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Chapter1 Solutions

Problem 12 *Parallel Parking*

Figure 1.1 is schematic to model the steering of a car. R_1 denotes the midpoint of the real axle while F_1 denotes the midpoint of the front axle and has coordinates (x_1, x_2). φ denotes the angle from the x_1 axis to the body axis $R_1 - F_1$ which has length ℓ. θ denotes the angle of the front wheels are pointing with respect to the body axis. The angle θ is an input whose value is set by the steering wheel. Figure 1.2 is a schematic

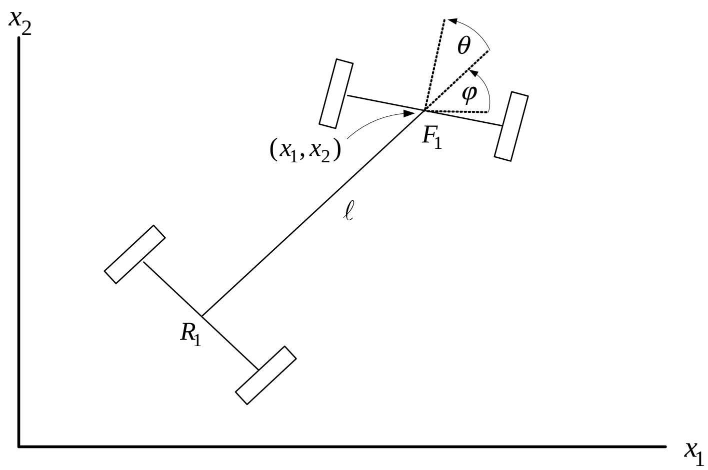

FIGURE 1.1. Schematic for steering a car.

indicating the motion of car of a small distance h in the direction the front wheels are pointing. The midpoint of the front axle goes from F_1 to F_2. The coordinates of F_2 are then

$$(x_1 + h\cos(\theta + \varphi), x_2 + h\sin(\theta + \varphi)).$$

The distance $F_2 - E$ is $h\sin(\theta)$ and with h small we have

$$\Delta\varphi = \frac{h\sin(\theta)}{\ell}.$$

The *drive* vector field g_D is

$$g_D \triangleq \cos(\theta + \varphi)\hat{x}_1 + \sin(\theta + \varphi))\hat{x}_2 + \frac{\sin(\theta)}{\ell}\hat{\varphi}$$

where $\hat{x}_1$ and $\hat{x}_2$ are unit vectors in the direction of increasing x_1 and x_2, respectively, and $\hat{\varphi}$ is a unit vector in the direction of increasing φ. This is the motion we get by applying power to the rear wheels to produce the velocity input u_1.

The *steer* vector field g_{Steer} is

$$g_{Steer} \triangleq \hat{\theta}$$

where $\hat{\theta}$ is a unit vector in the direction of increasing θ.

Define the *slide* vector field g_{Slide} to be

$$g_{Slide} \triangleq -\sin(\varphi)\hat{x}_1 + \cos(\varphi)\hat{x}_2$$

and the *rotate* vector field g_R to be

$$g_R \triangleq \frac{1}{\ell}\hat{\varphi}$$

where $\hat{\varphi}$ is a unit vector in the direction of increasing φ.

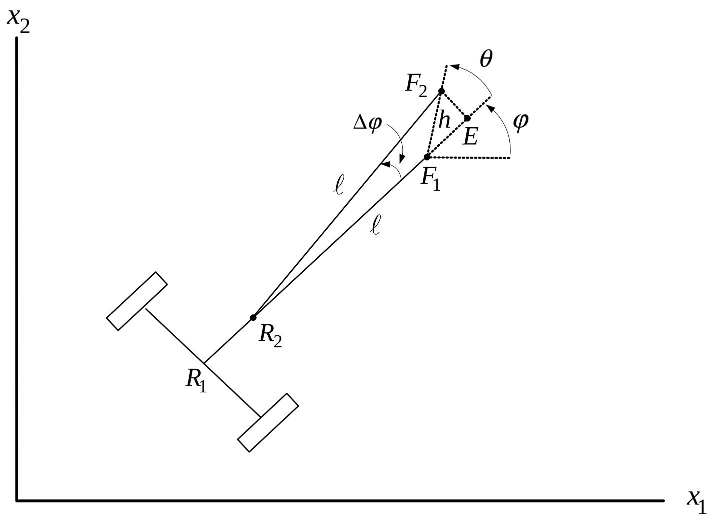

FIGURE 1.2. Computing drive.

We then have the model

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \theta \\ \varphi \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta+\varphi) \\ \sin(\theta+\varphi) \\ 0 \\ \sin(\theta)/\ell \end{bmatrix}}_{g_D} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{g_{Steer}} u_2$$

where u_1 is the first input which is the velocity of the vehicle in the direction g_D obtained by applying power to the rear wheels and u_2 is the second input which is the angular rate of the steering wheel.
(a) Let g_D, g_{Steer} denote the drive and steer vector fields, respectively. Define the *wriggle* vector field g_W to be

$$g_W \triangleq [g_{Steer}, g_D].$$

Compute g_W. Show that with $\theta = 0$ we have

$$g_W = g_{Slide} + g_R.$$

Remark Note that with $\theta = 0$ the vector field g_{Slide} is orthogonal to g_D. So trying to get your parked car out of a tight spot between two cars requires motion orthogonal to drive and a rotation which is given by wriggle. So to get wriggle one must: steer, drive, reverse steer, reverse drive and repeat.
(b) Let $z = \begin{bmatrix} x_1 & x_2 & \theta & \varphi \end{bmatrix}^T$.

$$\begin{aligned} g_W \triangleq [g_D, g_{Steer}] &= \frac{\partial g_D}{\partial z} g_{Steer} - \frac{\partial g_{Steer}}{\partial z} g_D \\ &= \begin{bmatrix} 0 & 0 & -\sin(\theta+\varphi) & -\sin(\theta+\varphi) \\ 0 & 0 & \cos(\theta+\varphi) & \cos(\theta+\varphi) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta)/\ell & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin(\theta+\varphi) \\ \cos(\theta+\varphi) \\ 0 \\ \cos(\theta)/\ell \end{bmatrix}. \end{aligned}$$

Then

$$g_W|_{\theta=0} \triangleq \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \\ 1/\ell \end{bmatrix} = g_{Slide} + g_R.$$

(c) Show

$$\begin{aligned} [g_{Steer}, g_W] &= -g_D \\ [g_W, g_D] &= g_{Slide}. \end{aligned}$$

$$\begin{aligned} [g_{Steer}, g_W] = \frac{\partial g_W}{\partial z} g_{Steer} - \frac{\partial g_{Steer}}{\partial z} g_W &= \begin{bmatrix} 0 & 0 & -\cos(\theta+\varphi) & -\cos(\theta+\varphi) \\ 0 & 0 & -\sin(\theta+\varphi) & -\sin(\theta+\varphi) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta)/\ell & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\cos(\theta+\varphi) \\ -\sin(\theta+\varphi) \\ 0 \\ -\sin\theta/\ell \end{bmatrix} \\ &= -g_D. \end{aligned}$$

$$\begin{aligned} [g_W, g_D] = & \begin{bmatrix} 0 & 0 & -\sin(\theta+\varphi) & -\sin(\theta+\varphi) \\ 0 & 0 & \cos(\theta+\varphi) & \cos(\theta+\varphi) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta)/\ell & 0 \end{bmatrix} \begin{bmatrix} -\sin(\theta+\varphi) \\ \cos(\theta+\varphi) \\ 0 \\ \cos(\theta)/\ell \end{bmatrix} - \\ & \begin{bmatrix} 0 & 0 & -\cos(\theta+\varphi) & -\cos(\theta+\varphi) \\ 0 & 0 & -\sin(\theta+\varphi) & -\sin(\theta+\varphi) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta)/\ell & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta+\varphi) \\ \sin(\theta+\varphi) \\ 0 \\ \sin\theta/\ell \end{bmatrix} \\ = & \frac{1}{\ell}\left(\begin{bmatrix} -\sin(\theta+\varphi)\cos\theta \\ \cos(\theta+\varphi)\cos\theta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos(\theta+\varphi)\sin\theta \\ \sin(\theta+\varphi)\sin\theta \\ 0 \\ 0 \end{bmatrix} \right) \\ = & \frac{1}{\ell} \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \\ 0 \end{bmatrix} \\ = & g_{Slide}. \end{aligned}$$