Chapter 4 Integral Manifolds and the Frobenius Theorem

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Contents

1

Integral Manifolds and the Frobenius Theorem

In this chapter the main tool for finding coordinate transformations to achieve feedback linearization as well as for finding transformations to design observers with linear error dynamics is presented. This tool is the Frobenius theorem and the notions of Lie brackets and vector fields on manifolds are developed to understand and prove this theorem. The fundamental existence theorem for differential equations is stated and the interpretation of a differential on a manifold is discussed. More specifically, a system of differential equations on a manifold are represented by its coordinates using a particular coordinate chart (patch) and so the system has a different representation in each chart. This suggests the idea of trying to find a particular coordinate system for which the resulting differential equations are easy to work with for control purposes (i.e., can be linearized using feedback). The Frobenius theorem gives explicit conditions that must hold for a feedback linearizing transformation to exist. These conditions are given in terms of the Lie brackets of the vector fields that define the control system.

1.1 Differential Equations

Let's review and quote some fundamental results from the theory of differential equations. Consider the system of differential equations given by

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(x_1(t),...,x_n(t),t), \quad x_1(t_0) = x_{01}\\ \vdots \; &= \qquad \vdots \\ \frac{dx_2}{dt} &= f_2(x_1(t),...,x_n(t),t), \quad x_n(t_0) = x_{0n}\end{aligned} \tag{1.1}$$

More compactly this can be written as

$$\frac{dx}{dt} = f(x(t),t), \quad x(t) = x_0. \tag{1.2}$$

A solution of this system differential equations is a function $\phi(t,x_0) : \mathbb{R}^1 \to \mathbb{R}^n$ such that

$$\frac{d\phi(t,x_0)}{dt} = f(\phi(t,x_0)), \quad \phi(t_0,x_0) = x_0. \tag{1.3}$$

We will denote $\phi(t,x_0)$ by $\phi_t(x_0)$ and often use the (imprecise) shorthand notation $x(t)$ for $\phi(t,x_0)$, i.e., $x(t) = \phi(t,x_0)$.

Theorem 1 *Existence Theorem for Ordinary Differential Equations*
Let $f(x,t) \in \mathbb{R}^n$ have continuous partial derivatives with respect to $x_i, i = 1,...,n$ and t in a neighborhood of (x_0,t_0). That is, the partial derivatives $\dfrac{\partial f_i}{\partial x_j}, i,j = 1,...n$ and $\dfrac{\partial f_i}{\partial t}, i = 1,...n$ all exist and are continuous in a neighborhood of (x_0,t_0). Then there is an open interval $\mathbf{I} \subset \mathbb{R}$ containing t_0 for which there is a unique solution $\phi(t,x_0)$ of $dx/dt = f(x(t),t)$ with $x(t_0) = \varphi(t_0,x_0) = x_0$.
Proof. omitted. ∎

Remark 1 The conditions of this theorem are sufficient, but *not* necessary.

Example 1 *Dependence on Initial Conditions*

Consider the differential equation

$$\frac{dx}{dt} = x^2, \quad x(0) = x_0$$

Note that with $f(x) = x^2$ we have

$$\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial t} = 0$$

so the conditions of the existence theorem hold.

To solve this differential equation we write

$$\frac{dx}{x^2} = dt$$

to obtain

$$-\frac{1}{x} = t + c$$

or

$$x(t) = -\frac{1}{t+c}.$$

The initial condition requires setting $c = -1/x_0$ so that

$$\phi(t, x_0) = -\frac{1}{t - 1/x_0} = -\frac{x_0}{x_0 t - 1}$$

(1) Let $x_0 > 0$. Then the solution is $\phi(t, x_0) = -\dfrac{x_0}{x_0 t - 1}$ is valid for $t \in \mathbf{I} = (-\infty, 1/x_0)$.

(2) Let $x_0 < 0$. Then the solution is $\phi(t, x_0) = -\dfrac{x_0}{x_0 t - 1}$ is valid for $t \in \mathbf{I} = (1/x_0, \infty)$.

(3) Let $x_0 = 0$. Then the solution is $\phi(t, x_0) = -\dfrac{x_0}{x_0 t + 1} = 0$ is valid for $t \in \mathbf{I} = (-\infty, \infty)$.

Example 2 *Linear Time-Invariant Differential Equations*

$$\frac{dx}{dt} = Ax, \quad x(t_0) = x_0, \quad A \in \mathbb{R}^{n \times n}.$$

Then with $f(x) = Ax \in \mathbb{R}^n$ we have

$$\begin{aligned} \frac{\partial f_i}{\partial x_i} &= a_{ij} \text{ for } i, j = 1, ..., n \\ \frac{\partial f_i}{\partial t} &= 0 \text{ for } i = 1, ..., n. \end{aligned}$$

so the conditions of the existence theorem hold.

The solution is

$$x(t) = \phi(t, x_0) = e^{A(t - t_0)} x_0$$

and is valid for all $x_0 \in \mathbb{R}^n$ and $-\infty < t < \infty$.

Example 3 *Linear Differential Equations*

$$\frac{dx}{dt} = A(t)x, \quad x(t_0) = x_0, \quad A \in \mathbb{R}^{n \times n}.$$

Then with $f(x) = A(t)x \in \mathbb{R}^n$ we have

$$\frac{\partial f_i}{\partial x_i} = a_{ij}(t) \text{ for } i, j = 1, ..., n$$
$$\frac{\partial f_i}{\partial t} = \sum_{j=1}^{n} \dot{a}_{ij}(t) x_j \text{ for } i = 1, ..., n.$$

If the $a_{ij}(t)$ are continuously differentiable then the conditions of the existence theorem hold. It turns out that a solution then exists which is valid for all $x_0 \in \mathbb{R}^n$ and $-\infty < t < \infty$.

Example 4 *Nonlinear Differentiable Equation*

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}, \quad x(t_0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}.$$

Then straightforwardly we have $x_1(t) = x_{01}$ and $x_2(t) = t - t_0 + x_{02}$. To find $x_3(t)$ we write

$$\frac{dx_3}{dt} = \frac{x_3}{t - t_0 + x_{02}}$$

so

$$\frac{dx_3}{x_3} = \frac{dt}{t - t_0 + x_{02}}$$

which gives $\ln x_3 = \ln(t - t_0 + x_{02}) + c$ or $x_3 = e^c(t - t_0 + x_{02})$. The initial condition requires $x_3(t) = x_{03}/x_{02}(t - t_0 + x_{02})$. Let

$$\phi(t, x_0) = \begin{bmatrix} x_{01} \\ t - t_0 + x_{02} \\ \dfrac{x_{03}}{x_{02}}(t - t_0 + x_{02}). \end{bmatrix}$$

Exercise 1 These questions pertain to the previous examples.

(1) Verify $\phi(t, x_0)$ satisfies the differential equation for all initial conditions except $x_{02} = 0$.

(2) What is the solution if x_{02} is zero?

Exercise 2 *Differential Equation Without a Unique Solution* [1]
Show that the nonlinear first-order differential equation

$$\frac{dx}{dt} = \sqrt{x}, \quad x(0) = 0$$

has the two solutions

$$x(t) \equiv 0$$

and

$$x(t) = \begin{cases} t^2/4, & t \geq 0 \\ -t^2/4, & t < 0. \end{cases}$$

That is, it does not have a unique solution. Does this differential equation satisfy the sufficient conditions of the existence theorem? Explain.

1.2 Properties of the Flow of a Vector Field

In the following it is assumed the sufficient conditions of the existence theorem hold. We have denoted the solution to $dx/dt = f(x)$ with $x(0) = x_0$ by $\phi(t, x_0)$ or $\phi_t(x_0)$ with the solution guaranteed to exist (at least) for $|t| < \varepsilon$ for some $\varepsilon > 0$. The solution $\phi(t, x_0)$ is called the *flow* of the vector field $f(x)$. $\phi(t, x_0)$ is a function from $\mathbb{R}^{n+1} \to \mathbb{R}^n$ taking the point $(t, x_0) \in \mathbb{R}^{n+1}$ to the point $\phi(t, x_0) \in \mathbb{R}^n$. On the other hand, for each fixed t we can consider $\phi(t, x_0)$ as a map from $\mathbb{R}^n \to \mathbb{R}^n$ taking x_0 to $x(t) = \phi(t, x_0)$. In words, starting at x_0 the state moves in the direction specified by $f(x)$ for a time t and ends up at $\phi(t, x_0)$.

Now let $x_0' = \phi(t, x_0)$ and then follow the same vector field $f(x)$ for a time s so we have $\phi(s, x_0') = \phi(s, \phi(t, x_0))$. By the uniqueness of solutions to differential equations this must be the same as starting at x_0 and following $f(x)$ for a time $t + s$ which is $\phi(t + s, x_0)$. That is,

$$\phi(t + s, x_0) = \phi(s, \phi(t, x_0))$$

or equivalently

$$\phi_{t+s}(x_0) = \phi_s(\phi_t(x_0)) = \phi_s \circ \phi_t(x_0).$$

It must also be true that

$$\phi_{s+t}(x_0) = \phi_t(\phi_s(x_0)) = \phi_t \circ \phi_s(x_0)$$

so that

$$\phi_s \circ \phi_t(x_0) = \phi_t \circ \phi_s(x_0). \tag{1.4}$$

The solutions $\phi_t(x_0) = \phi(t, x_0), \phi_s(x_0) = \phi(s, x_0), \phi_{t+s}(x_0) = \phi(t+s, x_0)$ all exist for $|t| < \varepsilon, |s| < \varepsilon, |t+s| < \varepsilon$, respectively.

Consider the following situation where $f(x)$ is followed for a time t to reach $x_0' = \phi(t, x_0)$. Then from x_0' follow $f(x)$ for a time $-t$ to reach $\phi(-t, x_0')$, that is, go backwards. By following the vector field backwards for the same amount of time the vector field was followed forwards it would be expected to end up in the same place. This is indeed the case as

$$\phi(-t, x_0') = \phi(-t, \phi(t, x_0)) = \phi(t + (-t), x_0) = \phi(0, x_0) = x_0.$$

This is also written as

$$\phi_{-t}(x_0') = \phi_{-t}(\phi_t(x_0)) = \phi_{t-t}(x_0) = \phi_0(x_0) = x_0.$$

Example 5 *Flow of a Linear System*

Let $f(x) = Ax$ with $A \in \mathbb{R}^{n\times n}$ and consider

$$\frac{dx}{dt} = f(x),\ \ x(0) = x_0.$$

Then $\phi(t, x_0) = e^{At}x_0$

$$\phi_s \circ \phi_t(x_0) = \phi(s, \phi(t, x_0)) = e^{As}e^{At}x_0 = e^{At}e^{As}x_0 = \phi(t, \phi(s, x_0)) = \phi_t \circ \phi_s(x_0).$$

With $x_0' = \phi(t, x_0) = e^{At}x_0$ we have

$$\phi(-t, \phi(t, x_0)) = \phi(-t, x_0') = e^{-At}x_0' = e^{-At}e^{At}x_0 = x_0.$$

Example 6 *Flow of a Nonlinear System*

Let

$$f(x) = \begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}.$$

With $x_{02} \neq 0$ the flow is

$$\phi(t, x_0) = \begin{bmatrix} x_{01} \\ x_{02} + t \\ (x_{02} + t)\dfrac{x_{03}}{x_{02}} \end{bmatrix}$$

for $x_{02} + t \neq 0$. With $x'_0 = \begin{bmatrix} x'_{01} \\ x'_{02} \\ x'_{03} \end{bmatrix}$ we have $\phi(-t, x'_0) = \begin{bmatrix} x'_{01} \\ x'_{02} - t \\ (x'_{02} - t)\dfrac{x'_{03}}{x'_{02}} \end{bmatrix}$. Then

$$\phi(-t, \phi(t, x_0)) = \phi(-t, x'_0)_{x'_0 = \phi(t, x_0)} = \begin{bmatrix} x'_{01} \\ x'_{02} - t \\ (x'_{02} - t)\dfrac{x'_{03}}{x'_{02}} \end{bmatrix}_{|x'_0 = \phi(t, x_0)} = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}.$$

Now we consider t fixed and look at the mapping $\phi_t(x) = \phi(t, x) : \mathbb{R}^n \to \mathbb{R}^n$. This map takes the point $x \in \mathbb{R}^n$ to the new point $x \in \mathbb{R}^n$ by following the vector field $f(x)$ for the fixed time t. By the definition of $\phi(t, x)$ we have $\phi(0, x) = x$ so that

$$\frac{\partial}{\partial x}\phi(t, x)_{|t=0} = \frac{\partial}{\partial x}\phi(0, x) = \frac{\partial}{\partial x}x = I_{n\times n}.$$

With the flow existing for $|t| < \epsilon$, $\phi_t(x) = \phi(t, x)$ has inverse $\phi_{-t}(x) = \phi(-t, x)$. In other words

$$\begin{aligned} \phi\left(-t, \phi(t, x)\right) &= \phi_{-t}(\phi_t(x)) \equiv x \\ \phi\left(t, \phi(-t, x)\right) &= \phi_t(\phi_{-t}(x)) \equiv x. \end{aligned}$$

So, for fixed t, the mapping $\phi_t(x) = \phi(t, x) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible in a neighborhood of x. Using the chain rule we have

$$\frac{\partial}{\partial x}\phi_{-t}(\phi_t(x)) = \frac{\partial}{\partial x'}\phi_{-t}(x')\Big|_{|x' = \phi_t(x)} \frac{\partial}{\partial x}(\phi_t(x)) = I_{n\times n}.$$

Example 7 *Flow of a Vector Field*

Let $f(x) = \begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}$ so $\phi(t, x) = \begin{bmatrix} x_1 \\ x_2 + t \\ (x_2 + t)\dfrac{x_3}{x_2} \end{bmatrix}$ with $\phi(0, x) = x$

and $\phi(-t, x') = \begin{bmatrix} x'_1 \\ x'_2 - t \\ (x'_2 - t)\dfrac{x'_3}{x'_2} \end{bmatrix}$ with $\phi(0, x') = x'$.

$$\begin{aligned} \frac{\partial}{\partial x'}\phi_{-t}(x')\Big|_{x' = \phi_t(x)} &= \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t\dfrac{x'_3}{(x'_2)^2} & \dfrac{x'_2 - t}{x'_2} \end{bmatrix}\right|_{x' = \phi_t(x) = \begin{bmatrix} x_1 \\ x_2 + t \\ (x_2 + t)\dfrac{x_3}{x_2} \end{bmatrix}} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t\dfrac{x_3/x_2}{x_2 + t} & \dfrac{x_2}{x_2 + t} \end{bmatrix}. \end{aligned}$$

As

$$\frac{\partial}{\partial x}\phi_t(x) = \phi(t,x) = \frac{\partial}{\partial x}\begin{bmatrix} x_1 \\ x_2 + t \\ (x_2+t)\dfrac{x_3}{x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t\dfrac{x_3}{x_2^2} & \dfrac{x_2+t}{x_2} \end{bmatrix}$$

we have

$$\frac{\partial}{\partial x'}\phi_{-t}(x')\Big|_{x'=\phi_t(x)} \frac{\partial}{\partial x}\phi_t(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t\dfrac{x_3/x_2}{x_2+t} & \dfrac{x_2}{x_2+t} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t\dfrac{x_3}{x_2^2} & \dfrac{x_2+t}{x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.3 Integral Manifolds

Let's start with the manifold $\mathbf{E}^3$ with the Cartesian coordinate system so

$$\mathbf{z}(x_1,x_2,x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{E}^3.$$

Suppose at each point of $\mathbf{E}^3$ there are two linearly independent vector fields of the form

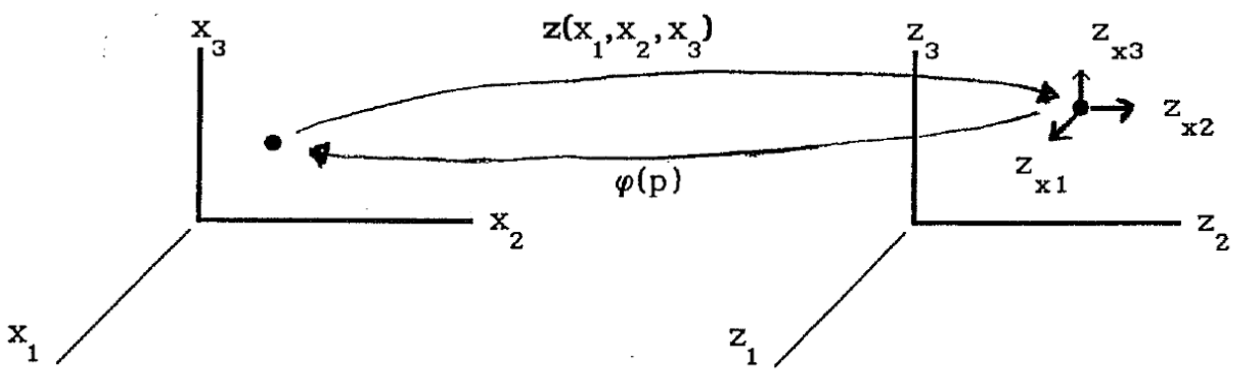


FIGURE 1.1. Cartesian coordinate system for $\mathbf{E}^3$.

$$f^{(1)}(x_1,x_2,x_3) = \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1,x_2) \end{bmatrix}, \quad f^{(2)}(x_1,x_2,x_3) = \begin{bmatrix} 0 \\ 1 \\ f_3^{(2)}(x_1,x_2) \end{bmatrix}$$

With $\mathcal{U}$ an open subset of $\mathbf{E}^3$ and $p_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} \in \mathcal{U}$ is there a two-dimensional surface (submanifold) containing p_0 and whose tangent vectors at each point of the surface are $f^{(1)}(x_1,x_2,x_3)$ and $f^{(2)}(x_1,x_2,x_3)$? That is, with $\mathcal{D} \subset \mathbb{R}^2$ an open set containing (x_{01},x_{02}) we are looking for a surface $S(x_1,x_2) : \mathcal{D} \to \mathcal{U}$ of the form

$$S(x_1,x_2) = \begin{bmatrix} x_1 \\ x_2 \\ s(x_1,x_2) \end{bmatrix}$$

such that $S(x_{01}, x_{02}) = \begin{bmatrix} x_{01} \\ x_{02} \\ s_3(x_{01}, x_{02}) \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$ and

$$\frac{\partial S}{\partial x_1} = \begin{bmatrix} 1 \\ 0 \\ \dfrac{\partial s(x_1, x_2)}{\partial x_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1, x_2) \end{bmatrix} = f^{(1)}(x_1, x_2, x_3)$$
$$\frac{\partial S}{\partial x_2} = \begin{bmatrix} 1 \\ 0 \\ \dfrac{\partial s(x_1, x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ f_3^{(2)}(x_1, x_2) \end{bmatrix} = f^{(2)}(x_1, x_2, x_3).$$

In this case the problem reduces to finding a scalar function $s_3(x_1, x_2)$ such that $s(x_{01}, x_{02}) = x_{03}$ and

$$\frac{\partial s(x_1, x_2)}{\partial x_1} = f_3^{(1)}(x_1, x_2)$$
$$\frac{\partial s(x_1, x_2)}{\partial x_2} = f_3^{(2)}(x_1, x_2).$$

If $s(x_1, x_2)$ exists then

$$\frac{\partial^2 s(x_1, x_2)}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2)$$
$$\frac{\partial^2 s(x_1, x_2)}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2)$$

or

$$\frac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2) = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2).$$

Exercise 3 *Lie Bracket*

Show that the Lie bracket of $[f^{(1)}, f^{(2)}] = 0$ for $x \in \mathcal{U}$ if and only if $\frac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2) = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2)$ for $x \in \mathcal{U}$.

Exercise 4 *Surface Specified by Tangent Vectors*

Let $f^{(1)} = \begin{bmatrix} 1 & 0 & x_2 \end{bmatrix}^T$ and $f^{(2)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ for all for $x \in \mathbf{E}^3$. Is there a surface $S(x_1, x_2) = \begin{bmatrix} x_1 & x_2 & s(x_1, x_2) \end{bmatrix}^T$ such that $\frac{\partial S}{\partial x_1} = f^{(1)}(x_1, x_2, x_3)$ and $\frac{\partial S}{\partial x_2} = f^{(2)}(x_1, x_2, x_3)$.

We now show that

$$\frac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2) = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2)$$

is a *sufficient* condition for a surface to exist.

Theorem 2 *Integrability of Vector Fields*

Let

$$f^{(1)}(x_1, x_2, x_3) = \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1, x_2) \end{bmatrix}, \quad f^{(2)}(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 1 \\ f_3^{(2)}(x_1, x_2) \end{bmatrix}$$

be vector fields defined on an open set $\mathcal{U} \subset \mathbf{E}^3$. Let $x_0 = \begin{bmatrix} x_{01} & x_{02} & x_{03} \end{bmatrix}^T \in \mathcal{U}$ and suppose that

$$\frac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2) = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2)$$

holds for all (x_1, x_2) in some neighborhood of (x_{01}, x_{02}). Then there is a surface $S(x_1, x_2)$ of the form

$$S(x_1,x_2)=\begin{bmatrix} x_1 \\ x_2 \\ s(x_1,x_2) \end{bmatrix} \text{ with } S(x_{01},x_{02})=\begin{bmatrix} x_{01} \\ x_{02} \\ s(x_{01},x_{02}) \end{bmatrix}=\begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

such that

$$\frac{\partial S}{\partial x_1}=f^{(1)}(x_1,x_2,x_3),\quad \frac{\partial S}{\partial x_2}=f^{(2)}(x_1,x_2,x_3).$$

Proof. With x_{02} fixed let

$$s(x_1,x_{02})\triangleq x_{03}+\int_{x_{01}}^{x_1} f_3^{(1)}(u,x_{02})du$$

to obtain the *curve*

$$S(x_1,x_{02})=\begin{bmatrix} x_1 \\ x_{02} \\ s(x_1,x_{02}) \end{bmatrix}=\begin{bmatrix} x_1 \\ x_{02} \\ x_{03}+\int_{x_{01}}^{x_1} f_3^{(1)}(u,x_{02})du \end{bmatrix}.$$

This satisfies

$$S(x_{01},x_{02})=\begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

and

$$\frac{\partial}{\partial x_1}S(x_1,x_{02})=\frac{\partial}{\partial x_1}\begin{bmatrix} x_1 \\ x_{02} \\ s(x_1,x_{02}) \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1,x_{02}) \end{bmatrix}=f^{(1)}(x_1,x_{02}).$$

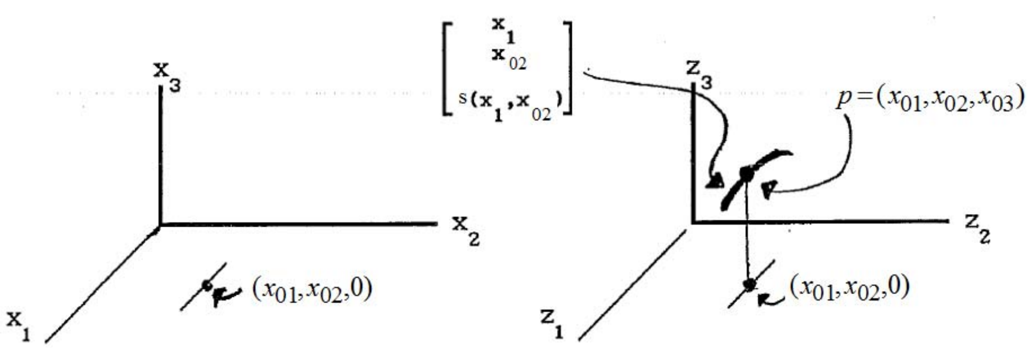


FIGURE 1.2. Follow the vector field $f^{(1)}$ from x_0.

Next define

$$s(x_1,x_2)\triangleq x_{03}+\int_{x_{01}}^{x_1} f_3^{(1)}(u,x_{02})du+\int_{x_{02}}^{x_2} f_3^{(2)}(x_1,v)dv.$$

Figure 1.3 illustrates the construction of this surface.

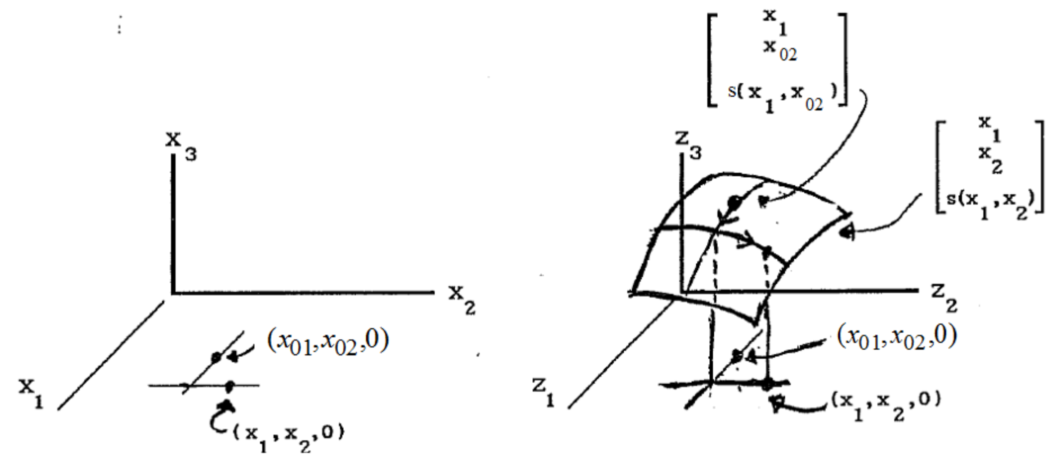


FIGURE 1.3. Follow the vector field $f^{(2)}$ from $\begin{bmatrix} x_1 & x_{02} & x_{03} \end{bmatrix}^T$.

Then

$$\begin{aligned}
\frac{\partial}{\partial x_1} S(x_1, x_2) &= \frac{\partial}{\partial x_1} \begin{bmatrix} x_1 \\ x_2 \\ x_{03} + \int_{x_{01}}^{x_1} f_3^{(1)}(u, x_{02}) du + \int_{x_{02}}^{x_2} f_3^{(2)}(x_1, v) dv \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1, x_{02}) + \int_{x_{02}}^{x_2} \dfrac{\partial f_3^{(2)}(x_1, v)}{\partial x_1} dv \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1, x_{02}) + \int_{x_{02}}^{x_2} \dfrac{\partial f_3^{(1)}(x_1, v)}{\partial v} dv \end{bmatrix} \quad \text{as } \frac{\partial}{\partial x_2} f_3^{(1)}(x_1, x_2) = \frac{\partial}{\partial x_1} f_3^{(2)}(x_1, x_2) \\
&= \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1, x_{02}) + f_3^{(1)}(x_1, v)\Big|_{v = x_{02}}^{v = x_2} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ f_3^{(1)}(x_1, x_{02}) + f_3^{(1)}(x_1, x_2) - f_3^{(1)}(x_1, x_{02}) \end{bmatrix} \\
&= f^{(1)}(x_1, x_2).
\end{aligned}$$

Further

$$\begin{aligned}
\frac{\partial}{\partial x_2} S(x_1, x_2) = \frac{\partial}{\partial x_2} \begin{bmatrix} x_1 \\ x_2 \\ x_{03} + \int_{x_{01}}^{x_1} f_3^{(1)}(u, x_{02}) du + \int_{x_{02}}^{x_2} f_3^{(2)}(x_1, v) dv \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \\ \dfrac{\partial}{\partial x_2} \int_{x_{02}}^{x_2} f_3^{(2)}(x_1, v) dv \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \\ f_3^{(2)}(x_1, x_2) \end{bmatrix} \\
&= f^{(2)}(x_1, x_2, x_3).
\end{aligned}$$

■

Example 8 *Integrability of Vector Fields*
Let

$$f^{(1)}(x_1,x_2,x_3) = \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix}, \quad f^{(2)}(x_1,x_2,x_3) = \begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix}.$$

Computing

$$\frac{\partial}{\partial x_2}(2x_1x_2) = 2x_1$$
$$\frac{\partial}{\partial x_1}(x_1^2) = 2x_1$$

shows the integrability conditions are satisfied. Then $s(x_1,x_2)$ is given by

$$s(x_1,x_2) = x_{03} + \int_{x_{01}}^{x_1} 2ux_{02}du + \int_{x_{02}}^{x_2} x_1^2 dv = x_{03} + (x_1^2 - x_{01}^2)x_{02} + x_1^2(x_2 - x_{02}) = x_{03} + x_1^2x_2 - x_{01}^2x_{02}$$

and the surface is

$$S(x_1,x_2) = \begin{bmatrix} x_1 \\ x_2 \\ x_{03} + x_1^2x_2 - x_{01}^2x_{02} \end{bmatrix}.$$

To check we have

$$S(x_1,x_2) = \begin{bmatrix} x_1 \\ x_2 \\ x_{03} + x_1^2x_2 - x_{01}^2x_{02} \end{bmatrix}\Bigg|_{x=x_0} = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

and

$$\frac{\partial}{\partial x_1}S(x_1,x_2) = \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix} = f^{(1)}(x_1,x_2), \quad \frac{\partial}{\partial x_2}S(x_1,x_2) = \begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix} = f^{(2)}(x_1,x_2).$$

Let's take another look at the previous example. The idea is to start at any desired point $\begin{bmatrix} x_{01} & x_{02} & x_{03} \end{bmatrix}^T$ which the surface is to pass through. Next follow the vector field $f^{(1)}$ for a time t_1 and then follow the vector field $f^{(2)}$ for a time t_2.

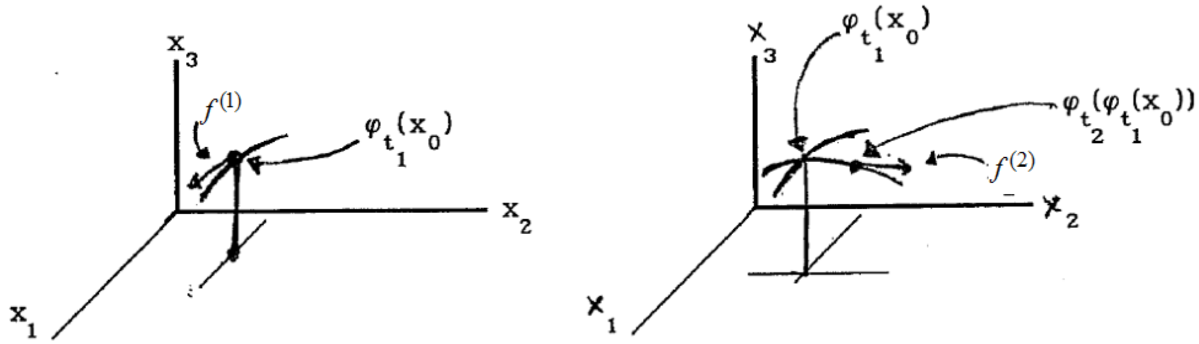


FIGURE 1.4. Follow $f^{(1)}$ for a time t_1 and then $f^{(2)}$ for a time t_2.

We solve

$$\frac{d}{dt_1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix}}_{f^{(1)}}, \quad x(0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

to obtain

$$\phi_{t_1}(x_0) = \begin{bmatrix} t + x_{01} \\ x_{02} \\ x_{03} + (t_1 + x_{01})^2 x_{02} - x_{01}^2 x_{02} \end{bmatrix}.$$

It is straightforward to check that

$$\frac{\partial}{\partial t_1}\phi_{t_1}(x_0) = f^{(1)}(\phi_{t_1}(x_0)) \quad \text{with} \quad \phi_0(x_0) = x_0.$$

Now from $x_0' = \phi_{t_1}(x_0)$ we follow $f^{(2)}$ by solving

$$\frac{d}{dt_2}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix}}_{f^{(2)}}, \quad x(0) = x_0' \triangleq \phi_{t_1}(x_0).$$

The solution is

$$\phi_{t_2}(x_0) = \begin{bmatrix} x_{01}' \\ t + x_{02}' \\ x_{03}' + (x_{01}')^2 t_2 \end{bmatrix}.$$

Straightforwardly it can be checked that

$$\frac{\partial}{\partial t_2}\phi_{t_2}(x_0) = f^{(1)}(\phi_{t_2}(x_0')) \quad \text{with} \quad \phi_0(x_0') = x_0'.$$

Form the surface as the composition of these two solutions given by

$$\begin{aligned}
\phi_{t_2}(\phi_{t_1}(x_0)) &= \begin{bmatrix} x_{01}' \\ t_2 + x_{02}' \\ x_{03}' + (x_{01}')^2 t_2 \end{bmatrix}\Bigg|_{x_0' = \phi_{t_1}(x_0) = \begin{bmatrix} t_1 + x_{01} \\ x_{02} \\ x_{03} + (t_1 + x_{01})^2 x_{02} - x_{01}^2 x_{02} \end{bmatrix}} \\
&= \begin{bmatrix} t_1 + x_{01} \\ t_2 + x_{02} \\ x_{03} + (t_1 + x_{01})^2 x_{02} - x_{01}^2 x_{02} + (t_1 + x_{01})^2 t_2 \end{bmatrix} \\
&= \begin{bmatrix} t_1 + x_{01} \\ t_2 + x_{02} \\ x_{03} + (t_1 + x_{01})^2 (x_{02} + t_2) - x_{01}^2 x_{02} \end{bmatrix}.
\end{aligned}$$

So the surface $S(t_1, t_2)$ is defined by

$$S(t_1, t_2) = \begin{bmatrix} x_1(t_1, t_2) \\ x_2(t_1, t_2) \\ x_3(t_1, t_2) \end{bmatrix} = \phi_{t_2}(\phi_{t_1}(x_0)) = \begin{bmatrix} t_1 + x_{01} \\ t_2 + x_{02} \\ x_{03} + (t_1 + x_{01})^2 (x_{02} + t_2) - x_{01}^2 x_{02} \end{bmatrix}$$

with tangent vectors to the surface given by

$$\begin{aligned}
\frac{\partial S}{\partial t_1} &= \begin{bmatrix} 1 \\ 0 \\ 2(t_1 + x_{01})(x_{02} + t_2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2x_1(t_1)x_2(t_2) \end{bmatrix} \\
\frac{\partial S}{\partial t_2} &= \begin{bmatrix} 0 \\ 1 \\ (t_1 + x_{01})^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ x_1^2(t_1) \end{bmatrix}.
\end{aligned}$$

Exercise 5 *Lie Brackets and Integrability of Vector Fields*

Starting at $x_0 = \begin{bmatrix} x_{01} & x_{02} & x_{03} \end{bmatrix}^T$ follow the vector field $f^{(2)} = \begin{bmatrix} 0 & 1 & x_1^2 \end{bmatrix}^T$ for t_2 units of time and then follow $f^{(1)} = \begin{bmatrix} 1 & 0 & 2x_1x_2 \end{bmatrix}^T$ for t_1 units of time. That is, compute $\phi_{t_1}(\phi_{t_2}(x_0))$ as shown above. If you do it right you should find that $\phi_{t_1}(\phi_{t_2}(x_0)) = \phi_{t_2}(\phi_{t_1}(x_0))$. It will be show later that this is guaranteed by the fact that $[f^{(1)}, f^{(2)}] \equiv 0$.

Exercise 6 *Lie Brackets and Integrability of Vector Fields*

Consider the two vector fields on $\mathbf{E}^3$ given by

$$f^{(1)}(x_1, x_2) = \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 + x_2^2 \end{bmatrix}, \quad f^{(2)}(x_1, x_2) = \begin{bmatrix} 0 \\ 1 \\ x_1^2 + 3x_1x_2^2 \end{bmatrix}.$$

With x_0 the initial state, compute $\phi_{t_2}(\phi_{t_1}(x_0))$ and show that the surface $S(t_1, t_2) \triangleq \phi_{t_2}(\phi_{t_1}(x_0))$ satisfies $S(0,0) = x_0$ and $\dfrac{\partial S}{\partial t_1} = f^{(1)}, \dfrac{\partial S}{\partial t_2} = f^{(2)}$. Also show that $[f^{(1)}, f^{(2)}] = 0$ for all $x \in \mathbf{E}^3$.

Exercise 7 *Lie Brackets and Integrability of Vector Fields*

Consider the two vector fields on $\mathbf{E}^3$ given by

$$f^{(1)}(x_1, x_2) = \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix}, \quad f^{(2)}(x_1, x_2) = \begin{bmatrix} 0 \\ 1 \\ x_1^2x_2 \end{bmatrix}.$$

(a) Show that $[f^{(1)}, f^{(2)}] \neq 0$ on $\mathbf{E}^3$ as $\dfrac{\partial}{\partial x_2}(2x_1x_2) \neq \dfrac{\partial}{\partial x_1}(x_1^2x_2)$.

(b) With x_0 the initial state, compute $\phi_{t_2}(\phi_{t_1}(x_0))$ and show that $\phi_{t_2}(\phi_{t_1}(x_0))$ satisfies $S(0,0) = x_0$ and $\dfrac{\partial S}{\partial t_2} = f^{(2)}$, but $\dfrac{\partial S}{\partial t_1} \neq f^{(1)}$ unless $t_2 = 0$.

(c) With x_0 the initial state, compute $\phi_{t_1}(\phi_{t_2}(x_0))$ and show that $\phi_{t_1}(\phi_{t_2}(x_0))$ satisfies $S(0,0) = x_0$ and $\dfrac{\partial S}{\partial t_1} = f^{(1)}$, but $\dfrac{\partial S}{\partial t_2} \neq f^{(2)}$ unless $t_1 = 0$.

Let's consider another example.

Example 9 *Integrability of Vector Fields*

Consider the two vector fields on $\mathbf{E}^3$ given by

$$f^{(1)}(x_1, x_2, x_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad f^{(2)}(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}.$$

With the initial state x_0 follow the vector field $f^{(1)}$ by solving

$$\frac{d}{dt_1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{f^{(1)}} \text{ with } x(0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}.$$

The solution is

$$\phi_{t_1}(x_0) = \begin{bmatrix} t_1 + x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}.$$

Then with the initial condition $x_0' = \phi_{t_1}(x_0)$ we follow $f^{(2)}$ by solving

$$\frac{d}{dt_1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}}_{f^{(2)}} \text{ with } x(0) = \begin{bmatrix} x_{01}' \\ x_{02}' \\ x_{03}' \end{bmatrix}.$$

This has solution

$$\phi_{t_2}(x_0') = \begin{bmatrix} x_{01}' \\ t_2 + x_{02}' \\ (x_{03}'/x_{02}')(t_2 + x_{02}') \end{bmatrix}.$$

The resulting surface is

$$S(t_1,t_2) = \begin{bmatrix} x_1(t_1,t_2) \\ x_2(t_1,t_2) \\ x_3(t_1,t_2) \end{bmatrix} = \phi_{t_2}(\phi_{t_1}(x_0)) = \begin{bmatrix} x_{01}' \\ t_2 + x_{02}' \\ (x_{03}'/x_{02}')(t_2 + x_{02}') \end{bmatrix}\Bigg|_{x_0'=\phi_{t_1}(x_0)=\begin{bmatrix} t_1 + x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}}$$

$$= \begin{bmatrix} t_1 + x_{01} \\ t_2 + x_{02} \\ (x_{03}/x_{02})(t_2 + x_{02}) \end{bmatrix}.$$

It is straightforward to check that $S(0,0) = x_0$ and

$$\frac{\partial S}{\partial t_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\partial S}{\partial t_2} = \begin{bmatrix} 0 \\ 1 \\ x_{03}/x_{02} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}.$$

Also note that $[f^{(1)}, f^{(2)}] = 0$ for all $x \in \mathbf{E}^3$.

Exercise 8 *Integrability of Vector Fields*
In the previous example show, by direct computation, that $\phi_{t_1}(\phi_{t_2}(x_0)) = \phi_{t_2}(\phi_{t_1}(x_0))$. This is guaranteed to hold because $[f^{(1)}, f^{(2)}] \equiv 0$.

We now make a generalization of Theorem 2. With the Cartesian coordinate system on $\mathbf{E}^3$ consider the two vector fields defined on an open set $\mathcal{U} \subset \mathbf{E}^3$ given by

$$f^{(1)}(x_1,x_2,x_3) = \begin{bmatrix} f_1^{(1)}(x_1,x_2,x_3) \\ f_2^{(1)}(x_1,x_2,x_3) \\ f_3^{(1)}(x_1,x_2,x_3) \end{bmatrix}, \quad f^{(2)}(x_1,x_2,x_3) = \begin{bmatrix} f_1^{(2)}(x_1,x_2,x_3) \\ f_2^{(2)}(x_1,x_2,x_3) \\ f_3^{(2)}(x_1,x_2,x_3) \end{bmatrix}. \tag{1.5}$$

Given any point $x_0 \in \mathcal{U}$ and the vector fields $f^{(1)}, f^{(2)}$ we want to a surface $S(t_1,t_2)$ of the form

$$S(t_1,t_2) = \begin{bmatrix} s_1(t_1,t_2) \\ s_2(t_1,t_2) \\ s_3(t_1,t_2) \end{bmatrix} \tag{1.6}$$

with $S(0,0) = x_0$ and satisfying

$$\frac{\partial}{\partial t_1}S(t_1,t_2) = \begin{bmatrix} \frac{\partial}{\partial t_1}s_1(t_1,t_2) \\ \frac{\partial}{\partial t_1}s_2(t_1,t_2) \\ \frac{\partial}{\partial t_1}s_3(t_1,t_2) \end{bmatrix} = \begin{bmatrix} f_1^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \\ f_2^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \\ f_3^{(1)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \end{bmatrix} \tag{1.7}$$

$$\frac{\partial}{\partial t_2}S(t_1,t_2) = \begin{bmatrix} \frac{\partial}{\partial t_2}s_1(t_1,t_2) \\ \frac{\partial}{\partial t_2}s_2(t_1,t_2) \\ \frac{\partial}{\partial t_2}s_3(t_1,t_2) \end{bmatrix} = \begin{bmatrix} f_1^{(2)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \\ f_2^{(2)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \\ f_3^{(2)}(s_1(t_1,t_2), s_2(t_1,t_2), s_3(t_1,t_2)) \end{bmatrix}. \tag{1.8}$$

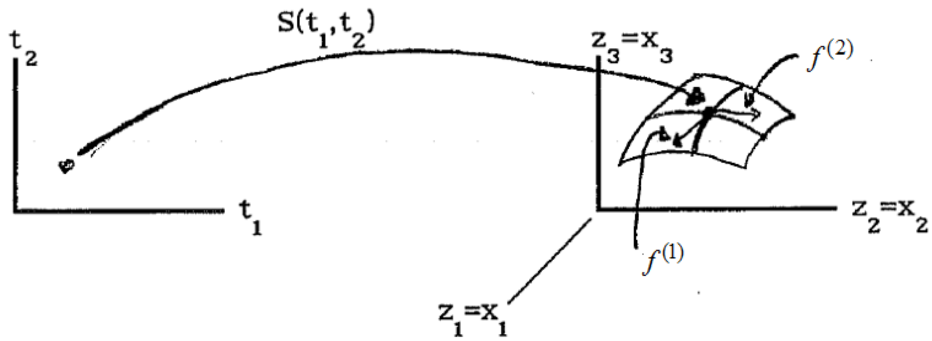


FIGURE 1.5. Integrability of vector fields.

We first show that a *necessary* condition for such a surface to exist is that $[f^{(1)},f^{(2)}] = \dfrac{\partial f^{(1)}}{\partial x}f^{(2)} - \dfrac{\partial f^{(2)}}{\partial x}f^{(1)} = 0_{3\times 1}$ for $x \in \mathcal{U}$. We do this by showing that $[f^{(1)},f^{(2)}] = 0$ is equivalent to

$$\frac{\partial^2 S(t_1,t_2)}{\partial t_2 \partial t_1} - \frac{\partial^2 S(t_1,t_2)}{\partial t_1 \partial t_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To proceed suppose the surface exists so we have

$$\frac{\partial}{\partial t_1} S(t_1,t_2) = \begin{bmatrix} f_1^{(1)}(s_1(t_1,t_2),s_2(t_1,t_2),s_3(t_1,t_2)) \\ f_2^{(1)}(s_1(t_1,t_2),s_2(t_1,t_2),s_3(t_1,t_2)) \\ f_3^{(1)}(s_1(t_1,t_2),s_2(t_1,t_2),s_3(t_1,t_2)) \end{bmatrix}$$

and therefore

$$\begin{aligned}\frac{\partial^2 S(t_1,t_2)}{\partial t_2 \partial t_1} &= \begin{bmatrix} \frac{\partial f_1^{(1)}}{\partial x_1} & \frac{\partial f_1^{(1)}}{\partial x_2} & \frac{\partial f_1^{(1)}}{\partial x_3} \\ \frac{\partial f_2^{(1)}}{\partial x_1} & \frac{\partial f_2^{(1)}}{\partial x_2} & \frac{\partial f_2^{(1)}}{\partial x_3} \\ \frac{\partial f_3^{(1)}}{\partial x_1} & \frac{\partial f_3^{(1)}}{\partial x_2} & \frac{\partial f_3^{(1)}}{\partial x_3} \end{bmatrix}_{|x=S(t_1,t_2)} \begin{bmatrix} \frac{\partial s_1(t_1,t_2)}{\partial t_2} \\ \frac{\partial s_2(t_1,t_2)}{\partial t_2} \\ \frac{\partial s_3(t_1,t_2)}{\partial t_2} \end{bmatrix} \\ &= \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)} f^{(2)}(s_1(t_1,t_2),s_2(t_1,t_2),s_3(t_1,t_2)).\end{aligned}$$

Similarly

$$\frac{\partial^2 S(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)} f^{(1)}(s_1(t_1,t_2),s_2(t_1,t_2),s_3(t_1,t_2)).$$

Consequently

$$\frac{\partial^2 S(t_1,t_2)}{\partial t_2 \partial t_1} - \frac{\partial^2 S(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial f^{(1)}}{\partial x}f^{(2)} - \frac{\partial f^{(2)}}{\partial x}f^{(1)}\Big|_{x=(s_1(t_1,t_2),s_2(t_1,t_2),s_3(t_1,t_2))} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is not quite the condition we claimed because it is in terms of $S(t_1,t_2)$ which is unknown! However we require that given any point $x_0 \in \mathcal{U} \subset \mathbf{E}^3$ there is a surface $S(t_1,t_2)$ through it. In particular at $(t_1,t_2) = (0,0)$

we have $S(0,0)=x_0$ and $\left.\dfrac{\partial f^{(1)}}{\partial x}f^{(2)}-\dfrac{\partial f^{(2)}}{\partial x}f^{(1)}\right|_{x_0}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. As x_0 is an arbitrary point of $\mathcal{U}$ we must have

$$[f^{(1)},f^{(2)}]=\frac{\partial f^{(1)}}{\partial x}f^{(2)}-\frac{\partial f^{(2)}}{\partial x}f^{(1)}\equiv\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{1.9}$$

for all $x_0\in\mathcal{U}\subset\mathbf{E}^3$.

We now want to $[f^{(1)},f^{(2)}]\equiv 0$ is sufficient for such a surface to exist.

Theorem 3 $[f^{(1)},f^{(2)}]\equiv 0$ *Implies an Integral Manifold Exists*
Let $f^{(1)}$ and $f^{(2)}$ be vector fields defined on a open set $\mathcal{U}\subset\mathbf{E}^3$ with $[f^{(1)},f^{(2)}]\equiv 0$ on $\mathcal{U}$. Then for any point $x_0\in\mathcal{U}$ there exists a neighborhood $\mathcal{D}\subset\mathbb{R}^2$ containing the origin $(0,0)$ for which a surface $S(t_1,t_2):\mathcal{D}\rightarrow\mathcal{U}$ exists satisfying $S(0,0)=x_0$ and

$$\frac{\partial}{\partial t_1}S(t_1,t_2)=f^{(1)}(s_1(t_1,t_2),s_2(t_1,t_2),s_3(t_1,t_2)) \tag{1.10}$$

$$\frac{\partial}{\partial t_2}S(t_1,t_2)=f^{(2)}(s_1(t_1,t_2),s_2(t_1,t_2),s_3(t_1,t_2)). \tag{1.11}$$

Proof. Consider the differential equation defined by

$$\frac{d}{dt_1}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}f_1^{(1)}(x_1,x_2,x_3)\\f_2^{(1)}(x_1,x_2,x_3)\\f_3^{(1)}(x_1,x_2,x_3)\end{bmatrix}\quad\text{with}\;\;x(0)=x_0.$$

Let the solution $\phi_{t_1}(x_0)$ exist for $|t_1|<\epsilon_1$ for some $\epsilon_1>0$. So $\dfrac{d}{dt_1}\phi_{t_1}(x_0)=f^{(1)}(\phi_{t_1}(x_0))$ with $\phi_0(x_0)=x_0$. This is the solution at time t_1 starting from x_0 and moving in the direction $f^{(1)}$. Now consider the differential equation defined by the vector field $f^{(2)}$ given by

$$\frac{d}{dt_2}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}f_1^{(2)}(x_1,x_2,x_3)\\f_2^{(2)}(x_1,x_2,x_3)\\f_3^{(2)}(x_1,x_2,x_3)\end{bmatrix}\quad\text{with}\;\;x(0)=x_0'=\phi_{t_1}(x_0).$$

Let the solution $\phi_{t_2}(x_0')$ exist for $|t_2|<\epsilon_2$ for some $\epsilon_2>0$. So $\dfrac{d}{dt_2}\phi_{t_2}(x_0)=f^{(2)}(\phi_{t_2}(x_0'))$ with $\phi_0(x_0')=x_0'$. Next define

$$S(t_1,t_2)=\begin{bmatrix}x_1(t_1,t_2)\\x_2(t_1,t_2)\\x_3(t_1,t_2)\end{bmatrix}\triangleq\phi_{t_2}(\phi_{t_1}(x_0))$$

where by the construction of $\phi_{t_2}(\phi_{t_1}(x_0))$ we have

$$\frac{\partial}{\partial t_2}S(t_1,t_2)=\frac{\partial}{\partial t_2}\phi_{t_2}(\phi_{t_1}(x_0))=f^{(2)}(\phi_{t_2}(\phi_{t_1}(x_0))).$$

Further, at $t_2=0$ we have

$$\left.\frac{\partial}{\partial t_1}S(t_1,t_2)\right|_{t_2=0}=\frac{\partial}{\partial t_1}\phi_0(\phi_{t_1}(x_0))=\frac{\partial}{\partial t_1}\phi_{t_1}(x_0)=f^{(1)}(\phi_{t_1}(x_0))=f^{(1)}(S(t_1,0)).$$

We now need to show that

$$\frac{\partial}{\partial t_1}S(t_1,t_2)=f^{(1)}(S(t_1,t_2))$$

for all $|t_2| < \epsilon_2$ (not just $t_2 = 0$). Define

$$g(t_2) = \frac{\partial}{\partial t_1}S(t_1,t_2) - f^{(1)}(S(t_1,t_2)) \in \mathbf{E}^3$$

where $g(0) = 0_{3\times 1}$. Differentiating $g(t_2)$ we have

$$\begin{aligned}
\frac{d}{dt_2}g(t_2) &= \frac{\partial}{\partial t_2}\left(\frac{\partial}{\partial t_1}S(t_1,t_2)\right) - \frac{\partial}{\partial t_2}f^{(1)}(S(t_1,t_2))\\
&= \frac{\partial}{\partial t_1}\left(\frac{\partial}{\partial t_2}S(t_1,t_2)\right) - \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)}\frac{\partial S(t_1,t_2)}{\partial t_2} \ \text{ as } \ \frac{\partial^2 S(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial^2 S(t_1,t_2)}{\partial t_2 \partial t_1}\\
&= \frac{\partial}{\partial t_1}f^{(2)}(S(t_1,t_2)) - \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)}f^{(2)}(S(t_1,t_2))\\
&= \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}\frac{\partial S(t_1,t_2)}{\partial t_1} - \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)}f^{(2)}(S(t_1,t_2))\\
&= \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}\Big(g(t_2) + f^{(1)}(S(t_1,t_2)\Big) - \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)}f^{(2)}(S(t_1,t_2))\\
&= \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}g(t_2) + \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}f^{(1)}(S(t_1,t_2) - \frac{\partial f^{(1)}}{\partial x}_{|x=S(t_1,t_2)}f^{(2)}(S(t_1,t_2))\\
&= \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}g(t_2) - \left(\frac{\partial f^{(1)}}{\partial x}f^{(2)}(x) - \frac{\partial f^{(2)}}{\partial x}f^{(1)}(x)\right)_{|x=S(t_1,t_2)}\\
&= \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)}g(t_2) \ \text{ as } \ [f^{(1)},f^{(2)}] \equiv 0.
\end{aligned}$$

We have

$$\frac{d}{dt_2}g(t_2) = A_{t_1}(t_2)g(t_2) \ \text{ with } \ g(0) = 0 \tag{1.12}$$

where $A_{t_1}(t_2) \triangleq \frac{\partial f^{(2)}}{\partial x}_{|x=S(t_1,t_2)} \in \mathbb{R}^{3\times 3}$. For each fixed t_1 this is a *linear time varying* differential equation in t_2. This has the unique solution $g(t_2) \equiv 0$ as the components of $\frac{\partial f^{(2)}}{\partial x}$ are assumed to be continuously differentiable. ∎

Remark We defined $S(t_1,t_2) \triangleq \phi_{t_2}(\phi_{t_1}(x_0))$ and it is always true that $\frac{\partial}{\partial t_2}S(t_1,t_2) = \frac{\partial}{\partial t_2}\phi_{t_2}(\phi_{t_1}(x_0)) = f^{(2)}(\phi_{t_2}(\phi_{t_1}(x_0)))$ by the definition of ϕ_{t_2} whether or not the Lie bracket of $f^{(1)}$ and $f^{(2)}$ is zero. However, $\frac{\partial}{\partial t_1}S(t_1,t_2) = \frac{\partial}{\partial t_1}\phi_{t_2}(\phi_{t_1}(x_0)) = f^{(1)}(\phi_{t_2}(\phi_{t_1}(x_0)))$ if and only if $[f^{(1)},f^{(2)}] \equiv 0$ on $\mathcal{U}$.

On the other hand, if we had defined $S(t_1,t_2) \triangleq \phi_{t_1}(\phi_{t_2}(x_0))$ then it is always true that $\frac{\partial}{\partial t_1}S(t_1,t_2) = \frac{\partial}{\partial t_1}\phi_{t_1}(\phi_{t_2}(x_0)) = f^{(1)}(\phi_{t_1}(\phi_{t_2}(x_0)))$ by the definition of ϕ_{t_1} whether or not the Lie bracket of $f^{(1)}$ and $f^{(2)}$ is zero. However, $\frac{\partial}{\partial t_2}S(t_1,t_2) = \frac{\partial}{\partial t_2}\phi_{t_1}(\phi_{t_2}(x_0)) = f^{(2)}(\phi_{t_1}(\phi_{t_2}(x_0)))$ if and only if $[f^{(1)},f^{(2)}] \equiv 0$ on $\mathcal{U}$.

Corollary 1 $[f^{(1)},f^{(2)}] \equiv 0 \Longrightarrow \phi_{t_2}(\phi_{t_1}(x_0)) = \phi_{t_1}(\phi_{t_2}(x_0))$

Let $f^{(1)}$ and $f^{(2)}$ be vector fields defined on a open set $\mathcal{U} \subset \mathbf{E}^3$ with $[f^{(1)},f^{(2)}] \equiv 0$ on $\mathcal{U}$. Then

$$\phi_{t_2}(\phi_{t_1}(x_0)) = \phi_{t_1}(\phi_{t_2}(x_0))$$

for $-\epsilon < t_1, t_2 < \epsilon$ and some $\epsilon > 0$.

Proof. As $[f^{(1)}, f^{(2)}] \equiv 0$ on $\mathcal{U}$. For any fixed t_1 with $-\epsilon < t_1 < \epsilon$ Theorem 3 tells us that

$$\frac{d}{dt_2}\phi_{t_2}(\phi_{t_1}(x_0)) = f^{(2)}(\phi_{t_2}(\phi_{t_1}(x_0))) \text{ with } \phi_{t_2}(\phi_{t_1}(x_0))_{|t_2=0} = \phi_{t_1}(x_0)$$

and

$$\frac{d}{dt_2}\phi_{t_1}(\phi_{t_2}(x_0)) = f^{(2)}(\phi_{t_1}(\phi_{t_2}(x_0))) \text{ with } \phi_{t_1}(\phi_{t_2}(x_0))_{|t_2=0} = \phi_{t_1}(x_0).$$

That is, for each t_1, $\phi_{t_2}(\phi_{t_1}(x_0))$ and $\phi_{t_1}(\phi_{t_2}(x_0))$ both satisfy the same differential equation with the small initial state. By the uniqueness of the solution to this differential equation it follows that

$$\phi_{t_2}(\phi_{t_1}(x_0)) = \phi_{t_1}(\phi_{t_2}(x_0)).$$

∎

Example 10 *Integrability of Vector Fields*

Consider the two vector fields

$$f^{(1)}(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad f^{(2)}(x) = \begin{bmatrix} -x_2 \\ x_1 \\ x_3 \end{bmatrix}.$$

where a straightforward calculation shows $[f^{(1)}, f^{(2)}] \equiv 0$. The solution to

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{with} \quad x(0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

is

$$\varphi_{t_1}(x_0) = \begin{bmatrix} x_{01}e^{t_1} \\ x_{02}e^{t_1} \\ x_{03}e^{t_1} \end{bmatrix}.$$

The solution to

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \\ x_3 \end{bmatrix} \quad \text{with} \quad x'(0) = \begin{bmatrix} x'_{01} \\ x'_{02} \\ x'_{03} \end{bmatrix}$$

is

$$\varphi_{t_2}(x'_0) = \begin{bmatrix} x'_{01}\cos(t_2) - x'_{02}\sin(t_2) \\ x'_{01}\sin(t_2) + x'_{02}\cos(t_2) \\ x'_{03}e^{t_2} \end{bmatrix}.$$

Then

$$\begin{aligned} S(t_1,t_2) = \varphi_{t_2}(\varphi_{t_1}(x_0)) &= \left.\begin{bmatrix} x'_{01}\cos(t_2) - x'_{02}\sin(t_2) \\ x'_{01}\sin(t_2) + x'_{02}\cos(t_2) \\ x'_{03}e^{t_2} \end{bmatrix}\right|_{x'_0=\varphi_{t_1}(x_0)=\begin{bmatrix} x_{01}e^{t_1} \\ x_{02}e^{t_1} \\ x_{03}e^{t_1} \end{bmatrix}} \\ &= \begin{bmatrix} \big(x_{01}\cos(t_2) - x_{02}\sin(t_2)\big)e^{t_1} \\ \big(x_{01}\sin(t_2) + x_{02}\cos(t_2)\big)e^{t_1} \\ x_{03}e^{t_1+t_2} \end{bmatrix}. \end{aligned}$$

It is straightforward to see that $S(0,0) = x_0$, and

$$\frac{\partial}{\partial t_1} S(t_1,t_2) = f^{(1)}(S(t_1,t_2)) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{|x=S(t_1,t_2)}$$

$$\frac{\partial}{\partial t_2} S(t_1,t_2) = f^{(2)}(S(t_1,t_2)) = \begin{bmatrix} -x_2 \\ x_1 \\ x_3 \end{bmatrix}_{|x=S(t_1,t_2)} = \begin{bmatrix} -\big(x_{01}\sin(t_2) + x_{02}\cos(t_2)\big)e^{t_1} \\ \big(x_{01}\cos(t_2) - x_{02}\sin(t_2)\big)e^{t_1} \\ x_{03}e^{t_1+t_2} \end{bmatrix}$$

Exercise 9 *Integrability of Vector Fields*
In the previous example show that $\phi_{t_1}(\phi_{t_2}(x_0)) = \phi_{t_2}(\phi_{t_1}(x_0))$ by directly computing $\phi_{t_1}(\phi_{t_2}(x_0))$.

Example 11 *Integrability of Vector Fields*
Consider the two vector fields on $\mathbf{E}^3$ given by

$$f^{(1)}(x) = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}, \quad f^{(2)}(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

It is easy to check that $[f^{(1)}, f^{(2)}] \neq 0$ anywhere in $\mathbf{E}^3$. Then

$$\phi_{t_2}(\phi_{t_1}(x_0)) = \begin{bmatrix} x'_{01} \\ x'_{02} + t_2 \\ x'_{03} \end{bmatrix}_{x=\begin{bmatrix} x_{01}+t_1 \\ x_{02} \\ x_{03}+x_{02}t_1 \end{bmatrix}} = \begin{bmatrix} x_{01}+t_1 \\ x_{02}+t_2 \\ x_{03}+x_{02}t_1 \end{bmatrix}$$

while

$$\phi_{t_1}(\phi_{t_2}(x_0)) = \begin{bmatrix} x'_{01}+t_1 \\ x'_{02} \\ x'_{03}+x'_{02}t_1 \end{bmatrix}_{x=\begin{bmatrix} x_{01} \\ x_{02}+t_2 \\ x_{03} \end{bmatrix}} = \begin{bmatrix} x_{01}+t_1 \\ x_{02}+t_2 \\ x_{03}+(x_{02}+t_2)t_1 \end{bmatrix}$$

showing that $\phi_{t_2}(\phi_{t_1}(x_0)) \neq \phi_{t_1}(\phi_{t_2}(x_0))$.

Exercise 10 *Integrability of Vector Fields*
Consider the two vector fields

$$f^{(1)}(x) = \begin{bmatrix} x_1+x_2 \\ 2x_2 \\ x_3 \end{bmatrix}, \quad f^{(2)}(x) = \begin{bmatrix} 2x_1+3x_2 \\ x_2 \\ x_3 \end{bmatrix}.$$

(a) Show $[f^{(1)}, f^{(2)}] \equiv 0$.

(b) Compute $\phi_{t_1}(x_0)$.

(c) Compute $S(t_1,t_2) = \phi_{t_2}(\phi_{t_1}(x_0))$.

(d) Show $\dfrac{\partial}{\partial t_2} S(t_1,t_2) = f^{(2)}(S(t_1,t_2))$.

(e) $\dfrac{\partial}{\partial t_1} S(t_1,t_2) \neq f^{(1)}(S(t_1,t_2))$ unless $t_2 = 0$.

(f) Repeat part (c) with $S(t_1,t_2) = \phi_{t_1}(\phi_{t_2}(x_0))$. How do parts (d) and (e) change?

Exercise 11 *Integrability of Linear Vector Fields*
With $x \in \mathbb{R}^3, A_1 \in \mathbb{R}^{3\times 3}, A_2 \in \mathbb{R}^{3\times 3}$ let $f^{(1)} = A_1 x$ and $f^{(2)} = A_2 x$. Show $[f^{(1)}, f^{(2)}] \equiv 0$ if and only if $A_1A_2 - A_2A_1 = 0_{3\times 3}$, i.e., if and only if A_1 and A_2 commute.

Exercise 12 *Integrability of Linear Vector Fields*
With $x \in \mathbb{R}^3, A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3$ let $f^{(1)} = A_1 x$ and $f^{(2)} = b$. Show $[f^{(1)}, f^{(2)}] \equiv 0$ if and only if $Ab = 0_{3\times 1}$.

Special Coordinate System

Let $f^{(1)}, f^{(2)}, f^{(3)}$ be three linearly independent vector fields on some open set $\mathcal{U} \subset \mathbf{E}^3$ with $[f^{(2)}, f^{(3)}] \equiv 0$ on $\mathcal{U}$. Let $x_0 \in \mathcal{U}$. We now show how a coordinate system can be constructed in which $f^{(2)}$ and $f^{(3)}$ have a very simple representations in the new coordinates. Define the map from $\mathbb{R}^3 \to \mathbf{E}^3$ by

$$S(t_1, t_2, t_3) = \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))) \tag{1.13}$$

where $\varphi_{t_1}(x_0)$ is the solution to $dx/dt_1 = f^{(1)}(x)$ with $x(0) = x_0, \varphi_{t_2}(x_0)$ is the solution to $dx/dt_2 = f^{(2)}(x)$ with $x(0) = x_0'$, and $\varphi_{t_3}(x_0)$ is the solution to $dx/dt_3 = f^{(3)}(x)$ with $x(0) = x_0''$. Writing

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1(t_1,t_2,t_3) \\ s_2(t_1,t_2,t_3) \\ s_3(t_1,t_2,t_3) \end{bmatrix} \tag{1.14}$$

with $S(0,0,0) = x_0$ this can be a coordinate transformation if and only if it is invertible. The Jacobian of this transformation is

$$\frac{\partial x}{\partial t} = \begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \frac{\partial x_1}{\partial t_3} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & \frac{\partial x_2}{\partial t_3} \\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} & \frac{\partial x_3}{\partial t_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial s_1}{\partial t_1} & \frac{\partial s_1}{\partial t_2} & \frac{\partial s_1}{\partial t_3} \\ \frac{\partial s_2}{\partial t_1} & \frac{\partial s_2}{\partial t_2} & \frac{\partial s_2}{\partial t_3} \\ \frac{\partial s_3}{\partial t_1} & \frac{\partial s_3}{\partial t_2} & \frac{\partial s_3}{\partial t_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial S}{\partial t_1} & \frac{\partial S}{\partial t_2} & \frac{\partial S}{\partial t_3} \end{bmatrix}. \tag{1.15}$$

As

$$\begin{aligned}
\frac{\partial S}{\partial t_1}_{|t=0} &= \frac{\partial}{\partial t_1}\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))\Big|_{t=0} = \frac{\partial}{\partial t_1}\phi_0(\varphi_0(\varphi_{t_1}(x_0)))\Big|_{t_1=0} = \frac{\partial}{\partial t_1}\varphi_{t_1}(x_0)\Big|_{t_1=0} = f^{(1)}(\varphi_0(x_0)) = f^{(1)}(x_0) \\
\frac{\partial S}{\partial t_2}_{|t=0} &= \frac{\partial}{\partial t_2}\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))\Big|_{t=0} = \frac{\partial}{\partial t_2}\phi_0(\varphi_{t_2}(\varphi_0(x_0)))\Big|_{t_2=0} = \frac{\partial}{\partial t_2}\varphi_{t_2}(x_0)\Big|_{t_2=0} = f^{(2)}(\varphi_0(x_0)) = f^{(2)}(x_0) \\
\frac{\partial S}{\partial t_3}_{|t=0} &= \frac{\partial}{\partial t_3}\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))\Big|_{t=0} = \frac{\partial}{\partial t_3}\phi_{t_{30}}(\varphi_0(\varphi_0(x_0)))\Big|_{t_3=0} = \frac{\partial}{\partial t_3}\varphi_{t_3}(x_0)\Big|_{t_3=0} = f^{(3)}(\varphi_0(x_0)) = f^{(3)}(x_0)
\end{aligned}$$

it follows that

$$\det\left(\frac{\partial x}{\partial t}\right)_{|t=0} = \det\begin{bmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & f^{(3)}(x_0) \end{bmatrix} \neq 0 \tag{1.16}$$

as $f^{(1)}(x_0), f^{(2)}(x_0)$, and $f^{(3)}(x_0)$ are linearly independent. As the vector fields $\frac{\partial S}{\partial t_1}, \frac{\partial S}{\partial t_2}, \frac{\partial S}{\partial t_3}$ are continuous functions of $t = (t_1, t_2, t_3)$ it follows that there is a neighborhood $\mathcal{D} \subset \mathbb{R}^3$ containing $(0,0,0)$ such that for all $(t_1, t_2, t_3) \in \mathcal{D}$ we have

$$\det\left(\frac{\partial x}{\partial t}\right) = \det\begin{bmatrix} \frac{\partial S}{\partial t_1} & \frac{\partial S}{\partial t_2} & \frac{\partial S}{\partial t_3} \end{bmatrix} \neq 0.$$

Consequently, $(x_1, x_2, x_3) = S(t_1, t_2, t_3) : \mathcal{D} \to \mathcal{U} \subset \mathbf{E}^3$ is invertible with $S(0,0,0) = x_0$. Denote the inverse by $T(x) \triangleq S^{-1}(x) : \mathcal{U} \to \mathcal{D}$ written out as

$$\begin{aligned} t_1 &= T_1(x_1, x_2, x_3) \\ t_2 &= T_2(x_1, x_2, x_3) \\ t_3 &= T_3(x_1, x_2, x_3) \end{aligned} \tag{1.17}$$

with $T(x_0) = T(x_{01}, x_{02}, x_{03}) = (0,0,0)$. As S and T are inverses we have

$$\begin{aligned} t_1 &= T_1(s_1(t_1,t_2,t_3), s_2(t_1,t_2,t_3), s_3(t_1,t_2,t_3)) \\ t_2 &= T_2(s_1(t_1,t_2,t_3), s_2(t_1,t_2,t_3), s_3(t_1,t_2,t_3)) \\ t_3 &= T_3(s_1(t_1,t_2,t_3), s_2(t_1,t_2,t_3), s_3(t_1,t_2,t_3)). \end{aligned} \tag{1.18}$$

By the chain rule

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{\partial}{\partial t} T(S(t)) &= \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} & \frac{\partial T_2}{\partial x_3} \\ \frac{\partial T_3}{\partial x_1} & \frac{\partial T_3}{\partial x_2} & \frac{\partial T_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} \frac{\partial s_1}{\partial t_1} & \frac{\partial s_1}{\partial t_2} & \frac{\partial s_1}{\partial t_3} \\ \frac{\partial s_2}{\partial t_1} & \frac{\partial s_2}{\partial t_2} & \frac{\partial s_2}{\partial t_3} \\ \frac{\partial s_3}{\partial t_1} & \frac{\partial s_3}{\partial t_2} & \frac{\partial s_3}{\partial t_3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} & \frac{\partial T_2}{\partial x_3} \\ \frac{\partial T_3}{\partial x_1} & \frac{\partial T_3}{\partial x_2} & \frac{\partial T_3}{\partial x_3} \end{bmatrix}_{|x=S(t_1,t_2,t_3)} \begin{bmatrix} \frac{\partial S}{\partial t_1} & \frac{\partial S}{\partial t_2} & \frac{\partial S}{\partial t_3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} & \frac{\partial T_2}{\partial x_3} \\ \frac{\partial T_3}{\partial x_1} & \frac{\partial T_3}{\partial x_2} & \frac{\partial T_3}{\partial x_3} \end{bmatrix}_{|x=S(t_1,t_2,t_3)} \begin{bmatrix} \frac{\partial S}{\partial t_1} & f^{(2)}(S(t_1,t_2,t_3)) & f^{(3)}(S(t_1,t_2,t_3)) \end{bmatrix} \end{aligned} \tag{1.19}$$

where in the last line we used

$$\frac{\partial S}{\partial t_3} = \frac{\partial}{\partial t_3} \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))) = f^{(3)}(\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))) = f^{(3)}(S(t_1,t_2,t_3))$$

and

$$\begin{aligned} \frac{\partial S}{\partial t_2} = \frac{\partial}{\partial t_2} \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))) &= \frac{\partial}{\partial t_2} \phi_{t_2}(\varphi_{t_3}(\varphi_{t_1}(x_0))) \quad \text{as} \quad [f^{(2)}, f^{(3)}] \equiv 0 \\ &= f^{(2)}(\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))) \\ &= f^{(2)}(S(t_1,t_2,t_3)). \end{aligned}$$

Then new coordinates are $t = (t_1, t_2, t_3)$ with t_1 the amount of time $f^{(1)}$ is followed, t_2 is the amount of time $f^{(2)}$ is followed, and t_3 is the amount of time $f^{(3)}$ is followed.

Let's change the notation to $x_1^* = t_1, x_2^* = t_2, x_3^* = t_3$ so as not to confuse $t = (t_1, t_2, t_3)$ with the scalar time t. We write

$$\begin{aligned} x_1^* &= T_1(x_1, x_2, x_3) \\ x_2^* &= T_2(x_1, x_2, x_3) \\ x_3^* &= T_3(x_1, x_2, x_3). \end{aligned} \tag{1.20}$$

The representation of $\dfrac{dx}{dt} = f^{(2)}(x)$ in the x^* coordinate system is

$$\frac{dx^*}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} = \frac{\partial T}{\partial x} f^{(2)}(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where the last equality follows by (1.19). The representation for $\dfrac{dx}{dt} = f^{(3)}(x)$ in the x^* coordinate system is

$$\frac{dx^*}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} = \frac{\partial T}{\partial x} f^{(3)}(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where the last equality also follows by (1.19). In the x^* coordinates $f^{(2)}$ and $f^{(3)}$ have these very simple representations. Let's look at the geometric picture of what is going on here. Starting from x_0 at time $t_1 = 0$ the trajectory $\varphi_{t_1}(x_0)$ is followed till some time t_{01}. Holding t_{01} fixed we vary t_2 and t_3 to produce a surface in $\mathbf{E}^3$. In more detail, from $\phi_{t_{01}}(x_0)$ follow $f^{(2)}$ for a time t_{02} to end up at $\phi_{t_{02}}(\phi_{t_{01}}(x_0))$. Then from $\phi_{t_{02}}(\phi_{t_{01}}(x_0))$ follow $f^{(3)}$ for a time t_{03} to end up at $\phi_{t_{03}}(\phi_{t_{02}}(\phi_{t_{01}}(x_0)))$. See Figure 1.6. The mapping S takes the coordinate curve (t_{01}, t_2, t_{03}) for $|t_2| < \epsilon$ in $\mathbb{R}^3$ to the curve $S(t_{01}, t_2, t_{03})$ in $\mathbf{E}^3$ and, as $[f^{(2)}, f^{(3)}] \equiv 0$, we have $\left.\dfrac{\partial S(t_{01}, t_2, t_{03})}{\partial t_2}\right|_{t_{02}} = f^{(2)}(x)\big|_{x=\phi_{t_{03}}(\phi_{t_{02}}(\phi_{t_{01}}(x_0)))}$. Further the coordinate curve mapping (t_{01}, t_{02}, t_3) for $|t_3| < \epsilon$ in $\mathbb{R}^3$ to the curve $S(t_{01}, t_{02}, t_3)$ in $\mathbf{E}^3$ with $\left.\dfrac{\partial S(t_{01}, t_{02}, t_3)}{\partial t_3}\right|_{t_{03}} = f^{(3)}(x)\big|_{x=\phi_{t_{03}}(\phi_{t_{02}}(\phi_{t_{01}}(x_0)))}$.

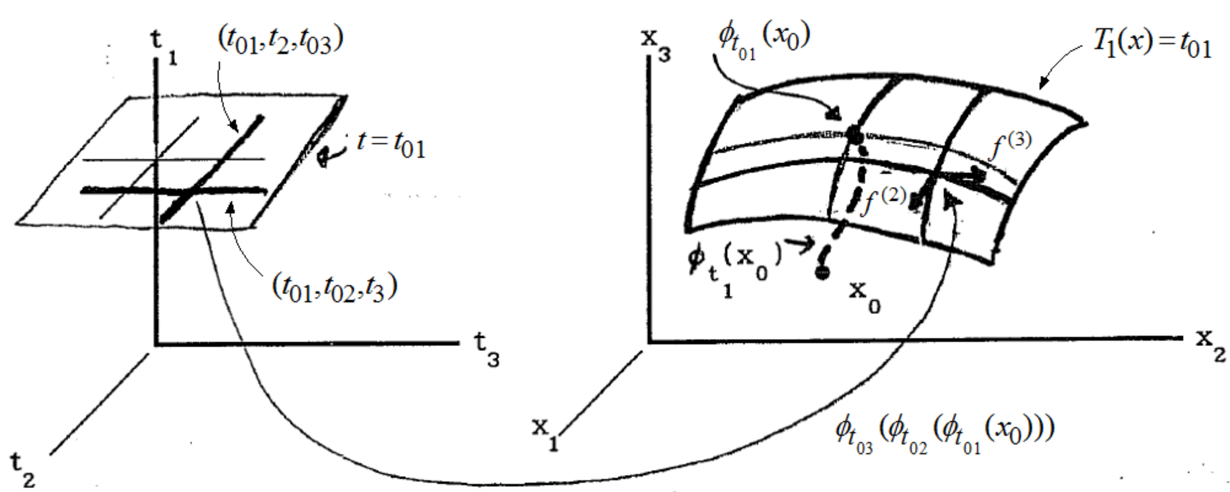


FIGURE 1.6. Coordinates (t_1, t_2, t_3) constructed by following $f^{(1)}$ for a time t_1, $f^{(2)}$ for a time t_2, $f^{(3)}$ for a time t_3.

We have just shown that with t_1 constant, $S(t_1,t_2,t_3)$ sweeps out a surface as (t_2,t_3) is varied around $(0,0)$ with the tangent to this surface spanned by

$$\frac{\partial S(t_1,t_2,t_3)}{\partial t_2} = f^{(2)}(S(t_1,t_2,t_3)) \text{ and } \frac{\partial S(t_1,t_2,t_3)}{\partial t_3} = f^{(3)}(S(t_1,t_2,t_3)).$$

In particular, for $t_1 = 0$, the surface $S(0,t_2,t_3)$ contains x_0 as $S(0,0,0) = x_0$. The set $\{x \in \mathbf{E}^3 | \; T_1(x) = t_{01}\}$ implicitly gives a surface in $\mathbf{E}^3$ whose tangent plane at each point is spanned by $f^{(2)}(x)$ and $f^{(3)}(x)$. The gradient $dT_1 = \dfrac{\partial T_1}{\partial x}$ is normal (perpendicular) to this surface so $\mathcal{L}_{f^{(2)}}(T_1) = \langle dT_1, f^{(2)}\rangle = 0$ and $\mathcal{L}_{f^{(3)}}(T_1) = \langle dT_1, f^{(3)}\rangle = 0$ on this surface. Further, as $f^{(1)}$ is linearly independent of $f^{(2)}$ and $f^{(3)}$ we have must have $\mathcal{L}_{f^{(1)}}(T_1) = \langle dT_1, f^{(1)}\rangle \neq 0$ on the surface.

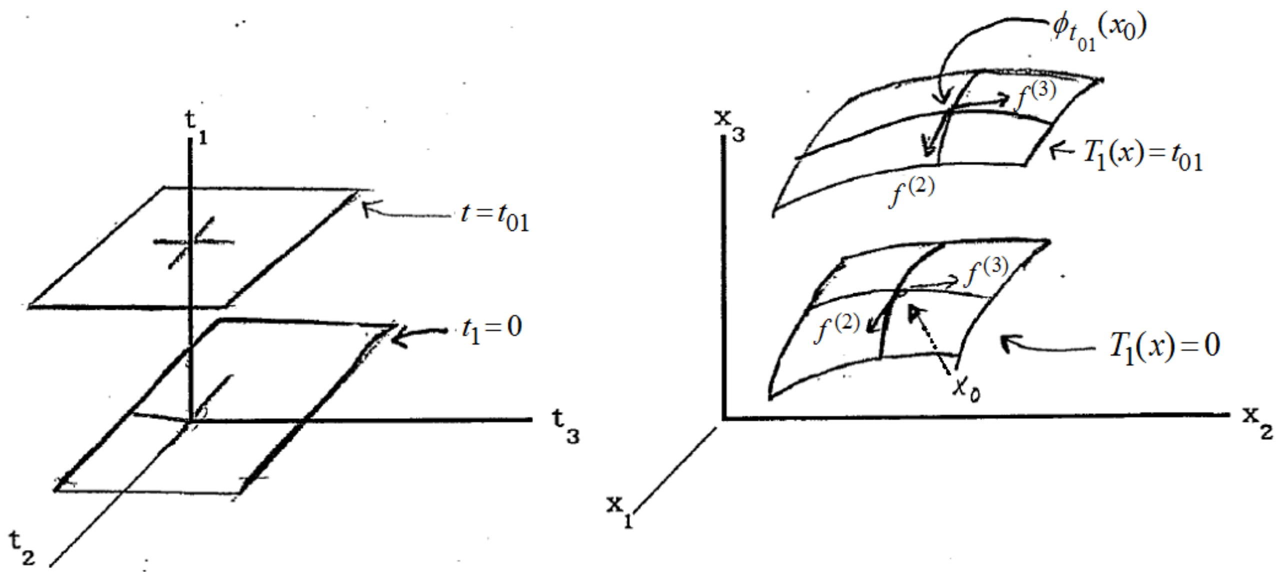


FIGURE 1.7. Surface containing x_0 with its tangent space spanned by $f^{(2)}$ and $f^{(3)}$.

Example 12 *Special Coordinate System*
Consider the three vector fields on $\mathbf{E}^3$ given by

$$f^{(1)}(x_1,x_2,x_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad f^{(2)}(x_1,x_2,x_3) = \begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix}, \quad f^{(3)}(x_1,x_2,x_3) = \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix}.$$

The solution to $\dfrac{dx}{dt} = f^{(1)}(x), x(0) = x_0$ is

$$\phi_{t_1}(x_0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} + t_1 \end{bmatrix}.$$

The solution to $\dfrac{dx}{dt_2} = f^{(2)}(x), x(0) = x_0'$ is

$$\phi_{t_2}(x_0) = \begin{bmatrix} x_{01}' \\ x_{02}' + t_2 \\ x_{03}' + (x_{01}')^2 t_2 \end{bmatrix}.$$

The solution to $\dfrac{dx}{dt_3} = f^{(3)}(x), x(0) = x_0''$ is

$$\phi_{t_3}(x_0) = \begin{bmatrix} x_{01}'' + t_3 \\ x_{02}'' \\ x_{03}'' + (x_{01}'' + t_3)^2 x_{02}'' - (x_{01}'')^2 x_{02}'' \end{bmatrix}.$$

Then with $x_0'' = \varphi_{t_2}(\varphi_{t_1}(x_0))$ and $x_0' = \varphi_{t_1}(x_0)$ we have

$$S(t_1, t_2, t_3) \triangleq \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))) = \begin{bmatrix} x_{01}'' + t_3 \\ x_{02}'' \\ x_{03}'' + (x_{01}'' + t_3)^2 x_{02}'' - (x_{01}'')^2 x_{02}'' \end{bmatrix}\Bigg|_{x_0''=\varphi_{t_2}(x_0')=\begin{bmatrix} x_{01}' \\ x_{02}' + t_2 \\ x_{03}' + (x_{01}')^2 t_2 \end{bmatrix}}$$

$$= \begin{bmatrix} x_{01}' + t_3 \\ x_{02}' + t_2 \\ x_{03}' + (x_{01}')^2 t_2 + (x_{01}' + t_3)^2 (x_{02}' + t_2) - (x_{01}')^2 (x_{02}' + t_2) \end{bmatrix}\Bigg|_{x_0'=\varphi_{t_1}(x_0)=\begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} + t_1 \end{bmatrix}}$$

$$= \begin{bmatrix} x_{01} + t_3 \\ x_{02} + t_2 \\ x_{03} + t_1 + (x_{01})^2 t_2 + (x_{01} + t_3)^2 (x_{02} + t_2) - (x_{01})^2 (x_{02} + t_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_{01} + t_3 \\ x_{02} + t_2 \\ x_{03} + t_1 + (x_{01} + t_3)^2 (x_{02} + t_2) - (x_{01})^2 x_{02} \end{bmatrix}$$

In Example 8 it was shown that $[f^{(2)}, f^{(3)}] \equiv 0$ so we know that $\dfrac{\partial S(t_1, t_2, t_3)}{\partial t_2} = f^{(2)}(t_1, t_2, t_3)$, $\dfrac{\partial S(t_1, t_2, t_3)}{\partial t_3} = f^{(3)}(t_1, t_2, t_3)$. Further $\{f^{(1)}, f^{(2)}, f^{(3)}\}$ are linearly independent. In fact

$$\det\left(\frac{\partial S}{\partial t}\right)_{|t=(0,0,0)} = \det\left(\begin{bmatrix} \dfrac{\partial S}{\partial t_1} & \dfrac{\partial S}{\partial t_2} & \dfrac{\partial S}{\partial t_3} \end{bmatrix}\right)_{|t=(0,0,0)} = \det\begin{bmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & f^{(3)}(x_0) \end{bmatrix} = -1 \neq 0.$$

As $\det\left(\dfrac{\partial S}{\partial t}\right)$ is a continuos function of $t = (t_1, t_2, t_3)$ it is non zero in a neighborhood of $(0, 0, 0)$. By the inverse function theorem we know the map

$$x = S(t_1, t_2, t_3) = \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))$$

has an inverse in a neighborhood of $S(0, 0, 0) = x_0$ which we denote as

$$t = S^{-1}(x) = T(x).$$

Setting $(t_1, t_2, t_3) = (x_1^*, x_2^*, x_3^*)$ we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_{01} + x_3^* \\ x_{02} + x_2^* \\ x_{03} + x_1^* + (x_{01} + x_3^*)^2 (x_{02} + x_2^*) - (x_{01})^2 x_{02} \end{bmatrix}$$

and see that it has a global inverse given by

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} x_3 - x_{03} - x_1^2 x_2 - x_{01}^2 x_{02} \\ x_2 - x_{02} \\ x_1 - x_{01} \end{bmatrix}.$$

In the x^* coordinates the differential equation $dx/dt = f^{(2)}(x)$ becomes

$$\frac{dx^*}{dt} = \frac{\partial x^*}{\partial x}\frac{dx}{dt} = \frac{\partial x^*}{\partial x} f^{(2)}(x) = \begin{bmatrix} -2x_1x_2 & -x_1^2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = f^{*(2)}(x^*).$$

That is,

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

In the x^* coordinates the differential equation $dx/dt = f^{(3)}(x)$ becomes

$$\frac{dx^*}{dt} = \frac{\partial x^*}{\partial x}\frac{dx}{dt} = \frac{\partial x^*}{\partial x} f^{(3)}(x) = \begin{bmatrix} -2x_1x_2 & -x_1^2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = f^{*(3)}(x^*).$$

That is,

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, the gradient of $T_1(x)$ is given by

$$dT_1 = \frac{\partial}{\partial x}\left(x_3 - x_{03} - x_1^2 x_2 - x_{01}^2 x_{02}\right) = \begin{bmatrix} -2x_1x_2 & -x_1^2 & 1 \end{bmatrix}$$

and

$$\mathcal{L}_{f^{(2)}}(T_1) = \left\langle dT_1, f^{(2)} \right\rangle = \begin{bmatrix} -2x_1x_2 & -x_1^2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ x_1^2 \end{bmatrix} = 0$$

$$\mathcal{L}_{f^{(3)}}(T_1) = \left\langle dT_1, f^{(3)} \right\rangle = \begin{bmatrix} -2x_1x_2 & -x_1^2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2x_1x_2 \end{bmatrix} = 0.$$

1.4 The Frobenius Theorem

We have seen that if we have two linearly independent vector fields $f^{(1)}, f^{(2)}$ defined on some open set $\mathcal{U} \subset \mathbf{E}^3$ satisfying $[f^{(1)}, f^{(2)}] \equiv 0$, then through any point of $\mathcal{U}$ there is a surface that goes through that point and whose tangent vectors are $f^{(1)}, f^{(2)}$. The Frobenius theorem is a generalization of this result. To proceed we define the notion of a *distribution* of vectors. Let $f^{(1)}$ and $f^{(2)}$ be two vector fields on an open set $\mathcal{U} \subset \mathbf{E}^3$. As in the previous sections of this chapter, we identify the Cartesian coordinates for $\mathbf{E}^3$ with the points of $\mathbf{E}^3$, that is, the coordinates (x_1, x_2, x_3) and the point $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ with $z_1 = x_1, z_2 = x_2, z_3 = x_3$ to be the same thing. At each point $p \in \mathcal{U}$ define the vector space Δ_p as

$$\Delta_p \triangleq \left\{ r_1 f^{(1)}(x) + r_2 f^{(2)}(x) \middle|\ r_1, r_2 \in \mathbb{R},\ p = (x_1, x_2, x_3) \in \mathcal{U} \right\}. \tag{1.21}$$

Δ_p is the vector space spanned by $f^{(1)}$ and $f^{(2)}$. The distribution Δ is the collection of all these vector spaces for all $p \in \mathcal{U}$, that is,

$$\Delta = \cup_{p \in \mathcal{U}} \Delta_p. \tag{1.22}$$

We want to find a surface $S(t_1,t_2) \in \mathbf{E}^3$ whose tangent vectors at any point of the surface are in Δ. That is, instead of requiring $\dfrac{\partial S}{\partial t_1} = f^{(1)}, \dfrac{\partial S}{\partial t_2} = f^{(2)}$ it is just required that $\dfrac{\partial S}{\partial t_1}$ and $\dfrac{\partial S}{\partial t_2}$ be linear combinations of $f^{(1)}$ and $f^{(2)}$ at each point of the surface. Let's first state the theorem.

Theorem 4 *Frobenius First Version (Sufficiency)*
Let $\mathcal{U} \subset \mathbf{E}^3$ be an open set with the two linearly independent vector fields $f^{(1)}$ and $f^{(2)}$ defined on $\mathcal{U}$. If for all points $p = (x_1, x_2, x_3)$ in $\mathcal{U}$ we have

$$[f^{(1)}, f^{(2)}] = \alpha_1(x) f^{(1)}(x) + \alpha_2(x) f^{(2)}(x) \in \Delta \tag{1.23}$$

then given any point $p = (x_{01}, x_{02}, x_{03}) \in \mathcal{U}$ there is a surface $S(t_1,t_2)$ such that $S(0,0) = (x_{01}, x_{02}, x_{03})$ and, for all $|t_1| < \epsilon, |t_2| < \epsilon$ for some $\epsilon > 0$, we have

$$\frac{\partial S}{\partial t_1}, \frac{\partial S}{\partial t_2} \in \Delta. \tag{1.24}$$

Definition 1 *Involutive*
Let $f^{(1)}$ and $f^{(2)}$ be two vector fields defined on $\mathcal{U} \subset \mathbf{E}^3$ with Δ defined as in (1.22). If $f^{(1)}$ and $f^{(2)}$ satisfy (1.23) then they are said to be *involutive.*

We first prove four lemmas before proving the Frobenius theorem. We follow the approach of Allendoerfer [1].

Lemma 1 Let $a_1(x), a_2(x)$ be two scalar functions and $f^{(1)}(x), f^{(2)}(x)$ be two vector fields defined an open set $\mathcal{U} \subset \mathbf{E}^3$. Then

$$[a_1 f^{(1)}, a_2 f^{(2)}] = a_1(x) a_2(x) [f^{(1)}, f^{(2)}] + \alpha_2(x) \mathcal{L}_{f^{(2)}}(a_1) f^{(1)}(x) - \alpha_1(x) \mathcal{L}_{f^{(1)}}(a_2) f^{(2)}(x). \tag{1.25}$$

Proof. By the definition of Lie bracket we have

$$[a_1 f^{(1)}, a_2 f^{(2)}] = \frac{\partial(a_2 f^{(2)})}{\partial x} a_1 f^{(1)} - \frac{\partial(a_1 f^{(1)})}{\partial x} a_2 f^{(2)}.$$

We compute

$$\frac{\partial(a_2 f^{(2)})}{\partial x} = \begin{bmatrix} \dfrac{\partial}{\partial x}(a_2 f_1^{(2)}) \\ \dfrac{\partial}{\partial x}(a_2 f_2^{(2)}) \\ \dfrac{\partial}{\partial x}(a_2 f_3^{(2)}) \end{bmatrix} = \begin{bmatrix} a_2 \dfrac{\partial}{\partial x} f_1^{(2)} \\ a_2 \dfrac{\partial}{\partial x} f_2^{(2)} \\ a_2 \dfrac{\partial}{\partial x} f_3^{(2)} \end{bmatrix} + \begin{bmatrix} f_1^{(2)} \dfrac{\partial a_2}{\partial x} \\ f_2^{(2)} \dfrac{\partial a_2}{\partial x} \\ f_3^{(2)} \dfrac{\partial a_2}{\partial x} \end{bmatrix}.$$

Then

$$\begin{aligned} \frac{\partial(a_2 f^{(2)})}{\partial x} a_1 f^{(1)} &= a_1 a_2 \begin{bmatrix} \dfrac{\partial}{\partial x} f_1^{(2)} \\ \dfrac{\partial}{\partial x} f_2^{(2)} \\ \dfrac{\partial}{\partial x} f_3^{(2)} \end{bmatrix} f^{(1)} + a_1 \begin{bmatrix} f_1^{(2)} \dfrac{\partial a_2}{\partial x} \\ f_2^{(2)} \dfrac{\partial a_2}{\partial x} \\ f_3^{(2)} \dfrac{\partial a_2}{\partial x} \end{bmatrix} f^{(1)} = a_1 a_2 \frac{\partial f^{(2)}}{\partial x} f^{(1)} + a_1 \begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \\ f_3^{(2)} \end{bmatrix} \mathcal{L}_{f^{(1)}}(a_2) \\ &= a_1 a_2 \frac{\partial f^{(2)}}{\partial x} f^{(1)} + a_1 \mathcal{L}_{f^{(1)}}(a_2) f^{(2)} \end{aligned}$$

Similarly

$$\frac{\partial(a_1 f^{(1)})}{\partial x} a_2 f^{(2)} = a_1 a_2 \frac{\partial f^{(1)}}{\partial x} f^{(2)} + a_2 \mathcal{L}_{f^{(2)}}(a_1) f^{(1)}.$$

Subtracting these two expressions we have our result:

$$\begin{aligned}
[a_1 f^{(1)}, a_2 f^{(2)}] &= a_1 a_2 \frac{\partial f^{(2)}}{\partial x} f^{(1)} + a_1 \mathcal{L}_{f^{(1)}}(a_2) f^{(2)} - \left(a_1 a_2 \frac{\partial f^{(1)}}{\partial x} f^{(2)} + a_2 \mathcal{L}_{f^{(2)}}(a_1) f^{(1)} \right) \\
&= a_1 a_2 [f^{(1)}, f^{(2)}] + \alpha_1 \mathcal{L}_{f^{(1)}}(a_2) f^{(2)} - \alpha_2 \mathcal{L}_{f^{(2)}}(a_1) f^{(1)}.
\end{aligned}$$

■

Lemma 2 Let $f^{(1)}, f^{(2)}, g^{(1)}, g^{(2)}$ be a set of vector fields on $\mathcal{U} \subset \mathbf{E}^3$. Then

(a) $[f^{(1)} + f^{(2)}, g^{(1)}] = [f^{(1)}, g^{(1)}] + [f^{(2)}, g^{(1)}]$

(b) $[f^{(1)}, g^{(1)} + g^{(2)}] = [f^{(1)}, g^{(1)}] + [f^{(1)}, g^{(2)}]$

(c) $[f^{(1)}, g^{(1)}] = -[g^{(1)}, f^{(1)}]$

(d) $[f^{(1)}, f^{(1)}] = 0$

Proof. Exercise ■

Lemma 3 Given two linearly independent vector fields $f^{(1)}$ and $f^{(2)}$, define two new vector fields $f^{*(1)}, f^{*(2)}$ by

$$f^{*(1)} \triangleq a_{11}(x) f^{(1)}(x) + a_{12}(x) f^{(2)}(x) \tag{1.26}$$

$$f^{*(2)} \triangleq a_{21}(x) f^{(1)}(x) + a_{22}(x) f^{(2)}(x) \tag{1.27}$$

with

$$\det \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix} \neq 0$$

on some open subset $\mathcal{U} \subset \mathbf{E}^3$. (The determinant not being zero ensures that $f^{*(1)}$ and $f^{*(2)}$ are linearly independent.). If for all $x \in \mathcal{U}$ we have

$$[f^{(1)}, f^{(2)}] = \alpha_1(x) f^{(1)}(x) + \alpha_2(x) f^{(2)}(x)$$

with α_1, α_2 scalar functions, then

$$[f^{*(1)}, f^{*(2)}] = \alpha_1^*(x) f^{*(1)}(x) + \alpha_2^*(x) f^{*(2)}(x) \tag{1.28}$$

for some scalar functions α_1^*, α_2^*.

Proof.

$$\begin{aligned}
[f^{*(1)}, f^{*(2)}] =& [a_{11}(x) f^{(1)}(x) + a_{12}(x) f^{(2)}(x), a_{21}(x) f^{(1)}(x) + a_{22}(x) f^{(2)}(x)] \\
=& [a_{11}(x) f^{(1)}(x), a_{21}(x) f^{(1)}(x) + a_{22}(x) f^{(2)}(x)] + \\
& [a_{12}(x) f^{(2)}(x), a_{21}(x) f^{(1)}(x) + a_{22}(x) f^{(2)}(x)] \\
=& [a_{11}(x) f^{(1)}(x), a_{21}(x) f^{(1)}(x)] + [a_{11}(x) f^{(1)}(x), a_{22}(x) f^{(2)}(x)] + \\
& [a_{12}(x) f^{(2)}(x), a_{21}(x) f^{(1)}(x)] + [a_{12}(x) f^{(2)}(x), a_{22}(x) f^{(2)}(x)] \\
=& a_{11} a_{21} [f^{(1)}, f^{(1)}] + \alpha_{21} \mathcal{L}_{f^{(1)}}(a_{11}) f^{(1)} - \alpha_{11} \mathcal{L}_{f^{(1)}}(a_{21}) f^{(1)} + \\
& a_{11} a_{22} [f^{(1)}, f^{(2)}] + \alpha_{22} \mathcal{L}_{f^{(2)}}(a_{11}) f^{(1)} - \alpha_{11} \mathcal{L}_{f^{(1)}}(a_{22}) f^{(2)} + \\
& a_{12} a_{21} [f^{(2)}, f^{(1)}] + \alpha_{21} \mathcal{L}_{f^{(1)}}(a_{12}) f^{(2)} - \alpha_{12} \mathcal{L}_{f^{(2)}}(a_{21}) f^{(1)} + \\
& a_{12} a_{22} [f^{(2)}, f^{(2)}] + \alpha_{22} \mathcal{L}_{f^{(2)}}(a_{12}) f^{(2)} - \alpha_{12} \mathcal{L}_{f^{(2)}}(a_{22}) f^{(2)}.
\end{aligned}$$

Rearranging this last expression gives

$$\begin{aligned}[f^{*(1)}, f^{*(2)}] = (a_{11}a_{22} - a_{12}a_{21})[f^{(1)}, f^{(2)}] + \big(\alpha_{21}\mathcal{L}_{f^{(1)}}(a_{11}) - \alpha_{11}\mathcal{L}_{f^{(1)}}(a_{21}) + \alpha_{22}\mathcal{L}_{f^{(2)}}(a_{11}) - \alpha_{12}\mathcal{L}_{f^{(2)}}(a_{21})\big)\, f^{(1)}\\ \big(-\alpha_{11}\mathcal{L}_{f^{(1)}}(a_{22}) + \alpha_{21}\mathcal{L}_{f^{(1)}}(a_{12}) + \alpha_{22}\mathcal{L}_{f^{(2)}}(a_{12}) - \alpha_{12}\mathcal{L}_{f^{(2)}}(a_{22})\big)\, f^{(2)}.\end{aligned}$$

By assumption $[f^{(1)}, f^{(2)}] = \alpha_1(x)f^{(1)}(x) + \alpha_2(x)f^{(2)}(x)$ so this last expression may be written in the form

$$[f^{*(1)}, f^{*(2)}] = \gamma_1(x)f^{(1)} + \gamma_2(x)f^{(2)}(x).$$

Next rewrite (1.26) and (1.27) as

$$[f^{*(1)}, f^{*(2)}] \triangleq \begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix} \begin{bmatrix} a_{11}(x) & a_{21}(x) \\ a_{12}(x) & a_{22}(x) \end{bmatrix}.$$

Then with

$$\begin{bmatrix} b_{11}(x) & b_{21}(x) \\ b_{12}(x) & b_{22}(x) \end{bmatrix} \triangleq \begin{bmatrix} a_{11}(x) & a_{21}(x) \\ a_{11}(x) & a_{22}(x) \end{bmatrix}^{-1}$$

so that

$$[f^{*(1)}, f^{*(2)}] \begin{bmatrix} b_{11}(x) & b_{21}(x) \\ b_{12}(x) & b_{22}(x) \end{bmatrix} = \begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix} \begin{bmatrix} a_{11}(x) & a_{21}(x) \\ a_{12}(x) & a_{22}(x) \end{bmatrix} \begin{bmatrix} b_{11}(x) & b_{21}(x) \\ b_{12}(x) & b_{22}(x) \end{bmatrix} = \begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix}$$

or

$$\begin{aligned} f^{(1)} &\triangleq b_{11}(x)f^{*(1)}(x) + b_{12}(x)f^{*(2)}(x)\\ f^{(2)} &\triangleq b_{21}(x)f^{*(1)}(x) + b_{22}(x)f^{*(2)}(x). \end{aligned}$$

The Lie bracket $[f^{*(1)}, f^{*(2)}]$ then has the form

$$\begin{aligned}[f^{*(1)}, f^{*(2)}] &= \gamma_1(x)\Big(b_{11}(x)f^{*(1)}(x) + b_{12}(x)f^{*(2)}(x)\Big) + \gamma_2(x)\Big(b_{21}(x)f^{*(1)}(x) + b_{22}(x)f^{*(2)}(x)\Big)\\ &= \alpha_1^*(x)f^{*(1)}(x) + \alpha_2^*(x)f^{*(2)}(x).\end{aligned}$$

■

Lemma 4 Let $f^{(1)}, f^{(2)}$ be two linearly independent involutive vector fields $f^{(1)}, f^{(2)}$ on some open subset $\mathcal{U} \subset \mathbf{E}^3$ so that

$$[f^{(1)}, f^{(2)}] = \alpha_1(x)f^{(1)}(x) + \alpha_2(x)f^{(2)}(x)$$

for some scalar fields $\alpha_1(x), \alpha_2(x) \in \mathbb{R}$. Then there exists a nonsingular transformation of the vector fields given by ($a_{ij}(x) \in \mathbb{R}$)

$$\begin{aligned} f^{*(1)} &\triangleq a_{11}(x)f^{(1)}(x) + a_{12}(x)f^{(2)}(x)\\ f^{*(2)} &\triangleq a_{21}(x)f^{(1)}(x) + a_{22}(x)f^{(2)}(x) \end{aligned}$$

such that $[f^{*(1)}, f^{*(2)}] \equiv 0$ on $\mathcal{U}$.

Proof. Form the 3×2 matrix from the vector fields $f^{(1)}, f^{(2)}$ by

$$\begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix} = \begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \\ f_3^{(1)} & f_3^{(2)} \end{bmatrix}.$$

As $f^{(1)}, f^{(2)}$ are linearly independent, the 3×2 matrix $\begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix}$ must have a nonzero minor. Let's assume the minor is

$$\begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \end{bmatrix}, \quad \text{i.e., } \det \begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \end{bmatrix} \neq 0.$$

Define

$$\begin{bmatrix} a_{11}(x) & a_{21}(x) \\ a_{12}(x) & a_{22}(x) \end{bmatrix} \triangleq \begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \end{bmatrix}^{-1}.$$

Then

$$[f^{*(1)}, f^{*(2)}] \triangleq \begin{bmatrix} f^{(1)} & f^{(2)} \end{bmatrix} \begin{bmatrix} a_{11}(x) & a_{21}(x) \\ a_{12}(x) & a_{22}(x) \end{bmatrix} = \begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \\ f_3^{(1)} & f_3^{(2)} \end{bmatrix} \begin{bmatrix} f_1^{(1)} & f_1^{(2)} \\ f_2^{(1)} & f_2^{(2)} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \mu_1(x) & \mu_2(x) \end{bmatrix}$$

with the obvious definitions for $\mu_1(x), \mu_2(x) \in \mathbb{R}$.
Then

$$[f^{*(1)}, f^{*(2)}] = \frac{\partial f^{*(2)}}{\partial x} f^{*(1)} - \frac{\partial f^{*(1)}}{\partial x} f^{*(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{\partial \mu_2}{\partial x_1} & \dfrac{\partial \mu_2}{\partial x_2} & \dfrac{\partial \mu_2}{\partial x_3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \mu_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{\partial \mu_1}{\partial x_1} & \dfrac{\partial \mu_1}{\partial x_2} & \dfrac{\partial \mu_1}{\partial x_3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}$$

with the obvious definition for γ. By lemma 3 we know there exists α_1^*, α_2^* such that

$$[f^{*(1)}, f^{*(2)}] = \alpha_1^*(x) f^{*(1)}(x) + \alpha_2^*(x) f^{*(2)}(x).$$

As a consequence we have

$$\begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha_1^* \\ 0 \\ \alpha_1^*\mu_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2^* \\ \alpha_2^*\mu_2 \end{bmatrix}$$

which shows $\alpha_1^* = 0, \alpha_2^* = 0$ and $\gamma = 0$. ■

Proof of the Frobenius Theorem (Sufficiency)

Proof. The Frobenius theorem follows from this last lemma and Theorem 3. Specifically, we are given two linear independent vector fields $f^{(1)}, f^{(2)}$ which for all $p = (x_1, x_2, x_3)$ in $\mathcal{U} \subset \mathbf{E}^3$ satisfy

$$[f^{(1)}, f^{(2)}] = \alpha_1(x) f^{(1)}(x) + \alpha_2(x) f^{(2)}(x).$$

Then by Lemma 4 we can construct two linearly independent vectors $f^{*(1)}, f^{*(2)}$ which are linear combinations of $f^{(1)}, f^{(2)}$ and satisfy $[f^{*(1)}, f^{*(2)}] = 0$ on $\mathcal{U}$. By Theorem 3, for any given point $p = (x_{01}, x_{02}, x_{03}) \in \mathcal{U}$, a surface $S(t_1, t_2)$ can be constructed in $\mathcal{U}$ with $S(0,0) = (x_{01}, x_{02}, x_{03})$ and such that for all $|t_1| < \epsilon, |t_2| < \epsilon$ and some $\epsilon > 0$ we have $\dfrac{\partial S}{\partial t_1} = f^{*(1)}, \dfrac{\partial S}{\partial t_2} = f^{*(2)}$ and thus

$$\left.\frac{\partial S}{\partial t_1}\right|_p, \left.\frac{\partial S}{\partial t_2}\right|_p \in \Delta_p = \left\{ r_1 f^{(1)}(x) + r_2 f^{(2)}(x) | \; r_1, r_2 \in \mathbb{R} \right\}$$

for all $p = (x_1, x_2, x_3) \in \mathcal{U}$. ■

Example 13 *Special Coordinate System - Again*

Let

$$f^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; f^{(3)} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

A simple calculation shows that $[f^{(2)}, f^{(3)}] = f^{(2)}$ showing the pair $\{f^{(2)}, f^{(3)}\}$ is involutive. Then

$$\begin{bmatrix} f^{(2)} & f^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix}.$$

With the assumption that $p = (x_{01}, x_{02}, x_{03}) \in \mathcal{U}$ with $x_{02} \neq 0$, the minor

$$\begin{bmatrix} 1 & x_1 \\ 0 & x_2 \end{bmatrix}$$

is invertible. Let

$$\begin{bmatrix} f^{*(2)} & f^{*(3)} \end{bmatrix} \triangleq \begin{bmatrix} 1 & x_1 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 0 & x_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & x_1 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -x_1/x_2 \\ 0 & 1/x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x_3/x_2 \end{bmatrix}.$$

Another simple calculation shows that $[f^{*(2)}, f^{*(3)}] \equiv 0$. The solution to $dx/dt_2 = f^{*(2)}(x)$ with $x(0) = x_0'$ is

$$\phi^*_{t_2}(x_0') = \begin{bmatrix} x_{01}' + t_2 \\ x_{02}' \\ x_{03}' \end{bmatrix}$$

and the solution to $dx/dt_3 = f^{*(3)}(x)$ with $x(0) = x_0''$ is

$$\phi^*_{t_3}(x_0'') = \begin{bmatrix} x_{01}'' \\ x_{02}'' + t_3 \\ (x_{03}''/x_{02}'')(x_{02}'' + t_3) \end{bmatrix}.$$

For $f^{*(1)}$ choose a vector that is linearly independent of $f^{*(2)}, f^{*(3)}$. Here we chose

$$f^{*(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the solution to $dx/dt_1 = f^{*(1)}(x)$ with $x(0) = x_0$ is

$$\phi^*_{t_1}(x_0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} + t_1 \end{bmatrix}.$$

Then set

$$\begin{aligned} S(t_1, t_2, t_3) = \phi^*_{t_3}(\phi^*_{t_2}(\phi^*_{t_1}(x_0))) &= \left[\begin{bmatrix} x_{01}'' \\ x_{02}'' + t_3 \\ (x_{03}''/x_{02}'')(x_{02}'' + t_3) \end{bmatrix} \right|_{x_0'' = \phi^*_{t_2}(x_0') ==} \begin{bmatrix} x_{01}' + t_2 \\ x_{02}' \\ x_{03}' \end{bmatrix} \\ &= \left[\begin{bmatrix} x_{01}' + t_2 \\ x_{02}' + t_3 \\ (x_{03}'/x_{02}')(x_{02}' + t_3) \end{bmatrix} \right|_{x_0' = \phi^*_{t_1}(x_0) =} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} + t_1 \end{bmatrix} \\ &= \begin{bmatrix} x_{01} + t_2 \\ x_{02} + t_3 \\ (x_{03} + t_1)(x_{02} + t_3)/x_{02} \end{bmatrix}. \end{aligned}$$

We calculate

$$\frac{\partial S}{\partial t_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = f^{*(2)}, \quad \frac{\partial S}{\partial t_3} = \begin{bmatrix} 0 \\ 1 \\ (x_{03} + t_1)/x_{02} \end{bmatrix} = f^{*(3)}.$$

The inverse of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1(t_1,t_2,t_3) \\ s_2(t_1,t_2,t_3) \\ s_3(t_1,t_2,t_3) \end{bmatrix} = \begin{bmatrix} x_{01}+t_2 \\ x_{02}+t_3 \\ (x_{03}+t_1)(x_{02}+t_3)/x_{02} \end{bmatrix}$$

is given by

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \underbrace{\begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \end{bmatrix}}_{T(x)} = \begin{bmatrix} \dfrac{x_3}{x_2}x_{02} - x_{03} \\ x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix}$$

where it is seen that $T(x_0) = 0_{3\times 1}$.

With

$$T_1(x) = \frac{x_3}{x_2}x_{02} - x_{03}$$

we have

$$dT_1 = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix}$$

and

$$\mathcal{L}_{f^{*(2)}}(T_1) = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{f^{*(2)}} = 0$$

$$\mathcal{L}_{f^{*(3)}}(T_1) = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 1 \\ x_3/x_2 \end{bmatrix}}_{f^{*(3)}} = 0.$$

Further, as $f^{(2)}$ and $f^{(3)}$ are linear combinations of $f^{*(2)}, f^{*(3)}$, we also have $\mathcal{L}_{f^{(2)}}(T_1) = 0$ and $\mathcal{L}_{f^{(3)}}(T_1) = 0$. Explicitly we have

$$\mathcal{L}_{f^{(2)}}(T_1) = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{f^{(2)}} = 0$$

$$\mathcal{L}_{f^{(3)}}(T_1) = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{f^{(2)}} = 0.$$

Finally, as $f^{*(1)}$ is linearly independent of $f^{*(2)}$ and $f^{*(3)}$ we must have $\mathcal{L}_{f^{*(1)}}(T_1) = \left\langle dT_1, f^{*(1)} \right\rangle \neq 0$. Specifically this is

$$\mathcal{L}_{f^{*(1)}}(T_1) = \begin{bmatrix} 0 & -x_{02}x_3/x_2^2 & x_{02}/x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{f^{*(1)}} = x_{02}/x_2.$$

In light of this example we reformulate the Frobenius theorem.

Theorem 5 *Frobenius Theorem (Necessity)*
Let $f^{(1)}, f^{(2)}, f^{(3)}$ be three linearly independent vector fields on an open set $\mathcal{U} \subset \mathbf{E}^3$. There exists a coordinate system $S : \mathcal{D} \subset \mathbb{R}^3 \to S(\mathcal{D}) \subset \mathcal{U}$ given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1(t_1,t_2,t_3) \\ s_2(t_1,t_2,t_3) \\ s_3(t_1,t_2,t_3) \end{bmatrix}, \quad \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} = \begin{bmatrix} s_1(t_{01},t_{02},t_{03}) \\ s_2(t_{01},t_{02},t_{03}) \\ s_3(t_{01},t_{02},t_{03}) \end{bmatrix}$$

with inverse $T : S(\mathcal{D}) \to \mathcal{D}$

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_1(x_1,x_2,x_3) \\ T_2(x_1,x_2,x_3) \\ T_3(x_1,x_2,x_3) \end{bmatrix}$$

for which the two-dimensional manifolds (surfaces) defined by

$$\left\{x \in \mathcal{U} \subset \mathbf{E}^3 \middle| \; T_1(x) = t_{01}\right\}$$

for a fixed t_{01} have $f^{(2)}, f^{(3)}$ spanning their tangent planes if and only if

$$[f^{(2)}, f^{(3)}] = \alpha_2 f^{(2)} + \alpha_3 f^{(3)}$$

for some scalar valued functions α_2 and α_3.

Proof. We have already shown that if the involutive condition holds then such a coordinate system exists. We now show necessity, i.e., if there exists such a coordinate system then $[f^{(2)}, f^{(3)}] = \alpha_2 f^{(2)} + \alpha_3 f^{(3)}$. Let

$$\mathcal{M} \triangleq \left\{x \in \mathcal{U} \subset \mathbf{E}^3 \middle| \; T_1(x) = t_{01}\right\}.$$

We are given that $f^{(2)}$ and $f^{(3)}$ span the tangent plane at each point of $\mathcal{M}$. We also know that the gradient of $T_1(x)$ is perpendicular to $\mathcal{M}$ so for all $x \in \mathcal{M}$ we have

$$\frac{\partial T_1}{\partial x} f^{(2)} = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \\ f_3^{(2)} \end{bmatrix} \equiv 0$$

$$\frac{\partial T_1}{\partial x} f^{(3)} = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1^{(3)} \\ f_2^{(3)} \\ f_3^{(3)} \end{bmatrix} \equiv 0.$$

As this is true for some neighborhood of t_{01} (say for all $-\varepsilon < t_{01} < \varepsilon$) these two equations must hold for all x in a neighborhood of $x_0 = \begin{bmatrix} x_{01} & x_{02} & x_{03} \end{bmatrix}^T$. Using Equation (**??**) of Chapter **??** the gradient of $\frac{\partial T_1}{\partial x} f^{(2)}$ is given by

$$\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f^{(2)}\right) = \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} \frac{\partial T_1}{\partial x_i} f_i^{(2)}\right) = (f^{(2)})^T \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial T_1}{\partial x}\frac{\partial f^{(2)}}{\partial x} \equiv \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

where $\frac{\partial^2 T_1}{\partial x^2}$ is the Hessian matrix. Multiply this last expression by $f^{(3)}$ on the right to obtain

$$\left(\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f^{(2)}\right)\right) f^{(3)} = (f^{(2)})^T \frac{\partial^2 T_1}{\partial x^2} f^{(3)} + \frac{\partial T_1}{\partial x}\frac{\partial f^{(2)}}{\partial x} f^{(3)} \equiv \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} f^{(3)} \equiv 0. \tag{1.29}$$

Similarly we have

$$\left(\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f^{(3)}\right)\right) f^{(2)} = (f^{(3)})^T \frac{\partial^2 T_1}{\partial x^2} f^{(2)} + \frac{\partial T_1}{\partial x}\frac{\partial f^{(3)}}{\partial x} f^{(2)} \equiv \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} f^{(2)} \equiv 0. \tag{1.30}$$

Subtracting (1.29) from (1.30) we have

$$\frac{\partial T_1}{\partial x}\frac{\partial f^{(3)}}{\partial x}f^{(2)} - \frac{\partial T_1}{\partial x}\frac{\partial f^{(2)}}{\partial x}f^{(3)} = \frac{\partial T_1}{\partial x}[f^{(2)}, f^{(3)}] \equiv 0.$$

This shows $[f^{(2)}, f^{(3)}]$ is normal to $\dfrac{\partial T_1}{\partial x}$. The set of vectors normal to $\dfrac{\partial T_1}{\partial x}$ is the tangent space of the manifold $\mathcal{M} \triangleq \left\{x \in \mathcal{U} \subset \mathbf{E}^3 \middle|\ T_1(x) = t_{01}\right\}$ which is spanned by $f^{(2)}$ and $f^{(3)}$. That is, $[f^{(2)}, f^{(3)}] = \alpha_2 f^{(2)} + \alpha_3 f^{(3)}$.

■

Exercise 13 Show $(f^{(2)})^T \dfrac{\partial^2 T_1}{\partial x^2} f^{(3)} = (f^{(3)})^T \dfrac{\partial^2 T_1}{\partial x^2} f^{(2)}$.

We give a more general definition of involutive.

Definition 2 *Involutive Vector Fields*
Let $f^{(1)}, f^{(2)}, ..., f^{(k)}$ be k vector fields on $\mathcal{U} \subset \mathbf{E}^3$ satisfying

$$[f^{(i)}, f^{(j)}] = \sum_{i=1}^{k} \alpha_k f^{(k)}$$

for all $i, j = 1, ..., k$ and some scalar valued functions $\alpha_m(x)$ for $m = 1, 2, ..., k$. Then the set of vectors $\{f^{(1)}, f^{(2)}, ..., f^{(k)}\}$ is said to be involutive.

Theorem 6 *Frobenius Theorem - Third Version*
Let $f^{(1)}, f^{(2)}, f^{(3)}$ be three linearly independent vector fields on an open set $\mathcal{U} \subset \mathbf{E}^3$ with the pair $\{f^{(2)}, f^{(3)}\}$ involutive, i.e., $[f^{(2)}, f^{(3)}] = \alpha_2 f^{(2)} + \alpha_3 f^{(3)}$ on $\mathcal{U}$. For any $x_0 \in \mathcal{U}$ let $\phi_{t_1}(x_0)$ be the solution to $dx/dt_1 = f^{(1)}(x), x(0) = x_0, \phi_{t_2}(x_0')$ be the solution to $dx/dt_2 = f^{(2)}(x), x(0) = x_0'$, and $\phi_{t_3}(x_0'')$ be the solution to $dx/dt_3 = f^{(3)}(x), x(0) = x_0''$. We take these solutions to exist for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$ for some $\epsilon > 0$. Define the transformation

$$x(t_1, t_2, t_3) = S(t_1, t_2, t_3) \triangleq \phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0))).$$

Then $S(0, 0, 0) = x_0$ and for all $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$ this transformation is invertible with

$$\frac{\partial S}{\partial t_1}, \frac{\partial S}{\partial t_2} \in \Delta_p = \left\{r_1 f^{(1)}(S(t)) + r_2 f^{(2)}(S(t)) \middle|\ r_1, r_2 \in \mathbb{R}\right\}$$

In words,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1(t_1, t_2, t_3) \\ s_2(t_1, t_2, t_3) \\ s_3(t_1, t_2, t_3) \end{bmatrix}$$

is a coordinate map with inverse

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_1(x_1, x_2, x_3) \\ T_2(x_1, x_2, x_3) \\ T_3(x_1, x_2, x_3) \end{bmatrix}$$

and the two dimensional submanifolds of $\mathbf{E}^3$ defined by

$$\left\{x \in \mathcal{U} \subset \mathbf{E}^3 \middle|\ T_1(x) = t_{01}\right\}$$

have $f^{(2)}, f^{(3)}$ spanning their tangent planes.

Proof. Omitted. ■

Remark 2 The distinction between this version of the Frobenius theorem and the two previous one is in the construction of the manifold such that $f^{(2)}, f^{(3)}$ span its tangent plane. Previously we constructed two new vectors $f^{*(2)}, f^{*(3)}$ as linear combinations of $f^{(2)}, f^{(3)}$ satisfying $[f^{*(2)}, f^{*(3)}] \equiv 0$. This version says the coordinate system can be constructed directly from $f^{(2)}$ and $f^{(3)}$.

Example 14 *Construction of a Coordinate System Using an Involutive Distribution*

Let

$$f^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, f^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, f^{(3)} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

defined on $\mathbf{E}^3$ with a vector space defined at $p = (x_1, x_2, x_3)$ by

$$\Delta_p = \left\{ r_2 f^{(2)}(x) + r_3 f^{(3)}(x) | \; r_2, r_3 \in \mathbb{R} \right\}.$$

Define the distribution Δ by

$$\Delta = \cup_{p \in \mathbf{E}^3} \Delta_p = \cup_{p \in \mathbf{E}^3} \left\{ r_2 f^{(2)}(p) + r_3 f^{(3)}(p) | \; r_2, r_3 \in \mathbb{R} \right\}.$$

A straightforward calculation shows that $[f^{(2)}, f^{(3)}] = f^{(2)}$ for all of $\mathbf{E}^3$ showing that the distribution is involutive.

The solution $dx/dt_1 = f^{(1)}(x), x(0) = x_0$ is

$$\phi_{t_1}(x_0) = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} + t_1 \end{bmatrix}.$$

The solution $dx/dt_2 = f^{(2)}(x), x(0) = x_0'$ is

$$\phi_{t_2}(x_0') = \begin{bmatrix} x_{01}' + t_2 \\ x_{02}' \\ x_{03}' \end{bmatrix}.$$

The solution $dx/dt_3 = f^{(3)}(x), x(0) = x_0''$ is

$$\phi_{t_3}(x_0'') = \begin{bmatrix} x_{01}'' e^{t_3} \\ x_{02}'' e^{t_3} \\ x_{03}'' e^{t_3} \end{bmatrix}.$$

The change of coordinates is then given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S(t_1, t_2, t_3) \triangleq \phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0))) = \begin{bmatrix} (x_{01} + t_2) e^{t_3} \\ x_{02} e^{t_3} \\ (x_{03} + t_1) e^{t_3} \end{bmatrix}.$$

Then $S(0,0,0) = x_0$ and

$$\frac{\partial S}{\partial t_2} = \begin{bmatrix} e^{t_3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 / x_{02} \\ 0 \\ 0 \end{bmatrix} = (x_2/x_{02}) f^{(2)} \in \Delta$$

$$\frac{\partial S}{\partial t_3} = \begin{bmatrix} (x_{01} + t_2) e^{t_3} \\ x_{02} e^{t_3} \\ (x_{03} + t_1) e^{t_3} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = f^{(3)} \in \Delta.$$

The inverse of the transformation is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \end{bmatrix} = \begin{bmatrix} x_3x_{02}/x_2 - x_{03} \\ x_1x_{02}/x_2 - x_{01} \\ \ln(x_2/x_{02}) \end{bmatrix}$$

where it is seen that $T(x_0) = 0$. The gradient of $T_1(x) = x_3x_{02}/x_2 - x_{03}$ is

$$dT_1 = \begin{bmatrix} 0 & -x_3x_{02}/x_2^2 & x_{02}/x_2 \end{bmatrix}.$$

We have

$$\mathcal{L}_{f^{(2)}}(T_1) = \left\langle dT_1, f^{(2)} \right\rangle = \begin{bmatrix} 0 & -x_3x_{02}/x_2^2 & x_{02}/x_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\mathcal{L}_{f^{(3)}}(T_1) = \left\langle dT_1, f^{(3)} \right\rangle = \begin{bmatrix} 0 & -x_3x_{02}/x_2^2 & x_{02}/x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Further, as $f^{(1)}$ is linearly independent of $f^{(2)}$ and $f^{(3)}$ we must have $\mathcal{L}_{f^{(1)}}(T_1) \neq 0$. In fact,

$$\mathcal{L}_{f^{(1)}}(T_1) = \begin{bmatrix} 0 & -x_3x_{02}/x_2^2 & x_{02}/x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_{02}/x_2.$$

There is a nice geometric picture when working in the t-coordinates. Let

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_1(x_1, x_2, x_3) \\ T_2(x_1, x_2, x_3) \\ T_3(x_1, x_2, x_3) \end{bmatrix}$$

be the inverse of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S(t_1, t_2, t_3) = \begin{bmatrix} s_1(t_1, t_2, t_3) \\ s_2(t_1, t_2, t_3) \\ s_3(t_1, t_2, t_3) \end{bmatrix}.$$

With t_1 held constant and varying t_2, t_3, $S(t_1, t_2, t_3)$ sweeps out a surface and by the Frobenius theorem we have

$$\frac{\partial S}{\partial t_1}, \frac{\partial S}{\partial t_2} \in \Delta_p = \left\{ r_2 f^{(2)}(p) + r_3 f^{(3)}(p) |\ r_2, r_3 \in \mathbb{R} \right\}$$

for all points $p = S(t_1, t_2, t_3)$. With $t_1 = 0$ the corresponding surface contains x_0. In general $\left\{ x \in \mathcal{U} \subset \mathbf{E}^3 |\ T_1(x) = t_{01} \right\}$ implicitly gives a surface in $\mathcal{U} \subset \mathbf{E}^3$ whose tangent plane spanned by $f^{(2)}, f^{(3)}$. As the gradient $dT_1 = \dfrac{\partial T_1}{\partial x}$ is normal to the surface, if follows that

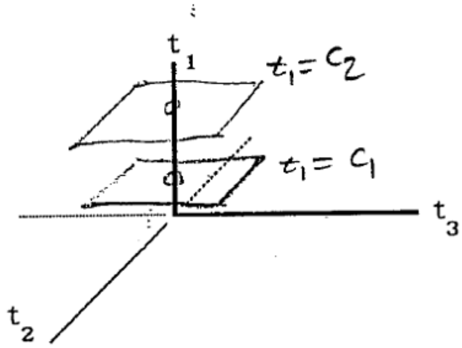
$$\mathcal{L}_{f^{(2)}}(T_1) = \left\langle dT_1, f^{(2)} \right\rangle \equiv 0$$

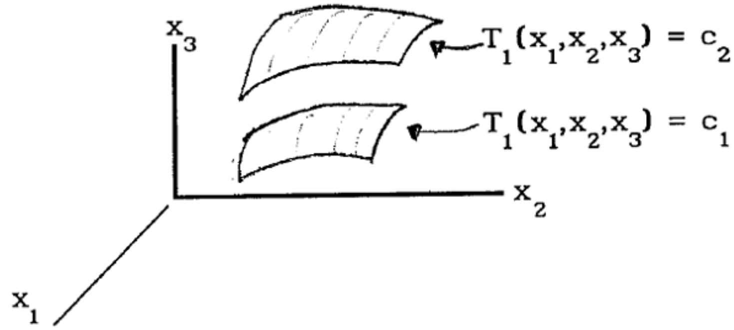
$$\mathcal{L}_{f^{(3)}}(T_1) = \left\langle dT_1, f^{(3)} \right\rangle \equiv 0$$

for all points on the surface. Further, as $f^{(1)}$ is linearly independent of $f^{(2)}$ and $f^{(3)}$ it follows that

$$\mathcal{L}_{f^{(1)}}(T_1) = \left\langle dT_1, f^{(1)} \right\rangle \neq 0$$

in a neighborhood of x_0.





Definition 3 *Lie Derivative of a Gradient*
Let h be defined on an open set $\mathcal{U} \subset \mathbb{R}^3$. Then

$$\mathcal{L}_f(dh) \triangleq d\mathcal{L}_f(h) = \frac{\partial}{\partial x}\mathcal{L}_f(h) = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} f\right) = \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} \frac{\partial h}{\partial x_i} f_i\right).$$

Exercise 14 Show that

$$\mathcal{L}_f(dh) = \left(\frac{\partial (dh)^T}{\partial x} f\right)^T + dh\frac{\partial f}{\partial x} = f^T \frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial x}\frac{\partial f}{\partial x}$$

where $\dfrac{\partial^2 h}{\partial x^2}$ is the Hessian of h.

Theorem 7 $\mathcal{L}_{[f,g]}(h) = \mathcal{L}_f(\mathcal{L}_g(h)) - \mathcal{L}_g(\mathcal{L}_f(h))$
Let f, g be two vector fields on an open subset of $\mathcal{U} \subset \mathbf{E}^3$ Then for any differentiable function h on $\mathcal{U}$ we have

$$\mathcal{L}_{[f,g]}(h) = \mathcal{L}_f(\mathcal{L}_g(h)) - \mathcal{L}_g(\mathcal{L}_f(h)).$$

Proof. This is the same as Theorem **??** of Chapter **??** on page **??**. ■

Corollary 2 *Leibniz Identity*
The following is an identity:

$$\mathcal{L}_f(\langle dh, g\rangle) = \langle \mathcal{L}_f(dh), g\rangle - \langle dh, [f,g]\rangle$$

Proof. Exercise. ■

Definition 4 *Repeated Lie Brackets*
We have the following definition of repeated Lie brackets.

$$\begin{aligned}
ad_f^0 g &\triangleq g \\
ad_f^1 g &\triangleq [f,g] \\
ad_f^2 g &\triangleq [f, ad_f g] \\
ad_f^3 g &\triangleq [f, ad_f^2 g] \\
\vdots\; &\triangleq\; \vdots \\
ad_f^k g &\triangleq [f, ad_f^{k-1} g]
\end{aligned}$$

Example 15 Let $f(x) = Ax, A \in \mathbb{R}^{n\times n}$ and $g \in \mathbb{R}^n$, then $ad_f^k g = (-1)A^k b$.

Exercise 15 *Repeated Lie Derivatives*
It was shown in Theorem 7 that

$$\mathcal{L}_{[f,g]}(h) = \mathcal{L}_f(\mathcal{L}_g(h)) - \mathcal{L}_g(\mathcal{L}_f(h)). \tag{1.31}$$

Replacing g by $ad_f g$ in this expression results in

$$\mathcal{L}_{ad_f^2 g}(h) = \mathcal{L}_{[f,ad_f^1 g]}(h) = \mathcal{L}_f(\mathcal{L}_{[f,g]}(h)) - \mathcal{L}_{[f,g]}(\mathcal{L}_f(h)). \tag{1.32}$$

Next replace h by $\mathcal{L}_f(h)$ in (1.31) gives

$$\mathcal{L}_{[f,g]}(\mathcal{L}_f(h)) = \mathcal{L}_f(\mathcal{L}_g(\mathcal{L}_f(h))) - \mathcal{L}_g(\mathcal{L}_f(\mathcal{L}_f(h))). \tag{1.33}$$

Substituting this expression for $\mathcal{L}_{[f,g]}(\mathcal{L}_f(h))$ in (1.33) into the right side of (1.32) we obtain

$$\begin{aligned}\mathcal{L}_{ad_f^2 g}(h) &= \mathcal{L}_f(\mathcal{L}_{[f,g]}(h)) - (\mathcal{L}_f(\mathcal{L}_g(\mathcal{L}_f(h))) - \mathcal{L}_g(\mathcal{L}_f(\mathcal{L}_f(h))))\\ &= \mathcal{L}_f(\mathcal{L}_{[f,g]}(h)) - \mathcal{L}_f(\mathcal{L}_g(\mathcal{L}_f(h))) + \mathcal{L}_g(\mathcal{L}_f(\mathcal{L}_f(h))).\end{aligned} \tag{1.34}$$

Finally, substituting the expression for $\mathcal{L}_{[f,g]}(h)$ in (1.31) into (1.34) we have

$$\begin{aligned}\mathcal{L}_{ad_f^2 g}(h) &= \mathcal{L}_f\Big(\mathcal{L}_f(\mathcal{L}_g(h)) \ - \mathcal{L}_g(\mathcal{L}_f(h))\Big) - \mathcal{L}_f(\mathcal{L}_g(\mathcal{L}_f(h))) + \mathcal{L}_g(\mathcal{L}_f(\mathcal{L}_f(h)))\\ &= \mathcal{L}_f^2\mathcal{L}_g(h)) - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f(h) + \mathcal{L}_g\mathcal{L}_f^2(h).\end{aligned} \tag{1.35}$$

(a) Show that

$$\mathcal{L}_{ad_f^3 g}(h) = \mathcal{L}_{[f,ad_f^2 g]}(h) = \mathcal{L}_f^3\mathcal{L}_g(h)) - 3\mathcal{L}_f^2\mathcal{L}_g\mathcal{L}_f(h) + 3\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f^2(h) - \mathcal{L}_g\mathcal{L}_f^3(h). \tag{1.36}$$

(b) Use an induction argument to shown

$$\mathcal{L}_{ad_f^k g}(h) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\mathcal{L}_f^i\mathcal{L}_g\mathcal{L}_f^{k-i}(h) \tag{1.37}$$

where $\binom{k}{i} = \frac{k!}{(k-i)!i!}$.

1.5 The Modern Abstract Approach to Tangent Vectors and Lie Brackets

We now look at how the Lie bracket fits into the modern approach to tangent vectors. Consider the manifold $\mathbf{E}^4$, that is, Euclidean four-space with the Cartesian coordinate system. Let the following three vector fields on $\mathbf{E}^4$ given by

$$\mathbf{f}^{(1)} = \begin{bmatrix} -z_2\\ z_1\\ z_4\\ -z_3\end{bmatrix},\ \mathbf{f}^{(2)} = \begin{bmatrix} -z_3\\ -z_4\\ z_1\\ z_2\end{bmatrix},\ \mathbf{f}^{(3)} = \begin{bmatrix} -z_4\\ z_3\\ -z_2\\ z_1\end{bmatrix}.$$

The vector field

$$\mathbf{f}^{(4)} = \begin{bmatrix} z_1\\ z_2\\ z_3\\ z_4\end{bmatrix}$$

points in the radial direction in $\mathbf{E}^4$ and is orthogonal to $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$, $\mathbf{f}^{(3)}$. This observation tell us that $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$, $\mathbf{f}^{(3)}$ are tangent to three-dimensional spherical submanifolds defined by

$$\mathbf{S}^3_r \triangleq \{\mathbf{z} \in \mathbf{E}^4 | z_1^2 + z_2^2 + z_3^2 + z_4^4 = r^2\}.$$

If $r = 1$ then we write

$$\mathbf{S}^3 \triangleq \{\mathbf{z} \in \mathbf{E}^4 | z_1^2 + z_2^2 + z_3^2 + z_4^4 = 1\}.$$

Exercise 16 Show $[\mathbf{f}^{(1)}, \mathbf{f}^{(2)}] = 2\mathbf{f}^{(3)}$, $[\mathbf{f}^{(2)}, \mathbf{f}^{(3)}] = 2\mathbf{f}^{(1)}$, $[\mathbf{f}^{(3)}, \mathbf{f}^{(1)}] = 2\mathbf{f}^{(2)}$

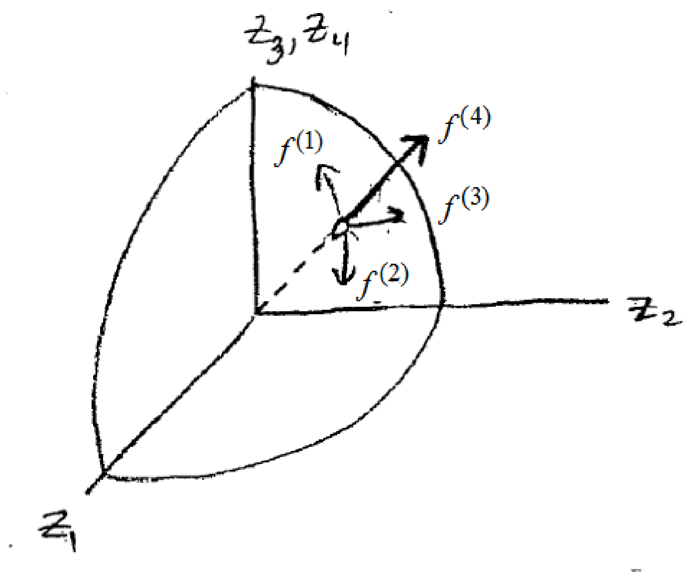


FIGURE 1.8. The submanifold $\mathbf{S}^3 = \{\mathbf{z} \in \mathbf{E}^4 |\ z_1^2 + z_2^2 + z_3^2 + z_4^4 = 1\}$

In $\mathbf{S}^3$ define the north pole to be the point $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$. A coordinate chart for the northern hemisphere of $\mathbf{S}^3$ is $\varphi^{-1} : \mathcal{D} \to \mathcal{U}$ given by

$$\varphi^{-1}(x_1, x_2, x_3) = \mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \sqrt{1 - (x_1^2 + x_2^2 + x_3^2)} \end{bmatrix}$$

where $\mathcal{D} = \{x \in \mathbb{R}^3 |\ x_1^2 + x_2^2 + x_3^2 < 1\}$ and $\mathcal{U} \triangleq \{\mathbf{z} \in \mathbf{S}^3 |\ z_4 > 0\}$. In this northern hemisphere coordinate chart a basis for the tangent space is $\mathbf{T}_p(\mathbf{S}^3)$ is

$$\mathbf{z}_{x_1} = \frac{\partial \mathbf{z}}{\partial x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \dfrac{-x_1}{d(x)} \end{bmatrix}, \quad \mathbf{z}_{x_2} = \frac{\partial \mathbf{z}}{\partial x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \dfrac{-x_2}{d(x)} \end{bmatrix}, \quad \mathbf{z}_{x_3} = \frac{\partial \mathbf{z}}{\partial x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \dfrac{-x_3}{d(x)} \end{bmatrix}$$

where $d(x) = \sqrt{1-(x_1^2+x_2^2+x_3^2)}$. Note that

$$\mathbf{f}^{(1)}(\varphi^{-1}(x)) = \begin{bmatrix} -z_2 \\ z_1 \\ z_4 \\ -z_3 \end{bmatrix}\Bigg|_{\mathbf{z}=\varphi^{-1}(x)=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \sqrt{1-(x_1^2+x_2^2+x_3^2)} \end{bmatrix}} = \begin{bmatrix} -x_2 \\ x_1 \\ d(x) \\ -x_3 \end{bmatrix} = -x_2\mathbf{z}_{x_1} + x_1\mathbf{z}_{x_2} + d(x)\mathbf{z}_{x_3}$$

Similarly we have

$$\mathbf{f}^{(2)}(\varphi^{-1}(x)) = \begin{bmatrix} -z_3 \\ -z_4 \\ z_1 \\ z_2 \end{bmatrix}\Bigg|_{\mathbf{z}=\varphi^{-1}(x)=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \sqrt{1-(x_1^2+x_2^2+x_3^2)} \end{bmatrix}} = \begin{bmatrix} -x_3 \\ -d(x) \\ x_1 \\ x_2 \end{bmatrix} = -x_3\mathbf{z}_{x_1} - d(x)\mathbf{z}_{x_2} + x_1\mathbf{z}_{x_3}$$

$$\mathbf{f}^{(3)}(\varphi^{-1}(x)) = \begin{bmatrix} -z_4 \\ z_3 \\ -z_2 \\ z_1 \end{bmatrix}\Bigg|_{\mathbf{z}=\varphi^{-1}(x)=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \sqrt{1-(x_1^2+x_2^2+x_3^2)} \end{bmatrix}} = \begin{bmatrix} -d(x) \\ x_3 \\ -x_2 \\ x_1 \end{bmatrix} = -d(x)\mathbf{z}_{x_1} + x_3\mathbf{z}_{x_2} - x_2\mathbf{z}_{x_3}.$$

Considering $\mathbf{S}^3$ embedded in $\mathbf{E}^3$ and with h a differentiable function on $\mathbf{E}^4$ we write

$$\begin{aligned}
\left\langle dh, \mathbf{f}^{(1)} \right\rangle &= \begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} & \dfrac{\partial h}{\partial z_4} \end{bmatrix} \begin{bmatrix} -z_2 \\ z_1 \\ z_4 \\ -z_3 \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} & \dfrac{\partial h}{\partial z_4} \end{bmatrix} \left(-x_2\mathbf{z}_{x_1} + x_1\mathbf{z}_{x_2} + d(x)\mathbf{z}_{x_3} \right) \\
&= \begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} & \dfrac{\partial h}{\partial z_4} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{x_1} & \mathbf{z}_{x_2} & \mathbf{z}_{x_3} \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \\ d(x) \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} & \dfrac{\partial h}{\partial z_4} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \varphi^{-1}}{\partial x_1} & \dfrac{\partial \varphi^{-1}}{\partial x_2} & \dfrac{\partial \varphi^{-1}}{\partial x_3} \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \\ d(x) \end{bmatrix} \\
&= -x_2\frac{\partial h \circ \varphi^{-1}}{\partial x_1} + x_1\frac{\partial h \circ \varphi^{-1}}{\partial x_2} + \sqrt{1-(x_1^2+x_2^2+x_3^2)}\frac{\partial h \circ \varphi^{-1}}{\partial x_3}
\end{aligned}$$

These calculations motivated the new definition of a tangent vector given as follow: Let $h \in \mathcal{F}(\mathcal{U})$ be a differentiable function on $\mathcal{U} \triangleq \{\mathbf{z} \in \mathbf{S}^3 |\ z_4 > 0\} \subset \mathbf{S}^3$, that is, $h \circ \varphi^{-1}(x_1, x_2, x_3) = h(\varphi^{-1}(x_1, x_2, x_3))$ is a differentiable function on $\mathcal{D} = \left\{x \in \mathbb{R}^3 |\ x_1^2 + x_2^2 + x_3^2 < 1\right\}$. Then the tangent vector $\mathbf{f}^{(1)}$ above is replaced by the mapping $\mathbf{f}^{(1)} : \mathcal{F}(\mathcal{U}) \to \mathbb{R}$ given by

$$\mathbf{f}^{(1)} : h \to -x_2\frac{\partial h \circ \varphi^{-1}}{\partial x_1} + x_1\frac{\partial h \circ \varphi^{-1}}{\partial x_2} + \sqrt{1-(x_1^2+x_2^2+x_3^2)}\frac{\partial h \circ \varphi^{-1}}{\partial x_3} \tag{1.38}$$

or we write

$$\mathbf{f}^{(1)} = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + \sqrt{1-(x_1^2+x_2^2+x_3^2)}\frac{\partial}{\partial x_3}. \tag{1.39}$$

This last expression is bewildering since we have dropped the φ^{-1} and it is the $\dfrac{\partial \varphi^{-1}}{\partial x_i}$ which were our original notion of a tangent vector. However, in (1.39) the partial derivative operators $\dfrac{\partial}{\partial x_i}$ are respect to a particular coordinate system with coordinates (x_1, x_2, x_3) which is this example is the northern hemisphere. The components of the tangent vector $\mathbf{f}^{(1)}$ given by $(-x_2, x_1, \sqrt{1-(x_1^2+x_2^2+x_3^2)})$ are only valid in this coordinate system. Similarly, in the northern hemisphere coordinate chart, we have

$$\mathbf{f}^{(2)} = -x_3\frac{\partial}{\partial x_1} - \sqrt{1-(x_1^2+x_2^2+x_3^2)}\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} \tag{1.40}$$

$$\mathbf{f}^{(3)} = -\sqrt{1-(x_1^2+x_2^2+x_3^2)}\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}. \tag{1.41}$$

In terms of the components we have

$$f^{(1)} = \begin{bmatrix} -x_2 \\ x_1 \\ \sqrt{1-(x_1^2+x_2^2+x_3^2)} \end{bmatrix}, \quad f^{(2)} = \begin{bmatrix} -x_3 \\ -\sqrt{1-(x_1^2+x_2^2+x_3^2)} \\ x_1 \end{bmatrix}, \quad f^{(3)} = \begin{bmatrix} -\sqrt{1-(x_1^2+x_2^2+x_3^2)} \\ x_3 \\ -x_2 \end{bmatrix}$$

These components of the tangent vectors are in $\mathcal{D} = \left\{x \in \mathbb{R}^3 |\ x_1^2+x_2^2+x_3^2 < 1\right\}$.

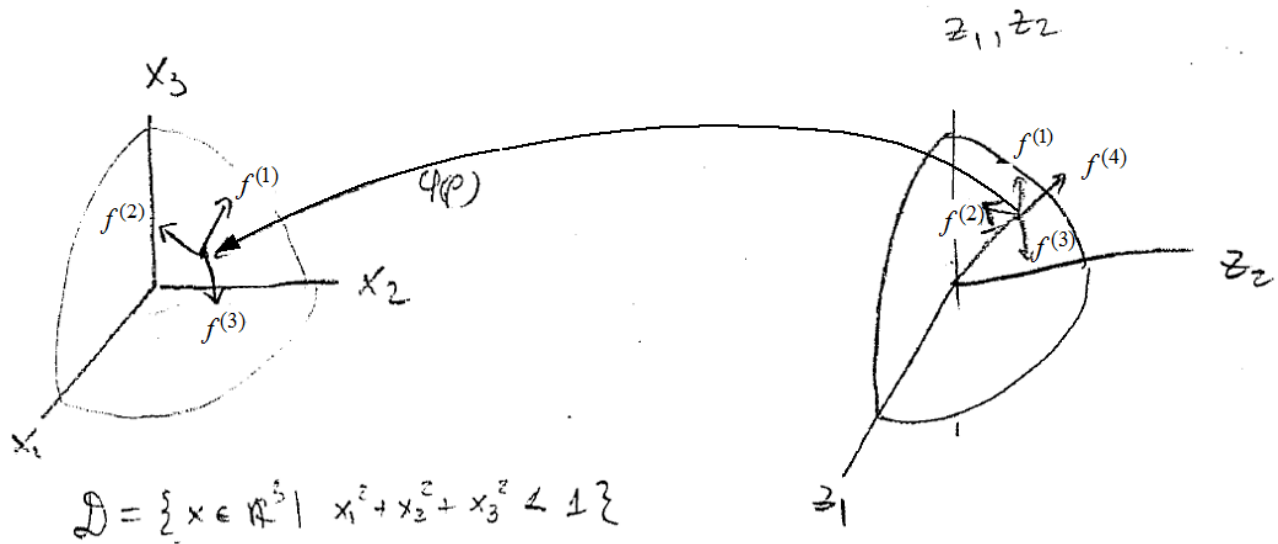


FIGURE 1.9. φ is a 1-1 and onto map from the northern hemisphere $\mathcal{U}$ of S^3 to $\mathcal{D} = \{x \in \mathbb{R}^3 |\ x_1^2+x_2^2+x_3^2 < 1\}$.

Exercise 17 Is $\mathbf{f}^{(4)}$ a linear combination of $\mathbf{z}_{x_1}, \mathbf{z}_{x_2}, \mathbf{z}_{x_3}$? Explain.

We now compute the Lie bracket of $[f^{(1)}, f^{(2)}]$ in the northern hemisphere coordinate chart. We have

$$\begin{aligned}
[f^{(1)}, f^{(2)}] &= \begin{bmatrix} 0 & 0 & -1 \\ x_1/d(x) & x_2/d(x) & x_3/d(x) \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \\ d(x) \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -x_1/d(x) & -x_2/d(x) & -x_3/d(x) \end{bmatrix} \begin{bmatrix} -x_3 \\ -d(x) \\ x_1 \end{bmatrix} \\
&= \begin{bmatrix} -d(x) \\ x_3 \\ -x_2 \end{bmatrix} - \begin{bmatrix} d(x) \\ -x_3 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} -d(x) \\ x_3 \\ -x_2 \end{bmatrix} = 2f^{(3)}.
\end{aligned}$$

Similarly we have $[f^{(1)}, f^{(3)}] = 2f^{(2)}$, $[f^{(2)}, f^{(3)}] = 2f^{(1)}$.

Why should we expect this? We now show that

$$[f^{(1)}, f^{(2)}] = 2f^{(3)} \iff [\mathbf{f}^{(1)}, \mathbf{f}^{(2)}] = 2\mathbf{f}^{(3)}.$$

Recall the above discussion were we wrote

$$\begin{bmatrix} \dfrac{\partial h}{\partial z_1} & \dfrac{\partial h}{\partial z_2} & \dfrac{\partial h}{\partial z_3} & \dfrac{\partial h}{\partial z_4} \end{bmatrix} \underbrace{\begin{bmatrix} -z_2 \\ z_1 \\ z_4 \\ -z_3 \end{bmatrix}}_{\mathbf{f}^{(1)}(z)} = \left(\begin{bmatrix} \dfrac{\partial h \circ \varphi^{-1}}{\partial x_1} & \dfrac{\partial h \circ \varphi^{-1}}{\partial x_2} & \dfrac{\partial h \circ \varphi^{-1}}{\partial x_3} \end{bmatrix} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \\ d(x) \end{bmatrix}}_{f^{(1)}} \right)_{|x=\varphi(z)}$$

With $\mathfrak{h} \triangleq h \circ \varphi^{-1}(x)$ we may rewrite this as

$$\mathcal{L}_{\mathbf{f}^{(1)}}(h) = \mathcal{L}_{f^{(1)}}(h \circ \varphi^{-1})_{|x=\varphi(z)} = \mathcal{L}_{f^{(1)}}(\mathfrak{h}) \circ \varphi.$$

Repeating this we have

$$\mathcal{L}_{\mathbf{f}^{(2)}}\left(\mathcal{L}_{\mathbf{f}^{(1)}}(h)\right) = \left(\mathcal{L}_{f^{(2)}}\left(\mathcal{L}_{f^{(1)}}(h \circ \varphi^{-1})\right)\right) \circ \varphi = \mathcal{L}_{f^{(2)}}\left(\mathcal{L}_{f^{(1)}}(\mathfrak{h})\right) \circ \varphi.$$

Then

$$\mathcal{L}_{\mathbf{f}^{(2)}}\left(\mathcal{L}_{\mathbf{f}^{(1)}}(h)\right) - \mathcal{L}_{\mathbf{f}^{(1)}}\left(\mathcal{L}_{\mathbf{f}^{(2)}}(h)\right) = \left(\mathcal{L}_{f^{(2)}}\left(\mathcal{L}_{f^{(1)}}(h \circ \varphi^{-1})\right) - \mathcal{L}_{f^{(1)}}\left(\mathcal{L}_{f^{(2)}}(h \circ \varphi^{-1})\right)\right) \circ \varphi$$

or

$$\mathcal{L}_{[\mathbf{f}^{(2)}, \mathbf{f}^{(1)}]}(h) = \mathcal{L}_{[f^{(2)}, f^{(1)}]}(\mathfrak{h}) \circ \varphi.$$

1.6 Problems

Problem 1 *Tangent Vectors as Derivations*

Let $\mathcal{D} \subset \mathbb{R}^n$ be an open subset and denote by $\mathcal{C}^\infty(\mathcal{D})$ the infinitely differentiable functions from $\mathcal{D}$ to $\mathbb{R}$. A *derivation* is a map D from $\mathcal{C}^\infty(\mathcal{D}) \to \mathbb{R}$ such that for any $h_1, h_2 \in \mathcal{C}^\infty(\mathcal{D})$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ it satisfies

(i) Linearity
$D(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 D(h_1) + \alpha_2 D(h_2)$
(ii) Product Rule
$D(h_1 h_2) = h_1 D(h_2) + h_2 D(h_1)$

(a) Let $\mathcal{U} \subset \mathbf{S}^2$ and $\varphi : \mathcal{D} \to \varphi(\mathcal{U})$ be a coordinate chart on $\mathbf{S}^2$. Let h be a function defined on $\mathcal{U}$ so that with $\varphi(p) = (x_1, x_2)$ we have $h(p) = h \circ \varphi^{-1}(x_1, x_2) = \mathfrak{h}(x_1, x_2)$ is defined on $\mathcal{D} = \varphi(\mathcal{U})$. Define $\dfrac{\partial}{\partial x_1}$: $\mathcal{C}^\infty(\mathbf{S}^2) \to \mathbb{R}$ by

$$\frac{\partial}{\partial x_1} : h \to \frac{\partial}{\partial x_1} h \circ \varphi^{-1}\Big|_{\varphi(p)} = \frac{\partial \mathfrak{h}}{\partial x_1}\Big|_{\varphi(p)} \in \mathbb{R}.$$

Show that this satisfies (i) and (ii), i.e., it is a derivation.

(b) Recall the modern definition of a tangent vector as a mapping $\mathbf{z}_p : \mathcal{C}^\infty(\mathbf{S}^2) \to \mathbb{R}$ given by

$$\mathbf{z}_p : h \to \mathbf{z}_p(h) \triangleq f_1(x_1, x_2)\, \frac{\partial}{\partial x_1} h \circ \varphi^{-1}\Big|_{\varphi(p)} + f_2(x_1, x_2)\, \frac{\partial}{\partial x_2} h \circ \varphi^{-1}\Big|_{\varphi(p)}$$

where $\varphi(p) = (x_1, x_2)$ and $f_1(x_1, x_2), f_2(x_1, x_2)$ are the *components* of the vector. Show that $\mathbf{z}_p$ is a derivation.

Also recall that we might "prefer" to look at this mapping as

$$\begin{aligned}\mathbf{z}_p(h) &= \begin{bmatrix} \dfrac{\partial h(z)}{\partial z_1} & \dfrac{\partial h(z)}{\partial z_2} & \dfrac{\partial h(z)}{\partial z_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \\ \dfrac{\partial z_3}{\partial x_1} & \dfrac{\partial z_3}{\partial x_2} \end{bmatrix}_{(x_{01}, x_{02})} \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \\ &= dh \begin{bmatrix} \mathbf{z}_{x_1} & \mathbf{z}_{x2} \end{bmatrix} \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}\end{aligned}$$

with $f_1(x_1, x_2)\mathbf{z}_{x_1} + f_2(x_1, x_2)\mathbf{z}_{x_2}$ the "tangent vector". However, this does not make sense as h is defined only on $\mathbf{S}^2$ (so $\partial h(z)/\partial z_1$, etc. are not defined) and $\mathbf{z}_{x_1}, \mathbf{z}_{x2}$ stick off of $\mathbf{S}^2$.

(c) Let $f = (f_1(x_1, x_2), f_2(x_1, x_2))$ and $g = (g_1(x_1, x_2), g_2(x_1, x_2))$ be the components of two vector fields defined on $\mathcal{D}$. With $\mathfrak{h} = h \circ \varphi^{-1}(x_1, x_2)$ and by Theorem **??** of Chapter **??** (page **??**)

$$\mathcal{L}_f(\mathcal{L}_g(\mathfrak{h})) = g^T \frac{\partial^2 \mathfrak{h}}{\partial x^2} f + \frac{\partial \mathfrak{h}}{\partial x} \frac{\partial g}{\partial x} f.$$

Is this a derivation, i.e., does it satisfy (i) and (ii) above? Explain.

(d) Show that $D \triangleq \mathcal{L}_f \mathcal{L}_g - \mathcal{L}_g \mathcal{L}_f : \mathcal{C}^\infty(\mathbf{S}^2) \to \mathbb{R}$ given by

$$\mathfrak{h} \to \Big(\mathcal{L}_f(\mathcal{L}_g(\mathfrak{h})) - \mathcal{L}_g(\mathcal{L}_f(\mathfrak{h}))\Big)_{|\varphi(p)=(x_1,x_2)}$$

is a derivation. Equivalently, $\mathcal{L}_{[f,g]}(\mathfrak{h}) = \mathcal{L}_f(\mathcal{L}_g(\mathfrak{h})) - \mathcal{L}_g(\mathcal{L}_f(\mathfrak{h}))$ is a derivation.

Problem 2 *The Frobenius Theorem* [1]

Consider the following system of partial differential equations

$$\frac{\partial S}{\partial u_1} = f^{(1)}(u_1, u_2, s_1(u_1, u_2), s_2(u_1, u_2), s_3(u_1, u_2)) \tag{1.42}$$

$$\frac{\partial S}{\partial u_2} = f^{(2)}(u_1, u_2, s_1(u_1, u_2), s_2(u_1, u_2), s_3(u_1, u_2)) \tag{1.43}$$

where

$$S(u_1, u_2) = \begin{bmatrix} s_1(u_1, u_2) \\ s_2(u_1, u_2) \\ s_3(u_1, u_2) \end{bmatrix}$$

and

$$f^{(1)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} f_1^{(1)}(u_1, u_2, x_1, x_2, x_3) \\ f_2^{(1)}(u_1, u_2, x_1, x_2, x_3) \\ f_3^{(1)}(u_1, u_2, x_1, x_2, x_3) \end{bmatrix}, \quad f^{(2)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} f_1^{(2)}(u_1, u_2, x_1, x_2, x_3) \\ f_2^{(2)}(u_1, u_2, x_1, x_2, x_3) \\ f_3^{(2)}(u_1, u_2, x_1, x_2, x_3) \end{bmatrix}.$$

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^3$, $\mathcal{D}$ an open subset of $\mathbb{R}^2$. Suppose, given any point $x_0 \in \mathcal{U}$ and any $u_0 \in (u_{01}, u_{02}) \in \mathcal{D}$, there is a surface $S(u_1, u_2)$ satisfying (1.42) and (1.43) in some neighborhood of (u_{01}, u_{02}) with $S(u_{01}, u_{02}) = x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$.

(a) Show that

$$\frac{\partial f^{(1)}(u,x)}{\partial u_2} - \frac{\partial f^{(2)}(u,x)}{\partial u_1} + \frac{\partial f^{(1)}(u,x)}{\partial x} f^{(2)} - \frac{\partial f^{(2)}(u,x)}{\partial x} f^{(1)} \equiv 0 \tag{1.44}$$

for all $u \triangleq (u_1, u_2) \in \mathcal{D}$ and $x \triangleq (x_1, x_2, x_3) \in \mathcal{U}$.

(b) Let

$$f^{(1)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} x_1 - u_1 - u_2 \\ -x_2 + u_1 + u_2 \\ 0 \end{bmatrix}, \quad f^{(2)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} x_1^2 - u_1^2 - u_2^2 - 2u_1 - 2u_2 - 2u_1u_2 \\ 1 \\ 0 \end{bmatrix}. \tag{1.45}$$

Are the integrability conditions (1.44) satisfied for these vector fields.

(c) Let

$$S(u_1, u_2) = \begin{bmatrix} u_1 + u_2 + 1 \\ u_1 + u_2 - 1 \\ x_{03} \end{bmatrix}. \tag{1.46}$$

Does this satisfy the partial differential equations (1.42) (1.43) with $f^{(1)}, f^{(2)}$ given by (1.45)? Does (1.46) satisfy

$$\left(\frac{\partial f^{(1)}(u,x)}{\partial u_2} - \frac{\partial f^{(2)}(u,x)}{\partial u_1} + \frac{\partial f^{(1)}(u,x)}{\partial x} f^{(2)} - \frac{\partial f^{(2)}(u,x)}{\partial x} f^{(1)} \right)_{x = s(u_1, u_2)} \equiv 0 \ ?$$

Is there any contradiction with your answer to part (b)? Explain.

Problem 3 *The Frobenius Theorem*

Consider the following system of partial differential equations

$$\frac{\partial S}{\partial u_1} = f^{(1)}(u_1, u_2, s_1(u_1, u_2), s_2(u_1, u_2), s_3(u_1, u_2)) \tag{1.47}$$

$$\frac{\partial S}{\partial u_2} = f^{(2)}(u_1, u_2, s_1(u_1, u_2), s_2(u_1, u_2), s_3(u_1, u_2)) \tag{1.48}$$

where

$$S(u_1, u_2) = \begin{bmatrix} s_1(u_1, u_2) \\ s_2(u_1, u_2) \\ s_3(u_1, u_2) \end{bmatrix}$$

and

$$f^{(1)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} f_1^{(1)}(u_1, u_2, x_1, x_2, x_3) \\ f_2^{(1)}(u_1, u_2, x_1, x_2, x_3) \\ f_3^{(1)}(u_1, u_2, x_1, x_2, x_3) \end{bmatrix}, \quad f^{(2)}(u_1, u_2, x_1, x_2, x_3) = \begin{bmatrix} f_1^{(2)}(u_1, u_2, x_1, x_2, x_3) \\ f_2^{(2)}(u_1, u_2, x_1, x_2, x_3) \\ f_3^{(2)}(u_1, u_2, x_1, x_2, x_3) \end{bmatrix}.$$

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^3$, $\mathcal{D}$ an open subset of $\mathbb{R}^2$ and suppose that

$$\frac{\partial f^{(1)}}{\partial u_2} - \frac{\partial f^{(2)}}{\partial u_1} + \frac{\partial f^{(1)}}{\partial x} f^{(2)} - \frac{\partial f^{(2)}}{\partial x} f^{(1)} \equiv 0$$

for all $x \in \mathcal{U}$ and $u \in \mathcal{D}$.

Prove that, given any point $x_0 \in \mathcal{U}$ and any $u_0 \in (u_{01}, u_{02}) \in \mathcal{D}$, there is a surface $S(u_1, u_2)$ satisfying (1.47) and (1.48) in some neighborhood of (u_{01}, u_{02}) with $S(u_{01}, u_{02}) = x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$.

Hint: Mimic the proof in this chapter which was done for the case where the vector fields were "time invariant", that is, $f^{(1)}, f^{(2)}$ depend only x and not u.

Problem 4 *The Frobenius Theorem*

Consider the manifold $\mathbf{E}^4$, that is, Euclidean four-space with the Cartesian coordinate system. Let the following three vector fields on $\mathbf{E}^4$ given by

$$\mathbf{f}^{(1)} = \begin{bmatrix} -z_2 \\ z_1 \\ z_4 \\ -z_3 \end{bmatrix}, \quad \mathbf{f}^{(2)} = \begin{bmatrix} -z_3 \\ -z_4 \\ z_1 \\ z_2 \end{bmatrix}, \quad \mathbf{f}^{(3)} = \begin{bmatrix} -z_4 \\ z_3 \\ -z_2 \\ z_1 \end{bmatrix}.$$

(a) Is the set of vector fields $\{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}\}$ involutive? Are they orthogonal to each other?

(b) Let

$$\mathbf{f}^{(4)} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

be a fourth vector field. Is $\mathbf{f}^{(4)}$ orthogonal to the set of vectors $\{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}\}$?

(c) What does the Frobenius theorem guarantee in terms of the a surface whose tangent space spanned by $\{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}\}$. Can you explicitly find the surface? Hint: Think about the geometry of this problem instead of trying to crank out calculations. In particular, $\mathbf{f}^{(4)}$ points in the radial direction in $\mathbf{E}^4$.

(d) Use your answer in part (c) to explicitly find a coordinate transformation $x^* = T(x)$ such that in the new coordinates $x_1^* = T_1(x) = c$ (constant) corresponds to a surface in the x coordinates for which the tangent space is spanned by $\{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}\}$.

Problem 5 *The Lie Derivative of a Vector* $\mathcal{L}_f g$

Consider two vector fields f and g in $\mathcal{U} \subset \mathbf{E}^3$. Let $\phi^f(t,x) = \phi_t^f(x)$ and $\phi^g(t,x) = \phi_t^g(x)$ be the flows of f and g, respectively. That is,

$$\begin{aligned}\frac{d}{dt}\phi^f(t,x) &= f(\phi^f(t,x)) \text{ with } \phi^f(0,x) = x\\ \frac{d}{dt}\phi^g(t,x) &= g(\phi^g(t,x)) \text{ with } \phi^g(0,x) = x.\end{aligned}$$

For each fixed t, $\phi^f(t,x)$ represents starting at x and moving in the direction specified by f for the time t to reach the point $x' \triangleq \phi^f(t,x)$. That is, for each fixed t, $\phi^f(t,\cdot) : \mathbf{E}^3 \to \mathbf{E}^3$ that takes x to $x' = \phi^f(t,x)$. Further, starting at $x' = \phi^f(t,x)$ and following the vector field f for a time $-t$ results in coming back to x, that is,

$$x = \phi^f(-t,x') = \phi^f(-t,\phi^f(t,x)).$$

Then

$$\frac{\partial}{\partial x}x = \frac{\partial}{\partial x}\phi^f(-t,\phi^f(t,x)) = \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)} \frac{\partial}{\partial x}\phi^f(t,x) = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

(a) Use this relationship to show that

$$\left(\frac{d}{dt}\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\right)_{t=0} = -\frac{\partial f(x)}{\partial x}. \tag{1.49}$$

(b) With

$$\left(\phi^f_{-t}\right)_* \triangleq \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}$$

define

$$\mathcal{L}_f g \triangleq \left(\frac{d}{dt}\left(\left(\phi^f_{-t}\right)_* g(\phi^f(t,x))\right)\right)_{t=0}$$

and show that

$$\mathcal{L}_f g = [f,g].$$

Remark A more enlightening way to evaluate the definition of $\mathcal{L}_f g$ is shown in [2] (page 61) as follows. Using Equation (1.49) a two term Taylor series expansion of

$$\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}$$

about $t=0$ is

$$\begin{aligned}\left(\phi^f_{-t}\right)_* \triangleq \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)} &\approx \left(\frac{\partial}{\partial x'}\phi^f(-0,x')\right)_{|x'=\phi^f(0,x)} + \left(\frac{d}{dt}\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\right)_{t=0} t\\ &= I_{3\times 3} - \frac{\partial f(x)}{\partial x}t.\end{aligned}$$

A two term Taylor series expansion of $g(\phi^f(t,x))$ about $t=0$ is

$$\begin{aligned}g(\phi^f(t,x)) \approx g(\phi^f(0,x)) + \left(\frac{d}{dt}g(\phi^f(t,x))\bigg|_{t=0}\right)t &= g(\phi^f(0,x)) + \left(\frac{\partial g}{\partial x}_{|\phi^f(0,x)}\right)\left(\frac{d\phi^f(t,x))}{dt}\bigg|_{t=0}\right)t\\ &= g(\phi^f(0,x)) + \frac{\partial g}{\partial x}\left(f(\phi^f(t,x))_{|t=0}\right)t\\ &= g(x)) + \left(\frac{\partial g}{\partial x}f(x)\right)t\end{aligned}$$

Then

$$\mathcal{L}_f g \triangleq \left(\frac{d}{dt}\left(\left(\phi^f_{-t}\right)_* g(\phi^f(t,x))\right)\right)_{t=0} = \lim_{t\to 0} \frac{\left(I_{3\times 3} - \frac{\partial f(x)}{\partial x}t\right)\left(g(x) + \left(\frac{\partial g}{\partial x}f(x)\right)t\right) - g(x)}{t}$$

$$= \lim_{t\to 0} \frac{g(x) + \left(\frac{\partial g}{\partial x}f(x)\right)t - \frac{\partial f(x)}{\partial x}t\left(g(x) + \left(\frac{\partial g}{\partial x}f(x)\right)t\right) - g(x)}{t}$$

$$= \lim_{t\to 0}\left(\frac{\partial g}{\partial x}f(x) - \frac{\partial f(x)}{\partial x}\left(g(x) + \left(\frac{\partial g}{\partial x}f(x)\right)t\right)\right)$$

$$= \frac{\partial g}{\partial x}f(x) - \frac{\partial f(x)}{\partial x}g(x).$$

Problem 6 *Jacobi Identify*
Let $f^{(1)}, f^{(2)}, f^{(3)}$ be three vector fields in $\mathbf{E}^3$.

(a) Show that

$$[[f^{(1)}, f^{(2)}], f^{(3)}] + [[f^{(2)}, f^{(3)}], f^{(1)}] + [[f^{(3)}, f^{(1)}], f^{(2)}] \equiv 0.$$

This is known as the *Jacobi identity*.

(b) Let f and g be vector fields in $\mathbf{E}^3$. Use the Jacobi identity to show that the conditions

$$[ad_f^k g, ad_f^\ell g] \equiv 0 \quad \text{for} \quad k+\ell = 0,1,2,3,4,5$$

are equivalent to the conditions

$$[g, ad_f^k g] \equiv 0 \quad \text{for} \quad k = 1,3,5.$$

(c) Let f and g be vector fields in $\mathbf{E}^n$. Use the Jacobi identity to show that the conditions

$$[ad_f^k g, ad_f^\ell g] \equiv 0 \quad \text{for} \quad k+\ell = 0,1,2,...,2n-1$$

are equivalent to the conditions

$$[g, ad_f^k g] \equiv 0 \quad \text{for} \quad k = 1,3,5,...,2n-1.$$

Problem 7 *Lie Brackets and Integrability of Vector Fields*
Let $f^{(1)}, f^{(2)}$, and $f^{(3)}$ be three linearly independent fields on an open set $\mathcal{U} \subset \mathbf{E}^3$. Let $x_0 \in \mathcal{U}$ and define the map $S(t_1,t_2,t_3) \triangleq \phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0)))$ where the solution to $dx/dt_1 = f^{(1)}(x), x(0) = x_0$ is $\phi_{t_1}(x_0)$, the solution to $dx/dt_2 = f^{(2)}(x), x(0) = x_0'$ is $\phi_{t_2}(x_0')$ and the solution to $dx/dt_3 = f^{(3)}(x), x(0) = x_0''$ is $\phi_{t_3}(x_0'')$. Assume that $S(t_1,t_2,t_3) = \phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0)))$ exits for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$ for some $\epsilon > 0$. It should be clear that $S(0,0,0) = x_0$.

(a) Suppose $[f^{(2)}, f^{(3)}] \neq 0$ for $x \in \mathcal{U}$.

Does $\dfrac{\partial S}{\partial t_3} = f^{(3)}(x)_{|x=S(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

Does $\dfrac{\partial S}{\partial t_2} = f^{(2)}(x)_{|x=S(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

(b) Suppose $[f^{(2)}, f^{(3)}] = 0$ for $x \in \mathcal{U}$.

Does $\dfrac{\partial S}{\partial t_3} = f^{(3)}(x)_{|x=S(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

Does $\dfrac{\partial S}{\partial t_2} = f^{(2)}(x)_{|x=S(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

Problem 8 *Lie Brackets and Integrability of Vector Fields*

Let $f^{(1)}, f^{(2)}$, and $f^{(3)}$ be three linearly independent fields on an open set $\mathcal{U} \subset \mathbf{E}^3$. Let $x_0 \in \mathcal{U}$ and define the map $R(t_1, t_2, t_3) \triangleq \phi_{t_2}(\phi_{t_3}(\phi_{t_1}(x_0)))$ where the solution to $dx/dt_1 = f^{(1)}(x), x(0) = x_0$ is $\phi_{t_1}(x_0)$, the solution to $dx/dt_2 = f^{(2)}(x), x(0) = x_0'$ is $\phi_{t_2}(x_0')$ and the solution to $dx/dt_3 = f^{(3)}(x), x(0) = x_0''$ is $\phi_{t_3}(x_0'')$. Assume that $S(t_1, t_2, t_3) = \phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0)))$ exits for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$ for some $\epsilon > 0$. It should be clear that $R(0, 0, 0) = x_0$.

(a) Suppose $[f^{(2)}, f^{(3)}] \neq 0$ for $x \in \mathcal{U}$.

Does $\dfrac{\partial R}{\partial t_3} = f^{(3)}(x)_{|x=R(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

Does $\dfrac{\partial R}{\partial t_2} = f^{(2)}(x)_{|x=R(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

(b) Suppose $[f^{(2)}, f^{(3)}] = 0$ for $x \in \mathcal{U}$.

Does $\dfrac{\partial R}{\partial t_3} = f^{(3)}(x)_{|x=R(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

Does $\dfrac{\partial R}{\partial t_2} = f^{(2)}(x)_{|x=R(t_1,t_2,t_3)}$ for $|t_1| < \epsilon, |t_2| < \epsilon, |t_3| < \epsilon$. Answer yes or no and explain briefly.

1.7 References

[1] Carl B. Allendoerfer, *Calculus of Several Variables and Differnentiable Manifolds*, Copyright by Carl B. Allendoerfer, 1973.

[2] Harry G. Kwatny and Gilmer L. Blankenship, *Nonlinear Control and Analytical Mechanics, A Computational Approach*, Birkhäuser, 2000.