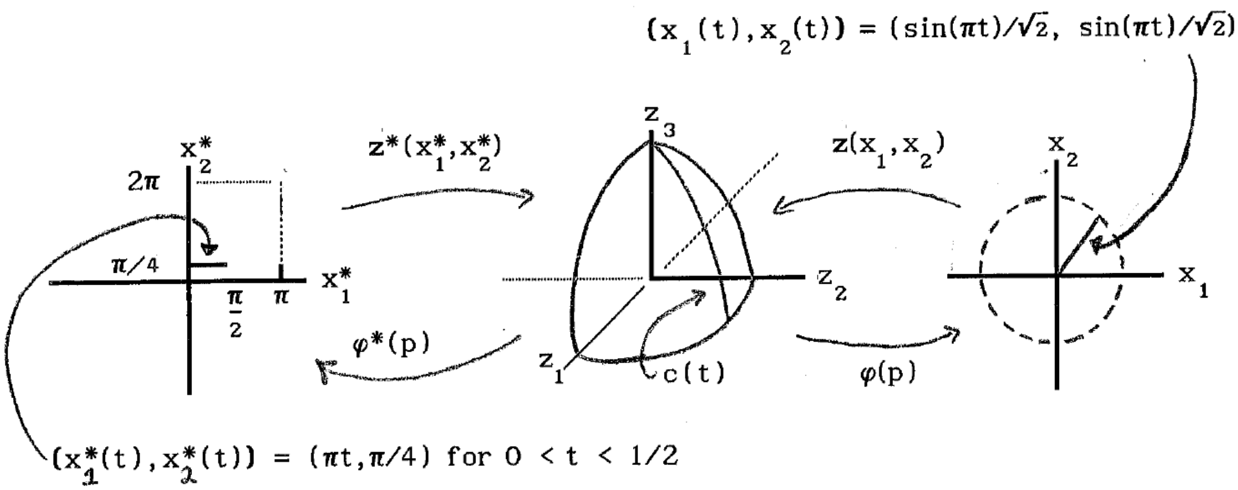


FIGURE 2.46. $c(t)$ in spherical and northern hemisphere coordinates.

We want to now find the relationship between the components of the tangent vector $\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}\right)$ in the northern hemisphere coordinates and the components of the tangent vector $\left(\frac{dx_1^*}{dt}, \frac{dx_2^*}{dt}\right)$ in spherical coordinates.

Recall with $x_1^* = \theta, x_2^* = \varphi$ and

$$\begin{aligned}
\mathcal{D}_1 &= \left\{(x_1^*, x_2^*) \in \mathbb{R}^3 \middle|\ 0 < x_1^* < \pi, 0 < x_2^* < 2\pi\right\} \\
\mathcal{U}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 \middle|\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\},
\end{aligned}$$

the spherical coordinate map $\varphi^{*-1}: \mathcal{D}_1 \to \mathcal{U}_1$ for $\mathbf{S}^2$ given by

$$\varphi^{*-1}(x_1^*, x_2^*) = \begin{bmatrix} \sin(x_1^*)\cos(x_2^*) \\ \sin(x_1^*)\sin(x_2^*) \\ \cos(x_1^*) \end{bmatrix} \in \mathbf{S}^2.$$

Further with

$$\begin{aligned} \mathcal{U}_2 &\triangleq \left\{ \mathbf{z} \in \mathbf{S}^2 \middle| \text{ If } z_1 > 0 \text{ the } z_2 \neq 0 \right\} \\ \mathcal{D}_2 &= \left\{ (x_1, x_2) \in \mathbb{R}^3 \middle| \ 0 < x_1 < \pi, 0 < x_2 < 2\pi \right\}. \end{aligned}$$

the northern hemisphere coordinate map $\varphi: \mathcal{U}_2 \to \mathcal{D}_2$ for $\mathbf{S}^2$ given by

$$\varphi\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right) = (z_1, z_2) = (x_1, x_2).$$

Then $\varphi \circ \varphi^{*-1}: \varphi^*(\mathcal{U}_1 \cap \mathcal{U}_2) \to \varphi(\mathcal{U}_1 \cap \mathcal{U}_2)$ given by

$$\varphi \circ \varphi^{*-1}(x_1^*, x_2^*) = \varphi\left(\varphi^{*-1}(x_1^*, x_2^*)\right) = (\sin(x_1^*)\cos(x_2^*), \sin(x_1^*)\sin(x_2^*)) = (x_1, x_2)$$

is the coordinate transformation from the spherical to the northern hemisphere coordinates. We may also write this as

$$\begin{aligned} x_1 &= \sin(x_1^*)\cos(x_2^*) \\ x_2 &= \sin(x_1^*)\sin(x_2^*). \end{aligned}$$

A curve $(x_1^*(t), x_2^*(t))$ in spherical coordinates gives the curve

$$\varphi^{*-1}(x_1^*(t), x_2^*(t)) = \begin{bmatrix} \sin(x_1^*(t))\cos(x_2^*(t)) \\ \sin(x_1^*(t))\sin(x_2^*(t)) \\ \cos(x_1^*(t)) \end{bmatrix} \in \mathbf{S}^2.$$

This is represented in the northern hemisphere coordinates by

$$(x_1(t), x_2(t)) = (\sin(x_1^*(t))\cos(x_2^*(t)), \sin(x_1^*(t))\cos(x_2^*(t))).$$

By the chain rule for partial differentiation we have

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{\partial x_1}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial x_1}{\partial x_2^*}\frac{dx_2^*}{dt} = \cos(x_1^*(t))\cos(x_2^*(t))\frac{dx_1^*}{dt} - \sin(x_1^*(t))\sin(x_2^*(t))\frac{dx_2^*}{dt} \\ \frac{dx_2}{dt} &= \frac{\partial x_2}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial x_2}{\partial x_2^*}\frac{dx_2^*}{dt} = \cos(x_1^*(t))\sin(x_2^*(t))\frac{dx_1^*}{dt} + \sin(x_1^*(t))\cos(x_2^*(t))\frac{dx_2^*}{dt} \end{aligned}$$

or in matrix notation

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \cos(x_1^*(t))\cos(x_2^*(t)) & -\sin(x_1^*(t))\sin(x_2^*(t)) \\ \\ \cos(x_1^*(t))\sin(x_2^*(t)) & \sin(x_1^*(t))\cos(x_2^*(t)) \end{bmatrix} \begin{bmatrix} \dfrac{dx_1^*}{dt} \\ \\ \dfrac{dx_2^*}{dt} \end{bmatrix}.$$

Exercise 12 *Northern Hemisphere Coordinates to Spherical Coordinates*
Show that the coordinate transformation for $\mathbf{S}^2$ from the northern hemisphere coordinates (x_1, x_2) to the spherical coordinates (x_1^*, x_2^*) is

$$\varphi^* \circ \varphi^{-1}(x_1, x_2) = \left(\sin^{-1}\left(\sqrt{x_1^2 + x_2^2}\right), \tan^{-1}(x_1, x_2) \right)$$

or

$$\begin{aligned} x_1^* &= \sin^{-1}\left(\sqrt{x_1^2+x_2^2}\right) \\ x_2^* &= \tan^{-1}(x_1,x_2). \end{aligned}$$

Show that

$$\begin{bmatrix} \frac{\partial x_1^*}{\partial x_1} & \frac{\partial x_1^*}{\partial x_2} \\ \frac{\partial x_2^*}{\partial x_1} & \frac{\partial x_2^*}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial x_1^*} & \frac{\partial x_1}{\partial x_2^*} \\ \frac{\partial x_2}{\partial x_1^*} & \frac{\partial x_2}{\partial x_2^*} \end{bmatrix}^{-1}_{x^*=\varphi^*(\varphi(x))}$$

This is equivalent to

$$\sum_{j=1}^{2} \frac{\partial x_\ell}{\partial x_j^*}\frac{\partial x_j^*}{\partial x_k} = \delta_k^\ell.$$

Exercise 13 *Northern Hemisphere Coordinates to Eastern Hemisphere Coordinates*

On $\mathbf{S}^2$ let $\mathbf{z}(x_1,x_2) : \mathcal{D}_1 = \left\{(x_1,x_2) \in \mathbb{R}^2 |\ 0 \le x_1^2 + x_2^2 < 1\right\} \to \mathcal{U}_1 \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}$ be the northern hemisphere coordinate map and $\bar{\mathbf{z}}(\bar{x}_1,\bar{x}_2) : \mathcal{D}_2 = \left\{(\bar{x}_1,\bar{x}_2) \in \mathbb{R}^2 |\ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1\right\} \to \mathcal{U}_2 = \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}$ be the eastern hemisphere coordinate map. Using the change of coordinates from (x_1,x_2) to $(\bar{x}_1,\bar{x}_2)$ find the change of coordinates for the components (a_1,a_2) of a tangent vector represented in the northern hemisphere patch to the components $(\bar{a}_1,\bar{a}_2)$ of the same tangent vector represented in an eastern hemisphere coordinate patch.

Tangent vectors on $\mathbf{E}^3$

We continue to look at tangent vectors, but now on the manifold $\mathbf{E}^3$.

Example 24 *Spherical Coordinates on* $\mathbf{E}^3$

With

$$\begin{aligned} \mathcal{D}_1 &= \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 |\ x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\} \\ \mathcal{U}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\}, \end{aligned}$$

recall the spherical coordinate system $\mathbf{z}(x_1,x_2,x_3) : \mathcal{D}_1 \to \mathcal{U}_1$ given by

$$\mathbf{z}(x_1,x_2,x_3) = \begin{bmatrix} x_1\sin(x_2)\cos(x_3) \\ x_1\sin(x_2)\sin(x_3) \\ x_1\cos(x_2) \end{bmatrix} \in \mathbf{E}^3.$$

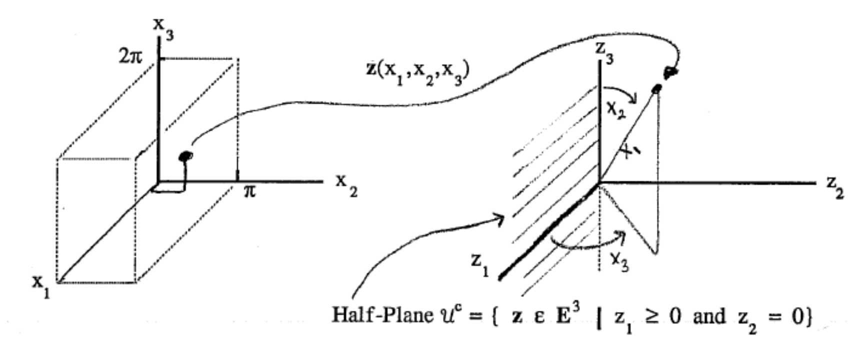


FIGURE 2.47. Spherical coordinates on $\mathbf{E}^3$

Let $(x_1(t), x_2(t), x_3(t))$ be a curve in $\mathcal{D}_1$ corresponding to the curve $\mathbf{z}(x_1(t), x_2(t), x_3(t))$ on $\mathcal{U}_1$ in $\mathbf{E}^3$. The tangent vector at time t is

$$\begin{aligned}\frac{d}{dt}\mathbf{z}(x_1(t), x_2(t), x_3(t)) &= \frac{\partial \mathbf{z}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{z}}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \mathbf{z}}{\partial x_3}\frac{dx_3}{dt} \\ &= \begin{bmatrix} \sin(x_2)\cos(x_3) \\ \sin(x_2)\sin(x_3) \\ \cos(x_2) \end{bmatrix}\frac{dx_1}{dt} + \begin{bmatrix} x_1\cos(x_2)\cos(x_3) \\ x_1\cos(x_2)\sin(x_3) \\ -x_1\sin(x_2) \end{bmatrix}\frac{dx_2}{dt} + \begin{bmatrix} -x_1\sin(x_2)\sin(x_3) \\ x_1\sin(x_2)\cos(x_3) \\ -x_1\sin(x_2) \end{bmatrix}\frac{dx_3}{dt} \\ &= \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x_2}\frac{dx_2}{dt} + \mathbf{z}_{x_3}\frac{dx_3}{dt}\end{aligned}$$

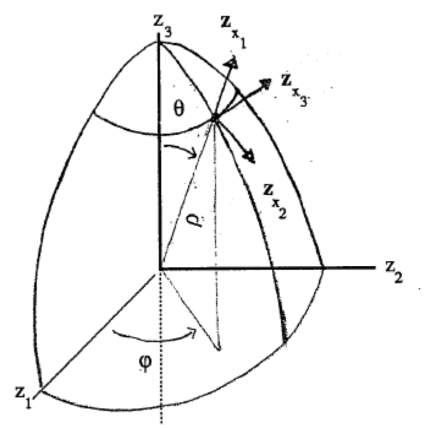


FIGURE 2.48. Tangent vectors on $\mathbf{E}^3$ using spherical coordinates

Any curve through a point $p \in \mathcal{U}_1 \subset \mathbf{E}^3$ with local spherical coordinates $(x_1(t), x_2(t), x_3(t))$ has a tangent vector which is a linear combination of $\mathbf{z}_{x_1}(x_1, x_2, x_3)$, $\mathbf{z}_{x_2}(x_1, x_2, x_3)$, and $\mathbf{z}_{x_3}(x_1, x_2, x_3)$. The set of all possible tangent vectors at a point p of the manifold of $\mathbf{E}^3$ is the tangent space $\mathbf{T}_p(\mathbf{E}^3)$ given by

$$\mathbf{T}_p(\mathbf{E}^3) = \left\{a_1\mathbf{z}_{x_1} + a_2\mathbf{z}_{x_2} + a_3\mathbf{z}_{x_3} \middle| \, (a_1, a_2, a_3) \in \mathbb{R}^3\right\}.$$

The tangent vectors $\mathbf{z}_{x_1}, \mathbf{z}_{x_2}$, and $\mathbf{z}_{x3}$ form a basis for the three dimensional vector space $\mathbf{T}_p(\mathbf{E}^3)$ with (a_1, a_2, a_3) the components of the vector $a_1\mathbf{z}_{x_1} + a_2\mathbf{z}_{x_2} + a_3\mathbf{z}_{x_3}$.

Example 25 *Cylindrical Coordinates on* $\mathbf{E}^3$

With

$$\begin{aligned}\mathcal{D}_2 &= \left\{(x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3 \middle| \, x_1^* > 0, 0 < x_2^* < 2\pi, -\infty < x_3^* < \infty\right\} \\ \mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 \middle| \text{ If } z_1 > 0 \text{ the } z_2 \neq 0\right\},\end{aligned}$$

the cylindrical coordinate system $\mathbf{z}(x_1^*, x_2^*, x_3^*) : \mathcal{D}_2 \to \mathcal{U}_2$ is

$$\mathbf{z}^*(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} x_1^*\cos(x_2^*) \\ x_1^*\sin(x_2^*) \\ x_3^* \end{bmatrix} \in \mathbf{E}^3.$$

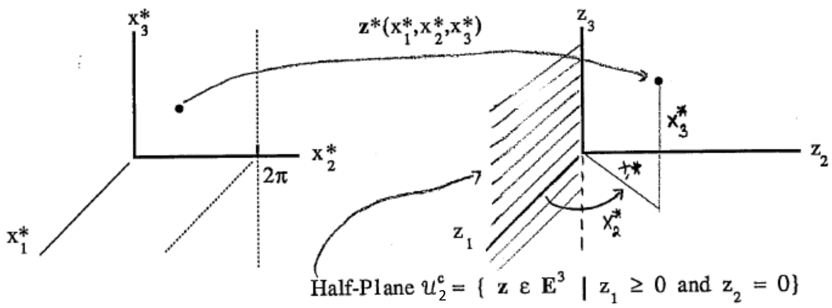


FIGURE 2.49. $\mathcal{D}_2 = \left\{(x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3 | \ x_1^* > 0, 0 < x_2^* < 2\pi, -\infty < x_3^* < \infty\right\}, \mathcal{U}_2 \triangleq \left\{\vec{z} \in \mathbf{E}^3 | \text{ If } z_1 > 0 \text{ the } z_2 \neq 0\right\}$

Let $(x_1^*(t), x_2^*(t), x_3^*(t))$ be a curve in $\mathcal{D}_2$ with corresponding curve $\mathbf{z}^*(x_1^*(t), x_2^*(t), x_3^*(t))$ on $\mathcal{U}_2$ in $\mathbf{E}$. The tangent vector to this curve at time t is

$$
\begin{aligned}
\frac{d}{dt}\mathbf{z}(x_1^*(t), x_2^*(t), x_3^*(t)) &= \frac{\partial \mathbf{z}^*}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_2^*}\frac{dx_2^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_3^*}\frac{dx_3^*}{dt} \\
&= \begin{bmatrix} \sin(x_2^*) \\ \sin(x_2^*) \\ 0 \end{bmatrix} \frac{dx_1^*}{dt} + \begin{bmatrix} -x_1^* \sin(x_2^*) \\ x_1^* \cos(x_2^*) \\ 0 \end{bmatrix} \frac{dx_2^*}{dt} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{dx_3^*}{dt} \\
&= \mathbf{z}^*_{x_1^*}\frac{dx_1^*}{dt} + \mathbf{z}^*_{x_2^*}\frac{dx_2^*}{dt} + \mathbf{z}^*_{x_3^*}\frac{dx_3^*}{dt}
\end{aligned}
$$

The curve $(x_1^*(t), x_2^*(t), x_3^*(t)) = (x_{01}^*, t, x_{03}^*)$ for $0 < t < 2\pi$ maps to $c(t) = \mathbf{z}^*(x_{01}^*, t, x_{03}^*)$ which is a circle in $\mathbf{E}^3$ in the $z_1 - z_2$ plane as shown in Figure 2.50. The tangent vector to the curve is

$$
\frac{d}{dt}c(t) = \frac{d}{dt}\mathbf{z}^*(x_{01}^*, t, x_{03}^*) = \mathbf{z}^*_{x_2^*}(x_{01}^*, t, x_{03}^*).
$$

.

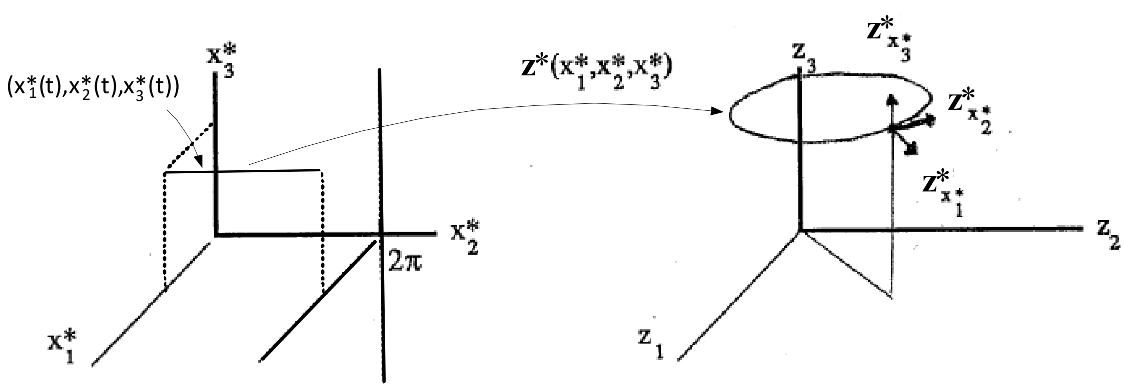


FIGURE 2.50. Curve in cylindrical coordinates.

Any curve through a point $p \in \mathcal{U}_2 \subset \mathbf{E}^3$ with local cylindrical coordinates $(x_1(t), x_2(t), x_3(t))$ has a tangent vector which is a linear combination of $\mathbf{z}^*_{x_1^*}(x_1^*, x_2^*, x_3^*)$, $\mathbf{z}^*_{x_2^*}(x_1^*, x_2^*, x_3^*)$, and $\mathbf{z}^*_{x_3^*}(x_1^*, x_2^*, x_3^*)$. The set of all possible tangent vectors at a point $p = \mathbf{z}^*(x_1^*, x_2^*, x_3^*)$ of the manifold of $\mathbf{E}^3$ is the tangent space $\mathbf{T}_p(\mathbf{E}^3)$

given by $\mathbf{T}_p(\mathbf{E}^3) = \left\{ a_1^* \mathbf{z}^*_{x_1^*} + a_2^* \mathbf{z}^*_{x_2^*} + a_3^* \mathbf{z}^*_{x_3^*} \middle| \; (a_1^*, a_2^*, a_3^*) \in \mathbb{R}^3 \right\}$. The tangent vector $a_1^* \mathbf{z}^*_{x_1^*} + a_2^* \mathbf{z}^*_{x_2^*} + a_3^* \mathbf{z}^*_{x_3^*}$ at $p = \mathbf{z}^*(x_1^*, x_2^*, x_3^*)$ has components a_1^*, a_2^*, and a_3^*.

Example 26 *Cartesian Coordinates on* $\mathbf{E}^3$
With

$$\begin{aligned}
\mathcal{D}_3 &= \left\{ (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 \middle| \; -\infty < \bar{x}_1 < \infty, -\infty < \bar{x}_2 < \infty, -\infty < \bar{x}_3 < \infty \right\} \\
\mathcal{U}_3 &\triangleq \left\{ \mathbf{z} \in \mathbf{E}^3 \middle| \text{ If } z_1 > 0 \text{ the } z_2 \neq 0 \right\},
\end{aligned}$$

the Cartesian coordinates $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) : \mathcal{D}_3 \to \mathcal{U}_3$ is

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} \in \mathbf{E}^3.$$

Let $(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$ be a curve in $\mathcal{D}_3$ corresponding to the curve $\bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$ on $\mathcal{U}_3$ in $\mathbf{E}^3$. The tangent vector at time t is

$$\begin{aligned}
\frac{d}{dt}\bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t)) &= \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_1}\frac{d\bar{x}_1}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_2}\frac{d\bar{x}_2}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_3}\frac{d\bar{x}_3}{dt} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\frac{d\bar{x}_1}{dt} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\frac{d\bar{x}_2}{dt} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\frac{d\bar{x}_3}{dt} \\
&= \bar{\mathbf{z}}_{\bar{x}_1}\frac{d\bar{x}_1}{dt} + \bar{\mathbf{z}}_{\bar{x}_2}\frac{d\bar{x}_2}{dt} + \bar{\mathbf{z}}_{\bar{x}_3}\frac{d\bar{x}_3}{dt}.
\end{aligned}$$

Any curve through a point $p \in \mathcal{U}_3 = \mathbf{E}^3$ with local Cartesian coordinates $(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$ has a tangent vector which is a linear combination of $\bar{\mathbf{z}}_{\bar{x}_1}, \bar{\mathbf{z}}_{\bar{x}_2}$, and $\bar{\mathbf{z}}_{\bar{x}_3}$. The set of all possible tangent vectors at a point $p = \bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ of the manifold of $\mathbf{E}^3$ is the tangent space $\mathbf{T}_p(\mathbf{E}^3)$ given by

$$\mathbf{T}_p(\mathbf{E}^3) = \left\{ \bar{a}_1 \bar{\mathbf{z}}_{\bar{x}_1} + \bar{a}_2 \bar{\mathbf{z}}_{\bar{x}_2} + \bar{a}_3 \bar{\mathbf{z}}_{\bar{x}_3} \middle| \; (\bar{a}_1, \bar{a}_2, \bar{a}_3) \in \mathbb{R}^3 \right\}.$$

The tangent vector $\bar{a}_1 \bar{\mathbf{z}}_{\bar{x}_1} + \bar{a}_2 \bar{\mathbf{z}}_{\bar{x}_2} + \bar{a}_3 \bar{\mathbf{z}}_{\bar{x}_3}$ at $p = \bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ has components $\bar{a}_1, \bar{a}_2$, and $\bar{a}_3$.

Let's now consider the representation of a helical curve in $\mathbf{E}^3$ using each of the three coordinate systems above.

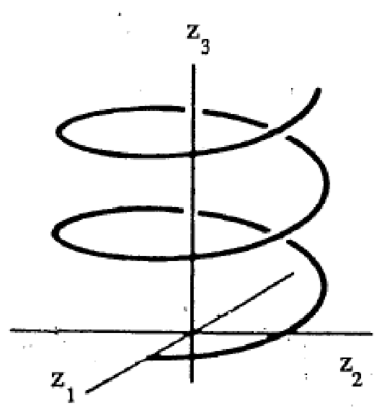


FIGURE 2.51. A helix in $\mathbf{E}^3$.

In *spherical coordinates* with $t > 0$ the helix is

$$(x_1(t), x_2(t), x_3(t)) = \left(\sqrt{1+t^2}, \tan^{-1}(1/t), 2\pi t\right)$$

as $\mathbf{z}(x_1(t), x_2(t), x_3(t))$ is given by

$$c(t) = \mathbf{z}(\sqrt{1+t^2}, \tan(1/t), 2\pi t) = \begin{bmatrix} \sqrt{1+t^2}\sin(\tan^{-1}(1/t))\cos(2\pi t) \\ \sqrt{1+t^2}\sin(\tan^{-1}(1/t))\sin(2\pi t) \\ \sqrt{1+t^2}\cos(\tan^{-1}(1/t)) \end{bmatrix} = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ t \end{bmatrix} \in \mathbf{E}^3$$

where $\sin(\tan^{-1}(1/t)) = 1/\sqrt{1+t^2}$ and $\cos(\tan^{-1}(1/t)) = t/\sqrt{1+t^2}$ was used. The components of the tangent vector are

$$\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\right) = \left(\frac{d}{dt}\sqrt{1+t^2}, \frac{d}{dt}\tan^{-1}(1/t), \frac{d}{dt}2\pi t\right) = \left(\frac{t}{\sqrt{1+t^2}}, -\frac{1}{1+t^2}, 2\pi\right).$$

In *cylindrical coordinates* with $t > 0$ the helix is

$$(x_1^*(t), x_2^*(t), x_3^*(t)) = (1, 2\pi t, t)$$

as $\mathbf{z}^*(x_1^*(t), x_2^*(t), x_3^*(t))$ is given by

$$c(t) = \mathbf{z}^*(x_1^*(t), x_2^*(t), x_3^*(t)) = \begin{bmatrix} x_1^*\cos(x_2^*) \\ x_1^*\sin(x_2^*) \\ x_3^* \end{bmatrix}_{x^*(t)=(1,2\pi t,t)} = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ t \end{bmatrix} \in \mathbf{E}^3.$$

The components of the tangent vector are

$$\left(\frac{dx_1^*}{dt}, \frac{dx_2^*}{dt}, \frac{dx_3^*}{dt}\right) = \left(\frac{d}{dt}1, \frac{d}{dt}2\pi t, \frac{d}{dt}t\right) = (0, 2\pi, 1).$$

In *Cartesian coordinates* with $t > 0$ the helix is

$$(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t)) = (\cos(2\pi t), \sin(2\pi t), t)$$

as $\bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$ is given by

$$c(t) = \bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t)) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \end{bmatrix}_{x^*(t)=(\cos(2\pi t),\sin(2\pi t),t)} = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ t \end{bmatrix} \in \mathbf{E}^3.$$

$$\left(\frac{d\bar{x}_1}{dt}, \frac{d\bar{x}_2}{dt}, \frac{d\bar{x}_3}{dt}\right) = \left(\frac{d}{dt}\cos(2\pi t), \frac{d}{dt}\sin(2\pi t), \frac{d}{dt}t\right) = (-2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1).$$

In summary, the local coordinate representation of the helical curve $c(t)$ and the components of its tangent vector are quite different in each of the coordinate systems.

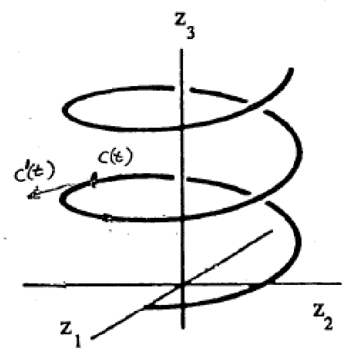


FIGURE 2.52. $\dfrac{dc}{dt} = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ t \end{bmatrix} \in \mathbf{E}^3.$

Representing the curve $c(t) \in \mathbf{E}^3$ in any of the local coordinates will still result in the *same* tangent vector $\dfrac{dc}{dt} \in \mathbf{E}^3$. This is seen explicitly as

$$\mathbf{z}(x_1(t), x_2(t), x_3(t)) = \mathbf{z}^*(x_1^*(t), x_2^*(t), x_3^*(t)) = \bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$$

we have

$$\begin{aligned} \frac{dc}{dt} &= \frac{\partial \mathbf{z}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{z}}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \mathbf{z}}{\partial x_3}\frac{dx_3}{dt} = \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x_2}\frac{dx_2}{dt} + \mathbf{z}_{x_3}\frac{dx_3}{dt} \\ &= \frac{\partial \mathbf{z}^*}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_2^*}\frac{dx_2^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_3^*}\frac{dx_3^*}{dt} = \mathbf{z}_{x_1^*}\frac{dx_1^*}{dt} + \mathbf{z}_{x_2^*}\frac{dx_2^*}{dt} + \mathbf{z}_{x_3^*}\frac{dx_3^*}{dt} \\ &= \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_1}\frac{d\bar{x}_1}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_2}\frac{d\bar{x}_2}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_3}\frac{d\bar{x}_3}{dt} = \bar{\mathbf{z}}_{\bar{x}_1}\frac{d\bar{x}_1}{dt} + \bar{\mathbf{z}}_{\bar{x}_2}\frac{d\bar{x}_2}{dt} + \bar{\mathbf{z}}_{\bar{x}_3}\frac{d\bar{x}_3}{dt}. \end{aligned}$$

Transformation of the Components of a Tangent Vector in $\mathbf{E}^3$

We now show how the components of the *same* tangent vector in different coordinate systems of $\mathbf{E}^3$ are related. To see how this is done consider the transformation between spherical and cylindrical coordinates charts as illustrated in Figure 2.53.

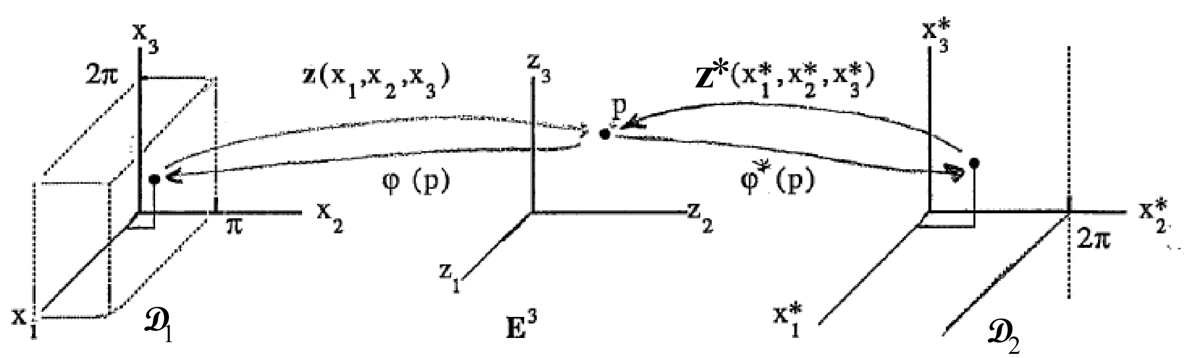


FIGURE 2.53. Transformation between spherical and cylindrical coordinates in $\mathbf{E}^3$.

The mapping from a point $\mathbf{z} \in \mathbf{E}^3$ to its spherical coordinates is

$$\varphi(p) = \varphi\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \left(\sqrt{z_1^2 + z_2^2 + z_3^2}, \tan^{-1}\left(z_3, \sqrt{z_1^2 + z_2^2}\right), \tan^{-1}(z_1, z_2)\right) = (x_1, x_2, x_3)$$

while the mapping from the spherical coordinates (x_1, x_2, x_3) to $\mathbf{E}^3$ is

$$\varphi^{-1}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3.$$

Further the mapping from a point $\mathbf{z} \in \mathbf{E}^3$ to its cylindrical coordinates is.

$$\varphi^*(p) = \varphi^*\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \left(\sqrt{z_1^2 + z_2^2}, \tan^{-1}(z_1, z_2), z_3\right) = (x_1^*, x_2^*, x_3^*)$$

while the mapping from the cylindrical coordinates (x_1^*, x_2^*, x_3^*) to $\mathbf{E}^3$ is

$$\varphi^{*-1}(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} x_1^* \sin(x_2^*) \\ x_1^* \cos(x_2^*) \\ x_3^* \end{bmatrix}.$$

The coordinate transformation between the spherical coordinates on

$$\mathcal{D}_1 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\}$$

to the cylindrical coordinates on

$$\mathcal{D}_2 = \left\{(x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3 |\ x_1^* > 0, 0 < x_2^* < 2\pi, -\infty < x_3^* < \infty\right\}$$

is

$$\begin{aligned} (x_1^*, x_2^*, x_3^*) = \varphi^* \circ \varphi^{-1}(x_1, x_2, x_3) &= \varphi^*(\varphi^{-1}(x_1, x_2, x_3)) \\ &= \varphi^*\left(\begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix}\right) \\ &= \left(x_1 \sin(x_2), x_3,\ x_1 \cos(x_2)\right) \end{aligned}$$

or

$$\begin{aligned} x_1^* &= x_1 \sin(x_2) \\ x_2^* &= x_3 \\ x_3^* &= x_1 \cos(x_2). \end{aligned}$$

A curve $(x_1(t), x_2(t), x_3(t))$ in the spherical coordinate system gives the curve $\varphi^{-1}(x_1(t), x_2(t), x_3(t))$ is represented in cylindrical coordinates by

$$(x_1^*(t), x_2^*(t), x_3^*(t)) = \left(x_1(t)\sin(x_2(t)), x_3(t),\ x_1(t)\cos(x_2(t))\right).$$

The components of the tangent vectors in these two coordinate charts are

$$\begin{pmatrix} \dfrac{dx_1^*}{dt} \\ \dfrac{dx_2^*}{dt} \\ \dfrac{dx_3^*}{dt} \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{\partial x_1^*}{\partial x_1} & \dfrac{\partial x_1^*}{\partial x_2} & \dfrac{\partial x_1^*}{\partial x_3} \\ \dfrac{\partial x_2^*}{\partial x_1} & \dfrac{\partial x_2^*}{\partial x_2} & \dfrac{\partial x_2^*}{\partial x_3} \\ \dfrac{\partial x_3^*}{\partial x_1} & \dfrac{\partial x_3^*}{\partial x_2} & \dfrac{\partial x_3^*}{\partial x_3} \end{pmatrix}}_{\left[\frac{\partial x_i^*}{\partial x_j}\right]} \begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} \sin(x_2) & -x_1\cos(x_2) & 0 \\ 0 & 0 & 1 \\ \cos(x_2) & -x_1\sin(x_2) & 0 \end{pmatrix} \begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{pmatrix}.$$

Exercise 14 *Cylindrical to Spherical Coordinate Transformation in* $\mathbf{E}^3$

(a) Compute the change of coordinates $\varphi \circ \varphi^{*-1}$ from cylindrical to spherical coordinates.

(b) Use your answer in part (a) to find the transformation matrix $\left[\dfrac{\partial x_i}{\partial x_j^*}\right]$ of the components of a tangent vector in spherical coordinates in terms of the components of the same tangent vector in cylindrical coordinates.

(c) Show that $\left[\dfrac{\partial x_i}{\partial x_j^*}\right]^{-1} = \left[\dfrac{\partial x_i^*}{\partial x_j}\right]$ which is equivalent to $\sum_{\ell=1}^{3} \dfrac{\partial x_i}{\partial x_\ell^*}\dfrac{\partial x_\ell^*}{\partial x_j} = \delta_j^i$.

Exercise 15 *Linear Manifold and its Tangent Vectors*
Consider the linear manifold defined by

$$\mathcal{M} = \left\{\mathbf{z} \in \mathbf{E}^3 \middle|\ z_2 - z_3 = -1\right\}$$

with the two coordinate charts given by $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathcal{M}$ given by

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + 1 \end{bmatrix}$$

and $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) : \mathbb{R}^2 \to \mathcal{M}$

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_1 + \bar{x}_2 \\ \bar{x}_1 \\ \bar{x}_1 + 1 \end{bmatrix}.$$

(a) Find a basis for the tangent space $\mathbf{T}_p(\mathcal{M})$ using the coordinate chart $\mathbf{z}(x_1, x_2)$.

(b) Find a basis for the tangent space $\mathbf{T}_p(\mathcal{M})$ using the coordinate chart $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2)$.

(c) Consider the curve $c(t) : \mathbb{R} \to \mathcal{M}$ given by

$$c(t) = \begin{bmatrix} t^3 \\ t^2 \\ t^2 - 1 \end{bmatrix}.$$

Show $c(t) \in \mathcal{M}$ for all t. Show that $(x_1(t), x_2(t)) = (t^2, t^3)$ and $(\bar{x}_1(t), \bar{x}_2(t)) = (t^2, t^3 - t^2)$ are the coordinate representations of $c(t)$ in the two coordinate system respectively.

(d) Compute the tangent vector to $c(t)$ in terms of $\mathbf{z}_{x_1}, \mathbf{z}_{x_2}$ as well as in terms of $\bar{\mathbf{z}}_{\bar{x}_1}, \bar{\mathbf{z}}_{\bar{x}_2}$.

(e) Compute the transformation from the components of the tangent vector in terms of $\mathbf{z}_{x_1}, \mathbf{z}_{x_2}$ to the components of the same tangent vector in terms of $\bar{\mathbf{z}}_{\bar{x}_1}, \bar{\mathbf{z}}_{\bar{x}_2}$.

Exercise 16 *Linear Manifold and its Tangent Vector*
Consider the linear manifold defined by

$$\mathcal{M} = \mathbf{E}^3$$

with the two coordinate charts given by $\mathbf{z}(x_1, x_2, x_3) : \mathbb{R}^3 \to \mathbf{E}^3$

$$\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_3 \end{bmatrix}$$

and $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) | \ \mathbb{R}^2 \to \mathcal{M}$

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \begin{bmatrix} \bar{x}_1 + \bar{x}_2 \\ \bar{x}_1 \\ \bar{x}_1 + \bar{x}_3 \end{bmatrix}.$$

Answer $(a), (b), (c), (d)$, and(e) of the previous exercise.

2.5 Problems

Problem 1 *Inverse Function Theorem*
Define a nonlinear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned} y_1 &= e^{x_1}\cos(x_2) \\ y_2 &= e^{x_1}\sin(x_2). \end{aligned}$$

(a) Compute the Jacobian matrix and its determinant.

(b) Is this transformation one-to-one?

(c) If the domain is restricted to $\left\{(x_1, x_2) \in \mathbb{R}^2 | \; 0 < x_2 < 2\pi\right\}$ show that transformation is one-to-one and find its inverse.

Problem 2 *Inverse Function Theorem*
Define a nonlinear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned} y_1 &= x_1^2 \\ y_2 &= x_2/x_1 \end{aligned}$$

(a) Compute the Jacobian matrix and its determinant.

(b) With $x_1 > 0$ for what region in the (x_1, x_2) plane is this transformation invertible. For this region find the corresponding range (image) of this transformation.

Problem 3 *Field Controlled DC Motor* [16]
The equations describing a separately excited DC motor are

$$\begin{aligned} J\frac{d\omega}{dt} &= K_T L_f i_f i_a - \tau_L \\ L\frac{di_a}{dt} &= -Ri_a - K_b L_f i_f \omega + V_{a0} \\ L_f\frac{di_f}{dt} &= -R_f i_f + V_f. \end{aligned}$$

Here ω is the rotor angular speed, V_{a0} is the (constant) armature voltage, i_a is the armature current, V_f is the field voltage, i_f is the field current, τ_L is the load torque, K_T is the torque constant, and K_b is the back-emf constant. The armature resistance and armature inductance are denoted by R and L,respectively, and the field resistance and field inductance are R_f and L_f, respectively.

Historically field controlled DC motors were used in mills for rolling out sheets of steel. This application required armature currents of $1000-2000$ Amperes to obtain the torques need to roll out the steel. However, there were not *variable* voltage sources available that could handle that amount of current. So a *constant* voltage source for the armature was used which could supply the large current. The field current i_f was on the order of only 25 Amperes and there were voltage sources that could provide a *varying* voltage while supplying that amount of current. By varying the field voltage V_f the current i_f and speed ω could then be controlled.

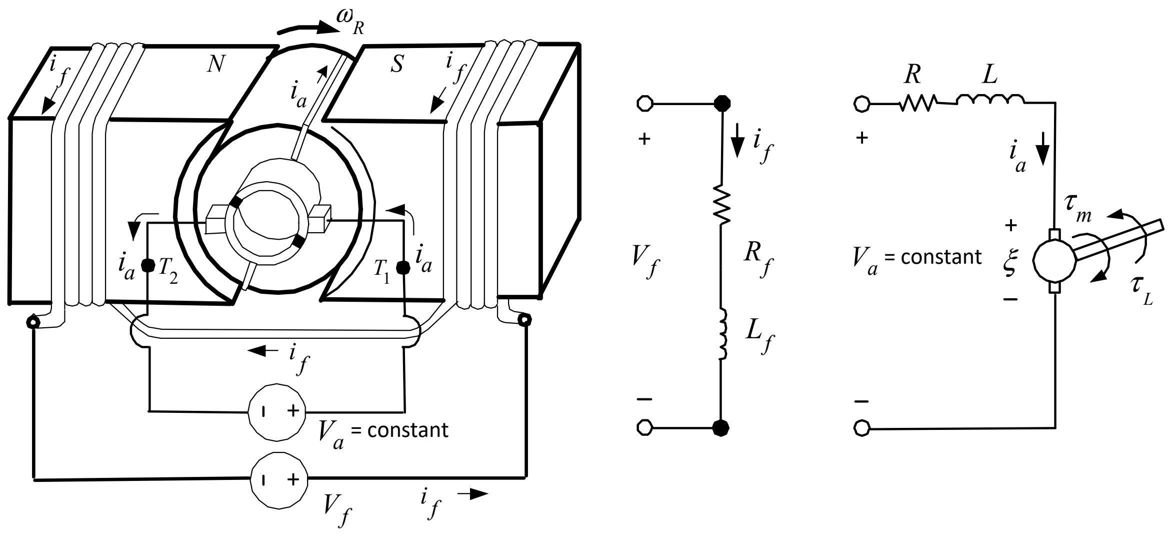


FIGURE 2.54. Field controlled DC motor. $\xi = K_b L_f i_f$ and $\tau_m = K_T L_f i_f i_a$.

Let $x_1 = i_f, x_2 = i_a, x_3 = \omega, u = V_f/L_f$, and define the constants $c_0 = V_{a0}/L, c_1 = R_f/L_f, c_2 = R/L, c_3 = K_b L_f/L, c_4 \triangleq K_T L_f/J, c_5 = 1/L$. The equations describing the field controlled DC motor are then

$$\begin{aligned} \frac{dx_1}{dt} &= -c_1 x_1 + u \\ \frac{dx_2}{dt} &= -c_2 x_2 - c_3 x_1 x_3 + c_0 \\ \frac{dx_3}{dt} &= c_4 x_1 x_2 - \tau_L/J \end{aligned} \tag{2.12}$$

or

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -c_1 x_1 \\ -c_2 x_2 - c_3 x_1 x_3 + c_0 \\ c_4 x_1 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1/J \end{bmatrix}}_{p} \tau_L.$$

(a) Define $T_1(x) = \dfrac{c_4}{c_3} x_2^2 + x_3^2 = (L i_a^2 + J\omega^2)/J$. With $\tau_L = 0$ and

$$\begin{aligned} x_1^* &= T_1(x) \\ x_2^* &= \mathcal{L}_f(T_1) \\ x_3^* &= \mathcal{L}_f^2(T_1) \end{aligned}$$

compute dx^*/dt.

(b) Use feedback linearization so that in the x^* coordinates the system is linear. What conditions on the state variables x_1, x_2, x_3 are needed to use this feedback?

3

Vector Fields and Differential Equations

Recall the northern hemisphere patch $\mathbf{z}(x_1, x_2) : \mathcal{D}_1 \to \mathcal{U}_1$ for $\mathbf{S}^2$

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1-(x_1^2+x_2^2)} \end{bmatrix} \in \mathbf{S}^2$$

where $\mathcal{D}_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 \le x_1^2 + x_2^2 < 1\right\}$ and $\mathcal{U}_1 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}$. Consider the system of differential equations defined on the northern hemisphere coordinates given by

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2 \\ \frac{dx_2}{dt} &= x_1 \end{aligned}$$

with $(x_1(0), x_2(0)) = (1/2, 0)$. The solution is

$$\begin{aligned} x_1(t) &= \frac{1}{2}\cos(t) \\ x_2(t) &= \frac{1}{2}\sin(t) \end{aligned}$$

and is shown in Figure 3.1.

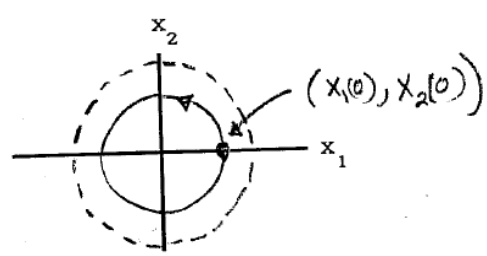


FIGURE 3.1. The northern hemisphere curve $(x_1(t), x_2(t)) = (\frac{1}{2}\cos(t), \frac{1}{2}\sin(t))$.

This curve in the northern hemisphere coordinates results in the curve on $\mathbf{S}^2$ given by

$$\mathbf{z}(x_1(t), x_2(t)) = \begin{bmatrix} \frac{1}{2}\cos(t) \\ \frac{1}{2}\sin(t) \\ \sqrt{\frac{3}{2}} \end{bmatrix} \in \mathbf{S}^2.$$

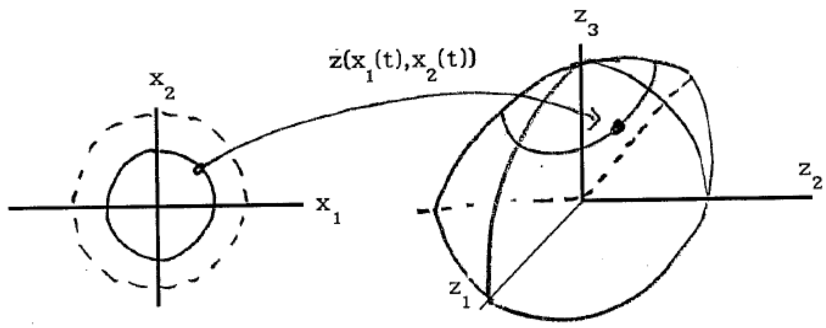


FIGURE 3.2. The curve $\begin{bmatrix} \frac{1}{2}\cos(t) \\ \frac{1}{2}\sin(t) \\ \sqrt{\frac{3}{2}} \end{bmatrix} \in \mathbf{S}^2$.

What is the differential equation describing this curve in the eastern hemisphere coordinate chart? The eastern hemisphere patch $\mathbf{z}^*(x_1^*, \hat{x_2}) : \mathcal{D}_2 \to \mathcal{U}_2$ for $\mathbf{S}^2$ is given by

$$\mathbf{z}^*(x_1^*, x_2^*) = \begin{bmatrix} x_2^* \\ \sqrt{1 - ((x_1^*)^2 + (x_2^*)^2)} \\ x_1^* \end{bmatrix} \in \mathbf{S}^2.$$

Here $\mathcal{D}_2 = \{(x_1^*, x_2^*) \in \mathbb{R}^2 |\ 0 \le (x_1^*)^2 + (x_2^*)^2 < 1\}$ and $\mathcal{U}_2 \triangleq \{\mathbf{z} \in \mathbf{S}^2 | z_2 > 0\}$. For $-\pi/2 < t < \pi/2$ this curve in $\mathcal{U}_1 \cap \mathcal{U}_2 \subset \mathbf{S}^2$, that is, it is in both the northern and eastern coordinate charts. The change of coordinates from the northern hemisphere patch to the eastern hemisphere patch is

$$\begin{aligned} x_1^* &= \sqrt{1 - (x_1^2 + x_2^2)} \\ x_2^* &= x_1 \end{aligned}$$

with inverse

$$\begin{aligned} x_1 &= x_2^* \\ x_2 &= \sqrt{1 - ((x_1^*)^2 + (x_2^*)^2)}. \end{aligned}$$

Then

$$\begin{aligned} \frac{dx_1^*}{dt} &= \frac{1}{2}\frac{-2x_1}{\sqrt{1 - (x_1^2 + x_2^2)}}\frac{dx_1}{dt} + \frac{1}{2}\frac{-2x_2}{\sqrt{1 - (x_1^2 + x_2^2)}}\frac{dx_2}{dt} = \frac{-x_1}{\sqrt{1 - (x_1^2 + x_2^2)}}(-x_2) + \frac{-x_2}{\sqrt{1 - (x_1^2 + x_2^2)}}x_1 = 0 \\ \frac{dx_2^*}{dt} &= \frac{dx_1}{dt} = -x_2 = -\sqrt{1 - ((x_1^*)^2 + (x_2^*)^2)} \end{aligned}$$

or

$$\begin{aligned} \frac{dx_1^*}{dt} &= 0 \\ \frac{dx_2^*}{dt} &= -\sqrt{1 - ((x_1^*)^2 + (x_2^*)^2)} \end{aligned}$$

with initial conditions $(x_1^*(0), x_2^*(0)) = \left(\sqrt{1 - (x_1^2(0) + x_2^2(0))}, x_1(0)\right) = (\sqrt{3}/2, 1/2)$.

Note that the differential equations in the x^* (eastern hemisphere) coordinates are nonlinear while in the x (northern hemisphere) coordinates they are linear. However, they describe the *same* curve on the manifold $\mathbf{S}^2$.

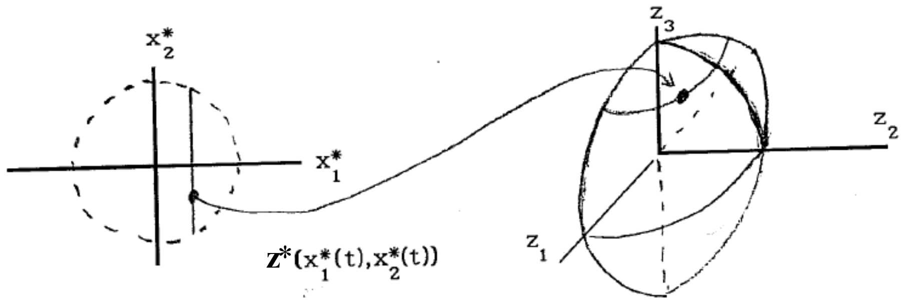


FIGURE 3.3. The curve $\begin{bmatrix} \frac{1}{2}\cos(t) \\ \frac{1}{2}\sin(t) \\ \sqrt{\frac{3}{2}} \end{bmatrix} \in \mathbf{S}^2$.

In contrast, suppose we started on the eastern hemisphere coordinate chart with the differential equations

$$\begin{aligned} \frac{dx_1^*}{dt} &= -x_2^* \\ \frac{dx_2^*}{dt} &= x_1^* \end{aligned}$$

and $(x_1^*(0), x_2^*(0)) = (1/2, 0)$. The solution is

$$\begin{aligned} x_1^*(t) &= \frac{1}{2}\cos(t) \\ x_2^*(t) &= \frac{1}{2}\sin(t) \end{aligned}$$

resulting in the curve

$$\mathbf{z}^*(x_1^*(t), x_2^*(t)) = \begin{bmatrix} x_2^*(t) \\ \sqrt{1-((x_1^*(t))^2+(x_2^*(t))^2)} \\ x_1^*(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sin(t) \\ \sqrt{\frac{3}{2}} \\ \frac{1}{2}\cos(t) \end{bmatrix} \in \mathbf{S}^2.$$

As shown in Figure 3.4 below this is a *different* curve on the manifold from that just considered above.

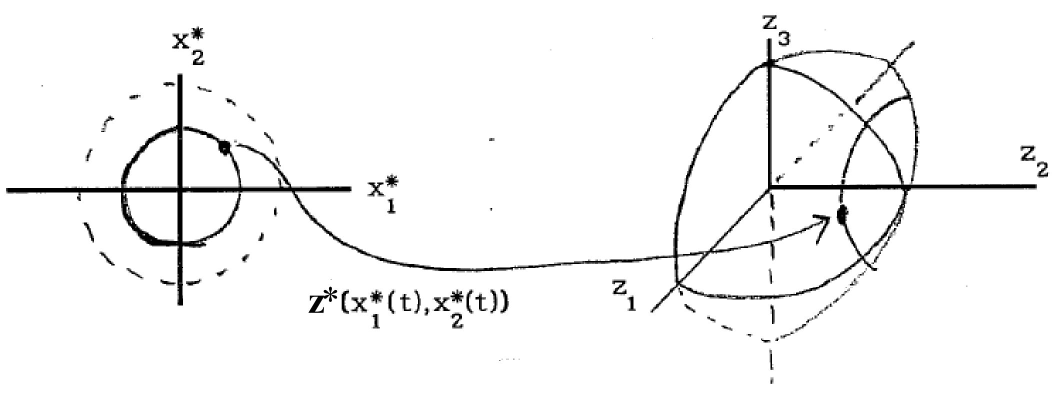


FIGURE 3.4. The curve $\begin{bmatrix} \frac{1}{2}\sin(t) \\ \sqrt{\frac{3}{2}} \\ \frac{1}{2}\cos(t) \end{bmatrix} \in \mathbf{S}^2$.

3.1 Vector Fields and Tangent Vectors

We again consider the differential equation

$$\begin{aligned}\frac{dx_1}{dt} &= -x_2\\ \frac{dx_2}{dt} &= x_1\end{aligned}$$

in the northern hemisphere coordinate system. With the initial conditions $(x_1(0), x_2(0)) = (x_{01}, x_{02})$ the solution is

$$\begin{aligned}x_1(t) &= x_{01}\cos(t) - x_{02}\sin(t)\\ x_2(t) &= x_{01}\sin(t) + x_{02}\cos(t).\end{aligned}$$

The corresponding curve $\mathbf{z}(x_1(t), x_2(t)) : \mathcal{D}_1 \to \mathcal{U}_1$ on $\mathbf{S}^2$ is given by

$$\mathbf{z}(x_1(t), x_2(t)) = \begin{bmatrix} x_1(t)\\ x_2(t)\\ \sqrt{1-(x_1^2(t)+x_2^2(t))}\end{bmatrix} = \begin{bmatrix} x_{01}\cos(t) - x_{02}\sin(t)\\ x_{01}\sin(t) + x_{02}\cos(t)\\ \sqrt{1-(x_{01}^2+x_{02}^2)}\end{bmatrix} \in \mathbf{S}^2.$$

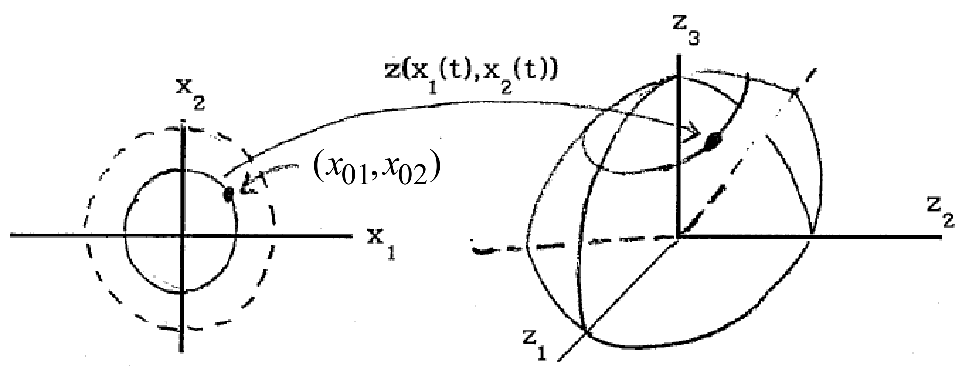


FIGURE 3.5.

The tangent to the curve at time t is

$$\begin{aligned}\frac{d}{dt}\mathbf{z}(x_1(t), x_2(t)) &= \frac{\partial \mathbf{z}(x_1,x_2)}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{z}(x_1,x_2)}{\partial x_2}\frac{dx_2}{dt}\\ &= \underbrace{\begin{bmatrix} 1\\ 0\\ \dfrac{-x_1}{\sqrt{1-(x_1^2+x_2^2)}}\end{bmatrix}}_{\mathbf{z}_{x_1}}(-x_2) + \underbrace{\begin{bmatrix} 0\\ 1\\ \dfrac{-x_2}{\sqrt{1-(x_1^2+x_2^2)}}\end{bmatrix}}_{\mathbf{z}_{x_2}}(x_1).\end{aligned}$$

The pair of coordinates $(x_1, x_2) \in \mathcal{D}_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 \middle|\ 0 \le x_1^2 + x_2^2 < 1\right\}$ corresponds to the point

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1\\ x_2\\ \sqrt{1-(x_1^2+x_2^2)}\end{bmatrix} \in \mathbf{S}^2$$

with the *unique* tangent vector at that point given by

$$(-x_2)\mathbf{z}_{x_1} + (x_1)\mathbf{z}_{x_2} \in \mathbf{T}_p(\mathbf{S}^2).$$

The component of this tangent vector are $-x_2$ and x_1. In other words, for each point $p \in \mathcal{U}_1$ of the northern hemisphere there is a unique tangent vector whose *components* are specified by the differential equation

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2 \\ \frac{dx_2}{dt} &= \quad x_1. \end{aligned}$$

This set of vectors is called a vector field. Specifically, a *vector field* on an open subset $\mathcal{U}$ of a manifold $\mathcal{M}$ is an assignment to each point of $\mathcal{U}$ a unique tangent from the tangent space $\mathbf{T}_p(\mathcal{M})$. If the components of the vector field are differentiable functions of the coordinates then it is a *smooth* vector field. However, in all that follows the term *vector field* will be taken to mean a *smooth* vector field.

Consider a general differential equation on northern hemisphere coordinates given by

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2) \\ \frac{dx_2}{dt} &= f_2(x_1, x_2). \end{aligned}$$

This specifies the unique tangent vector $\in \mathbf{T}_p(\mathbf{S}^2)$ given by

$$f_1(x_1, x_2)\mathbf{z}_{x_1} + f_2(x_1, x_2)\mathbf{z}_{x_2}$$

at each point $p \in \mathcal{U}_1$ with local coordinates $\varphi(p) = (x_1, x_2)$. For any solution $(x_1(t), x_2(t))$ of the above differential equation we have

$$\frac{d}{dt}\mathbf{z}(x_1(t), x_2(t)) = \frac{\partial \mathbf{z}(x_1, x_2)}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{z}(x_1, x_2)}{\partial x_2}\frac{dx_2}{dt} = f_1(x_1, x_2)\mathbf{z}_{x_1} + f_2(x_1, x_2)\mathbf{z}_{x_2}.$$

Change of Basis Vectors for the Tangent Space $\mathbf{T}_p(\mathbf{S}^2)$

We look at how the basis vectors of the tangent space $\mathbf{T}_p(\mathbf{S}^2)$ transform going from one coordinate system to another. We will rediscover how the components of a tangent vector change going between coordinate systems. To do this consider the change of coordinates between the spherical and northern hemisphere patches on $\mathbf{T}_p(\mathbf{S}^2)$.

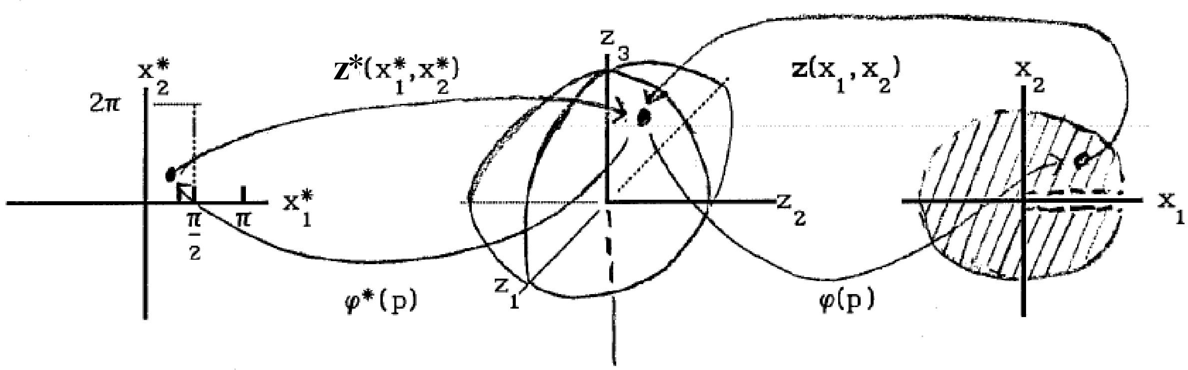


FIGURE 3.6. Spherical and northen hemisphere patches on $\mathbf{T}_p(\mathbf{S}^2)$

The spherical coordinate map $\mathbf{z}^*(x_1^*, x_2^*) : \mathcal{D}_1 \to \mathcal{U}_1$ for $\mathbf{S}^2$ is

$$\mathbf{z}^*(x_1^*, x_2^*) = \begin{bmatrix} \sin(x_1^*)\cos(x_2^*) \\ \sin(x_1^*)\sin(x_2^*) \\ \cos(x_1^*) \end{bmatrix} \in \mathbf{S}^2$$

where $\mathcal{D}_1 = \left\{(x_1^*, x_2^*) \in \mathbb{R}^2 |\ 0 < x_1^* < \pi, 0 < x_2^* < 2\pi\right\}$ and $\mathcal{U}_1 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ \text{If } z_1 > 0 \text{ then } z_2 \neq 0\right\}$.

The northern hemisphere coordinate map $\mathbf{z}(x_1, x_2) : \mathcal{D}_2 \to \mathcal{U}_2$ for $\mathbf{S}^2$ is

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} \in \mathbf{S}^2$$

where $\mathcal{D}_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 \le x_1^2 + x_2^2 < 1\right\}$ and $\mathcal{U}_2 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ z_3 > 0\right\}$.

The coordinate transformation from spherical to northern hemisphere coordinates $\varphi \circ \varphi^{*-1}$ is

$$\begin{aligned} x_1 &= \sin(x_1^*)\cos(x_2^*) \\ x_2 &= \sin(x_1^*)\sin(x_2^*) \end{aligned}$$

with inverse $\varphi^* \circ \varphi^{-1}$ given by

$$\begin{aligned} x_1^* &= \sqrt{x_1^2 + x_2^2} \\ x_2^* &= \tan^{-1}(x_1, x_2). \end{aligned}$$

Consider the calculation

$$\mathbf{z}(x_1, x_2)\Big|_{\substack{x_1 = \sin(x_1^*)\cos(x_2^*) \\ x_2 = \sin(x_1^*)\sin(x_2^*)}} = \left[\begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix}\right]\Bigg|_{\substack{x_1 = \sin(x_1^*)\cos(x_2^*) \\ x_2 = \sin(x_1^*)\sin(x_2^*)}} = \begin{bmatrix} \sin(x_1^*)\cos(x_2^*) \\ \sin(x_1^*)\sin(x_2^*) \\ \cos(x_1^*) \end{bmatrix} = \mathbf{z}^*(x_1^*, x_2^*).$$

The basis vectors $\mathbf{z}^*_{x_1^*}$ and $\mathbf{z}^*_{x_2^*}$ in terms of the basis vectors $\mathbf{z}_{x_1}$ and $\mathbf{z}_{x_2}$ are given by

$$\begin{aligned} \mathbf{z}^*_{x_1^*} &= \frac{\partial \mathbf{z}^*(x_1^*, x_2^*)}{\partial x_1^*} = \frac{\partial \mathbf{z}(x_1, x_2)}{\partial x_1}\frac{\partial x_1}{\partial x_1^*} + \frac{\partial \mathbf{z}(x_1, x_2)}{\partial x_2}\frac{\partial x_2}{\partial x_1^*} = \mathbf{z}_{x_1}\frac{\partial x_1}{\partial x_1^*} + \mathbf{z}_{x_2}\frac{\partial x_2}{\partial x_1^*} \\ \mathbf{z}^*_{x_2^*} &= \frac{\partial \mathbf{z}^*(x_1^*, x_2^*)}{\partial x_2^*} = \frac{\partial \mathbf{z}(x_1, x_2)}{\partial x_1}\frac{\partial x_1}{\partial x_2^*} + \frac{\partial \mathbf{z}(x_1, x_2)}{\partial x_2}\frac{\partial x_2}{\partial x_2^*} = \mathbf{z}_{x_1}\frac{\partial x_1}{\partial x_2^*} + \mathbf{z}_{x_2}\frac{\partial x_2}{\partial x_2^*}. \end{aligned}$$

In matrix form we can write

$$\begin{bmatrix} \mathbf{z}^*_{x_1^*} \\ \mathbf{z}^*_{x_2^*} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x_1}{\partial x_1^*} & \dfrac{\partial x_2}{\partial x_1^*} \\ \dfrac{\partial x_1}{\partial x_2^*} & \dfrac{\partial x_2}{\partial x_2^*} \end{bmatrix}}_{\text{Change of Basis Matrix}} \begin{bmatrix} \mathbf{z}_{x_1} \\ \mathbf{z}_{x_2} \end{bmatrix} = \begin{bmatrix} \cos(x_1^*)\cos(x_2^*) & \cos(x_1^*)\sin(x_2^*) \\ -\sin(x_1^*)\sin(x_2^*) & \sin(x_1^*)\cos(x_2^*) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{x_1} \\ \mathbf{z}_{x_2} \end{bmatrix}.$$

Let $a_1^*\mathbf{z}^*_{x_1^*} + a_2^*\mathbf{z}^*_{x_2^*}$ be a tangent vector from the tangent space $\mathbf{T}_p(\mathbf{S}^2)$ represented in the spherical coordinate patch. What is the representation of this vector in the northern hemisphere coordinate patch? The relationships between basis vectors $\left\{\mathbf{z}^*_{x_1^*}, \mathbf{z}^*_{x_2^*}\right\}$ and the basis vectors $\{\mathbf{z}_{x_1}, \mathbf{z}_{x_2}\}$ may showns as

$$\begin{array}{ccc} \mathbf{z}^*_{x_1^*} & & \mathbf{z}^*_{x_2^*} \\ \| & & \| \\ \dfrac{\partial x_1}{\partial x_1^*}\mathbf{z}_{x_1} & & \dfrac{\partial x_1}{\partial x_2^*}\mathbf{z}_{x_1} \\ + & & + \\ \dfrac{\partial x_2}{\partial x_1^*}\mathbf{z}_{x_2} & & \dfrac{\partial x_2}{\partial x_2^*}\mathbf{z}_{x_2} \end{array}$$

With $a_1^*\mathbf{z}_{x_1^*} + a_2^*\mathbf{z}_{x_2^*} = a_1\mathbf{z}_{x_1} + a_2\mathbf{z}_{x_2}$ and, using the above relationships between basis vectors, we have the diagram

$$\begin{array}{ccccc}
 & & a_1^*\mathbf{z}_{x_1^*} & + & a_2^*\mathbf{z}_{x_2^*} \\
 & & & \| & \\
a_1\mathbf{z}_{x_1} & = & a_1^*\dfrac{\partial x_1}{\partial x_1^*}\mathbf{z}_{x_1} & + & a_2^*\dfrac{\partial x_1}{\partial x_2^*}\mathbf{z}_{x_1} \\
+ & & + & & + \\
a_2\mathbf{z}_{x_2} & = & a_1^*\dfrac{\partial x_2}{\partial x_1^*}\mathbf{z}_{x_2} & + & a_2^*\dfrac{\partial x_2}{\partial x_2^*}\mathbf{z}_{x_2}
\end{array}$$

From the diagram we have

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x_1}{\partial x_1^*} & \dfrac{\partial x_1}{\partial x_2^*} \\ \dfrac{\partial x_2}{\partial x_1^*} & \dfrac{\partial x_2}{\partial x_2^*} \end{bmatrix}}_{\text{change of components matrix}} \begin{bmatrix} a_1^* \\ a_2^* \end{bmatrix} = \begin{bmatrix} \cos(x_1^*)\cos(x_2^*) & -\sin(x_1^*)\sin(x_2^*) \\ \cos(x_1^*)\sin(x_2^*) & \sin(x_1^*)\cos(x_2^*) \end{bmatrix} \begin{bmatrix} a_1^* \\ a_2^* \end{bmatrix}.$$

We derived these change of components matrix previously using a curve $(x_1^*(t), x_2^*(t))$ given in the spherical coordinates which has coordinates

$$(x_1(t), x_2(t)) = \Big(\sin(x_1^*(t)\cos(x_2^*(t)),\ \sin(x_1^*(t))\sin(x_2^*(t))\Big).$$

There we differentiated both sides to obtain

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x_1}{\partial x_1^*} & \dfrac{\partial x_1}{\partial x_2^*} \\ \dfrac{\partial x_2}{\partial x_1^*} & \dfrac{\partial x_2}{\partial x_2^*} \end{bmatrix}}_{\text{change of components matrix}} \begin{bmatrix} \dfrac{dx_1^*}{dt} \\ \dfrac{dx_2^*}{dt} \end{bmatrix} = \begin{bmatrix} \cos(x_1^*)\cos(x_2^*) & -\sin(x_1^*)\sin(x_2^*) \\ \cos(x_1^*)\sin(x_2^*) & \sin(x_1^*)\cos(x_2^*) \end{bmatrix} \begin{bmatrix} \dfrac{dx_1^*}{dt} \\ \dfrac{dx_2^*}{dt} \end{bmatrix}.$$

To summarize, we write $a_1^*\mathbf{z}_{x_1^*} + a_2^*\mathbf{z}_{x_2^*} = a_1\mathbf{z}_{x_1} + a_2\mathbf{z}_{x_2}$ to mean the same vector represented in two different coordinate systems. Their components and basis vectors must be related as shown above for this to hold.

Exercise 17 *Northern Hemisphere to Spherical Coordinates*

Show that the spherical coordinates of a tangent vector (a_1^*, a_2^*) are related to its northern hemisphere components (a_1, a_2) by

$$\begin{bmatrix} a_1^* \\ a_2^* \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x_1^*}{\partial x_1} & \dfrac{\partial x_1^*}{\partial x_2} \\ \dfrac{\partial x_2^*}{\partial x_1} & \dfrac{\partial x_2^*}{\partial x_2} \end{bmatrix}}_{\text{Change of components matrix}} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Show that the basis vectors $(\mathbf{z}^*_{x_1^*}, \mathbf{z}^*_{x_2^*})$ in spherical coordinates are related to the basis vectors $(\mathbf{z}_{x_1}, \mathbf{z}_{x_2})$ in the northern hemisphere by

$$\begin{bmatrix} \mathbf{z}_{x_1} \\ \mathbf{z}_{x_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x_1^*}{\partial x_1} & \dfrac{\partial x_2^*}{\partial x_1} \\ \dfrac{\partial x_1^*}{\partial x_2} & \dfrac{\partial x_2^*}{\partial x_2} \end{bmatrix}}_{\text{Change of basis matrix}} \begin{bmatrix} \mathbf{z}^*_{x_1^*} \\ \mathbf{z}^*_{x_2^*} \end{bmatrix}.$$

Example 1 *Series Connected DC Motor*

Recall the equations describing the series connected DC motor given by

$$\begin{aligned}
\frac{di}{dt} &= -\frac{R}{L}i - \frac{K_b L_f}{L} i\omega + \frac{V_S}{L} \\
\frac{d\omega}{dt} &= \frac{K_T L_f}{J} i^2 - \frac{\tau_L}{J} \\
\frac{d\theta}{dt} &= \omega.
\end{aligned}$$

Set $z_1 = \theta, z_2 = \omega, z_3 = i$, and $u = V_S/L$ to obtain

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underbrace{\begin{bmatrix} z_2 \\ c_1 z_3^2 - d \\ -c_2 z_3 - c_3 z_3 z_2 \end{bmatrix}}_{f(z)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g(z)} u$$

with $c_1 \triangleq K_T L_f / J, c_2 \triangleq R/L$, $c_3 \triangleq K_b L_f / L$, and $d \triangleq \tau_L / J$. Let

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \theta \\ \omega \\ i \end{bmatrix}$$

be considered as points in the manifold $\mathbf{E}^3$ and consider two different coordinate systems for it. The first coordinate system is the Cartesian coordinate map $\mathbf{z}(x_1, x_2, x_3) : \mathbb{R}^3 \to \mathbf{E}^3$ is

$$\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{E}^3.$$

In Example 3 of Chapter 1 it was shown that the change of coordinates given by

$$\begin{aligned}
x_1^* &= x_1 = z_1 \\
x_2^* &= x_2 = z_2 \\
x_3^* &= c_1 x_3^2 - d = c_1 z_3^2 - d
\end{aligned}$$

was the feedback linearizing transformation. So we take the inverse of this as the second coordinate system $\varphi^{*-1}(x^*) = \mathbf{z}^*(x^*) : \mathcal{D} \to \mathcal{U} \subset \mathbf{E}^3$ given by

$$\varphi^{*-1}(x^*) = \mathbf{z}^*(x^*) = \begin{bmatrix} x_1^* \\ x_2^* \\ \sqrt{(x_3^* + d)/c_1} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{E}^3$$

where $\mathcal{D} = \left\{ x^* \in \mathbb{R}^3 | \ x_3^* + d > 0 \right\}$ and $\mathcal{U} = \left\{ \mathbf{z} \in \mathbf{E}^3 | \ z_3 > 0 \right\}$. Then $\varphi^* = \mathbf{z}^{*-1} : \mathcal{U} \to \mathcal{D}$ is written as

$$\varphi^*\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right) = (z_1, z_2, c_1 z_3^2 - d) = (x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3.$$

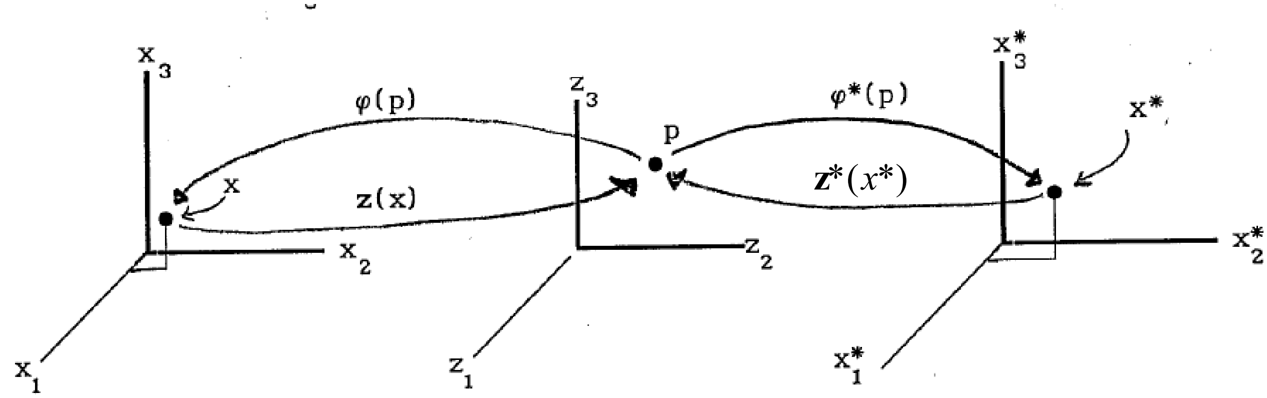


FIGURE 3.7. Coordinate systems for the series connected DC motor.

For each $u \in \mathbb{R}$ the tangent vector specified by the series connected DC motor is

$$\frac{dz}{dt} = f(z) + g(z)u \in \mathbf{T}_p(\mathbf{E}^3)$$

where $p = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$. The representation of the tangent vector $f(z)$ in Cartesian coordinates is

$$f_1(x)\mathbf{z}_{x_1} + f_2(x)\mathbf{z}_{x_2} + (f_3(x) + u)\mathbf{z}_{x_3}$$

with components $f_1(x) = x_2, f_2(x) = c_1 x_3^2 - d, f_3(x) = -c_2 x_3 - c_3 x_3 x_2$ and basis vectors

$$\mathbf{z}_{x_1} = \frac{\partial}{\partial x_1}\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{z}_{x_2} = \frac{\partial}{\partial x_2}\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{z}_{x_3} = \frac{\partial}{\partial x_3}\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

To determine the representation of this tangent vector in the x^* coordinates first note that the basis vectors in this coordinate system are

$$\mathbf{z}^*_{x_1^*} = \frac{\partial}{\partial x_1^*}\mathbf{z}^*(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{z}^*_{x_2^*} = \frac{\partial}{\partial x_2^*}\mathbf{z}^*(x_1^*, x_2^* x_3^*) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{z}^*_{x_3} = \frac{\partial}{\partial x_3^*}\mathbf{z}^*(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{2c_1\sqrt{(x_3^* + d)/c_1}} \end{bmatrix}.$$

The change of coordinates $T = \varphi^* \circ \varphi^{-1} : \left\{(x_1, x_2, x_3) \subset \mathbb{R}^3 |\ x_3 > 0\right\} \to \mathcal{D}$ is

$$(x_1^*, x_2^*, x_3^*) = \varphi^* \circ \varphi^{-1}(x_1, x_2, x_3) = \varphi^*(\varphi^{-1}(x_1, x_2, x_3)) = (x_1, x_2, c_1 x_3^2 - d)$$

or

$$\begin{aligned} x_1^* &= T_1(x_1) = x_1 \\ x_2^* &= T_2(x) = x_2 \\ x_3^* &= T_3(x) = c_1 x_3^2 - d. \end{aligned}$$

Then

$$\begin{aligned}
\begin{bmatrix} f_1^*(x^*) \\ f_2^*(x^*) \\ f_3^*(x^*) \end{bmatrix} &= \begin{bmatrix} \dfrac{\partial x_1^*}{\partial x_1} & \dfrac{\partial x_1^*}{\partial x_2} & \dfrac{\partial x_1^*}{\partial x_3} \\ \dfrac{\partial x_2^*}{\partial x_1} & \dfrac{\partial x_2^*}{\partial x_2} & \dfrac{\partial x_2^*}{\partial x_3} \\ \dfrac{\partial x_2^*}{\partial x_3} & \dfrac{\partial x_2^*}{\partial x_3} & \dfrac{\partial x_3^*}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2c_1x_3 \end{bmatrix} \begin{bmatrix} x_2 \\ c_1x_3^2 - d \\ -c_2x_3 - c_3x_3x_2 \end{bmatrix} \Bigg|_{x=\varphi(\varphi^{*-1}(x^*))=\begin{bmatrix} x_1^* \\ x_2^* \\ \sqrt{(x_3^*+d)/c_1} \end{bmatrix}} \\
&= \begin{bmatrix} x_2^* \\ x_3^* \\ -2c_2(x_3^*+d) - 2c_3x_2^*(x_3^*+d) \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{bmatrix} g_1^*(x^*) \\ g_2^*(x^*) \\ g_3^*(x^*) \end{bmatrix} &= \begin{bmatrix} \dfrac{\partial x_1^*}{\partial x_1} & \dfrac{\partial x_1^*}{\partial x_2} & \dfrac{\partial x_1^*}{\partial x_3} \\ \dfrac{\partial x_2^*}{\partial x_1} & \dfrac{\partial x_2^*}{\partial x_2} & \dfrac{\partial x_2^*}{\partial x_3} \\ \dfrac{\partial x_2^*}{\partial x_3} & \dfrac{\partial x_2^*}{\partial x_3} & \dfrac{\partial x_3^*}{\partial x_3} \end{bmatrix} \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2c_1x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Bigg|_{x=\varphi(\varphi^{*-1}(x^*))=\begin{bmatrix} x_1^* \\ x_2^* \\ \sqrt{(x_3^*+d)/c_1} \end{bmatrix}} \\
&= \begin{bmatrix} 0 \\ 0 \\ 2c_1\sqrt{(x_3^*+d)/c_1} \end{bmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
f_1(x^*)\mathbf{z}_{x_1^*} + f_2(x^*)\mathbf{z}_{x_2^*} + f_3(x^*)\mathbf{z}_{x_3^*} &= x_2^*\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3^*\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \Big(-2c_2(x_3^*+d) - 2c_3x_2^*(x_3^*+d)\Big)\begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{2c_1\sqrt{(x_3^*+d)/c_1}} \end{bmatrix} \\
&= \begin{bmatrix} x_2^* \\ x_3^* \\ (-c_2 - c_3x_2^*)\sqrt{(x_3^*+d)/c_1} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
g_1(x^*)\mathbf{z}_{x_1^*} + g_2(x^*)\mathbf{z}_{x_2^*} + g_3(x^*)\mathbf{z}_{x_3^*} &= 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2c_1\sqrt{(x_3^*+d)/c_1}\begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{2c_1\sqrt{(x_3^*+d)/c_1}} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Exercise 18 Show that

$$\begin{aligned} f_1(x^*)\mathbf{z}_{x_1^*} + f_2(x^*)\mathbf{z}_{x_2^*} + f_3(x^*)\mathbf{z}_{x_3^*} + \Big(g_1(x^*)\mathbf{z}_{x_1^*} + g_2(x^*)\mathbf{z}_{x_2^*} + g_3(x^*)\mathbf{z}_{x_3^*}\Big)u \\ = \begin{bmatrix} x_2^* \\ x_3^* \\ (-c_2 - c_3 x_2^*)\sqrt{(x_3^* + d)/c_1} + u \end{bmatrix}_{x^* = \varphi^*\varphi^{-1}(x))} \\ = \begin{bmatrix} x_2 \\ c_1 x_3^2 - d \\ -c_2 x_3 - c_3 x_3 x_2 + u \end{bmatrix} \\ = f_1(x)\mathbf{z}_{x_1} + f_2(x)\mathbf{z}_{x_2} + (f_3(x) + u)\mathbf{z}_{x_3} + \Big(g_1(x)\mathbf{z}_{x_1} + g_2(x)\mathbf{z}_{x_2} + g_3(x)\mathbf{z}_{x_3}\Big)u \end{aligned}$$

3.2 How Mathematicians View Tangent Vectors

We now introduce the modern (abstract) approach to differential geometry. Once again consider tangent vectors on $\mathbf{S}^2$ using the northern hemisphere coordinate system. The northern hemisphere chart $\mathbf{z}(x_1, x_2) : \mathcal{D}_1 \to \mathcal{U}_1 \subset \mathbf{S}^2$ is

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} \in \mathbf{S}^2$$

where $\mathcal{D}_1 = \big\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 \le x_1^2 + x_2^2 < 1\big\}$ and $\mathcal{U}_1 \triangleq \big\{\mathbf{z} \in \mathbf{S}^2 |\ z_3 > 0\big\}$. With $p = \mathbf{z}(x_1, x_2)$, a set of basis vectors for the tangent space $\mathbf{T}_p(\mathbf{S}^2)$ is

$$\mathbf{z}_{x_1} = \frac{\partial}{\partial x_1}\mathbf{z}(x_1, x_2) = \begin{bmatrix} 1 \\ 0 \\ \dfrac{-x_1}{\sqrt{1 - (x_1^2 + x_2^2)}} \end{bmatrix}, \ \mathbf{z}_{x_2} = \frac{\partial}{\partial x_2}\mathbf{z}(x_1, x_2) = \begin{bmatrix} 0 \\ 1 \\ \dfrac{-x_2}{\sqrt{1 - (x_1^2 + x_2^2)}} \end{bmatrix}$$

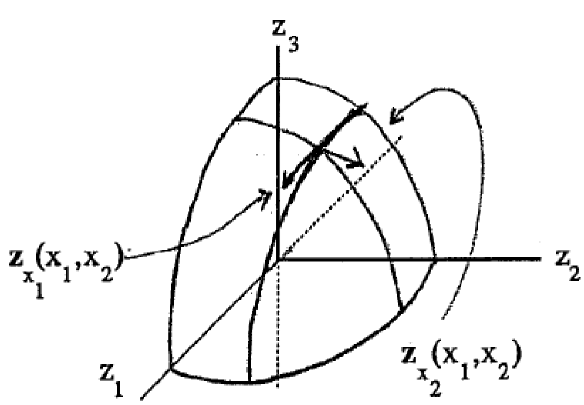


FIGURE 3.8. Northern hemisphere coordinate chart for $\mathbf{S}^2$ with the tangent vectors $\vec{z}_{x_1}$ and $\vec{z}_{x_2}$.

As previously mentioned, any tangent vector to the manifold for a point $p \in \mathcal{U}_1$ is a linear combination of $\mathbf{z}_{x_1}$ and $\mathbf{z}_{x_2}$. However, there is a problem with this description of a tangent space. To explain, imagine there are two-dimensional (2-D) people living on the surface $\mathbf{S}^2$. By 2-D people is meant that they can move anywhere on the surface $\mathbf{S}^2$, but not off of it. In fact, they are not even aware of that the direction perpendicular to $\mathbf{S}^2$ exists. In this case the tangent vectors $\mathbf{z}_{x_1}$ and $\mathbf{z}_{x_2}$ don't make sense as they stick out off the surface as illustrated in Figure 3.8. The 2-D people cannot draw $\mathbf{z}_{x_1}$ and $\mathbf{z}_{x_2}$ as they cannot go off the surface. We need to reconcile this problem so that tangent vectors are part of the manifold $\mathbf{S}^2$ in such a way to that the 2-D people have knowledge of them. This is done by redefining tangent vectors in such a way that the 2-D people can use them and that they are, in some sense, equivalent to the old definition.

Remark Instead of the manifold $\mathbf{S}^2$ consider the manifold $\mathbf{E}^2$. In this case the Cartesian coordinate system $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathbf{E}^2$ is

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with tangent vectors at each point of the manifold given by

$$\mathbf{z}_{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{z}_{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which lie in the manifold.

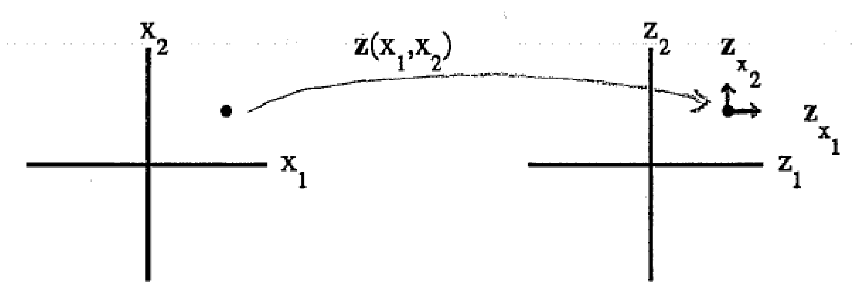


FIGURE 3.9. Euclidean space $\mathbf{E}^2$.

In this case 2-D people living on $\mathbf{E}^2$ can use $\{\mathbf{z}_{x_1}, \mathbf{z}_{x_2}\}$.

Digression Gradients

Before going forward with the modern definition of tangent vectors we digress to discuss gradients. Let $\mathcal{C}^k(\mathbb{R}^3)$ be the set of k-times differentiable functions on $\mathbb{R}^3$ and $\mathcal{C}^\infty(\mathbb{R}^3)$ be all infinitely differentiable functions on $\mathbb{R}^3$. Let $\mathcal{C}^k(p)$ and $\mathcal{C}^\infty(p)$ denote, respectively, all k-times differentiable and infinitely differentiable functions in a *neighborhood*[1] of a point p. For example, with $p = (x_{01}, x_{02}, x_{03}) \in \mathbb{R}^3$ let $h(x_1, x_2, x_3) : \mathbb{R}^3 \to \mathbb{R}$ be in $\mathcal{C}^1(p)$ so that $\partial h(x)/\partial x_1, \partial h(x)/\partial x_2, \partial h(x)/\partial x_2$ all exist and are continuous in a neighborhood of p. The *gradient* is the operator $d : \mathcal{C}^1(p) \to \mathbb{R}^{1\times 3}$ given by

$$dh = \begin{bmatrix} \dfrac{\partial h(x)}{\partial x_1} & \dfrac{\partial h(x)}{\partial x_2} & \dfrac{\partial h(x)}{\partial x_3} \end{bmatrix}.$$

[1] Just take neighborhood of p to mean an open set that contains p.

Let $(x_1(t), x_2(t), x_3(t))$ be a smooth (differentiable) curve in $\mathbb{R}^3$ with $(x_1(0), x_2(0), x_3(0)) = (x_{01}, x_{02}, x_{03}) = p$. The scalar function $h \circ x(t) = h(x(t))$ has derivative

$$\frac{dh}{dt}\bigg|_{t=0} = \frac{\partial h(x)}{\partial x_1}\bigg|_p \frac{dx_1}{dt}\bigg|_{t=0} + \frac{\partial h(x)}{\partial x_2}\bigg|_p \frac{dx_2}{dt}\bigg|_{t=0} + \frac{\partial h(x)}{\partial x_3}\bigg|_p \frac{dx_3}{dt}\bigg|_{t=0} = \begin{bmatrix} \frac{\partial h(x)}{\partial x_1} & \frac{\partial h(x)}{\partial x_2} & \frac{\partial h(x)}{\partial x_3} \end{bmatrix}\bigg|_p \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix}\Bigg|_{t=0}$$

$$= \left\langle dh, \frac{dx}{dt} \right\rangle\bigg|_{t=0}$$

where $\langle \cdot, \cdot \rangle$ is the dual product. $\frac{dh}{dt}\Big|_{t=0}$ is the rate of change of h at p in the direction $\frac{dx}{dt}\Big|_{t=0} = (\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt})\Big|_{t=0}$. If $\frac{dx}{dt}\Big|_{t=0}$ points in the same direction as dh then the rate of change is at a maximum. Now consider the mapping (function) that takes $h \in \mathcal{C}^\infty(p)$ to the real number $\left\langle dh, \frac{dx}{dt} \right\rangle\Big|_{t=0}$. Specifically, define $X_p(h) : \mathcal{C}^\infty(p) \to \mathbb{R}$ by

$$X_p(h) \triangleq \left\langle dh, \frac{dx}{dt} \right\rangle\bigg|_{t=0}.$$

For example, let $h(x_1, x_2, x_3) = x_1$ so

$$X_p(h) \triangleq \left\langle dh, \frac{dx}{dt} \right\rangle\bigg|_{t=0} = \left\langle \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \frac{dx}{dt}\bigg|_{t=0} \right\rangle = \frac{dx_1}{dt}\bigg|_{t=0}.$$

Similarly, $h(x_1, x_2, x_3) = x_2$ gives $X_p(h) = \frac{dx_2}{dt}\Big|_{t=0}$ and $h(x_1, x_2, x_3) = x_3$ gives $X_p(h) = \frac{dx_3}{dt}\Big|_{t=0}$.

This mapping $X_p(h) : \mathcal{C}^\infty(p) \to \mathbb{R}$ is completely determined by the three numbers $\frac{dx_1}{dt}, \frac{dx_2}{dt}$, and $\frac{dx_3}{dt}$ (assumed to evaluated at $t = 0$). In other words, suppose there are three numbers a, b, c such that for all $h \in \mathcal{C}^\infty(p)$ we have

$$\begin{bmatrix} \frac{\partial h(x)}{\partial x_1} & \frac{\partial h(x)}{\partial x_2} & \frac{\partial h(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial h(x)}{\partial x_1} & \frac{\partial h(x)}{\partial x_2} & \frac{\partial h(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Then $a = \frac{dx_1}{dt}, b = \frac{dx_2}{dt}$, and $c = \frac{dx_3}{dt}$. To show this simply let $h(x)$ equal successively $h(x) = x_1, h(x) = x_2$, and $h(x) = x_3$.

A New Interpretation of the Tangent Vector on $\mathbf{S}^2$

We now go to the manifold $\mathbf{S}^2$ with the northern hemisphere coordinate chart and see how this new notion of a tangent vector can be used by 2-D people on $\mathbf{S}^2$. A 2-D person on $\mathbf{S}^2$ is only aware of functions defined on $\mathbf{S}^2$. The northern hemisphere patch $\mathbf{z}(x_1, x_2) : \mathcal{D}_1 \to \mathcal{U}_1 \subset \mathbf{S}^2$ is

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} \in \mathbf{S}^2$$

where $\mathcal{D}_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 \le x_1^2 + x_2^2 < 1\right\}$ and $\mathcal{U}_1 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ z_3 > 0\right\}$.

Let $\mathbf{z}_0 = \mathbf{z}(x_{01}, x_{02}) = \begin{bmatrix} x_{01} & x_{02} & \sqrt{1-(x_{01}^2+x_{02}^2)} \end{bmatrix}^T$ so it has local coordinates (x_{01}, x_{02}). The coordinate mapping $\varphi\left(\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T\right) = (z_1, z_2)$ is only defined for those $\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$ with $z_1^2+z_2^2+z_3^2 = 1$, that is, $\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$ must be on the manifold $\mathbf{S}^2$. Let $h(p) = h(z_1, z_2, z_3)$ be a function defined in a neighborhood of $p = \mathbf{z}_0$. By this is meant $h(z)$ is defined only for $\mathbf{z} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$ with $z_1^2 + z_2^2 + z_3^2 = 1$ which are close to $\mathbf{z}_0 = \mathbf{z}(x_{01}, x_{02})$.

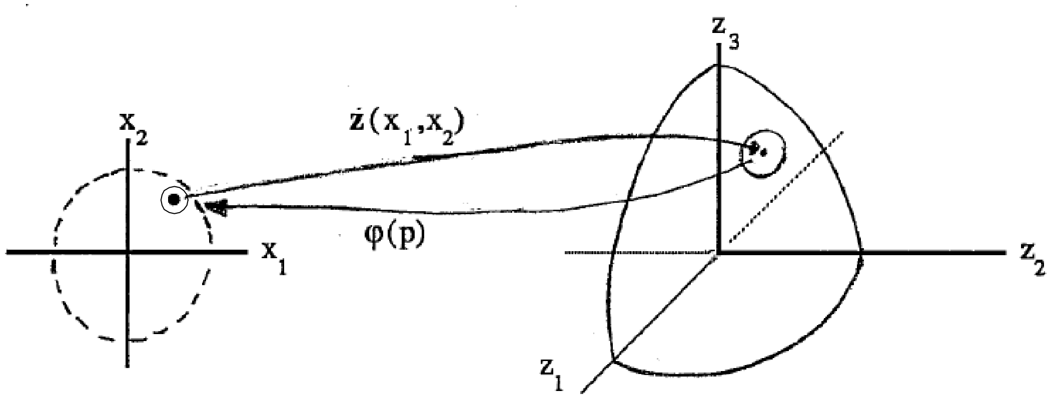


FIGURE 3.10. Northern hemisphere coordinate chart for $\mathbf{S}^2$. Left: Neighborhood of $x_0 = (x_{01}, x_{02})$. Right: Neighborhod of $\mathbf{z}_0 = \mathbf{z}(x_{01}, x_{02})$.

Now to the tangent vector. Let $(x_1(t), x_2(t))$ be a curve in $\mathcal{D}_1$ in a neighborhood of $x_0 = (x_{01}, x_{02})$ with $(x_1(0), x_2(0)) = (x_{01}, x_{02})$. Then $\mathbf{z}(x_1(t), x_2(t))$ is a curve on $\mathbf{S}^2$ going through $p = \begin{bmatrix} x_{01} & x_{02} & \sqrt{1-(x_{01}^2+x_{02}^2)} \end{bmatrix}^T$ at $t = 0$. Define

$$h(t) \triangleq h(z_1(x_1(t), x_2(t)), z_2(x_1(t), x_2(t)), z_3(x_1(t), x_2(t)))$$

which is an ordinary function of time. The rate of change of h at p determined by the curve $\mathbf{z}(x(t), x(t))$ is

$$\begin{aligned} \frac{dh}{dt}_{|t=0} = \frac{d}{dt} h \circ \mathbf{z}(x_1(t), x_2(t))\Big|_{t=0} &= \frac{d}{dt} h(\mathbf{z}(x_1(t), x_2(t)))\Big|_{t=0} \\ &= \frac{\partial h(\mathbf{z}(x_1, x_2))}{\partial x_1}\Big|_{(x_{01}, x_{02})} \frac{dx_1}{dt}_{|t=0} + \frac{\partial h(\mathbf{z}(x_1, x_2))}{\partial x_2}\Big|_{(x_{01}, x_{02})} \frac{dx_2}{dt}_{|t=0}. \end{aligned}$$

Note that we do *not* write

$$\begin{aligned} \frac{dh}{dt}_{|t=0} &= \begin{bmatrix} \dfrac{\partial h(z)}{\partial z_1} & \dfrac{\partial h(z)}{\partial z_2} & \dfrac{\partial h(z)}{\partial z_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \\ \dfrac{\partial z_3}{\partial x_1} & \dfrac{\partial z_3}{\partial x_2} \end{bmatrix}_{(x_{01}, x_{02})} \begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix}_{t=0} \\ &= dh \begin{bmatrix} \mathbf{z}_{x_1} & \mathbf{z}_{x2} \end{bmatrix} \begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix}_{|t=0} \\ &= \left\langle dh, \frac{d\mathbf{z}(t)}{dt} \right\rangle \text{ where } \frac{d\mathbf{z}(t)}{dt} = \mathbf{z}_{x_1} \frac{dx_1}{dt} + \mathbf{z}_{x_2} \frac{dx_2}{dt}. \end{aligned}$$

This is because the partial derivatives $\dfrac{\partial h(z)}{\partial z_1}, \dfrac{\partial h(z)}{\partial z_2}, \dfrac{\partial h(z)}{\partial z_3}$ as well as "tangent vectors" $\mathbf{z}_{x_1}, \mathbf{z}_{x_2}$ do not make sense! To explain, $\dfrac{\partial h(z)}{\partial z_2}$ is calculated as the limit

$$\frac{\partial h(z)}{\partial z_2} = \lim_{\Delta z \to 0} \frac{h(z_{01}, z_{02} + \Delta z, z_{03}) - h(z_{01}, z_{02}, z_{03})}{\Delta z}.$$

As illustrated in Figure 3.11 the point $[z_{01}, z_{02} + \Delta z, z_{03}]^T$ is not on $\mathbf{S}^2$ so $h(z_{01}, z_{02} + \Delta z, z_{03})$ is not defined (does not exist) for $\Delta z \neq 0$.

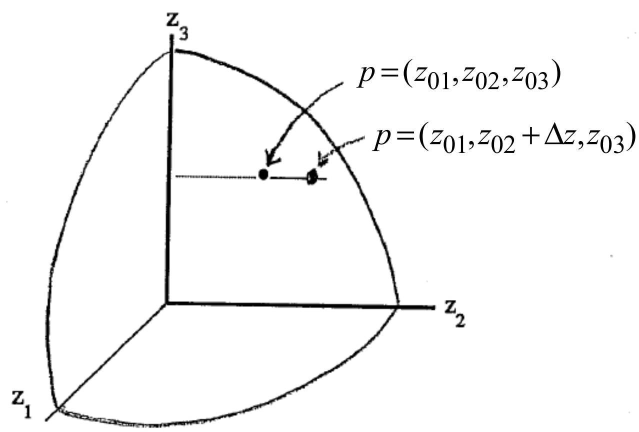


FIGURE 3.11. Calculation of $\dfrac{\partial h(z)}{\partial z_2} = \lim_{\Delta z \to 0} \frac{h(z_{01}, z_{02}+\Delta z, z_{03}) - h(z_{01}, z_{02}, z_{03})}{\Delta z}$.

In contrast,

$$\frac{\partial h \circ \varphi^{-1}(x_1, x_2)}{\partial x_1} = \frac{\partial h(\mathbf{z}(x_1, x_2))}{\partial x_1} = \frac{\partial}{\partial x_1} h(z_1(x_1, x_2), z_2(x_1, x_2), z_3(x_1, x_2))$$

is well defined as

$$\begin{aligned}
&\frac{\partial h \circ \varphi^{-1}(x_1, x_2)}{\partial x_1} \\
&= \lim_{\Delta x_1 \to 0} \frac{h(z_1(x_1 + \Delta x_1, x_2), z_2(x_1 + \Delta x_1, x_2), z_3(x_1 + \Delta x_1, x_2)) - h(z_1(x_1, x_2), z_2(x_1, x_2), z_3(x_1, x_2))}{\Delta x_1}.
\end{aligned}$$

In summary a 2-D person $\mathbf{S}^2$ is not aware of vectors like $\mathbf{z}_{x_1}$ $\mathbf{z}_{x2}$ as they stick off the manifold and only has knowledge of functions defined on $\mathbf{S}^2$.

Definition 1 *Differentiable Functions on* $\mathbf{S}^2$
Let $p = \mathbf{z}(x_{01}, x_{02}) \in \mathbf{S}^2$ for some coordinate system $\mathbf{z}(x_1, x_2)$. A function h on $\mathbf{S}^2$ is differentiable at $p \in \mathbf{S}^2$ if for every such coordinate system

$$h \circ \mathbf{z}(x_1, x_2)) = h(\mathbf{z}(x_1, x_2)))$$

is a differentiable function x_1 and x_2.

Definition 2 *Tangent Vectors on* $\mathbf{S}^2$

Let $\mathcal{F}(\mathbf{S}^2)$ denote the differentiable functions on $\mathbf{S}^2$, $\mathbf{z}(x_1, x_2) : \mathcal{D} \to \mathcal{U} \subset \mathbf{S}^2$ a coordinate system for $\mathbf{S}^2$, and $(x_1(t), x_2(t))$ be a curve in $\mathcal{D}$ with $(x_1(0), x_2(0)) = (x_{01}, x_{02})$. The tangent vector at $p = \mathbf{z}(x_{01}, x_{02})$ determined by the curve $\mathbf{z}(x_1(t), x_2(t))$ is the mapping $\mathbf{z}_p : \mathcal{F}(\mathbf{S}^2) \to \mathbb{R}$ given by

$$\begin{aligned}
\mathbf{z}_p : h \to \mathbf{z}_p(h) &\triangleq \frac{d}{dt}(h \circ \mathbf{z}(x_1(t), x_2(t)))\Big|_{t=0} \\
&= \frac{d}{dt}h(\mathbf{z}(x_1(t), x_2(t))\Big|_{t=0} \\
&= \frac{\partial h \circ \mathbf{z}(x_1, x_2)}{\partial x_1}\Big|_{(x_{01}, x_{02})} \frac{dx_1}{dt}_{|t=0} + \frac{\partial h \circ \mathbf{z}(x_1, x_2)}{\partial x_2}\Big|_{(x_{01}, x_{02})} \frac{dx_2}{dt}_{|t=0}.
\end{aligned}$$

The two mappings

$$\frac{\partial}{\partial x_1} : h \in \mathcal{F}(\mathbf{S}^2) \to \frac{\partial h \circ \mathbf{z}(x_1, x_2)}{\partial x_1} \in \mathbb{R}$$

$$\frac{\partial}{\partial x_2} : h \in \mathcal{F}(\mathbf{S}^2) \to \frac{\partial h \circ \mathbf{z}(x_1, x_2)}{\partial x_2} \in \mathbb{R}$$

are *basis vectors* for the tangent space $\mathbf{T}_p(\mathbf{S}^2)$.

$\frac{dx_1}{dt}_{|t=0}$ and $\frac{dx_2}{dt}_{|t=0}$ are the *components* of the tangent vector.

Recall $p = \mathbf{z}(x_1, x_2)$ and $\varphi(p) = (x_1, x_2)$ are inverses of each other. The northern hemisphere patch $\varphi^{-1}(x_1, x_2) = \mathbf{z}(x_1, x_2) : \mathcal{D}_1 \to \mathcal{U}_1 \subset \mathbf{S}^2$ is

$$\varphi^{-1}(x_1, x_2) = \mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} \in \mathbf{S}^2$$

where $\mathcal{D}_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 \le x_1^2 + x_2^2 < 1\right\}$ and $\mathcal{U}_1 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ z_3 > 0\right\}$. We have $\varphi : \mathcal{U}_1 \subset \mathbf{S}^2 \to \mathcal{D}_1$ given by

$$\varphi(p) = \left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (z_1, z_2).$$

Mathematicians use the notation $\varphi^{-1}(x_1, x_2)$ instead of $\mathbf{z}(x_1, x_2)$! In this case we write the basis vectors as

$$\frac{\partial}{\partial x_1} : h \in \mathcal{F}(\mathbf{S}^2) \to \frac{\partial h \circ \varphi^{-1}(x_1, x_2)}{\partial x_1} \in \mathbb{R}$$

$$\frac{\partial}{\partial x_2} : h \in \mathcal{F}(\mathbf{S}^2) \to \frac{\partial h \circ \varphi^{-1}(x_1, x_2)}{\partial x_2} \in \mathbb{R}.$$

A tangent vector $\mathbf{z}_p$ at a point $p = \varphi^{-1}(x_1, x_2) \in \mathbf{S}^2$ is the mapping $\mathbf{z}_p : \mathcal{F}(\mathbf{S}^2) \to \mathbb{R}$

$$\mathbf{z}_p : h \to \mathbf{z}_p(h) \triangleq a_1 \frac{\partial h \circ \varphi^{-1}(x_1, x_2)}{\partial x_1} + a_2 \frac{\partial h \circ \varphi^{-1}(x_1, x_2)}{\partial x_2}$$

where a_1 and a_2 are the components of $\mathbf{z}_p$.

A New Interpretation of the Tangent Vector on $\mathbf{E}^3$

We now look at this new formulation of the tangent vector on the manifold $\mathbf{E}^3$. To so, we consider the spherical coordinate system on $\mathbf{E}^3$. With

$$\begin{aligned}\mathcal{D} &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\}\\ \mathcal{U} &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\},\end{aligned}$$

recall the spherical coordinate system $\mathbf{z}(x_1, x_2, x_3) : \mathcal{D} \to \mathcal{U}$ given by

$$\varphi^{-1}(x_1, x_2, x_3) = \mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3.$$

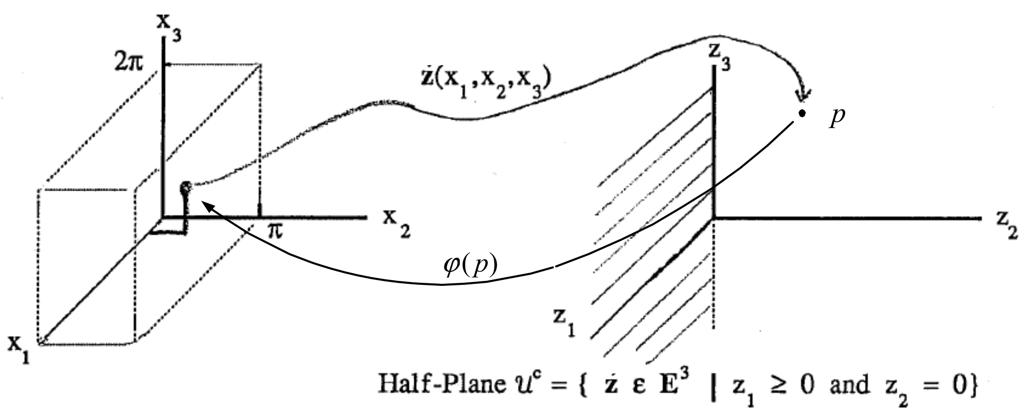


FIGURE 3.12. Spherical coordinates on $\mathbf{E}^3$.

Let $\mathcal{F}(\mathcal{U})$ denote the differentiable functions on $\mathcal{U} \subset \mathbf{E}^3$. Let $h(z_1, z_2, z_3) \in \mathcal{F}(\mathcal{U})$ with $p \in \mathcal{U}$. Let $(x_1(t), x_2(t), x_3(t))$ be a curve in $\mathcal{D}$ resulting in the curve $\varphi^{-1}(x_1(t), x_2(t), x_3(t)) = \mathbf{z}(x_1(t), x_2(t), x_3(t))$ in $\mathcal{U}$ with $\varphi^{-1}(x_1(0), x_2(0), x_3(0)) = p$. The tangent to the curve is the mapping $\mathbf{z}_p : \mathcal{F}(\mathcal{U}) \to \mathbb{R}$ that takes $h \in \mathcal{F}(\mathcal{U})$ to the number

$$\begin{aligned}\mathbf{z}_p : h \to \mathbf{z}_p(h) &= \left.\frac{d}{dt} h \circ \varphi^{-1}(x_1(t), x_2(t), x_3(t))\right|_{t=0}\\ &= \frac{\partial h \circ \varphi^{-1}(x_1, x_2, x_3)}{\partial x_1} \frac{dx_1}{dt}_{|t=0} + \frac{\partial h \circ \varphi^{-1}(x_1, x_2, x_3)}{\partial x_2} \frac{dx_2}{dt}_{|t=0} + \frac{\partial h \circ \varphi^{-1}(x_1, x_2, x_3)}{\partial x_3} \frac{dx_3}{dt}_{|t=0}.\end{aligned}$$

The components of this tangent vector (mapping) are $\frac{dx_1}{dt}_{|t=0}$, $\frac{dx_2}{dt}_{|t=0}$, and $\frac{dx_3}{dt}_{|t=0}$.

A basis for the tangent space at p are the three mappings

$$\begin{aligned}\frac{\partial}{\partial x_1} &: h \to \frac{\partial}{\partial x_1}(h \circ \varphi^{-1})\\ \frac{\partial}{\partial x_2} &: h \to \frac{\partial}{\partial x_2}(h \circ \varphi^{-1})\\ \frac{\partial}{\partial x_3} &: h \to \frac{\partial}{\partial x_3}(h \circ \varphi^{-1}).\end{aligned}$$

For this particular manifold $\mathbf{E}^3$ we can expand the above tangent vector to obtain

$$\begin{aligned}
\mathbf{z}_p(h) &= \left.\frac{d}{dt}(h\circ\varphi^{-1})(x_1(t),x_2(t),x_3(t))\right|_{t=0} \\
&= \begin{bmatrix} \frac{\partial h}{\partial z_1} & \frac{\partial h}{\partial z_2} & \frac{\partial h}{\partial z_3} \end{bmatrix}
\begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_2}{\partial x_3} \\ \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} \end{bmatrix}
\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial h}{\partial z_1} & \frac{\partial h}{\partial z_2} & \frac{\partial h}{\partial z_3} \end{bmatrix}
\underbrace{\begin{bmatrix} \mathbf{z}_{x_1} & \mathbf{z}_{x_2} & \mathbf{z}_{x_3} \end{bmatrix}
\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix}}_{\frac{d\mathbf{z}}{dt}}
\end{aligned}$$

With $\frac{d\mathbf{z}}{dt} = \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x_2}\frac{dx_2}{dt} + \mathbf{z}_{x_3}\frac{dx_3}{dt}$ (linear combination of $\mathbf{z}_{x_1}, \mathbf{z}_{x_2}$, and $\mathbf{z}_{x_3}$) we can write

$$\frac{d}{dt}(h\circ\varphi^{-1}) = \frac{d}{dt}(h\circ\mathbf{z}) = \left\langle dh, \frac{d\mathbf{z}}{dt}\right\rangle$$

and consider the tangent vector to be the mapping

$$\mathbf{z}_p : h \to \left\langle dh, \frac{d\mathbf{z}}{dt}\right\rangle$$

instead of $\frac{d\mathbf{z}}{dt} = \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x_2}\frac{dx_2}{dt} + \mathbf{z}_{x_3}\frac{dx_3}{dt}$. We make the correspondence

$$\begin{aligned}
\frac{\partial}{\partial x_1} &\leftrightarrow \mathbf{z}_{x_1} = \frac{\partial \mathbf{z}}{\partial x_1} \\
\frac{\partial}{\partial x_2} &\leftrightarrow \mathbf{z}_{x_2} = \frac{\partial \mathbf{z}}{\partial x_2} \\
\frac{\partial}{\partial x_3} &\leftrightarrow \mathbf{z}_{x_3} = \frac{\partial \mathbf{z}}{\partial x_3}.
\end{aligned}$$

That is, for any function $h(z_1, z_2, z_3)$ we have

$$\begin{aligned}
\frac{\partial(h\circ\varphi^{-1})}{\partial x_1} &= \left\langle dh, \frac{\partial \mathbf{z}}{\partial x_1}\right\rangle \\
\frac{\partial(h\circ\varphi^{-1})}{\partial x_2} &= \left\langle dh, \frac{\partial \mathbf{z}}{\partial x_2}\right\rangle \\
\frac{\partial(h\circ\varphi^{-1})}{\partial x_3} &= \left\langle dh, \frac{\partial \mathbf{z}}{\partial x_3}\right\rangle.
\end{aligned}$$

For any tangent vector we have the correspondence

$$\mathbf{z}_p \triangleq a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3} \leftrightarrow a_1\mathbf{z}_{x_1} + a_2\mathbf{z}_{x_2} + a_3\mathbf{z}_{x_3}.$$

We reiterate: $\mathbf{z}_p \triangleq a_1\dfrac{\partial}{\partial x_1} + a_2\dfrac{\partial}{\partial x_2} + a_3\dfrac{\partial}{\partial x_3}$ is the mapping that takes $h \in \mathcal{F}(\mathcal{U})^2 \to \mathbb{R}$ given by

$$\mathbf{z}_p(h) \triangleq a_1\frac{\partial(h\circ\varphi^{-1})}{\partial x_1} + a_2\frac{\partial(h\circ\varphi^{-1})}{\partial x_2} + a_3\frac{\partial(h\circ\varphi^{-1})}{\partial x_3}.$$

Because this is the manifold $\mathbf{E}^3$ we can write $\mathbf{z}_p(h)$ as

$$\begin{aligned}
\mathbf{z}_p(h) &\triangleq a_1\frac{\partial(h\circ\varphi^{-1})}{\partial x_1} + a_2\frac{\partial(h\circ\varphi^{-1})}{\partial x_2} + a_3\frac{\partial(h\circ\varphi^{-1})}{\partial x_3} \\
&= \begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{z}}{\partial x_1} & \dfrac{\partial \mathbf{z}}{\partial x_2} & \dfrac{\partial \mathbf{z}}{\partial x_3} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\
&= \left\langle dh, \frac{d\mathbf{z}}{dt} \right\rangle
\end{aligned}$$

where $\dfrac{d\mathbf{z}}{dt} = a_1\mathbf{z}_{x_1} + a_2\mathbf{z}_{x_2} + a_3\mathbf{z}_{x_3}$.

3.3 Lie Derivatives and Tangent Vectors

We now make the connection between Lie derivatives and tangent vectors. In fact, we show they are one and the same! To explain, once again consider the northern hemisphere coordinate chart $\varphi^{-1}(x_1,x_2) = \mathbf{z}(x_1,x_2) : \mathcal{D} \to \mathcal{U}$ given by

$$\varphi^{-1}(x_1,x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1-(x_1^2+x_2^2)} \end{bmatrix} \in \mathbf{S}^2$$

where $\mathcal{D} = \{(x_1,x_2)\in\mathbb{R}^2 |\ 0 \le x_1^2+x_2^2 < 1\}$ and $\mathcal{U} \triangleq \{\mathbf{z}\in\mathbf{S}^2 |\ z_3 > 0\}$. Also $\varphi : \mathcal{U} \subset S^2 \to \mathcal{D}$

$$\varphi\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (z_1, z_2).$$

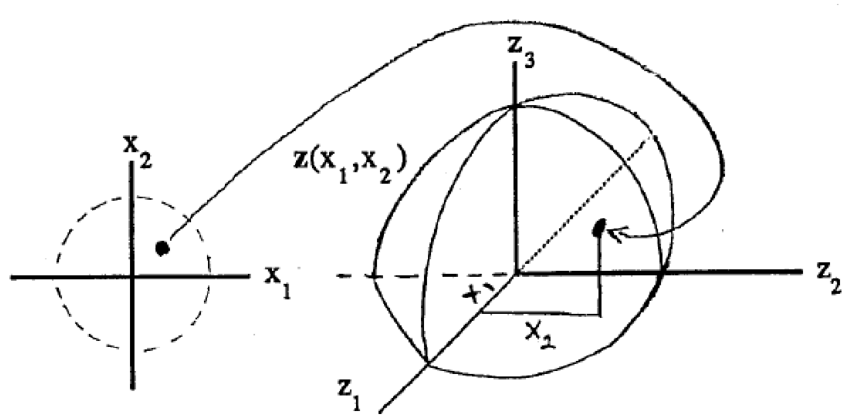


FIGURE 3.13. Northern hemisphere coordinate chart.

[2] $\mathcal{F}(\mathcal{U})$ are the differentiable functions on $\mathcal{U} \subset \mathbf{E}^3$.

Let h be a function defined on $\mathbf{S}^2$ so that

$$(h \circ \varphi^{-1})(x_1, x_2) = h\Big(x_1, x_2, \sqrt{1-(x_1^2+x_2^2)}\Big)$$

is defined on the coordinate patch $\mathcal{D}$. For any curve $(x_1(t), x_2(t))$ in $\mathcal{D}$, the curve $\varphi^{-1}(x_1(t), x_2(t))$ is on $\mathbf{S}^2$ and we have

$$\frac{d}{dt}(h \circ \varphi^{-1})(x_1(t), x_2(t)) = \frac{\partial(h \circ \varphi^{-1})}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial(h \circ \varphi^{-1})}{\partial x_2}\frac{dx_2}{dt}.$$

Recall that the tangent vector is now defined to be the map

$$\mathbf{z}_p : h \to \frac{\partial(h \circ \varphi^{-1})}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial(h \circ \varphi^{-1})}{\partial x_2}\frac{dx_2}{dt}.$$

More generally, with $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ scalar functions defined on $\mathcal{D}$, the tangent vector at the point $p = \varphi^{-1}(x_1, x_2) \in \mathbf{S}^2$ is given by the mapping

$$f_1(x_1,x_2)\frac{\partial}{\partial x_1} + f_2(x_1,x_2)\frac{\partial}{\partial x_2} : h \to f_1(x_1,x_2)\frac{\partial(h \circ \varphi^{-1})}{\partial x_1} + f_2(x_1,x_2)\frac{\partial(h \circ \varphi^{-1})}{\partial x_2}.$$

Further, with $\mathfrak{h}(x_1, x_2) \triangleq (h \circ \varphi^{-1})(x_1, x_2)$ and $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ we have the Lie derivative of $\mathfrak{h}(x_1, x_2)$ with respect to $f(x_1, x_2)$ given by

$$\mathcal{L}_f \mathfrak{h} = \frac{\partial \mathfrak{h}}{\partial x_1} f_1(x_1, x_2) + \frac{\partial \mathfrak{h}}{\partial x_2} f_2(x_1, x_2).$$

That is the Lie derivative operator $\mathcal{L}_f : \mathfrak{h} \to \mathcal{L}_f \mathfrak{h}$ is the tangent vector mapping.

Remark 1 Writing the tangent vector as $f_1(x_1,x_2)\dfrac{\partial}{\partial x_1} + f_2(x_1,x_2)\dfrac{\partial}{\partial x_2}$ tell us the local coordinates are (x_1, x_2) with a corresponding coordinate map φ^{-1} taking these coordinates to the point $\varphi^{-1}(x_1, x_2)$ on the manifold $\mathbf{S}^2$. Further, given a function h on the manifold the function $\mathfrak{h}(x_1, x_2) \triangleq (h \circ \varphi^{-1})(x_1, x_2)$ is known as φ^{-1} is known.

Remark 2 The mapping

$$\mathbf{z}_p : h \to \mathbf{z}_p(h) = f_1(x_1,x_2)\frac{\partial(h \circ \varphi^{-1})}{\partial x_1} + f_2(x_1,x_2)\frac{\partial(h \circ \varphi^{-1})}{\partial x_2}$$

is our new definition of tangent vector. Recall that we would prefer to look at the mapping as given by

$$\begin{aligned}
\mathbf{z}_p : h \to \mathbf{z}_p(h) &= \begin{bmatrix} \dfrac{\partial h(z)}{\partial z_1} & \dfrac{\partial h(z)}{\partial z_2} & \dfrac{\partial h(z)}{\partial z_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \\ \dfrac{\partial z_3}{\partial x_1} & \dfrac{\partial z_3}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} \\
&= dh \begin{bmatrix} \mathbf{z}_{x_1} & \mathbf{z}_{x_2} \end{bmatrix} \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} \\
&= \langle dh, f_1(x_1,x_2)\mathbf{z}_{x_1} + f_2(x_1,x_2)\mathbf{z}_{x_2} \rangle
\end{aligned}$$

with $f_1(x_1, x_2)\mathbf{z}_{x_1} + f_2(x_1, x_2)\mathbf{z}_{x2}$ the tangent vector. However, this doesn't make sense because h is only defined on $\mathbf{S}^2$ ($\partial h/\partial z_1$, etc. are not defined) and $\mathbf{z}_{x_1}, \mathbf{z}_{x2}$ stick off the manifold.

Let's do a similar example in $\mathbf{E}^3$. Recall the spherical coordinate system $\varphi^{-1}(x_1, x_2, x_3)|\mathcal{D} \to \mathcal{U}$ given by

$$\varphi^{-1}(x_1, x_2, x_3) = \mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3$$

where

$$\begin{aligned} \mathcal{D} &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\} \\ \mathcal{U} &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\}. \end{aligned}$$

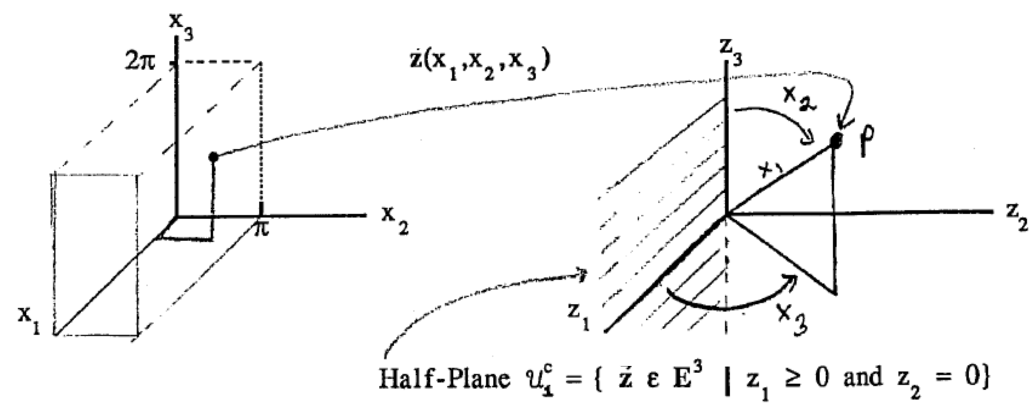


FIGURE 3.14. Spherical coordinates in $\mathbf{E}^3$.

Let $h(z_1, z_2, z_3)$ be a function on $\mathbf{E}^3$ so that

$$\mathfrak{h}(x_1, x_2, x_3) \triangleq (h \circ \varphi^{-1})(x_1, x_2, x_3) = h\left(\begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix}\right)$$

is the representation of h in spherical coordinates. For any curve $(x_1(t), x_2(t), x_3(t))$ in $\mathcal{D}$ at the point $p = \varphi^{-1}(x_1(t), x_2(t), x_3(t)) \in \mathcal{U} \subset \mathbf{E}^3$ we have the tangent vector mapping

$$\mathbf{z}_p : h \to \frac{d(h \circ \varphi^{-1})}{dt} = \frac{\partial(h \circ \varphi^{-1})}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial(h \circ \varphi^{-1})}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial(h \circ \varphi^{-1})}{\partial x_3}\frac{dx_3}{dt}.$$

That is, $\mathbf{z}_p$ is the map that takes a function h to the number $\dfrac{d(h \circ \varphi^{-1})}{dt}$ and is our new definition of tangent vector. The components of this tangent vector are $\dfrac{dx_1}{dt}$, $\dfrac{dx_2}{dt}$, and $\dfrac{dx_3}{dt}$. A set of basis vectors for the tangent space at $p = \varphi^{-1}(x_1(t), x_2(t), x_3(t))$ is

$$\begin{aligned} \frac{\partial}{\partial x_1} &: \ h \to \frac{\partial(h \circ \varphi^{-1})}{\partial x_1} \\ \frac{\partial}{\partial x_2} &: \ h \to \frac{\partial(h \circ \varphi^{-1})}{\partial x_2} \\ \frac{\partial}{\partial x_3} &: \ h \to \frac{\partial(h \circ \varphi^{-1})}{\partial x_3}. \end{aligned}$$

Given $f(x) = (f_1(x), f_2(x), f_3(x))$ with $x \in \mathcal{D} \subset \mathbb{R}^3$ and $h(z_1, z_2, z_3)$ at function on $\mathbf{E}^3$ we have the tangent vector

$$f_1(x)\frac{\partial}{\partial x_1} + f_2(x)\frac{\partial}{\partial x_2} + f_3(x)\frac{\partial}{\partial x_3} : h \to f_1(x)\frac{\partial(h \circ \varphi^{-1})}{\partial x_1} + f_2(x)\frac{\partial(h \circ \varphi^{-1})}{\partial x_2} + f_3(x)\frac{\partial(h \circ \varphi^{-1})}{\partial x_3}$$

at the point $p = \varphi^{-1}(x_1, x_2, x_3)$. This is the same as the Lie derivative of $\mathfrak{h} \triangleq h \circ \varphi^{-1}$ with respect to f written as

$$\mathcal{L}_f : \mathfrak{h} \to \mathcal{L}_f\mathfrak{h} = \frac{\partial \mathfrak{h}}{\partial x_1} f_1(x) + \frac{\partial \mathfrak{h}}{\partial x_2} f_2(x) + \frac{\partial \mathfrak{h}}{\partial x_3} f_3(x)$$

Remark 3 Writing the tangent vector as $f_1(x)\dfrac{\partial}{\partial x_1} + f_2(x)\dfrac{\partial}{\partial x_2} + f_3(x)\dfrac{\partial}{\partial x_3}$ tell us the local coordinates are (x_1, x_2, x_3) with a corresponding coordinate map φ^{-1} taking these coordinates to the point $\varphi^{-1}(x_1, x_2, x_3)$ on the manifold $\mathbf{S}^2$. Further, given a function h on the manifold the function $\mathfrak{h}(x_1, x_2, x_3) \triangleq (h \circ \varphi^{-1})(x_1, x_2, x_3)$ is known as φ^{-1} is known.

Remark 4 For this particular manifold $\mathbf{E}^3$ we *can* rewrite

$$\mathbf{z}_p : h \to \mathbf{z}_p(h) = f_1(x)\frac{\partial(h \circ \varphi^{-1})}{\partial x_1} + f_2(x)\frac{\partial(h \circ \varphi^{-1})}{\partial x_2} + f_3(x)\frac{\partial(h \circ \varphi^{-1})}{\partial x_3}$$

as

$$\begin{aligned}
\mathbf{z}_p(h) &= \begin{bmatrix} \dfrac{\partial h(z)}{\partial z_1} & \dfrac{\partial h(z)}{\partial z_2} & \dfrac{\partial h(z)}{\partial z_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} & \dfrac{\partial z_1}{\partial x_3} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} & \dfrac{\partial z_2}{\partial x_3} \\ \dfrac{\partial z_3}{\partial x_1} & \dfrac{\partial z_3}{\partial x_2} & \dfrac{\partial z_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{bmatrix} \\
&= dh \begin{bmatrix} \mathbf{z}_{x_1} & \mathbf{z}_{x_2} & \mathbf{z}_{x_3} \end{bmatrix} \begin{bmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{bmatrix} \\
&= \langle dh, f_1(x_1, x_2, x_3)\mathbf{z}_{x_1} + f_2(x_1, x_2, x_3)\mathbf{z}_{x_2} + f_2(x_1, x_2, x_3)\mathbf{z}_{x_3} \rangle
\end{aligned}$$

and think of $f_1(x_1, x_2, x_3)\mathbf{z}_{x_1} + f_2(x_1, x_2, x_3)\mathbf{z}_{x_2} + f_2(x_1, x_2, x_3)\mathbf{z}_{x_3}$ as the tangent vector.

Transformation Law for Gradients

Consider two different coordinate systems for $\mathbf{z}(x) = \varphi^{-1}(x)$ and $\mathbf{z}(\bar{x}) = \bar{\varphi}^{-1}(\bar{x})$ for $\mathbf{E}^3$ as indicated in Figure 3.15. We can write the change of coordinates from x to $\bar{x}$ and vice versa as

$$\begin{aligned}
x &= (\varphi \circ \bar{\varphi}^{-1})(\bar{x}) = \varphi(\bar{\varphi}^{-1}(\bar{x})) \\
\bar{x} &= (\bar{\varphi} \circ \varphi^{-1})(x) = \bar{\varphi}(\varphi^{-1}(x)).
\end{aligned}$$

We also write this change of coordinates as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1(\bar{x}_1, \bar{x}_2, \bar{x}_3) \\ x_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) \\ x_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) \end{pmatrix} \tag{3.1}$$

and

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} \bar{x}_1(x_1, x_2, x_3) \\ \bar{x}_2(x_1, x_2, x_3) \\ \bar{x}_3(x_1, x_2, x_3) \end{pmatrix} \tag{3.2}$$

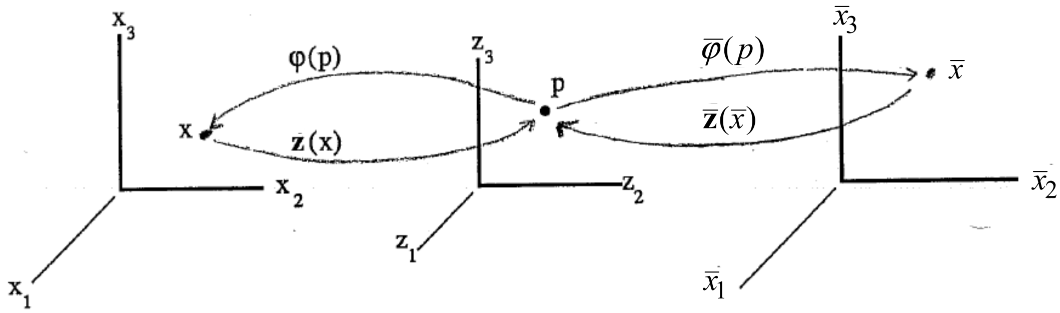


FIGURE 3.15. Two coordinate maps φ and $\bar\varphi$ for $\mathbf{E}^2$.

With h a function on $\mathbf{E}^3$, h in the x coordinates is $\mathfrak{h} = h \circ \varphi^{-1}(x)$ with gradient

$$d\mathfrak{h} = \left[\ \frac{\partial}{\partial x_1}(h \circ \varphi^{-1}) \quad \frac{\partial}{\partial x_2}(h \circ \varphi^{-1}) \quad \frac{\partial}{\partial x_3}(h \circ \varphi^{-1})\ \right]. \tag{3.3}$$

In the $\bar x$ coordinates h is represented by $\bar{\mathfrak{h}} = h \circ \bar\varphi^{-1}(\bar x)$ with gradient

$$d\bar{\mathfrak{h}} = \left[\ \frac{\partial}{\partial \bar x_1}\left(h \circ \bar\varphi^{-1}\right) \quad \frac{\partial}{\partial \bar x_2}\left(h \circ \bar\varphi^{-1}\right) \quad \frac{\partial}{\partial \bar x_3}\left(h \circ \bar\varphi^{-1}\right)\ \right]. \tag{3.4}$$

How are these two gradients related? Well we have

$$\begin{aligned} \mathfrak{h}(x) &= h \circ \varphi^{-1}(x) \\ \bar{\mathfrak{h}}(\bar x) &= h \circ \bar\varphi^{-1}(\bar x) \end{aligned}$$

so that

$$\bar{\mathfrak{h}}(\bar x)\big|_{\bar x = (\bar\varphi\circ\varphi^{-1})(x)} = h \circ \bar\varphi^{-1}(\bar x)\big|_{\bar x = (\bar\varphi\circ\varphi^{-1})(x)} = h(\bar\varphi^{-1}(\bar\varphi(\varphi^{-1}(x))) = h(\varphi^{-1}(x)) = \mathfrak{h}(x).$$

By the chain rule we can write

$$\frac{\partial}{\partial x_1}\mathfrak{h}(x) = \frac{\partial}{\partial x_1}\left(\bar{\mathfrak{h}}(\bar x)\big|_{\bar x = (\bar\varphi\circ\varphi^{-1})(x)}\right) = \sum_{j=1}^{3} \frac{\partial \bar{\mathfrak{h}}(\bar x)}{\partial \bar x_j}\frac{\partial \bar x_j}{\partial x_1}$$

or

$$\begin{aligned} d\mathfrak{h} &= \left[\ \frac{\partial \bar{\mathfrak{h}}(\bar x)}{\partial \bar x_1} \quad \frac{\partial \bar{\mathfrak{h}}(\bar x)}{\partial \bar x_2} \quad \frac{\partial \bar{\mathfrak{h}}(\bar x)}{\partial \bar x_3}\ \right] \begin{bmatrix} \frac{\partial \bar x_1}{\partial x_1} & \frac{\partial \bar x_1}{\partial x_2} & \frac{\partial \bar x_1}{\partial x_3} \\ \frac{\partial \bar x_2}{\partial x_1} & \frac{\partial \bar x_2}{\partial x_2} & \frac{\partial \bar x_2}{\partial x_3} \\ \frac{\partial \bar x_3}{\partial x_1} & \frac{\partial \bar x_3}{\partial x_2} & \frac{\partial \bar x_3}{\partial x_3} \end{bmatrix} && (3.5) \\ &= d\bar{\mathfrak{h}}\frac{\partial \bar x}{\partial x}. && (3.6) \end{aligned}$$

Let $(\bar x_1(t), \bar x_2(t), \bar x_3(t))$ be a curve in the $\bar x$ coordinates which corresponds to the curve

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} x_1(\bar x_1(t), \bar x_2(t), \bar x_3(t)) \\ x_2(\bar x_1(t), \bar x_2(t), \bar x_3(t)) \\ x_3(\bar x_1(t), \bar x_2(t), \bar x_3(t)) \end{pmatrix}. \tag{3.7}$$

Then

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{pmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial \bar{x}_1} & \dfrac{\partial x_1}{\partial \bar{x}_2} & \dfrac{\partial x_1}{\partial \bar{x}_3} \\ \dfrac{\partial x_2}{\partial \bar{x}_1} & \dfrac{\partial x_2}{\partial \bar{x}_2} & \dfrac{\partial x_2}{\partial \bar{x}_3} \\ \dfrac{\partial x_3}{\partial \bar{x}_1} & \dfrac{\partial x_3}{\partial \bar{x}_2} & \dfrac{\partial x_3}{\partial \bar{x}_3} \end{bmatrix} \begin{pmatrix} \dfrac{d\bar{x}_1}{dt} \\ \dfrac{d\bar{x}_2}{dt} \\ \dfrac{d\bar{x}_3}{dt} \end{pmatrix} \tag{3.8}$$

showing how the components of the tangent vector transform going from the $\bar{x}$ coordinates to the x coordinates. More generally we have

$$\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial \bar{x}_1} & \dfrac{\partial x_1}{\partial \bar{x}_2} & \dfrac{\partial x_1}{\partial \bar{x}_3} \\ \dfrac{\partial x_2}{\partial \bar{x}_1} & \dfrac{\partial x_2}{\partial \bar{x}_2} & \dfrac{\partial x_2}{\partial \bar{x}_3} \\ \dfrac{\partial x_3}{\partial \bar{x}_1} & \dfrac{\partial x_3}{\partial \bar{x}_2} & \dfrac{\partial x_3}{\partial \bar{x}_3} \end{bmatrix} \begin{pmatrix} \bar{f}_1(\bar{x}) \\ \bar{f}_2(\bar{x}) \\ \bar{f}_3(\bar{x}) \end{pmatrix} \tag{3.9}$$

or compactly

$$f = \frac{\partial x}{\partial \bar{x}} \bar{f}. \tag{3.10}$$

Equation (3.5) is the transformation of the gradient in the $\bar{x}$ coordinates to the gradient in the x coordinates while Equation (3.10) is the transformation of the components of the tangent vector in the $\bar{x}$ coordinates to the components in the x coordinates. Note that these matrix transformations are inverses of each other. A quantity that transforms like the components of a tangent vector is called a *contravariant* vector while a quantity that transforms like a gradient is called a *covector*.

Invariance of the Lie Derivative Under Coordinate Transformations

Let $f(x) = (f_1(x), f_2(x), f_3(x))$ be the components of a tangent vector in the x coordinate system and $\bar{f}(\bar{x}) = \big(\bar{f}_1(\bar{x}), \bar{f}_2(\bar{x}), \bar{f}_3(\bar{x})\big)$ be the components of this same tangent vector in the $\bar{x}$ coordinates. With h a function defined on $\mathbf{E}^3$ it is represented in the x coordinates as $h \circ \varphi^{-1}$ and in the $\bar{x}$ coordinates as $h \circ \bar{\varphi}^{-1}$. Then

$$\begin{aligned} \mathcal{L}_f(h \circ \varphi^{-1}) &= \begin{bmatrix} \dfrac{\partial (h \circ \varphi^{-1})}{\partial x_1} & \dfrac{\partial (h \circ \varphi^{-1})}{\partial x_2} & \dfrac{\partial (h \circ \varphi^{-1})}{\partial x_3} \end{bmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} \\ &= \begin{bmatrix} \dfrac{\partial (h \circ \bar{\varphi}^{-1})}{\partial \bar{x}_1} & \dfrac{\partial (h \circ \bar{\varphi}^{-1})}{\partial \bar{x}_2} & \dfrac{\partial (h \circ \bar{\varphi}^{-1})}{\partial \bar{x}_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \bar{x}_1}{\partial x_1} & \dfrac{\partial \bar{x}_1}{\partial x_2} & \dfrac{\partial \bar{x}_1}{\partial x_3} \\ \dfrac{\partial \bar{x}_2}{\partial x_1} & \dfrac{\partial \bar{x}_2}{\partial x_2} & \dfrac{\partial \bar{x}_2}{\partial x_3} \\ \dfrac{\partial \bar{x}_3}{\partial x_1} & \dfrac{\partial \bar{x}_3}{\partial x_2} & \dfrac{\partial \bar{x}_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x_1}{\partial \bar{x}_1} & \dfrac{\partial x_1}{\partial \bar{x}_2} & \dfrac{\partial x_1}{\partial \bar{x}_3} \\ \dfrac{\partial x_2}{\partial \bar{x}_1} & \dfrac{\partial x_2}{\partial \bar{x}_2} & \dfrac{\partial x_2}{\partial \bar{x}_3} \\ \dfrac{\partial x_3}{\partial \bar{x}_1} & \dfrac{\partial x_3}{\partial \bar{x}_2} & \dfrac{\partial x_3}{\partial \bar{x}_3} \end{bmatrix} \begin{pmatrix} \bar{f}_1(\bar{x}) \\ \bar{f}_2(\bar{x}) \\ \bar{f}_3(\bar{x}) \end{pmatrix} \\ &= \mathcal{L}_{\bar{f}}(h \circ \bar{\varphi}^{-1}) \end{aligned} \tag{3.11}$$

showing that the value of Lie derivative (tangent vector) is the same in all coordinate systems.

3.4 Submanifolds and the Implicit Function Theorem

Let's work in the manifold $\mathbf{E}^3$ with the Cartesian coordinate system $\varphi^{-1} : \mathbb{R}^3 \to \mathbf{E}^3$ given by

$$\varphi^{-1}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{E}^3. \tag{3.12}$$

With c a constant, consider the implicit equation $F(z_1, z_2, z_3) = c$ defined on $\mathbf{E}^3$. For example, let $F(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2$ and consider the points $z \in \mathbf{E}^3$ satisfying $F(z_1, z_2, z_3) = c$ with $c > 0$. This implicit relationship allows us to solve for one of the variables in terms of the other two variables. Specifically, for all $z \in \mathbf{E}^3$ with $z_1^2 + z_2^2 \leq \sqrt{c}$, we have

$$\begin{aligned} z_3 &= \sqrt{c - (z_1^2 + z_2^2)} \\ z_3 &= -\sqrt{c - (z_1^2 + z_2^2)}. \end{aligned}$$

We use the function F to define a new manifold (submanifold) $\mathcal{M} \subset \mathbf{E}^3$ given by

$$\mathcal{M} \triangleq \left\{ \mathbf{z} \in \mathbf{E}^3 \middle| \; F(z_1, z_2, z_3) - c = z_1^2 + z_2^2 + z_3^2 - c = 0 \right\} \tag{3.13}$$

which is just the surface of a sphere of radius $\sqrt{c}$. To make it into a manifold we need to have coordinate charts for it. We can define a coordinate chart by $\mathbf{z}(x_1, x_2) : \mathcal{D}_1 \to \mathcal{U}_1 \subset \mathbf{E}^3$ by

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{c - (x_1^2 + x_2^2)} \end{bmatrix} \in \mathbf{E}^3$$

where $\mathcal{D}_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \; 0 \leq x_1^2 + x_2^2 < 1 \right\}$ and $\mathcal{U}_1 \triangleq \{\mathbf{z} \in \mathcal{M} | \; z_3 > 0\}$.

Similarly, a coordinate chart for points of $\mathcal{M}$ with $z_3 < 0$ is given by $\mathbf{z}(\bar{x}_1, \bar{x}_2) : \mathcal{D}_2 \to \mathcal{U}_2 \subset \mathbf{E}^3$

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ -\sqrt{c - (\bar{x}_1^2 + \bar{x}_2^2)} \end{bmatrix} \in \mathbf{E}^3 \tag{3.14}$$

where $\mathcal{D}_2 = \left\{ (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 \middle| \; 0 \leq \bar{x}_1^2 + \bar{x}_2^2 < 1 \right\}$ and $\mathcal{U} \triangleq \{\mathbf{z} \in \mathcal{M} | \; z_3 < 0\}$. To get coordinate charts for points of $\mathcal{M}$ where $z_3 = 0$ one must solve for z_1 or z_2 in terms of the remaining variables.

We now generalize this example by considering $\mathbf{E}^3$ to be made up of spherically shaped submanifolds. To do this consider the coordinate system $\varphi^* : \mathcal{D}^* \to \mathcal{U}^*$ on $\mathbf{E}^3$ given by

$$\varphi^* \left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right) = \left(z_1, z_2, z_1^2 + z_2^2 + z_3^2 \right) = (x_1^*, x_2^*, x_3^*) \tag{3.15}$$

where $\mathcal{D}^* = \left\{ (x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3 \middle| \; 0 < (x_1^*)^2 + (x_2^*)^2 < x_3^* \right\}$ and $\mathcal{U}^* \triangleq \{\mathbf{z} \in \mathbf{E}^3 | \; z_3 > 0\}$. The inverse is

$$\varphi^{*-1}(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} x_1^* \\ x_2^* \\ \sqrt{x_3^* - ((x_1^*)^2 + (x_2^*)^2)} \end{bmatrix}. \tag{3.16}$$

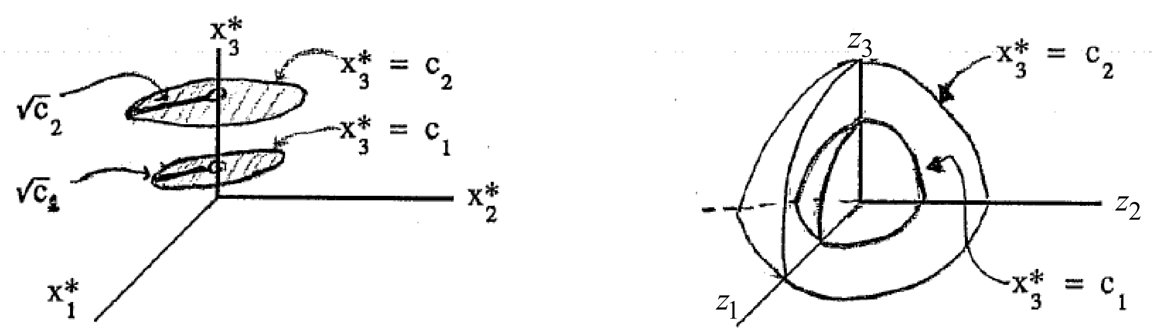


FIGURE 3.16. x and x^* coordinate systems.

With (x_1, x_2, x_3) the Cartesian coordinates for $\mathbf{E}^3$ the change of coordinates from x^* to x is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \\ \sqrt{x_3^* - \left((x_1^*)^2 + (x_2^*)^2\right)} \end{pmatrix} \tag{3.17}$$

with inverse

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_1^2 + x_2^2 + x_3^2 \end{pmatrix}. \tag{3.18}$$

It turns out to be very common to define submanifolds of $\mathbf{E}^3$ implicitly as just described. The main tool needed to use this approach is the implicit theorem.

Theorem 1 *Implicit Function Theorem*
Let $F(x_1, x_2, x_3) : \mathbb{R}^3 \to \mathbb{R}$ be a scalar valued continuously differentiable function. Let $\mathcal{M}$ be the subset of $\mathbb{R}^3$ defined by

$$\mathcal{M} \triangleq \left\{ x \in \mathbb{R}^3 |\ F(x_1, x_2, x_3) - c = 0 \right\}.$$

Suppose

(1) $\mathcal{M}$ is nonempty.

(2) $\left.\dfrac{\partial F(x_1, x_2, x_3)}{\partial x_3}\right|_{x_0} \neq 0$ for $x_0 = (x_{01}, x_{02}, x_{03}) \in \mathcal{M}$.

Then there exists a neighborhood $\mathcal{D}$ of (x_{01}, x_{02}) and a scalar function $s(x_1, x_2) : \mathcal{D} \to \mathcal{M}$ such that

$$F(x_1, x_2, s(x_1, x_2)) - c = 0.$$

Proof. Define the transformation

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ F(x_1, x_2, x_3) - c \end{bmatrix}. \tag{3.19}$$

with

$$\begin{bmatrix} x_{01}' \\ x_{02}' \\ x_{03}' \end{bmatrix} \triangleq \begin{bmatrix} x_{01} \\ x_{02} \\ F(x_{01}, x_{02}, x_{03}) - c \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \\ 0 \end{bmatrix}. \tag{3.20}$$

The Jacobian of this transformation is

$$J(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{\partial F(x_1, x_2, x_3)}{\partial x_1} & \dfrac{\partial F(x_1, x_2, x_3)}{\partial x_2} & \dfrac{\partial F(x_1, x_2, x_3)}{\partial x_3} \end{bmatrix} \tag{3.21}$$

which is nonsingular at x_0 as $\det J(x_0) = \left.\dfrac{\partial F(x_1, x_2, x_3)}{\partial x_3}\right|_{x_0} \neq 0$. By the inverse function theorem this transformation has a unique inverse about the point $x_0' = \begin{bmatrix} x_{01} & x_{02} & 0 \end{bmatrix}^T$ of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \triangleq \begin{bmatrix} x_1' \\ x_2' \\ h(x_1', x_2', x_3') \end{bmatrix}. \tag{3.22}$$

Substitute (3.22) into (3.19) to obtain

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \triangleq \begin{bmatrix} x_1' \\ x_2' \\ F(x_1', x_2', h(x_1', x_2', x_3')) - c \end{bmatrix}. \tag{3.23}$$

This is true for all (x_1', x_2', x_3') in some neighborhood $\mathcal{U}'$ which contains the point $(x_{01}, x_{02}, 0)$. This is illustrated in Figure 3.17.

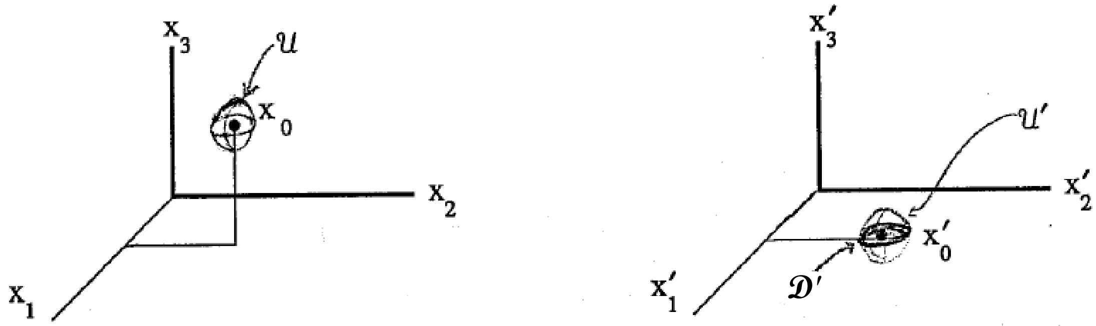


FIGURE 3.17. Transformation to the x' coordinate system.

Consequently Equation (3.23) holds for all $(x_1', x_2', 0)$ in a neighborhood $(x_{01}, x_{02}, 0)$. Specifically, let $\mathcal{D}' \triangleq \{(x_1', x_2') | \; (x_1', x_2', 0) \in \mathcal{U}'\}$ and set $x_3' = 0$ in (3.23) to obtain

$$0 = F(x_1', x_2', h(x_1', x_2', 0)) - c$$

for all $(x_1', x_2') \in \mathcal{D}'$ of (x_{01}, x_{02}). Define $s(x_1, x_2) \triangleq h(x_1, x_2, 0)$ and, as $x_1 = x_1', x_2 = x_2'$ we have

$$F(x_1, x_2, s(x_1, x_2)) - c \equiv 0.$$

■

Remark An immediate result of this theorem is that

$$\varphi^{-1}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ s(x_1, x_2) \end{bmatrix} \in \mathcal{M}$$

is a coordinate map for $\mathcal{M}$ around the point $p = \varphi^{-1}(x_{01}, x_{02}) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$.

Gradients, Tangent Vectors, and Manifolds

With the manifold $\mathbf{E}^3$ we can consider the Cartesian coordinates $(x_1, x_2, x_3) \in \mathbb{R}^3$ and the point $\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$ to be the same thing with $z_1 = x_1, z_2 = x_2, z_3 = x_3$. With this understanding, we now again consider a submanifold of $\mathbf{E}^3$ defined implicitly by a function $F(x_1, x_2, x_3) : \mathbb{R}^3 \to \mathbb{R}$, that is, $\mathcal{M} \triangleq \{x \in \mathbb{R}^3 | \; F(x_1, x_2, x_3) - c = 0\}$ for some constant c. If $\left.\dfrac{\partial F(x_1, x_2, x_3)}{\partial x_3}\right|_{x_0} \neq 0$ for some point x_0 satisfying $F(x_0) - c = 0$ the implicit function theorem ensures we can find a surface

$$S(u_1, u_2) = \begin{bmatrix} s_1(u_1, u_2) \\ s_2(u_1, u_2) \\ s_3(u_1, u_2) \end{bmatrix} \tag{3.24}$$

with

$$S(u_{01}, u_{02}) = x_0 \tag{3.25}$$

and

$$F(s_1(u_1,u_2), s_2(u_1,u_2), s_3(u_1,u_2)) - c \equiv 0. \tag{3.26}$$

(In the proof of the implicit function theorem we took $u_1 = x_1, u_2 = x_2$ and $s_1(u_1,u_2) = u_1, s_2(u_1,u_2) = u_2$.)

Using the chain rule on Equation (3.26) we have

$$\begin{aligned}
\frac{\partial}{\partial u_1} F(s_1(u_1,u_2), s_2(u_1,u_2), s_3(u_1,u_2)) &= \frac{\partial F}{\partial x_1}\frac{\partial s_1}{\partial u_1} + \frac{\partial F}{\partial x_2}\frac{\partial s_2}{\partial u_1} + \frac{\partial F}{\partial x_3}\frac{\partial s_3}{\partial u_1} \equiv 0 \\
\frac{\partial}{\partial u_2} F(s_1(u_1,u_2), s_2(u_1,u_2), s_3(u_1,u_2)) &= \frac{\partial F}{\partial x_1}\frac{\partial s_1}{\partial u_2} + \frac{\partial F}{\partial x_2}\frac{\partial s_2}{\partial u_2} + \frac{\partial F}{\partial x_3}\frac{\partial s_3}{\partial u_2} \equiv 0.
\end{aligned}$$

These can be written in a geometric fashion via the dual product

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial S}{\partial u_1} \right\rangle \equiv 0 \tag{3.27}$$

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial S}{\partial u_2} \right\rangle \equiv 0 \tag{3.28}$$

where $\dfrac{\partial S}{\partial u_1} \in \mathbb{R}^3, \dfrac{\partial S}{\partial u_2} \in \mathbb{R}^3$ are contravariant (column) vectors and $\dfrac{\partial F}{\partial x} \in \mathbb{R}^{1\times 3}$ is a (row) covector.

The tangent space $\mathbf{T}_p(\mathcal{M})$ to the surface (submanifold) at the point $p = [s_1(u_1,u_2), s_2(u_1,u_2), s_3(u_1,u_2)]^T$ is spanned by $\dfrac{\partial S}{\partial u_1}$ and $\dfrac{\partial S}{\partial u_2}$. The above relationships show that the gradient $\left.\dfrac{\partial F}{\partial x}\right|_{p\in\mathcal{M}}$ is orthogonal (normal/perpendicular) to the tangent space $\mathbf{T}_p(\mathcal{M})$. In other words, if a manifold is defined implicitly by a function $F(x_1,x_2,x_3) : \mathbb{R}^3 \to \mathbb{R}$ as $\mathcal{M} \triangleq \left\{x \in \mathbb{R}^3 | \; F(x_1,x_2,x_3) - c = 0\right\}$, then the gradient $\left.\dfrac{\partial F}{\partial x}\right|_{p\in\mathcal{M}}$ is perpendicular to $\mathbf{T}_p(\mathcal{M})$.

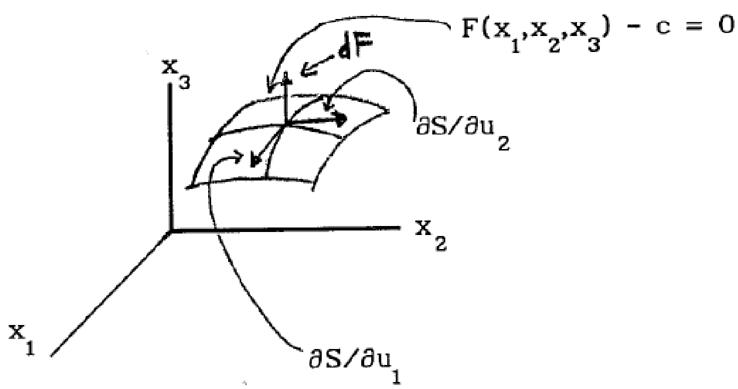


FIGURE 3.18. $\dfrac{\partial S}{\partial u_1}, \dfrac{\partial S}{\partial u_2}$ are tangent to the submanifold and $\dfrac{\partial F}{\partial x}$ is normal to the manifold.

Remark 5 The above discussion is really no surprise from a basic calculus course. Again let $F(x_1,x_2,x_3)$: $\mathbb{R}^3 \to \mathbb{R}$ be a continuously differentiable function. Next consider a curve $(x_1(t), x_2(t), x_3(t))$ with $(x_1(0), x_2(0), x_3(0)) = (x_{01}, x_{02}, x_{03})$ and let $f(t) \triangleq F(x_1(t), x_2(t), x_3(t))$ so that we have

$$\frac{d}{dt}f(t) = \frac{d}{dt}F(x_1(t), x_2(t), x_3(t)) = \frac{\partial F}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial F}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial F}{\partial x_3}\frac{dx_3}{dt}$$

and, at $t = 0$,

$$f'(0) = \left\langle \left.\frac{\partial F}{\partial x}\right|_{x_0}, \dot{x}(0) \right\rangle.$$

The gradient $\left.\frac{\partial F}{\partial x}\right|_{x_0}$ points in the direction of the *maximum* rate of change of the function $F(x)$ at x_0. If the curve lies on the level curve defined by $F(x_1, x_2, x_3) - c = 0$ then $F(x_1(t), x_2(t), x_3(t)) \equiv c$ and it value does not change on $x(t)$. Thus $f'(0) = 0$ so $\dot{x}(0) = (\dot{x}_1(0), \dot{x}_2(0), \dot{x}_3(0))$ must be orthogonal to $\left.\frac{\partial F}{\partial x}\right|_{x_0}$. Further, as $(x_1(t), x_2(t), x_3(t)) \in \mathcal{M} = \{x \in \mathbb{R}^3 |\ F(x_1, x_2, x_3) - c = 0\}$ for all t, it must be that $\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t))$ is tangent to $\mathcal{M}$ for all t.

We now state the implicit function theorem for the general case.

Theorem 2 *Implicit Function Theorem - General Case*
Let $x \in \mathbb{R}^{n+m}$ and consider the m differentiable functions $F_1(x), F_2(x), ..., F_m(x)$. We look for solutions to the m equations

$$\begin{aligned} F_1(x_1, ..., x_n, x_{n+1}, ..., x_{n+m}) &= 0 \\ F_2(x_1, ..., x_n, x_{n+1}, ..., x_{n+m}) &= 0 \\ \vdots \quad &= \vdots \\ F_m(x_1, ..., x_n, x_{n+1}, ..., x_{n+m}) &= 0. \end{aligned} \tag{3.29}$$

Suppose at $x_0 = (x_{01}, ..., x_{0n}, x_{0,n+1}, ..., x_{0,n+m})$ we have $F_1(x_0) = 0, F_2(x_0) = 0, ..., F_m(x_0) = 0$ and

$$\det \begin{bmatrix} \frac{\partial F_1}{\partial x_{n+1}} & \cdots & \frac{\partial F_1}{\partial x_{n+m}} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_{n+1}} & \cdots & \frac{\partial F_m}{\partial x_{n+m}} \end{bmatrix}_{|x_0} \neq 0. \tag{3.30}$$

Then there exists m functions $s_1(x_1, ..., x_n), s_2(x_1, ..., x_n), ..., s_m(x_1, ..., x_n)$ such that

$$\begin{aligned} F_1(x_1, ..., x_n, s_1(x_1, ..., x_n), s_2(x_1, ..., x_n), ..., s_m(x_1, ..., x_n)) &= 0 \\ F_2(x_1, ..., x_n, s_1(x_1, ..., x_n), s_2(x_1, ..., x_n), ..., s_m(x_1, ..., x_n)) &= 0 \\ \vdots \quad &= \vdots \\ F_m(x_1, ..., x_n, s_1(x_1, ..., x_n), s_2(x_1, ..., x_n), ..., s_m(x_1, ..., x_n)) &= 0 \end{aligned} \tag{3.31}$$

for all $(x_1, ..., x_n)$ in a neighborhood of $(x_{01}, ..., x_{0n})$ with

$$\begin{aligned} x_{0,n+1} &= s_1(x_{01}, ..., x_{0n}) \\ x_{0,n+2} &= s_2(x_{01}, ..., x_{0n}) \\ \vdots &= \vdots \\ x_{0,n+m} &= s_m(x_{01}, ..., x_{0n}) \end{aligned} \tag{3.32}$$

Proof. Exercise - Just use the inverse function theorem. ■

Remark 6 The implicit function theorem shows tells that if $x_0 \in \mathbb{R}^{n+m}$ satisfies

$$\mathcal{M} \triangleq \left\{x \in \mathbb{R}^{n+m} |\ F_1(x) = 0, ..., F_m(x) = 0\right\}$$

with

$$\det \begin{bmatrix} \dfrac{\partial F_1}{\partial x_{n+1}} & \cdots & \dfrac{\partial F_1}{\partial x_{n+m}} \\ \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial x_{n+1}} & \cdots & \dfrac{\partial F_m}{\partial x_{n+m}} \end{bmatrix}_{x_0} \neq 0$$

then there is a neighborhood of points of x_0 in $\mathcal{M}$ for which we have a coordinate chart given by

$$\varphi^{-1}(x_1, ..., x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ s_1(x_1, ..., x_n) \\ \vdots \\ s_m(x_1, ..., x_n) \end{bmatrix} \in \mathcal{M}.$$

We say $\mathcal{M}$ is n dimensional as it has n independent coordinates.

3.5 Feedback Linearizing Transformations and Integral Manifolds

Let's go back and take a look at the idea of feedback linearization and its relationship to tangent vectors and manifolds.

Consider the control system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{bmatrix} + \begin{bmatrix} g_1(x_1, x_2, x_3) \\ g_2(x_1, x_2, x_3) \\ g_3(x_1, x_2, x_3) \end{bmatrix} u \tag{3.33}$$

or more compactly as

$$\frac{dx}{dt} = f(x) + g(x)u.$$

We want to find conditions for which a transformation of the form

$$\begin{aligned} x_1^* &= T_1(x_1, x_2, x_3) \\ x_2^* &= T_2(x_1, x_2, x_3) \\ x_3^* &= T_3(x_1, x_2, x_3) \end{aligned} \tag{3.34}$$

exists such that in the new coordinates the control system is given by

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} x_2^* \\ x_3^* \\ f_3^*(x_1^*, x_2^*, x_3^*) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_3^*(x_1^*, x_2^*, x_3^*) \end{bmatrix} u. \tag{3.35}$$

We have

$$\begin{aligned} \frac{dx_1^*}{dt} &= \mathcal{L}_f(T_1) + u\mathcal{L}_g(T_1) \\ \frac{dx_2^*}{dt} &= \mathcal{L}_f(T_2) + u\mathcal{L}_g(T_2) \\ \frac{dx_3^*}{dt} &= \mathcal{L}_f(T_3) + u\mathcal{L}_g(T_3) \end{aligned} \tag{3.36}$$

In order for (3.36) to have the form (3.35) we define must define x_2^* as

$$x_2^* \triangleq \mathcal{L}_f(T_1)$$

with T_1 satisfying

$$\mathcal{L}_g(T_1) = 0$$

and we must define x_3^* as

$$x_3^* \triangleq \mathcal{L}_f(T_2) = \mathcal{L}_f^2(T_1)$$

with T_1 also satisfying

$$\mathcal{L}_g(T_2) = \mathcal{L}_g(\mathcal{L}_f(T_1)) = 0.$$

That is, we must find $T_1(x)$ such that

$$\begin{aligned} \mathcal{L}_g(T_1) &= 0 &(3.37)\\ \mathcal{L}_g(\mathcal{L}_f(T_1)) &= 0. &(3.38) \end{aligned}$$

These are second-order nonlinear partial differential equations in the unknown T_1. We can simplify these equations a bit by using the identity $\mathcal{L}_{[f,g]}(h) = \mathcal{L}_g\mathcal{L}_f(h) - \mathcal{L}_f\mathcal{L}_g(h)$ shown in Chapter 1. The conditions (3.37) and (3.38) become

$$\begin{aligned} \mathcal{L}_g(T_1) &= 0 \\ \mathcal{L}_{[f,g]}(T_1) &= \mathcal{L}_g\mathcal{L}_f(T_1) - \mathcal{L}_f\mathcal{L}_g(T_1) = 0 \end{aligned}$$

or

$$dT_1 \begin{bmatrix} g & ad_f g \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

where it is recalled that $ad_f g \triangleq [f, g]$. This has a nice geometric interpretation because it says that the gradient dT_1 must be perpendicular to $g, ad_f g$. The coordinate function $T_1(x)$ is a mapping from $\mathbb{R}^3 \to \mathbb{R}$. If we consider the submanifold $\mathcal{M} \triangleq \left\{x \in \mathbb{R}^3 | \; T_1(x_1, x_2, x_3) = c_1\right\}$ then we know that dT_1 is orthogonal to this two dimensional manifold. So the key problem to finding a feedback linearizing function is to find a function $T_1 : \mathbb{R}^3 \to \mathbb{R}$ such that for any c_1 the tangent space to $\mathcal{M} \triangleq \left\{x \in \mathbb{R}^3 | \; T_1(x_1, x_2, x_3) = c_1\right\}$ is spanned by $\{g, ad_f g\}$. The implicit function theorem tells us if certain conditions hold there is an explicit representation of the surface given by

$$S(u_1, u_2) = \begin{bmatrix} s_1(u_1, u_2) \\ s_2(u_1, u_2) \\ s_3(u_1, u_2) \end{bmatrix}$$

mapping a subset of $\mathbb{R}^2$ to $\mathcal{M}$. The tangent vectors to M are $\dfrac{\partial S}{\partial u_1}$ and $\dfrac{\partial S}{\partial u_2}$ and the feedback linearizing conditions tells us that

$$\{g, ad_f g\} \in span\left\{\frac{\partial S}{\partial u_1}, \frac{\partial S}{\partial u_2}\right\} = \left\{r_1\frac{\partial S}{\partial u_1} + r_2\frac{\partial S}{\partial u_2} \mid \text{with } r_1, r_2 \in \mathbb{R}\right\}.$$

We show how this all works out in the next chapter.

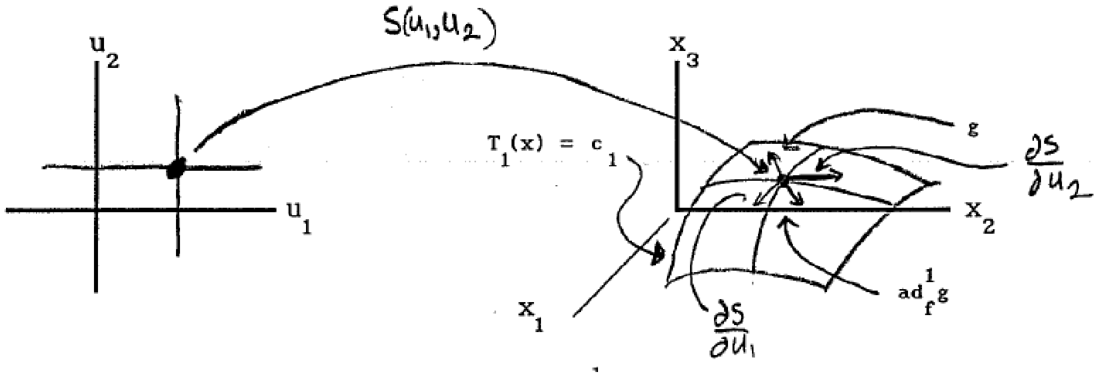


FIGURE 3.19. The vectors $\dfrac{\partial S}{\partial u_1}$ and $\dfrac{\partial S}{\partial u_2}$ span the same plane as the vectors g and $ad_f g$.

3.6 Problems

Problem 1 *Implicit Function Theorem*

Let $F_1(x_1, x_2, x_3, x_4, x_5)$ and $F_2(x_1, x_2, x_3, x_4, x_5)$ be continuously differentiable functions from an open set $\mathcal{U} \subset \mathbb{R}^5 \to \mathbb{R}$. Further suppose $x_0 = (x_{01}, x_{02}, x_{03}, x_{04}, x_{05}) \in \mathcal{U}$ satisfying $F_1(x_0) = F_2(x_0) = 0$ and

$$\left.\frac{\partial(F_1, F_2)}{\partial x_4 \partial x_5}\right|_{x_0} \triangleq \det \left.\begin{bmatrix} \dfrac{\partial F_1}{\partial x_4} & \dfrac{\partial F_1}{\partial x_5} \\ \dfrac{\partial F_2}{\partial x_4} & \dfrac{\partial F_2}{\partial x_5} \end{bmatrix}\right|_{x_0} \neq 0.$$

(a) Using the inverse function theorem show there exists functions $s_1(x_1, x_2, x_3)$ and $s_2(x_1, x_2, x_3)$ defined on a neighborhood $\mathcal{D} \subset \mathbb{R}^3$ containing (x_{01}, x_{02}, x_{03}) such that

$$\begin{aligned} x_{04} &= s_1(x_{01}, x_{02}, x_{03}) \\ x_{05} &= s_2(x_{01}, x_{02}, x_{03}) \end{aligned}$$

and for all $(x_1, x_2, x_3) \in \mathcal{D}$

$$\begin{aligned} F_1(x_1, x_2, x_3, s_1(x_1, x_2, x_3), s_2(x_1, x_2, x_3)) &\equiv 0 \\ F_2(x_1, x_2, x_3, s_1(x_1, x_2, x_3), s_2(x_1, x_2, x_3)) &\equiv 0. \end{aligned}$$

(b) Use part (a) to construct a coordinate chart for the manifold defined by

$$\mathcal{M} \triangleq \left\{x \in \mathbb{R}^5 \middle|\ F_1(x_1, x_2, x_3, x_4, x_5) = 0,\ F_2(x_1, x_2, x_3, x_4, x_5) = 0\right\}$$

which contains x_0.

(c) Consider the system of equations

$$\begin{aligned} F_1(x) &= x_1^2 - x_2 x_4 = 0 \\ F_2(x) &= x_1 x_2 + x_4 x_5 = 0. \end{aligned}$$

Show that $x_0 = (-1, 1, 0, 1, 1)$ satisfies this system of equations. Show that $\left.\dfrac{\partial(F_1, F_2)}{\partial x_4 \partial x_5}\right|_{x_0} \neq 0$. Explicitly find $x_4 = s_1(x_1, x_2, x_3), x_5 = s_2(x_1, x_2, x_3)$ which are valid in a neighborhood of $(-1, 1, 0)$. Use s_1, s_2 to define a coordinate chart for $\mathcal{M}$.

4

Integral Manifolds and the Frobenius Theorem

In this chapter the main tool for finding coordinate transformations to achieve feedback linearization as well as for finding transformations to design observers with linear error dynamics is presented. This tool is the Frobenius theorem and the notions of Lie brackets and vector fields on manifolds are developed to understand and prove this theorem. The fundamental existence theorem for differential equations is stated and the interpretation of a differential on a manifold is discussed. More specifically, a system of differential equations on a manifold are represented by its coordinates using a particular coordinate chart (patch) and so the system has a different representation in each chart. This suggests the idea of trying to find a particular coordinate system for which the resulting differential equations are easy to work with for control purposes (i.e., can be linearized using feedback). The Frobenius theorem gives explicit conditions that must hold for a feedback linearizing transformation to exist. These conditions are given in terms of the Lie brackets of the vector fields that define the control system.

4.1 Differential Equations

Let's review and quote some fundamental results from the theory of differential equations. Consider the system of differential equations given by

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1(t),...,x_n(t),t), \quad x_1(t_0) = x_{01} \\ \vdots &= \qquad\quad \vdots \\ \frac{dx_2}{dt} &= f_2(x_1(t),...,x_n(t),t), \quad x_n(t_0) = x_{0n} \end{aligned} \tag{4.1}$$

More compactly this can be written as

$$\frac{dx}{dt} = f(x(t),t), \quad x(t) = x_0. \tag{4.2}$$

A solution of this system differential equations is a function $\phi(t,x_0) : \mathbb{R}^1 \to \mathbb{R}^n$ such that

$$\frac{d\phi(t,x_0)}{dt} = f(\phi(t,x_0)), \quad \phi(t_0,x_0) = x_0. \tag{4.3}$$

We will denote $\phi(t,x_0)$ by $\phi_t(x_0)$ and often use the (imprecise) shorthand notation $x(t)$ for $\phi(t,x_0)$, i.e., $x(t) = \phi(t,x_0)$.

Theorem 1 *Existence Theorem for Ordinary Differential Equations*
Let $f(x,t) \in \mathbb{R}^n$ have continuous partial derivatives with respect to $x_i, i = 1,...,n$ and t in a neighborhood of (x_0,t_0). That is, the partial derivatives $\dfrac{\partial f_i}{\partial x_j}, i,j = 1,...n$ and $\dfrac{\partial f_i}{\partial t}, i = 1,...n$ all exist and are continuous in a neighborhood of (x_0,t_0). Then there is an open interval $\mathbf{I} \subset \mathbb{R}$ containing t_0 for which there is a unique solution $\phi(t,x_0)$ of $dx/dt = f(x(t),t)$ with $x(t_0) = \varphi(t_0,x_0) = x_0$.

Proof. omitted. ■

Remark 1 The conditions of this theorem are sufficient, but *not* necessary.

Example 1 *Dependence on Initial Conditions*

Consider the differential equation

$$\frac{dx}{dt} = x^2, \quad x(0) = x_0$$

Note that with $f(x) = x^2$ we have

$$\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial t} = 0$$

so the conditions of the existence theorem hold.

To solve this differential equation we write

$$\frac{dx}{x^2} = dt$$

to obtain

$$-\frac{1}{x} = t + c$$

or

$$x(t) = -\frac{1}{t+c}.$$

The initial condition requires setting $c = -1/x_0$ so that

$$\phi(t, x_0) = -\frac{1}{t - 1/x_0} = -\frac{x_0}{x_0 t - 1}$$

(1) Let $x_0 > 0$. Then the solution is $\phi(t, x_0) = -\dfrac{x_0}{x_0 t - 1}$ is valid for $t \in \mathbf{I} = (-\infty, 1/x_0)$.

(2) Let $x_0 < 0$. Then the solution is $\phi(t, x_0) = -\dfrac{x_0}{x_0 t - 1}$ is valid for $t \in \mathbf{I} = (1/x_0, \infty)$.

(3) Let $x_0 = 0$. Then the solution is $\phi(t, x_0) = -\dfrac{x_0}{x_0 t + 1} = 0$ is valid for $t \in \mathbf{I} = (-\infty, \infty)$.

Example 2 *Linear Time-Invariant Differential Equations*

$$\frac{dx}{dt} = Ax, \quad x(t_0) = x_0, \quad A \in \mathbb{R}^{n \times n}.$$

Then with $f(x) = Ax \in \mathbb{R}^n$ we have

$$\begin{aligned} \frac{\partial f_i}{\partial x_i} &= a_{ij} \text{ for } i, j = 1, ..., n \\ \frac{\partial f_i}{\partial t} &= 0 \text{ for } i = 1, ..., n. \end{aligned}$$

so the conditions of the existence theorem hold.

The solution is

$$x(t) = \phi(t, x_0) = e^{A(t-t_0)} x_0$$

and is valid for all $x_0 \in \mathbb{R}^n$ and $-\infty < t < \infty$.

Example 3 *Linear Differential Equations*

$$\frac{dx}{dt} = A(t)x, \quad x(t_0) = x_0, \quad A \in \mathbb{R}^{n \times n}.$$

Then with $f(x) = A(t)x \in \mathbb{R}^n$ we have

$$\frac{\partial f_i}{\partial x_i} = a_{ij}(t) \text{ for } i, j = 1, ..., n$$
$$\frac{\partial f_i}{\partial t} = \sum_{j=1}^{n} \dot{a}_{ij}(t) x_j \text{ for } i = 1, ..., n.$$

If the $a_{ij}(t)$ are continuously differentiable then the conditions of the existence theorem hold. It turns out that a solution then exists which is valid for all $x_0 \in \mathbb{R}^n$ and $-\infty < t < \infty$.

Example 4 *Nonlinear Differentiable Equation*

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}, \quad x(t_0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}.$$

Then straightforwardly we have $x_1(t) = x_{01}$ and $x_2(t) = t - t_0 + x_{02}$. To find $x_3(t)$ we write

$$\frac{dx_3}{dt} = \frac{x_3}{t - t_0 + x_{02}}$$

so

$$\frac{dx_3}{x_3} = \frac{dt}{t - t_0 + x_{02}}$$

which gives $\ln x_3 = \ln(t - t_0 + x_{02}) + c$ or $x_3 = e^c(t - t_0 + x_{02})$. The initial condition requires $x_3(t) = x_{03}/x_{02}(t - t_0 + x_{02})$. Let

$$\phi(t, x_0) = \begin{bmatrix} x_{01} \\ t - t_0 + x_{02} \\ \dfrac{x_{03}}{x_{02}}(t - t_0 + x_{02}). \end{bmatrix}$$

Exercise 19 These questions pertain to the previous examples.

(1) Verify $\phi(t, x_0)$ satisfies the differential equation for all initial conditions except $x_{02} = 0$.

(2) What is the solution if x_{02} is zero?

Exercise 20 *Differential Equation Without a Unique Solution* [23]
Show that the nonlinear first-order differential equation

$$\frac{dx}{dt} = \sqrt{x}, \quad x(0) = 0$$

has the two solutions

$$x(t) \equiv 0$$

and

$$x(t) = \begin{cases} t^2/4, & t \geq 0 \\ -t^2/4, & t < 0. \end{cases}$$

That is, it does not have a unique solution. Does this differential equation satisfy the sufficient conditions of the existence theorem? Explain.

4.2 Properties of the Flow of a Vector Field

In the following it is assumed the sufficient conditions of the existence theorem hold. We have denoted the solution to $dx/dt = f(x)$ with $x(0) = x_0$ by $\phi(t, x_0)$ or $\phi_t(x_0)$ with the solution guaranteed to exist (at least) for $|t| < \varepsilon$ for some $\varepsilon > 0$. The solution $\phi(t, x_0)$ is called the *flow* of the vector field $f(x)$. $\phi(t, x_0)$ is a function from $\mathbb{R}^{n+1} \to \mathbb{R}^n$ taking the point $(t, x_0) \in \mathbb{R}^{n+1}$ to the point $\phi(t, x_0) \in \mathbb{R}^n$. On the other hand, for each fixed t we can consider $\phi(t, x_0)$ as a map from $\mathbb{R}^n \to \mathbb{R}^n$ taking x_0 to $x(t) = \phi(t, x_0)$. In words, starting at x_0 the state moves in the direction specified by $f(x)$ for a time t and ends up at $\phi(t, x_0)$.

Now let $x_0' = \phi(t, x_0)$ and then follow the same vector field $f(x)$ for a time s so we have $\phi(s, x_0') = \phi(s, \phi(t, x_0))$. By the uniqueness of solutions to differential equations this must be the same as starting at x_0 and following $f(x)$ for a time $t + s$ which is $\phi(t + s, x_0)$. That is,

$$\phi(t + s, x_0) = \phi(s, \phi(t, x_0))$$

or equivalently

$$\phi_{t+s}(x_0) = \phi_s(\phi_t(x_0)) = \phi_s \circ \phi_t(x_0).$$

It must also be true that

$$\phi_{s+t}(x_0) = \phi_t(\phi_s(x_0)) = \phi_t \circ \phi_s(x_0)$$

so that

$$\phi_s \circ \phi_t(x_0) = \phi_t \circ \phi_s(x_0). \tag{4.4}$$

The solutions $\phi_t(x_0) = \phi(t, x_0), \phi_s(x_0) = \phi(s, x_0), \phi_{t+s}(x_0) = \phi(t+s, x_0)$ all exist for $|t| < \varepsilon, |s| < \varepsilon, |t+s| < \varepsilon$, respectively.

Consider the following situation where $f(x)$ is followed for a time t to reach $x_0' = \phi(t, x_0)$. Then from x_0' follow $f(x)$ for a time $-t$ to reach $\phi(-t, x_0')$, that is, go backwards. By following the vector field backwards for the same amount of time the vector field was followed forwards it would be expected to end up in the same place. This is indeed the case as

$$\phi(-t, x_0') = \phi(-t, \phi(t, x_0)) = \phi(t + (-t), x_0) = \phi(0, x_0) = x_0.$$

This is also written as

$$\phi_{-t}(x_0') = \phi_{-t}(\phi_t(x_0)) = \phi_{t-t}(x_0) = \phi_0(x_0) = x_0.$$

Example 5 *Flow of a Linear System*

Let $f(x) = Ax$ with $A \in \mathbb{R}^{n \times n}$ and consider

$$\frac{dx}{dt} = f(x),\ \ x(0) = x_0.$$

Then $\phi(t, x_0) = e^{At}x_0$

$$\phi_s \circ \phi_t(x_0) = \phi(s, \phi(t, x_0)) = e^{As}e^{At}x_0 = e^{At}e^{As}x_0 = \phi(t, \phi(s, x_0)) = \phi_t \circ \phi_s(x_0).$$

With $x_0' = \phi(t, x_0) = e^{At}x_0$ we have

$$\phi(-t, \phi(t, x_0)) = \phi(-t, x_0') = e^{-At}x_0' = e^{-At}e^{At}x_0 = x_0.$$

Example 6 *Flow of a Nonlinear System*

Let

$$f(x) = \begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}.$$

With $x_{02} \neq 0$ the flow is

$$\phi(t, x_0) = \begin{bmatrix} x_{01} \\ x_{02} + t \\ (x_{02} + t)\dfrac{x_{03}}{x_{02}} \end{bmatrix}$$

for $x_{02} + t \neq 0$. With $x_0' = \begin{bmatrix} x_{01}' \\ x_{02}' \\ x_{03}' \end{bmatrix}$ we have $\phi(-t, x_0') = \begin{bmatrix} x_{01}' \\ x_{02}' - t \\ (x_{02}' - t)\dfrac{x_{03}'}{x_{02}'} \end{bmatrix}$. Then

$$\phi(-t, \phi(t, x_0)) = \phi(-t, x_0')_{x_0' = \phi(t, x_0)} = \begin{bmatrix} x_{01}' \\ x_{02}' - t \\ (x_{02}' - t)\dfrac{x_{03}'}{x_{02}'} \end{bmatrix}_{|x_0' = \phi(t, x_0)} = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}.$$

Now we consider t fixed and look at the mapping $\phi_t(x) = \phi(t, x) : \mathbb{R}^n \to \mathbb{R}^n$. This map takes the point $x \in \mathbb{R}^n$ to the new point $x \in \mathbb{R}^n$ by following the vector field $f(x)$ for the fixed time t. By the definition of $\phi(t, x)$ we have $\phi(0, x) = x$ so that

$$\frac{\partial}{\partial x}\phi(t, x)_{|t=0} = \frac{\partial}{\partial x}\phi(0, x) = \frac{\partial}{\partial x} x = I_{n \times n}.$$

With the flow existing for $|t| < \epsilon$, $\phi_t(x) = \phi(t, x)$ has inverse $\phi_{-t}(x) = \phi(-t, x)$. In other words

$$\begin{aligned} \phi\left(-t, \phi(t, x)\right) &= \phi_{-t}(\phi_t(x)) \equiv x \\ \phi\left(t, \phi(-t, x)\right) &= \phi_t(\phi_{-t}(x)) \equiv x. \end{aligned}$$

So, for fixed t, the mapping $\phi_t(x) = \phi(t, x) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible in a neighborhood of x. Using the chain rule we have

$$\frac{\partial}{\partial x}\phi_{-t}(\phi_t(x)) = \frac{\partial}{\partial x'}\phi_{-t}(x')\Big|_{|x' = \phi_t(x)} \frac{\partial}{\partial x}(\phi_t(x)) = I_{n \times n}.$$

Example 7 *Flow of a Vector Field*

Let $f(x) = \begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}$ so $\phi(t, x) = \begin{bmatrix} x_1 \\ x_2 + t \\ (x_2 + t)\dfrac{x_3}{x_2} \end{bmatrix}$ with $\phi(0, x) = x$

and $\phi(-t, x') = \begin{bmatrix} x_1' \\ x_2' - t \\ (x_2' - t)\dfrac{x_3'}{x_2'} \end{bmatrix}$ with $\phi(0, x') = x'$.

$$\begin{aligned} \frac{\partial}{\partial x'}\phi_{-t}(x')\Big|_{x' = \phi_t(x)} &= \left.\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t\dfrac{x_3'}{(x_2')^2} & \dfrac{x_2' - t}{x_2'} \end{bmatrix}\right|_{x' = \phi_t(x) = \begin{bmatrix} x_1 \\ x_2 + t \\ (x_2 + t)\dfrac{x_3}{x_2} \end{bmatrix}} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t\dfrac{x_3/x_2}{x_2 + t} & \dfrac{x_2}{x_2 + t} \end{bmatrix}. \end{aligned}$$

As

$$\frac{\partial}{\partial x}\phi_t(x) = \phi(t,x) = \frac{\partial}{\partial x}\begin{bmatrix} x_1 \\ x_2+t \\ (x_2+t)\dfrac{x_3}{x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t\dfrac{x_3}{x_2^2} & \dfrac{x_2+t}{x_2} \end{bmatrix}$$

we have

$$\frac{\partial}{\partial x'}\phi_{-t}(x')\Big|_{x'=\phi_t(x)}\frac{\partial}{\partial x}\phi_t(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t\dfrac{x_3/x_2}{x_2+t} & \dfrac{x_2}{x_2+t} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t\dfrac{x_3}{x_2^2} & \dfrac{x_2+t}{x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4.3 Integral Manifolds

Let's start with the manifold $\mathbf{E}^3$ with the Cartesian coordinate system so

$$\mathbf{z}(x_1,x_2,x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{E}^3.$$

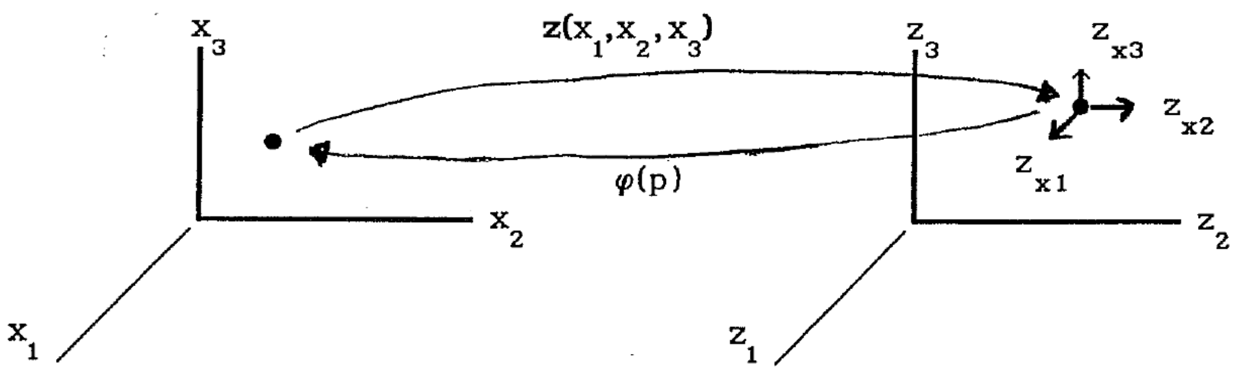


FIGURE 4.1. Cartesian coordinate system for $\mathbf{E}^3$.

Suppose at each point of $\mathbf{E}^3$ there are two linearly independent vector fields of the form

$$f^{(1)}(x_1,x_2,x_3) = \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1,x_2) \end{bmatrix}, \quad f^{(2)}(x_1,x_2,x_3) = \begin{bmatrix} 0 \\ 1 \\ f_3^{(2)}(x_1,x_2) \end{bmatrix}$$

With $\mathcal{U}$ an open subset of $\mathbf{E}^3$ and $p_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} \in \mathcal{U}$ is there a two-dimensional surface (submanifold) containing p_0 and whose tangent vectors at each point of the surface are $f^{(1)}(x_1,x_2,x_3)$ and $f^{(2)}(x_1,x_2,x_3)$? That is, with $\mathcal{D} \subset \mathbb{R}^2$ an open set containing (x_{01},x_{02}) we are looking for a surface $S(x_1,x_2) : \mathcal{D} \to \mathcal{U}$ of the form

$$S(x_1,x_2) = \begin{bmatrix} x_1 \\ x_2 \\ s(x_1,x_2) \end{bmatrix}$$

such that $S(x_{01}, x_{02}) = \begin{bmatrix} x_{01} \\ x_{02} \\ s_3(x_{01}, x_{02}) \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$ and

$$\frac{\partial S}{\partial x_1} = \begin{bmatrix} 1 \\ 0 \\ \dfrac{\partial s(x_1, x_2)}{\partial x_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1, x_2) \end{bmatrix} = f^{(1)}(x_1, x_2, x_3)$$

$$\frac{\partial S}{\partial x_2} = \begin{bmatrix} 1 \\ 0 \\ \dfrac{\partial s(x_1, x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ f_3^{(2)}(x_1, x_2) \end{bmatrix} = f^{(2)}(x_1, x_2, x_3).$$

In this case the problem reduces to finding a scalar function $s_3(x_1, x_2)$ such that $s(x_{01}, x_{02}) = x_{03}$ and

$$\frac{\partial s(x_1, x_2)}{\partial x_1} = f_3^{(1)}(x_1, x_2)$$
$$\frac{\partial s(x_1, x_2)}{\partial x_2} = f_3^{(2)}(x_1, x_2).$$

If $s(x_1, x_2)$ exists then

$$\frac{\partial^2 s(x_1, x_2)}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2)$$
$$\frac{\partial^2 s(x_1, x_2)}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2)$$

or

$$\frac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2) = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2).$$

Exercise 21 *Lie Bracket*

Show that the Lie bracket of $[f^{(1)}, f^{(2)}] = 0$ for $x \in \mathcal{U}$ if and only if $\dfrac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2) = \dfrac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2)$ for $x \in \mathcal{U}$.

Exercise 22 *Surface Specified by Tangent Vectors*

Let $f^{(1)} = \begin{bmatrix} 1 & 0 & x_2 \end{bmatrix}^T$ and $f^{(2)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ for all for $x \in \mathbf{E}^3$. Is there a surface $S(x_1, x_2) = \begin{bmatrix} x_1 & x_2 & s(x_1, x_2) \end{bmatrix}^T$ such that $\dfrac{\partial S}{\partial x_1} = f^{(1)}(x_1, x_2, x_3)$ and $\dfrac{\partial S}{\partial x_2} = f^{(2)}(x_1, x_2, x_3)$.

We now show that

$$\frac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2) = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2)$$

is a *sufficient* condition for a surface to exist.

Theorem 2 *Integrability of Vector Fields*

Let

$$f^{(1)}(x_1, x_2, x_3) = \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1, x_2) \end{bmatrix}, \quad f^{(2)}(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 1 \\ f_3^{(2)}(x_1, x_2) \end{bmatrix}$$

be vector fields defined on an open set $\mathcal{U} \subset \mathbf{E}^3$. Let $x_0 = \begin{bmatrix} x_{01} & x_{02} & x_{03} \end{bmatrix}^T \in \mathcal{U}$ and suppose that

$$\frac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2) = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2)$$

holds for all (x_1, x_2) in some neighborhood of (x_{01}, x_{02}). Then there is a surface $S(x_1, x_2)$ of the form

$$S(x_1,x_2) = \begin{bmatrix} x_1 \\ x_2 \\ s(x_1,x_2) \end{bmatrix} \text{ with } S(x_{01},x_{02}) = \begin{bmatrix} x_{01} \\ x_{02} \\ s(x_{01},x_{02}) \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

such that

$$\frac{\partial S}{\partial x_1} = f^{(1)}(x_1,x_2,x_3), \quad \frac{\partial S}{\partial x_2} = f^{(2)}(x_1,x_2,x_3).$$

Proof. With x_{02} fixed let

$$s(x_1,x_{02}) \triangleq x_{03} + \int_{x_{01}}^{x_1} f_3^{(1)}(u,x_{02})du$$

to obtain the *curve*

$$S(x_1,x_{02}) = \begin{bmatrix} x_1 \\ x_{02} \\ s(x_1,x_{02}) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_{02} \\ x_{03} + \int_{x_{01}}^{x_1} f_3^{(1)}(u,x_{02})du \end{bmatrix}.$$

This satisfies

$$S(x_{01},x_{02}) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

and

$$\frac{\partial}{\partial x_1} S(x_1,x_{02}) = \frac{\partial}{\partial x_1} \begin{bmatrix} x_1 \\ x_{02} \\ s(x_1,x_{02}) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1,x_{02}) \end{bmatrix} = f^{(1)}(x_1,x_{02}).$$

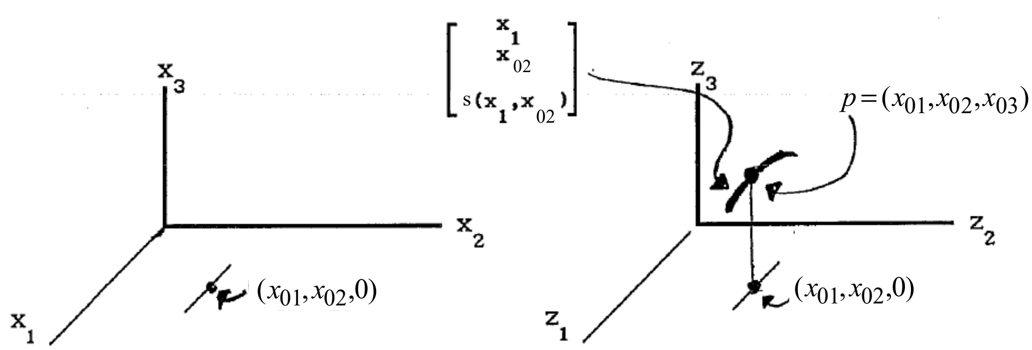


FIGURE 4.2. Follow the vector field $f^{(1)}$ from x_0.

Next define

$$s(x_1,x_2) \triangleq x_{03} + \int_{x_{01}}^{x_1} f_3^{(1)}(u,x_{02})du + \int_{x_{02}}^{x_2} f_3^{(2)}(x_1,v)dv.$$

Figure 4.3 illustrates the construction of this surface.

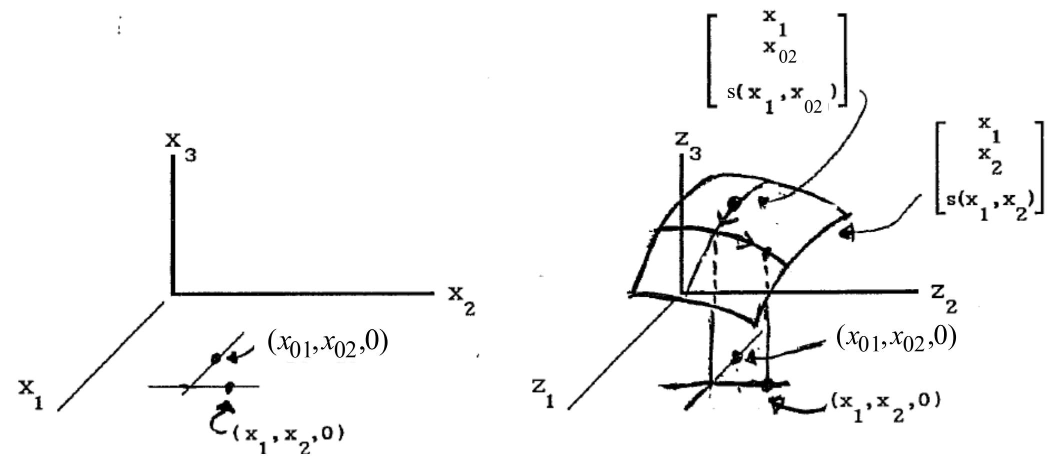


FIGURE 4.3. Follow the vector field $f^{(2)}$ from $\begin{bmatrix} x_1 & x_{02} & x_{03} \end{bmatrix}^T$.

Then

$$
\begin{aligned}
\frac{\partial}{\partial x_1} S(x_1,x_2) &= \frac{\partial}{\partial x_1} \begin{bmatrix} x_1 \\ x_2 \\ x_{03} + \int_{x_{01}}^{x_1} f_3^{(1)}(u,x_{02})du + \int_{x_{02}}^{x_2} f_3^{(2)}(x_1,v)dv \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1,x_{02}) + \int_{x_{02}}^{x_2} \dfrac{\partial f_3^{(2)}(x_1,v)}{\partial x_1} dv \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1,x_{02}) + \int_{x_{02}}^{x_2} \dfrac{\partial f_3^{(1)}(x_1,v)}{\partial v} dv \end{bmatrix} \quad \text{as } \frac{\partial}{\partial x_2} f_3^{(1)}(x_1,x_2) = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1,x_2) \\
&= \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1,x_{02}) + f_3^{(1)}(x_1,v)\Big|_{v=x_{02}}^{v=x_2} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1,x_{02}) + f_3^{(1)}(x_1,x_2) - f_3^{(1)}(x_1,x_{02}) \end{bmatrix} \\
&= f^{(1)}(x_1,x_2).
\end{aligned}
$$

Further

$$
\begin{aligned}
\frac{\partial}{\partial x_2} S(x_1,x_2) = \frac{\partial}{\partial x_2} \begin{bmatrix} x_1 \\ x_2 \\ x_{03} + \int_{x_{01}}^{x_1} f_3^{(1)}(u,x_{02})du + \int_{x_{02}}^{x_2} f_3^{(2)}(x_1,v)dv \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \\ \dfrac{\partial}{\partial x_2} \int_{x_{02}}^{x_2} f_3^{(2)}(x_1,v)dv \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \\ f_3^{(2)}(x_1,x_2) \end{bmatrix} \\
&= f^{(2)}(x_1,x_2,x_3).
\end{aligned}
$$

■

Example 8 *Integrability of Vector Fields*

Let

$$f^{(1)}(x_1,x_2,x_3) = \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix}, \quad f^{(2)}(x_1,x_2,x_3) = \begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix}.$$

Computing

$$\frac{\partial}{\partial x_2}(2x_1x_2) = 2x_1$$
$$\frac{\partial}{\partial x_1}(x_1^2) = 2x_1$$

shows the integrability conditions are satisfied. Then $s(x_1,x_2)$ is given by

$$s(x_1,x_2) = x_{03} + \int_{x_{01}}^{x_1} 2ux_{02}du + \int_{x_{02}}^{x_2} x_1^2 dv = x_{03} + (x_1^2 - x_{01}^2)x_{02} + x_1^2(x_2 - x_{02}) = x_{03} + x_1^2x_2 - x_{01}^2x_{02}$$

and the surface is

$$S(x_1,x_2) = \begin{bmatrix} x_1 \\ x_2 \\ x_{03} + x_1^2x_2 - x_{01}^2x_{02} \end{bmatrix}.$$

To check we have

$$S(x_1,x_2) = \left.\begin{bmatrix} x_1 \\ x_2 \\ x_{03} + x_1^2x_2 - x_{01}^2x_{02} \end{bmatrix}\right|_{x=x_0} = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

and

$$\frac{\partial}{\partial x_1}S(x_1,x_2) = \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix} = f^{(1)}(x_1,x_2), \quad \frac{\partial}{\partial x_2}S(x_1,x_2) = \begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix} = f^{(2)}(x_1,x_2).$$

Let's take another look at the previous example. The idea is to start at any desired point $\begin{bmatrix} x_{01} & x_{02} & x_{03} \end{bmatrix}^T$ which the surface is to pass through. Next follow the vector field $f^{(1)}$ for a time t_1 and then follow the vector field $f^{(2)}$ for a time t_2.

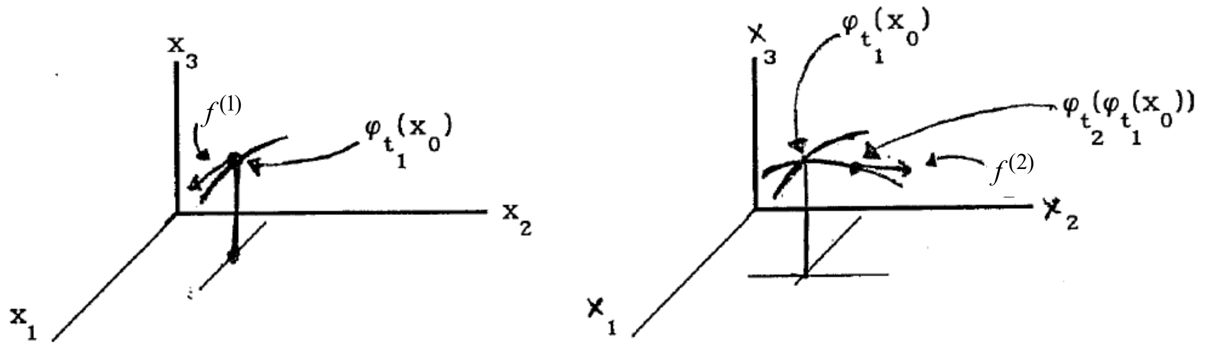


FIGURE 4.4. Follow $f^{(1)}$ for a time t_1 and then $f^{(2)}$ for a time t_2.

We solve

$$\frac{d}{dt_1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix}}_{f^{(1)}}, \quad x(0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

to obtain

$$\phi_{t_1}(x_0) = \begin{bmatrix} t + x_{01} \\ x_{02} \\ x_{03} + (t_1 + x_{01})^2 x_{02} - x_{01}^2 x_{02} \end{bmatrix}.$$

It is straightforward to check that

$$\frac{\partial}{\partial t_1}\phi_{t_1}(x_0) = f^{(1)}(\phi_{t_1}(x_0)) \;\text{ with }\; \phi_0(x_0) = x_0.$$

Now from $x_0' = \phi_{t_1}(x_0)$ we follow $f^{(2)}$ by solving

$$\frac{d}{dt_2}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix}}_{f^{(2)}}, \quad x(0) = x_0' \triangleq \phi_{t_1}(x_0).$$

The solution is

$$\phi_{t_2}(x_0) = \begin{bmatrix} x_{01}' \\ t + x_{02}' \\ x_{03}' + (x_{01}')^2 t_2 \end{bmatrix}.$$

Straightforwardly it can be checked that

$$\frac{\partial}{\partial t_2}\phi_{t_2}(x_0) = f^{(1)}(\phi_{t_2}(x_0')) \;\text{ with }\; \phi_0(x_0') = x_0'.$$

Form the surface as the composition of these two solutions given by

$$\begin{aligned} \phi_{t_2}(\phi_{t_1}(x_0)) &= \left.\begin{bmatrix} x_{01}' \\ t_2 + x_{02}' \\ x_{03}' + (x_{01}')^2 t_2 \end{bmatrix}\right|_{x_0'=\phi_{t_1}(x_0)=\begin{bmatrix} t_1 + x_{01} \\ x_{02} \\ x_{03} + (t_1 + x_{01})^2 x_{02} - x_{01}^2 x_{02} \end{bmatrix}} \\ &= \begin{bmatrix} t_1 + x_{01} \\ t_2 + x_{02} \\ x_{03} + (t_1 + x_{01})^2 x_{02} - x_{01}^2 x_{02} + (t_1 + x_{01})^2 t_2 \end{bmatrix} \\ &= \begin{bmatrix} t_1 + x_{01} \\ t_2 + x_{02} \\ x_{03} + (t_1 + x_{01})^2 (x_{02} + t_2) - x_{01}^2 x_{02} \end{bmatrix}. \end{aligned}$$

So the surface $S(t_1, t_2)$ is defined by

$$S(t_1, t_2) = \begin{bmatrix} x_1(t_1, t_2) \\ x_2(t_1, t_2) \\ x_3(t_1, t_2) \end{bmatrix} = \phi_{t_2}(\phi_{t_1}(x_0)) = \begin{bmatrix} t_1 + x_{01} \\ t_2 + x_{02} \\ x_{03} + (t_1 + x_{01})^2 (x_{02} + t_2) - x_{01}^2 x_{02} \end{bmatrix}$$

with tangent vectors to the surface given by

$$\frac{\partial S}{\partial t_1} = \begin{bmatrix} 1 \\ 0 \\ 2(t_1 + x_{01})(x_{02} + t_2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2x_1(t_1)x_2(t_2) \end{bmatrix}$$

$$\frac{\partial S}{\partial t_2} = \begin{bmatrix} 0 \\ 1 \\ (t_1 + x_{01})^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ x_1^2(t_1) \end{bmatrix}.$$

Exercise 23 *Lie Brackets and Integrability of Vector Fields*
Starting at $x_0 = \begin{bmatrix} x_{01} & x_{02} & x_{03} \end{bmatrix}^T$ follow the vector field $f^{(2)} = \begin{bmatrix} 0 & 1 & x_1^2 \end{bmatrix}^T$ for t_2 units of time and then follow $f^{(1)} = \begin{bmatrix} 1 & 0 & 2x_1x_2 \end{bmatrix}^T$ for t_1 units of time. That is, compute $\phi_{t_1}(\phi_{t_2}(x_0))$ as shown above. If you do it right you should find that $\phi_{t_1}(\phi_{t_2}(x_0)) = \phi_{t_2}(\phi_{t_1}(x_0))$. It will be show later that this is guaranteed by the fact that $[f^{(1)}, f^{(2)}] \equiv 0$.

Exercise 24 *Lie Brackets and Integrability of Vector Fields*
Consider the two vector fields on $\mathbf{E}^3$ given by

$$f^{(1)}(x_1, x_2) = \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 + x_2^2 \end{bmatrix}, \quad f^{(2)}(x_1, x_2) = \begin{bmatrix} 0 \\ 1 \\ x_1^2 + 3x_1x_2^2 \end{bmatrix}.$$

With x_0 the initial state, compute $\phi_{t_2}(\phi_{t_1}(x_0))$ and show that the surface $S(t_1, t_2) \triangleq \phi_{t_2}(\phi_{t_1}(x_0))$ satisfies $S(0,0) = x_0$ and $\dfrac{\partial S}{\partial t_1} = f^{(1)}$, $\dfrac{\partial S}{\partial t_2} = f^{(2)}$. Also show that $[f^{(1)}, f^{(2)}] = 0$ for all $x \in \mathbf{E}^3$.

Exercise 25 *Lie Brackets and Integrability of Vector Fields*
Consider the two vector fields on $\mathbf{E}^3$ given by

$$f^{(1)}(x_1, x_2) = \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix}, \quad f^{(2)}(x_1, x_2) = \begin{bmatrix} 0 \\ 1 \\ x_1^2x_2 \end{bmatrix}.$$

(a) Show that $[f^{(1)}, f^{(2)}] \neq 0$ on $\mathbf{E}^3$ as $\dfrac{\partial}{\partial x_2}(2x_1x_2) \neq \dfrac{\partial}{\partial x_1}(x_1^2x_2)$.

(b) With x_0 the initial state, compute $\phi_{t_2}(\phi_{t_1}(x_0))$ and show that $\phi_{t_2}(\phi_{t_1}(x_0))$ satisfies $S(0,0) = x_0$ and $\dfrac{\partial S}{\partial t_2} = f^{(2)}$, but $\dfrac{\partial S}{\partial t_1} \neq f^{(1)}$ unless $t_2 = 0$.

(c) With x_0 the initial state, compute $\phi_{t_1}(\phi_{t_2}(x_0))$ and show that $\phi_{t_1}(\phi_{t_2}(x_0))$ satisfies $S(0,0) = x_0$ and $\dfrac{\partial S}{\partial t_1} = f^{(1)}$, but $\dfrac{\partial S}{\partial t_2} \neq f^{(2)}$ unless $t_1 = 0$.

Let's consider another example.

Example 9 *Integrability of Vector Fields*
Consider the two vector fields on $\mathbf{E}^3$ given by

$$f^{(1)}(x_1, x_2, x_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad f^{(2)}(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}.$$

With the initial state x_0 follow the vector field $f^{(1)}$ by solving

$$\frac{d}{dt_1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{f^{(1)}} \text{ with } \; x(0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}.$$

The solution is

$$\phi_{t_1}(x_0) = \begin{bmatrix} t_1 + x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}.$$

Then with the initial condition $x'_0 = \phi_{t_1}(x_0)$ we follow $f^{(2)}$ by solving

$$\frac{d}{dt_1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}}_{f^{(2)}} \text{ with } x(0) = \begin{bmatrix} x'_{01} \\ x'_{02} \\ x'_{03} \end{bmatrix}.$$

This has solution

$$\phi_{t_2}(x'_0) = \begin{bmatrix} x'_{01} \\ t_2 + x'_{02} \\ (x'_{03}/x'_{02})(t_2 + x'_{02}) \end{bmatrix}.$$

The resulting surface is

$$S(t_1,t_2) = \begin{bmatrix} x_1(t_1,t_2) \\ x_2(t_1,t_2) \\ x_3(t_1,t_2) \end{bmatrix} = \phi_{t_2}(\phi_{t_1}(x_0)) = \begin{bmatrix} x'_{01} \\ t_2 + x'_{02} \\ (x'_{03}/x'_{02})(t_2 + x'_{02}) \end{bmatrix}\Bigg|_{x'_0=\phi_{t_1}(x_0)=\begin{bmatrix} t_1 + x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}}$$

$$= \begin{bmatrix} t_1 + x_{01} \\ t_2 + x_{02} \\ (x_{03}/x_{02})(t_2 + x_{02}) \end{bmatrix}.$$

It is straightforward to check that $S(0,0) = x_0$ and

$$\frac{\partial S}{\partial t_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\partial S}{\partial t_2} = \begin{bmatrix} 0 \\ 1 \\ x_{03}/x_{02} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}.$$

Also note that $[f^{(1)}, f^{(2)}] = 0$ for all $x \in \mathbf{E}^3$.

Exercise 26 *Integrability of Vector Fields*
In the previous example show, by direct computation, that $\phi_{t_1}(\phi_{t_2}(x_0)) = \phi_{t_2}(\phi_{t_1}(x_0))$. This is guaranteed to hold because $[f^{(1)}, f^{(2)}] \equiv 0$.

We now make a generalization of Theorem 2. With the Cartesian coordinate system on $\mathbf{E}^3$ consider the two vector fields defined on an open set $\mathcal{U} \subset \mathbf{E}^3$ given by

$$f^{(1)}(x_1,x_2,x_3) = \begin{bmatrix} f_1^{(1)}(x_1,x_2,x_3) \\ f_2^{(1)}(x_1,x_2,x_3) \\ f_3^{(1)}(x_1,x_2,x_3) \end{bmatrix}, \quad f^{(2)}(x_1,x_2,x_3) = \begin{bmatrix} f_1^{(2)}(x_1,x_2,x_3) \\ f_2^{(2)}(x_1,x_2,x_3) \\ f_3^{(2)}(x_1,x_2,x_3) \end{bmatrix}. \tag{4.5}$$

Given any point $x_0 \in \mathcal{U}$ and the vector fields $f^{(1)}, f^{(2)}$ we want to a surface $S(t_1,t_2)$ of the form

$$S(t_1,t_2) = \begin{bmatrix} s_1(t_1,t_2) \\ s_2(t_1,t_2) \\ s_3(t_1,t_2) \end{bmatrix} \tag{4.6}$$

with $S(0,0) = x_0$ and satisfying

$$\frac{\partial}{\partial t_1}S(t_1,t_2) = \begin{bmatrix} \frac{\partial}{\partial t_1}s_1(t_1,t_2) \\ \frac{\partial}{\partial t_1}s_2(t_1,t_2) \\ \frac{\partial}{\partial t_1}s_3(t_1,t_2) \end{bmatrix} = \begin{bmatrix} f_1^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \\ f_2^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \\ f_3^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \end{bmatrix} \tag{4.7}$$

$$\frac{\partial}{\partial t_2}S(t_1,t_2) = \begin{bmatrix} \frac{\partial}{\partial t_2}s_1(t_1,t_2) \\ \frac{\partial}{\partial t_2}s_2(t_1,t_2) \\ \frac{\partial}{\partial t_2}s_3(t_1,t_2) \end{bmatrix} = \begin{bmatrix} f_1^{(2)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \\ f_2^{(2)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \\ f_3^{(2)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \end{bmatrix}. \tag{4.8}$$

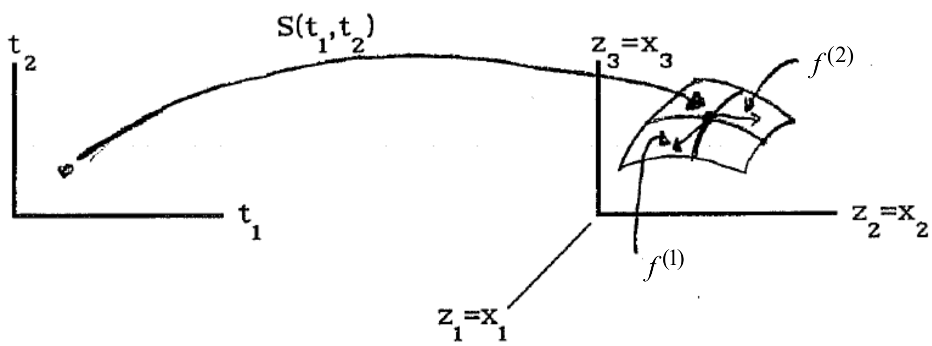


FIGURE 4.5. Integrability of vector fields.

We first show that a *necessary* condition for such a surface to exist is that $[f^{(1)}, f^{(2)}] = \dfrac{\partial f^{(1)}}{\partial x} f^{(2)} - \dfrac{\partial f^{(2)}}{\partial x} f^{(1)} = 0_{3\times 1}$ for $x \in \mathcal{U}$. We do this by showing that $[f^{(1)}, f^{(2)}] = 0$ is equivalent to

$$\frac{\partial^2 S(t_1,t_2)}{\partial t_2 \partial t_1} - \frac{\partial^2 S(t_1,t_2)}{\partial t_1 \partial t_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To proceed suppose the surface exists so we have

$$\frac{\partial}{\partial t_1}S(t_1,t_2) = \begin{bmatrix} f_1^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \\ f_2^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \\ f_3^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \end{bmatrix}$$

and therefore

$$\frac{\partial^2 S(t_1,t_2)}{\partial t_2 \partial t_1} = \begin{bmatrix} \dfrac{\partial f_1^{(1)}}{\partial x_1} & \dfrac{\partial f_1^{(1)}}{\partial x_2} & \dfrac{\partial f_1^{(1)}}{\partial x_3} \\ \dfrac{\partial f_2^{(1)}}{\partial x_1} & \dfrac{\partial f_2^{(1)}}{\partial x_2} & \dfrac{\partial f_2^{(1)}}{\partial x_3} \\ \dfrac{\partial f_3^{(1)}}{\partial x_1} & \dfrac{\partial f_3^{(1)}}{\partial x_2} & \dfrac{\partial f_3^{(1)}}{\partial x_3} \end{bmatrix}_{|x=S(t_1,t_2)} \begin{bmatrix} \dfrac{\partial s_1(t_1,t_2)}{\partial t_2} \\ \dfrac{\partial s_2(t_1,t_2)}{\partial t_2} \\ \dfrac{\partial s_3(t_1,t_2)}{\partial t_2} \end{bmatrix}$$

$$= \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)} f^{(2)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)).$$

Similarly

$$\frac{\partial^2 S(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)} f^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)).$$

Consequently

$$\frac{\partial^2 S(t_1,t_2)}{\partial t_2 \partial t_1} - \frac{\partial^2 S(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial f^{(1)}}{\partial x} f^{(2)} - \frac{\partial f^{(2)}}{\partial x} f^{(1)} \Big|_{x=(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2))} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is not quite the condition we claimed because it is in terms of $S(t_1,t_2)$ which is unknown! However we require that given any point $x_0 \in \mathcal{U} \subset \mathbf{E}^3$ there is a surface $S(t_1,t_2)$ through it. In particular at $(t_1,t_2) = (0,0)$ we have $S(0,0) = x_0$ and $\dfrac{\partial f^{(1)}}{\partial x} f^{(2)} - \dfrac{\partial f^{(2)}}{\partial x} f^{(1)} \Big|_{x_0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. As x_0 is an arbitrary point of $\mathcal{U}$ we must have

$$[f^{(1)}, f^{(2)}] = \frac{\partial f^{(1)}}{\partial x} f^{(2)} - \frac{\partial f^{(2)}}{\partial x} f^{(1)} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.9}$$

for all $x_0 \in \mathcal{U} \subset \mathbf{E}^3$.

We now want to $[f^{(1)}, f^{(2)}] \equiv 0$ is sufficient for such a surface to exist.

Theorem 3 $[f^{(1)}, f^{(2)}] \equiv 0$ *Implies an Integral Manifold Exists*

Let $f^{(1)}$ and $f^{(2)}$ be vector fields defined on a open set $\mathcal{U} \subset \mathbf{E}^3$ with $[f^{(1)}, f^{(2)}] \equiv 0$ on $\mathcal{U}$. Then for any point $x_0 \in \mathcal{U}$ there exists a neighborhood $\mathcal{D} \subset \mathbb{R}^2$ containing the origin $(0,0)$ for which a surface $S(t_1,t_2) : \mathcal{D} \to \mathcal{U}$ exists satisfying $S(0,0) = x_0$ and

$$\frac{\partial}{\partial t_1} S(t_1,t_2) = f^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \tag{4.10}$$

$$\frac{\partial}{\partial t_2} S(t_1,t_2) = f^{(2)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)). \tag{4.11}$$

Proof. Consider the differential equation defined by

$$\frac{d}{dt_1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1^{(1)}(x_1,x_2,x_3) \\ f_2^{(1)}(x_1,x_2,x_3) \\ f_3^{(1)}(x_1,x_2,x_3) \end{bmatrix} \quad \text{with} \quad x(0) = x_0.$$

Let the solution $\phi_{t_1}(x_0)$ exist for $|t_1| < \epsilon_1$ for some $\epsilon_1 > 0$. So $\dfrac{d}{dt_1}\phi_{t_1}(x_0) = f^{(1)}(\phi_{t_1}(x_0))$ with $\phi_0(x_0) = x_0$. This is the solution at time t_1 starting from x_0 and moving in the direction $f^{(1)}$. Now consider the differential equation defined by the vector field $f^{(2)}$ given by

$$\frac{d}{dt_2}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1^{(2)}(x_1,x_2,x_3) \\ f_2^{(2)}(x_1,x_2,x_3) \\ f_3^{(2)}(x_1,x_2,x_3) \end{bmatrix} \quad \text{with} \quad x(0) = x_0' = \phi_{t_1}(x_0).$$

Let the solution $\phi_{t_2}(x_0')$ exist for $|t_2| < \epsilon_2$ for some $\epsilon_2 > 0$. So $\dfrac{d}{dt_2}\phi_{t_2}(x_0) = f^{(2)}(\phi_{t_2}(x_0'))$ with $\phi_0(x_0') = x_0'$. Next define

$$S(t_1,t_2) = \begin{bmatrix} x_1(t_1,t_2) \\ x_2(t_1,t_2) \\ x_3(t_1,t_2) \end{bmatrix} \triangleq \phi_{t_2}(\phi_{t_1}(x_0))$$

where by the construction of $\phi_{t_2}(\phi_{t_1}(x_0))$ we have

$$\frac{\partial}{\partial t_2}S(t_1,t_2) = \frac{\partial}{\partial t_2}\phi_{t_2}(\phi_{t_1}(x_0)) = f^{(2)}(\phi_{t_2}(\phi_{t_1}(x_0))).$$

Further, at $t_2 = 0$ we have

$$\left.\frac{\partial}{\partial t_1}S(t_1,t_2)\right|_{t_2=0} = \frac{\partial}{\partial t_1}\phi_0(\phi_{t_1}(x_0)) = \frac{\partial}{\partial t_1}\phi_{t_1}(x_0) = f^{(1)}(\phi_{t_1}(x_0)) = f^{(1)}(S(t_1,0)).$$

We now need to show that

$$\frac{\partial}{\partial t_1}S(t_1,t_2) = f^{(1)}(S(t_1,t_2))$$

for all $|t_2| < \epsilon_2$ (not just $t_2 = 0$). Define

$$g(t_2) = \frac{\partial}{\partial t_1}S(t_1,t_2) - f^{(1)}(S(t_1,t_2)) \in \mathbf{E}^3$$

where $g(0) = 0_{3\times 1}$. Differentiating $g(t_2)$ we have

$$\begin{aligned}
\frac{d}{dt_2}g(t_2) &= \frac{\partial}{\partial t_2}\left(\frac{\partial}{\partial t_1}S(t_1,t_2)\right) - \frac{\partial}{\partial t_2}f^{(1)}(S(t_1,t_2)) \\
&= \frac{\partial}{\partial t_1}\left(\frac{\partial}{\partial t_2}S(t_1,t_2)\right) - \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)}\frac{\partial S(t_1,t_2)}{\partial t_2} \quad \text{as} \quad \frac{\partial^2 S(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial^2 S(t_1,t_2)}{\partial t_2 \partial t_1} \\
&= \frac{\partial}{\partial t_1}f^{(2)}(S(t_1,t_2)) - \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)}f^{(2)}(S(t_1,t_2)) \\
&= \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}\frac{\partial S(t_1,t_2)}{\partial t_1} - \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)}f^{(2)}(S(t_1,t_2)) \\
&= \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}\Big(g(t_2) + f^{(1)}(S(t_1,t_2)\Big) - \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)}f^{(2)}(S(t_1,t_2)) \\
&= \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}g(t_2) + \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}f^{(1)}(S(t_1,t_2) - \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)}f^{(2)}(S(t_1,t_2)) \\
&= \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}g(t_2) - \left(\frac{\partial f^{(1)}}{\partial x}f^{(2)}(x) - \frac{\partial f^{(2)}}{\partial x}f^{(1)}(x)\right)_{|x=S(t_1,t_2)} \\
&= \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}g(t_2) \quad \text{as} \quad [f^{(1)}, f^{(2)}] \equiv 0.
\end{aligned}$$

We have

$$\frac{d}{dt_2}g(t_2) = A_{t_1}(t_2)g(t_2) \quad \text{with} \quad g(0) = 0 \tag{4.12}$$

where $A_{t_1}(t_2) \triangleq \dfrac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)} \in \mathbb{R}^{3\times 3}$. For each fixed t_1 this is a *linear time varying* differential equation in t_2. This has the unique solution $g(t_2) \equiv 0$ as the components of $\dfrac{\partial f^{(2)}}{\partial x}$ are assumed to be continuously differentiable. ∎

Remark We defined $S(t_1,t_2) \triangleq \phi_{t_2}(\phi_{t_1}(x_0))$ and it is always true that $\dfrac{\partial}{\partial t_2}S(t_1,t_2) = \dfrac{\partial}{\partial t_2}\phi_{t_2}(\phi_{t_1}(x_0)) = f^{(2)}(\phi_{t_2}(\phi_{t_1}(x_0)))$ by the definition of ϕ_{t_2} whether or not the Lie bracket of $f^{(1)}$ and $f^{(2)}$ is zero. However, $\dfrac{\partial}{\partial t_1}S(t_1,t_2) = \dfrac{\partial}{\partial t_1}\phi_{t_2}(\phi_{t_1}(x_0)) = f^{(1)}(\phi_{t_2}(\phi_{t_1}(x_0)))$ if and only if $[f^{(1)}, f^{(2)}] \equiv 0$ on $\mathcal{U}$.

On the other hand, if we had defined $S(t_1,t_2) \triangleq \phi_{t_1}(\phi_{t_2}(x_0))$ then it is always true that $\dfrac{\partial}{\partial t_1}S(t_1,t_2) = \dfrac{\partial}{\partial t_1}\phi_{t_1}(\phi_{t_2}(x_0)) = f^{(1)}(\phi_{t_1}(\phi_{t_2}(x_0)))$ by the definition of ϕ_{t_1} whether or not the Lie bracket of $f^{(1)}$ and $f^{(2)}$ is zero. However, $\dfrac{\partial}{\partial t_2}S(t_1,t_2) = \dfrac{\partial}{\partial t_2}\phi_{t_1}(\phi_{t_2}(x_0)) = f^{(2)}(\phi_{t_1}(\phi_{t_2}(x_0)))$ if and only if $[f^{(1)}, f^{(2)}] \equiv 0$ on $\mathcal{U}$.

Corollary 1 $[f^{(1)}, f^{(2)}] \equiv 0 \Longrightarrow \phi_{t_2}(\phi_{t_1}(x_0)) = \phi_{t_1}(\phi_{t_2}(x_0))$

Let $f^{(1)}$ and $f^{(2)}$ be vector fields defined on a open set $\mathcal{U} \subset \mathbf{E}^3$ with $[f^{(1)}, f^{(2)}] \equiv 0$ on $\mathcal{U}$. Then

$$\phi_{t_2}(\phi_{t_1}(x_0)) = \phi_{t_1}(\phi_{t_2}(x_0))$$

for $-\epsilon < t_1, t_2 < \epsilon$ and some $\epsilon > 0$.

Proof. As $[f^{(1)}, f^{(2)}] \equiv 0$ on $\mathcal{U}$. For any fixed t_1 with $-\epsilon < t_1 < \epsilon$ Theorem 3 tells us that

$$\frac{d}{dt_2}\phi_{t_2}(\phi_{t_1}(x_0)) = f^{(2)}(\phi_{t_2}(\phi_{t_1}(x_0))) \text{ with } \phi_{t_2}(\phi_{t_1}(x_0))_{|t_2=0} = \phi_{t_1}(x_0)$$

and

$$\frac{d}{dt_2}\phi_{t_1}(\phi_{t_2}(x_0)) = f^{(2)}(\phi_{t_1}(\phi_{t_2}(x_0))) \text{ with } \phi_{t_1}(\phi_{t_2}(x_0))_{|t_2=0} = \phi_{t_1}(x_0).$$

That is, for each t_1, $\phi_{t_2}(\phi_{t_1}(x_0))$ and $\phi_{t_1}(\phi_{t_2}(x_0))$ both satisfy the same differential equation with the small initial state. By the uniqueness of the solution to this differential equation it follows that

$$\phi_{t_2}(\phi_{t_1}(x_0)) = \phi_{t_1}(\phi_{t_2}(x_0)).$$

∎

Example 10 *Integrability of Vector Fields*

Consider the two vector fields

$$f^{(1)}(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad f^{(2)}(x) = \begin{bmatrix} -x_2 \\ x_1 \\ x_3 \end{bmatrix}.$$

where a straightforward calculation shows $[f^{(1)}, f^{(2)}] \equiv 0$. The solution to

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{with} \quad x(0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

is

$$\varphi_{t_1}(x_0) = \begin{bmatrix} x_{01}e^{t_1} \\ x_{02}e^{t_1} \\ x_{03}e^{t_1} \end{bmatrix}.$$

The solution to

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \\ x_3 \end{bmatrix} \quad \text{with} \quad x'(0) = \begin{bmatrix} x'_{01} \\ x'_{02} \\ x'_{03} \end{bmatrix}$$

is

$$\varphi_{t_2}(x'_0) = \begin{bmatrix} x'_{01}\cos(t_2) - x'_{02}\sin(t_2) \\ x'_{01}\sin(t_2) + x'_{02}\cos(t_2) \\ x'_{03}e^{t_2} \end{bmatrix}.$$

Then

$$\begin{aligned} S(t_1, t_2) = \varphi_{t_2}(\varphi_{t_1}(x_0)) &= \left.\begin{bmatrix} x'_{01}\cos(t_2) - x'_{02}\sin(t_2) \\ x'_{01}\sin(t_2) + x'_{02}\cos(t_2) \\ x'_{03}e^{t_2} \end{bmatrix}\right|_{x'_0 = \varphi_{t_1}(x_0) = \begin{bmatrix} x_{01}e^{t_1} \\ x_{02}e^{t_1} \\ x_{03}e^{t_1} \end{bmatrix}} \\ &= \begin{bmatrix} \big(x_{01}\cos(t_2) - x_{02}\sin(t_2)\big)e^{t_1} \\ \big(x_{01}\sin(t_2) + x_{02}\cos(t_2)\big)e^{t_1} \\ x_{03}e^{t_1+t_2} \end{bmatrix}. \end{aligned}$$

It is straightforward to see that $S(0,0) = x_0$, and

$$\frac{\partial}{\partial t_1}S(t_1,t_2) = f^{(1)}(S(t_1,t_2)) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{|x=S(t_1,t_2)}$$

$$\frac{\partial}{\partial t_2}S(t_1,t_2) = f^{(2)}(S(t_1,t_2)) = \begin{bmatrix} -x_2 \\ x_1 \\ x_3 \end{bmatrix}_{|x=S(t_1,t_2)} = \begin{bmatrix} -\big(x_{01}\sin(t_2) + x_{02}\cos(t_2)\big)e^{t_1} \\ \big(x_{01}\cos(t_2) - x_{02}\sin(t_2)\big)e^{t_1} \\ x_{03}e^{t_1+t_2} \end{bmatrix}$$

Exercise 27 *Integrability of Vector Fields*
In the previous example show that $\phi_{t_1}(\phi_{t_2}(x_0)) = \phi_{t_2}(\phi_{t_1}(x_0))$ by directly computing $\phi_{t_1}(\phi_{t_2}(x_0))$.

Example 11 *Integrability of Vector Fields*
Consider the two vector fields on $\mathbf{E}^3$ given by

$$f^{(1)}(x) = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}, \quad f^{(2)}(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

It is easy to check that $[f^{(1)}, f^{(2)}] \neq 0$ anywhere in $\mathbf{E}^3$. Then

$$\phi_{t_2}(\phi_{t_1}(x_0)) = \begin{bmatrix} x'_{01} \\ x'_{02} + t_2 \\ x'_{03} \end{bmatrix}_{x = \begin{bmatrix} x_{01} + t_1 \\ x_{02} \\ x_{03} + x_{02}t_1 \end{bmatrix}} = \begin{bmatrix} x_{01} + t_1 \\ x_{02} + t_2 \\ x_{03} + x_{02}t_1 \end{bmatrix}$$

while

$$\phi_{t_1}(\phi_{t_2}(x_0)) = \begin{bmatrix} x'_{01} + t_1 \\ x'_{02} \\ x'_{03} + x'_{02}t_1 \end{bmatrix}_{x = \begin{bmatrix} x_{01} \\ x_{02} + t_2 \\ x_{03} \end{bmatrix}} = \begin{bmatrix} x_{01} + t_1 \\ x_{02} + t_2 \\ x_{03} + (x_{02} + t_2)t_1 \end{bmatrix}$$

showing that $\phi_{t_2}(\phi_{t_1}(x_0)) \neq \phi_{t_1}(\phi_{t_2}(x_0))$.

Exercise 28 *Integrability of Vector Fields*
Consider the two vector fields

$$f^{(1)}(x) = \begin{bmatrix} x_1 + x_2 \\ 2x_2 \\ x_3 \end{bmatrix}, \quad f^{(2)}(x) = \begin{bmatrix} 2x_1 + 3x_2 \\ x_2 \\ x_3 \end{bmatrix}.$$

(a) Show $[f^{(1)}, f^{(2)}] \equiv 0$.

(b) Compute $\phi_{t_1}(x_0)$.

(c) Compute $S(t_1, t_2) = \phi_{t_2}(\phi_{t_1}(x_0))$.

(d) Show $\dfrac{\partial}{\partial t_2} S(t_1, t_2) = f^{(2)}(S(t_1, t_2))$.

(e) $\dfrac{\partial}{\partial t_1} S(t_1, t_2) \neq f^{(1)}(S(t_1, t_2))$ unless $t_2 = 0$.

(f) Repeat part (c) with $S(t_1, t_2) = \phi_{t_1}(\phi_{t_2}(x_0))$. How do parts (d) and (e) change?

Exercise 29 *Integrability of Linear Vector Fields*
With $x \in \mathbb{R}^3, A_1 \in \mathbb{R}^{3\times 3}, A_2 \in \mathbb{R}^{3\times 3}$ let $f^{(1)} = A_1 x$ and $f^{(2)} = A_2 x$. Show $[f^{(1)}, f^{(2)}] \equiv 0$ if and only if $A_1 A_2 - A_2 A_1 = 0_{3\times 3}$, i.e., if and only if A_1 and A_2 commute.

Exercise 30 *Integrability of Linear Vector Fields*
With $x \in \mathbb{R}^3, A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3$ let $f^{(1)} = A_1 x$ and $f^{(2)} = b$. Show $[f^{(1)}, f^{(2)}] \equiv 0$ if and only if $Ab = 0_{3\times 1}$.

Special Coordinate System

Let $f^{(1)}, f^{(2)}, f^{(3)}$ be three linearly independent vector fields on some open set $\mathcal{U} \subset \mathbf{E}^3$ with $[f^{(2)}, f^{(3)}] \equiv 0$ on $\mathcal{U}$. Let $x_0 \in \mathcal{U}$. We now show how a coordinate system can be constructed in which $f^{(2)}$ and $f^{(3)}$ have a very simple representations in the new coordinates. Define the map from $\mathbb{R}^3 \to \mathbf{E}^3$ by

$$S(t_1, t_2, t_3) = \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))) \tag{4.13}$$

where $\varphi_{t_1}(x_0)$ is the solution to $dx/dt_1 = f^{(1)}(x)$ with $x(0) = x_0, \varphi_{t_2}(x_0)$ is the solution to $dx/dt_2 = f^{(2)}(x)$ with $x(0) = x_0'$, and $\varphi_{t_3}(x_0)$ is the solution to $dx/dt_3 = f^{(3)}(x)$ with $x(0) = x_0''$. Writing

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1(t_1, t_2, t_3) \\ s_2(t_1, t_2, t_3) \\ s_3(t_1, t_2, t_3) \end{bmatrix} \tag{4.14}$$

with $S(0, 0, 0) = x_0$ this can be a coordinate transformation if and only if it is invertible. The Jacobian of this transformation is

$$\frac{\partial x}{\partial t} = \begin{bmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_1}{\partial t_2} & \dfrac{\partial x_1}{\partial t_3} \\[2ex] \dfrac{\partial x_2}{\partial t_1} & \dfrac{\partial x_2}{\partial t_2} & \dfrac{\partial x_2}{\partial t_3} \\[2ex] \dfrac{\partial x_3}{\partial t_1} & \dfrac{\partial x_3}{\partial t_2} & \dfrac{\partial x_3}{\partial t_3} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial s_1}{\partial t_1} & \dfrac{\partial s_1}{\partial t_2} & \dfrac{\partial s_1}{\partial t_3} \\[2ex] \dfrac{\partial s_2}{\partial t_1} & \dfrac{\partial s_2}{\partial t_2} & \dfrac{\partial s_2}{\partial t_3} \\[2ex] \dfrac{\partial s_3}{\partial t_1} & \dfrac{\partial s_3}{\partial t_2} & \dfrac{\partial s_3}{\partial t_3} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial S}{\partial t_1} & \dfrac{\partial S}{\partial t_2} & \dfrac{\partial S}{\partial t_3} \end{bmatrix}. \tag{4.15}$$

As

$$\frac{\partial S}{\partial t_1}_{|t=0} = \frac{\partial}{\partial t_1}\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))\Big|_{t=0} = \frac{\partial}{\partial t_1}\phi_0(\varphi_0(\varphi_{t_1}(x_0)))\Big|_{t_1=0} = \frac{\partial}{\partial t_1}\varphi_{t_1}(x_0)\Big|_{t_1=0} = f^{(1)}(\varphi_0(x_0)) = f^{(1)}(x_0)$$
$$\frac{\partial S}{\partial t_2}_{|t=0} = \frac{\partial}{\partial t_2}\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))\Big|_{t=0} = \frac{\partial}{\partial t_2}\phi_0(\varphi_{t_2}(\varphi_0(x_0)))\Big|_{t_2=0} = \frac{\partial}{\partial t_2}\varphi_{t_2}(x_0)\Big|_{t_2=0} = f^{(2)}(\varphi_0(x_0)) = f^{(2)}(x_0)$$
$$\frac{\partial S}{\partial t_3}_{|t=0} = \frac{\partial}{\partial t_3}\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))\Big|_{t=0} = \frac{\partial}{\partial t_3}\phi_{t_{30}}(\varphi_0(\varphi_0(x_0)))\Big|_{t_3=0} = \frac{\partial}{\partial t_3}\varphi_{t_3}(x_0)\Big|_{t_3=0} = f^{(3)}(\varphi_0(x_0)) = f^{(3)}(x_0)$$

it follows that

$$\det\left(\frac{\partial x}{\partial t}\right)_{|t=0} = \det\left[\begin{array}{ccc} f^{(1)}(x_0) & f^{(2)}(x_0) & f^{(3)}(x_0) \end{array}\right] \neq 0 \tag{4.16}$$

as $f^{(1)}(x_0), f^{(2)}(x_0)$, and $f^{(3)}(x_0)$ are linearly independent. As the vector fields $\frac{\partial S}{\partial t_1}, \frac{\partial S}{\partial t_2}, \frac{\partial S}{\partial t_3}$ are continuous functions of $t = (t_1, t_2, t_3)$ it follows that there is a neighborhood $\mathcal{D} \subset \mathbb{R}^3$ containing $(0,0,0)$ such that for all $(t_1, t_2, t_3) \in \mathcal{D}$ we have

$$\det\left(\frac{\partial x}{\partial t}\right) = \det\left[\begin{array}{ccc} \frac{\partial S}{\partial t_1} & \frac{\partial S}{\partial t_2} & \frac{\partial S}{\partial t_3} \end{array}\right] \neq 0.$$

Consequently, $(x_1, x_2, x_3) = S(t_1, t_2, t_3) : \mathcal{D} \to \mathcal{U} \subset \mathbf{E}^3$ is invertible with $S(0,0,0) = x_0$. Denote the inverse by $T(x) \triangleq S^{-1}(x) : \mathcal{U} \to \mathcal{D}$ written out as

$$\begin{aligned} t_1 &= T_1(x_1, x_2, x_3) \\ t_2 &= T_2(x_1, x_2, x_3) \\ t_3 &= T_3(x_1, x_2, x_3) \end{aligned} \tag{4.17}$$

with $T(x_0) = T(x_{01}, x_{02}, x_{03}) = (0,0,0)$. As S and T are inverses we have

$$\begin{aligned} t_1 &= T_1(s_1(t_1,t_2,t_3), s_2(t_1,t_2,t_3), s_3(t_1,t_2,t_3)) \\ t_2 &= T_2(s_1(t_1,t_2,t_3), s_2(t_1,t_2,t_3), s_3(t_1,t_2,t_3)) \\ t_3 &= T_3(s_1(t_1,t_2,t_3), s_2(t_1,t_2,t_3), s_3(t_1,t_2,t_3)). \end{aligned} \tag{4.18}$$

By the chain rule

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{\partial}{\partial t}T(S(t)) &= \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \\ \dfrac{\partial T_2}{\partial x_1} & \dfrac{\partial T_2}{\partial x_2} & \dfrac{\partial T_2}{\partial x_3} \\ \dfrac{\partial T_3}{\partial x_1} & \dfrac{\partial T_3}{\partial x_2} & \dfrac{\partial T_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial s_1}{\partial t_1} & \dfrac{\partial s_1}{\partial t_2} & \dfrac{\partial s_1}{\partial t_3} \\ \dfrac{\partial s_2}{\partial t_1} & \dfrac{\partial s_2}{\partial t_2} & \dfrac{\partial s_2}{\partial t_3} \\ \dfrac{\partial s_3}{\partial t_1} & \dfrac{\partial s_3}{\partial t_2} & \dfrac{\partial s_3}{\partial t_3} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \\ \dfrac{\partial T_2}{\partial x_1} & \dfrac{\partial T_2}{\partial x_2} & \dfrac{\partial T_2}{\partial x_3} \\ \dfrac{\partial T_3}{\partial x_1} & \dfrac{\partial T_3}{\partial x_2} & \dfrac{\partial T_3}{\partial x_3} \end{bmatrix}_{|x=S(t_1,t_2,t_3)} \begin{bmatrix} \dfrac{\partial S}{\partial t_1} & \dfrac{\partial S}{\partial t_2} & \dfrac{\partial S}{\partial t_3} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \\ \dfrac{\partial T_2}{\partial x_1} & \dfrac{\partial T_2}{\partial x_2} & \dfrac{\partial T_2}{\partial x_3} \\ \dfrac{\partial T_3}{\partial x_1} & \dfrac{\partial T_3}{\partial x_2} & \dfrac{\partial T_3}{\partial x_3} \end{bmatrix}_{|x=S(t_1,t_2,t_3)} \begin{bmatrix} \dfrac{\partial S}{\partial t_1} & f^{(2)}(S(t_1,t_2,t_3)) & f^{(3)}(S(t_1,t_2,t_3)) \end{bmatrix}
\end{aligned} \tag{4.19}
$$

where in the last line we used

$$\frac{\partial S}{\partial t_3} = \frac{\partial}{\partial t_3}\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))) = f^{(3)}(\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))) = f^{(3)}(S(t_1,t_2,t_3))$$

and

$$
\begin{aligned}
\frac{\partial S}{\partial t_2} = \frac{\partial}{\partial t_2}\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))) &= \frac{\partial}{\partial t_2}\phi_{t_2}(\varphi_{t_3}(\varphi_{t_1}(x_0))) \quad \text{as} \quad [f^{(2)}, f^{(3)}] \equiv 0 \\
&= f^{(2)}(\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))) \\
&= f^{(2)}(S(t_1,t_2,t_3)).
\end{aligned}
$$

Then new coordinates are $t = (t_1, t_2, t_3)$ with t_1 the amount of time $f^{(1)}$ is followed, t_2 is the amount of time $f^{(2)}$ is followed, and t_3 is the amount of time $f^{(3)}$ is followed.

Let's change the notation to $x_1^* = t_1, x_2^* = t_2, x_3^* = t_3$ so as not to confuse $t = (t_1, t_2, t_3)$ with the scalar time t. We write

$$
\begin{aligned}
x_1^* &= T_1(x_1, x_2, x_3) \\
x_2^* &= T_2(x_1, x_2, x_3) \\
x_3^* &= T_3(x_1, x_2, x_3).
\end{aligned} \tag{4.20}
$$

The representation of $\dfrac{dx}{dt} = f^{(2)}(x)$ in the x^* coordinate system is

$$\frac{dx^*}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} = \frac{\partial T}{\partial x}f^{(2)}(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where the last equality follows by (4.19). The representation for $\dfrac{dx}{dt} = f^{(3)}(x)$ in the x^* coordinate system is

$$\frac{dx^*}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} = \frac{\partial T}{\partial x} f^{(3)}(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where the last equality also follows by (4.19). In the x^* coordinates $f^{(2)}$ and $f^{(3)}$ have these very simple representations. Let's look at the geometric picture of what is going on here. Starting from x_0 at time $t_1 = 0$ the trajectory $\varphi_{t_1}(x_0)$ is followed till some time t_{01}. Holding t_{01} fixed we vary t_2 and t_3 to produce a surface in $\mathbf{E}^3$. In more detail, from $\phi_{t_{01}}(x_0)$ follow $f^{(2)}$ for a time t_{02} to end up at $\phi_{t_{02}}(\phi_{t_{01}}(x_0))$. Then from $\phi_{t_{02}}(\phi_{t_{01}}(x_0))$ follow $f^{(3)}$ for a time t_{03} to end up at $\phi_{t_{03}}(\phi_{t_{02}}(\phi_{t_{01}}(x_0)))$. See Figure 4.6. The mapping S takes the coordinate curve (t_{01}, t_2, t_{03}) for $|t_2| < \epsilon$ in $\mathbb{R}^3$ to the curve $S(t_{01}, t_2, t_{03})$ in $\mathbf{E}^3$ and, as $[f^{(2)}, f^{(3)}] \equiv 0$, we have $\left.\dfrac{\partial S(t_{01}, t_2, t_{03})}{\partial t_2}\right|_{t_{02}} = f^{(2)}(x)\big|_{x=\phi_{t_{03}}(\phi_{t_{02}}(\phi_{t_{01}}(x_0)))}$. Further the coordinate curve mapping (t_{01}, t_{02}, t_3) for $|t_3| < \epsilon$ in $\mathbb{R}^3$ to the curve $S(t_{01}, t_{02}, t_3)$ in $\mathbf{E}^3$ with $\left.\dfrac{\partial S(t_{01}, t_{02}, t_3)}{\partial t_3}\right|_{t_{03}} = f^{(3)}(x)\big|_{x=\phi_{t_{03}}(\phi_{t_{02}}(\phi_{t_{01}}(x_0)))}$.

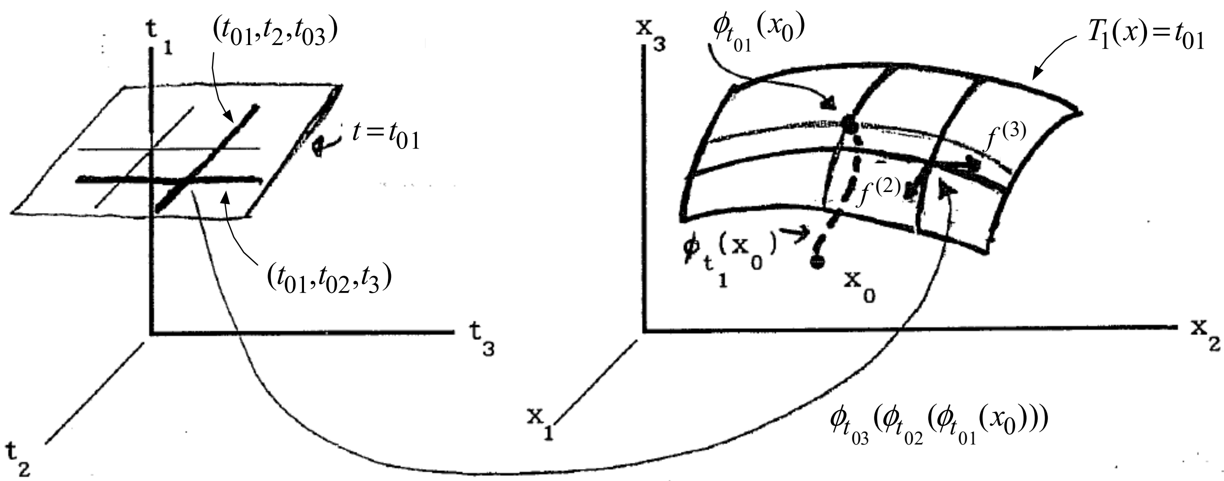


FIGURE 4.6. Coordinates (t_1, t_2, t_3) constructed by following $f^{(1)}$ for a time t_1, $f^{(2)}$ for a time t_2, $f^{(3)}$ for a time t_3.

We have just shown that with t_1 constant, $S(t_1, t_2, t_3)$ sweeps out a surface as (t_2, t_3) is varied around $(0, 0)$ with the tangent to this surface spanned by

$$\frac{\partial S(t_1, t_2, t_3)}{\partial t_2} = f^{(2)}(S(t_1, t_2, t_3)) \text{ and } \frac{\partial S(t_1, t_2, t_3)}{\partial t_3} = f^{(3)}(S(t_1, t_2, t_3)).$$

In particular, for $t_1 = 0$, the surface $S(0, t_2, t_3)$ contains x_0 as $S(0, 0, 0) = x_0$. The set $\left\{x \in \mathbf{E}^3 \mid T_1(x) = t_{01}\right\}$ implicitly gives a surface in $\mathbf{E}^3$ whose tangent plane at each point is spanned by $f^{(2)}(x)$ and $f^{(3)}(x)$. The gradient $dT_1 = \dfrac{\partial T_1}{\partial x}$ is normal (perpendicular) to this surface so $\mathcal{L}_{f^{(2)}}(T_1) = \left\langle dT_1, f^{(2)}\right\rangle = 0$ and $\mathcal{L}_{f^{(3)}}(T_1) = \left\langle dT_1, f^{(3)}\right\rangle = 0$ on this surface. Further, as $f^{(1)}$ is linearly independent of $f^{(2)}$ and $f^{(3)}$ we have must have $\mathcal{L}_{f^{(1)}}(T_1) = \left\langle dT_1, f^{(1)}\right\rangle \neq 0$ on the surface.

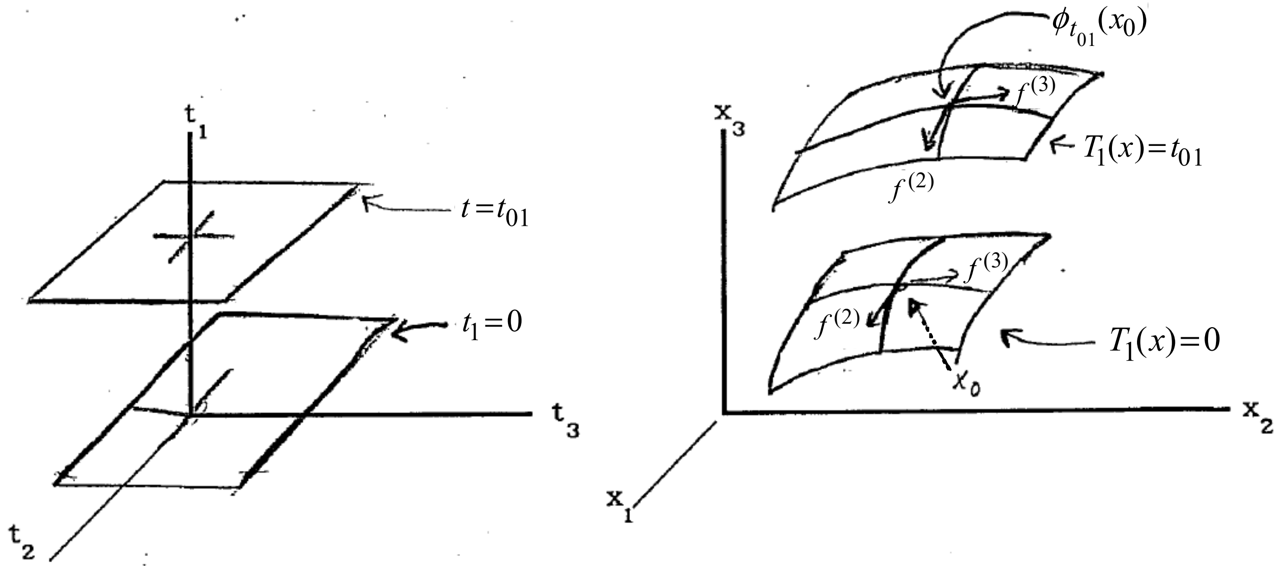


FIGURE 4.7. Surface containing x_0 with its tangent space spanned by $f^{(2)}$ and $f^{(3)}$.

Example 12 *Special Coordinate System*
Consider the three vector fields on $\mathbf{E}^3$ given by

$$f^{(1)}(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad f^{(2)}(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix}, \quad f^{(3)}(x_1, x_2, x_3) = \begin{bmatrix} 1 \\ 0 \\ 2x_1 x_2 \end{bmatrix}.$$

The solution to $\dfrac{dx}{dt} = f^{(1)}(x), x(0) = x_0$ is

$$\phi_{t_1}(x_0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} + t_1 \end{bmatrix}.$$

The solution to $\dfrac{dx}{dt_2} = f^{(2)}(x), x(0) = x_0'$ is

$$\phi_{t_2}(x_0) = \begin{bmatrix} x_{01}' \\ x_{02}' + t_2 \\ x_{03}' + (x_{01}')^2 t_2 \end{bmatrix}.$$

The solution to $\dfrac{dx}{dt_3} = f^{(3)}(x), x(0) = x_0''$ is

$$\phi_{t_3}(x_0) = \begin{bmatrix} x_{01}'' + t_3 \\ x_{02}'' \\ x_{03}'' + (x_{01}'' + t_3)^2 x_{02}'' - (x_{01}'')^2 x_{02}'' \end{bmatrix}.$$

Then with $x_0'' = \varphi_{t_2}(\varphi_{t_1}(x_0))$ and $x_0' = \varphi_{t_1}(x_0)$ we have

$$S(t_1,t_2,t_3) \triangleq \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))) = \left.\begin{bmatrix} x_{01}''+t_3 \\ x_{02}'' \\ x_{03}''+(x_{01}''+t_3)^2x_{02}''-(x_{01}'')^2x_{02}'' \end{bmatrix}\right|_{x_0''=\varphi_{t_2}(x_0')=\begin{bmatrix} x_{01}' \\ x_{02}'+t_2 \\ x_{03}'+(x_{01}')^2t_2 \end{bmatrix}}$$

$$= \left.\begin{bmatrix} x_{01}'+t_3 \\ x_{02}'+t_2 \\ x_{03}'+(x_{01}')^2t_2+(x_{01}'+t_3)^2(x_{02}'+t_2)-(x_{01}')^2(x_{02}'+t_2) \end{bmatrix}\right|_{x_0'=\varphi_{t_1}(x_0)=\begin{bmatrix} x_{01} \\ x_{02} \\ x_{03}+t_1 \end{bmatrix}}$$

$$= \begin{bmatrix} x_{01}+t_3 \\ x_{02}+t_2 \\ x_{03}+t_1+(x_{01})^2t_2+(x_{01}+t_3)^2(x_{02}+t_2)-(x_{01})^2(x_{02}+t_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_{01}+t_3 \\ x_{02}+t_2 \\ x_{03}+t_1+(x_{01}+t_3)^2(x_{02}+t_2)-(x_{01})^2x_{02} \end{bmatrix}$$

In Example 8 it was shown that $[f^{(2)},f^{(3)}] \equiv 0$ so we know that $\dfrac{\partial S(t_1,t_2,t_3)}{\partial t_2} = f^{(2)}(t_1,t_2,t_3)$, $\dfrac{\partial S(t_1,t_2,t_3)}{\partial t_3} = f^{(3)}(t_1,t_2,t_3)$. Further $\{f^{(1)},f^{(2)},f^{(3)}\}$ are linearly independent. In fact

$$\det\left(\frac{\partial S}{\partial t}\right)_{|t=(0,0,0)} = \det\left(\begin{bmatrix} \dfrac{\partial S}{\partial t_1} & \dfrac{\partial S}{\partial t_2} & \dfrac{\partial S}{\partial t_3} \end{bmatrix}\right)_{|t=(0,0,0)} = \det\begin{bmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & f^{(3)}(x_0) \end{bmatrix} = -1 \neq 0.$$

As $\det\left(\dfrac{\partial S}{\partial t}\right)$ is a continuos function of $t=(t_1,t_2,t_3)$ it is non zero in a neighborhood of $(0,0,0)$. By the inverse function theorem we know the map

$$x = S(t_1,t_2,t_3) = \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))$$

has an inverse in a neighborhood of $S(0,0,0)=x_0$ which we denote as

$$t = S^{-1}(x) = T(x).$$

Setting $(t_1,t_2,t_3)=(x_1^*,x_2^*,x_3^*)$ we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_{01}+x_3^* \\ x_{02}+x_2^* \\ x_{03}+x_1^*+(x_{01}+x_3^*)^2(x_{02}+x_2^*)-(x_{01})^2x_{02} \end{bmatrix}$$

and see that it has a global inverse given by

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} x_3-x_{03}-x_1^2x_2-x_{01}^2x_{02} \\ x_2-x_{02} \\ x_1-x_{01} \end{bmatrix}.$$

In the x^* coordinates the differential equation $dx/dt = f^{(2)}(x)$ becomes

$$\frac{dx^*}{dt} = \frac{\partial x^*}{\partial x}\frac{dx}{dt} = \frac{\partial x^*}{\partial x}f^{(2)}(x) = \begin{bmatrix} -2x_1x_2 & -x_1^2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = f^{*(2)}(x^*).$$

That is,

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

In the x^* coordinates the differential equation $dx/dt = f^{(3)}(x)$ becomes

$$\frac{dx^*}{dt} = \frac{\partial x^*}{\partial x}\frac{dx}{dt} = \frac{\partial x^*}{\partial x}f^{(3)}(x) = \begin{bmatrix} -2x_1x_2 & -x_1^2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = f^{*(3)}(x^*).$$

That is,

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, the gradient of $T_1(x)$ is given by

$$dT_1 = \frac{\partial}{\partial x}\left(x_3 - x_{03} - x_1^2x_2 - x_{01}^2x_{02}\right) = \begin{bmatrix} -2x_1x_2 & -x_1^2 & 1 \end{bmatrix}$$

and

$$\mathcal{L}_{f^{(2)}}(T_1) = \left\langle dT_1, f^{(2)}\right\rangle = \begin{bmatrix} -2x_1x_2 & -x_1^2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix} = 0$$

$$\mathcal{L}_{f^{(3)}}(T_1) = \left\langle dT_1, f^{(3)}\right\rangle = \begin{bmatrix} -2x_1x_2 & -x_1^2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix} = 0.$$

4.4 The Frobenius Theorem

We have seen that if we have two linearly independent vector fields $f^{(1)}, f^{(2)}$ defined on some open set $\mathcal{U} \subset \mathbf{E}^3$ satisfying $[f^{(1)}, f^{(2)}] \equiv 0$, then through any point of $\mathcal{U}$ there is a surface that goes through that point and whose tangent vectors are $f^{(1)}, f^{(2)}$. The Frobenius theorem is a generalization of this result. To proceed we define the notion of a *distribution* of vectors. Let $f^{(1)}$ and $f^{(2)}$ be two vector fields on an open set $\mathcal{U} \subset \mathbf{E}^3$. As in the previous sections of this chapter, we identify the Cartesian coordinates for $\mathbf{E}^3$ with the points of $\mathbf{E}^3$, that is, the coordinates (x_1, x_2, x_3) and the point $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ with $z_1 = x_1, z_2 = x_2, z_3 = x_3$ to be the same thing. At each point $p \in \mathcal{U}$ define the vector space Δ_p as

$$\Delta_p \triangleq \left\{ r_1 f^{(1)}(x) + r_2 f^{(2)}(x) \middle|\ r_1, r_2 \in \mathbb{R},\ p = (x_1, x_2, x_3) \in \mathcal{U} \right\}. \tag{4.21}$$

Δ_p is the vector space spanned by $f^{(1)}$ and $f^{(2)}$. The distribution Δ is the collection of all these vector spaces for all $p \in \mathcal{U}$, that is,

$$\Delta = \cup_{p\in\mathcal{U}}\Delta_p. \tag{4.22}$$

We want to find a surface $S(t_1, t_2) \in \mathbf{E}^3$ whose tangent vectors at any point of the surface are in Δ. That is, instead of requiring $\dfrac{\partial S}{\partial t_1} = f^{(1)}, \dfrac{\partial S}{\partial t_2} = f^{(2)}$ it is just required that $\dfrac{\partial S}{\partial t_1}$ and $\dfrac{\partial S}{\partial t_2}$ be linear combinations of $f^{(1)}$ and $f^{(2)}$ at each point of the surface. Let's first state the theorem.

Theorem 4 *Frobenius First Version (Sufficiency)*
Let $\mathcal{U} \subset \mathbf{E}^3$ be an open set with the two linearly independent vector fields $f^{(1)}$ and $f^{(2)}$ defined on $\mathcal{U}$. If for all points $p = (x_1, x_2, x_3)$ in $\mathcal{U}$ we have

$$[f^{(1)}, f^{(2)}] = \alpha_1(x) f^{(1)}(x) + \alpha_2(x) f^{(2)}(x) \in \Delta \tag{4.23}$$

then given any point $p = (x_{01}, x_{02}, x_{03}) \in \mathcal{U}$ there is a surface $S(t_1, t_2)$ such that $S(0,0) = (x_{01}, x_{02}, x_{03})$ and, for all $|t_1| < \epsilon, |t_2| < \epsilon$ for some $\epsilon > 0$, we have

$$\frac{\partial S}{\partial t_1}, \frac{\partial S}{\partial t_2} \in \Delta. \tag{4.24}$$

Definition 1 *Involutive*
Let $f^{(1)}$ and $f^{(2)}$ be two vector fields defined on $\mathcal{U} \subset \mathbf{E}^3$ with Δ defined as in (4.22). If $f^{(1)}$ and $f^{(2)}$ satisfy (4.23) then they are said to be *involutive*.

We first prove four lemmas before proving the Frobenius theorem. We follow the approach of Allendoerfer [23].

Lemma 1 Let $a_1(x), a_2(x)$ be two scalar functions and $f^{(1)}(x), f^{(2)}(x)$ be two vector fields defined an open set $\mathcal{U} \subset \mathbf{E}^3$. Then

$$[a_1 f^{(1)}, a_2 f^{(2)}] = a_1(x) a_2(x) [f^{(1)}, f^{(2)}] + \alpha_2(x) \mathcal{L}_{f^{(2)}}(a_1) f^{(1)}(x) - \alpha_1(x) \mathcal{L}_{f^{(1)}}(a_2) f^{(2)}(x). \tag{4.25}$$

Proof. By the definition of Lie bracket we have

$$[a_1 f^{(1)}, a_2 f^{(2)}] = \frac{\partial (a_2 f^{(2)})}{\partial x} a_1 f^{(1)} - \frac{\partial (a_1 f^{(1)})}{\partial x} a_2 f^{(2)}.$$

We compute

$$\frac{\partial (a_2 f^{(2)})}{\partial x} = \begin{bmatrix} \frac{\partial}{\partial x}(a_2 f_1^{(2)}) \\ \frac{\partial}{\partial x}(a_2 f_2^{(2)}) \\ \frac{\partial}{\partial x}(a_2 f_3^{(2)}) \end{bmatrix} = \begin{bmatrix} a_2 \frac{\partial}{\partial x} f_1^{(2)} \\ a_2 \frac{\partial}{\partial x} f_2^{(2)} \\ a_2 \frac{\partial}{\partial x} f_3^{(2)} \end{bmatrix} + \begin{bmatrix} f_1^{(2)} \frac{\partial a_2}{\partial x} \\ f_2^{(2)} \frac{\partial a_2}{\partial x} \\ f_3^{(2)} \frac{\partial a_2}{\partial x} \end{bmatrix}.$$

Then

$$\begin{aligned}\frac{\partial (a_2 f^{(2)})}{\partial x} a_1 f^{(1)} &= a_1 a_2 \begin{bmatrix} \frac{\partial}{\partial x} f_1^{(2)} \\ \frac{\partial}{\partial x} f_2^{(2)} \\ \frac{\partial}{\partial x} f_3^{(2)} \end{bmatrix} f^{(1)} + a_1 \begin{bmatrix} f_1^{(2)} \frac{\partial a_2}{\partial x} \\ f_2^{(2)} \frac{\partial a_2}{\partial x} \\ f_3^{(2)} \frac{\partial a_2}{\partial x} \end{bmatrix} f^{(1)} = a_1 a_2 \frac{\partial f^{(2)}}{\partial x} f^{(1)} + a_1 \begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \\ f_3^{(2)} \end{bmatrix} \mathcal{L}_{f^{(1)}}(a_2) \\ &= a_1 a_2 \frac{\partial f^{(2)}}{\partial x} f^{(1)} + a_1 \mathcal{L}_{f^{(1)}}(a_2) f^{(2)}\end{aligned}$$

Similarly

$$\frac{\partial (a_1 f^{(1)})}{\partial x} a_2 f^{(2)} = a_1 a_2 \frac{\partial f^{(1)}}{\partial x} f^{(2)} + a_2 \mathcal{L}_{f^{(2)}}(a_1) f^{(1)}.$$

Subtracting these two expressions we have our result:

$$\begin{aligned}[a_1 f^{(1)}, a_2 f^{(2)}] &= a_1 a_2 \frac{\partial f^{(2)}}{\partial x} f^{(1)} + a_1 \mathcal{L}_{f^{(1)}}(a_2) f^{(2)} - \left(a_1 a_2 \frac{\partial f^{(1)}}{\partial x} f^{(2)} + a_2 \mathcal{L}_{f^{(2)}}(a_1) f^{(1)} \right) \\ &= a_1 a_2 [f^{(1)}, f^{(2)}] + \alpha_1 \mathcal{L}_{f^{(1)}}(a_2) f^{(2)} - \alpha_2 \mathcal{L}_{f^{(2)}}(a_1) f^{(1)}.\end{aligned}$$

■

Lemma 2 Let $f^{(1)}, f^{(2)}, g^{(1)}, g^{(2)}$ be a set of vector fields on $\mathcal{U} \subset \mathbf{E}^3$. Then

(a) $[f^{(1)} + f^{(2)}, g^{(1)}] = [f^{(1)}, g^{(1)}] + [f^{(2)}, g^{(1)}]$

(b) $[f^{(1)}, g^{(1)} + g^{(2)}] = [f^{(1)}, g^{(1)}] + [f^{(1)}, g^{(2)}]$

(c) $[f^{(1)}, g^{(1)}] = -[g^{(1)}, f^{(1)}]$

(d) $[f^{(1)}, f^{(1)}] = 0$

Proof. Exercise ■

Lemma 3 Given two linearly independent vector fields $f^{(1)}$ and $f^{(2)}$, define two new vector fields $f^{*(1)}, f^{*(2)}$ by

$$f^{*(1)} \triangleq a_{11}(x)f^{(1)}(x) + a_{12}(x)f^{(2)}(x) \tag{4.26}$$
$$f^{*(2)} \triangleq a_{21}(x)f^{(1)}(x) + a_{22}(x)f^{(2)}(x) \tag{4.27}$$

with

$$\det \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix} \neq 0$$

on some open subset $\mathcal{U} \subset \mathbf{E}^3$. (The determinant not being zero ensures that $f^{*(1)}$ and $f^{*(2)}$ are linearly independent.). If for all $x \in \mathcal{U}$ we have

$$[f^{(1)}, f^{(2)}] = \alpha_1(x)f^{(1)}(x) + \alpha_2(x)f^{(2)}(x)$$

with α_1, α_2 scalar functions, then

$$[f^{*(1)}, f^{*(2)}] = \alpha_1^*(x)f^{*(1)}(x) + \alpha_2^*(x)f^{*(2)}(x) \tag{4.28}$$

for some scalar functions α_1^*, α_2^*.

Proof.

$$\begin{aligned}
[f^{*(1)}, f^{*(2)}] =& [a_{11}(x)f^{(1)}(x) + a_{12}(x)f^{(2)}(x), a_{21}(x)f^{(1)}(x) + a_{22}(x)f^{(2)}(x)] \\
=& [a_{11}(x)f^{(1)}(x), a_{21}(x)f^{(1)}(x) + a_{22}(x)f^{(2)}(x)] + \\
& [a_{12}(x)f^{(2)}(x), a_{21}(x)f^{(1)}(x) + a_{22}(x)f^{(2)}(x)] \\
=& [a_{11}(x)f^{(1)}(x), a_{21}(x)f^{(1)}(x)] + [a_{11}(x)f^{(1)}(x), a_{22}(x)f^{(2)}(x)] + \\
& [a_{12}(x)f^{(2)}(x), a_{21}(x)f^{(1)}(x)] + [a_{12}(x)f^{(2)}(x), a_{22}(x)f^{(2)}(x)] \\
=& a_{11}a_{21}[f^{(1)}, f^{(1)}] + \alpha_{21}\mathcal{L}_{f^{(1)}}(a_{11})f^{(1)} - \alpha_{11}\mathcal{L}_{f^{(1)}}(a_{21})f^{(1)} + \\
& a_{11}a_{22}[f^{(1)}, f^{(2)}] + \alpha_{22}\mathcal{L}_{f^{(2)}}(a_{11})f^{(1)} - \alpha_{11}\mathcal{L}_{f^{(1)}}(a_{22})f^{(2)} + \\
& a_{12}a_{21}[f^{(2)}, f^{(1)}] + \alpha_{21}\mathcal{L}_{f^{(1)}}(a_{12})f^{(2)} - \alpha_{12}\mathcal{L}_{f^{(2)}}(a_{21})f^{(1)} + \\
& a_{12}a_{22}[f^{(2)}, f^{(2)}] + \alpha_{22}\mathcal{L}_{f^{(2)}}(a_{12})f^{(2)} - \alpha_{12}\mathcal{L}_{f^{(2)}}(a_{22})f^{(2)}.
\end{aligned}$$

Rearranging this last expression gives

$$\begin{aligned}
[f^{*(1)}, f^{*(2)}] = & (a_{11}a_{22} - a_{12}a_{21})[f^{(1)}, f^{(2)}] + \left(\alpha_{21}\mathcal{L}_{f^{(1)}}(a_{11}) - \alpha_{11}\mathcal{L}_{f^{(1)}}(a_{21}) + \alpha_{22}\mathcal{L}_{f^{(2)}}(a_{11}) - \alpha_{12}\mathcal{L}_{f^{(2)}}(a_{21})\right) f^{(1)} \\
& \left(-\alpha_{11}\mathcal{L}_{f^{(1)}}(a_{22}) + \alpha_{21}\mathcal{L}_{f^{(1)}}(a_{12}) + \alpha_{22}\mathcal{L}_{f^{(2)}}(a_{12}) - \alpha_{12}\mathcal{L}_{f^{(2)}}(a_{22})\right) f^{(2)}.
\end{aligned}$$

By assumption $[f^{(1)}, f^{(2)}] = \alpha_1(x)f^{(1)}(x) + \alpha_2(x)f^{(2)}(x)$ so this last expression may be written in the form

$$[f^{*(1)}, f^{*(2)}] = \gamma_1(x)f^{(1)} + \gamma_2(x)f^{(2)}(x).$$

Next rewrite (4.26) and (4.27) as

$$[f^{*(1)}, f^{*(2)}] \triangleq \begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix} \begin{bmatrix} a_{11}(x) & a_{21}(x) \\ a_{12}(x) & a_{22}(x) \end{bmatrix}.$$

Then with

$$\begin{bmatrix} b_{11}(x) & b_{21}(x) \\ b_{12}(x) & b_{22}(x) \end{bmatrix} \triangleq \begin{bmatrix} a_{11}(x) & a_{21}(x) \\ a_{11}(x) & a_{22}(x) \end{bmatrix}^{-1}$$

so that

$$[f^{*(1)}, f^{*(2)}] \begin{bmatrix} b_{11}(x) & b_{21}(x) \\ b_{12}(x) & b_{22}(x) \end{bmatrix} = \begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix} \begin{bmatrix} a_{11}(x) & a_{21}(x) \\ a_{12}(x) & a_{22}(x) \end{bmatrix} \begin{bmatrix} b_{11}(x) & b_{21}(x) \\ b_{12}(x) & b_{22}(x) \end{bmatrix} = \begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix}$$

or

$$\begin{aligned} f^{(1)} &\triangleq b_{11}(x) f^{*(1)}(x) + b_{12}(x) f^{*(2)}(x) \\ f^{(2)} &\triangleq b_{21}(x) f^{*(1)}(x) + b_{22}(x) f^{*(2)}(x). \end{aligned}$$

The Lie bracket $[f^{*(1)}, f^{*(2)}]$ then has the form

$$\begin{aligned} [f^{*(1)}, f^{*(2)}] &= \gamma_1(x)\Big(b_{11}(x) f^{*(1)}(x) + b_{12}(x) f^{*(2)}(x)\Big) + \gamma_2(x)\Big(b_{21}(x) f^{*(1)}(x) + b_{22}(x) f^{*(2)}(x)\Big) \\ &= \alpha_1^*(x) f^{*(1)}(x) + \alpha_2^*(x) f^{*(2)}(x). \end{aligned}$$

■

Lemma 4 Let $f^{(1)}, f^{(2)}$ be two linearly independent involutive vector fields $f^{(1)}, f^{(2)}$ on some open subset $\mathcal{U} \subset \mathbf{E}^3$ so that

$$[f^{(1)}, f^{(2)}] = \alpha_1(x) f^{(1)}(x) + \alpha_2(x) f^{(2)}(x)$$

for some scalar fields $\alpha_1(x), \alpha_2(x) \in \mathbb{R}$. Then there exists a nonsingular transformation of the vector fields given by $(a_{ij}(x) \in \mathbb{R})$

$$\begin{aligned} f^{*(1)} &\triangleq a_{11}(x) f^{(1)}(x) + a_{12}(x) f^{(2)}(x) \\ f^{*(2)} &\triangleq a_{21}(x) f^{(1)}(x) + a_{22}(x) f^{(2)}(x) \end{aligned}$$

such that $[f^{*(1)}, f^{*(2)}] \equiv 0$ on $\mathcal{U}$.

Proof. Form the 3×2 matrix from the vector fields $f^{(1)}, f^{(2)}$ by

$$\begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix} = \begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \\ f_3^{(1)} & f_3^{(2)} \end{bmatrix}.$$

As $f^{(1)}, f^{(2)}$ are linearly independent, the 3×2 matrix $\begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix}$ must have a nonzero minor. Let's assume the minor is

$$\begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \end{bmatrix}, \text{ i.e., } \det \begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \end{bmatrix} \neq 0.$$

Define

$$\begin{bmatrix} a_{11}(x) & a_{21}(x) \\ a_{12}(x) & a_{22}(x) \end{bmatrix} \triangleq \begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \end{bmatrix}^{-1}.$$

Then

$$[f^{*(1)}, f^{*(2)}] \triangleq \begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix} \begin{bmatrix} a_{11}(x) & a_{21}(x) \\ a_{12}(x) & a_{22}(x) \end{bmatrix} = \begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \\ f_3^{(1)} & f_3^{(2)} \end{bmatrix} \begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \mu_1(x) & \mu_2(x) \end{bmatrix}$$

with the obvious definitions for $\mu_1(x), \mu_2(x) \in \mathbb{R}$.

Then

$$[f^{*(1)}, f^{*(2)}] = \frac{\partial f^{*(2)}}{\partial x} f^{*(1)} - \frac{\partial f^{*(1)}}{\partial x} f^{*(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{\partial \mu_2}{\partial x_1} & \dfrac{\partial \mu_2}{\partial x_2} & \dfrac{\partial \mu_2}{\partial x_3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \mu_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{\partial \mu_1}{\partial x_1} & \dfrac{\partial \mu_1}{\partial x_2} & \dfrac{\partial \mu_1}{\partial x_3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}$$

with the obvious definition for γ. By lemma 3 we know there exists α_1^*, α_2^* such that

$$[f^{*(1)}, f^{*(2)}] = \alpha_1^*(x) f^{*(1)}(x) + \alpha_2^*(x) f^{*(2)}(x).$$

As a consequence we have

$$\begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha_1^* \\ 0 \\ \alpha_1^* \mu_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2^* \\ \alpha_2^* \mu_2 \end{bmatrix}$$

which shows $\alpha_1^* = 0, \alpha_2^* = 0$ and $\gamma = 0$. ∎

Proof of the Frobenius Theorem (Sufficiency)

Proof. The Frobenius theorem follows from this last lemma and Theorem 3. Specifically, we are given two linear independent vector fields $f^{(1)}, f^{(2)}$ which for all $p = (x_1, x_2, x_3)$ in $\mathcal{U} \subset \mathbf{E}^3$ satisfy

$$[f^{(1)}, f^{(2)}] = \alpha_1(x) f^{(1)}(x) + \alpha_2(x) f^{(2)}(x).$$

Then by Lemma 4 we can construct two linearly independent vectors $f^{*(1)}, f^{*(2)}$ which are linear combinations of $f^{(1)}, f^{(2)}$ and satisfy $[f^{*(1)}, f^{*(2)}] = 0$ on $\mathcal{U}$. By Theorem 3, for any given point $p = (x_{01}, x_{02}, x_{03}) \in \mathcal{U}$, a surface $S(t_1, t_2)$ can be constructed in $\mathcal{U}$ with $S(0,0) = (x_{01}, x_{02}, x_{03})$ and such that for all $|t_1| < \epsilon, |t_2| < \epsilon$ and some $\epsilon > 0$ we have $\dfrac{\partial S}{\partial t_1} = f^{*(1)}, \dfrac{\partial S}{\partial t_2} = f^{*(2)}$ and thus

$$\left.\frac{\partial S}{\partial t_1}\right|_p, \left.\frac{\partial S}{\partial t_2}\right|_p \in \Delta_p = \left\{ r_1 f^{(1)}(x) + r_2 f^{(2)}(x) \middle|\ r_1, r_2 \in \mathbb{R} \right\}$$

for all $p = (x_1, x_2, x_3) \in \mathcal{U}$. ∎

Example 13 *Special Coordinate System - Again*

Let

$$f^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ f^{(3)} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

A simple calculation shows that $[f^{(2)}, f^{(3)}] = f^{(2)}$ showing the pair $\{f^{(2)}, f^{(3)}\}$ is involutive. Then

$$\begin{bmatrix} f^{(2)} & f^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix}.$$

With the assumption that $p = (x_{01}, x_{02}, x_{03}) \in \mathcal{U}$ with $x_{02} \neq 0$, the minor

$$\begin{bmatrix} 1 & x_1 \\ 0 & x_2 \end{bmatrix}$$

is invertible. Let

$$\begin{bmatrix} f^{*(2)} & f^{*(3)} \end{bmatrix} \triangleq \begin{bmatrix} 1 & x_1 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 0 & x_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & x_1 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -x_1/x_2 \\ 0 & 1/x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x_3/x_2 \end{bmatrix}.$$

Another simple calculation shows that $[f^{*(2)}, f^{*(3)}] \equiv 0$. The solution to $dx/dt_2 = f^{*(2)}(x)$ with $x(0) = x_0'$ is

$$\phi^*_{t_2}(x_0') = \begin{bmatrix} x_{01}' + t_2 \\ x_{02}' \\ x_{03}' \end{bmatrix}$$

and the solution to $dx/dt_3 = f^{*(3)}(x)$ with $x(0) = x_0''$ is

$$\phi^*_{t_3}(x_0'') = \begin{bmatrix} x_{01}'' \\ x_{02}'' + t_3 \\ (x_{03}''/x_{02}'')(x_{02}'' + t_3) \end{bmatrix}.$$

For $f^{*(1)}$ choose a vector that is linearly independent of $f^{*(2)}, f^{*(3)}$. Here we chose

$$f^{*(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the solution to $dx/dt_1 = f^{*(1)}(x)$ with $x(0) = x_0$ is

$$\phi^*_{t_1}(x_0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} + t_1 \end{bmatrix}.$$

Then set

$$\begin{aligned} S(t_1, t_2, t_3) = \phi^*_{t_3}(\phi^*_{t_2}(\phi^*_{t_1}(x_0))) &= \begin{bmatrix} x_{01}'' \\ x_{02}'' + t_3 \\ (x_{03}''/x_{02}'')(x_{02}'' + t_3) \end{bmatrix} \Bigg|_{x_0'' = \phi^*_{t_2}(x_0') = = \begin{bmatrix} x_{01}' + t_2 \\ x_{02}' \\ x_{03}' \end{bmatrix}} \\ &= \begin{bmatrix} x_{01}' + t_2 \\ x_{02}' + t_3 \\ (x_{03}'/x_{02}')(x_{02}' + t_3) \end{bmatrix} \Bigg|_{x_0' = \phi^*_{t_1}(x_0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} + t_1 \end{bmatrix}} \\ &= \begin{bmatrix} x_{01} + t_2 \\ x_{02} + t_3 \\ (x_{03} + t_1)(x_{02} + t_3)/x_{02} \end{bmatrix}. \end{aligned}$$

We calculate

$$\frac{\partial S}{\partial t_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = f^{*(2)}, \quad \frac{\partial S}{\partial t_3} = \begin{bmatrix} 0 \\ 1 \\ (x_{03} + t_1)/x_{02} \end{bmatrix} = f^{*(3)}.$$

The inverse of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1(t_1,t_2,t_3) \\ s_2(t_1,t_2,t_3) \\ s_3(t_1,t_2,t_3) \end{bmatrix} = \begin{bmatrix} x_{01}+t_2 \\ x_{02}+t_3 \\ (x_{03}+t_1)(x_{02}+t_3)/x_{02} \end{bmatrix}$$

is given by

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \underbrace{\begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \end{bmatrix}}_{T(x)} = \begin{bmatrix} \dfrac{x_3}{x_2}x_{02}-x_{03} \\ x_1-x_{01} \\ x_2-x_{02} \end{bmatrix}$$

where it is seen that $T(x_0)=0_{3\times 1}$.

With

$$T_1(x) = \frac{x_3}{x_2}x_{02}-x_{03}$$

we have

$$dT_1 = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix}$$

and

$$\mathcal{L}_{f^{*(2)}}(T_1) = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{f^{*(2)}} = 0$$

$$\mathcal{L}_{f^{*(3)}}(T_1) = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}}_{f^{*(3)}} = 0.$$

Further, as $f^{(2)}$ and $f^{(3)}$ are linear combinations of $f^{*(2)}, f^{*(3)}$, we also have $\mathcal{L}_{f^{(2)}}(T_1)=0$ and $\mathcal{L}_{f^{(3)}}(T_1)=0$. Explicitly we have

$$\mathcal{L}_{f^{(2)}}(T_1) = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{f^{(2)}} = 0$$

$$\mathcal{L}_{f^{(3)}}(T_1) = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{f^{(2)}} = 0.$$

Finally, as $f^{*(1)}$ is linearly independent of $f^{*(2)}$ and $f^{*(3)}$ we must have $\mathcal{L}_{f^{*(1)}}(T_1) = \left\langle dT_1, f^{*(1)} \right\rangle \neq 0$. Specifically this is

$$\mathcal{L}_{f^{*(1)}}(T_1) = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{f^{*(1)}} = x_{02}/x_2.$$

In light of this example we reformulate the Frobenius theorem.

Theorem 5 *Frobenius Theorem (Necessity)*
Let $f^{(1)}, f^{(2)}, f^{(3)}$ be three linearly independent vector fields on an open set $\mathcal{U} \subset \mathbf{E}^3$. There exists a coordinate system $S : \mathcal{D} \subset \mathbb{R}^3 \to S(\mathcal{D}) \subset \mathcal{U}$ given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1(t_1, t_2, t_3) \\ s_2(t_1, t_2, t_3) \\ s_3(t_1, t_2, t_3) \end{bmatrix}, \quad \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} = \begin{bmatrix} s_1(t_{01}, t_{02}, t_{03}) \\ s_2(t_{01}, t_{02}, t_{03}) \\ s_3(t_{01}, t_{02}, t_{03}) \end{bmatrix}$$

with inverse $T : S(\mathcal{D}) \to \mathcal{D}$

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_1(x_1, x_2, x_3) \\ T_2(x_1, x_2, x_3) \\ T_3(x_1, x_2, x_3) \end{bmatrix}$$

for which the two-dimensional manifolds (surfaces) defined by

$$\left\{ x \in \mathcal{U} \subset \mathbf{E}^3 \middle| \, T_1(x) = t_{01} \right\}$$

for a fixed t_{01} have $f^{(2)}, f^{(3)}$ spanning their tangent planes if and only if

$$[f^{(2)}, f^{(3)}] = \alpha_2 f^{(2)} + \alpha_3 f^{(3)}$$

for some scalar valued functions α_2 and α_3.

Proof. We have already shown that if the involutive condition holds then such a coordinate system exists. We now show necessity, i.e., if there exists such a coordinate system then $[f^{(2)}, f^{(3)}] = \alpha_2 f^{(2)} + \alpha_3 f^{(3)}$. Let

$$\mathcal{M} \triangleq \left\{ x \in \mathcal{U} \subset \mathbf{E}^3 \middle| \, T_1(x) = t_{01} \right\}.$$

We are given that $f^{(2)}$ and $f^{(3)}$ span the tangent plane at each point of $\mathcal{M}$. We also know that the gradient of $T_1(x)$ is perpendicular to $\mathcal{M}$ so for all $x \in \mathcal{M}$ we have

$$\frac{\partial T_1}{\partial x} f^{(2)} = \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \\ f_3^{(2)} \end{bmatrix} \equiv 0$$

$$\frac{\partial T_1}{\partial x} f^{(3)} = \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1^{(3)} \\ f_2^{(3)} \\ f_3^{(3)} \end{bmatrix} \equiv 0.$$

As this is true for some neighborhood of t_{01} (say for all $-\varepsilon < t_{01} < \varepsilon$) these two equations must hold for all x in a neighborhood of $x_0 = \begin{bmatrix} x_{01} & x_{02} & x_{03} \end{bmatrix}^T$. Using Equation (1.73) of Chapter 1 the gradient of $\dfrac{\partial T_1}{\partial x} f^{(2)}$ is given by

$$\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f^{(2)} \right) = \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} \frac{\partial T_1}{\partial x_i} f_i^{(2)} \right) = (f^{(2)})^T \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial T_1}{\partial x}\frac{\partial f^{(2)}}{\partial x} \equiv \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

where $\dfrac{\partial^2 T_1}{\partial x^2}$ is the Hessian matrix. Multiply this last expression by $f^{(3)}$ on the right to obtain

$$\left(\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f^{(2)} \right) \right) f^{(3)} = (f^{(2)})^T \frac{\partial^2 T_1}{\partial x^2} f^{(3)} + \frac{\partial T_1}{\partial x}\frac{\partial f^{(2)}}{\partial x} f^{(3)} \equiv \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} f^{(3)} \equiv 0. \tag{4.29}$$

Similarly we have

$$\left(\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f^{(3)} \right) \right) f^{(2)} = (f^{(3)})^T \frac{\partial^2 T_1}{\partial x^2} f^{(2)} + \frac{\partial T_1}{\partial x}\frac{\partial f^{(3)}}{\partial x} f^{(2)} \equiv \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} f^{(2)} \equiv 0. \tag{4.30}$$

Subtracting (4.29) from (4.30) we have

$$\frac{\partial T_1}{\partial x}\frac{\partial f^{(3)}}{\partial x}f^{(2)} - \frac{\partial T_1}{\partial x}\frac{\partial f^{(2)}}{\partial x}f^{(3)} = \frac{\partial T_1}{\partial x}[f^{(2)}, f^{(3)}] \equiv 0.$$

This shows $[f^{(2)}, f^{(3)}]$ is normal to $\dfrac{\partial T_1}{\partial x}$. The set of vectors normal to $\dfrac{\partial T_1}{\partial x}$ is the tangent space of the manifold $\mathcal{M} \triangleq \left\{x \in \mathcal{U} \subset \mathbf{E}^3 |\ T_1(x) = t_{01}\right\}$ which is spanned by $f^{(2)}$ and $f^{(3)}$. That is, $[f^{(2)}, f^{(3)}] = \alpha_2 f^{(2)} + \alpha_3 f^{(3)}$.

■

Exercise 31 Show $(f^{(2)})^T \dfrac{\partial^2 T_1}{\partial x^2} f^{(3)} = (f^{(3)})^T \dfrac{\partial^2 T_1}{\partial x^2} f^{(2)}$.

We give a more general definition of involutive.

Definition 2 *Involutive Vector Fields*

Let $f^{(1)}, f^{(2)}, ..., f^{(k)}$ be k vector fields on $\mathcal{U} \subset \mathbf{E}^3$ satisfying

$$[f^{(i)}, f^{(j)}] = \sum_{i=1}^{k} \alpha_k f^{(k)}$$

for all $i, j = 1, ..., k$ and some scalar valued functions $\alpha_m(x)$ for $m = 1, 2, ..., k$. Then the set of vectors $\{f^{(1)}, f^{(2)}, ..., f^{(k)}\}$ is said to be involutive.

Theorem 6 *Frobenius Theorem - Third Version*

Let $f^{(1)}, f^{(2)}, f^{(3)}$ be three linearly independent vector fields on an open set $\mathcal{U} \subset \mathbf{E}^3$ with the pair $\{f^{(2)}, f^{(3)}\}$ involutive, i.e., $[f^{(2)}, f^{(3)}] = \alpha_2 f^{(2)} + \alpha_3 f^{(3)}$ on $\mathcal{U}$. For any $x_0 \in \mathcal{U}$ let $\phi_{t_1}(x_0)$ be the solution to $dx/dt_1 = f^{(1)}(x), x(0) = x_0, \phi_{t_2}(x_0')$ be the solution to $dx/dt_2 = f^{(2)}(x), x(0) = x_0'$, and $\phi_{t_3}(x_0'')$ be the solution to $dx/dt_3 = f^{(3)}(x), x(0) = x_0''$. We take these solutions to exist for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$ for some $\epsilon > 0$. Define the transformation

$$x(t_1, t_2, t_3) = S(t_1, t_2, t_3) \triangleq \phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0))).$$

Then $S(0, 0, 0) = x_0$ and for all $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$ this transformation is invertible with

$$\frac{\partial S}{\partial t_1}, \frac{\partial S}{\partial t_2} \in \Delta_p = \left\{r_1 f^{(1)}(S(t)) + r_2 f^{(2)}(S(t))|\ r_1, r_2 \in \mathbb{R}\right\}$$

In words,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1(t_1, t_2, t_3) \\ s_2(t_1, t_2, t_3) \\ s_3(t_1, t_2, t_3) \end{bmatrix}$$

is a coordinate map with inverse

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_1(x_1, x_2, x_3) \\ T_2(x_1, x_2, x_3) \\ T_3(x_1, x_2, x_3) \end{bmatrix}$$

and the two dimensional submanifolds of $\mathbf{E}^3$ defined by

$$\left\{x \in \mathcal{U} \subset \mathbf{E}^3 |\ T_1(x) = t_{01}\right\}$$

have $f^{(2)}, f^{(3)}$ spanning their tangent planes.

Proof. Omitted. ■

Remark 2 The distinction between this version of the Frobenius theorem and the two previous one is in the construction of the manifold such that $f^{(2)}, f^{(3)}$ span its tangent plane. Previously we constructed two new vectors $f^{*(2)}, f^{*(3)}$ as linear combinations of $f^{(2)}, f^{(3)}$ satisfying $[f^{*(2)}, f^{*(3)}] \equiv 0$. This version says the coordinate system can be constructed directly from $f^{(2)}$ and $f^{(3)}$.

Example 14 *Construction of a Coordinate System Using an Involutive Distribution*

Let

$$f^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, f^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, f^{(3)} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

defined on $\mathbf{E}^3$ with a vector space defined at $p = (x_1, x_2, x_3)$ by

$$\Delta_p = \left\{ r_2 f^{(2)}(x) + r_3 f^{(3)}(x) | \; r_2, r_3 \in \mathbb{R} \right\}.$$

Define the distribution Δ by

$$\Delta = \cup_{p \in \mathbf{E}^3} \Delta_p = \cup_{p \in \mathbf{E}^3} \left\{ r_2 f^{(2)}(p) + r_3 f^{(3)}(p) | \; r_2, r_3 \in \mathbb{R} \right\}.$$

A straightforward calculation shows that $[f^{(2)}, f^{(3)}] = f^{(2)}$ for all of $\mathbf{E}^3$ showing that the distribution is involutive.

The solution $dx/dt_1 = f^{(1)}(x), x(0) = x_0$ is

$$\phi_{t_1}(x_0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} + t_1 \end{bmatrix}.$$

The solution $dx/dt_2 = f^{(2)}(x), x(0) = x_0'$ is

$$\phi_{t_2}(x_0') = \begin{bmatrix} x_{01}' + t_2 \\ x_{02}' \\ x_{03}' \end{bmatrix}.$$

The solution $dx/dt_3 = f^{(3)}(x), x(0) = x_0''$ is

$$\phi_{t_3}(x_0'') = \begin{bmatrix} x_{01}'' e^{t_3} \\ x_{02}'' e^{t_3} \\ x_{03}'' e^{t_3} \end{bmatrix}.$$

The change of coordinates is then given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S(t_1, t_2, t_3) \triangleq \phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0))) = \begin{bmatrix} (x_{01} + t_2) e^{t_3} \\ x_{02} e^{t_3} \\ (x_{03} + t_1) e^{t_3} \end{bmatrix}.$$

Then $S(0,0,0) = x_0$ and

$$\frac{\partial S}{\partial t_2} = \begin{bmatrix} e^{t_3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2/x_{02} \\ 0 \\ 0 \end{bmatrix} = (x_2/x_{02}) f^{(2)} \in \Delta$$

$$\frac{\partial S}{\partial t_3} = \begin{bmatrix} (x_{01} + t_2) e^{t_3} \\ x_{02} e^{t_3} \\ (x_{03} + t_1) e^{t_3} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = f^{(3)} \in \Delta.$$

The inverse of the transformation is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \end{bmatrix} = \begin{bmatrix} x_3 x_{02}/x_2 - x_{03} \\ x_1 x_{02}/x_2 - x_{01} \\ \ln(x_2/x_{02}) \end{bmatrix}$$

where it is seen that $T(x_0) = 0$. The gradient of $T_1(x) = x_3 x_{02}/x_2 - x_{03}$ is

$$dT_1 = \begin{bmatrix} 0 & -x_3 x_{02}/x_2^2 & x_{02}/x_2 \end{bmatrix}.$$

We have

$$\begin{aligned} \mathcal{L}_{f^{(2)}}(T_1) &= \left\langle dT_1, f^{(2)} \right\rangle = \begin{bmatrix} 0 & -x_3 x_{02}/x_2^2 & x_{02}/x_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \\ \mathcal{L}_{f^{(3)}}(T_1) &= \left\langle dT_1, f^{(3)} \right\rangle = \begin{bmatrix} 0 & -x_3 x_{02}/x_2^2 & x_{02}/x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0. \end{aligned}$$

Further, as $f^{(1)}$ is linearly independent of $f^{(2)}$ and $f^{(3)}$ we must have $\mathcal{L}_{f^{(1)}}(T_1) \neq 0$. In fact,

$$\mathcal{L}_{f^{(1)}}(T_1) = \begin{bmatrix} 0 & -x_3 x_{02}/x_2^2 & x_{02}/x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_{02}/x_2.$$

There is a nice geometric picture when working in the t-coordinates. Let

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_1(x_1, x_2, x_3) \\ T_2(x_1, x_2, x_3) \\ T_3(x_1, x_2, x_3) \end{bmatrix}$$

be the inverse of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S(t_1, t_2, t_3) = \begin{bmatrix} s_1(t_1, t_2, t_3) \\ s_2(t_1, t_2, t_3) \\ s_3(t_1, t_2, t_3) \end{bmatrix}.$$

With t_1 held constant and varying t_2, t_3, $S(t_1, t_2, t_3)$ sweeps out a surface and by the Frobenius theorem we have

$$\frac{\partial S}{\partial t_1}, \frac{\partial S}{\partial t_2} \in \Delta_p = \left\{ r_2 f^{(2)}(p) + r_3 f^{(3)}(p) | \; r_2, r_3 \in \mathbb{R} \right\}$$

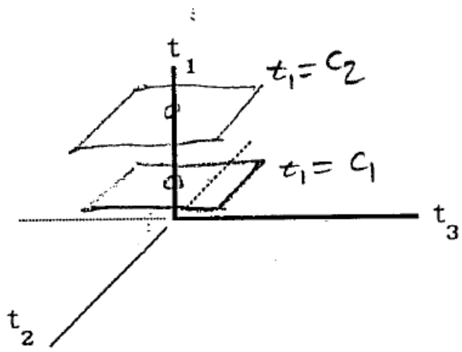
for all points $p = S(t_1, t_2, t_3)$. With $t_1 = 0$ the corresponding surface contains x_0. In general $\left\{ x \in \mathcal{U} \subset \mathbf{E}^3 | \; T_1(x) = t_{01} \right\}$ implicitly gives a surface in $\mathcal{U} \subset \mathbf{E}^3$ whose tangent plane spanned by $f^{(2)}, f^{(3)}$. As the gradient $dT_1 = \dfrac{\partial T_1}{\partial x}$ is normal to the surface, if follows that

$$\begin{aligned} \mathcal{L}_{f^{(2)}}(T_1) &= \left\langle dT_1, f^{(2)} \right\rangle \equiv 0 \\ \mathcal{L}_{f^{(3)}}(T_1) &= \left\langle dT_1, f^{(3)} \right\rangle \equiv 0 \end{aligned}$$

for all points on the surface. Further, as $f^{(1)}$ is linearly independent of $f^{(2)}$ and $f^{(3)}$ it follows that

$$\mathcal{L}_{f^{(1)}}(T_1) = \left\langle dT_1, f^{(1)} \right\rangle \neq 0$$

in a neighborhood of x_0.



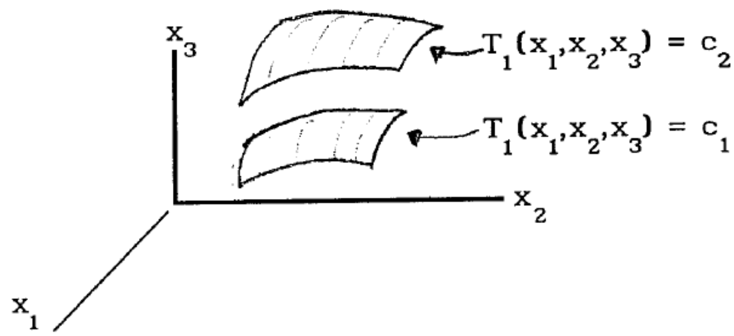


FIGURE 4.8.

Definition 3 *Lie Derivative of a Gradient*
Let h be defined on an open set $\mathcal{U} \subset \mathbb{R}^3$. Then

$$\mathcal{L}_f(dh) \triangleq d\mathcal{L}_f(h) = \frac{\partial}{\partial x}\mathcal{L}_f(h) = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} f\right) = \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} \frac{\partial h}{\partial x_i} f_i\right).$$

Exercise 32 Show that

$$\mathcal{L}_f(dh) = \left(\frac{\partial (dh)^T}{\partial x} f\right)^T + dh\frac{\partial f}{\partial x} = f^T \frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial x}\frac{\partial f}{\partial x}$$

where $\dfrac{\partial^2 h}{\partial x^2}$ is the Hessian of h.

Theorem 7 $\mathcal{L}_{[f,g]}(h) = \mathcal{L}_f(\mathcal{L}_g(h)) - \mathcal{L}_g(\mathcal{L}_f(h))$
Let f, g be two vector fields on an open subset of $\mathcal{U} \subset \mathbf{E}^3$ Then for any differentiable function h on $\mathcal{U}$ we have

$$\mathcal{L}_{[f,g]}(h) = \mathcal{L}_f(\mathcal{L}_g(h)) - \mathcal{L}_g(\mathcal{L}_f(h)).$$

Proof. This is the same as Theorem 1 of Chapter 1 on page 37. ∎

Corollary 2 *Leibniz Identity*
The following is an identity:

$$\mathcal{L}_f(\langle dh, g\rangle) = \langle \mathcal{L}_f(dh), g\rangle - \langle dh, [f,g]\rangle$$

Proof. Exercise. ∎

Definition 4 *Repeated Lie Brackets*
We have the following definition of repeated Lie brackets.

$$\begin{aligned}
ad_f^0 g &\triangleq g \\
ad_f^1 g &\triangleq [f,g] \\
ad_f^2 g &\triangleq [f, ad_f g] \\
ad_f^3 g &\triangleq [f, ad_f^2 g] \\
\vdots\ &\triangleq\ \vdots \\
ad_f^k g &\triangleq [f, ad_f^{k-1} g]
\end{aligned}$$

Example 15 Let $f(x) = Ax, A \in \mathbb{R}^{n\times n}$ and $g \in \mathbb{R}^n$, then $ad_f^k g = (-1)A^k b$.

Exercise 33 *Repeated Lie Derivatives*
It was shown in Theorem 7 that

$$\mathcal{L}_{[f,g]}(h) = \mathcal{L}_f(\mathcal{L}_g(h)) - \mathcal{L}_g(\mathcal{L}_f(h)). \tag{4.31}$$

Replacing g by $ad_f g$ in this expression results in

$$\mathcal{L}_{ad_f^2 g}(h) = \mathcal{L}_{[f,ad_f^1 g]}(h) = \mathcal{L}_f(\mathcal{L}_{[f,g]}(h)) - \mathcal{L}_{[f,g]}(\mathcal{L}_f(h)). \tag{4.32}$$

Next replace h by $\mathcal{L}_f(h)$ in (4.31) gives

$$\mathcal{L}_{[f,g]}(\mathcal{L}_f(h)) = \mathcal{L}_f(\mathcal{L}_g(\mathcal{L}_f(h))) - \mathcal{L}_g(\mathcal{L}_f(\mathcal{L}_f(h))). \tag{4.33}$$

Substituting this expression for $\mathcal{L}_{[f,g]}(\mathcal{L}_f(h))$ in (4.33) into the right side of (4.32) we obtain

$$\begin{aligned}\mathcal{L}_{ad_f^2 g}(h) &= \mathcal{L}_f(\mathcal{L}_{[f,g]}(h)) - (\mathcal{L}_f(\mathcal{L}_g(\mathcal{L}_f(h))) - \mathcal{L}_g(\mathcal{L}_f(\mathcal{L}_f(h))))\\ &= \mathcal{L}_f(\mathcal{L}_{[f,g]}(h)) - \mathcal{L}_f(\mathcal{L}_g(\mathcal{L}_f(h))) + \mathcal{L}_g(\mathcal{L}_f(\mathcal{L}_f(h))).\end{aligned} \tag{4.34}$$

Finally, substituting the expression for $\mathcal{L}_{[f,g]}(h)$ in (4.31) into (4.34) we have

$$\begin{aligned}\mathcal{L}_{ad_f^2 g}(h) &= \mathcal{L}_f\Big(\mathcal{L}_f(\mathcal{L}_g(h)) \ - \mathcal{L}_g(\mathcal{L}_f(h))\Big) - \mathcal{L}_f(\mathcal{L}_g(\mathcal{L}_f(h))) + \mathcal{L}_g(\mathcal{L}_f(\mathcal{L}_f(h)))\\ &= \mathcal{L}_f^2\mathcal{L}_g(h)) - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f(h) + \mathcal{L}_g\mathcal{L}_f^2(h).\end{aligned} \tag{4.35}$$

(a) Show that

$$\mathcal{L}_{ad_f^3 g}(h) = \mathcal{L}_{[f,ad_f^2 g]}(h) = \mathcal{L}_f^3\mathcal{L}_g(h)) - 3\mathcal{L}_f^2\mathcal{L}_g\mathcal{L}_f(h) + 3\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f^2(h) - \mathcal{L}_g\mathcal{L}_f^3(h). \tag{4.36}$$

(b) Use an induction argument to shown

$$\mathcal{L}_{ad_f^k g}(h) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\mathcal{L}_f^i\mathcal{L}_g\mathcal{L}_f^{k-i}(h) \tag{4.37}$$

where $\binom{k}{i} = \frac{k!}{(k-i)!i!}$.

4.5 The Modern Abstract Approach to Tangent Vectors and Lie Brackets

We now look at how the Lie bracket fits into the modern approach to tangent vectors. Consider the manifold $\mathbf{E}^4$, that is, Euclidean four-space with the Cartesian coordinate system. Let the following three vector fields on $\mathbf{E}^4$ given by

$$\mathbf{f}^{(1)} = \begin{bmatrix}-z_2\\ z_1\\ z_4\\ -z_3\end{bmatrix},\ \mathbf{f}^{(2)} = \begin{bmatrix}-z_3\\ -z_4\\ z_1\\ z_2\end{bmatrix},\ \mathbf{f}^{(3)} = \begin{bmatrix}-z_4\\ z_3\\ -z_2\\ z_1\end{bmatrix}.$$

The vector field

$$\mathbf{f}^{(4)} = \begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix}$$

points in the radial direction in $\mathbf{E}^4$ and is orthogonal to $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$, $\mathbf{f}^{(3)}$. This observation tell us that $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$, $\mathbf{f}^{(3)}$ are tangent to three-dimensional spherical submanifolds defined by

$$\mathbf{S}_r^3 \triangleq \{\mathbf{z} \in \mathbf{E}^4 | z_1^2 + z_2^2 + z_3^2 + z_4^4 = r^2\}.$$

If $r = 1$ then we write

$$\mathbf{S}^3 \triangleq \{\mathbf{z} \in \mathbf{E}^4 | z_1^2 + z_2^2 + z_3^2 + z_4^4 = 1\}.$$

Exercise 34 Show $[\mathbf{f}^{(1)}, \mathbf{f}^{(2)}] = 2\mathbf{f}^{(3)}$, $[\mathbf{f}^{(2)}, \mathbf{f}^{(3)}] = 2\mathbf{f}^{(1)}$, $[\mathbf{f}^{(3)}, \mathbf{f}^{(1)}] = 2\mathbf{f}^{(2)}$

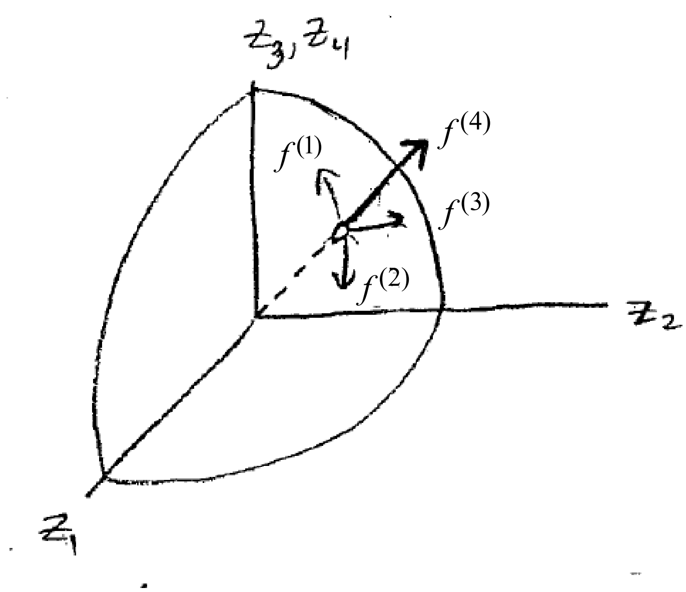


FIGURE 4.9. The submanifold $\mathbf{S}^3 = \{\mathbf{z} \in \mathbf{E}^4 |\ z_1^2 + z_2^2 + z_3^2 + z_4^4 = 1\}$

In $\mathbf{S}^3$ define the north pole to be the point $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$. A coordinate chart for the northern hemisphere of $\mathbf{S}^3$ is $\varphi^{-1} : \mathcal{D} \to \mathcal{U}$ given by

$$\varphi^{-1}(x_1, x_2, x_3) = \mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \sqrt{1 - (x_1^2 + x_2^2 + x_3^2} \end{bmatrix}$$

where $\mathcal{D} = \{x \in \mathbb{R}^3 |\ x_1^2 + x_2^2 + x_3^2 < 1\}$ and $\mathcal{U} \triangleq \{\mathbf{z} \in \mathbf{S}^3 |\ z_4 > 0\}$. In this northern hemisphere coordinate chart a basis for the tangent space is $\mathbf{T}_p(\mathbf{S}^3)$ is

$$\mathbf{z}_{x_1} = \frac{\partial \mathbf{z}}{\partial x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \dfrac{-x_1}{d(x)} \end{bmatrix}, \quad \mathbf{z}_{x_2} = \frac{\partial \mathbf{z}}{\partial x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \dfrac{-x_2}{d(x)} \end{bmatrix}, \quad \mathbf{z}_{x_3} = \frac{\partial \mathbf{z}}{\partial x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \dfrac{-x_3}{d(x)} \end{bmatrix}$$

where $d(x) = \sqrt{1-(x_1^2+x_2^2+x_3^2)}$. Note that

$$\mathbf{f}^{(1)}(\varphi^{-1}(x)) = \begin{bmatrix} -z_2 \\ z_1 \\ z_4 \\ -z_3 \end{bmatrix}\Bigg|_{\mathbf{z}=\varphi^{-1}(x)=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \sqrt{1-(x_1^2+x_2^2+x_3^2)} \end{bmatrix}} = \begin{bmatrix} -x_2 \\ x_1 \\ d(x) \\ -x_3 \end{bmatrix} = -x_2\mathbf{z}_{x_1} + x_1\mathbf{z}_{x_2} + d(x)\mathbf{z}_{x_3}$$

Similarly we have

$$\mathbf{f}^{(2)}(\varphi^{-1}(x)) = \begin{bmatrix} -z_3 \\ -z_4 \\ z_1 \\ z_2 \end{bmatrix}\Bigg|_{\mathbf{z}=\varphi^{-1}(x)=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \sqrt{1-(x_1^2+x_2^2+x_3^2)} \end{bmatrix}} = \begin{bmatrix} -x_3 \\ -d(x) \\ x_1 \\ x_2 \end{bmatrix} = -x_3\mathbf{z}_{x_1} - d(x)\mathbf{z}_{x_2} + x_1\mathbf{z}_{x_3}$$

$$\mathbf{f}^{(3)}(\varphi^{-1}(x)) = \begin{bmatrix} -z_4 \\ z_3 \\ -z_2 \\ z_1 \end{bmatrix}\Bigg|_{\mathbf{z}=\varphi^{-1}(x)=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \sqrt{1-(x_1^2+x_2^2+x_3^2)} \end{bmatrix}} = \begin{bmatrix} -d(x) \\ x_3 \\ -x_2 \\ x_1 \end{bmatrix} = -d(x)\mathbf{z}_{x_1} + x_3\mathbf{z}_{x_2} - x_2\mathbf{z}_{x_3}.$$

Considering $\mathbf{S}^3$ embedded in $\mathbf{E}^3$ and with h a differentiable function on $\mathbf{E}^4$ we write

$$\begin{aligned}
\left\langle dh, \mathbf{f}^{(1)} \right\rangle &= \begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} & \dfrac{\partial h}{\partial z_4} \end{bmatrix} \begin{bmatrix} -z_2 \\ z_1 \\ z_4 \\ -z_3 \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} & \dfrac{\partial h}{\partial z_4} \end{bmatrix} \left(-x_2\mathbf{z}_{x_1} + x_1\mathbf{z}_{x_2} + d(x)\mathbf{z}_{x_3} \right) \\
&= \begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} & \dfrac{\partial h}{\partial z_4} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{x_1} & \mathbf{z}_{x_2} & \mathbf{z}_{x_3} \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \\ d(x) \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} & \dfrac{\partial h}{\partial z_4} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \varphi^{-1}}{\partial x_1} & \dfrac{\partial \varphi^{-1}}{\partial x_2} & \dfrac{\partial \varphi^{-1}}{\partial x_3} \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \\ d(x) \end{bmatrix} \\
&= -x_2 \frac{\partial h \circ \varphi^{-1}}{\partial x_1} + x_1 \frac{\partial h \circ \varphi^{-1}}{\partial x_2} + \sqrt{1-(x_1^2+x_2^2+x_3^2)} \frac{\partial h \circ \varphi^{-1}}{\partial x_3}
\end{aligned}$$

These calculations motivated the new definition of a tangent vector given as follow: Let $h \in \mathcal{F}(\mathcal{U})$ be a differentiable function on $\mathcal{U} \triangleq \{\mathbf{z} \in \mathbf{S}^3 |\ z_4 > 0\} \subset \mathbf{S}^3$, that is, $h \circ \varphi^{-1}(x_1, x_2, x_3) = h(\varphi^{-1}(x_1, x_2, x_3))$ is a differentiable function on $\mathcal{D} = \left\{x \in \mathbb{R}^3 |\ x_1^2 + x_2^2 + x_3^2 < 1\right\}$. Then the tangent vector $\mathbf{f}^{(1)}$ above is replaced by the mapping $\mathbf{f}^{(1)} : \mathcal{F}(\mathcal{U}) \to \mathbb{R}$ given by

$$\mathbf{f}^{(1)} : h \to -x_2 \frac{\partial h \circ \varphi^{-1}}{\partial x_1} + x_1 \frac{\partial h \circ \varphi^{-1}}{\partial x_2} + \sqrt{1-(x_1^2+x_2^2+x_3^2)} \frac{\partial h \circ \varphi^{-1}}{\partial x_3} \tag{4.38}$$

or we write

$$\mathbf{f}^{(1)} = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + \sqrt{1-(x_1^2+x_2^2+x_3^2)} \frac{\partial}{\partial x_3}. \tag{4.39}$$

This last expression is bewildering since we have dropped the φ^{-1} and it is the $\dfrac{\partial \varphi^{-1}}{\partial x_i}$ which were our original notion of a tangent vector. However, in (4.39) the partial derivative operators $\dfrac{\partial}{\partial x_i}$ are respect to a particular coordinate system with coordinates (x_1, x_2, x_3) which is this example is the northern hemisphere. The components of the tangent vector $\mathbf{f}^{(1)}$ given by $(-x_2, x_1, \sqrt{1-(x_1^2+x_2^2+x_3^2)})$ are only valid in this coordinate system. Similarly, in the northern hemisphere coordinate chart, we have

$$\mathbf{f}^{(2)} = -x_3\frac{\partial}{\partial x_1} - \sqrt{1-(x_1^2+x_2^2+x_3^2)}\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} \tag{4.40}$$

$$\mathbf{f}^{(3)} = -\sqrt{1-(x_1^2+x_2^2+x_3^2)}\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}. \tag{4.41}$$

In terms of the components we have

$$f^{(1)} = \begin{bmatrix} -x_2 \\ x_1 \\ \sqrt{1-(x_1^2+x_2^2+x_3^2)} \end{bmatrix},\quad f^{(2)} = \begin{bmatrix} -x_3 \\ -\sqrt{1-(x_1^2+x_2^2+x_3^2)} \\ x_1 \end{bmatrix},\quad f^{(3)} = \begin{bmatrix} -\sqrt{1-(x_1^2+x_2^2+x_3^2)} \\ x_3 \\ -x_2 \end{bmatrix}$$

These components of the tangent vectors are in $\mathcal{D} = \left\{x \in \mathbb{R}^3 |\ x_1^2+x_2^2+x_3^2 < 1\right\}$.

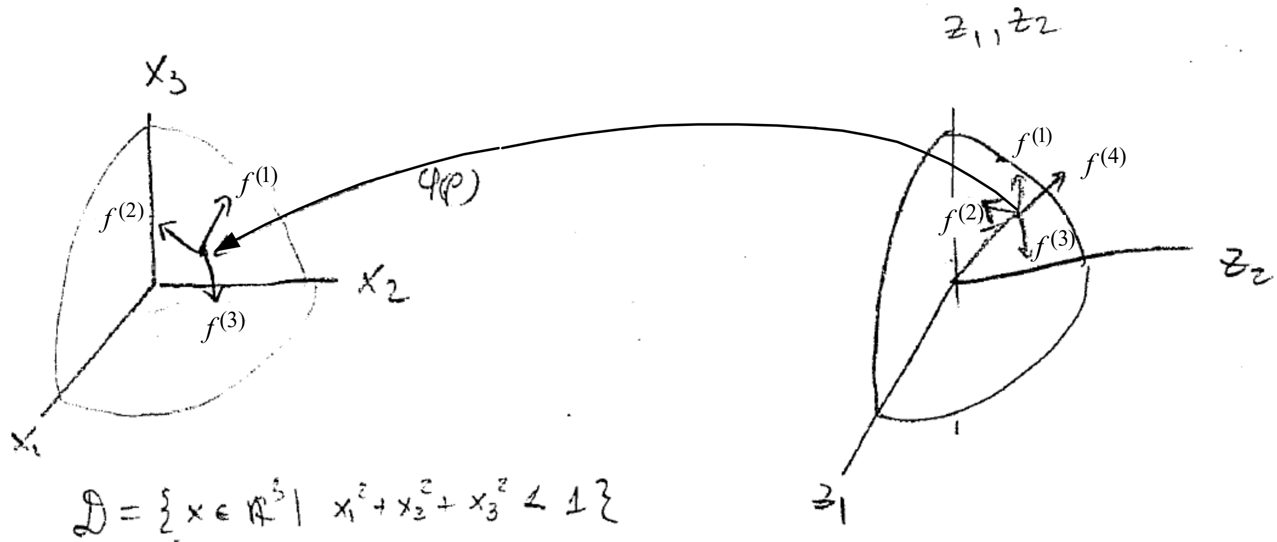


FIGURE 4.10. φ is a 1-1 and onto map from the northern hemisphere $\mathcal{U}$ of S^3 to $\mathcal{D} = \{x \in \mathbb{R}^3 |\ x_1^2+x_2^2+x_3^2 < 1\}$.

Exercise 35 Is $\mathbf{f}^{(4)}$ a linear combination of $\mathbf{z}_{x_1}, \mathbf{z}_{x_2}, \mathbf{z}_{x_3}$? Explain.

We now compute the Lie bracket of $[f^{(1)}, f^{(2)}]$ in the northern hemisphere coordinate chart. We have

$$\begin{aligned}
[f^{(1)}, f^{(2)}] &= \begin{bmatrix} 0 & 0 & -1 \\ x_1/d(x) & x_2/d(x) & x_3/d(x) \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -x_2 \\ x_1 \\ d(x) \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -x_1/d(x) & -x_2/d(x) & -x_3/d(x) \end{bmatrix}\begin{bmatrix} -x_3 \\ -d(x) \\ x_1 \end{bmatrix} \\
&= \begin{bmatrix} -d(x) \\ x_3 \\ -x_2 \end{bmatrix} - \begin{bmatrix} d(x) \\ -x_3 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} -d(x) \\ x_3 \\ -x_2 \end{bmatrix} = 2f^{(3)}.
\end{aligned}$$

Similarly we have $[f^{(1)}, f^{(3)}] = 2f^{(2)}$, $[f^{(2)}, f^{(3)}] = 2f^{(1)}$.

Why should we expect this? We now show that

$$[f^{(1)}, f^{(2)}] = 2f^{(3)} \Longleftrightarrow [\mathbf{f}^{(1)}, \mathbf{f}^{(2)}] = 2\mathbf{f}^{(3)}.$$

Recall the above discussion were we wrote

$$\begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} & \dfrac{\partial h}{\partial z_4} \end{bmatrix} \underbrace{\begin{bmatrix} -z_2 \\ z_1 \\ z_4 \\ -z_3 \end{bmatrix}}_{\mathbf{f}^{(1)}(z)} = \left(\begin{bmatrix} \dfrac{\partial h \circ \varphi^{-1}}{\partial x_1} & \dfrac{\partial h \circ \varphi^{-1}}{\partial x_2} & \dfrac{\partial h \circ \varphi^{-1}}{\partial x_3} \end{bmatrix} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \\ d(x) \end{bmatrix}}_{f^{(1)}} \right)_{|x=\varphi(z)}$$

With $\mathfrak{h} \triangleq h \circ \varphi^{-1}(x)$ we may rewrite this as

$$\mathcal{L}_{\mathbf{f}^{(1)}}(h) = \mathcal{L}_{f^{(1)}}(h \circ \varphi^{-1})_{|x=\varphi(z)} = \mathcal{L}_{f^{(1)}}(\mathfrak{h}) \circ \varphi.$$

Repeating this we have

$$\mathcal{L}_{\mathbf{f}^{(2)}}\left(\mathcal{L}_{\mathbf{f}^{(1)}}(h)\right) = \left(\mathcal{L}_{f^{(2)}}\left(\mathcal{L}_{f^{(1)}}(h \circ \varphi^{-1})\right)\right) \circ \varphi = \mathcal{L}_{f^{(2)}}\left(\mathcal{L}_{f^{(1)}}(\mathfrak{h})\right) \circ \varphi.$$

Then

$$\mathcal{L}_{\mathbf{f}^{(2)}}\left(\mathcal{L}_{\mathbf{f}^{(1)}}(h)\right) - \mathcal{L}_{\mathbf{f}^{(1)}}\left(\mathcal{L}_{\mathbf{f}^{(2)}}(h)\right) = \left(\mathcal{L}_{f^{(2)}}\left(\mathcal{L}_{f^{(1)}}(h \circ \varphi^{-1})\right) - \mathcal{L}_{f^{(1)}}\left(\mathcal{L}_{f^{(2)}}(h \circ \varphi^{-1})\right)\right) \circ \varphi$$

or

$$\mathcal{L}_{[\mathbf{f}^{(2)}, \mathbf{f}^{(1)}]}(h) = \mathcal{L}_{[f^{(2)}, f^{(1)}]}(\mathfrak{h}) \circ \varphi.$$

4.6 Problems

Problem 1 *Tangent Vectors as Derivations*

Let $\mathcal{D} \subset \mathbb{R}^n$ be an open subset and denote by $\mathcal{C}^\infty(\mathcal{D})$ the infinitely differentiable functions from $\mathcal{D}$ to $\mathbb{R}$. A *derivation* is a map D from $\mathcal{C}^\infty(\mathcal{D}) \to \mathbb{R}$ such that for any $h_1, h_2 \in \mathcal{C}^\infty(\mathcal{D})$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ it satisfies

(i) Linearity

$D(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 D(h_1) + \alpha_2 D(h_2)$

(ii) Product Rule

$D(h_1 h_2) = h_1 D(h_2) + h_2 D(h_1)$

(a) Let $\mathcal{U} \subset \mathbf{S}^2$ and $\varphi : \mathcal{D} \to \varphi(\mathcal{U})$ be a coordinate chart on $\mathbf{S}^2$. Let h be a function defined on $\mathcal{U}$ so that with $\varphi(p) = (x_1, x_2)$ we have $h(p) = h \circ \varphi^{-1}(x_1, x_2) = \mathfrak{h}(x_1, x_2)$ is defined on $\mathcal{D} = \varphi(\mathcal{U})$. Define $\dfrac{\partial}{\partial x_1}$: $\mathcal{C}^\infty(\mathbf{S}^2) \to \mathbb{R}$ by

$$\frac{\partial}{\partial x_1} : h \to \frac{\partial}{\partial x_1} h \circ \varphi^{-1}\Big|_{\varphi(p)} = \frac{\partial \mathfrak{h}}{\partial x_1}\Big|_{\varphi(p)} \in \mathbb{R}.$$

Show that this satisfies (i) and (ii), i.e., it is a derivation.

(b) Recall the modern definition of a tangent vector as a mapping $\mathbf{z}_p : \mathcal{C}^\infty(\mathbf{S}^2) \to \mathbb{R}$ given by

$$\mathbf{z}_p : h \to \mathbf{z}_p(h) \triangleq f_1(x_1, x_2)\, \frac{\partial}{\partial x_1} h \circ \varphi^{-1}\Big|_{\varphi(p)} + f_2(x_1, x_2)\, \frac{\partial}{\partial x_2} h \circ \varphi^{-1}\Big|_{\varphi(p)}$$

where $\varphi(p) = (x_1, x_2)$ and $f_1(x_1, x_2), f_2(x_1, x_2)$ are the *components* of the vector. Show that $\mathbf{z}_p$ is a derivation.

Also recall that we might "prefer" to look at this mapping as

$$\begin{aligned}\mathbf{z}_p(h) &= \begin{bmatrix} \dfrac{\partial h(z)}{\partial z_1} & \dfrac{\partial h(z)}{\partial z_2} & \dfrac{\partial h(z)}{\partial z_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \\ \dfrac{\partial z_3}{\partial x_1} & \dfrac{\partial z_3}{\partial x_2} \end{bmatrix}_{(x_{01}, x_{02})} \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \\ &= dh \begin{bmatrix} \mathbf{z}_{x_1} & \mathbf{z}_{x2} \end{bmatrix} \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}\end{aligned}$$

with $f_1(x_1, x_2)\mathbf{z}_{x_1} + f_2(x_1, x_2)\mathbf{z}_{x_2}$ the "tangent vector". However, this does not make sense as h is defined only on $\mathbf{S}^2$ (so $\partial h(z)/\partial z_1$, etc. are not defined) and $\mathbf{z}_{x_1}, \mathbf{z}_{x2}$ stick off of $\mathbf{S}^2$.

(c) Let $f = (f_1(x_1, x_2), f_2(x_1, x_2))$ and $g = (g_1(x_1, x_2), g_2(x_1, x_2))$ be the components of two vector fields defined on $\mathcal{D}$. With $\mathfrak{h} = h \circ \varphi^{-1}(x_1, x_2)$ and by Theorem 1 of Chapter 1 (page 37)

$$\mathcal{L}_f(\mathcal{L}_g(\mathfrak{h})) = g^T \frac{\partial^2 \mathfrak{h}}{\partial x^2} f + \frac{\partial \mathfrak{h}}{\partial x} \frac{\partial g}{\partial x} f.$$

Is this a derivation, i.e., does it satisfy (i) and (ii) above? Explain.

(d) Show that $D \triangleq \mathcal{L}_f \mathcal{L}_g - \mathcal{L}_g \mathcal{L}_f : \mathcal{C}^\infty(\mathbf{S}^2) \to \mathbb{R}$ given by

$$\mathfrak{h} \to \Big(\mathcal{L}_f(\mathcal{L}_g(\mathfrak{h})) \; - \mathcal{L}_g(\mathcal{L}_f(\mathfrak{h}))\Big)_{|\varphi(p) = (x_1, x_2)}$$

is a derivation. Equivalently, $\mathcal{L}_{[f,g]}(\mathfrak{h}) = \mathcal{L}_f(\mathcal{L}_g(\mathfrak{h})) - \mathcal{L}_g(\mathcal{L}_f(\mathfrak{h}))$ is a derivation.

Problem 2 *The Frobenius Theorem* [23]

Consider the following system of partial differential equations

$$\frac{\partial S}{\partial u_1} = f^{(1)}(u_1, u_2, s_1(u_1, u_2), s_2(u_1, u_2), s_3(u_1, u_2)) \tag{4.42}$$

$$\frac{\partial S}{\partial u_2} = f^{(2)}(u_1, u_2, s_1(u_1, u_2), s_2(u_1, u_2), s_3(u_1, u_2)) \tag{4.43}$$

where

$$S(u_1, u_2) = \begin{bmatrix} s_1(u_1, u_2) \\ s_2(u_1, u_2) \\ s_3(u_1, u_2) \end{bmatrix}$$

and

$$f^{(1)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} f_1^{(1)}(u_1, u_2, x_1, x_2, x_3) \\ f_2^{(1)}(u_1, u_2, x_1, x_2, x_3) \\ f_3^{(1)}(u_1, u_2, x_1, x_2, x_3) \end{bmatrix}, \quad f^{(2)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} f_1^{(2)}(u_1, u_2, x_1, x_2, x_3) \\ f_2^{(2)}(u_1, u_2, x_1, x_2, x_3) \\ f_3^{(2)}(u_1, u_2, x_1, x_2, x_3) \end{bmatrix}.$$

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^3$, $\mathcal{D}$ an open subset of $\mathbb{R}^2$. Suppose, given any point $x_0 \in \mathcal{U}$ and any $u_0 \in (u_{01}, u_{02}) \in \mathcal{D}$, there is a surface $S(u_1, u_2)$ satisfying (4.42) and (4.43) in some neighborhood of (u_{01}, u_{02}) with $S(u_{01}, u_{02}) = x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$.

(a) Show that

$$\frac{\partial f^{(1)}(u, x)}{\partial u_2} - \frac{\partial f^{(2)}(u, x)}{\partial u_1} + \frac{\partial f^{(1)}(u, x)}{\partial x} f^{(2)} - \frac{\partial f^{(2)}(u, x)}{\partial x} f^{(1)} \equiv 0 \tag{4.44}$$

for all $u \triangleq (u_1, u_2) \in \mathcal{D}$ and $x \triangleq (x_1, x_2, x_3) \in \mathcal{U}$.

(b) Let

$$f^{(1)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} x_1 - u_1 - u_2 \\ -x_2 + u_1 + u_2 \\ 0 \end{bmatrix}, \quad f^{(2)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} x_1^2 - u_1^2 - u_2^2 - 2u_1 - 2u_2 - 2u_1 u_2 \\ 1 \\ 0 \end{bmatrix}. \tag{4.45}$$

Are the integrability conditions (4.44) satisfied for these vector fields.

(c) Let

$$S(u_1, u_2) = \begin{bmatrix} u_1 + u_2 + 1 \\ u_1 + u_2 - 1 \\ x_{03} \end{bmatrix} \tag{4.46}$$

Does this satisfy the partial differential equations (4.42) (4.43) with $f^{(1)}, f^{(2)}$ given by (4.45)? Does (4.46) satisfy

$$\left(\frac{\partial f^{(1)}(u, x)}{\partial u_2} - \frac{\partial f^{(2)}(u, x)}{\partial u_1} + \frac{\partial f^{(1)}(u, x)}{\partial x} f^{(2)} - \frac{\partial f^{(2)}(u, x)}{\partial x} f^{(1)} \right)_{x = s(u_1, u_2)} \equiv 0 \text{ ?}$$

Is there any contradiction with your answer to part (b)? Explain.

Problem 3 *The Frobenius Theorem*

Consider the following system of partial differential equations

$$\frac{\partial S}{\partial u_1} = f^{(1)}(u_1, u_2, s_1(u_1, u_2), s_2(u_1, u_2), s_3(u_1, u_2)) \tag{4.47}$$
$$\frac{\partial S}{\partial u_2} = f^{(2)}(u_1, u_2, s_1(u_1, u_2), s_2(u_1, u_2), s_3(u_1, u_2)) \tag{4.48}$$

where

$$S(u_1, u_2) = \begin{bmatrix} s_1(u_1, u_2) \\ s_2(u_1, u_2) \\ s_3(u_1, u_2) \end{bmatrix}$$

and

$$f^{(1)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} f_1^{(1)}(u_1, u_2, x_1, x_2, x_3) \\ f_2^{(1)}(u_1, u_2, x_1, x_2, x_3) \\ f_3^{(1)}(u_1, u_2, x_1, x_2, x_3) \end{bmatrix}, \quad f^{(2)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} f_1^{(2)}(u_1, u_2, x_1, x_2, x_3) \\ f_2^{(2)}(u_1, u_2, x_1, x_2, x_3) \\ f_3^{(2)}(u_1, u_2, x_1, x_2, x_3) \end{bmatrix}.$$

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^3$, $\mathcal{D}$ an open subset of $\mathbb{R}^2$ and suppose that

$$\frac{\partial f^{(1)}}{\partial u_2} - \frac{\partial f^{(2)}}{\partial u_1} + \frac{\partial f^{(1)}}{\partial x} f^{(2)} - \frac{\partial f^{(2)}}{\partial x} f^{(1)} \equiv 0$$

for all $x \in \mathcal{U}$ and $u \in \mathcal{D}$.

Prove that, given any point $x_0 \in \mathcal{U}$ and any $u_0 \in (u_{01}, u_{02}) \in \mathcal{D}$, there is a surface $S(u_1, u_2)$ satisfying (4.47) and (4.48) in some neighborhood of (u_{01}, u_{02}) with $S(u_{01}, u_{02}) = x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$.

Hint: Mimic the proof in this chapter which was done for the case where the vector fields were "time invariant", that is, $f^{(1)}, f^{(2)}$ depend only x and not u.

Problem 4 *The Frobenius Theorem*

Consider the manifold $\mathbf{E}^4$, that is, Euclidean four-space with the Cartesian coordinate system. Let the following three vector fields on $\mathbf{E}^4$ given by

$$\mathbf{f}^{(1)} = \begin{bmatrix} -z_2 \\ z_1 \\ z_4 \\ -z_3 \end{bmatrix}, \quad \mathbf{f}^{(2)} = \begin{bmatrix} -z_3 \\ -z_4 \\ z_1 \\ z_2 \end{bmatrix}, \quad \mathbf{f}^{(3)} = \begin{bmatrix} -z_4 \\ z_3 \\ -z_2 \\ z_1 \end{bmatrix}.$$

(a) Is the set of vector fields $\{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}\}$ involutive? Are they orthogonal to each other?

(b) Let

$$\mathbf{f}^{(4)} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

be a fourth vector field. Is $\mathbf{f}^{(4)}$ orthogonal to the set of vectors $\{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}\}$?

(c) What does the Frobenius theorem guarantee in terms of the a surface whose tangent space spanned by $\{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}\}$. Can you explicitly find the surface? Hint: Think about the geometry of this problem instead of trying to crank out calculations. In particular, $\mathbf{f}^{(4)}$ points in the radial direction in $\mathbf{E}^4$.

(d) Use your answer in part (c) to explicitly find a coordinate transformation $x^* = T(x)$ such that in the new coordinates $x_1^* = T_1(x) = c$ (constant) corresponds to a surface in the x coordinates for which the tangent space is spanned by $\{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}\}$.

Problem 5 *The Lie Derivative of a Vector $\mathcal{L}_f g$*

Consider two vector fields f and g in $\mathcal{U} \subset \mathbf{E}^3$. Let $\phi^f(t,x) = \phi^f_t(x)$ and $\phi^g(t,x) = \phi^g_t(x)$ be the flows of f and g, respectively. That is,

$$\frac{d}{dt}\phi^f(t,x) = f(\phi^f(t,x)) \text{ with } \phi^f(0,x) = x$$
$$\frac{d}{dt}\phi^g(t,x) = g(\phi^g(t,x)) \text{ with } \phi^g(0,x) = x.$$

For each fixed t, $\phi^f(t,x)$ represents starting at x and moving in the direction specified by f for the time t to reach the point $x' \triangleq \phi^f(t,x)$. That is, for each fixed t, $\phi^f(t,\cdot) : \mathbf{E}^3 \to \mathbf{E}^3$ that takes x to $x' = \phi^f(t,x)$. Further, starting at $x' = \phi^f(t,x)$ and following the vector field f for a time $-t$ results in coming back to x, that is,

$$x = \phi^f(-t,x') = \phi^f(-t,\phi^f(t,x)).$$

Then

$$\frac{\partial}{\partial x}x = \frac{\partial}{\partial x}\phi^f(-t,\phi^f(t,x)) = \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\frac{\partial}{\partial x}\phi^f(t,x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Use this relationship to show that

$$\left(\frac{d}{dt}\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\right)_{t=0} = -\frac{\partial f(x)}{\partial x}. \tag{4.49}$$

(b) With

$$\left(\phi^f_{-t}\right)_* \triangleq \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}$$

define

$$\mathcal{L}_f g \triangleq \left(\frac{d}{dt}\left(\left(\phi^f_{-t}\right)_* g(\phi^f(t,x))\right)\right)_{t=0}$$

and show that

$$\mathcal{L}_f g = [f,g].$$

Remark A more enlightening way to evaluate the definition of $\mathcal{L}_f g$ is shown in [18] (page 61) as follows. Using Equation (4.49) a two term Taylor series expansion of

$$\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}$$

about $t=0$ is

$$\begin{aligned}\left(\phi^f_{-t}\right)_* \triangleq \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)} &\approx \left(\frac{\partial}{\partial x'}\phi^f(-0,x')\right)_{|x'=\phi^f(0,x)} + \left(\frac{d}{dt}\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\right)_{t=0} t \\ &= I_{3\times 3} - \frac{\partial f(x)}{\partial x}t.\end{aligned}$$

A two term Taylor series expansion of $g(\phi^f(t,x))$ about $t=0$ is

$$\begin{aligned}g(\phi^f(t,x)) \approx g(\phi^f(0,x)) + \left(\frac{d}{dt}g(\phi^f(t,x))\Big|_{t=0}\right)t &= g(\phi^f(0,x)) + \left(\frac{\partial g}{\partial x}_{|\phi^f(0,x)}\right)\left(\frac{d\phi^f(t,x))}{dt}\Big|_{t=0}\right)t \\ &= g(\phi^f(0,x)) + \frac{\partial g}{\partial x}\left(f(\phi^f(t,x))_{|t=0}\right)t \\ &= g(x)) + \left(\frac{\partial g}{\partial x}f(x)\right)t\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{L}_f g \triangleq \left(\frac{d}{dt}\left(\left(\phi^f_{-t}\right)_* g(\phi^f(t,x))\right)\right)_{t=0} &= \lim_{t\to 0}\frac{\left(I_{3\times 3} - \dfrac{\partial f(x)}{\partial x}t\right)\left(g(x) + \left(\dfrac{\partial g}{\partial x}f(x)\right)t\right) - g(x)}{t}\\
&= \lim_{t\to 0}\frac{g(x) + \left(\dfrac{\partial g}{\partial x}f(x)\right)t - \dfrac{\partial f(x)}{\partial x}t\left(g(x) + \left(\dfrac{\partial g}{\partial x}f(x)\right)t\right) - g(x)}{t}\\
&= \lim_{t\to 0}\left(\frac{\partial g}{\partial x}f(x) - \frac{\partial f(x)}{\partial x}\left(g(x) + \left(\frac{\partial g}{\partial x}f(x)\right)t\right)\right)\\
&= \frac{\partial g}{\partial x}f(x) - \frac{\partial f(x)}{\partial x}g(x).
\end{aligned}$$

Problem 6 *Jacobi Identify*
Let $f^{(1)}, f^{(2)}, f^{(3)}$ be three vector fields in $\mathbf{E}^3$.

(a) Show that

$$[[f^{(1)}, f^{(2)}], f^{(3)}] + [[f^{(2)}, f^{(3)}], f^{(1)}] + [[f^{(3)}, f^{(1)}], f^{(2)}] \equiv 0.$$

This is known as the *Jacobi identity.*

(b) Let f and g be vector fields in $\mathbf{E}^3$. Use the Jacobi identity to show that the conditions

$$[ad_f^k g, ad_f^\ell g] \equiv 0 \quad \text{for} \quad k + \ell = 0, 1, 2, 3, 4, 5$$

is equivalent to the conditions

$$[g, ad_f^k g] \equiv 0 \quad \text{for} \quad k = 1, 3, 5.$$

(c) Let f and g be vector fields in $\mathbf{E}^n$. Use the Jacobi identity to show that the conditions

$$[ad_f^k g, ad_f^\ell g] \equiv 0 \quad \text{for} \quad k + \ell = 0, 1, 2, ..., 2n-1$$

is equivalent to the conditions

$$[g, ad_f^k g] \equiv 0 \quad \text{for} \quad k = 1, 3, 5, ..., 2n-1.$$

Problem 7 *Lie Brackets and Integrability of Vector Fields*
Let $f^{(1)}, f^{(2)}$, and $f^{(3)}$ be three linearly independent fields on an open set $\mathcal{U} \subset \mathbf{E}^3$. Let $x_0 \in \mathcal{U}$ and define the map $S(t_1, t_2, t_3) \triangleq \phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0)))$ where the solution to $dx/dt_1 = f^{(1)}(x), x(0) = x_0$ is $\phi_{t_1}(x_0)$, the solution to $dx/dt_2 = f^{(2)}(x), x(0) = x_0'$ is $\phi_{t_2}(x_0')$ and the solution to $dx/dt_3 = f^{(3)}(x), x(0) = x_0''$ is $\phi_{t_3}(x_0'')$. Assume that $S(t_1, t_2, t_3) = \phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0)))$ exits for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$ for some $\epsilon > 0$. It should be clear that $S(0,0,0) = x_0$.

(a) Suppose $[f^{(2)}, f^{(3)}] \neq 0$ for $x \in \mathcal{U}$.

Does $\dfrac{\partial S}{\partial t_3} = f^{(3)}(x)_{|x=S(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

Does $\dfrac{\partial S}{\partial t_2} = f^{(2)}(x)_{|x=S(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

(b) Suppose $[f^{(2)}, f^{(3)}] = 0$ for $x \in \mathcal{U}$.

Does $\dfrac{\partial S}{\partial t_3} = f^{(3)}(x)_{|x=S(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

Does $\dfrac{\partial S}{\partial t_2} = f^{(2)}(x)_{|x=S(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

Problem 8 *Lie Brackets and Integrability of Vector Fields*

Let $f^{(1)}, f^{(2)}$, and $f^{(3)}$ be three linearly independent fields on an open set $\mathcal{U} \subset \mathbf{E}^3$. Let $x_0 \in \mathcal{U}$ and define the map $R(t_1, t_2, t_3) \triangleq \phi_{t_2}(\phi_{t_3}(\phi_{t_1}(x_0)))$ where the solution to $dx/dt_1 = f^{(1)}(x), x(0) = x_0$ is $\phi_{t_1}(x_0)$, the solution to $dx/dt_2 = f^{(2)}(x), x(0) = x_0'$ is $\phi_{t_2}(x_0')$ and the solution to $dx/dt_3 = f^{(3)}(x), x(0) = x_0''$ is $\phi_{t_3}(x_0'')$. Assume that $S(t_1, t_2, t_3) = \phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0)))$ exits for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$ for some $\epsilon > 0$. It should be clear that $R(0,0,0) = x_0$.

(a) Suppose $[f^{(2)}, f^{(3)}] \neq 0$ for $x \in \mathcal{U}$.

Does $\dfrac{\partial R}{\partial t_3} = f^{(3)}(x)_{|x=R(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

Does $\dfrac{\partial R}{\partial t_2} = f^{(2)}(x)_{|x=R(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

(b) Suppose $[f^{(2)}, f^{(3)}] = 0$ for $x \in \mathcal{U}$.

Does $\dfrac{\partial R}{\partial t_3} = f^{(3)}(x)_{|x=R(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

Does $\dfrac{\partial R}{\partial t_2} = f^{(2)}(x)_{|x=R(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

5

Transformation of Nonlinear Systems to Feedback Linearizing Form

5.1 SISO Nonlinear Control Systems

Consider the single-input single-output (SISO) nonlinear control system given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \\ g_4(x) \end{bmatrix}}_{g(x)} u \in \mathbb{R}^4. \tag{5.1}$$

Under what conditions does there exist an invertible transformation $x^* = T(x) : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$\begin{aligned} x_1^* &= T_1(x_1, x_2, x_3, x_4) \\ x_2^* &= T_2(x_1, x_2, x_3, x_4) \\ x_3^* &= T_3(x_1, x_2, x_3, x_4) \\ x_n^* &= T_n(x_1, x_2, x_3, x_4) \end{aligned}$$

such that in the x^* coordinates this nonlinear control system has the form

$$\frac{dx_1^*}{dt} = x_2^* \tag{5.2}$$

$$\frac{dx_2^*}{dt} = x_3^* \tag{5.3}$$

$$\frac{dx_3^*}{dt} = x_4^* \tag{5.4}$$

$$\frac{dx_4^*}{dt} = f_4^*(x_1^*, x_2^*, x_3^*, x_n^*) + g_4^*(x_1^*, x_2^*, x_3^*, x_n^*)u. \tag{5.5}$$

To find the conditions we proceed as follows. By the chain rule we have

$$\frac{dx_1^*}{dt} = \mathcal{L}_{f+gu}(T_1) = \mathcal{L}_f(T_1) + u\mathcal{L}_g(T_1) \tag{5.6}$$

$$\frac{dx_2^*}{dt} = \mathcal{L}_{f+gu}(T_2) = \mathcal{L}_f(T_2) + u\mathcal{L}_g(T_2) \tag{5.7}$$

$$\frac{dx_3^*}{dt} = \mathcal{L}_{f+gu}(T_3) = \mathcal{L}_f(T_3) + u\mathcal{L}_g(T_3) \tag{5.8}$$

$$\frac{dx_4^*}{dt} = \mathcal{L}_{f+gu}(T_4) = \mathcal{L}_f(T_4) + u\mathcal{L}_g(T_4). \tag{5.9}$$

We want Equations (5.6)-(5.9) to have the form of Equations (5.2)-(5.5) which requires

$$\begin{aligned} \mathcal{L}_f(T_1) &= T_2 \text{ and } \mathcal{L}_g(T_1) = 0 \\ \mathcal{L}_f(T_2) &= T_3 \text{ and } \mathcal{L}_g(T_2) = 0 \\ \mathcal{L}_f(T_3) &= T_4 \text{ and } \mathcal{L}_g(T_3) = 0 \\ \mathcal{L}_f(T_4) &= T_5 \text{ and } \mathcal{L}_g(T_4) \neq 0. \end{aligned}$$

This is the same as finding T_1 that satisfies

$$\begin{array}{ll} T_2 \triangleq \mathcal{L}_f(T_1) & \mathcal{L}_g(T_1) = 0 \\ T_3 \triangleq \mathcal{L}_f^2(T_1) & \mathcal{L}_g\mathcal{L}_f(T_1) = 0 \\ T_4 \triangleq \mathcal{L}_f^3(T_1) & \mathcal{L}_g\mathcal{L}_f^2(T_1) = 0 \end{array} \tag{5.10}$$

and

$$\mathcal{L}_g\mathcal{L}_f^3(T_1) \neq 0. \tag{5.11}$$

These conditions show that "only" T_1 needs to be found. However, these conditions involve T_1 and it derivatives up to order $n-1$. We next develop equivalent conditions which involve only the first order derivatives of T_1. In Exercise 33 of Chapter 4 (page 175) you were asked to show that

$$\begin{aligned} \mathcal{L}_{ad_f^1 g} &= \mathcal{L}_f\mathcal{L}_g - \mathcal{L}_g\mathcal{L}_f \\ \mathcal{L}_{ad_f^2 g} &= \mathcal{L}_g\mathcal{L}_f^2 - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f + \mathcal{L}_f^2\mathcal{L}_g \\ \mathcal{L}_{ad_f^3 g} &= \mathcal{L}_g\mathcal{L}_f^3 - 3\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f^2 + 3\mathcal{L}_f^2\mathcal{L}_g\mathcal{L}_f - \mathcal{L}_f^3\mathcal{L}_g. \end{aligned}$$

Using the right most column of (5.10) we have

$$\begin{aligned} \mathcal{L}_{ad_f^1 g}(T_1) &= \mathcal{L}_f\mathcal{L}_g(T_1) - \mathcal{L}_g\mathcal{L}_f(T_1) = 0 \\ \mathcal{L}_{ad_f^2 g}(T_1) &= \mathcal{L}_g\mathcal{L}_f^2(T_1) - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f(T_1) + \mathcal{L}_f^2\mathcal{L}_g(T_1) = 0 \\ \mathcal{L}_{ad_f^3 g}(T_1) &= \mathcal{L}_g\mathcal{L}_f^3(T_1) - 3\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f^2(T_1) + 3\mathcal{L}_f^2\mathcal{L}_g\mathcal{L}_f(T_1) - \mathcal{L}_f^3\mathcal{L}_g(T_1) = \mathcal{L}_g\mathcal{L}_f^3(T_1). \end{aligned}$$

Summarizing the conditions are

$$dT_1 \begin{bmatrix} g & ad_f g & ad_f^2 g & ad_f^3 g \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \beta(x) \end{bmatrix}$$

with $\beta(x) \neq 0$. The matrix

$$\mathcal{C} \triangleq \begin{bmatrix} g & ad_f g & ad_f^2 g & ad_f^3 g \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

plays the role of the controllability matrix.

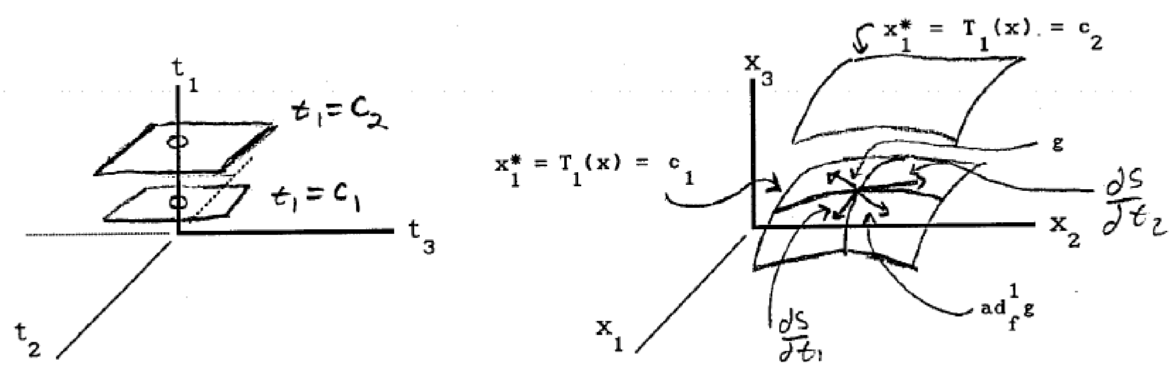


FIGURE 5.1. Illustrated for $n = 3$. $\{g, ad_f g\}$ and $\{\partial S/\partial t_3, \partial S/\partial t_2\}$ have the same spans at each point of the surface $t_1 =$ constant.

Feedback Linearization and Involutiveness

In the previous chapter we constructed the coordinate transformation as follows. Define the map from $\mathbb{R}^4 \to \mathbf{E}^4$ by

$$S(t_1, t_2, t_3, t_4) = \phi_{t_4}(\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))) \tag{5.12}$$

where $\varphi_{t_1}(x_0)$ is the solution to $dx/dt_1 = ad_f^3 g(x)$ with $x(0) = x_0$, $\varphi_{t_2}(x_0)$ is the solution to $dx/dt_2 = ad_f^2 g(x)$ with $x(0) = x_0'$, $\varphi_{t_3}(x_0)$ is the solution to $dx/dt_3 = ad_f g$ with $x(0) = x_0''$ and, finally, $\varphi_{t_4}(x_0)$ is the solution to $dx/dt_4 = g$ with $x(0) = x_0'''$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = S(t_1,t_2,t_3,t_4) = \begin{bmatrix} s_1(t_1,t_2,t_3,t_4) \\ s_2(t_1,t_2,t_3,t_4) \\ s_3(t_1,t_2,t_3,t_4) \\ s_4(t_1,t_2,t_3,t_4) \end{bmatrix} \tag{5.13}$$

As the set $\{g, ad_f g, ad_f^2 g\}$ is involutive we know that for all (t_1,t_2,t_3,t_4) in a neighborhood of $(0,0,0,0)$ that

$$\frac{\partial S}{\partial t_4}, \frac{\partial S}{\partial t_3}, \frac{\partial S}{\partial t_2} \in \Delta_{x=S(t_1,t_2,t_3,t_4)} \triangleq \left\{ r_1 g(x) + r_2 ad_f g(x) + r_3 ad_f^2 g |\ x = S(t_1,t_2,t_3,t_4), \text{ and } r_1, r_2, r_3 \in \mathbb{R} \right\}.$$

That is, for any *fixed* t_1, varying t_2, t_3, t_4 sweeps out a three-dimensional surface with any tangent to this surface being a linear combination of $g(x)_{|x=S(t_1,t_2,t_3,t_4)}$, $ad_f g(x)_{|x=S(t_1,t_2,t_3,t_4)}$, and $ad_f^2 g(x)_{|x=S(t_1,t_2,t_3,t_4)}$. Inverting the transformation (5.13) we have

$$\begin{aligned} t_1 &= T_1(x) \\ t_2 &= T_2(x) \\ t_3 &= T_3(x) \\ t_4 &= T_4(x). \end{aligned}$$

Consequently for any fixed t_1 the tangent vectors to the three-dimensional surface

$$\left\{ x \in \mathbb{R}^4 |\ T_1(x) = t_1 \right\}$$

are linear combinations of $g, ad_f g$, and $ad_f^2 g$. We also know that the gradient

$$dT_1 = \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} & \dfrac{\partial T_1}{\partial x_4} \end{bmatrix}$$

is normal to this surface so

$$dT_1 \begin{bmatrix} g & ad_f g & ad_f^2 g \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \tag{5.14}$$

We showed above that these conditions (4.14) are equivalent to the conditions (5.14), i.e., $\mathcal{L}_g(T_1) = 0, \mathcal{L}_g\mathcal{L}_f(T_1) = 0, \mathcal{L}_g\mathcal{L}_f^2(T_1) = 0$.

Where did the involutiveness condition come in? We find the components of the gradient

$$dT_1 = \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} & \dfrac{\partial T_1}{\partial x_4} \end{bmatrix}$$

by computing $\omega_1(x), \omega_2(x), \omega_3(x), \omega_4(x)$ that satisfy

$$\begin{bmatrix} \omega_1(x) & \omega_2(x) & \omega_3(x) & \omega_4(x) \end{bmatrix} \begin{bmatrix} g & ad_f g & ad_f^2 g \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \tag{5.15}$$

However, it is not enough to just specify the quantities $\omega_1(x), \omega_2(x), \omega_3(x), \omega_4(x)$ that satisfy (5.15). The quantities $\omega_1(x), \omega_2(x), \omega_3(x), \omega_4(x)$ must also satisfy

$$\begin{aligned}
\frac{\partial^2 T_1}{\partial x_2 \partial x_1} &= \frac{\partial \omega_1}{\partial x_2} = \frac{\partial \omega_2}{\partial x_1} = \frac{\partial^2 T_1}{\partial x_1 \partial x_2} \\
\frac{\partial^2 T_1}{\partial x_3 \partial x_1} &= \frac{\partial \omega_1}{\partial x_3} = \frac{\partial \omega_3}{\partial x_1} = \frac{\partial^2 T_1}{\partial x_1 \partial x_3} \\
\frac{\partial^2 T_1}{\partial x_4 \partial x_1} &= \frac{\partial \omega_1}{\partial x_4} = \frac{\partial \omega_4}{\partial x_1} = \frac{\partial^2 T_1}{\partial x_1 \partial x_4} \\
\frac{\partial^2 T_1}{\partial x_3 \partial x_2} &= \frac{\partial \omega_2}{\partial x_3} = \frac{\partial \omega_3}{\partial x_2} = \frac{\partial^2 T_1}{\partial x_2 \partial x_3} \\
\frac{\partial^2 T_1}{\partial x_4 \partial x_2} &= \frac{\partial \omega_2}{\partial x_4} = \frac{\partial \omega_4}{\partial x_2} = \frac{\partial^2 T_1}{\partial x_2 \partial x_4} \\
\frac{\partial^2 T_1}{\partial x_4 \partial x_3} &= \frac{\partial \omega_3}{\partial x_4} = \frac{\partial \omega_4}{\partial x_3} = \frac{\partial^2 T_1}{\partial x_3 \partial x_4}.
\end{aligned}$$

The involutiveness of $\{g, ad_f g, ad_f^2 g\}$ is necessary and sufficient for a T_1 to exist satisfying these conditions.

Example 1 *Linear Control System*

Let $f(x) = Ax, g(x) = b$ with $A \in \mathbb{R}^{4\times 4}, b \in \mathbb{R}^4$ where simple computations show that $ad_f^k g = (-1)^k A^k b$ for $k = 0, 1, 2, 3$. Then the conditions for the existence of the $T_1(x)$ is

$$dT_1 \begin{bmatrix} b & -Ab & A^2 b & -A^3 b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}$$

where we have chosen $\beta(x) = -1$. This is, of course, equivalent to

$$dT_1 \underbrace{\begin{bmatrix} b & Ab & A^2 b & A^3 b \end{bmatrix}}_{\mathcal{C}} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

With the pair (A, b) controllable, let

$$q \triangleq \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}$$

which is the last row of $\mathcal{C}^{-1}$. We then have

$$\begin{aligned}
x_1^* &= T_1(x) = qx \\
x_2^* &= T_2(x) = qAx \\
x_3^* &= T_3(x) = qA^2 x \\
x_4^* &= T_4(x) = qA^3 x
\end{aligned}$$

or

$$x^* = \underbrace{\begin{bmatrix} q \\ qA \\ qA^2 \\ qA^3 \end{bmatrix}}_{\mathbf{T}} x \triangleq \mathbf{T}x.$$

In the x^* coordinate system we have

$$\begin{aligned}
\frac{dx_1^*}{dt} &= q(Ax + bu) = qAx = x_2^* \\
\frac{dx_2^*}{dt} &= qA(Ax + bu) = qA^2 x = x_2^* \\
\frac{dx_3^*}{dt} &= qA^2(Ax + bu) = qA^3 x + qA^2 bu = x_4^* \\
\frac{dx_4^*}{dt} &= qA^3(Ax + bu) = qA^4 x + qA^3 bu = qA^4 x + u.
\end{aligned}$$

With

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix} \triangleq -qA^4$$

this becomes

$$\frac{dx_1}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} x^* + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

Example 2 *Series Connected DC Motor*

In Chapter 1 we consider the series connected DC motor model given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}}_{p} \tau_L$$

where $x_1 = \theta, x_2 = \omega, x_3 = i$, and $u = V_S/L$.

$$ad_f g = [f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = -\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2c_3 x_3 \\ 0 & -c_3 x_3 & -c_2 - c_3 x_2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2c_3 x_3 \\ c_2 + c_3 x_2 \end{bmatrix}$$

$$\begin{aligned} ad_f^2 g = [f, ad_f g] &= \frac{\partial ad_f g}{\partial x} f - \frac{\partial f}{\partial x} ad_f g \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2c_3 \\ 0 & c_2 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2c_3 x_3 \\ 0 & -c_3 x_3 & -c_2 - c_3 x_2 \end{bmatrix}\begin{bmatrix} 0 \\ -2c_3 x_3 \\ c_2 + c_3 x_2 \end{bmatrix} \\ &= \begin{bmatrix} 2c_3 x_3 \\ 0 \\ (c_2 + c_3 x_2)^2 - 2c_3^2 x_3^2 + c_1 c_2 x_3^2 \end{bmatrix} \end{aligned}$$

Then

$$\begin{aligned} \mathcal{C} &\triangleq \begin{bmatrix} g & ad_f g & ad_f^2 g \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 2c_3 x_3 \\ 0 & -2c_3 x_3 & 0 \\ 1 & c_2 + c_3 x_2 & (c_2 + c_3 x_2)^2 - 2c_3^2 x_3^2 + c_1 c_2 x_3^2 \end{bmatrix} \in \mathbb{R}^{4\times 4} \end{aligned}$$

and

$$\det \mathcal{C} = 4c_3^2 x_3^2.$$

We need to solve

$$dT_1 \begin{bmatrix} g & ad_f g & ad_f^2 g \end{bmatrix} = \begin{bmatrix} 0 & 0 & \beta(x) \end{bmatrix}.$$

Check the involutiveness of the two vectors $\{g, ad_f g\}$. Computing

$$[g, ad_f g] = \frac{\partial ad_f g}{\partial x} g - \frac{\partial g}{\partial x} ad_f g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2c_3 \\ 0 & c_3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2c_3 \\ 0 \end{bmatrix} = \frac{1}{x_3} ad_f g - \frac{c_2 + c_3 x_2}{x_3} g$$

shows that the pair $\{g, ad_f g\}$ is involutive for $x_3 = i \neq 0$.

To obtain T_1 we solve

$$\begin{aligned}
dT_1 &= \begin{bmatrix} 0 & 0 & \beta(x) \end{bmatrix} \mathcal{C}^{-1} \\
&= \begin{bmatrix} 0 & 0 & \beta(x) \end{bmatrix} \frac{1}{4c_3^2x_3^2} \begin{bmatrix} -2c_2^2c_3x_3 - 4c_2c_3^2x_2x_3 - 2c_1c_2c_3x_3^3 - 2c_3^3x_2^2x_3 + 4c_3^3x_3^3 & 2x_2x_3c_3^2 + 2c_2x_3c_3 & 4c_3^2x_3^2 \\ 0 & -2c_3x_3 & 0 \\ 2c_3x_3 & 0 & 0 \end{bmatrix} \\
&= \beta(x) \begin{bmatrix} \dfrac{1}{2c_3x_3} & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Choosing $\beta(x) = 2c_3x_3$ we have

$$dT_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned}
x_1^* &= T_1(x) = x_1 \\
x_2^* &= T_2(x) = \mathcal{L}_f(T_1) = \langle dT_1, f \rangle = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ c_1x_3^2 \\ -c_2x_3 - c_3x_3x_2 \end{bmatrix} = x_2 \\
x_3^* &= T_3(x) = \mathcal{L}_f(T_2) = \langle dT_2, f \rangle = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ c_1x_3^2 \\ -c_2x_3 - c_3x_3x_2 \end{bmatrix} = c_1x_3^2.
\end{aligned}$$

or

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_2(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ c_1x_3^2 \end{bmatrix}.$$

will transform the equations of the series connected DC motor into a form where the nonlinearities can be canceled out by feedback. Explicitly we compute

$$\begin{aligned}
\frac{dx_1^*}{dt} &= \mathcal{L}_{f+gu+p\tau_L}(T_1) = \mathcal{L}_f(T_1) = x_2 = x_2^* \\
\frac{dx_2^*}{dt} &= \mathcal{L}_{f+gu+p\tau_L}(T_2) = \mathcal{L}_f(T_2) + \tau_L\mathcal{L}_p(T_2) = -c_2x_3 - c_3x_3x_2 - \tau_L/J. = x_2^* - \tau_L/J \\
\frac{dx_3^*}{dt} &= \mathcal{L}_{f+gu+p\tau_L}(T_3) = \mathcal{L}_f(T_3) + u\mathcal{L}_g(T_3) + \tau_L\mathcal{L}_p(T_2) = -2c_1c_2x_3^2 - 2c_1c_3x_2x_3^2 + 2c_1x_3u - \tau_L/J.
\end{aligned}$$

More succinctly we have

$$\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^* \\
\frac{dx_2^*}{dt} &= x_3^* - \tau_L/J \\
\frac{dx_3^*}{dt} &= \underbrace{-2c_1c_2x_3^2 - 2c_1c_3x_2x_3^2}_{a(x)} + \underbrace{2c_1x_3}_{b(x)}u.
\end{aligned}$$

Exercise 36 In the previous example suppose we chose $\beta(x) = 1$ to solve

$$dT_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathcal{C}^{-1} = \begin{bmatrix} \dfrac{1}{2c_3x_3} & 0 & 0 \end{bmatrix}.$$

Would this work? Hint: With $\begin{bmatrix} \omega_1(x) & \omega_2(x) & \omega_3(x) \end{bmatrix} \triangleq \begin{bmatrix} \dfrac{1}{2c_3x_3} & 0 & 0 \end{bmatrix}$, does $\dfrac{\partial \omega_1}{\partial x_3} = \dfrac{\partial \omega_3}{\partial x_3}$?

Example 3 *Series Connected DC Motor*

Now suppose we want to find $T_1(x)$ for the series connected DC motor by computing

$$S(t_1,t_2,t_3) = \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))).$$

$\varphi_{t_1}(x_0)$ is the solution to

$$\frac{dx}{dt_1} = ad_f^2 g = \begin{bmatrix} 2c_3x_3 \\ 0 \\ (c_2+c_3x_2)^2 - 2c_3^2x_3^2 + c_1c_2x_3^2 \end{bmatrix} \text{ with } x(0) = x_0.$$

$\varphi_{t_2}(x_0)$ is the solution to

$$\frac{dx}{dt_2} = ad_f^1 g = \begin{bmatrix} 0 \\ -2c_3x_3 \\ c_2+c_3x_2 \end{bmatrix} \text{ with } x(0) = x_0'.$$

$\varphi_{t_3}(x_0)$ is the solution to

$$\frac{dx}{dt_3} = g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ with } x(0) = x_0''.$$

After these computations we would then have to invert $x = S(t_1,t_2,t_3) = \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))$ to obtain $T_1(x)$. Good luck with all that! However there is a way to use this method. Simply choose two linearly independent vectors that span the same space as $\{g, ad_f^1 g\}$. It is straightforward to see that

$$span\{g, ad_f^1 g\} = span\left\{ f^{(3)} \triangleq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, f^{(2)} \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

for $x_3 \neq 0$. Further, instead of using $ad_f^2 g$ we just use a vector field which is normal to $\{g, ad_f^1 g\}$. We choose

$$f^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The solution to $dx/dt_1 = f^{(1)}$ with $x(0) = x_0$ is

$$\varphi_{t_1}(x_0) = \begin{bmatrix} x_{01}+t_1 \\ x_{02} \\ x_{03} \end{bmatrix}.$$

The solution to $dx/dt_2 = f^{(2)}$ with $x(0) = x_0'$

$$\varphi_{t_2}(x_0) = \begin{bmatrix} x_{01}' \\ x_{02}'+t_2 \\ x_{03}' \end{bmatrix}.$$

The solution to $dx/dt_3 = f^{(3)}$ with $x(0) = x_0''$ is

$$\varphi_{t_3}(x_0) = \begin{bmatrix} x_{01}'' \\ x_{02}'' \\ x_{03}''+t_3 \end{bmatrix}.$$

Then

$$x = S(t_1,t_2,t_3) = \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))) = \begin{bmatrix} x_{01}+t_1 \\ x_{02}+t_2 \\ x_{03}+t_3 \end{bmatrix}$$

with inverse

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = T(x_1, x_2, x_3) = \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \\ x_3 - x_{03} \end{bmatrix}.$$

We set

$$\begin{aligned} x_1^* &\triangleq T_1(x) = x_1 - x_{01} \\ x_2^* &\triangleq \mathcal{L}_f T_1(x) \\ x_3^* &\triangleq \mathcal{L}_f^2 T_1(x). \end{aligned}$$

5.2 MIMO Nonlinear Control Systems

We now look at finding feedback linearizing transformations for multi-input nonlinear control systems. To do this we first need to look at the structure of multi-input linear time-invariant systems. Let

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}. \tag{5.16}$$

The input matrix $B \in \mathbb{R}^{n\times m}$ is assumed to be of full rank m ($m \le n$). The system is also assumed to be controllable, that is, the controllability matrix defined by

$$\mathcal{C} \triangleq \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times mn} \tag{5.17}$$

has rank n.

Definition 1 *r Indices*
Let

$$r_0 = rank[B] = m \tag{5.18}$$

and, for $j = 1, 2, ..., n-1$, set

$$r_j = rank \begin{bmatrix} B & AB & \cdots & A^jB \end{bmatrix} - rank \begin{bmatrix} B & AB & \cdots & A^{j-1}B \end{bmatrix}. \tag{5.19}$$

Exercise 37 Show that $0 \le r_j \le m$ and $\sum_{j=0}^{n-1} r_j = n$.

Example 4 *r Indices*
Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A^2B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^3B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so

$$\begin{aligned} r_0 &= rank[B] = 2 \\ r_1 &= rank \begin{bmatrix} B & AB \end{bmatrix} - rank[B] = 3 - 2 = 1 \\ r_2 &= rank \begin{bmatrix} B & AB & A^2B \end{bmatrix} - rank \begin{bmatrix} B & AB \end{bmatrix} = 4 - 3 = 1 \\ r_3 &= rank \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} - rank \begin{bmatrix} B & AB & A^2B \end{bmatrix} = 4 - 4 = 0 \end{aligned}$$

Example 5 *r Indices*
Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^2B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^3B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so

$$\begin{aligned} r_0 &= rank[B] = 2 \\ r_1 &= rank\begin{bmatrix} B & AB \end{bmatrix} - rank[B] = 4 - 2 = 2 \\ r_2 &= rank\begin{bmatrix} B & AB & A^2B \end{bmatrix} - rank\begin{bmatrix} B & AB \end{bmatrix} = 4 - 4 = 0 \\ r_3 &= rank\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} - rank\begin{bmatrix} B & AB & A^2B \end{bmatrix} = 4 - 4 = 0 \end{aligned}$$

Definition 2 *Controllability Indices*
Let r_j for $j = 1, 2, ..., n-1$ be the *r indices* for the controllable linear time-invariant system

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}.$$

It is assumed $rank[B] = m$. The *controllability indices* $\kappa_1, \kappa_2, ..., \kappa_m$ for this system are defined according to

$$\begin{aligned} \kappa_1 &\triangleq (\text{Number of } r_j \text{ greater than or equal to 1}) \\ \kappa_2 &\triangleq (\text{Number of } r_j \text{ greater than or equal to 2}) \\ &\vdots \\ \kappa_m &\triangleq (\text{Number of } r_j \text{ greater than or equal to m}). \end{aligned}$$

As the r indices satisfy $r_j \leq m$ for all j there can be no more than m controllability indices. Also

$$\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_m$$

and, as $\sum_{j=0}^{n-1} r_j = n$, we have

$$\sum_{i=1}^{n} \kappa_i = n.$$

Example 6 *Controllability Indices (Example 4 continued)*
In Example 4 the r indices were $r_0 = 2, r_1 = 1, r_2 = 1$, and $r_3 = 0$. Then

$$\begin{aligned} \kappa_1 &\triangleq (\text{Number of } r_j \text{ greater than or equal to 1}) = 3 \\ \kappa_2 &\triangleq (\text{Number of } r_j \text{ greater than or equal to 2}) = 1. \end{aligned}$$

Note that $\kappa_1 + \kappa_2 = 4$. As illustrated below, doing the sum $\sum_{j=0}^{3} r_j = 4$ can be seen as adding up the columns of the table first and then adding up the rows. On the other hand we can view $\sum_{i=1}^{2} \kappa_i = 4$ can be viewed as adding the rows of the table first and then the columns.

	r_0	r_1	r_2	r_3
κ_1	1	1	1	0
κ_2	1	0	0	0

Example 7 *Controllability Indices (Example 5 continued)*
In Example 5 the r indices were $r_0 = 2, r_1 = 2, r_2 = 0$, and $r_3 = 0$. Then

$$\kappa_1 \triangleq (\text{Number of } r_j \text{ greater than or equal to } 1) = 2$$
$$\kappa_2 \triangleq (\text{Number of } r_j \text{ greater than or equal to } 2) = 2$$

Note that $\kappa_1 + \kappa_2 = 4$. Again the sum $\sum_{j=0}^{3} r_j = 4$ can be seen as adding up the columns of the table first and then adding up the rows. On the other hand we can view $\sum_{i=1}^{2} \kappa_i = 4$ can be viewed as adding the rows of the table first then the rows.

	r_0	r_1	r_2	r_3
κ_1	1	1	0	0
κ_2	1	1	0	0

We next explain that the controllability indices are *invariant* under statespace transformations, input transformations, and state feedback. To do so, once again consider the controllable linear time-invariant system

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$$

with $rank[B] = m$. Using the state feedback

$$u = -Kx + v, \quad K \in \mathbb{R}^{m\times n}$$

the system becomes

$$\frac{dx}{dt} = (A - BK)x + Bv.$$

Under the statespace transformation $x^* = Tx$ we then have

$$\frac{d}{dt}x^* = T(A - BK)T^{-1}x^* + TBv.$$

Finally, under a change of input variables $v = Uv^*$ the system is given by

$$\frac{d}{dt}x^* = T(A - BK)T^{-1}x^* + TBUv^*.$$

With this background we can now state the following theorem.

Theorem 1 *Brunovsky Canonical Form* [24][25]
Consider the controllable linear time-invariant

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m} \tag{5.20}$$

with $rank[B] = m$. With $T \in \mathbb{R}^{n\times n}, U \in \mathbb{R}^{m\times m}$ nonsingular matrices and $K \in \mathbb{R}^{m\times n}$ the controllability indices of the pair

$$(A^*, B^*) \triangleq (T(A - BK)T^{-1}, TBU) \tag{5.21}$$

are the same as the controllability indices of the pair (A, B).
Further, there exist nonsingular transformations $T^* \in \mathbb{R}^{n\times n}$ and $U^* \in \mathbb{R}^{m\times m}$ and a feedback matrix $K^* \in \mathbb{R}^{m\times n}$ such that the system

$$\frac{dx^*}{dt} = A^*x^* + B^*v^* \tag{5.22}$$

with $x^* = Tx, v^* = U^{-1}v$ has the form

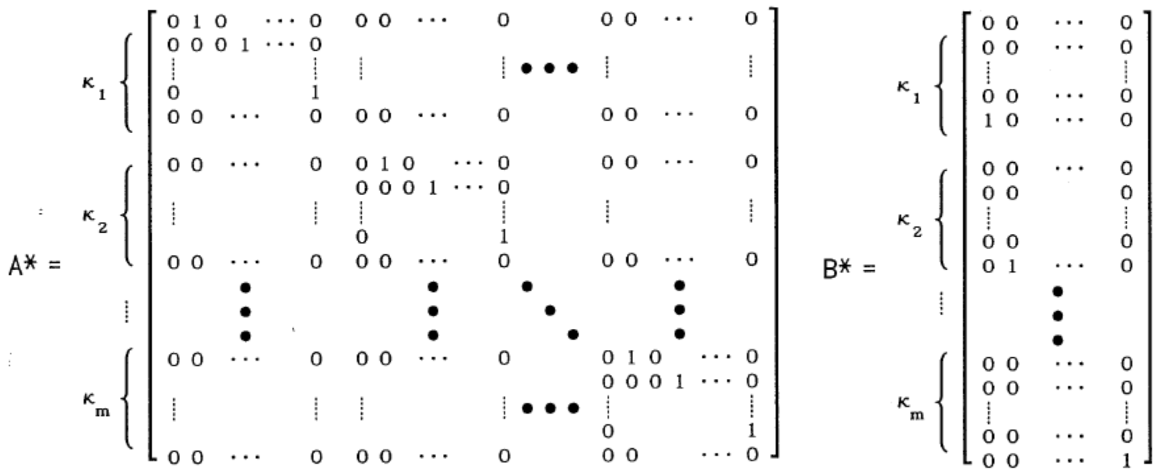


FIGURE 5.2.

That is, the control system is decoupled into m single-input systems of orders $\kappa_1, \kappa_2, ..., \kappa_m$, respectively all in control canonical form.

Proof. Omitted. ■

This theorem has interesting consequences on what can be achieved by feedback. For example, consider a controllable two-input four state variables system given by

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^4, u \in \mathbb{R}^2, A \in \mathbb{R}^{4\times 4}, B \in \mathbb{R}^{4\times 2}$$

with $rank[B] = 2$. Using state feedback along with a statespace and an input transformation the system can be transformed into either

$$\frac{d}{dt}x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v^*$$

or

$$\frac{d}{dt}x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} v^*.$$

To determine which form just simply compute the controllability indices using the original pair (A, B).

Another result using controllability indices is the following theorem.

Theorem 2 *Controllability Matrix*

Consider the controllable linear time-invariant

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$$

with $rank[B] = m$. The controllability matrix $\mathcal{C} \in \mathbb{R}^{n\times mn}$ is

$$\mathcal{C} \triangleq \begin{bmatrix} b_1 & b_2 & \cdots & b_m & Ab_1 & Ab_2 & \cdots & Ab_m & A^2b_1 & A^2b_2 & \cdots & A^2b_m & \cdots & A^{n-1}b_1 & A^{n-1}b_2 & \cdots & A^{n-1}b_m \end{bmatrix}. \tag{5.23}$$

Search $\mathcal{C}$ from left to right to find the first n linearly independent columns. These n linearly independent columns will be of the form

$$b_1, Ab_1, \cdots, A^{d_1-1}b_1, b_2, Ab_2, \cdots, A^{d_2-1}b_2, \cdots, b_m, Ab_m, \cdots, A^{d_m-1}b_m \tag{5.24}$$

for some positive integers $d_1, d_2, ..., d_m$ with $d_1 + d_2 + \cdots + d_m = n$. If we assume the b_i are rearranged so that $d_1 \geq d_2 \geq \cdots \geq d_m$ then $d_i = \kappa_i$ for $i = 1, 2, ..., m$. With this assumption the matrix defined by

$$C \triangleq \begin{bmatrix} b_1 & Ab_1 & \cdots & A^{\kappa_1-1}b_1 & b_2 & Ab_2 & \cdots & A^{\kappa_2-1}b_2 & \cdots & b_m & Ab_m & \cdots & A^{\kappa_m-1}b_m \end{bmatrix} \tag{5.25}$$

has full rank n and therefore is invertible.

Proof. Exercise. ∎

To see the importance of this theorem we look at an example.

Example 8 *Control Canonical Form*

Consider the controllable linear time-invariant system

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^4, u \in \mathbb{R}^2, A \in \mathbb{R}^{4\times 4}, B \in \mathbb{R}^{4\times 2} \tag{5.26}$$

with $rank[B] = 2$. Suppose $\kappa_1 = 3$ and $\kappa_2 = 1$ where

$$\begin{aligned} r_0 &= rank \begin{bmatrix} b_1 & b_2 \end{bmatrix} = 2 \\ r_1 &= rank \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 \end{bmatrix} - rank \begin{bmatrix} b_1 & b_2 \end{bmatrix} = 1 \\ r_2 &= rank \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 \end{bmatrix} - rank \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 \end{bmatrix} = 1. \end{aligned}$$

From the computation of r_1 either $rank \begin{bmatrix} b_1 & b_2 & Ab_1 \end{bmatrix} = 3$ or $rank \begin{bmatrix} b_1 & b_2 & Ab_2 \end{bmatrix} = 3$, but not both. Let's suppose

$$rank \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 \end{bmatrix} = rank \begin{bmatrix} b_1 & b_2 & Ab_1 \end{bmatrix}$$

so that

$$Ab_2 = \beta_1 b_1 + \beta_2 b_2 + \beta_3 Ab_1 \tag{5.27}$$

which implies

$$A^2b_2 = \beta_1 Ab_1 + \beta_2 Ab_2 + \beta_3 A^2b_1.$$

In the computation of r_2 we have

$$r_2 = rank \begin{bmatrix} b_1 & b_2 & Ab_1 & \beta_1 b_1 + \beta_2 b_2 + \beta_3 Ab_1 & A^2b_1 & \beta_1 Ab_1 + \beta_2 Ab_2 + \beta_3 A^2b_1 \end{bmatrix} - rank \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 \end{bmatrix} = 1$$

which shows that

$$C \triangleq \begin{bmatrix} b_1 & Ab_1 & A^2b_1 & b_2 \end{bmatrix}$$

has rank 4 and is therefore invertible.

Let q_1 be the $\kappa_1 = 3$ row of C^{-1} and q_2 be the $\kappa_1 + \kappa_2 = 4$ row of C^{-1} and define the transformation

$$T \triangleq \begin{bmatrix} q_1 \\ q_1 A \\ q_1 A^2 \\ q_2 \end{bmatrix}.$$

Then

$$TA = \begin{bmatrix} q_1 \\ q_1 A \\ q_1 A^2 \\ q_2 \end{bmatrix} A = \begin{bmatrix} q_1 A \\ q_1 A^2 \\ q_1 A^3 \\ q_2 A \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} \begin{bmatrix} q_1 \\ q_1 A \\ q_1 A^2 \\ q_2 \end{bmatrix}$$

$$TB = \begin{bmatrix} q_1 \\ q_1 A \\ q_1 A^2 \\ q_2 \end{bmatrix} B = \begin{bmatrix} q_1 b_1 & q_1 b_2 \\ q_1 Ab_1 & q_1 Ab_2 \\ q_1 A^2 b_1 & q_1 A^2 b_2 \\ q_2 b_1 & q_2 b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & q_1 Ab_2 \\ 1 & q_1 A^2 b_2 \\ 0 & 1 \end{bmatrix}$$

Using (5.27) we see that

$$q_1 A b_2 = \beta_1 q_1 b_1 + \beta_2 q_1 b_2 + \beta_3 q_1 A b_1 = 0$$

Thus we can write

$$TB = \begin{bmatrix} q_1 \\ q_1 A \\ q_1 A^2 \\ q_2 \end{bmatrix} B = \begin{bmatrix} q_1 b_1 & q_1 b_2 \\ q_1 A b_1 & q_1 A b_2 \\ q_1 A^2 b_1 & q_2 A^2 b_2 \\ q_2 b_1 & q_2 b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & q_2 A^2 b_2 \\ 0 & 1 \end{bmatrix}.$$

Let

$$u = \underbrace{\begin{bmatrix} 1 & -q_2 A^2 b_2 \\ 0 & 0 \end{bmatrix}}_{U} v^*$$

so that

$$TA = \begin{bmatrix} q_1 \\ q_1 A \\ q_1 A^2 \\ q_2 \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} \begin{bmatrix} q_1 \\ q_1 A \\ q_1 A^2 \\ q_2 \end{bmatrix}$$

$$TBU = \begin{bmatrix} q_1 \\ q_1 A \\ q_1 A^2 \\ q_2 \end{bmatrix} BU = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & q_2 A^2 b_2 \\ 0 & 1 \end{bmatrix} U = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

With $x^* = Tx$ and $u = Uv$ we have

$$\frac{d}{dt} x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v.$$

With the feedback

$$v = -\begin{bmatrix} \alpha_{21} & \alpha_{22} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} x^* + v$$

we finally have

$$\frac{d}{dt} x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v^*$$

Example 9 *Permanent Magnet Synchronous Motor*
In Chapter 1 we considered the model of a two-phase permanent magnet motor given by

$$\begin{aligned} L_S \frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m \sin(n_p\theta)\omega + u_{Sa} \\ L_S \frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m \cos(n_p\theta)\omega + u_{Sb} \\ J\frac{d\omega}{dt} &= K_m(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)) - \tau_L \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

With $x_1 = i_{Sa}, x_2 = i_{Sb}, x_3 = \omega, x_4 = \theta, u_1 = u_{Sa}/L, u_2 = u_{Sb}/L$ and $c_1 = R_S/L_S, c_2 = K_m/L_S, c_3 = K_m/J$ we may rewrite this as

$$\begin{aligned}
\frac{dx_1}{dt} &= -c_1x_1 + c_2x_3\sin(n_px_4) + u_1\\
\frac{dx_2}{dt} &= -c_1x_2 - c_2x_3\cos(n_px_4) + u_2\\
\frac{dx_3}{dt} &= -c_3x_1\sin(n_px_4) + c_3x_2\cos(n_px_4)\\
\frac{dx_4}{dt} &= x_3
\end{aligned}$$

or

$$\frac{dx}{dt} = \underbrace{\begin{bmatrix} -c_1x_1 + c_2x_3\sin(n_px_4)\\ -c_1x_2 - c_2x_3\cos(n_px_4)\\ -c_3x_1\sin(n_px_4) + c_3x_2\cos(n_px_4)\\ x_3 \end{bmatrix}}_{f} + \underbrace{\begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}}_{g_1} u_1 + \underbrace{\begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}}_{g_2} u_2.$$

We write

$$\mathcal{C} = \left[\begin{array}{cccccccc} g_1 & g_2 & ad_fg_1 & ad_fg_2 & ad_f^2g_1 & ad_f^2g_2 & ad_f^3g_1 & ad_f^3g_2 \end{array}\right]$$

where

$$ad_fg_1 = \begin{bmatrix} c_1\\ 0\\ c_3\sin(n_px_4)\\ 0 \end{bmatrix}, ad_fg_1 = \begin{bmatrix} 0\\ c_1\\ -c_3\cos(n_px_4)\\ 0 \end{bmatrix}$$

$$ad_f^2g_1 = \begin{bmatrix} c_1^2 - c_2c_3\sin^2(n_px_4)\\ c_2c_3\sin(n_px_4)\cos(n_px_4)\\ c_3n_px_3\cos(n_px_4) + c_3(c_1+c_4)\sin(n_px_4)\\ -c_3\sin(n_px_4) \end{bmatrix}, ad_f^2g_1 = \begin{bmatrix} c_2c_3\sin(n_px_4)\cos(n_px_4)\\ c_1^2 - c_2c_3\cos^2(n_px_4)\\ c_3n_px_3\sin(n_px_4) - c_3(c_1+c_4)\cos(n_px_4)\\ c_3\cos(n_px_4) \end{bmatrix}.$$

By inspection $rank[\mathcal{C}] = 4$ for all x. Note that

$$\begin{aligned}
ad_fg_1 &= -c_1\tan(n_px_4)g_2 - c_1g_1 - \tan(n_px_4)ad_fg_2\\
ad_fg_2 &= -c_1\cot(n_px_4)g_1 - c_1g_2 - \cot(n_px_4)ad_fg_1\\
ad_f^2g_1 &= -\cot(n_px_4)ad_f^2g_1 + n_px_3\cot(n_px_4)ad_fg_2 - n_px_3ad_fg_1 + (c_1^2\cot(n_px_4) - c_1n_px_3)g_1 + (c_1^2 + c_1n_px_3\cot(n_px_4))g_2
\end{aligned}$$

$ad_f^3g_1, ad_f^3g_2$ are not needed and so are not computed. Then

$$\begin{aligned}
r_0 &= rank\left[\begin{array}{cc} g_1 & g_2 \end{array}\right] = 2\\
r_1 &= rank\left[\begin{array}{cccc} g_1 & g_2 & ad_fg_1 & ad_fg_2 \end{array}\right] - rank\left[\begin{array}{cc} g_1 & g_2 \end{array}\right] = 1\\
r_2 &= rank\left[\begin{array}{cccccc} g_1 & g_2 & ad_fg_1 & ad_fg_2 & ad_f^2g_1 & ad_f^2g_2 \end{array}\right] - rank\left[\begin{array}{cccc} g_1 & g_2 & ad_fg_1 & ad_fg_2 \end{array}\right] = 1
\end{aligned}$$

with corresponding controllability indices

$$\begin{aligned}
\kappa_1 &= \text{number of } r_j \geq 1 = 3\\
\kappa_2 &= \text{number of } r_j \geq 2 = 1.
\end{aligned}$$

Feedback Linearzing Transformation for the PM Synchronous Motor

Let's find a feedback linearization transformation for the PM synchronous machine given by

$$\frac{dx}{dt} = \underbrace{\begin{bmatrix} -c_1x_1 + c_2x_3\sin(n_px_4) \\ -c_1x_2 - c_2x_3\cos(n_px_4) \\ -c_3x_1\sin(n_px_4) + c_3x_2\cos(n_px_4) \\ x_3 \end{bmatrix}}_{f} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{g_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{g_2} u_2. \tag{5.28}$$

A general nonlinear change of coordinates is given by

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \\ T_4(x) \end{bmatrix} \tag{5.29}$$

and

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f(T_1) + u_1\mathcal{L}_{g_1}(T_1) + u_2\mathcal{L}_{g_2}(T_1) \\ \mathcal{L}_f(T_2) + u_1\mathcal{L}_{g_1}(T_2) + u_2\mathcal{L}_{g_2}(T_2) \\ \mathcal{L}_f(T_3) + u_1\mathcal{L}_{g_1}(T_3) + u_2\mathcal{L}_{g_2}(T_3) \\ \mathcal{L}_f(T_4) + u_1\mathcal{L}_{g_1}(T_4) + u_2\mathcal{L}_{g_2}(T_4) \end{bmatrix}. \tag{5.30}$$

As shown in the above example the PM synchronous machine has controllability indices $\kappa_1 = 3, \kappa_2 = 1$. This requires

$$\mathcal{L}_{g_1}(T_1) = 0, \mathcal{L}_{g_2}(T_1) = 0 \tag{5.31}$$

$$\mathcal{L}_{g_1}(T_2) = 0, \mathcal{L}_{g_2}(T_2) = 0 \tag{5.32}$$

and

$$x_2^* = T_2 \triangleq \mathcal{L}_f(T_1), \quad x_3^* = T_3 \triangleq \mathcal{L}_f(T_2) \tag{5.33}$$

so that the system of equations (5.30) have the form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f(T_1) \\ \mathcal{L}_f(T_2) \\ \mathcal{L}_f(T_3) \\ \mathcal{L}_f(T_4) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathcal{L}_{g_1}(T_3) & \mathcal{L}_{g_2}(T_3) \\ \mathcal{L}_{g_1}(T_4) & \mathcal{L}_{g_2}(T_4) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{5.34}$$

Recalling the identity $\mathcal{L}_{[f,g]} = \mathcal{L}_g\mathcal{L}_f - \mathcal{L}_f\mathcal{L}_g$ the conditions (5.31), (5.32) and (5.33) are equivalent to

$$\mathcal{L}_{g_1}(T_1) = 0, \mathcal{L}_{[f,g_1]}(T_1) = 0, \mathcal{L}_{g_2}(T_1) = 0, \mathcal{L}_{[f,g_2]}(T_1) = 0$$

or

$$dT_1 \begin{bmatrix} g_1 & ad_fg_1 & g_2 & ad_fg_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also using the identity (See 4.35 of Chapter 1 on page 175)

$$\mathcal{L}_{ad_f^2g_i}(T_1) = \mathcal{L}_f^2\mathcal{L}_{g_i}(T_1)) - 2\mathcal{L}_f\mathcal{L}_{g_i}\mathcal{L}_f(T_1) + \mathcal{L}_{g_i}\mathcal{L}_f^2(T_1) = \mathcal{L}_{g_i}\mathcal{L}_f^2(T_1).$$

we may write

$$\mathcal{L}_{g_1}(T_3) = \mathcal{L}_{g_1}(\mathcal{L}_f^2(T_1)) = \mathcal{L}_{ad_f^2g_1}(T_1)$$

$$\mathcal{L}_{g2}(T_3) = \mathcal{L}_{g_2}(\mathcal{L}_f^2(T_1)) = \mathcal{L}_{ad_f^2g_2}(T_1)$$

so that (5.34) becomes

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f(T_1) \\ \mathcal{L}_f^2(T_1) \\ \mathcal{L}_f^3(T_1) \\ \mathcal{L}_f(T_4) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{g_1}(T_4) & \mathcal{L}_{g_2}(T_4) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{5.35}$$

We require the input matrix has full rank (otherwise the system is not controllable), that is,

$$\det\begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{g_1}(T_4) & \mathcal{L}_{g_2}(T_4) \end{bmatrix} \neq 0. \tag{5.36}$$

The conditions (5.34) and (5.36) are necessary conditions on the unknown functions $T_1(x)$ and $T_4(x)$. Note that these conditions only involve the first order derivatives (gradient) of $T_1(x)$ and $T_4(x)$. Let's look for a solution to (5.34) by writing it out explicitly as

$$\begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} & \frac{\partial T_1}{\partial x_4} \end{bmatrix}\left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} c_1 \\ 0 \\ c_3\sin(n_p x_4) \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ -c_3\cos(n_p x_4) \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

By inspection we must have $\dfrac{\partial T_1}{\partial x_1} = 0, \dfrac{\partial T_1}{\partial x_3} = 0$. Taking $\dfrac{\partial T_1}{\partial x_3} = 0$ leaves us with

$$dT_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial T_1}{\partial x_4} \end{bmatrix}$$

with $\dfrac{\partial T_1}{\partial x_4}$ still to be determined. With this choice for the gradient $T_1(x)$ is only a function of x_4. To determine (5.36) we compute

$$\mathcal{L}_{ad_f^2 g_1}(T_1) = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial T_1}{\partial x_4} \end{bmatrix}\begin{bmatrix} c_1^2 - c_2 c_3 \sin^2(n_p x_4) \\ c_2 c_3 \sin(n_p x_4)\cos(n_p x_4) \\ c_3 n_p x_3 \cos(n_p x_4) + c_3(c_1 + c_4)\sin(n_p x_4) \\ -c_3 \sin(n_p x_4) \end{bmatrix} = -c_3 \frac{\partial T_1}{\partial x_4}\sin(n_p x_4)$$

$$\mathcal{L}_{ad_f^2 g_2}(T_1) = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial T_1}{\partial x_4} \end{bmatrix}\begin{bmatrix} c_2 c_3 \sin(n_p x_4)\cos(n_p x_4) \\ c_1^2 - c_2 c_3 \cos^2(n_p x_4) \\ c_3 n_p x_3 \sin(n_p x_4) - c_3(c_1 + c_4)\cos(n_p x_4) \\ c_3 \cos(n_p x_4) \end{bmatrix} = c_3 \frac{\partial T_1}{\partial x_4}\cos(n_p x_4)$$

$$\mathcal{L}_{g_1}(T_4) = \begin{bmatrix} \frac{\partial T_4}{\partial x_1} & \frac{\partial T_4}{\partial x_2} & \frac{\partial T_4}{\partial x_3} & \frac{\partial T_4}{\partial x_4} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{\partial T_4}{\partial x_1}$$

$$\mathcal{L}_{g_2}(T_4) = \begin{bmatrix} \frac{\partial T_4}{\partial x_1} & \frac{\partial T_4}{\partial x_2} & \frac{\partial T_4}{\partial x_3} & \frac{\partial T_4}{\partial x_4} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{\partial T_4}{\partial x_2}.$$

Then

$$\begin{aligned} \det\begin{bmatrix} \mathcal{L}_{ad_f^2 g}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{g_1}(T_4) & \mathcal{L}_{g_2}(T_4) \end{bmatrix} &= \det\begin{bmatrix} -c_3\frac{\partial T_1}{\partial x_4}\sin(n_p x_4) & c_3\frac{\partial T_1}{\partial x_4}\cos(n_p x_4) \\ \frac{\partial T_4}{\partial x_1} & \frac{\partial T_4}{\partial x_2} \end{bmatrix} \\ &= -c_3\frac{\partial T_1}{\partial x_4}\left(\frac{\partial T_4}{\partial x_2}\sin(n_p x_4) + \frac{\partial T_4}{\partial x_1}\cos(n_p x_4)\right). \end{aligned}$$

This suggests setting

$$\frac{\partial T_4}{\partial x_2} = \sin(n_p x_4)$$
$$\frac{\partial T_4}{\partial x_1} = \cos(n_p x_4)$$
$$\frac{\partial T_1}{\partial x_4} = 1$$

so that

$$\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{g_1}(T_4) & \mathcal{L}_{g_2}(T_4) \end{bmatrix} = -c_3 \neq 0.$$

Then

$$T_1(x) = x_4$$
$$T_4(x) = x_1 \cos(n_p x_4) + x_2 \sin(n_p x_4)$$

The full transformation is

$$x_1^* = T_1 = x_4$$
$$x_2^* = \mathcal{L}_f(T_1) = x_3$$
$$x_3^* = \mathcal{L}_f^3(T_1) = c_3\Big(-x_1 \sin(n_p x_4) + \ x_2 \cos(n_p x_4)\Big)$$
$$x_4^* = T_4 = x_1 \cos(n_p x_4) + x_2 \sin(n_p x_4).$$

In the new coordinates we have

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} x_2^* \\ x_3^* \\ \mathcal{L}_f^3(T_1) \\ \mathcal{L}_f(T_4) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\sin(n_p x_4) & \cos(n_p x_4) \\ \cos(n_p x_4) & \sin(n_p x_4) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5.37}$$

With

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sin(n_p x_4) & \cos(n_p x_4) \\ \cos(n_p x_4) & -\sin(n_p x_4) \end{bmatrix} \begin{bmatrix} v_1^* - \mathcal{L}_f^3(T_1) \\ v_2^* - \mathcal{L}_f(T_4) \end{bmatrix}$$

this becomes

$$\frac{d}{dt}x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v^*.$$

Remark $\mathcal{L}_f^3(T_1)$ and $\mathcal{L}_f(T_4)$ are given by

$$\begin{aligned}
\mathcal{L}_f^3(T_1) &= \begin{bmatrix} -c_3\sin(n_px_4) & c_3\cos(n_px_4) & 0 & -c_3n_p(x_1\cos(n_px_4)+x_2\sin(n_px_4)) \end{bmatrix} \begin{bmatrix} -c_1x_1+c_2x_3\sin(n_px_4) \\ -c_1x_2-c_2x_3\cos(n_px_4) \\ -c_3x_1\sin(n_px_4)+c_3x_2\cos(n_px_4) \\ x_3 \end{bmatrix} \\
&= c_1c_3(-x_1\sin(n_px_4)+x_2\cos(n_px_4)) - c_3c_2x_3 - c_3n_px_3(x_1\cos(n_px_4)+x_2\sin(n_px_4)) \\
&= c_3(c_1i_q - c_2\omega - n_p\omega i_d) \\
\mathcal{L}_f(T_4) &= \begin{bmatrix} \cos(n_px_4) & \sin(n_px_4) & 0 & -n_p(-x_1\sin(n_px_4)+x_2\cos(n_px_4)) \end{bmatrix} \begin{bmatrix} -c_1x_1+c_2x_3\sin(n_px_4) \\ -c_1x_2-c_2x_3\cos(n_px_4) \\ -c_3x_1\sin(n_px_4)+c_3x_2\cos(n_px_4) \\ x_3 \end{bmatrix} \\
&= -c_1(x_1\cos(n_px_4)+x_2\sin(n_px_4)) + n_px_3(-x_1\sin(n_px_4)+x_2\cos(n_px_4)) \\
&= -c_1i_d + n_p\omega i_q.
\end{aligned}$$

i_d, i_q, u_d, u_q are defined as in Equations (1.27) and (1.28) of Chapter 1 (page 21) showing that the system model (5.37) above is the same as the system (1.29) - (1.32) of Chapter 1.

Theorem 3 *Multi-Input Exact Linearization Problem*
Given the vector fields $f, g_1, ..., g_m$ on an open set $\mathcal{U} \subset \mathbf{E}^n$ consider the nonlinear control system

$$\frac{dx}{dt} = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m. \tag{5.38}$$

Define

$$\mathcal{C} \triangleq \begin{bmatrix} g_1 & \cdots & g_2 & ad_fg_1 & \cdots & ad_fg_m & \cdots & ad_f^{n-1}g_1 & \cdots & ad_f^{n-1}g_m \end{bmatrix} \tag{5.39}$$

and

$$\begin{aligned}
G_0 &\triangleq span\{g_1, ..., g_m\} \\
G_1 &\triangleq span\{g_1, ..., g_m, ad_fg_1, ..., ad_fg_m\} \\
&\vdots \\
G_{n-2} &\triangleq span\{g_1, ..., g_m, ad_fg_1, ..., ad_fg_m, ..., ad_f^{n-2}g_1, ..., ad_f^{n-2}g_m\} \\
G_{n-1} &\triangleq span\{g_1, ..., g_m, ad_fg_1, ..., ad_fg_m, ..., ad_f^{n-1}g_1, ..., ad_f^{n-1}g_m\}
\end{aligned} \tag{5.40}$$

Note that $G_{n-1} = \mathcal{C}$.
Let $x_0 \in \mathcal{U}$. Suppose in a neighborhood of x_0 we have

$$\begin{aligned} rank[\mathcal{C}] &= n \\ rank[G_0] &= m \end{aligned}$$

and, for $i = 1, ..., n-2$, the G_i have *constant* rank.
Further suppose that each of the distributions $G_0, G_1, ..., G_{n-2}$ are *involutive*. Define the nonlinear con-

trollability indices by

$$\begin{aligned}
r_0 &= rank[G_0] \\
r_1 &= rank\left[\begin{array}{cc} G_0 & G_1 \end{array}\right] - rank[G_0] \\
r_2 &= rank\left[\begin{array}{ccc} G_0 & G_1 & G_2 \end{array}\right] - rank\left[\begin{array}{cc} G_0 & G_1 \end{array}\right] \\
&\vdots \\
r_{n-1} &= rank\left[\begin{array}{cccc} G_0 & G_1 & \cdots & G_{n-1} \end{array}\right] - rank\left[\begin{array}{cccc} G_0 & G_1 & \cdots & G_{n-2} \end{array}\right].
\end{aligned}$$

Let $\kappa_1, \kappa_2, ..., \kappa_m$ be the corresponding controllability indices determined by these controllability indices.

Then there exists an invertible statespace transformation $T(x)$ defined in a neighborhood of x_0 given by

$$\begin{aligned}
x_1^* &= T_1(x) \\
x_2^* &= T_2(x) \\
&\vdots \\
x_n^* &= T_n(x)
\end{aligned}$$

and an invertible input transformation

$$\begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_m^* \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha_1(x) \\ \alpha_2(x) \\ \vdots \\ \alpha_m(x) \end{bmatrix}}_{\alpha(x)} + \underbrace{\begin{bmatrix} \beta_{11}(x) & \beta_{12}(x) & \cdots & \beta_{1m}(x) \\ \beta_{21}(x) & \beta_{22}(x) & \cdots & \beta_{2m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1}(x) & \beta_{m2}(x) & \cdots & \beta_{mm}(x) \end{bmatrix}}_{\beta(x)} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

such that in the new x^* coordinates and the new v^* inputs the system has the form

$$\frac{dx^*}{dt} = A^*x^* + B^*v^*$$

where A^* and B^* have the form given in Figure 5.2.

Proof. Special case of $\kappa_1 = 3$ and $\kappa_2 = 2$.

We have

$$\frac{dx}{dt} = f(x) + G(x)u, \quad x \in \mathbb{R}^5, u \in \mathbb{R}^2 f \in \mathbb{R}^5, \ G(x) \in \mathbb{R}^{5\times 2}.$$

With $x^* = T(x)$ we have

$$\begin{aligned}
\frac{dx_1^*}{dt} &= \mathcal{L}_f(T_1) + u_1\mathcal{L}_{g_1}(T_1) + u_2\mathcal{L}_{g_2}(T_1) \\
\frac{dx_2^*}{dt} &= \mathcal{L}_f(T_2) + u_1\mathcal{L}_{g_1}(T_2) + u_2\mathcal{L}_{g_2}(T_2) \\
\frac{dx_3^*}{dt} &= \mathcal{L}_f(T_3) + u_1\mathcal{L}_{g_1}(T_3) + u_2\mathcal{L}_{g_2}(T_3) \\
\frac{dx_4^*}{dt} &= \mathcal{L}_f(T_4) + u_1\mathcal{L}_{g_1}(T_4) + u_2\mathcal{L}_{g_2}(T_4) \\
\frac{dx_5^*}{dt} &= \mathcal{L}_f(T_5) + u_1\mathcal{L}_{g_1}(T_5) + u_2\mathcal{L}_{g_2}(T_5).
\end{aligned}$$

As the controllability indices are $\kappa_1 = 3$ and $\kappa_2 = 2$ we require

$$\begin{aligned}
&T_2 = \mathcal{L}_f(T_1), \quad \mathcal{L}_{g_1}(T_1) = 0, \quad \mathcal{L}_{g2}(T_1) = 0 \\
&T_3 = \mathcal{L}_f(T_2), \quad \mathcal{L}_{g_1}(T_2) = \mathcal{L}_{g_1}(\mathcal{L}_f(T_1)) = 0, \quad \mathcal{L}_{g2}(T_2) = \mathcal{L}_{g2}(\mathcal{L}_f(T_1)) = 0
\end{aligned}$$

and

$$T_5 = \mathcal{L}_f(T_4), \quad \mathcal{L}_{g_1}(T_4) = 0, \quad \mathcal{L}_{g2}(T_4) = 0.$$

Using

$$\begin{aligned}
\mathcal{L}_{ad_f g}(h) &= \mathcal{L}_{[f,g]}(h) = \mathcal{L}_f(\mathcal{L}_g(h)) - \mathcal{L}_g(\mathcal{L}_f(h)) \\
\mathcal{L}_{ad_f^2 g}(h) &= \mathcal{L}_f^2\mathcal{L}_g(h)) - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f(h) + \mathcal{L}_g\mathcal{L}_f^2(h)
\end{aligned}$$

this becomes

$$\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^* \\
\frac{dx_2^*}{dt} &= x_3^* \\
\frac{dx_3^*}{dt} &= \mathcal{L}_f^2(T_1) + u_1\mathcal{L}_{ad_f^2 g_1}(T_1) + u_2\mathcal{L}_{ad_f^2 g_2}(T_1) \\
\frac{dx_4^*}{dt} &= x_5^* \\
\frac{dx_5^*}{dt} &= \mathcal{L}_f^2(T_4) + u_1\mathcal{L}_{ad_f g_1}(T_4) + u_2\mathcal{L}_{ad_f g_2}(T_4).
\end{aligned}$$

This reduces the problem to finding T_1 and T_4 such that

$$dT_1 \begin{bmatrix} g_1 & g_2 & ad_f g_1 & ad_f g_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.41}$$

$$dT_2 \begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{5.42}$$

and

$$\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{ad_f g_1}(T_4) & \mathcal{L}_{ad_f g_2}(T_4) \end{bmatrix} \neq 0. \tag{5.43}$$

Conditions (5.41) and (5.42) require that

$$\begin{aligned}
G_0 &= \{g_1, g_2\} \\
G_1 &= \{g_1, g_2, ad_f g_1, ad_f g_2\}
\end{aligned}$$

be involutive. The controllability matrix $\mathcal{C}$

$$\mathcal{C} \triangleq G_{n-1} \triangleq \begin{bmatrix} g_1 & g_2 & ad_f g_1 & ad_f g_2 & ad_f^2 g_1 & ad_f^2 g_2 & ad_f^3 g_1 & ad_f^3 g_2 \end{bmatrix} \in \mathbb{R}^{5\times 8}$$

has rank 5 in a neighborhood of x_0 which implies that either

$$C_1 \triangleq \begin{bmatrix} g_1 & ad_f g_1 & ad_f^2 g_1 & g_2 & ad_f g_2 \end{bmatrix} \in \mathbb{R}^{5\times 5}$$

or

$$C_2 \triangleq \begin{bmatrix} g_1 & ad_f g_1 & g_2 & ad_f g_2 & ad_f^2 g_2 \end{bmatrix} \in \mathbb{R}^{5\times 5}$$

or both have rank 5. We have to construct an *invertible* transformation $T(x)$ satisfying (5.41), (5.42) and (5.43).

We now proceed to constructing the required transformation. Define

$$S(t_1, t_2, t_3, t_4, t_5) = \phi_{t_5}(\phi_{t_4}(\phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0)))))$$

where
$\phi_{t_1}(x_0)$ is the solution to

$$\frac{dx}{dt} = ad_f^2 g_1(x), \; x(0) = x_0$$

$\phi_{t_2}(x_0)$ is the solution to

$$\frac{dx}{dt} = ad_f g_1(x),\ x(0) = x_0'$$

$\phi_{t_3}(x_0)$ is the solution to

$$\frac{dx}{dt} = ad_f g_2(x),\ x(0) = x_0''$$

$\phi_{t_4}(x_0)$ is the solution to

$$\frac{dx}{dt} = g_1(x),\ x(0) = x_0'''$$

$\phi_{t_5}(x_0)$ is the solution to

$$\frac{dx}{dt} = g_2(x),\ x(0) = x_0''''.$$

At $t = (0,0,0,0,0)$ we have

$$\frac{\partial S}{\partial t}_{|t=(0,0,0,0,0)} = \left[\begin{array}{ccccc} ad_f^2 g_1 & ad_f g_1 & ad_f g_2 & g_1 & g_2 \end{array}\right]_{|x_0}.$$

By the inverse function theorem this has an inverse defined in a neighborhood of $t = 0$. Denote the inverse of

$$\begin{aligned} x_1 &= s_1(t_1,t_2,t_3,t_4,t_5) \\ x_2 &= s_2(t_1,t_2,t_3,t_4,t_5) \\ x_3 &= s_3(t_1,t_2,t_3,t_4,t_5) \\ x_4 &= s_4(t_1,t_2,t_3,t_4,t_5) \\ x_5 &= s_5(t_1,t_2,t_3,t_4,t_5) \end{aligned} \tag{5.44}$$

by

$$\begin{aligned} t_1 &= T_1(x_1,x_2,x_3,x_4,x_5) \\ t_2 &= T_2(x_1,x_2,x_3,x_4,x_5) \\ t_3 &= T_3(x_1,x_2,x_3,x_4,x_5) \\ t_4 &= T_4(x_1,x_2,x_3,x_4,x_5) \\ t_5 &= T_5(x_1,x_2,x_3,x_4,x_5). \end{aligned} \tag{5.45}$$

If $t_1 = t_{01}$ is held constant, then $S(t_{01},t_2,t_3,t_4,t_5) = \phi_{t_5}(\phi_{t_4}(\phi_{t_3}(\phi_{t_2}(\phi_{t_{01}}(x_0)))))$ sweeps out a four dimensional surface in $\mathbb{R}^5$ as t_2, t_3, t_4, and t_5 are varied. As the vector fields

$$\{g_1, g_2, ad_f g_1, ad_f g_2\}$$

are involutive, Frobenius' theorem tells us that the vectors

$$\left[\begin{array}{cccc} g_1 & g_2 & ad_f g_1 & ad_f g_2 \end{array}\right]_{x=S(t_{01},t_2,t_3,t_4,t_5)}$$

span tangent space to the surface

$$\{x \in \mathbb{R}^5 |\ T_1(x) = t_{01}\}. \tag{5.46}$$

That is,

$$\mathcal{L}_{g_1}(T_1) = 0,\ \mathcal{L}_{g_2}(T_1) = 0,\ \mathcal{L}_{ad_f g_1}(T_1) = 0,\ \mathcal{L}_{ad_f g_2}(T_1) = 0. \tag{5.47}$$

Further

$$\mathcal{L}_{ad_f g_2}(T_1) \neq 0 \tag{5.48}$$

as $ad_f g_2$ is linearly independent of $\{g_1, g_2, ad_f g_1, ad_f g_2\}$ and therefore cannot be in the tangent space of the surface.

Now if $t_1 = t_{01}, t_2 = t_{02}, t_3 = t_{03}$ are held constant, then $S(t_{01}, t_{02}, t_{03}, t_4, t_5) = \phi_{t_5}(\phi_{t_4}(\phi_{t_{03}}(\phi_{t_{02}}(\phi_{t_{01}}(x_0)))))$ sweeps out a two dimensional surface in $\mathbb{R}^5$. As the set

$$\{g_1, g_2\}$$

is involutive, Frobenius' theorem tells us that

$$\frac{\partial S}{\partial t_4}, \frac{\partial S}{\partial t_5} \in \Delta_{x=S(t_{01},t_{02},t_{03},t_4,t_5)} \triangleq \{r_1 g_1(x) + r_2 g_2(x) |\ x = S(t_{01}, t_{02}, t_{03}, t_4, t_5) \text{ and } r_1, r_2 \in \mathbb{R}\}.$$

That is, $\{g_1, g_2\}$ span the tangent plane of the surface (submanifold)

$$\{x \in \mathbb{R}^5 |\ T_1(x) = t_{01}, T_2(x) = t_{02}, T_{03}(x) = t_{03}\} \tag{5.49}$$

for $x = S(t_{01}, t_{02}, t_{03}, t_4, t_5)$. In particular we have

$$\mathcal{L}_{g_1}(T_3) = 0,\ \mathcal{L}_{g_2}(T_3) = 0 \qquad \text{(chapter5\_eq60b)}$$

in a neighborhood of x_0.

As $ad_f g_2$ is linearly independent of $\{g_1, g_2\}$ it is *not* tangent to the surface (5.49). By Equation (5.47) we know that $\mathcal{L}_{ad_f g_2}(T_1) = 0$ so $\mathcal{L}_{ad_f g_2}(T_2)$ and $\mathcal{L}_{ad_f g_2}(T_3)$ cannot both be zero since this would imply $ad_f g_2$ is in the tangent space of (5.49). It is next shown that $\mathcal{L}_{ad_f g_2}(T_3)_{|x_0} = 1$ implying $\mathcal{L}_{ad_f g_2}(T_3) \neq 0$ in a neighborhood of x_0. To proceed, $x = S(t)$ and $t = T(x)$ in (5.44) and (5.45) are inverses of each other so $t = T(S(t))$ which implies

$$I_{5\times 5} = \frac{\partial T}{\partial x}\frac{\partial S}{\partial t}. \tag{5.50}$$

The $(1,1)$ component (5.50) gives

$$1 = \frac{\partial T_1}{\partial x}_{|x_0} \frac{\partial S}{\partial t_1}_{|t=(0,0,0,0,0)} = \left\langle dT_1, ad_f^2 g_1 \right\rangle_{|x_0} = \mathcal{L}_{ad_f^2 g_1}(T_1)_{|x_0}.$$

The $(3,3)$ component of (5.50) gives

$$1 = \frac{\partial T_3}{\partial x}_{|x_0} \frac{\partial S}{\partial t_3}_{|t=(0,0,0,0,0)} = \langle dT_3, ad_f g_2 \rangle_{|x_0} = \mathcal{L}_{ad_f g_2}(T_3)_{|x_0}$$

and the $(3,2)$ component of (5.50) gives

$$0 = \frac{\partial T_3}{\partial x}_{|x_0} \frac{\partial S}{\partial t_2}_{|t=(0,0,0,0,0)} = \langle dT_3, ad_f g_1 \rangle_{|x_0} = \mathcal{L}_{ad_f g_1}(T_3)_{|x_0}.$$

Then

$$\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{ad_f g_1}(T_3) & \mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix}_{x_0} = \det \begin{bmatrix} 1 & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ 0 & 1 \end{bmatrix}_{|x_0} = 1$$

implying that

$$\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{ad_f g_1}(T_3) & \mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix}_{x_0} \neq 0$$

in a neighborhood of x_0.

Define a new transformation

$$\begin{aligned}
x_1^* &= T_1^*(x) \triangleq T_1(x) \\
x_2^* &= T_2^*(x) \triangleq \mathcal{L}_f T_1(x) \\
x_3^* &= T_3^*(x) \triangleq \mathcal{L}_f^2 T_1(x) \\
x_4^* &= T_4^*(x) \triangleq T_3(x) \\
x_5^* &= T_5^*(x) = \mathcal{L}_f T_3(x)
\end{aligned} \tag{5.51}$$

With $x^* = T^*(x)$ we have

$$\begin{aligned}
\frac{dx_1^*}{dt} &= \mathcal{L}_f(T_1) + u_1\mathcal{L}_{g_1}(T_1) + u_2\mathcal{L}_{g_2}(T_1) \\
\frac{dx_2^*}{dt} &= \mathcal{L}_f^2(T_1) + u_1\mathcal{L}_{g_1}\mathcal{L}_f(T_1) + u_2\mathcal{L}_{g_2}\mathcal{L}_f(T_1) \\
\frac{dx_3^*}{dt} &= \mathcal{L}_f^3(T_1) + u_1\mathcal{L}_{g_1}\mathcal{L}_f^2(T_1) + u_2\mathcal{L}_{g_2}\mathcal{L}_f^2(T_1) \\
\frac{dx_4^*}{dt} &= \mathcal{L}_f(T_3) + u_1\mathcal{L}_{g_1}(T_3) + u_2\mathcal{L}_{g_2}(T_3) \\
\frac{dx_5^*}{dt} &= \mathcal{L}_f(T_3) + u_1\mathcal{L}_{g_1}\mathcal{L}_f(T_3) + u_2\mathcal{L}_{g_2}\mathcal{L}_f(T_3)
\end{aligned}$$

which by all of the above reduces to

$$\begin{aligned}
\frac{dx_1^*}{dt} &= \mathcal{L}_f(T_1) \\
\frac{dx_2^*}{dt} &= \mathcal{L}_f^2(T_1) \\
\frac{dx_3^*}{dt} &= \mathcal{L}_f^3(T_1) + u_1\mathcal{L}_{ad_f^2 g_1}(T_1) + u_2\mathcal{L}_{ad_f^2 g_2}(T_1) \\
\frac{dx_4^*}{dt} &= \mathcal{L}_f(T_3) \\
\frac{dx_5^*}{dt} &= \mathcal{L}_f(T_3) + u_1\mathcal{L}_{ad_f g_1}(T_3) + u_2\mathcal{L}_{ad_f g_2}(T_3)
\end{aligned}$$

with

$$\det\begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{ad_f g_1}(T_3) & \mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix} \neq 0$$

in some neighborhood of x_0.

We still have to show that the transformation (5.51) is an *invertible* transformation. To do this we compute

$$\begin{aligned}
\frac{\partial T^*}{\partial x} C_1 &= \begin{bmatrix} dT_1^* \\ dT_2^* \\ dT_3^* \\ dT_4^* \\ dT_5^* \end{bmatrix} \begin{bmatrix} g_1 & ad_f g_1 & ad_f^2 g_1 & g_2 & ad_f g_2 \end{bmatrix} \\
&= \begin{bmatrix} dT_1 \\ d\mathcal{L}_f(T_1) \\ d\mathcal{L}_f^2(T_1) \\ dT_3 \\ d\mathcal{L}_f(T_3) \end{bmatrix} \begin{bmatrix} g_1 & ad_f g_1 & ad_f^2 g_1 & g_2 & ad_f g_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{L}_{g_1}(T_1) & \mathcal{L}_{ad_f g_1}(T_1) & \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{g_2}(T_1) & \mathcal{L}_{ad_f g_2}(T_1) \\
\mathcal{L}_{g_1}\mathcal{L}_f(T_1) & \mathcal{L}_{ad_f g_1}(\mathcal{L}_f(T_1)) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f(T_1) & \mathcal{L}_{g_2}\mathcal{L}_f(T_1) & \mathcal{L}_{ad_f g_2}\mathcal{L}_f(T_1) \\
\mathcal{L}_{g_1}\mathcal{L}_f^2(T_1) & \mathcal{L}_{ad_f g_1}(\mathcal{L}_f^2(T_1)) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f^2(T_1) & \mathcal{L}_{g_2}\mathcal{L}_f^2(T_1) & \mathcal{L}_{ad_f g_2}\mathcal{L}_f^2(T_1) \\
\mathcal{L}_{g_1}(T_3) & \mathcal{L}_{ad_f g_1}(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}(T_3) & \mathcal{L}_{g_2}(T_3) & \mathcal{L}_{ad_f g_2}(T_3) \\
\mathcal{L}_{g_1}\mathcal{L}_f(T_3) & \mathcal{L}_{ad_f g_1}\mathcal{L}_f(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}\mathcal{L}_f(T_3) & \mathcal{L}_{g_2}\mathcal{L}_f(T_3) & \mathcal{L}_{ad_f g_2}\mathcal{L}_f(T_3) \end{bmatrix}.
\end{aligned}$$

Using the identities

$$\begin{aligned}
\mathcal{L}_{ad_f g}(h) &= \mathcal{L}_{[f,g]}(h) = \mathcal{L}_f(\mathcal{L}_g(h)) - \mathcal{L}_g(\mathcal{L}_f(h)) \\
\mathcal{L}_{ad_f^2 g}(h) &= \mathcal{L}_f^2\mathcal{L}_g(h)) - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f(h) + \mathcal{L}_g\mathcal{L}_f^2(h)
\end{aligned}$$

$\dfrac{\partial T^*}{\partial x} C_1$ can be rewritten as

$$\frac{\partial T^*}{\partial x} C_1 = \begin{bmatrix} 0 & 0 & \mathcal{L}_{ad_f^2 g_1}(T_1) & 0 & 0 \\
0 & -\mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f(T_1) & 0 & -\mathcal{L}_{ad_f^2 g_2}(T_1) \\
\mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f g_1}(\mathcal{L}_f^2(T_1)) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f^2(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) & \mathcal{L}_{ad_f g_2}\mathcal{L}_f^2(T_1) \\
0 & \mathcal{L}_{ad_f g}(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}(T_3) & 0 & \mathcal{L}_{ad_f g_2}(T_3) \\
-\mathcal{L}_{ad_f g_1}(T_1) & \mathcal{L}_{ad_f g_1}\mathcal{L}_f(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}\mathcal{L}_f(T_3) & -\mathcal{L}_{ad_f g_2}(T_3) & \mathcal{L}_{ad_f g_2}\mathcal{L}_f(T_3) \end{bmatrix}. \tag{5.52}$$

We have shown

$$\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ -\mathcal{L}_{ad_f g_1}(T_3) & -\mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix} = -\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{ad_f g_1}(T_3) & \mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix} \neq 0$$

in a neighborhood of x_0. With

$$B \triangleq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ -\mathcal{L}_{ad_f g_1}(T_3) & -\mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix}^{-1}$$

so

$$\begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ -\mathcal{L}_{ad_f g_1}(T_3) & -\mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = I$$

we multiply both sides of (5.52) by

$$D_1 \triangleq \begin{bmatrix} b_{11} & 0 & 0 & b_{12} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ b_{21} & 0 & 0 & b_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

to obtain

$$\frac{\partial T^*}{\partial x} C_1 D_1 = \begin{bmatrix} 0 & 0 & \mathcal{L}_{ad_f^2 g_1}(T_1) & 0 & 0 \\ 0 & -\mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f(T_1) & 0 & -\mathcal{L}_{ad_f^2 g_2}(T_1) \\ 1 & \mathcal{L}_{ad_f g_1}(\mathcal{L}_f^2(T_1)) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f^2(T_1) & 0 & \mathcal{L}_{ad_f g_2}\mathcal{L}_f^2(T_1) \\ 0 & \mathcal{L}_{ad_f g}(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}(T_3) & 0 & \mathcal{L}_{ad_f g_2}(T_3) \\ 0 & \mathcal{L}_{ad_f g_1}\mathcal{L}_f(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}\mathcal{L}_f(T_3) & 1 & \mathcal{L}_{ad_f g_2}\mathcal{L}_f(T_3) \end{bmatrix}.$$

Next multiply this last result by

$$D_2 \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -b_{11} & 0 & 0 & -b_{12} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -b_{21} & 0 & 0 & -b_{22} \end{bmatrix}$$

we have

$$\frac{\partial T^*}{\partial x} C_1 D_1 D_2 = \begin{bmatrix} 0 & 0 & \mathcal{L}_{ad_f^2 g_1}(T_1) & 0 & 0 \\ 0 & 1 & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f(T_1) & 0 & 0 \\ 1 & \times & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f^2(T_1) & 0 & x \\ 0 & 0 & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}(T_3) & 0 & 1 \\ 0 & x & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}\mathcal{L}_f(T_3) & 1 & x \end{bmatrix}$$

where $\times$ denotes that is doesn't matter what is in those spots. It was shown above that $\mathcal{L}_{ad_f^2 g_1}(T_1) \neq 0$ in a neighborhood of x_0. By inspection the matrix $\frac{\partial T^*}{\partial x} C_1 D_1 D_2$ is invertible. As C_1, D_1, and D_2 are all invertible it follows that $\frac{\partial T^*}{\partial x}$ is invertible showing that $T^*(x)$ is an invertible transformation. ∎

5.3 Problems

Problem 1 *Controllability Matrix*
Consider the control system given by

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u.$$

(a) Compute the controllability matrix

$$\mathcal{C} \triangleq \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 & A^3b_1 & A^3b_2 & A^4b_1 & A^4b_2 \end{bmatrix}.$$

Searching $\mathcal{C}$ from left to right find the first 5 linearly independent columns of $\mathcal{C}$.

(b) With reference to Theorem 2 give the d_i and the controllability indices κ_i. Compute

$$C \triangleq \begin{bmatrix} b_1 & Ab_1 & \cdots & A^{\kappa_1 - 1}b_1 & b_2 & Ab_2 & \cdots & A^{\kappa_2 - 1}b_2 \end{bmatrix}.$$

(c) Let q_1 be the κ_1 row of C^{-1} and q_2 be the $\kappa_1 + \kappa_2$ row of C^{-1}. Use q_1, q_2 to find the transformation T that is used to put the system into control canonical form.

Problem 2 *Controllability Matrix*
Consider the control system given by

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u.$$

(a) Compute the controllability matrix

$$\mathcal{C} \triangleq \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 & A^3b_1 & A^3b_2 & A^4b_1 & A^4b_2 \end{bmatrix}.$$

(b) Searching $\mathcal{C}$ from left to right find the first 5 linearly independent columns of $\mathcal{C}$. With reference to Theorem 2 give the d_i and the controllability indices κ_i. Compute

$$C \triangleq \begin{bmatrix} b_1 & Ab_1 & \cdots & A^{\kappa_1 - 1}b_1 & b_2 & Ab_2 & \cdots & A^{\kappa_2 - 1}b_2 \end{bmatrix}.$$

(c) Let q_1 be the κ_1 row of C^{-1} and q_2 be the $\kappa_1 + \kappa_2$ row of C^{-1}. Use q_1, q_2 to find the transformation T that is used to put the system into control canonical form.

Problem 3 *Multi-Input Control Canonical Form*
Consider the controllable linear time-invariant control system

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^4, u \in \mathbb{R}^2, A \in \mathbb{R}^{4\times 4}, B \in \mathbb{R}^{4\times 2}$$

with $rank[B] = 2$. Suppose $\kappa_1 = 2$ and $\kappa_2 = 2$ so that by Theorem 2 it follows that

$$C \triangleq \begin{bmatrix} b_1 & Ab_1 & b_2 & Ab_2 \end{bmatrix}$$

is invertible. Transform this control system into the form

$$\frac{d}{dt}x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} v^*$$

by constructing the appropriate state and input transformations and the appropriate state feedback matrix.

Problem 4 *Exact Differentials*

Let $d\omega = \begin{bmatrix} \omega_1(x) & \omega_2(x) & \omega_3(x) \end{bmatrix} \in \mathbb{R}^{1\times 3}$ for all x in an open set $\mathcal{U} \subset \mathbb{R}^3$. Show that the necessary and sufficient condition for there to exist a scalar function $T_1(x)$ such that

$$\frac{\partial T_1}{\partial x_1} = \omega_1(x), \frac{\partial T_1}{\partial x_2} = \omega_2(x), \frac{\partial T_1}{\partial x_3} = \omega_3(x) \tag{5.53}$$

is that

$$\begin{aligned} \frac{\partial \omega_1(x)}{\partial x_2} &= \frac{\partial \omega_2(x)}{\partial x_1} \\ \frac{\partial \omega_1(x)}{\partial x_3} &= \frac{\partial \omega_3(x)}{\partial x_1} \\ \frac{\partial \omega_2(x)}{\partial x_3} &= \frac{\partial \omega_3(x)}{\partial x_2}. \end{aligned} \tag{5.54}$$

If $d\omega$ satisfies (5.54) it is said to be an *exact differential.* Hint: For sufficiency, show that

$$T_1(x) \triangleq \int_0^{x_1} \omega_1(x_1', 0, 0)dx_1' + \int_0^{x_2} \omega_1(x_1, x_2', 0)dx_2' + \int_0^{x_3} \omega_1(x_1, x_2, x_3')dx_3$$

will satisfy (5.53).

Problem 5 *Doubly Excited DC Motor*

Doubly Excited DC Motor

The equations describing a doubly excited DC motor are

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ J\frac{d\omega}{dt} &= K_T L_f i_f i_a - \tau_L \\ L\frac{di_a}{dt} &= -Ri_a - K_b L_f i_f \omega + V_a \\ L_f\frac{di_f}{dt} &= -R_f i_f + V_f. \end{aligned}$$

Here θ is the rotor angle, ω is the rotor angular speed, V_{a0} is the (constant) armature voltage, i_a is the armature current, V_f is the field voltage, i_f is the field current, τ_L is the load torque, K_T is the torque constant, and K_b is the back-emf constant. The armature resistance and armature inductance are denoted by R and L,respectively, and the field resistance and field inductance are R_f and L_f, respectively.

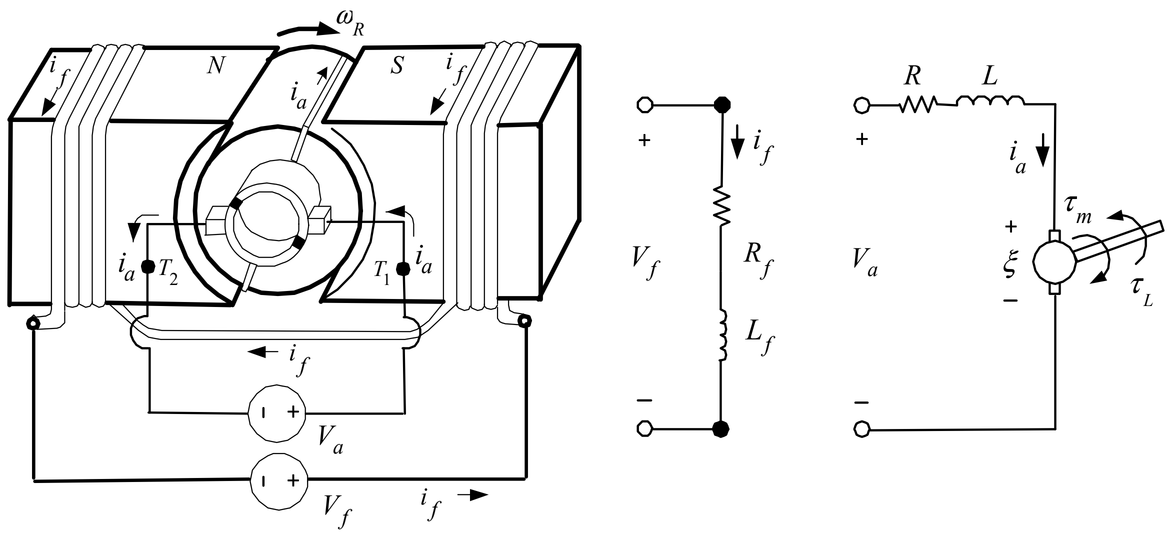


FIGURE 5.3. Field controlled DC motor. $\xi = K_b L_f i_f$ and $\tau_m = K_T L_f i_f i_a$.

Let $x_1 = i_f, x_2 = i_a, x_3 = \omega, x_4 = \theta, u_1 = V_a/L_a, u_2 = V_f/L_f$, and define the constants $c_1 = R_f/L_f, c_2 = R/L, c_3 = K_b L_f/L, c_4 \triangleq K_T L_f/J$. The equations describing the doubly excited DC motor are then

$$\begin{aligned}
\frac{dx_1}{dt} &= -c_1 x_1 + u_2 \\
\frac{dx_2}{dt} &= -c_2 x_2 - c_3 x_1 x_3 + u_1 \\
\frac{dx_3}{dt} &= c_4 x_1 x_2 - \tau_L/J \\
\frac{dx_4}{dt} &= x_3
\end{aligned}$$

or

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -c_1 x_1 \\ -c_2 x_2 - c_3 x_1 x_3 \\ c_4 x_1 x_2 \\ x_3 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{g_1(x)} u_1 + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{g_2(x)} u_2 + \begin{bmatrix} 0 \\ 0 \\ -1/J \\ 0 \end{bmatrix} \tau_L.$$

(a) Compute the controllability indices of this nonlinear system with $x_1 = i_f \neq 0$ and $i_a = x_2 \neq 0$.

(b) Can you find a feedback linearizing transformation? If so, do so. What conditions on the state variables x_1, x_2, x_3 are needed to use this feedback?

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