HW 2 Solutions

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Problem 8 *Magnetic Levitation*
Recall the equations of the current command magnetic levitation system given by

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= g - \frac{C}{m}\frac{u^2}{x_1^2}.\end{aligned}$$

Note that this nonlinear system is not of the form $dx/dt = f(x) + g(x)u$ as it is not linear in the input $u = i$ (current in the coil)

(a) Show that this system can be made linear by the appropriate choice of u. Design a state feedback controller to keep the steel ball at $x_1 = x_0$.

With $w \leq g$ set

$$w = g - \frac{C}{m}\frac{u^2}{x_1^2} \quad \text{or} \quad u = x_1\sqrt{\frac{m}{C}(g - w)}$$

so that the equations of this magnetic levitation system from the (new) input v to the state $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ are then

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w.$$

Figure 1.1 is a block diagram representation of the feedback linearizing controller.

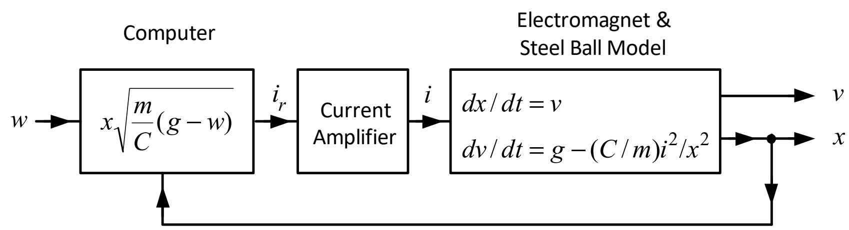

FIGURE 1.1. Feedback Linearization of the current command magnetic levitation system.

Figure 1.1 is then (at least mathematically) equivalent to the linear system (double integrator) shown in Figure 1.2.

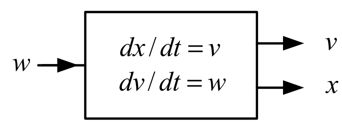

FIGURE 1.2. Equivalent model of Figure 1.1.

With

$$w = -k_1(x_1 - x_0) - k_2(x_2 - 0)$$

we have

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} k_1 x_0.$$

Let $r_1 > 0$ and $r_2 > 0$, set $k_2 = r_1 + r_2, k_1 = r_1 r_2$ to obtain

$$X_1(s) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & -1 \\ k_1 & s + k_2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{k_1 x_0}{s} = \frac{1}{(s + r_1)(s + r_2)}\frac{k_1 x_0}{s}.$$

Then $sX_1(s)$ is stable so by the final value theorem

$$\lim_{t\to\infty} x_1(t) = \lim_{s\to 0} sX_1(s) = x_0.$$

(b) Given that the position $x = x_1$ and coil current $u = i$ are measured, design an observer to estimate the velocity $v = x_2$.

Define the observer as

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g - \dfrac{C}{m}\dfrac{u^2}{x_1^2} \end{bmatrix} + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}(y - \hat{y}) \\ \hat{y} &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \end{aligned}$$

The estimation error satisfies

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \end{bmatrix} \\ &= \begin{bmatrix} -\ell_1 & 1 \\ -\ell_2 & 0 \end{bmatrix}\begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \end{bmatrix} \end{aligned}$$

with

$$\det\begin{bmatrix} s+\ell_1 & -1 \\ \ell_2 & s \end{bmatrix} = s^2+\ell_1 s+\ell_2.$$

The gain ℓ_1, ℓ_2 can chosen to force the estimation error to zero arbitrarily fast.

Problem 9 *DC motor State Estimation including Load Torque*
Recall the model of the DC motor given as

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega + V_S \\ J\frac{d\omega}{dt} &= K_T i - f\omega - \tau_L \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

With $x_1 = i, x_2 = \omega, x_3 = \theta$, and $u = V_S$ we may write

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 \\ K_T/J & -f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{b} u + \begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix}\tau_L \\ y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

where the rotor angle is taken as the output, i.e., it is measured. The load torque affects the speed and so it must be included in the observer. With the load torque taken to be constant and setting $x_4 = \tau_L/J$ the system is now modeled by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 & 0 \\ K_T/J & -f/J & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_a}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_a} u \\ y &= \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}}_{c_a}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \end{aligned}$$

Design an observer that estimates $x_1 = i, x_2 = \omega$, and $x_4 = \tau/J$ based on the measurement $y = \theta$. The observability matrix is

$$\mathcal{O} \triangleq \begin{bmatrix} c \\ cA \\ cA^2 \\ cA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ K_T/J & -f/J & 0 & -1 \\ -\dfrac{fK_T}{J^2} - \dfrac{RK_T}{JL} & \left(\dfrac{f}{J}\right)^2 - \dfrac{K_T K_b}{JL} & 0 & f/J \end{bmatrix}$$

with $\det \mathcal{O} = \dfrac{RK_T}{JL}$ so it is nonsingular. Let an observer be given by

$$\frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 & 0 \\ K_T/J & -f/J & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_a}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_a} u + \underbrace{\begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \end{bmatrix}}_{\ell_a}(y - \hat{y})$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}}_{c_a}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix}.$$

The error system is given by

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \\ x_3 - \hat{x}_3 \\ x_4 - \hat{x}_4 \end{bmatrix} &= \left(\begin{bmatrix} -R/L & -K_b/L & 0 & 0 \\ K_T/J & -f/J & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \\ x_3 - \hat{x}_3 \\ x_4 - \hat{x}_4 \end{bmatrix} \\ &= \begin{bmatrix} -R/L & -K_b/L & -\ell_1 & 0 \\ K_T/J & -f/J & -\ell_2 & -1 \\ 0 & 1 & -\ell_3 & 0 \\ 0 & 0 & -\ell_4 & 0 \end{bmatrix}\begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \\ x_3 - \hat{x}_3 \\ x_4 - \hat{x}_4 \end{bmatrix}\end{aligned}$$

The characteristic polynomial of $A_a - \ell_a c_a$ is then

$$\begin{aligned}\det\begin{bmatrix} s + R/L & K_b/L & \ell_1 & 0 \\ -K_T/J & s + f/J & \ell_2 & 1 \\ 0 & -1 & s + \ell_3 & 0 \\ 0 & 0 & \ell_4 & s \end{bmatrix} &= s^4 + \left(\frac{R}{L} + \frac{f}{J} + \ell_3\right)s^3 + \left(\frac{K_T K_b + Rf}{JL} + \ell_2 + \left(\frac{R}{L} + \frac{f}{J}\right)\ell_3\right)s^2 + \\ &\quad \left(\frac{R}{L}\ell_2 - \ell_4 + \frac{K_T}{J}\ell_1 + \ell_3\frac{K_T K_b + Rf}{JL}\right)s - \frac{R}{L}\ell_4.\end{aligned}$$

By inspection it is seen that the gains ℓ_1, ℓ_2, ℓ_3, and ℓ_4 can be chosen to assign the poles of $A_a - \ell_a c_a$ to desired set of values.

Problem 10 *Shunt Connected DC Motor*

A shunt connected DC motor has the field circuit and the armature circuit connected in parallel as illustrated in Figure 1.3. By connected in parallel is meant that the T_1 terminal of the armature is connected to the T_1' terminal of the field circuit and similarly for the T_2 and T_2'.

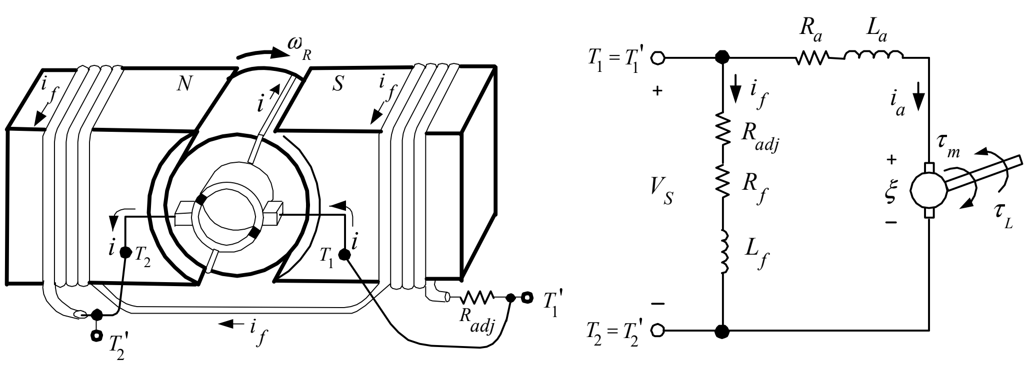

FIGURE 1.3. Shunt connected DC motor.

The equations describing the shunt motor are

$$\begin{aligned}
J\frac{d\omega}{dt} &= K_T L_f i_f i_a - \tau_L \\
L_a\frac{di_a}{dt} &= -R_a i_a - K_b L_f i_f \omega + V_S \\
L_f\frac{di_f}{dt} &= -(R_{adj} + R_f) i_f + V_S.
\end{aligned}$$

Here ω is the rotor angular speed, V_S is the terminal (source) voltage, i_a is the armature current, i_f is the field current, τ_L is the load torque, K_T is the torque constant, and K_b is the back-emf constant. The armature resistance and armature inductance are denoted by R_a and L_a,respectively, and the field resistance and field inductance are R_f and L_f, respectively. R_{adj} is an adjustable resistor so that the total field resistance $R_{adj} + R_f$ can be varied.

Let $x_1 = \omega, x_2 = i_a, x_3 = i_f, u = V_S$, and define the constants $c_1 \triangleq \dfrac{K_T L_f}{J}, c_2 = \dfrac{R_a}{L_a}, c_3 = \dfrac{K_b L_f}{L_a}, c_4 = \dfrac{1}{L_a}, c_5 = \dfrac{R_{adj} + R_f}{L_f}, c_6 = \dfrac{1}{L_f}$ so that the statespace model becomes

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 x_2 x_3 \\ -c_2 x_2 - c_3 x_1 x_3 \\ -c_f x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ c_4 \\ c_6 \end{bmatrix} u + \begin{bmatrix} -1/J \\ 0 \\ 0 \end{bmatrix} \tau_L.$$

The constant load torque is not known. Assuming that $x_1 = \omega, x_2 = i_a$, and $x_3 = i_f$ are measured this problem shows how to design an observer to estimate τ_L/J.

(a) Let $x_4 \triangleq \tau_L/J$ with $dx_4/dt = 0$ and suppose $x_1 = \omega, x_2 = i_a$, and $x_3 = i_f$ are all measured. Show the model of the shunt connected DC motor is

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} c_1 y_2 y_3 \\ -c_2 y_2 - c_3 y_1 y_3 \\ -c_f y_3 \\ 0 \end{bmatrix}}_{\varphi(y)} + \underbrace{\begin{bmatrix} 0 \\ c_4 \\ c_6 \\ 0 \end{bmatrix}}_{b} u \\
y &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
\end{aligned}$$

(b) Is the pair (C, A) observable?

Check the observability of the (C, A). We have

$$\begin{aligned}
C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
CA &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

showing that

$$rank\left(\begin{bmatrix} C \\ CA \end{bmatrix}\right) = 4.$$

(c) Let

$$T_o \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, T_o^{-1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and define the linear transformation $x^* = T_o x$. Transform the model given in part (a) into the x^* coordinate system.

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} + \\ &\quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 y_2 y_3 \\ -c_2 y_2 - c_3 y_1 y_3 \\ -c_f y_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ c_4 \\ c_6 \\ 0 \end{bmatrix} u \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} + \begin{bmatrix} -c_2 y_2 - c_3 y_1 y_3 \\ -y_3 c_f \\ 0 \\ -c_1 y_2 y_3 \end{bmatrix} + \begin{bmatrix} c_4 \\ c_6 \\ 0 \\ 0 \end{bmatrix} u. \end{aligned}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix}$$

(d) With $A_o \triangleq T_o A T_o^{-1}$ and $C_o = C T_o^{-1}$ let

$$L_o = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \\ \ell_{41} & \ell_{42} & \ell_{43} \end{bmatrix}$$

give the equations for an observer to estimate the state x^*.

$$\frac{d\hat{x}^*}{dt} = A_o \hat{x}^* + \begin{bmatrix} -c_2 y_2 - c_3 y_1 y_3 \\ -y_3 c_f \\ 0 \\ -c_1 y_2 y_3 \end{bmatrix} + \begin{bmatrix} c_4 \\ c_6 \\ 0 \\ 0 \end{bmatrix} u + L_o(x^* - \hat{x}^*)$$

(e) Give the equations for the estimate error $x^* - \hat{x}^*$ and show that the components of L_o can be chosen so that poles of the estimation error system can be put at $-r_1, -r_2, -r_3, -r_4$.

The equations describing the state estimation error are

$$\frac{d}{dt}(x^* - \hat{x}^*) = A_o(x^* - \hat{x}^*) + L_o(x^* - \hat{x}^*) = (A_o - L_o C_o)(x^* - \hat{x}^*).$$

The state estimation error matrix is

$$A_o - L_o C_o = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \\ \ell_{41} & \ell_{42} & \ell_{43} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\ell_{12} & -\ell_{13} & 0 & \ell_{11} \\ -\ell_{22} & -\ell_{23} & 0 & \ell_{21} \\ -\ell_{32} & -\ell_{33} & 0 & \ell_{31} \\ -\ell_{42} & -\ell_{43} & 1 & \ell_{41} \end{bmatrix}.$$

Set $\ell_{13} = 0, \ell_{11} = 0, \ell_{22} = 0, \ell_{21} = 0, \ell_{32} = 0, \ell_{33} = 0, \ell_{42} = 0$, and $\ell_{43} = 0$ so that

$$\det(A_o - L_o C_o) = \det \begin{bmatrix} s+\ell_{12} & 0 & 0 & 0 \\ 0 & s+\ell_{23} & 0 & 0 \\ 0 & 0 & s & -\ell_{31} \\ 0 & 0 & -1 & s-\ell_{41} \end{bmatrix} = (s+\ell_{12})\,(s+\ell_{23})\left(s^2 - s\ell_{41} - \ell_{31}\right).$$

With $\ell_{12} = r_1, \ell_{23} = r_2, \ell_{41} = -(r_3 + r_4)$, and $\ell_{31} = -r_1 r_2$ we have

$$\det(A_o - L_o C_o) = (s + r_1)(s + r_2)(s + r_3)(s + r_4).$$

2

Chapter 2

2.1 Problems Chapter 2

Problem 11 *Inverse Function Theorem*
Define a nonlinear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned} y_1 &= e^{x_1}\cos(x_2) \\ y_2 &= e^{x_1}\sin(x_2). \end{aligned}$$

(a) Compute the Jacobian matrix and its determinant.

$$\det\begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{bmatrix} = \det\begin{bmatrix} e^{x_1}\cos(x_2) & -e^{x_1}\sin(x_2) \\ e^{x_1}\sin(x_2) & e^{x_1}\cos(x_2) \end{bmatrix} = e^{2x_1}.$$

(b) Is this transformation one-to-one?

No, points of the form $(x_1, x_2 \pm 2\pi)$ all map to the same point (y_1, y_2).

(c) If the domain is restricted to $\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 < x_2 < 2\pi\}$ show that transformation is one-to-one and find its inverse.

If

$$\begin{aligned} e^{x_1}\cos(x_2) &= e^{x_1'}\cos(x_2') \\ e^{x_1}\sin(x_2) &= e^{x_1'}\sin(x_2') \end{aligned}$$

then

$$e^{2x_1} = \Big((e^{x_1}\cos(x_2)\Big)^2 + \Big(e^{x_1}\sin(x_2)\Big)^2 = \Big((e^{x_1'}\cos(x_2')\Big)^2 + \Big(e^{x_1'}\sin(x_2')\Big)^2 = e^{2x_1'}$$

or

$$x_1 = x_1'$$

as the function e^{2x} is a one-to-one function. It then follows that

$$\begin{aligned} \cos(x_2) &= \cos(x_2') \\ \sin(x_2) &= \sin(x_2'). \end{aligned}$$

As $0 < x_2, x_2' < 2\pi$ the equality $\cos(x_2) = \cos(x_2')$ implies $x_2' = x_2$ or $x_2' = 2\pi - x_2$. Again with $0 < x_2, x_2' < 2\pi$ the equality $\sin(x_2) = \sin(x_2')$ implies $x_2' = x_2$ or $x_2' = \pi - x_2$. Having to satisfy both conditions requires $x_2' = x_2$.The range is $\mathbb{R}^2 - \{(y_1, y_2)|\ y_2 = 0, y_1 \geq 0\}$ and the inverse function is

$$\begin{aligned} x_1 &= \ln\left(\sqrt{y_1^2 + y_2^2}\right) \\ x_2 &= \tan^{-1}(y_1, y_2). \end{aligned}$$

Problem 12 *Inverse Function Theorem*
Define a nonlinear transformation from $\mathcal{D} \triangleq \mathbb{R}^2 - \{(0, x_2) : x_2 \in \mathbb{R}\} \to \mathbb{R}^2$ by

$$\begin{aligned} y_1 &= x_1^2 \\ y_2 &= x_2/x_1 \end{aligned}$$

(a) Compute the Jacobian matrix and its determinant.

$$\det\begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{bmatrix} = \det\begin{bmatrix} 2x_1 & 0 \\ -x_2/x_1^2 & 1/x_1 \end{bmatrix} = 2 \text{ for } x_1 \neq 0.$$

(b) With $x_1 > 0$ show that the transformation is one-to-one. For this region find the corresponding range (image) of this transformation.

The equation $x_1^2 = x_1'^2$ implies $x_1 = x_1'$ as $x_1 > 0, x_1' > 0$. Then $x_2/x_1 = x_2'/x_1'$ implies $x_2 = x_2'$ as $x_1 = x_1'$.

The range is $\mathcal{R} \triangleq \{(y_1, y_2) \in \mathbb{R}^2 | \; y_1 > 0\}$ and the inverse function is

$$\begin{aligned} x_1 &= \sqrt{y_1} \\ x_2 &= \sqrt{y_1}y_2. \end{aligned}$$

The transformation in part (a) is one-to-one and onto between domain $\mathcal{D}$ and the range $\mathcal{R}$.

Problem 13 *Field Controlled DC Motor* [?]
The equations describing a separately excited DC motor are

$$\begin{aligned} J\frac{d\omega}{dt} &= K_T L_f i_f i_a - \tau_L \\ L\frac{di_a}{dt} &= -Ri_a - K_b L_f i_f \omega + V_{a0} \\ L_f\frac{di_f}{dt} &= -R_f i_f + V_f. \end{aligned}$$

Here ω is the rotor angular speed, V_{a0} is the (constant) armature voltage, i_a is the armature current, V_f is the field voltage, i_f is the field current, τ_L is the load torque, K_T is the torque constant, and K_b is the back-emf constant. The armature resistance and armature inductance are denoted by R and L,respectively, and the field resistance and field inductance are R_f and L_f, respectively.

Historically field controlled DC motors were used in mills for rolling out sheets of steel. This application required armature currents of $1000 - 2000$ Amperes to obtain the torques need to roll out the steel. However, there were not *variable* voltage sources available that could handle that amount of current. So a *constant* voltage source for the armature was used which could supply the large current. The field current i_f was on the order of only 25 Amperes and there were voltage sources that could provide a *varying* voltage while supplying that amount of current. By varying the field voltage V_f the current i_f and speed ω could then be controlled.

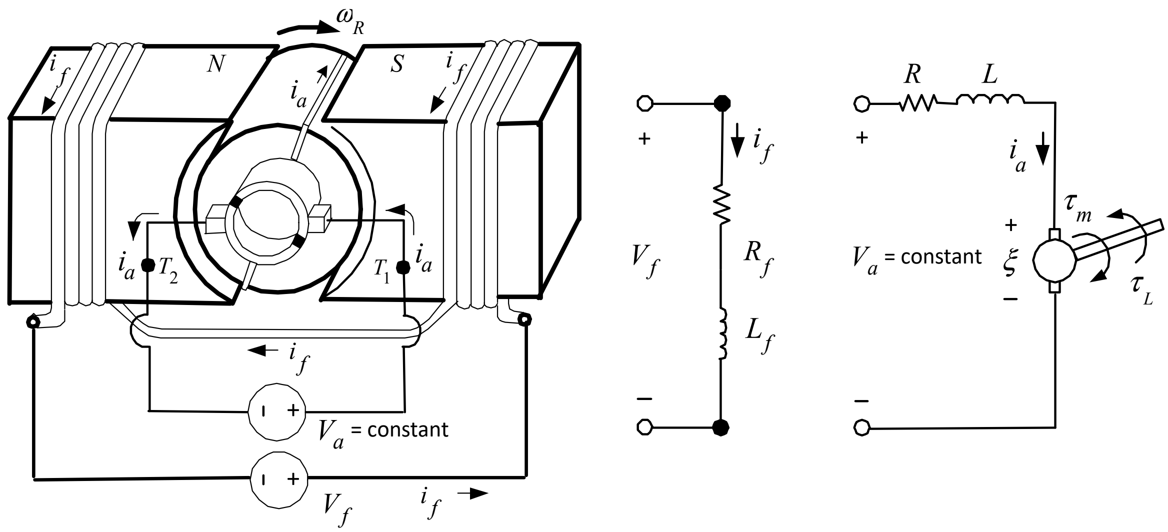

FIGURE 2.1. Field controlled DC motor. $\xi = K_b L_f i_f$ and $\tau_m = K_T L_f i_f i_a$.

Let $x_1 = i_f, x_2 = i_a, x_3 = \omega, u = V_f/L_f$, and define the constants $c_0 = V_{a0}/L, c_1 = R_f/L_f, c_2 = R/L, c_3 = K_b L_f/L, c_4 \triangleq K_T L_f/J, c_5 = 1/L$. The equations describing the field controlled DC motor are then

$$\begin{aligned}\frac{dx_1}{dt} &= -c_1x_1 + u\\ \frac{dx_2}{dt} &= -c_2x_2 - c_3x_1x_3 + c_0\\ \frac{dx_3}{dt} &= c_4x_1x_2 - \tau_L/J\end{aligned} \tag{2.1}$$

or

$$\frac{d}{dt}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} = \underbrace{\begin{bmatrix} -c_1x_1\\ -c_2x_2 - c_3x_1x_3 + c_0\\ c_4x_1x_2\end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{g(x)}u + \underbrace{\begin{bmatrix}0\\0\\-1/J\end{bmatrix}}_{p}\tau_L.$$

(a) Define $T_1(x) = \dfrac{c_4}{c_3}x_2^2 + x_3^2 = (Li_a^2 + J\omega^2)/J$. With

$$\begin{aligned}x_1^* &= T_1(x)\\ x_2^* &= \mathcal{L}_f(T_1)\\ x_3^* &= \mathcal{L}_f^2(T_1)\end{aligned}$$

compute dx^*/dt.

$$
\begin{aligned}
\frac{dx_1^*}{dt} &= \mathcal{L}_f T_1 + \mathcal{L}_g T_1 + \mathcal{L}_p T_1 \\
&= \begin{bmatrix} 0 & \frac{2c_4}{c_3} x_2 & 2x_3 \end{bmatrix} \left(\begin{bmatrix} -c_1 x_1 \\ -c_2 x_2 - c_3 x_1 x_3 + c_0 \\ c_4 x_1 x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ -1/J \end{bmatrix} \tau_L \right) \\
&= 2c_4 x_1 x_2 x_3 - \frac{2}{c_3} c_4 x_2 \left(c_2 x_2 - c_0 + c_3 x_1 x_3 \right) - (2x_3/J)\tau_L \\
&= \frac{2}{c_3} c_4 x_2 \left(c_0 - c_2 x_2 \right) - (2x_3/J)\tau_L \\
&= T_2(x) - (2x_3/J)\tau_L
\end{aligned}
$$

$$
\begin{aligned}
\frac{dx_2^*}{dt} &= \mathcal{L}_f T_2 + \mathcal{L}_g T_2 + \mathcal{L}_p T_2 \\
&= \begin{bmatrix} 0 & \frac{2c_0 c_4}{c_3} - \frac{4c_4 c_2}{c_3} x_2 & 0 \end{bmatrix} \left(\begin{bmatrix} -c_1 x_1 \\ -c_2 x_2 - c_3 x_1 x_3 + c_0 \\ c_4 x_1 x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ -1/J \end{bmatrix} \tau_L \right) \\
&= -\left(2\frac{c_0}{c_3} c_4 - 4\frac{c_2}{c_3} c_4 x_2 \right) \left(c_2 x_2 - c_0 + c_3 x_1 x_3 \right) \\
&= 2\frac{c_0^2}{c_3} c_4 - 2c_0 c_4 x_1 x_3 + 4\frac{c_2^2}{c_3} c_4 x_2^2 + 4c_2 c_4 x_1 x_2 x_3 - 6c_0 \frac{c_2}{c_3} c_4 x_2 \\
&= T_3(x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dx_3^*}{dt} &= \mathcal{L}_f T_3 + \mathcal{L}_g T_3 + \mathcal{L}_p T_3 \\
&= \begin{bmatrix} -2c_0 c_4 x_3 + 4c_4 c_2 x_2 x_3 & 8\frac{c_2^2}{c_3} c_4 x_2 + 4c_2 c_4 x_1 x_3 - 6c_0 \frac{c_2 c_4}{c_3} & -2c_0 c_4 x_1 + 4c_2 c_4 x_1 x_2 \end{bmatrix} \times \\
&\quad \left(\begin{bmatrix} -c_1 x_1 \\ -c_2 x_2 - c_3 x_1 x_3 + c_0 \\ c_4 x_1 x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ -1/J \end{bmatrix} \tau_L \right) \\
&= \mathcal{L}_f T_3 + (-2c_0 c_4 x_3 + 4c_2 c_4 x_2 x_3) u - (-2c_0 c_4 x_1 + 4c_2 c_4 x_1 x_2)(1/J)\tau_L
\end{aligned}
$$

(b) Use feedback linearization so that in the x^* coordinates the system is linear. What conditions on the state variables x_1, x_2, x_3 are needed to use this feedback?

With

$$u = \frac{w - \mathcal{L}_f T_3}{-2c_0 c_4 x_3 + 4c_2 c_4 x_2 x_3}$$

the system becomes

$$
\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^* - (2x_3/J)\tau_L \\
\frac{dx_2^*}{dt} &= x_3^* \\
\frac{dx_3^*}{dt} &= w - (-2c_0 c_4 x_1 + 4c_2 c_4 x_1 x_2)(1/J)\tau_L.
\end{aligned}
$$

This requires

$$-2c_0 c_4 x_3 + 4c_2 x_2 x_3 = x_3 c_4(-2c_0 + 4c_2 x_2) \neq 0$$

or

$$x_3 = i_f \neq 0 \text{ and } x_2 = i_a < \frac{2c_0}{4c_2} = \frac{V_{a0}}{2R}.$$