Chapter 5 Transformation of Nonlinear Systems For Control and State Estimation

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Transformation of Nonlinear Systems For Control and State Estimation

1.1 Single Input Nonlinear Control Systems

Consider the single-input nonlinear control system given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \\ g_4(x) \end{bmatrix}}_{g(x)} u \in \mathbb{R}^4. \tag{1.1}$$

Under what conditions does there exist an invertible transformation $x^* = T^*(x) : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$\begin{aligned} x_1^* &= T_1^*(x_1, x_2, x_3, x_4) \\ x_2^* &= T_2^*(x_1, x_2, x_3, x_4) \\ x_3^* &= T_3^*(x_1, x_2, x_3, x_4) \\ x_4^* &= T_4^*(x_1, x_2, x_3, x_4) \end{aligned}$$

such that in the x^* coordinates this nonlinear control system has the form

$$\frac{dx_1^*}{dt} = x_2^* \tag{1.2}$$

$$\frac{dx_2^*}{dt} = x_3^* \tag{1.3}$$

$$\frac{dx_3^*}{dt} = x_4^* \tag{1.4}$$

$$\frac{dx_4^*}{dt} = f_4^*(x_1^*, x_2^*, x_3^*, x_n^*) + g_4^*(x_1^*, x_2^*, x_3^*, x_n^*)u. \tag{1.5}$$

To find the conditions we proceed as follows. By the chain rule we have

$$\frac{dx_1^*}{dt} = \mathcal{L}_{f+gu}(T_1^*) = \mathcal{L}_f(T_1^*) + u\mathcal{L}_g(T_1^*) \tag{1.6}$$

$$\frac{dx_2^*}{dt} = \mathcal{L}_{f+gu}(T_2^*) = \mathcal{L}_f(T_2^*) + u\mathcal{L}_g(T_2^*) \tag{1.7}$$

$$\frac{dx_3^*}{dt} = \mathcal{L}_{f+gu}(T_3^*) = \mathcal{L}_f(T_3^*) + u\mathcal{L}_g(T_3^*) \tag{1.8}$$

$$\frac{dx_4^*}{dt} = \mathcal{L}_{f+gu}(T_4^*) = \mathcal{L}_f(T_4^*) + u\mathcal{L}_g(T_4^*). \tag{1.9}$$

We want Equations (1.6)-(1.9) to have the form of Equations (1.2)-(1.5) which requires

$$\begin{aligned} \mathcal{L}_f(T_1^*) = T_2^* &\text{ and } \mathcal{L}_g(T_1^*) = 0 \\ \mathcal{L}_f(T_2^*) = T_3^* &\text{ and } \mathcal{L}_g(T_2^*) = 0 \\ \mathcal{L}_f(T_3^*) = T_4^* &\text{ and } \mathcal{L}_g(T_3^*) = 0 \end{aligned}$$

and

$$\mathcal{L}_g(T_4^*) \neq 0.$$

This is the same as finding T_1^* that satisfies

$$\begin{array}{rr} T_2^* \triangleq \mathcal{L}_f(T_1^*) & \mathcal{L}_g(T_1^*) = 0 \\ T_3^* \triangleq \mathcal{L}_f^2(T_1^*) & \mathcal{L}_g\mathcal{L}_f(T_1^*) = 0 \\ T_4^* \triangleq \mathcal{L}_f^3(T_1^*) & \mathcal{L}_g\mathcal{L}_f^2(T_1^*) = 0 \end{array} \tag{1.10}$$

and

$$\mathcal{L}_g\mathcal{L}_f^3(T_1^*) \neq 0. \tag{1.11}$$

These conditions show that "only" T_1^* needs to be found. However, these conditions involve T_1^* and it derivatives up to order 3. We next develop equivalent conditions which involve only the first order derivatives of T_1^*. In Exercise **??** of Chapter **??** (page **??**) you were asked to show that

$$\begin{aligned} \mathcal{L}_{ad_f^1 g} &= \mathcal{L}_f\mathcal{L}_g - \mathcal{L}_g\mathcal{L}_f \\ \mathcal{L}_{ad_f^2 g} &= \mathcal{L}_g\mathcal{L}_f^2 - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f + \mathcal{L}_f^2\mathcal{L}_g \\ \mathcal{L}_{ad_f^3 g} &= \mathcal{L}_g\mathcal{L}_f^3 - 3\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f^2 + 3\mathcal{L}_f^2\mathcal{L}_g\mathcal{L}_f - \mathcal{L}_f^3\mathcal{L}_g. \end{aligned}$$

Using the right most column of (1.10) we have

$$\begin{aligned} \mathcal{L}_{ad_f^1 g}(T_1^*) &= \mathcal{L}_f\mathcal{L}_g(T_1^*) - \mathcal{L}_g\mathcal{L}_f(T_1^*) = 0 \\ \mathcal{L}_{ad_f^2 g}(T_1^*) &= \mathcal{L}_g\mathcal{L}_f^2(T_1^*) - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f(T_1^*) + \mathcal{L}_f^2\mathcal{L}_g(T_1^*) = 0 \\ \mathcal{L}_{ad_f^3 g}(T_1^*) &= \mathcal{L}_g\mathcal{L}_f^3(T_1^*) - 3\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f^2(T_1^*) + 3\mathcal{L}_f^2\mathcal{L}_g\mathcal{L}_f(T_1^*) - \mathcal{L}_f^3\mathcal{L}_g(T_1^*) = \mathcal{L}_g\mathcal{L}_f^3(T_1^*). \end{aligned}$$

The conditions become

$$dT_1^* \begin{bmatrix} g & ad_f g & ad_f^2 g \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

and

$$\left\langle dT_1^*, ad_f^3 g \right\rangle \neq 0.$$

To find the transformation we assume that the set of vectors $\{g, ad_f g, ad_f^2 g\}$ is involutive and the matrix

$$\mathcal{C} \triangleq \begin{bmatrix} g & ad_f g & ad_f^2 g & ad_f^3 g \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

is full rank.

To find T_1^* we first define the coordinate transformation $S(t)$: $\mathbb{R}^4 \to \mathbf{E}^4$ by

$$S(t_1, t_2, t_3, t_4) \triangleq \phi_{t_4}(\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))) \tag{1.12}$$

where $\varphi_{t_1}(x_0)$ is the solution to $dx/dt_1 = ad_f^3 g(x)$ with $x(0) = x_0$, $\varphi_{t_2}(x_0)$ is the solution to $dx/dt_2 = ad_f^2 g(x)$ with $x(0) = x_0'$, $\varphi_{t_3}(x_0)$ is the solution to $dx/dt_3 = ad_f g$ with $x(0) = x_0''$ and, finally, $\varphi_{t_4}(x_0)$ is the solution to $dx/dt_4 = g$ with $x(0) = x_0'''$. More explicitly

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = S(t_1, t_2, t_3, t_4) = \begin{bmatrix} s_1(t_1, t_2, t_3, t_4) \\ s_2(t_1, t_2, t_3, t_4) \\ s_3(t_1, t_2, t_3, t_4) \\ s_4(t_1, t_2, t_3, t_4) \end{bmatrix} = \phi_{t_4}(\phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))). \tag{1.13}$$

As the set $\{g, ad_f g, ad_f^2 g\}$ is involutive it follows for all (t_1, t_2, t_3, t_4) in a neighborhood of $(0,0,0,0)$ that

$$\frac{\partial S}{\partial t_4}, \frac{\partial S}{\partial t_3}, \frac{\partial S}{\partial t_2} \in \Delta_{x=S(t_1,t_2,t_3,t_4)} \triangleq \left\{ r_1 g(x) + r_2 ad_f g(x) + r_3 ad_f^2 g \mid x = S(t_1,t_2,t_3,t_4), \text{ and } r_1, r_2, r_3 \in \mathbb{R} \right\}.$$

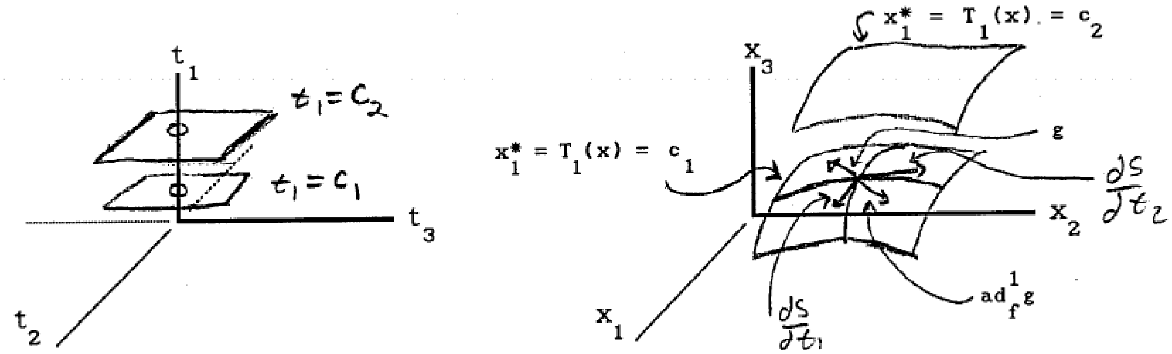

FIGURE 1.1. Illustrated for $n = 3$. $\{g, ad_f g\}$ and $\{\partial S/\partial t_3, \partial S/\partial t_2\}$ span the tangent space point of the surface $t_1 =$ constant.

That is, for any *fixed* t_1, varying t_2, t_3, t_4 sweeps out a three-dimensional surface with any tangent to this surface being a linear combination of $g(x)_{|x=S(t_1,t_2,t_3,t_4)}$, $ad_f g(x)_{|x=S(t_1,t_2,t_3,t_4)}$, and $ad_f^2 g(x)_{|x=S(t_1,t_2,t_3,t_4)}$. Inverting the transformation (1.13) gives

$$\begin{aligned} t_1 &= T_1(x) \\ t_2 &= T_2(x) \\ t_3 &= T_3(x) \\ t_4 &= T_4(x). \end{aligned}$$

Consequently for t_1 fixed the tangent vectors to the three-dimensional surface

$$\left\{x \in \mathbb{R}^4 | \; T_1(x) = t_1\right\}$$

are linear combinations of $g, ad_f g$, and $ad_f^2 g$. The gradient

$$dT_1 = \left[\begin{array}{cccc} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} & \frac{\partial T_1}{\partial x_4} \end{array} \right]$$

is normal to this surface so

$$dT_1 \left[\begin{array}{ccc} g & ad_f g & ad_f^2 g \end{array} \right] = \left[\begin{array}{ccc} 0 & 0 & 0 \end{array} \right]. \tag{1.14}$$

We showed above that these conditions (1.14) are equivalent to the conditions (1.10), i.e., $\mathcal{L}_g(T_1) = 0, \mathcal{L}_g\mathcal{L}_f(T_1) = 0, \mathcal{L}_g\mathcal{L}_f^2(T_1) = 0$. The feedback linearizing coordinate transformation $x^* = T^*(x)$ is then

$$\begin{aligned} x_1^* &= T_1^*(x) = T_1(x) \\ x_2^* &= T_2^*(x) = \mathcal{L}_f(T_1) \\ x_2^* &= T_3^*(x) = \mathcal{L}_f^2(T_1) \\ x_2^* &= T_3^*(x) = \mathcal{L}_f^3(T_1). \end{aligned}$$

To show that $T^*(x)$ is invertible, we compute

$$\frac{\partial x^*}{\partial x}\mathcal{C} = \left[\begin{array}{cccc} g & ad_f g & ad_f^2 g & ad_f^3 g \end{array} \right]$$

$$\begin{aligned}
\frac{\partial T^*}{\partial x}\mathcal{C} &= \begin{bmatrix} dT_1^* \\ dT_2^* \\ dT_3^* \\ dT_4^* \end{bmatrix} \begin{bmatrix} g & ad_f g & ad_f^2 g & ad_f^3 g \end{bmatrix} \\
&= \begin{bmatrix} dT_1 \\ d\mathcal{L}_f(T_1) \\ d\mathcal{L}_f^2(T_1) \\ d\mathcal{L}_f^3(T_1) \end{bmatrix} \begin{bmatrix} g & ad_f g & ad_f^2 g & ad_f^3 g \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{L}_g(T_1) & \mathcal{L}_{ad_f g}(T_1) & \mathcal{L}_{ad_f^2 g}(T_1) & \mathcal{L}_{ad_f^3 g}(T_1) \\ \mathcal{L}_g\mathcal{L}_f(T_1) & \mathcal{L}_{ad_f g}(\mathcal{L}_f(T_1)) & \mathcal{L}_{ad_f^2 g}\mathcal{L}_f(T_1) & \mathcal{L}_{ad_f^3 g}\mathcal{L}_f(T_1) \\ \mathcal{L}_g\mathcal{L}_f^2(T_1) & \mathcal{L}_{ad_f g}(\mathcal{L}_f^2(T_1)) & \mathcal{L}_{ad_f^2 g}\mathcal{L}_f^2(T_1) & \mathcal{L}_{ad_f^3 g}\mathcal{L}_f^2(T_1) \\ \mathcal{L}_g\mathcal{L}_f^3(T_1) & \mathcal{L}_{ad_f g}\mathcal{L}_f^3(T_1) & \mathcal{L}_{ad_f^2 g}\mathcal{L}_f^3(T_1) & \mathcal{L}_{ad_f^3 g}\mathcal{L}_f^3(T_1) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & \mathcal{L}_{ad_f^3 g}(T_1) \\ 0 & 0 & \mathcal{L}_{ad_f^2 g}\mathcal{L}_f(T_1) & \mathcal{L}_{ad_f^3 g}\mathcal{L}_f(T_1) \\ 0 & -\mathcal{L}_{ad_f g}(\mathcal{L}_f^2(T_1)) & \mathcal{L}_{ad_f^2 g}\mathcal{L}_f^2(T_1) & \mathcal{L}_{ad_f^3 g}\mathcal{L}_f^2(T_1) \\ \mathcal{L}_g\mathcal{L}_f^3(T_1) & \mathcal{L}_{ad_f g}\mathcal{L}_f^3(T_1) & \mathcal{L}_{ad_f^2 g}\mathcal{L}_f^3(T_1) & \mathcal{L}_{ad_f^3 g}\mathcal{L}_f^3(T_1) \end{bmatrix}.
\end{aligned}$$

Using $\mathcal{L}_{ad_f^1 g}(h) = \mathcal{L}_f\mathcal{L}_g(h) - \mathcal{L}_g\mathcal{L}_f(h)$ with $h = \mathcal{L}_f^2(T_1)$ gives

$$\mathcal{L}_{ad_f g}(\mathcal{L}_f^2(T_1)) = -\mathcal{L}_g\mathcal{L}_f^3(T_1).$$

Using $\mathcal{L}_{ad_f^2 g}(h) = \mathcal{L}_g\mathcal{L}_f^2(h) - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f(h) + \mathcal{L}_f^2\mathcal{L}_g(h)$ with $h = \mathcal{L}_f(T_1)$ gives

$$\mathcal{L}_{ad_f^2 g}\mathcal{L}_f(T_1) = \mathcal{L}_g\mathcal{L}_f^3(T_1).$$

Further, using $\mathcal{L}_{ad_f^3 g}(h) = \mathcal{L}_g\mathcal{L}_f^3(h) - 3\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f^2(h) + 3\mathcal{L}_f^2\mathcal{L}_g\mathcal{L}_f(h) - \mathcal{L}_f^3\mathcal{L}_g(h)$ with $h = T_1$ gives $\mathcal{L}_{ad_f^3 g}(T_1) = \mathcal{L}_g\mathcal{L}_f^3(T_1)$. These computations allow us rewrite $\dfrac{\partial T^*}{\partial x}\mathcal{C}$ as

$$\frac{\partial T^*}{\partial x}\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & \mathcal{L}_{ad_f^3 g}(T_1) \\ 0 & 0 & \mathcal{L}_{ad_f^3 g}(T_1) & \mathcal{L}_{ad_f^3 g}\mathcal{L}_f(T_1) \\ 0 & \mathcal{L}_{ad_f^3 g}(T_1) & \mathcal{L}_{ad_f^2 g}\mathcal{L}_f^2(T_1) & \mathcal{L}_{ad_f^3 g}\mathcal{L}_f^2(T_1) \\ \mathcal{L}_{ad_f^3 g}(T_1) & \mathcal{L}_{ad_f g}\mathcal{L}_f^3(T_1) & \mathcal{L}_{ad_f^2 g}\mathcal{L}_f^3(T_1) & \mathcal{L}_{ad_f^3 g}\mathcal{L}_f^3(T_1) \end{bmatrix}$$

which is invertible as $\mathcal{L}_{ad_f^3 g}(T_1) \neq 0$ in a neighborhood of x_0. As $\mathcal{C}$ is invertible in a neighborhood of x_0 it follows that $\dfrac{\partial T^*}{\partial x}$ invertible in a neighborhood of x_0 and therefore, by the inverse function theorem, $x^* = T^*(x)$ is invertible.

Remark

Just any row vector satisfying (1.14) will work. To explain, let $\omega_1(x), \omega_2(x), \omega_3(x), \omega_4(x)$ be four scalar functions that satisfy

$$\begin{bmatrix} \omega_1(x) & \omega_2(x) & \omega_3(x) & \omega_4(x) \end{bmatrix} \begin{bmatrix} g & ad_f g & ad_f^2 g \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \tag{1.15}$$

Is there a single scalar function $T_1(x)$ such that

$$dT_1 = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} & \frac{\partial T_1}{\partial x_4} \end{bmatrix} = \begin{bmatrix} \omega_1(x) & \omega_2(x) & \omega_3(x) & \omega_4(x) \end{bmatrix}? \tag{1.16}$$

If the quantities $\omega_1(x), \omega_2(x), \omega_3(x), \omega_4(x)$ are to be the components of the gradient dT_1 then they must also satisfy

$$\begin{aligned}
\frac{\partial^2 T_1}{\partial x_2 \partial x_1} &= \frac{\partial \omega_1}{\partial x_2} = \frac{\partial \omega_2}{\partial x_1} = \frac{\partial^2 T_1}{\partial x_1 \partial x_2} \\
\frac{\partial^2 T_1}{\partial x_3 \partial x_1} &= \frac{\partial \omega_1}{\partial x_3} = \frac{\partial \omega_3}{\partial x_1} = \frac{\partial^2 T_1}{\partial x_1 \partial x_3} \\
\frac{\partial^2 T_1}{\partial x_4 \partial x_1} &= \frac{\partial \omega_1}{\partial x_4} = \frac{\partial \omega_4}{\partial x_1} = \frac{\partial^2 T_1}{\partial x_1 \partial x_4} \\
\frac{\partial^2 T_1}{\partial x_3 \partial x_2} &= \frac{\partial \omega_2}{\partial x_3} = \frac{\partial \omega_3}{\partial x_2} = \frac{\partial^2 T_1}{\partial x_2 \partial x_3} \\
\frac{\partial^2 T_1}{\partial x_4 \partial x_2} &= \frac{\partial \omega_2}{\partial x_4} = \frac{\partial \omega_4}{\partial x_2} = \frac{\partial^2 T_1}{\partial x_2 \partial x_4} \\
\frac{\partial^2 T_1}{\partial x_4 \partial x_3} &= \frac{\partial \omega_3}{\partial x_4} = \frac{\partial \omega_4}{\partial x_3} = \frac{\partial^2 T_1}{\partial x_3 \partial x_4}.
\end{aligned}$$

The involutiveness of $\{g, ad_f g, ad_f^2 g\}$ is necessary and sufficient for a T_1 to exist satisfying these conditions.

Example 1 *Linear Control System*

Let $f(x) = Ax, g(x) = b$ with $A \in \mathbb{R}^{4\times 4}, b \in \mathbb{R}^4$ where simple computations show that $ad_f^k g = (-1)^k A^k b$ for $k = 0, 1, 2, 3$. Then the conditions for the existence of the $T_1(x)$ is

$$dT_1 \begin{bmatrix} b & -Ab & A^2 b & -A^3 b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}$$

where we have chosen $\beta(x) = -1$. This is, of course, equivalent to

$$dT_1 \underbrace{\begin{bmatrix} b & Ab & A^2 b & A^3 b \end{bmatrix}}_{\mathcal{C}} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

With the pair (A, b) controllable, let

$$dT_1 = q \triangleq \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}$$

which is the last row of $\mathcal{C}^{-1}$. We then have

$$\begin{aligned}
x_1^* &= T_1(x) = qx \\
x_2^* &= T_2(x) = qAx \\
x_3^* &= T_3(x) = qA^2 x \\
x_4^* &= T_4(x) = qA^3 x
\end{aligned}$$

or

$$x^* = \underbrace{\begin{bmatrix} q \\ qA \\ qA^2 \\ qA^3 \end{bmatrix}}_{\mathbf{T}} x \triangleq \mathbf{T}x.$$

In the x^* coordinate system we have

$$\begin{aligned}
\frac{dx_1^*}{dt} &= q(Ax+bu) = qAx = x_2^* \\
\frac{dx_2^*}{dt} &= qA(Ax+bu) = qA^2x = x_2^* \\
\frac{dx_3^*}{dt} &= qA^2(Ax+bu) = qA^3x + qA^2bu = x_4^* \\
\frac{dx_4^*}{dt} &= qA^3(Ax+bu) = qA^4x + qA^3bu = qA^4x + u.
\end{aligned}$$

With

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix} \triangleq -qA^4$$

this becomes

$$\frac{dx^*}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} x^* + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

Example 2 *Series Connected DC Motor*
In Chapter **??** we consider the series connected DC motor model given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ c_1x_3^2 \\ -c_2x_3 - c_3x_3x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}}_{p} \tau_L$$

where $x_1 = \theta, x_2 = \omega, x_3 = i$, and $u = V_S/L$.

$$ad_f g = [f,g] = \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g = -\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2c_3x_3 \\ 0 & -c_3x_3 & -c_2 - c_3x_2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2c_3x_3 \\ c_2 + c_3x_2 \end{bmatrix}$$

$$\begin{aligned}
ad_f^2 g = [f, ad_f g] &= \frac{\partial ad_f g}{\partial x}f - \frac{\partial f}{\partial x}ad_f g \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2c_3 \\ 0 & c_2 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ c_1x_3^2 \\ -c_2x_3 - c_3x_3x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2c_3x_3 \\ 0 & -c_3x_3 & -c_2 - c_3x_2 \end{bmatrix}\begin{bmatrix} 0 \\ -2c_3x_3 \\ c_2 + c_3x_2 \end{bmatrix} \\
&= \begin{bmatrix} 2c_3x_3 \\ 0 \\ (c_2 + c_3x_2)^2 - 2c_3^2x_3^2 + c_1c_2x_3^2 \end{bmatrix}
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{C} &\triangleq \begin{bmatrix} g & ad_f g & ad_f^2 g \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 2c_3x_3 \\ 0 & -2c_3x_3 & 0 \\ 1 & c_2 + c_3x_2 & (c_2 + c_3x_2)^2 - 2c_3^2x_3^2 + c_1c_2x_3^2 \end{bmatrix} \in \mathbb{R}^{4\times 4}
\end{aligned}$$

and

$$\det \mathcal{C} = 4c_3^2x_3^2.$$

We need to solve

$$dT_1 \begin{bmatrix} g & ad_f g & ad_f^2 g \end{bmatrix} = \begin{bmatrix} 0 & 0 & \beta(x) \end{bmatrix}.$$

Check the involutiveness of the two vectors $\{g, ad_f g\}$. Computing

$$[g, ad_f g] = \frac{\partial ad_f g}{\partial x} g - \frac{\partial g}{\partial x} ad_f g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2c_3 \\ 0 & c_3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2c_3 \\ 0 \end{bmatrix} = \frac{1}{x_3} ad_f g - \frac{c_2 + c_3 x_2}{x_3} g$$

shows that the pair $\{g, ad_f g\}$ is involutive for $x_3 = i \neq 0$.

To obtain T_1 we solve

$$\begin{aligned} dT_1 &= \begin{bmatrix} 0 & 0 & \beta(x) \end{bmatrix} \mathcal{C}^{-1} \\ &= \begin{bmatrix} 0 & 0 & \beta(x) \end{bmatrix} \frac{1}{4c_3^2 x_3^2} \begin{bmatrix} -2c_2^2 c_3 x_3 - 4c_2 c_3^2 x_2 x_3 - 2c_1 c_2 c_3 x_3^3 - 2c_3^3 x_2^2 x_3 + 4c_3^3 x_3^3 & 2x_2 x_3 c_3^2 + 2c_2 x_3 c_3 & 4c_3^2 x_3^2 \\ 0 & -2c_3 x_3 & 0 \\ 2c_3 x_3 & 0 & 0 \end{bmatrix} \\ &= \beta(x) \begin{bmatrix} \dfrac{1}{2c_3 x_3} & 0 & 0 \end{bmatrix}. \end{aligned}$$

Choosing $\beta(x) = 2c_3 x_3$ we have

$$dT_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} x_1^* &= T_1(x) = x_1 \\ x_2^* &= T_2(x) = \mathcal{L}_f(T_1) = \langle dT_1, f \rangle = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix} = x_2 \\ x_3^* &= T_3(x) = \mathcal{L}_f(T_2) = \langle dT_2, f \rangle = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix} = c_1 x_3^2. \end{aligned}$$

or

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ c_1 x_3^2 \end{bmatrix}.$$

will transform the equations of the series connected DC motor into a form where the nonlinearities can be canceled out by feedback. Explicitly we compute

$$\begin{aligned} \frac{dx_1^*}{dt} &= \mathcal{L}_{f+gu+p\tau_L}(T_1) = \mathcal{L}_f(T_1) = x_2 = x_2^* \\ \frac{dx_2^*}{dt} &= \mathcal{L}_{f+gu+p\tau_L}(T_2) = \mathcal{L}_f(T_2) + \tau_L \mathcal{L}_p(T_2) = -c_2 x_3 - c_3 x_3 x_2 - \tau_L/J = x_2^* - \tau_L/J \\ \frac{dx_3^*}{dt} &= \mathcal{L}_{f+gu+p\tau_L}(T_3) = \mathcal{L}_f(T_3) + u\mathcal{L}_g(T_3) + \tau_L \mathcal{L}_p(T_2) = -2c_1 c_2 x_3^2 - 2c_1 c_3 x_2 x_3^2 + 2c_1 x_3 u - \tau_L/J. \end{aligned}$$

More succinctly we have

$$\begin{aligned} \frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= x_3^* - \tau_L/J \\ \frac{dx_3^*}{dt} &= \underbrace{-2c_1 c_2 x_3^2 - 2c_1 c_3 x_2 x_3^2}_{a(x)} + \underbrace{2c_1 x_3}_{b(x)} u. \end{aligned}$$

Exercise 1 In the previous example suppose we chose $\beta(x) = 1$ to solve

$$dT_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathcal{C}^{-1} = \begin{bmatrix} \frac{1}{2c_3x_3} & 0 & 0 \end{bmatrix}.$$

Would this work? Hint: With $\begin{bmatrix} \omega_1(x) & \omega_2(x) & \omega_3(x) \end{bmatrix} \triangleq \begin{bmatrix} \frac{1}{2c_3x_3} & 0 & 0 \end{bmatrix}$, does $\dfrac{\partial \omega_1}{\partial x_3} = \dfrac{\partial \omega_3}{\partial x_1}$?

Example 3 *Series Connected DC Motor*

Now suppose we want to find $T_1(x)$ for the series connected DC motor by computing

$$S(t_1, t_2, t_3) = \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))).$$

$\varphi_{t_1}(x_0)$ is the solution to

$$\frac{dx}{dt_1} = ad_f^2 g = \begin{bmatrix} 2c_3x_3 \\ 0 \\ (c_2 + c_3x_2)^2 - 2c_3^2x_3^2 + c_1c_2x_3^2 \end{bmatrix} \text{ with } x(0) = x_0.$$

$\varphi_{t_2}(x_0)$ is the solution to

$$\frac{dx}{dt_2} = ad_f^1 g = \begin{bmatrix} 0 \\ -2c_3x_3 \\ c_2 + c_3x_2 \end{bmatrix} \text{ with } x(0) = x_0'.$$

$\varphi_{t_3}(x_0)$ is the solution to

$$\frac{dx}{dt_3} = g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ with } x(0) = x_0''.$$

After these computations we would then have to invert $x = S(t_1, t_2, t_3) = \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0)))$ to obtain $T_1(x)$. Good luck with all that! However there is a way to use this method. Simply choose two linearly independent vectors that span the same space as $\{g, ad_f^1 g\}$. It is straightforward to see that

$$span\{g, ad_f^1 g\} = span\left\{ f^{(3)} \triangleq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, f^{(2)} \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

for $x_3 \neq 0$. Further, instead of using $ad_f^2 g$ we just use a vector field which is normal to $\{g, ad_f^1 g\}$. We choose

$$f^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The solution to $dx/dt_1 = f^{(1)}$ with $x(0) = x_0$ is

$$\varphi_{t_1}(x_0) = \begin{bmatrix} x_{01} + t_1 \\ x_{02} \\ x_{03} \end{bmatrix}.$$

The solution to $dx/dt_2 = f^{(2)}$ with $x(0) = x_0'$

$$\varphi_{t_2}(x_0) = \begin{bmatrix} x_{01}' \\ x_{02}' + t_2 \\ x_{03}' \end{bmatrix}.$$

The solution to $dx/dt_3 = f^{(3)}$ with $x(0) = x_0''$ is

$$\varphi_{t_3}(x_0) = \begin{bmatrix} x_{01}'' \\ x_{02}'' \\ x_{03}'' + t_3 \end{bmatrix}.$$

Then

$$x = S(t_1, t_2, t_3) = \phi_{t_3}(\varphi_{t_2}(\varphi_{t_1}(x_0))) = \begin{bmatrix} x_{01} + t_1 \\ x_{02} + t_2 \\ x_{03} + t_3 \end{bmatrix}$$

with inverse

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = T(x_1, x_2, x_3) = \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \\ x_3 - x_{03} \end{bmatrix}.$$

We set

$$\begin{aligned} x_1^* &\triangleq T_1(x) = x_1 - x_{01} \\ x_2^* &\triangleq \mathcal{L}_f T_1(x) \\ x_3^* &\triangleq \mathcal{L}_f^2 T_1(x). \end{aligned}$$

1.2 Multi-Input Nonlinear Control Systems

We now look at finding feedback linearizing transformations for multi-input nonlinear control systems. To do this we first need to look at the structure of multi-input linear time-invariant systems. Let

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}. \tag{1.17}$$

The input matrix $B \in \mathbb{R}^{n\times m}$ is assumed to be of full rank m ($m \leq n$). The system is also assumed to be controllable, that is, the controllability matrix defined by

$$\mathcal{C} \triangleq \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times mn} \tag{1.18}$$

has rank n.

Definition 1 *r indices*
Let

$$r_0 = rank[B] = m \tag{1.19}$$

and, for $j = 1, 2, ..., n-1$, set

$$r_j = rank \begin{bmatrix} B & AB & \cdots & A^j B \end{bmatrix} - rank \begin{bmatrix} B & AB & \cdots & A^{j-1}B \end{bmatrix}. \tag{1.20}$$

Exercise 2 Show that $0 \leq r_j \leq m$ and $\sum_{j=0}^{n-1} r_j = n$.

Example 4 *r indices*
Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A^2B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^3B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so

$$\begin{aligned} r_0 &= rank[B] = 2 \\ r_1 &= rank\begin{bmatrix} B & AB \end{bmatrix} - rank[B] = 3 - 2 = 1 \\ r_2 &= rank\begin{bmatrix} B & AB & A^2B \end{bmatrix} - rank\begin{bmatrix} B & AB \end{bmatrix} = 4 - 3 = 1 \\ r_3 &= rank\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} - rank\begin{bmatrix} B & AB & A^2B \end{bmatrix} = 4 - 4 = 0 \end{aligned}$$

Example 5 *r indices*
Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^2B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^3B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so

$$\begin{aligned} r_0 &= rank[B] = 2 \\ r_1 &= rank\begin{bmatrix} B & AB \end{bmatrix} - rank[B] = 4 - 2 = 2 \\ r_2 &= rank\begin{bmatrix} B & AB & A^2B \end{bmatrix} - rank\begin{bmatrix} B & AB \end{bmatrix} = 4 - 4 = 0 \\ r_3 &= rank\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} - rank\begin{bmatrix} B & AB & A^2B \end{bmatrix} = 4 - 4 = 0 \end{aligned}$$

Definition 2 *Controllability Indices*
Let r_j for $j = 1, 2, ..., n-1$ be the *r indices* for the controllable linear time-invariant system

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}.$$

It is assumed $rank[B] = m$. The *controllability indices* $\kappa_1, \kappa_2, ..., \kappa_m$ for this system are defined according to

$$\begin{aligned} \kappa_1 &\triangleq (\text{Number of } r_j \text{ greater than or equal to 1}) \\ \kappa_2 &\triangleq (\text{Number of } r_j \text{ greater than or equal to 2}) \\ &\vdots \\ \kappa_m &\triangleq (\text{Number of } r_j \text{ greater than or equal to } m). \end{aligned}$$

As the r indices satisfy $r_j \le m$ for all j there can be no more than m controllability indices. As $\sum_{j=0}^{n-1} r_j = n$, we have

$$\sum_{i=1}^{n} \kappa_i = n.$$

Example 6 *Controllability Indices (Example 4 continued)*
In Example 4 the r indices were $r_0 = 2, r_1 = 1, r_2 = 1$, and $r_3 = 0$. Then

$$\kappa_1 \triangleq (\text{Number of } r_j \text{ greater than or equal to } 1) = 3$$
$$\kappa_2 \triangleq (\text{Number of } r_j \text{ greater than or equal to } 2) = 1.$$

Note that $\kappa_1 + \kappa_2 = 4$. As illustrated below, doing the sum $\sum_{j=0}^{3} r_j = 4$ can be seen as adding up the columns of the table first and then adding up the rows. On the other hand we can view $\sum_{i=1}^{2} \kappa_i = 4$ as adding up the rows of the table first and then the columns.

	r_0	r_1	r_2	r_3
κ_1	1	1	1	0
κ_2	1	0	0	0

Example 7 *Controllability Indices (Example 5 continued)*
In Example 5 the r indices were $r_0 = 2, r_1 = 2, r_2 = 0$, and $r_3 = 0$. Then

$$\kappa_1 \triangleq (\text{Number of } r_j \text{ greater than or equal to } 1) = 2$$
$$\kappa_2 \triangleq (\text{Number of } r_j \text{ greater than or equal to } 2) = 2$$

Note that $\kappa_1 + \kappa_2 = 4$. Again the sum $\sum_{j=0}^{3} r_j = 4$ can be seen as adding up the columns of the table first and then adding up the rows. On the other hand we can view $\sum_{i=1}^{2} \kappa_i = 4$ as adding up the rows of the table first and then the columns.

	r_0	r_1	r_2	r_3
κ_1	1	1	0	0
κ_2	1	1	0	0

We next explain that the controllability indices are *invariant* under statespace transformations, input transformations, and state feedback. To do so, once again consider the controllable linear time-invariant system

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$$

with $rank[B] = m$. Using the state feedback

$$u = -Kx + v, \quad K \in \mathbb{R}^{m\times n}$$

the system becomes

$$\frac{dx}{dt} = (A - BK)x + Bv.$$

Under the statespace transformation $x^* = Tx$ we then have

$$\frac{d}{dt}x^* = T(A - BK)T^{-1}x^* + TBv.$$

Finally, under a change of input variables $v = Uv^*$ the system is given by

$$\frac{d}{dt}x^* = T(A - BK)T^{-1}x^* + TBUv^*.$$

With this background we can now state the following theorem.

Theorem 1 *Brunovsky Canonical Form* [1][2]
Consider the controllable linear time-invariant system

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \tag{1.21}$$

with $rank[B] = m$. With $T \in \mathbb{R}^{n \times n}, U \in \mathbb{R}^{m \times m}$ nonsingular matrices and $K \in \mathbb{R}^{m \times n}$ the controllability indices of the pair

$$(A^*, B^*) \triangleq (T(A - BK)T^{-1}, TBU) \tag{1.22}$$

are the same as the controllability indices of the pair (A, B).

Further, there exist nonsingular transformations $T^* \in \mathbb{R}^{n \times n}$ and $U^* \in \mathbb{R}^{m \times m}$ and a feedback matrix $K^* \in \mathbb{R}^{m \times n}$ such that the system

$$\frac{dx^*}{dt} = A^* x^* + B^* v^* \tag{1.23}$$

with $x^* = Tx, v^* = U^{-1}v$ has the form

$$
A^* = \begin{array}{c}
\\ \\ \kappa_1\left\{ \\ \\ \\ \\ \kappa_2\left\{ \\ \\ \vdots \\ \\ \kappa_m\left\{ \\ \\
\end{array}
\left[\begin{array}{cccc}
0\,1\,0\,\cdots\,0 & 0\,0\,\cdots\,0 & & 0\,0\,\cdots\,0 \\
0\,0\,0\,1\,\cdots\,0 & \vdots & \cdots & \vdots \\
\vdots & & & \\
0\quad\quad 1 & & & \\
0\,0\,\cdots\,0 & 0\,0\,\cdots\,0 & & 0\,0\,\cdots\,0 \\
0\,0\,\cdots\,0 & 0\,1\,0\,\cdots\,0 & & 0\,0\,\cdots\,0 \\
 & 0\,0\,0\,1\,\cdots\,0 & & \\
\vdots & \vdots & & \vdots \\
 & 0\quad\quad 1 & & \\
0\,0\,\cdots\,0 & 0\,0\,\cdots\,0 & & 0\,0\,\cdots\,0 \\
\vdots & \vdots & \ddots & \vdots \\
0\,0\,\cdots\,0 & 0\,0\,\cdots\,0 & & 0\,1\,0\,\cdots\,0 \\
 & & & 0\,0\,0\,1\,\cdots\,0 \\
\vdots & \vdots & \cdots & \vdots \\
 & & & 0\quad\quad 1 \\
0\,0\,\cdots\,0 & 0\,0\,\cdots\,0 & & 0\,0\,\cdots\,0
\end{array}\right]
\qquad
B^* = \left[\begin{array}{c}
0\,0\,\cdots\,0 \\
0\,0\,\cdots\,0 \\
\vdots \\
0\,0\,\cdots\,0 \\
1\,0\,\cdots\,0 \\
0\,0\,\cdots\,0 \\
0\,0\quad\;\; 0 \\
\vdots \\
0\,0\quad\;\; 0 \\
0\,1\,\cdots\,0 \\
\vdots \\
0\,0\,\cdots\,0 \\
0\,0\,\cdots\,0 \\
\vdots \\
0\,0\,\cdots\,0 \\
0\,0\,\cdots\,1
\end{array}\right]
\begin{array}{l}
\\ \\ \}\,\kappa_1 \\ \\ \\ \\ \}\,\kappa_2 \\ \\ \\ \\ \\ \\ \}\,\kappa_m \\ \\ \\ \\
\end{array}
$$

That is, the control system is decoupled into m single-input systems of orders $\kappa_1, \kappa_2, ..., \kappa_m$, respectively, all in control canonical form.

Proof. Omitted. ∎

This theorem has interesting consequences on what can be achieved by feedback. For example, consider a controllable two-input four state variables system given by

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^4, u \in \mathbb{R}^2, A \in \mathbb{R}^{4 \times 4}, B \in \mathbb{R}^{4 \times 2}$$

with $rank[B] = 2$. Using state feedback along with a statespace and an input transformation the system can be transformed into either

$$\frac{d}{dt}x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v^*$$

or

$$\frac{d}{dt}x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} v^*.$$

To determine which form just simply compute the controllability indices using the original pair (A, B).

Another result using controllability indices is the following theorem.

Theorem 2 *Controllability Matrix*
Consider the controllable linear time-invariant

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$$

with $rank[B] = m$. The controllability matrix $\mathcal{C} \in \mathbb{R}^{n\times mn}$ is

$$\mathcal{C} \triangleq \begin{bmatrix} b_1 & b_2 & \cdots & b_m & Ab_1 & Ab_2 & \cdots & Ab_m & A^2b_1 & A^2b_2 & \cdots & A^2b_m & \cdots & A^{n-1}b_1 & A^{n-1}b_2 & \cdots & A^{n-1}b_m \end{bmatrix}. \tag{1.24}$$

Search $\mathcal{C}$ from left to right to find the first n linearly independent columns. These n linearly independent columns will be of the form

$$b_1, Ab_1, \cdots, A^{d_1-1}b_1, b_2, Ab_2, \cdots, A^{d_2-1}b_2, \cdots, b_m, Ab_m, \cdots, A^{d_m-1}b_m \tag{1.25}$$

for some positive integers $d_1, d_2, ..., d_m$ with $d_1 + d_2 + \cdots + d_m = n$. If the b_i are rearranged so that $d_1 \geq d_2 \geq \cdots \geq d_m$ then $d_i = \kappa_i$ for $i = 1, 2, ..., m$. The matrix defined by

$$C \triangleq \begin{bmatrix} b_1 & Ab_1 & \cdots & A^{d_1-1}b_1 & b_2 & Ab_2 & \cdots & A^{d_2-1}b_2 & \cdots & b_m & Ab_m & \cdots & A^{d_m-1}b_m \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{1.26}$$

has full rank n and therefore is invertible.
Proof. Exercise. ■

To see the importance of this theorem we look at an example.

Example 8 *Brunovsky Canonical Form*
Consider the controllable linear time-invariant system

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^4, u \in \mathbb{R}^2, A \in \mathbb{R}^{4\times 4}, B \in \mathbb{R}^{4\times 2} \tag{1.27}$$

with $rank[B] = 2$. Let's see how to put this system into Brunovsky canonical form.
Suppose $\kappa_1 = 3$ and $\kappa_2 = 1$ where

$$\begin{aligned} r_0 &= rank \begin{bmatrix} b_1 & b_2 \end{bmatrix} = 2 \\ r_1 &= rank \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 \end{bmatrix} - rank \begin{bmatrix} b_1 & b_2 \end{bmatrix} = 1 \\ r_2 &= rank \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 \end{bmatrix} - rank \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 \end{bmatrix} = 1. \end{aligned}$$

From the computation of r_1 either $rank \begin{bmatrix} b_1 & b_2 & Ab_1 \end{bmatrix} = 3$ or $rank \begin{bmatrix} b_1 & b_2 & Ab_2 \end{bmatrix} = 3$ (or both). Let's suppose

$$rank \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 \end{bmatrix} = rank \begin{bmatrix} b_1 & b_2 & Ab_1 \end{bmatrix}$$

so that

$$Ab_2 = \beta_1 b_1 + \beta_2 b_2 + \beta_3 Ab_1 \tag{1.28}$$

which implies

$$A^2b_2 = \beta_1 Ab_1 + \beta_2 Ab_2 + \beta_3 A^2b_1.$$

In the computation of r_2 we have

$$r_2 = rank \begin{bmatrix} b_1 & b_2 & Ab_1 & \beta_1 b_1 + \beta_2 b_2 + \beta_3 Ab_1 & A^2b_1 & \beta_1 Ab_1 + \beta_2 Ab_2 + \beta_3 A^2b_1 \end{bmatrix} - rank \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 \end{bmatrix} = 1$$

which shows that

$$C \triangleq \begin{bmatrix} b_1 & Ab_1 & A^2b_1 & b_2 \end{bmatrix} \in \mathbb{R}^4$$

has rank 4 and is therefore invertible.

Let q_3 be the $\kappa_1 = 3$ row of C^{-1} and q_4 be the $\kappa_1 + \kappa_2 = 4$ row of C^{-1} so that

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \begin{bmatrix} b_1 & Ab_1 & A^2b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.29}$$

Define the transformation

$$T \triangleq \begin{bmatrix} q_3 \\ q_3A \\ q_3A^2 \\ q_4 \end{bmatrix}.$$

Using (1.29) we have

$$\begin{aligned} TC = \begin{bmatrix} q_3 \\ q_3A \\ q_3A^2 \\ q_4 \end{bmatrix} \begin{bmatrix} b_1 & Ab_1 & A^2b_1 & b_2 \end{bmatrix} &= \begin{bmatrix} q_3b_1 & q_3Ab_1 & q_3A^2b_1 & q_3b_2 \\ q_3Ab_1 & q_3A^2b_1 & q_3A^3b_1 & q_3Ab_2 \\ q_3A^2b_1 & q_3A^3b_1 & q_3A^4b_1 & q_3A^2b_2 \\ q_4b_1 & q_4Ab_1 & q_4A^2b_1 & q_4b_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & q_3b_2 \\ 0 & 1 & q_3A^3b_1 & q_3Ab_2 \\ 1 & q_3A^3b_1 & q_3A^4b_1 & q_3A^2b_2 \\ q_4b_1 & q_4Ab_1 & q_4A^2b_1 & 1 \end{bmatrix}. \end{aligned}$$

This shows that TC is invertible and, as C is invertible, it follows that T is invertible.

Next we compute $A' \triangleq TAT^{-1}$ by solving $TA = A'T$ for A' and compute $B' = TB$ as well. We have

$$TA = \begin{bmatrix} q_3 \\ q_3A \\ q_3A^2 \\ q_4 \end{bmatrix} A = \begin{bmatrix} q_3A \\ q_3A^2 \\ q_3A^3 \\ q_4A \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} \begin{bmatrix} q_3 \\ q_3A \\ q_3A^2 \\ q_4 \end{bmatrix}$$

$$TB = \begin{bmatrix} q_3 \\ q_3A \\ q_3A^2 \\ q_4 \end{bmatrix} B = \begin{bmatrix} q_3b_1 & q_3b_2 \\ q_3Ab_1 & q_3Ab_2 \\ q_3A^2b_1 & q_3A^2b_2 \\ q_4b_1 & q_4b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & q_3Ab_2 \\ 1 & q_3A^2b_2 \\ 0 & 1 \end{bmatrix}$$

Using (1.28) we see that

$$q_3Ab_2 = \beta_1 q_3b_1 + \beta_2 q_3b_2 + \beta_3 q_3Ab_1 = 0$$

Thus we can write

$$TB = \begin{bmatrix} q_3 \\ q_3A \\ q_3A^2 \\ q_4 \end{bmatrix} B = \begin{bmatrix} q_3b_1 & q_3b_2 \\ q_3Ab_1 & q_3Ab_2 \\ q_3A^2b_1 & q_4A^2b_2 \\ q_4b_1 & q_4b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & q_4A^2b_2 \\ 0 & 1 \end{bmatrix}.$$

Let

$$u = \underbrace{\begin{bmatrix} 1 & -q_4A^2b_2 \\ 0 & 1 \end{bmatrix}}_{U} v$$

so that

$$
TA = \begin{bmatrix} q_3 \\ q_3 A \\ q_3 A^2 \\ q_4 \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} \begin{bmatrix} q_3 \\ q_3 A \\ q_3 A^2 \\ q_4 \end{bmatrix}
$$

$$
TBU = \begin{bmatrix} q_3 \\ q_3 A \\ q_3 A^2 \\ q_4 \end{bmatrix} BU = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & q_4 A^2 b_2 \\ 0 & 1 \end{bmatrix} U = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

With $x^* = Tx$ and $u = Uv$ we have

$$
\frac{d}{dt} x^* = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix}}_{TAT_{-}^{1}} x^* + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{TBU} v.
$$

With the feedback

$$
v = - \begin{bmatrix} \alpha_{21} & \alpha_{22} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} x^* + v^*
$$

we finally have

$$
\frac{d}{dt} x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v^*
$$

Example 9 *Permanent Magnet Synchronous Motor*
In Chapter **??** we considered the model of a two-phase permanent magnet motor given by

$$
\begin{aligned}
L_S \frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m \sin(n_p \theta)\omega + u_{Sa} \\
L_S \frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m \cos(n_p \theta)\omega + u_{Sb} \\
J \frac{d\omega}{dt} &= K_m(-i_{Sa} \sin(n_p \theta) + i_{Sb} \cos(n_p \theta)) - \tau_L \\
\frac{d\theta}{dt} &= \omega.
\end{aligned}
$$

With $x_1 = i_{Sa}, x_2 = i_{Sb}, x_3 = \omega, x_4 = \theta, u_1 = u_{Sa}/L, u_2 = u_{Sb}/L$ and $c_1 = R_S/L_S, c_2 = K_m/L_S, c_3 = K_m/J$ we may rewrite this as

$$
\begin{aligned}
\frac{dx_1}{dt} &= -c_1 x_1 + c_2 x_3 \sin(n_p x_4) + u_1 \\
\frac{dx_2}{dt} &= -c_1 x_2 - c_2 x_3 \cos(n_p x_4) + u_2 \\
\frac{dx_3}{dt} &= -c_3 x_1 \sin(n_p x_4) + c_3 x_2 \cos(n_p x_4) \\
\frac{dx_4}{dt} &= x_3
\end{aligned}
$$

or

$$\frac{dx}{dt} = \underbrace{\begin{bmatrix} -c_1x_1 + c_2x_3\sin(n_px_4) \\ -c_1x_2 - c_2x_3\cos(n_px_4) \\ -c_3x_1\sin(n_px_4) + c_3x_2\cos(n_px_4) \\ x_3 \end{bmatrix}}_{f} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{g_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{g_2} u_2.$$

We write

$$\mathcal{C} = \begin{bmatrix} g_1 & g_2 & ad_fg_1 & ad_fg_2 & ad_f^2g_1 & ad_f^2g_2 & ad_f^3g_1 & ad_f^3g_2 \end{bmatrix}$$

where

$$ad_fg_1 = \begin{bmatrix} c_1 \\ 0 \\ c_3\sin(n_px_4) \\ 0 \end{bmatrix}, ad_fg_2 = \begin{bmatrix} 0 \\ c_1 \\ -c_3\cos(n_px_4) \\ 0 \end{bmatrix}$$

and

$$ad_f^2g_1 = \begin{bmatrix} c_1^2 - c_2c_3\sin^2(n_px_4) \\ c_2c_3\sin(n_px_4)\cos(n_px_4) \\ c_3n_px_3\cos(n_px_4) + c_3(c_1+c_4)\sin(n_px_4) \\ -c_3\sin(n_px_4) \end{bmatrix}, ad_f^2g_2 = \begin{bmatrix} c_2c_3\sin(n_px_4)\cos(n_px_4) \\ c_1^2 - c_2c_3\cos^2(n_px_4) \\ c_3n_px_3\sin(n_px_4) - c_3(c_1+c_4)\cos(n_px_4) \\ c_3\cos(n_px_4) \end{bmatrix}.$$

By inspection $rank[\mathcal{C}] = 4$ for all x. Note that

$$\begin{aligned} ad_fg_1 &= c_1\tan(n_px_4)g_2 + c_1g_1 - \tan(n_px_4)ad_fg_2 \\ ad_fg_2 &= c_1\cot(n_px_4)g_1 + c_1g_2 - \cot(n_px_4)ad_fg_1 \\ ad_f^2g_2 &= -\cot(n_px_4)ad_f^2g_1 + (-n_px_3\tan(n_px_4) - n_px_3\cot(n_px_4))ad_fg_2 + \\ &\quad c_1^2\cot(n_px_4)g_1 + (c_1^2 + c_1n_p\tan(n_px_4) + c_1n_px_3\cot(n_px_4))g_2 \end{aligned}$$

$ad_f^3g_1, ad_f^3g_2$ are not needed and so are not computed. Then

$$\begin{aligned} r_0 &= rank\begin{bmatrix} g_1 & g_2 \end{bmatrix} = 2 \\ r_1 &= rank\begin{bmatrix} g_1 & g_2 & ad_fg_1 & ad_fg_2 \end{bmatrix} - rank\begin{bmatrix} g_1 & g_2 \end{bmatrix} = 1 \\ r_2 &= rank\begin{bmatrix} g_1 & g_2 & ad_fg_1 & ad_fg_2 & ad_f^2g_1 & ad_f^2g_2 \end{bmatrix} - rank\begin{bmatrix} g_1 & g_2 & ad_fg_1 & ad_fg_2 \end{bmatrix} = 1 \end{aligned}$$

with corresponding controllability indices

$$\begin{aligned} \kappa_1 &= \text{number of } r_j \geq 1 = 3 \\ \kappa_2 &= \text{number of } r_j \geq 2 = 1. \end{aligned}$$

Controllability Matrix $\mathcal{C}$

With $f(x) \in \mathbb{R}^4, g_1(x) \in \mathbb{R}^4, g_2(x) \in \mathbb{R}^4$ consider the control system

$$\frac{dx}{dt} = f(x) + g_1(x)u_1 + g_2(x)u_2.$$

Define the controllability matrix $\mathcal{C}$ by

$$\mathcal{C} \triangleq \begin{bmatrix} g_1 & g_2 & ad_fg_1 & ad_fg_2 & ad_f^2g_1 & ad_f^2g_2 & ad_f^3g_1 & ad_f^3g_2 \end{bmatrix} \in \mathbb{R}^{4\times 8}$$

and we assume $rank[\mathcal{C}] = 4$. Further suppose $rank\begin{bmatrix} g_1 & g_2 \end{bmatrix} = 2$. (If $rank\begin{bmatrix} g_1 & g_2 \end{bmatrix} = 1$ then it is really a single input system).

We now search this matrix from left to right to find four linearly independent columns. As $rank \begin{bmatrix} g_1 & g_2 \end{bmatrix} = 2$ we next look $ad_f g_1$ to see if it is linearly independent of $\{g_1, g_2\}$. Suppose it is *not* so that

$$ad_f g_1 = \alpha_1(x) g_1(x) + \alpha_2(x) g_2(x).$$

We next check $ad_f g_2$. Suppose it is linearly independent of $\{g_1, g_2\}$ so that

$$\begin{bmatrix} g_1 & g_2 & ad_f g_2 \end{bmatrix}$$

are three linearly independent vectors searching left to right. Now we come to $ad_f^2 g_1$ and compute (using Lemma **??** of **??**)

$$\begin{aligned} ad_f^2 g_1 = [f, ad_f g_1] = [f, \alpha_1(x) g_1(x) + \alpha_2(x) g_2(x)] &= \mathcal{L}_f(\alpha_1) g_1 + \alpha_1 [f, g_1] + \mathcal{L}_f(\alpha_2) g_2 + \alpha_1 [f, g_2] \\ &= \mathcal{L}_f(\alpha_1) g_1 + \alpha_1 ad_f g_1 + \mathcal{L}_f(\alpha_2) g_2 + \alpha_1 ad_f g_2 \\ &= \mathcal{L}_f(\alpha_1) g_1 + \alpha_1(\alpha_1 g_1 + \alpha_2 g_2) + \mathcal{L}_f(\alpha_2) g_2 + \alpha_1 ad_f g_2 \end{aligned}$$

show that $ad_f^2 g_1$ is linearly dependent on $\{g_1, g_2, ad_f g_2\}$. This same argument will show that $ad_f^3 g_1$ is linearly dependent on the column vectors of $\mathcal{C}$ to its left in $\mathcal{C}$. The next vector we encounter going left to right is $ad_f^2 g_2$ and this must be linearly independent of $\{g_1, g_2, ad_f g_2\}$. This is because $rank[\mathcal{C}] = 4$ and if $ad_f^2 g_2$ was linearly dependent on $\{g_1, g_2, ad_f g_2\}$ then $ad_f^3 g_2$ would be linearly independent of the column vectors of $\mathcal{C}$ to its left in $\mathcal{C}$. That is, $\mathcal{C}$ would not be full rank. Thus, in this example we have

$$C = \begin{bmatrix} g_1 & g_2 & ad_f g_2 & ad_f^2 g_2 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

has rank 4.

Feedback Linearization Transformation for the PM Synchronous Motor

Let's find a feedback linearization transformation for the PM synchronous machine given by

$$\frac{dx}{dt} = \underbrace{\begin{bmatrix} -c_1 x_1 + c_2 x_3 \sin(n_p x_4) \\ -c_1 x_2 - c_2 x_3 \cos(n_p x_4) \\ -c_3 x_1 \sin(n_p x_4) + c_3 x_2 \cos(n_p x_4) \\ x_3 \end{bmatrix}}_{f} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{g_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{g_2} u_2. \tag{1.30}$$

A general nonlinear change of coordinates is given by

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \\ T_4(x) \end{bmatrix} \tag{1.31}$$

and

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f(T_1) + u_1 \mathcal{L}_{g_1}(T_1) + u_2 \mathcal{L}_{g_2}(T_1) \\ \mathcal{L}_f(T_2) + u_1 \mathcal{L}_{g_1}(T_2) + u_2 \mathcal{L}_{g_2}(T_2) \\ \mathcal{L}_f(T_3) + u_1 \mathcal{L}_{g_1}(T_3) + u_2 \mathcal{L}_{g_2}(T_3) \\ \mathcal{L}_f(T_4) + u_1 \mathcal{L}_{g_1}(T_4) + u_2 \mathcal{L}_{g_2}(T_4) \end{bmatrix}. \tag{1.32}$$

As shown in the above example the PM synchronous machine has controllability indices $\kappa_1 = 3, \kappa_2 = 1$. This requires

$$\mathcal{L}_{g_1}(T_1) = 0, \mathcal{L}_{g_2}(T_1) = 0 \tag{1.33}$$

$$\mathcal{L}_{g_1}(T_2) = 0, \mathcal{L}_{g_2}(T_2) = 0 \tag{1.34}$$

and

$$x_2^* = T_2 \triangleq \mathcal{L}_f(T_1), \quad x_3^* = T_3 \triangleq \mathcal{L}_f(T_2) = \mathcal{L}_f^2(T_1) \tag{1.35}$$

so that the system of equations (1.32) have the form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f(T_1) \\ \mathcal{L}_f^2(T_1) \\ \mathcal{L}_f^3(T_1) \\ \mathcal{L}_f(T_4) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathcal{L}_{g_1}\mathcal{L}_f^2(T_1) & \mathcal{L}_{g_2}\mathcal{L}_f^2(T_1) \\ \mathcal{L}_{g_1}(T_4) & \mathcal{L}_{g_2}(T_4) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{1.36}$$

Recalling the identity $\mathcal{L}_{[f,g]} = \mathcal{L}_g\mathcal{L}_f - \mathcal{L}_f\mathcal{L}_g$ the conditions (1.33), (1.34) and (1.35) are equivalent to

$$\mathcal{L}_{g_1}(T_1) = 0, \mathcal{L}_{[f,g_1]}(T_1) = 0, \mathcal{L}_{g_2}(T_1) = 0, \mathcal{L}_{[f,g_2]}(T_1) = 0$$

or

$$dT_1\begin{bmatrix} g_1 & ad_f g_1 & g_2 & ad_f g_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also using the identity (See **??** of Chapter **??** on page **??**)

$$\mathcal{L}_{ad_f^2 g_i}(T_1) = \mathcal{L}_f^2\mathcal{L}_{g_i}(T_1)) - 2\mathcal{L}_f\mathcal{L}_{g_i}\mathcal{L}_f(T_1) + \mathcal{L}_{g_i}\mathcal{L}_f^2(T_1) = \mathcal{L}_{g_i}\mathcal{L}_f^2(T_1).$$

we may write

$$\begin{aligned} \mathcal{L}_{g_1}(\mathcal{L}_f^2(T_1)) &= \mathcal{L}_{ad_f^2 g_1}(T_1) \\ \mathcal{L}_{g_2}(\mathcal{L}_f^2(T_1)) &= \mathcal{L}_{ad_f^2 g_2}(T_1). \end{aligned}$$

so that (1.36) becomes

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f(T_1) \\ \mathcal{L}_f^2(T_1) \\ \mathcal{L}_f^3(T_1) \\ \mathcal{L}_f(T_4) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{g_1}(T_4) & \mathcal{L}_{g_2}(T_4) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{1.37}$$

We require the input matrix has full rank (otherwise the system is not controllable), that is,

$$\det\begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{g_1}(T_4) & \mathcal{L}_{g_2}(T_4) \end{bmatrix} \neq 0. \tag{1.38}$$

The conditions (1.36) and (1.38) are necessary conditions on the unknown functions $T_1(x)$ and $T_4(x)$. Note that these conditions only involve the first order derivatives (gradient) of $T_1(x)$ and $T_4(x)$. Let's look for a solution to (1.36) by writing it out explicitly as

$$\begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} & \frac{\partial T_1}{\partial x_4} \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} c_1 \\ 0 \\ c_3\sin(n_p x_4) \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ c_1 \\ -c_3\cos(n_p x_4) \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

By inspection we must have $\dfrac{\partial T_1}{\partial x_1} = 0, \dfrac{\partial T_1}{\partial x_2} = 0$. The fact that $c_3\sin(n_p x_4)$ and $-c_3\sin(n_p x_4)$ cannot both be zero requires $\dfrac{\partial T_1}{\partial x_3} = 0$. This leaves us with

$$dT_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial T_1}{\partial x_4} \end{bmatrix}$$

with $\dfrac{\partial T_1}{\partial x_4}$ still to be determined. With this choice for the gradient $T_1(x)$ is only a function of x_4. To determine (1.38) we compute

$$\mathcal{L}_{ad_f^2 g_1}(T_1) = \begin{bmatrix} 0 & 0 & 0 & \dfrac{\partial T_1}{\partial x_4} \end{bmatrix} \begin{bmatrix} c_1^2 - c_2 c_3 \sin^2(n_p x_4) \\ c_2 c_3 \sin(n_p x_4)\cos(n_p x_4) \\ c_3 n_p x_3 \cos(n_p x_4) + c_3(c_1 + c_4)\sin(n_p x_4) \\ -c_3 \sin(n_p x_4) \end{bmatrix} = -c_3 \frac{\partial T_1}{\partial x_4} \sin(n_p x_4)$$

$$\mathcal{L}_{ad_f^2 g_2}(T_1) = \begin{bmatrix} 0 & 0 & 0 & \dfrac{\partial T_1}{\partial x_4} \end{bmatrix} \begin{bmatrix} c_2 c_3 \sin(n_p x_4)\cos(n_p x_4) \\ c_1^2 - c_2 c_3 \cos^2(n_p x_4) \\ c_3 n_p x_3 \sin(n_p x_4) - c_3(c_1 + c_4)\cos(n_p x_4) \\ c_3 \cos(n_p x_4) \end{bmatrix} = c_3 \frac{\partial T_1}{\partial x_4} \cos(n_p x_4)$$

$$\mathcal{L}_{g_1}(T_4) = \begin{bmatrix} \dfrac{\partial T_4}{\partial x_1} & \dfrac{\partial T_4}{\partial x_2} & \dfrac{\partial T_4}{\partial x_3} & \dfrac{\partial T_4}{\partial x_4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{\partial T_4}{\partial x_1}$$

$$\mathcal{L}_{g_2}(T_4) = \begin{bmatrix} \dfrac{\partial T_4}{\partial x_1} & \dfrac{\partial T_4}{\partial x_2} & \dfrac{\partial T_4}{\partial x_3} & \dfrac{\partial T_4}{\partial x_4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{\partial T_4}{\partial x_2}.$$

Then

$$\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{g_1}(T_4) & \mathcal{L}_{g_2}(T_4) \end{bmatrix} = \det \begin{bmatrix} -c_3 \dfrac{\partial T_1}{\partial x_4} \sin(n_p x_4) & c_3 \dfrac{\partial T_1}{\partial x_4} \cos(n_p x_4) \\ \dfrac{\partial T_4}{\partial x_1} & \dfrac{\partial T_4}{\partial x_2} \end{bmatrix}$$
$$= -c_3 \frac{\partial T_1}{\partial x_4} \left(\frac{\partial T_4}{\partial x_2} \sin(n_p x_4) + \frac{\partial T_4}{\partial x_1} \cos(n_p x_4) \right).$$

This suggests setting

$$\frac{\partial T_4}{\partial x_2} = \sin(n_p x_4)$$
$$\frac{\partial T_4}{\partial x_1} = \cos(n_p x_4)$$
$$\frac{\partial T_1}{\partial x_4} = 1$$

so that

$$\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{g_1}(T_4) & \mathcal{L}_{g_2}(T_4) \end{bmatrix} = -c_3 \neq 0.$$

Then

$$T_1(x) = x_4$$
$$T_4(x) = x_1 \cos(n_p x_4) + x_2 \sin(n_p x_4).$$

The full transformation is

$$x_1^* = T_1 = x_4$$
$$x_2^* = \mathcal{L}_f(T_1) = x_3$$
$$x_3^* = \mathcal{L}_f^3(T_1) = c_3\Big(-x_1 \sin(n_p x_4) + \ x_2 \cos(n_p x_4)\Big)$$
$$x_4^* = T_4 = x_1 \cos(n_p x_4) + x_2 \sin(n_p x_4).$$

In the new coordinates we have

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} x_2^* \\ x_3^* \\ \mathcal{L}_f^3(T_1) \\ \mathcal{L}_f(T_4) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -c_3\sin(n_p x_4) & c_3\cos(n_p x_4) \\ \cos(n_p x_4) & \sin(n_p x_4) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1.39}$$

With

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\frac{1}{c_3}\begin{bmatrix} \sin(n_p x_4) & -c_3\cos(n_p x_4) \\ -\cos(n_p x_4) & -c_3\sin(n_p x_4) \end{bmatrix}\begin{bmatrix} v_1^* - \mathcal{L}_f^3(T_1) \\ v_2^* - \mathcal{L}_f(T_4) \end{bmatrix}$$

this becomes

$$\frac{d}{dt}x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v^*.$$

Remark $\mathcal{L}_f^3(T_1)$ and $\mathcal{L}_f(T_4)$ are given by

$$\begin{aligned}
\mathcal{L}_f^3(T_1) &= \begin{bmatrix} -c_3\sin(n_p x_4) & c_3\cos(n_p x_4) & 0 & -c_3 n_p(x_1\cos(n_p x_4) + x_2\sin(n_p x_4)) \end{bmatrix} \times \\
&\quad \begin{bmatrix} -c_1 x_1 + c_2 x_3 \sin(n_p x_4) \\ -c_1 x_2 - c_2 x_3\cos(n_p x_4) \\ -c_3 x_1 \sin(n_p x_4) + c_3 x_2 \cos(n_p x_4) \\ x_3 \end{bmatrix} \\
&= c_1 c_3(-x_1\sin(n_p x_4) + x_2\cos(n_p x_4)) - c_3 c_2 x_3 - c_3 n_p x_3(x_1\cos(n_p x_4) + x_2\sin(n_p x_4)) \\
&= c_3(c_1 i_q - c_2\omega - n_p\omega i_d) \\
\mathcal{L}_f(T_4) &= \begin{bmatrix} \cos(n_p x_4) & \sin(n_p x_4) & 0 & -n_p(-x_1\sin(n_p x_4) + x_2\cos(n_p x_4)) \end{bmatrix}\begin{bmatrix} -c_1 x_1 + c_2 x_3 \sin(n_p x_4) \\ -c_1 x_2 - c_2 x_3\cos(n_p x_4) \\ -c_3 x_1 \sin(n_p x_4) + c_3 x_2 \cos(n_p x_4) \\ x_3 \end{bmatrix} \\
&= -c_1(x_1\cos(n_p x_4) + x_2\sin(n_p x_4)) + n_p x_3(-x_1\sin(n_p x_4) + x_2\cos(n_p x_4)) \\
&= -c_1 i_d + n_p\omega i_q.
\end{aligned}$$

i_d, i_q, u_d, u_q are defined as in Equations (**??**) and (**??**) of Chapter **??** (page **??**) showing that the system model (1.39) above is the same as the system (**??**) - (**??**) of Chapter **??**.

Theorem 3 *Multi-Input Exact Linearization Problem*

Given the vector fields $f, g_1, ..., g_m$ on an open set $\mathcal{U} \subset \mathbf{E}^n$ consider the nonlinear control system

$$\frac{dx}{dt} = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m. \tag{1.40}$$

Define

$$\mathcal{C} \triangleq \begin{bmatrix} g_1 & \cdots & g_m & ad_f g_1 & \cdots & ad_f g_m & \cdots & ad_f^{n-1} g_1 & \cdots & ad_f^{n-1} g_m \end{bmatrix} \tag{1.41}$$

and

$$\begin{aligned}
G_0 &\triangleq span\{g_1, ..., g_m\} \\
G_1 &\triangleq span\{g_1, ..., g_m, ad_f g_1, ..., ad_f g_m\} \\
&\ \vdots \\
G_{n-2} &\triangleq span\{g_1, ..., g_m, ad_f g_1, ..., ad_f g_m, ..., ad_f^{n-2} g_1, ..., ad_f^{n-2} g_m\} \\
G_{n-1} &\triangleq span\{g_1, ..., g_m, ad_f g_1, ..., ad_f g_m, ..., ad_f^{n-1} g_1, ..., ad_f^{n-1} g_m\}
\end{aligned} \tag{1.42}$$

Note that $G_{n-1} = \mathcal{C}$.

Let $x_0 \in \mathcal{U}$. Suppose in a neighborhood of x_0 the G_i for $i = 0, ..., n-1$, the G_i have *constant* rank and

$$\begin{aligned} rank[G_0] &= m \\ rank[G_{n-1}] &= rank[\mathcal{C}] = n \end{aligned}$$

Further suppose that each of the distributions $G_0, G_1, ..., G_{n-2}$ are *involutive*.[1]

Define the r indices by

$$\begin{aligned} r_0 &= rank[G_0] \\ r_1 &= rank\,[G_1] - rank[G_0] \\ r_2 &= rank\,[G_2] - rank\,[G_1] \\ &\vdots \\ r_{n-1} &= rank\,[G_{n-1}] - rank\,[G_{n-2}]\,. \end{aligned}$$

Let $\kappa_1, \kappa_2, ..., \kappa_m$ be the corresponding controllability indices determined by these r indices.

Then there exists an invertible statespace transformation $T^*(x)$ defined in a neighborhood of x_0 given by

$$\begin{aligned} x_1^* &= T_1^*(x) \\ x_2^* &= T_2^*(x) \\ &\vdots \\ x_n^* &= T_n^*(x) \end{aligned}$$

and an invertible input transformation

$$\begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_m^* \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha_1(x) \\ \alpha_2(x) \\ \vdots \\ \alpha_m(x) \end{bmatrix}}_{\alpha(x)} + \underbrace{\begin{bmatrix} \beta_{11}(x) & \beta_{12}(x) & \cdots & \beta_{1m}(x) \\ \beta_{21}(x) & \beta_{22}(x) & \cdots & \beta_{2m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1}(x) & \beta_{m2}(x) & \cdots & \beta_{mm}(x) \end{bmatrix}}_{\beta(x)} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

such that in the new x^* coordinates and the new v^* inputs the system has the form

$$\frac{dx^*}{dt} = A^* x^* + B^* v^*$$

where A^* and B^* are in Brunovsky canonical form (see Figure 1).

Proof. Special case of $\kappa_1 = 3$ and $\kappa_2 = 2$.

We have

$$\frac{dx}{dt} = f(x) + G(x)u, \quad x \in \mathbb{R}^5, u \in \mathbb{R}^2, f \in \mathbb{R}^5, \;\; G(x) = \left[\begin{array}{cc} g_1(x) & g_2(x) \end{array}\right] \in \mathbb{R}^{5\times 2}.$$

[1] As $rank[G_{n-1}] = rank[\mathcal{C}] = n$, it follows that G_{n-1} is involutive.

With $x^* = T^*(x)$ we have

$$\begin{aligned}
\frac{dx_1^*}{dt} &= \mathcal{L}_f(T_1^*) + u_1\mathcal{L}_{g_1}(T_1^*) + u_2\mathcal{L}_{g_2}(T_1^*)\\
\frac{dx_2^*}{dt} &= \mathcal{L}_f(T_2^*) + u_1\mathcal{L}_{g_1}(T_2^*) + u_2\mathcal{L}_{g_2}(T_2^*)\\
\frac{dx_3^*}{dt} &= \mathcal{L}_f(T_3^*) + u_1\mathcal{L}_{g_1}(T_3^*) + u_2\mathcal{L}_{g_2}(T_3^*)\\
\frac{dx_4^*}{dt} &= \mathcal{L}_f(T_4^*) + u_1\mathcal{L}_{g_1}(T_4^*) + u_2\mathcal{L}_{g_2}(T_4^*)\\
\frac{dx_5^*}{dt} &= \mathcal{L}_f(T_5^*) + u_1\mathcal{L}_{g_1}(T_5^*) + u_2\mathcal{L}_{g_2}(T_5^*).
\end{aligned}$$

The controllability indices are $\kappa_1 = 3$ and $\kappa_2 = 2$ we require

$$\begin{aligned}
T_2^* &= \mathcal{L}_f(T_1^*), \quad \mathcal{L}_{g_1}(T_1^*) = 0, \quad \mathcal{L}_{g2}(T_1^*) = 0\\
T_3^* &= \mathcal{L}_f(T_2^*), \quad \mathcal{L}_{g_1}(T_2^*) = \mathcal{L}_{g_1}(\mathcal{L}_f(T_1^*)) = 0, \quad \mathcal{L}_{g_2}(T_2^*) = \mathcal{L}_{g_2}(\mathcal{L}_f(T_1^*)) = 0
\end{aligned}$$

and

$$T_5^* = \mathcal{L}_f(T_4^*), \quad \mathcal{L}_{g_1}(T_4^*) = 0, \quad \mathcal{L}_{g_2}(T_4^*) = 0.$$

Using

$$\begin{aligned}
\mathcal{L}_{ad_f g}(h) &= \mathcal{L}_{[f,g]}(h) = \mathcal{L}_f(\mathcal{L}_g(h)) - \mathcal{L}_g(\mathcal{L}_f(h))\\
\mathcal{L}_{ad_f^2 g}(h) &= \mathcal{L}_f^2\mathcal{L}_g(h)) - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f(h) + \mathcal{L}_g\mathcal{L}_f^2(h)
\end{aligned}$$

this becomes

$$\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^*\\
\frac{dx_2^*}{dt} &= x_3^*\\
\frac{dx_3^*}{dt} &= \mathcal{L}_f^2(T_1^*) + u_1\mathcal{L}_{ad_f^2 g_1}(T_1^*) + u_2\mathcal{L}_{ad_f^2 g_2}(T_1^*)\\
\frac{dx_4^*}{dt} &= x_5^*\\
\frac{dx_5^*}{dt} &= \mathcal{L}_f^2(T_4^*) + u_1(-\mathcal{L}_{ad_f g_1}(T_4^*)) + u_2(-\mathcal{L}_{ad_f g_2}(T_4^*)).
\end{aligned}$$

This reduces the problem to finding T_1^* and T_4^* such that

$$dT_1^* \begin{bmatrix} g_1 & g_2 & ad_f g_1 & ad_f g_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.43}$$

$$dT_4^* \begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{1.44}$$

and

$$\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1^*) & \mathcal{L}_{ad_f^2 g_2}(T_1^*) \\ \mathcal{L}_{ad_f g_1}(T_4^*) & \mathcal{L}_{ad_f g_2}(T_4^*) \end{bmatrix} \neq 0. \tag{1.45}$$

Conditions (1.43) and (1.44) require that

$$\begin{aligned}
G_0 &= \{g_1, g_2\}\\
G_1 &= \{g_1, g_2, ad_f g_1, ad_f g_2\}
\end{aligned}$$

be involutive. The matrix $\mathcal{C}$

$$\mathcal{C} \triangleq G_4 \triangleq \begin{bmatrix} g_1 & g_2 & ad_f g_1 & ad_f g_2 & ad_f^2 g_1 & ad_f^2 g_2 & ad_f^3 g_1 & ad_f^3 g_2 & ad_f^4 g_1 & ad_f^4 g_2 \end{bmatrix} \in \mathbb{R}^{5\times 10} \tag{1.46}$$

has rank 5 in a neighborhood of x_0. As $\kappa_1 = 3, \kappa_2 = 4$ define the matrices

$$C_1 \triangleq \begin{bmatrix} g_1 & ad_f g_1 & ad_f^2 g_1 & g_2 & ad_f g_2 \end{bmatrix} \in \mathbb{R}^{5\times 5}$$
$$C_2 \triangleq \begin{bmatrix} g_1 & ad_f g_1 & g_2 & ad_f g_2 & ad_f^2 g_2 \end{bmatrix} \in \mathbb{R}^{5\times 5}.$$

Using an argument similar the one given in the subsection C **Matrix** on 16 it follows that C_1 or C_2 (or both) have rank 5. [2]

To proceed, let's assume that C_1 has rank 5. We have to construct an *invertible* transformation $T^*(x)$ satisfying (1.43), (1.44) and (1.45).

To construct the required transformation $T^*(x)$ we first consider the transformation $T(x)$ given by

$$S(t_1, t_2, t_3, t_4, t_5) = \phi_{t_5}(\phi_{t_4}(\phi_{t_3}(\phi_{t_2}(\phi_{t_1}(x_0)))))$$

where
$\phi_{t_1}(x_0)$ is the solution to

$$\frac{dx}{dt} = ad_f^2 g_1(x),\ x(0) = x_0$$

$\phi_{t_2}(x_0)$ is the solution to

$$\frac{dx}{dt} = ad_f g_1(x),\ x(0) = x_0'$$

$\phi_{t_3}(x_0)$ is the solution to

$$\frac{dx}{dt} = ad_f g_2(x),\ x(0) = x_0''$$

$\phi_{t_4}(x_0)$ is the solution to

$$\frac{dx}{dt} = g_1(x),\ x(0) = x_0'''$$

$\phi_{t_5}(x_0)$ is the solution to

$$\frac{dx}{dt} = g_2(x),\ x(0) = x_0''''.$$

At $t = (0,0,0,0,0)$ we have

$$\frac{\partial S}{\partial t}_{|t=(0,0,0,0,0)} = \begin{bmatrix} ad_f^2 g_1 & ad_f g_1 & ad_f g_2 & g_1 & g_2 \end{bmatrix}_{|x_0}.$$

By the inverse function theorem $S(t)$ has an inverse defined in a neighborhood of $t = 0$. Denote the inverse of

$$\begin{aligned} x_1 &= s_1(t_1, t_2, t_3, t_4, t_5) \\ x_2 &= s_2(t_1, t_2, t_3, t_4, t_5) \\ x_3 &= s_3(t_1, t_2, t_3, t_4, t_5) \\ x_4 &= s_4(t_1, t_2, t_3, t_4, t_5) \\ x_5 &= s_5(t_1, t_2, t_3, t_4, t_5) \end{aligned} \tag{1.47}$$

[2] If $\kappa_1 = 4, \kappa_2 = 1$ then either

$$C_1 \triangleq \begin{bmatrix} g_1 & ad_f g_1 & ad_f^2 g_1 & ad_f^3 g_1 & g_2 \end{bmatrix} \in \mathbb{R}^{5\times 5}$$

or

$$C_2 \triangleq \begin{bmatrix} g_1 & g_2 & ad_f g_2 & ad_f^2 g_2 & ad_f^3 g_2 \end{bmatrix} \in \mathbb{R}^{5\times 5}$$

(or both) have rank 5.

by

$$\begin{aligned} t_1 &= T_1(x_1, x_2, x_3, x_4, x_5) \\ t_2 &= T_2(x_1, x_2, x_3, x_4, x_5) \\ t_3 &= T_3(x_1, x_2, x_3, x_4, x_5) \\ t_4 &= T_4(x_1, x_2, x_3, x_4, x_5) \\ t_5 &= T_5(x_1, x_2, x_3, x_4, x_5). \end{aligned} \tag{1.48}$$

If $t_1 = t_{01}$ is held constant, then $S(t_{01}, t_2, t_3, t_4, t_5) = \phi_{t_5}(\phi_{t_4}(\phi_{t_3}(\phi_{t_2}(\phi_{t_{01}}(x_0)))))$ sweeps out a four dimensional surface in $\mathbb{R}^5$ as t_2, t_3, t_4, and t_5 are varied. As the vector fields

$$\{g_1, g_2, ad_f g_1, ad_f g_2\}$$

are involutive, Frobenius' theorem tells us that the vectors

$$\left[\begin{array}{cccc} g_1 & g_2 & ad_f g_1 & ad_f g_2 \end{array}\right]_{x=S(t_{01},t_2,t_3,t_4,t_5)}$$

span tangent space to the surface

$$\{x \in \mathbb{R}^5 |\ T_1(x) = t_{01}\}. \tag{1.49}$$

That is,

$$\mathcal{L}_{g_1}(T_1) = 0,\ \mathcal{L}_{g_2}(T_1) = 0,\ \mathcal{L}_{ad_f g_1}(T_1) = 0,\ \mathcal{L}_{ad_f g_2}(T_1) = 0. \tag{1.50}$$

Further

$$\mathcal{L}_{ad_f^2 g_1}(T_1) \neq 0 \tag{1.51}$$

as $ad_f^2 g_1$ is linearly independent of $\{g_1, g_2, ad_f g_1, ad_f g_2\}$ and therefore cannot be in the tangent space of the surface.

Now if $t_1 = t_{01}, t_2 = t_{02}, t_3 = t_{03}$ are held constant, then $S(t_{01}, t_{02}, t_{03}, t_4, t_5) = \phi_{t_5}(\phi_{t_4}(\phi_{t_{03}}(\phi_{t_{02}}(\phi_{t_{01}}(x_0)))))$ sweeps out a two dimensional surface in $\mathbb{R}^5$. As the set

$$\{g_1, g_2\}$$

is involutive, Frobenius' theorem tells us that

$$\frac{\partial S}{\partial t_4}, \frac{\partial S}{\partial t_5} \in \Delta_{x=S(t_{01},t_{02},t_{03},t_4,t_5)} \triangleq \{r_1 g_1(x) + r_2 g_2(x) |\ x = S(t_{01}, t_{02}, t_{03}, t_4, t_5) \text{ and } r_1, r_2 \in \mathbb{R}\}.$$

That is, $\{g_1, g_2\}$ span the tangent plane of the surface (submanifold)

$$\{x \in \mathbb{R}^5 |\ T_1(x) = t_{01}, T_2(x) = t_{02}, T_3(x) = t_{03}\} \tag{1.52}$$

for $x = S(t_{01}, t_{02}, t_{03}, t_4, t_5)$. In particular we have

$$\mathcal{L}_{g_1}(T_3) = 0,\ \mathcal{L}_{g_2}(T_3) = 0 \tag{1.53}$$

in a neighborhood of x_0.

As $ad_f g_2$ is linearly independent of $\{g_1, g_2\}$ it is *not* tangent to the surface (1.52). That is, by Equation (1.50) we know that $\mathcal{L}_{ad_f g_2}(T_1) = 0$ so $\mathcal{L}_{ad_f g_2}(T_2)$ and $\mathcal{L}_{ad_f g_2}(T_3)$ cannot both be zero since this would imply $ad_f g_2$ is in the tangent space of (1.52). It is next shown that $\mathcal{L}_{ad_f g_2}(T_3)\big|_{x_0} = 1$ implying $\mathcal{L}_{ad_f g_2}(T_3) \neq 0$ in a neighborhood of x_0. To proceed, the transformations $x = S(t)$ and $t = T(x)$ in (1.47) and (1.48) are inverses of each other, that is, $t = T(S(t))$, which implies

$$I_{5\times 5} = \frac{\partial T}{\partial x}\frac{\partial S}{\partial t}. \tag{1.54}$$

The $(1,1)$ component (1.54) gives

$$1 = \frac{\partial T_1}{\partial x}_{|x_0} \frac{\partial S}{\partial t_1}_{|t=(0,0,0,0,0)} = \left\langle dT_1, ad_f^2 g_1 \right\rangle_{|x_0} = \mathcal{L}_{ad_f^2 g_1}(T_1)_{|x_0}.$$

The $(3,3)$ component of (1.54) gives

$$1 = \frac{\partial T_3}{\partial x}_{|x_0} \frac{\partial S}{\partial t_3}_{|t=(0,0,0,0,0)} = \left\langle dT_3, ad_f g_2 \right\rangle_{|x_0} = \mathcal{L}_{ad_f g_2}(T_3)_{|x_0}$$

and the $(3,2)$ component of (1.54) gives

$$0 = \frac{\partial T_3}{\partial x}_{|x_0} \frac{\partial S}{\partial t_2}_{|t=(0,0,0,0,0)} = \left\langle dT_3, ad_f g_1 \right\rangle_{|x_0} = \mathcal{L}_{ad_f g_1}(T_3)_{|x_0}.$$

Then

$$\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{ad_f g_1}(T_3) & \mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix}_{x_0} = \det \begin{bmatrix} 1 & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ 0 & 1 \end{bmatrix}_{|x_0} = 1$$

implying that

$$\det \begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{ad_f g_1}(T_3) & \mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix} \neq 0$$

in a neighborhood of x_0.

Define the feedback linearizing transformation $x^* = T^*(x)$ as

$$\begin{aligned} x_1^* &= T_1^*(x) \triangleq T_1(x) \\ x_2^* &= T_2^*(x) \triangleq \mathcal{L}_f T_1(x) \\ x_3^* &= T_3^*(x) \triangleq \mathcal{L}_f^2 T_1(x) \\ x_4^* &= T_4^*(x) \triangleq T_3(x) \\ x_5^* &= T_5^*(x) = \mathcal{L}_f T_3(x). \end{aligned} \tag{1.55}$$

With $x^* = T^*(x)$ we have

$$\begin{aligned} \frac{dx_1^*}{dt} &= \mathcal{L}_f(T_1) + u_1 \mathcal{L}_{g_1}(T_1) + u_2 \mathcal{L}_{g_2}(T_1) \\ \frac{dx_2^*}{dt} &= \mathcal{L}_f^2(T_1) + u_1 \mathcal{L}_{g_1}\mathcal{L}_f(T_1) + u_2 \mathcal{L}_{g_2}\mathcal{L}_f(T_1) \\ \frac{dx_3^*}{dt} &= \mathcal{L}_f^3(T_1) + u_1 \mathcal{L}_{g_1}\mathcal{L}_f^2(T_1) + u_2 \mathcal{L}_{g_2}\mathcal{L}_f^2(T_1) \\ \frac{dx_4^*}{dt} &= \mathcal{L}_f(T_3) + u_1 \mathcal{L}_{g_1}(T_3) + u_2 \mathcal{L}_{g_2}(T_3) \\ \frac{dx_5^*}{dt} &= \mathcal{L}_f(T_3) + u_1 \mathcal{L}_{g_1}\mathcal{L}_f(T_3) + u_2 \mathcal{L}_{g_2}\mathcal{L}_f(T_3) \end{aligned}$$

which by all of the above reduces to

$$\begin{aligned} \frac{dx_1^*}{dt} &= \mathcal{L}_f(T_1) \\ \frac{dx_2^*}{dt} &= \mathcal{L}_f^2(T_1) \\ \frac{dx_3^*}{dt} &= \mathcal{L}_f^3(T_1) + u_1 \mathcal{L}_{ad_f^2 g_1}(T_1) + u_2 \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \frac{dx_4^*}{dt} &= \mathcal{L}_f(T_3) \\ \frac{dx_5^*}{dt} &= \mathcal{L}_f(T_3) + u_1 \mathcal{L}_{ad_f g_1}(T_3) + u_2 \mathcal{L}_{ad_f g_2}(T_3) \end{aligned}$$

with

$$\det\begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{ad_f g_1}(T_3) & \mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix} \neq 0$$

in some neighborhood of x_0.

We still have to show that the transformation (1.55) is an *invertible* transformation. To do this we compute

$$\begin{aligned}
\frac{\partial T^*}{\partial x} C_1 &= \begin{bmatrix} dT_1^* \\ dT_2^* \\ dT_3^* \\ dT_4^* \\ dT_5^* \end{bmatrix} \begin{bmatrix} g_1 & ad_f g_1 & ad_f^2 g_1 & g_2 & ad_f g_2 \end{bmatrix} \\
&= \begin{bmatrix} dT_1 \\ d\mathcal{L}_f(T_1) \\ d\mathcal{L}_f^2(T_1) \\ dT_3 \\ d\mathcal{L}_f(T_3) \end{bmatrix} \begin{bmatrix} g_1 & ad_f g_1 & ad_f^2 g_1 & g_2 & ad_f g_2 \end{bmatrix} \\
&= \begin{bmatrix}
\mathcal{L}_{g_1}(T_1) & \mathcal{L}_{ad_f g_1}(T_1) & \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{g_2}(T_1) & \mathcal{L}_{ad_f g_2}(T_1) \\
\mathcal{L}_{g_1}\mathcal{L}_f(T_1) & \mathcal{L}_{ad_f g_1}(\mathcal{L}_f(T_1)) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f(T_1) & \mathcal{L}_{g_2}\mathcal{L}_f(T_1) & \mathcal{L}_{ad_f g_2}\mathcal{L}_f(T_1) \\
\mathcal{L}_{g_1}\mathcal{L}_f^2(T_1) & \mathcal{L}_{ad_f g_1}(\mathcal{L}_f^2(T_1)) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f^2(T_1) & \mathcal{L}_{g_2}\mathcal{L}_f^2(T_1) & \mathcal{L}_{ad_f g_2}\mathcal{L}_f^2(T_1) \\
\mathcal{L}_{g_1}(T_3) & \mathcal{L}_{ad_f g_1}(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}(T_3) & \mathcal{L}_{g_2}(T_3) & \mathcal{L}_{ad_f g_2}(T_3) \\
\mathcal{L}_{g_1}\mathcal{L}_f(T_3) & \mathcal{L}_{ad_f g_1}\mathcal{L}_f(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}\mathcal{L}_f(T_3) & \mathcal{L}_{g_2}\mathcal{L}_f(T_3) & \mathcal{L}_{ad_f g_2}\mathcal{L}_f(T_3)
\end{bmatrix}.
\end{aligned}$$

Using the identities

$$\begin{aligned}
\mathcal{L}_{ad_f g}(h) &= \mathcal{L}_{[f,g]}(h) = \mathcal{L}_f(\mathcal{L}_g(h)) - \mathcal{L}_g(\mathcal{L}_f(h)) \\
\mathcal{L}_{ad_f^2 g}(h) &= \mathcal{L}_f^2\mathcal{L}_g(h)) - 2\mathcal{L}_f\mathcal{L}_g\mathcal{L}_f(h) + \mathcal{L}_g\mathcal{L}_f^2(h)
\end{aligned}$$

$\dfrac{\partial T^*}{\partial x} C_1$ can be rewritten as

$$\frac{\partial T^*}{\partial x} C_1 = \begin{bmatrix}
0 & 0 & \mathcal{L}_{ad_f^2 g_1}(T_1) & 0 & 0 \\
0 & -\mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f(T_1) & 0 & -\mathcal{L}_{ad_f^2 g_2}(T_1) \\
\mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f g_1}(\mathcal{L}_f^2(T_1)) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f^2(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) & \mathcal{L}_{ad_f g_2}\mathcal{L}_f^2(T_1) \\
0 & \mathcal{L}_{ad_f g}(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}(T_3) & 0 & \mathcal{L}_{ad_f g_2}(T_3) \\
-\mathcal{L}_{ad_f g_1}(T_1) & \mathcal{L}_{ad_f g_1}\mathcal{L}_f(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}\mathcal{L}_f(T_3) & -\mathcal{L}_{ad_f g_2}(T_3) & \mathcal{L}_{ad_f g_2}\mathcal{L}_f(T_3)
\end{bmatrix}. \tag{1.56}$$

We have shown

$$\det\begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ -\mathcal{L}_{ad_f g_1}(T_3) & -\mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix} = -\det\begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ \mathcal{L}_{ad_f g_1}(T_3) & \mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix} \neq 0$$

in a neighborhood of x_0. With

$$B \triangleq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ -\mathcal{L}_{ad_f g_1}(T_3) & -\mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix}^{-1}$$

so

$$\begin{bmatrix} \mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_2}(T_1) \\ -\mathcal{L}_{ad_f g_1}(T_3) & -\mathcal{L}_{ad_f g_2}(T_3) \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = I$$

we multiply both sides of (1.56) by

$$D_1 \triangleq \begin{bmatrix} b_{11} & 0 & 0 & b_{12} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ b_{21} & 0 & 0 & b_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

to obtain

$$\frac{\partial T^*}{\partial x} C_1 D_1 = \begin{bmatrix} 0 & 0 & \mathcal{L}_{ad_f^2 g_1}(T_1) & 0 & 0 \\ 0 & -\mathcal{L}_{ad_f^2 g_1}(T_1) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f(T_1) & 0 & -\mathcal{L}_{ad_f^2 g_2}(T_1) \\ 1 & \mathcal{L}_{ad_f g_1}(\mathcal{L}_f^2(T_1)) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f^2(T_1) & 0 & \mathcal{L}_{ad_f g_2}\mathcal{L}_f^2(T_1) \\ 0 & \mathcal{L}_{ad_f g}(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}(T_3) & 0 & \mathcal{L}_{ad_f g_2}(T_3) \\ 0 & \mathcal{L}_{ad_f g_1}\mathcal{L}_f(T_3) & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}\mathcal{L}_f(T_3) & 1 & \mathcal{L}_{ad_f g_2}\mathcal{L}_f(T_3) \end{bmatrix}.$$

Next multiply this last result by

$$D_2 \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -b_{11} & 0 & 0 & -b_{12} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -b_{21} & 0 & 0 & -b_{22} \end{bmatrix}$$

we have

$$\frac{\partial T^*}{\partial x} C_1 D_1 D_2 = \begin{bmatrix} 0 & 0 & \mathcal{L}_{ad_f^2 g_1}(T_1) & 0 & 0 \\ 0 & 1 & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f(T_1) & 0 & 0 \\ 1 & \times & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_f^2(T_1) & 0 & \times \\ 0 & 0 & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}(T_3) & 0 & 1 \\ 0 & \times & \mathcal{L}_{ad_f^2 g_1}\mathcal{L}_{g_1}\mathcal{L}_f(T_3) & 1 & \times \end{bmatrix}$$

where $\times$ denotes that is doesn't matter what is in those spots. It was shown above that $\mathcal{L}_{ad_f^2 g_1}(T_1) \neq 0$ in a neighborhood of x_0. By inspection the matrix $\frac{\partial T^*}{\partial x} C_1 D_1 D_2$ is invertible. As C_1, D_1, and D_2 are all invertible it follows that $\frac{\partial T^*}{\partial x}$ is invertible showing that $T^*(x)$ is an invertible transformation. ■

1.3 Dynamic Feedback Linearization

What happens if a nonlinear control system is not feedback linearizable? One possibility is to try dynamic feedback linearization. Charlet et al. [3][4][5] have proposed inserting integrators in the input channels to see if the resulting system is feedback linearizable. We begin with an example.

Example 10 [5]

Consider the nonlinear control system given by

$$\frac{dx}{dt} = \underbrace{\begin{bmatrix} x_2 \\ 0 \\ 0 \\ x_3 \end{bmatrix}}_{f} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -x_3 \end{bmatrix}}_{g_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{g_2}.$$

Note that

$$[g_1, g_2] = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \notin span\{g_1, g_2\}$$

showing that $\{g_1, g_2\}$ is *not* involutive and so this system is not feedback linearizable. Add an integrator to the input u_1 by letting $x_5 \triangleq u_1\ dx_5/dt = w_1$ where w_1 is the new input. The extended 5^{th} order control system is then

$$\frac{dx}{dt} = \underbrace{\begin{bmatrix} x_2 \\ 0 \\ 0 \\ x_3 - x_3x_5 \\ 0 \end{bmatrix}}_{\tilde{f}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\tilde{g}_1} w_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\tilde{g}_2} u_2.$$

This system is feedback linearizable by simply checking the conditions.

Example 11 *Addition of an Integrator to the d-axis Input [6][7]*

Using current command in the direct-quadrature coordinate system the induction motor has the model given

$$\frac{d}{dt}\begin{bmatrix} \omega \\ \psi_d \\ \rho \end{bmatrix} = \begin{bmatrix} 0 \\ -\eta\psi_d \\ n_p\omega \end{bmatrix} + \begin{bmatrix} 0 \\ \eta M \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} \mu\psi_d \\ 0 \\ \eta M/\psi_d \end{bmatrix} u_2 + \begin{bmatrix} -1/J \\ 0 \\ 0 \end{bmatrix} \tau_L.$$

where ω is the rotor speed, ψ_d is the field flux, ρ is angle of the field flux, $u_1 = i_{dr}$ commanded direct current and $u_2 = i_{qr}$ is the command quadrature current. With $x_1 \triangleq \omega$, $x_2 \triangleq \psi_d$, $x_3 \triangleq \rho$ this system is rewritten as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\eta x_2 \\ n_p x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \eta M \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} \mu x_2 \\ 0 \\ \eta M/x_2 \end{bmatrix} u_2 + \begin{bmatrix} -1/J \\ 0 \\ 0 \end{bmatrix} \tau_L. \tag{1.57}$$

This is *not* feedback linearizable.

Let's add an integrator to u_1 by setting $x_4 \triangleq u_1$, $dx_4/dt \triangleq v_1$,and along with $v_2 \triangleq u_2$ to obtain the extended system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -\eta x_2 + \eta M x_4 \\ n_p x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v_1 + \begin{bmatrix} \mu x_2 \\ 0 \\ \eta M/x_2 \\ 0 \end{bmatrix} v_2 + \begin{bmatrix} -1/J \\ 0 \\ 0 \\ 0 \end{bmatrix} \tau_L.$$

With τ_L as a constant parameter this may be written compactly as

$$\frac{dx}{dt} = f(x) + g_1 v_1 + g_1 v_2 \tag{1.58}$$

with

$$f(x) = \begin{bmatrix} -\tau_L/J \\ -\eta x_2 + \eta M x_4 \\ n_p x_1 \\ 0 \end{bmatrix},\ g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ g_2 = \begin{bmatrix} \mu x_2 \\ 0 \\ \eta M/x_2 \\ 0 \end{bmatrix} \in \mathbb{R}^4.$$

The nonlinear state-space transformation given by

$$\begin{aligned}
z_1 &= x_2\\
z_2 &= -\eta x_2 + \eta M x_4\\
z_3 &= x_1 - \mu x_2^2 x_3/\eta M\\
z_4 &= \frac{2\mu}{M} x_2 x_3 (x_2 - M x_4) - \frac{\mu n_p}{\eta M} x_2^2 x_1 - \tau_L/J.
\end{aligned}$$

results in

$$\begin{aligned}
\frac{dz_1}{dt} &= z_2\\
\frac{dz_2}{dt} &= a_1(x) + v_1 b_{11} + v_2 b_{12}(x)\\
\frac{dz_3}{dt} &= z_4\\
\frac{dz_4}{dt} &= a_2(x) + v_1 b_{21} + v_2 b_{22}(x)
\end{aligned}$$

where

$$\begin{aligned}
a_1(x) &= -\eta(-\eta x_2 + \eta M x_4)\\
a_2(x) &= -2\eta M \mu x_3 x_4^2 + 6\eta\mu x_2 x_3 x_4 - 4\mu n_p x_1 x_2 x_4 + \frac{\mu n_p}{\eta M J}\tau_L x_2^2 + \frac{1}{M}(-4\eta\mu x_2^2 x_3 + 4\mu n_p x_1 x_2^2)\\
b_{11} &= \eta M,\ b_{12}(x) = 0,\ b_{21} = -2\mu x_2 x_3,\ b_{22}(x) = -2\mu(-\eta x_2 + \eta M x_4) - \frac{\mu^2 n_p}{\eta M} x_2^3.
\end{aligned}$$

Application of the feedback

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^{-1} \begin{bmatrix} w_1 - a_1(x) \\ w_2 - a_2(x) \end{bmatrix}$$

results in two decoupled second-order *linear* systems

$$\begin{aligned}
dz_1/dt &= z_2\\
dz_2/dt &= w_1\\
dz_3/dt &= z_4\\
dz_4/dt &= w_2.
\end{aligned}$$

The controller is singular if

$$\det\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = -2\eta M\mu(-\eta x_2 + \eta M x_4) - \mu^2 n_p x_2^3 = 0.$$

Rewriting this singularity condition in the dq coordinates it becomes

$$-(2\eta M\mu) d\psi_d/dt - \mu^2 n_p \psi_d^3 = 0$$

which is avoided if

$$\frac{d\psi_d}{dt} > -\frac{\mu n_p}{2\eta M}\psi_d^3.$$

ψ_d is easily controlled using the input w_1 to satisfy this condition.

If $x_0 \triangleq \theta$ with $dx_0/dt = x_1 = \omega$ is appended to the model (1.58) then it is straightforward to show the resulting system is *not* feedback linearizable.

1.4 Input-Output Linearization

Following Isidori [8], we first look at input-output linearization for linear systems. Let a system be described by the transfer function

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \tag{1.59}$$

with a statespace realization in control canonical form given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{b} u \tag{1.60}$$

$$y = \underbrace{\begin{bmatrix} b_0 & b_1 & b_2 & 0 & 0 \end{bmatrix}}_{c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}. \tag{1.61}$$

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} c \\ cA \\ cA^2 \\ cA^3 \\ cA^4 \end{bmatrix}$$

$$= \begin{bmatrix} b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \\ -a_0 b_2 & -a_1 b_2 & -a_2 b_2 & b_0 - a_3 b_2 & b_1 - a_4 b_2 \\ -a_0(b_1 - a_4 b_2) & -a_1(b_1 - a_4 b_2) - a_0 b_2 & -a_2(b_1 - a_4 b_2) - a_1 b_2 & -a_3(b_1 - a_4 b_2) - a_2 b_2 & b_0 - a_4(b_1 - a_4 b_2) - a_3 b_2 \end{bmatrix}. \tag{1.62}$$

With $y = h(x) \triangleq cx$, $f(x) = Ax$, and $g = b$ we differentiate the output $h(x) = cx$ until the input appears.

$$\begin{aligned} y &= h(x) = cx \\ dy/dt &= \mathcal{L}_{f+gu} h = \mathcal{L}_f h + u\mathcal{L}_g h = cAx \\ d^2y/dt^2 &= \mathcal{L}_{f+gu}(cAx) = \mathcal{L}_f(cAx) + u\mathcal{L}_g(cAx) = cA^2 x \\ d^3y/dt^3 &= \mathcal{L}_{f+gu}(cAx) = \mathcal{L}_f(cA^2x) + u\mathcal{L}_g(cA^2x) = cA^3x + \underbrace{cA^2g}_{b_2}u. \end{aligned}$$

Let's use this to define a new coordinate system with the first three new coordinates given by

$$z_1 \triangleq cx = b_0 x_1 + b_1 x_2 + b_2 x_3 \tag{1.63}$$

$$z_2 \triangleq cAx = b_0 x_2 + b_1 x_3 + b_2 x_4 \tag{1.64}$$

$$z_3 \triangleq cA^2x = b_0 x_3 + b_1 x_4 + b_2 x_5. \tag{1.65}$$

We need to define two more coordinates for the transformation. Let's take

$$z_4 = x_1 \tag{1.66}$$

$$z_5 = x_2. \tag{1.67}$$

Then

$$\frac{dz_4}{dt} = \frac{dx_1}{dt} = x_2 = z_5$$

and

$$\frac{dz_5}{dt} = \frac{dx_2}{dt} = x_3 = -(b_0/b_2)x_1 - (b_1/b_2)x_2 + (1/b_2)z_1 = -(b_0/b_2)z_4 - (b_1/b_2)z_5 + (1/b_2)z_1$$

where the expression for x_3 is from (1.63). The reason for this choice for z_4, z_5 is that $\mathcal{L}_g(x_1) = 0, \mathcal{L}_g(x_2) = 0$ and the transformation

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \underbrace{\begin{bmatrix} b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}. \tag{1.68}$$

is invertible.

Exercise 3 Explain why T is invertible.

In the new coordinates we have

$$\begin{aligned} \frac{dz_1}{dt} &= z_2 \\ \frac{dz_2}{dt} &= z_3 \\ \frac{dz_3}{dt} &= cA^3T^{-1}z + \underbrace{cA^2g}_{b_2}u \\ \frac{dz_4}{dt} &= z_5 \\ \frac{dz_5}{dt} &= -(b_0/b_2)z_4 - (b_1/b_2)z_5 + (1/b_2)z_1 \end{aligned} \tag{1.69}$$

where $cA^2g = b_2$ and $cA^3T^{-1}z$ has the form

$$cA^3T^{-1}z = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 + \alpha_4 z_4 + \alpha_5 z_5. \tag{1.70}$$

With the state feedback

$$u = -\frac{cA^3T^{-1}z + v}{cA^2g} \tag{1.71}$$

we obtain

$$\begin{aligned} \frac{dz_1}{dt} &= z_2 \\ \frac{dz_2}{dt} &= z_3 \\ \frac{dz_3}{dt} &= v \\ \frac{dz_4}{dt} &= z_5 \\ \frac{dz_5}{dt} &= -(b_0/b_2)z_4 - (b_1/b_2)z_5 + (1/b_2)z_1. \end{aligned} \tag{1.72}$$

Let (z_{d1}, z_{d2}, z_{d3}) and v_d be a reference trajectory and reference input, respectively, satisfying

$$\begin{aligned}\frac{dz_{d1}}{dt} &= z_{d2}\\ \frac{dz_{d2}}{dt} &= z_{d3}\\ \frac{dz_{d3}}{dt} &= v_d.\end{aligned} \tag{1.73}$$

Then, with an appropriate choice for k_1, k_2, and k_3, the control law $u = -k_1(z_{d1} - z_1) - k_2(z_{d2} - z_2) - k_3(z_{d3} - z_3) + v_d$ will force $(z_1, z_2, z_3) \to (z_{d1}, z_{d2}, z_{d3})$. What about the other two state variables z_4, z_5? We have

$$\frac{d}{dt}\begin{bmatrix} z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(b_0/b_2) & -(b_1/b_2) \end{bmatrix}\begin{bmatrix} z_4 \\ z_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/b_2 \end{bmatrix} z_1. \tag{1.74}$$

We are controlling z_1 and so it will be bounded. The system (1.74) is a second-order system driven by the bounded input z_1. The pair of state variables (z_4, z_5) will remain bounded if and only if

$$A_{z_4 z_5} \triangleq \begin{bmatrix} 0 & 1 \\ -(b_0/b_2) & -(b_1/b_2) \end{bmatrix}$$

is stable. The characteristic polynomial of $A_{z_4 z_5}$ is

$$\det(sI - A_{z_4 z_5}) = \begin{bmatrix} s & -1 \\ b_0/b_2 & s + b_1/b_2 \end{bmatrix} = s^2 + (b_1/b_2)s + (b_0/b_2) = (1/b_2)(b_2 s^2 + b_1 s + b_0)$$

from which it is clear that $A_{z_4 z_5}$ is stable if and only the zeros of the transfer function (1.59) are in the open left half-plane! The system of equations (1.74) are referred to as the *zero dynamics*.

Though we only considered a specific example this method will work for any controllable linear system whose zeros are in the open left half-plane. Of course, if a linear system is controllable then we can use state feedback to stabilize the full state whether or not it has right half-plane zeros. So there is no point to using this method to control linear systems. However, it does have use for nonlinear systems.

To get an idea of how input-output linearization is used for nonlinear control systems let's consider the following example from Isidori [8]. Consider the control system

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} u\\ y = h(x) &= x_3.\end{aligned}$$

Then we differentiate the output until the input appears which gives

$$\begin{aligned}dy/dt &= \mathcal{L}_{f+gu}(h) = \mathcal{L}_f h = x_2\\ d^2y/dt^2 &= \mathcal{L}_f^2(h) + \mathcal{L}_g \mathcal{L}_f(h) = x_1 x_2 + u.\end{aligned}$$

We get a new coordinate system by first setting

$$\begin{aligned}z_1 &= y = h = x_3\\ z_2 &= \dot{y} = \mathcal{L}_f h = x_2\end{aligned}$$

so that

$$\begin{aligned}\frac{dz_1}{dt} &= \frac{dx_3}{dt} = x_2 = z_2\\ \frac{dz_2}{dt} &= \frac{dx_2}{dt} = x_1 x_2 + u\end{aligned}$$

We define $z_3 = T_3(x)$ is such a way that

$$\mathcal{L}_g T_3 = 0$$

which means T_3 must satisfy

$$\frac{\partial T_3}{\partial x_1} e^{x_2} + \frac{\partial T_3}{\partial x_2} = 0.$$

Setting $\dfrac{\partial T_3}{\partial x_1} = 1$ and $\dfrac{\partial T_3}{\partial x_2} = -e^{x_2}$ leads to

$$T_3(x) = x_1 - e^{x_2} + 1$$

where the constant 1 is included so that $T_3(0) = 0$. Then with

$$\begin{aligned} z_1 &= T_1(x) = x_3 \\ z_2 &= T_2(x) = x_2 \\ z_3 &= T_3(x) = x_1 - e^{x_2} + 1 \end{aligned}$$

we have

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -e^{x_2} & 0 \end{bmatrix}}_{\frac{\partial T}{\partial x}} \left(\begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} u \right)_{x=T^{-1}(z)} \\ &= \begin{bmatrix} x_2 \\ x_1 x_2 \\ -x_1 - x_1 x_2 e^{x_2} \end{bmatrix}_{x=T^{-1}(z)} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \\ &= \begin{bmatrix} z_2 \\ (z_3 + e^{z_2} - 1) z_2 \\ -(z_3 + e^{z_2} - 1)(1 + z_2 e^{z_2}) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \end{aligned}$$

Letting $u = -(z_3 + e^{z_2} - 1)z_2 + v$ the equations for z_1, z_2 are then

$$\begin{aligned} \frac{dz_1}{dt} &= z_2 \\ \frac{dz_2}{dt} &= v. \end{aligned}$$

The *zero dynamics* are taken to be

$$\frac{dz_3}{dt} = -z_3(1 + z_2 e^{z_2}) - (e^{z_2} - 1)(1 + z_2 e^{z_2})$$

If v is used to force $z_1(t) \to z_{01}$ (constant) with the consequence $z_2(t) \to 0$ then, asymptotically, these zero dynamics become

$$\frac{dz_3}{dt} = -z_3$$

where it is clear that $z_3(t) \to 0$.

Ball and Beam

There is a classic control problem called the ball and beam. As illustrated in Figure 1.2 a ball is put on a beam with a sensor that measures the distance r of the ball from the center of rotation of the beam. A motor provides the capability to apply a torque τ (input) to the beam to control its angle θ with respect to horizontal with $\omega = d\theta/dt$. The control objective is to have the ball kept at a fixed location r_0 or perhaps

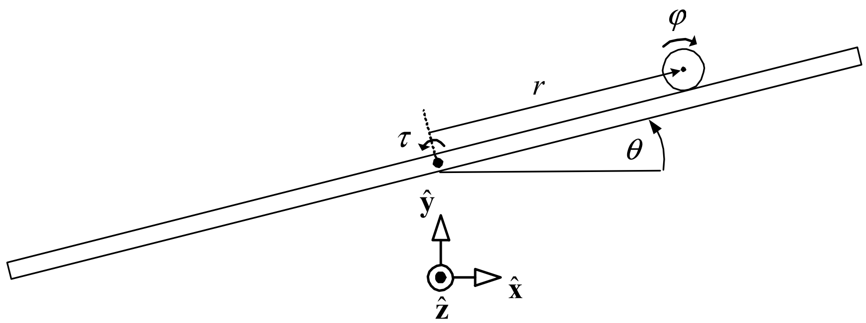

FIGURE 1.2. Ball and beam control system.

have it track a specified trajectory $r_d(t)$. Following [9] we use Lagrange's equations to derive the differential equation model. Let M_b denote the mass of the ball, R denote the radius of the ball, and J denote the moment of inertia of the beam about the $\hat{\boldsymbol{z}}$ axis. Let the axis of rotation of the ball be in the direction of the $\hat{\boldsymbol{z}}$ axis going through its center of mass and let J_b denote its moment of inertia about this axis. Assume the ball rolls without slip so we may write

$$\varphi = r/R.$$

The kinetic energy of the system is

$$K = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}M_b\dot{r}^2 + \frac{1}{2}M_b(r\dot{\theta})^2 + \frac{1}{2}J_b\dot{\varphi}^2 = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}M_b\dot{r}^2 + \frac{1}{2}M_b(r\dot{\theta})^2 + \frac{1}{2}J_b\left(\frac{\dot{r}}{R}\right)^2.$$

The potential energy is[3]

$$V = m\mathfrak{g}r\sin(\theta).$$

The Lagrangian is

$$L = K - V = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}M_b\dot{r}^2 + \frac{1}{2}M_b(r\dot{\theta})^2 + \frac{1}{2}J_b\left(\frac{\dot{r}}{R}\right)^2 - M_b\mathfrak{g}r\sin(\theta)$$

and the corresponding equations of motion are

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= \frac{d}{dt}\left(M_b\dot{r} + \frac{J_b}{R^2}\dot{r}\right) - M_b r\dot{\theta}^2 + M_b\mathfrak{g}\sin(\theta) = 0\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \frac{d}{dt}\left(J\dot{\theta} + M_b r^2\dot{\theta}\right) + M_b\mathfrak{g}r\cos(\theta) = \tau
\end{aligned}$$

or

$$\begin{gathered}
\left(M_b + \frac{J_b}{R^2}\right)\ddot{r} - M_b r\dot{\theta}^2 + M_b\mathfrak{g}\sin(\theta) = 0\\
\left(J + M_b r^2\right)\ddot{\theta} + 2M_b r\dot{r}\dot{\theta} + M_b\mathfrak{g}r\cos(\theta) = \tau.
\end{gathered}$$

Let $x_1 = r, x_2 = dr/dt, x_3 = \theta$, and $x_4 = d\theta/dt$. With $c_1 = \dfrac{M_b}{J_b/R^2 + M_b}$ a statespace model for the ball and beam system is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ c_1 x_1 x_4^2 - c_1\mathfrak{g}\sin(x_3) \\ x_4 \\ -\left(2M_b x_1 x_2 x_4 + rM_b\mathfrak{g}\sin(x_3)\right)/\left(M_b r^2 + J\right) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\left(M_b r^2 + J\right) \end{bmatrix}\tau.$$

[3] We use $\mathfrak{g}$ (fraktur font) for the acceleration due to gravity as g (latin font) is used for the input vector.

Set

$$\tau = 2M_b x_1 x_2 x_4 + rM_b \mathfrak{g}\sin(x_3) + (J + M_b r^2)u$$

so the equations simplify to

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ c_1 x_1 x_4^2 - c_1 \mathfrak{g}\sin(x_3) \\ x_4 \\ 0 \end{bmatrix}}_{f} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{g} u. \tag{1.75}$$

This is the setup considered in Hauser et al [10] where it is pointed out this control system is *not* feedback linearizable.

Exercise 4 Show that the control system 1.75 is not feedback linearizable.

Following [10], let's see why input-output linearization will not work either. To explain, let $y = h(x) = x_1$ and differentiate until the input appears. We have

$$\begin{aligned}
\frac{dy}{dt} &= \mathcal{L}_f h + u\mathcal{L}_g h = x_2 \\
\frac{d^2y}{dt^2} &= \mathcal{L}_f^2 h + u\mathcal{L}_g \mathcal{L}_f h = c_1 x_1 x_4^2 - c_1 \mathfrak{g}\sin(x_3) \\
\frac{d^3y}{dt^3} &= \mathcal{L}_f^3 h + u\mathcal{L}_g \mathcal{L}_f^2 h = c_1 x_2 x_4^2 - c_1 \mathfrak{g} x_4 \cos(x_3) + 2c_1 x_1 x_4 u.
\end{aligned}$$

With $y_1 = y, y_2 = x_2, y_3 = c_1 x_1 x_4^2 - c_1 \mathfrak{g}\sin(x_3)$ and the input

$$u = \frac{-c_1 x_2 x_4^2 + c_1 \mathfrak{g} x_4 \sin(x_3) + v}{2c_1 x_1 x_4}$$

we have

$$\begin{aligned}
\frac{dy_1}{dt} &= y_2 \\
\frac{dy_2}{dt} &= y_3 \\
\frac{dy_3}{dt} &= v.
\end{aligned}$$

The control objective is to keep the ball at some fixed r_0 with $\theta = 0$. However, to use this input-output linearization controller, it is necessary divide by $2c_1 x_1 x_4 = 2c_1 r\dot{\theta}$ so that u is singular exactly where the control is needed. Hauser et al [10] get around this problem by an approach they call *approximate* input-output linearization. Define

$$\begin{aligned}
z_1 &= T_1(x) = x_1 \\
z_2 &= T_2(x) = x_2 \\
z_3 &= T_3(x) = -c_1 \mathfrak{g}\sin(x_3) \\
z_4 &= T_4(x) = -c_1 \mathfrak{g} x_4 \cos(x_3)
\end{aligned}$$

so that

$$
\begin{aligned}
\frac{dz_1}{dt} &= \frac{dx_1}{dt} = x_2 = z_2 \\
\frac{dz_2}{dt} &= \frac{dx_2}{dt} = \underbrace{-c_1\mathfrak{g}\sin(x_3)}_{z_3} + \underbrace{c_1x_1x_4^2}_{\psi_2} = z_3 + \psi_2(x) \\
\frac{dz_3}{dt} &= -c_1\mathfrak{g}x_4\cos(x_3) = z_4 \\
\frac{dz_4}{dt} &= c_1\mathfrak{g}x_4^2\sin(x_3) - c_1\mathfrak{g}\cos(x_3)u
\end{aligned}
$$

The key trick here was to define $z_3 \triangleq -c_1\mathfrak{g}\sin(x_3)$ instead of setting z_3 equal $dz_2/dt = c_1x_1x_4^2 - c_1\mathfrak{g}\sin(x_3)$ so that z_3 does not have x_4. That is, the centripetal term $\psi_2(x) = c_1x_1x_4^2$ is ignored. With

$$
u = \frac{-c_1\mathfrak{g}x_4^2\sin(x_3) + v}{-c_1\mathfrak{g}\cos(x_3)}
$$

The statespace model in the z coordinates is then

$$
\begin{aligned}
\frac{dz_1}{dt} &= z_2 \\
\frac{dz_2}{dt} &= z_3 + \psi_2(z) \\
\frac{dz_3}{dt} &= z_4 \\
\frac{dz_4}{dt} &= v.
\end{aligned}
$$

where $\psi_2(z) \triangleq c_1x_1x_4^2\big|_{x=T^{-1}(z)}$. Using $z_3^2 + z_4^2/x_4 = c_1^2\mathfrak{g}^2$ so that $x_4 = \dfrac{z_4^2}{c_1^2\mathfrak{g}^2 - z_3^2} = \dfrac{z_4^2}{c_1^2\mathfrak{g}^2\cos^2(x_3)}$ showing that x_4 can be written in terms of z_3 and z_4 as long as the beam angle x_3 satisfies $|x_3| < \pi/2$. Then

$$
\psi_2(z) = \left. c_1x_1x_4^2\right|_{x=T^{-1}(z)} = c_1z_1\frac{z_4^4}{(c_1^2\mathfrak{g}^2 - z_3^2)^2}.
$$

Set

$$
z_0 = \begin{bmatrix} r_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

and

$$
v = -k_1(z_1 - r_0) - k_2z_2 - k_3z_3 - k_4z_4
$$

so the closed-loop system is

$$
\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 & -k_4 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_1 \end{bmatrix} r_0 + \begin{bmatrix} 0 \\ c_1z_1\dfrac{z_4^4}{(c_1^2\mathfrak{g}^2 - z_3^2)^2} \\ 0 \\ 0 \end{bmatrix}.
$$

If $z_4 = \omega$ is kept small then $z_4^4 = \omega^4$ is very small and the "perturbation" term $\psi_2(z)$ has little effect on the closed-loop dynamics with the result that $z(t) \to z_0$. Another way to view this is that ignoring the term $\psi_2(z) \triangleq c_1x_1x_4^2\big|_{x=T^{-1}(z)}$ the control system is feedback linearizable.

1.5 Input-Output Linearization Control of the Induction Motor

A statespace model of an n_p pole-pair two-phase induction motor[4] is given by (see Refs. [11][12])

$$\frac{d}{dt}\begin{bmatrix}\omega\\ \psi_{Ra}\\ \psi_{Rb}\\ i_{Sa}\\ i_{Sb}\end{bmatrix} = \begin{bmatrix}\mu\left(i_{Sb}\psi_{Ra}-i_{Sa}\psi_{Rb}\right)\\ -\eta\psi_{Ra}-n_p\omega\psi_{Rb}+\eta M i_{Sa}\\ -\eta\psi_{Rb}+n_p\omega\psi_{Ra}+\eta M i_{Sb}\\ \eta\beta\psi_{Ra}+\beta n_p\omega\psi_{Rb}-\gamma i_{Sa}\\ \eta\beta\psi_{Rb}-\beta n_p\omega\psi_{Ra}-\gamma i_{Sb}\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1/\sigma L_S\\0\end{bmatrix}u_{Sa} + \begin{bmatrix}0\\0\\0\\0\\1/\sigma L_S\end{bmatrix}u_{Sb} + \begin{bmatrix}-1/J\\0\\0\\0\\0\end{bmatrix}\tau_L \quad (1.76)$$

where i_{Sa}, i_{S_b} are the stator phase currents, i_{Ra}, i_{Rb} are the rotor phase currents, θ is the angular position of the rotor, ω is the angular speed of the rotor, J is the rotor's moment of inertia, τ_L is the load torque, L_S is the self-inductance coefficient of the stator phases, L_R is the self-inductance coefficient of the rotor phase, M is the coefficient of mutual inductance and

$$\eta \triangleq \frac{R_R}{L_R}, \beta \triangleq \frac{M}{\sigma L_R L_S}, \mu \triangleq \frac{n_p M}{J L_R}, \gamma \triangleq \frac{M^2 R_R}{\sigma L_R^2 L_S} + \frac{R_S}{\sigma L_S}. \quad (1.77)$$

With $x_1 = \omega, x_2 = \psi_{Ra}, x_3 = \psi_{Rb}, x_4 = i_{Sa}, x_5 = i_{Sb}, u_{Sa} = u_1$, and $u_{Sb} = u_2$ this is written as

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \underbrace{\begin{bmatrix}\mu\left(x_5x_2-x_4x_3\right)\\ -\eta x_2-n_px_1x_3+\eta M x_4\\ -\eta x_3+n_px_1x_2+\eta M x_5\\ \eta\beta x_2+\beta n_px_1x_3-\gamma x_4\\ \eta\beta x_3-\beta n_px_1x_2-\gamma x_5\end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix}0\\0\\0\\1/\sigma L_S\\0\end{bmatrix}}_{g_1}u_1 + \underbrace{\begin{bmatrix}0\\0\\0\\0\\1/\sigma L_S\end{bmatrix}}_{g_2}u_2 + \begin{bmatrix}-1/J\\0\\0\\0\\0\end{bmatrix}\tau_L \quad (1.78)$$

It is straightforward to check that this induction motor model does not satisfy the necessary and sufficient conditions to be feedback linearized. Following [11] consider the input-output linearizing (statespace) transformation given by

$$\begin{aligned}
x_1^* &= T_1(x) = x_1 = \omega\\
x_2^* &= T_2(x) = \mathcal{L}_f T_1 = \mu\left(x_5x_2 - x_4x_3\right) = \mu\left(i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}\right)\\
x_3^* &= T_3(x) = x_2^2 + x_3^2 = \psi_{Ra}^2 + \psi_{Rb}^2\\
x_4^* &= T_4(x) = \mathcal{L}_f T_3 = -2\eta(x_2^2+x_3^2) + 2\eta M(x_2x_4 + x_3x_5) = -2\eta(\psi_{Ra}^2 + \psi_{Rb}^2) + 2\eta M(\psi_{Ra} i_{sa} + \psi_{Rb} i_{Sb})\\
x_5^* &= T_5(x) = \tan^{-1}(x_3, x_2) = \tan^{-1}(\psi_{Rb}, \psi_{Ra})
\end{aligned} \quad (1.79)$$

where $\tan^{-1}(x_3, x_2)$ should be interpreted as $\texttt{atan2}(x_3, x_2)$. Then

$$\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^* - \tau_L/J\\
\frac{dx_2^*}{dt} &= \mathcal{L}_f^2 T_1 + u_1\underbrace{\mathcal{L}_{g_1}\mathcal{L}_f T_1}_{-\mu x_3} + u_2\underbrace{\mathcal{L}_{g_2}\mathcal{L}_f T_1}_{\mu x_2}\\
\frac{dx_3^*}{dt} &= x_4^*\\
\frac{dx_4^*}{dt} &= \mathcal{L}_f^2 T_3 + u_1\underbrace{\mathcal{L}_{g_1}\mathcal{L}_f T_3}_{2\eta M x_2} + u_2\underbrace{\mathcal{L}_{g_2}\mathcal{L}_f T_3}_{2\eta M x_3}\\
\frac{dx_5^*}{dt} &= n_p x_1 + \eta M\frac{x_5x_2 - x_4x_3}{x_2^2 + x_3^2} = n_p x_1^* + \frac{\eta M}{\mu}\frac{x_2^*}{x_3^*}
\end{aligned} \quad (1.80)$$

[4](Or, the two-phase equivalent model of a three-phase motor.)

The input matrix

$$D(x) \triangleq \begin{bmatrix} \mathcal{L}_{g_1}\mathcal{L}_f T_1 & \mathcal{L}_{g_2}\mathcal{L}_f T_1 \\ \mathcal{L}_{g_1}\mathcal{L}_f T_3 & \mathcal{L}_{g_2}\mathcal{L}_f T_3 \end{bmatrix} = \begin{bmatrix} -\mu x_3 & \mu x_2 \\ 2\eta M x_2 & 2\eta M x_3 \end{bmatrix}$$

has determinant

$$\det D = -2\eta M \mu (x_2^2 + x_3^2).$$

So, with $x_3^* = x_2^2 + x_3^2 > 0$ the feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = D^{-1}(x) \begin{bmatrix} -\mathcal{L}_f^2 T_1 + v_1 \\ -\mathcal{L}_f^2 T_3 + v_2 \end{bmatrix}$$

the system of equations (1.80) become

$$\begin{aligned} \frac{dx_1^*}{dt} &= x_2^* - \tau_L/J \\ \frac{dx_2^*}{dt} &= v_1 \\ \frac{dx_3^*}{dt} &= x_4^* \\ \frac{dx_4^*}{dt} &= v_2 \\ \frac{dx_5^*}{dt} &= n_p x_1 + \eta M \frac{x_5 x_2 - x_4 x_3}{x_2^2 + x_3^2} = n_p x_1^* + \frac{\eta M}{\mu}\frac{x_2^*}{x_3^*} \end{aligned} \tag{1.81}$$

where $\dfrac{dx_5^*}{dt}$ was computed as follows:

$$\begin{aligned} \frac{dx_5^*}{dt} = \frac{1}{1 + (x_3/x_2)^2}\frac{d}{dt}(x_3/x_2) &= \frac{1}{1 + (x_3/x_2)^2}\left(\frac{1}{x_2}(-\eta x_3 + n_p x_1 x_2 + \eta M x_5) - \frac{x_3}{x_2^2}(-\eta x_2 - n_p x_1 x_3 + \eta M x_4)\right) \\ &= n_p x_1 + \eta M \frac{x_5 x_2 - x_4 x_3}{x_2^2 + x_3^2}. \end{aligned}$$

The system of equations (1.81) is linear from the two inputs u_1 and u_2 to the two outputs $y_1 = x_1^* = \omega$ and $y_2 = x_3^* = \psi_{Ra}^2 + \psi_{Rb}^2$.The state variable $x_5^* = \tan^{-1}(x_3, x_2) = \tan^{-1}(\psi_{Rb}, \psi_{Ra})$ is an angle and will grow unbounded. However, it is reset every 2π radians so this is not a problem. The fluxes ψ_{Ra}, ψ_{Rb} need to be estimated and this is shown next.

Flux Observer

The dynamic equations for the flux linkages are given by

$$\begin{aligned} \frac{d}{dt}\psi_{Ra} &= -\eta\psi_{Ra} - n_p\omega\psi_{Rb} + \eta M i_{Sa} \\ \frac{d}{dt}\psi_{Rb} &= -\eta\psi_{Rb} + n_p\omega\psi_{Ra} + \eta M i_{Sb}. \end{aligned} \tag{1.82}$$

A straightforward way to estimate the flux linkages ψ_{Ra} and ψ_{Rb} is to simply implement a *real-time* simulation of the equations (1.82) on the controller processor. That is, the currents i_{Sa} and i_{Sb} are sampled from the motor through analog to digital (A/D) converters, the speed ω is known through a sensor and these quantities are then used to run the following *real-time* simulation of the flux linkage equations

$$\begin{aligned} \frac{d}{dt}\hat{\psi}_{Ra} &= -\eta\hat{\psi}_{Ra} - n_p\omega\hat{\psi}_{Rb} + \eta M i_{Sa} \\ \frac{d}{dt}\hat{\psi}_{Rb} &= -\eta\hat{\psi}_{Rb} + n_p\omega\hat{\psi}_{Ra} + \eta M i_{Sb} \end{aligned} \tag{1.83}$$

on the controller processor. The solutions to these equations are then used as the *estimates* of the fluxes for use in the feedback control algorithm. To show the convergence subtract (1.83) from (1.82) to obtain error system

$$\begin{aligned}\dot{\varepsilon}_{Ra} &= -\eta\varepsilon_{Ra} - n_p\omega\varepsilon_{Rb} \\ \dot{\varepsilon}_{Rb} &= -\eta\varepsilon_{Rb} + n_p\omega\varepsilon_{Ra}\end{aligned} \tag{1.84}$$

where $\varepsilon_{Ra} \triangleq \psi_{Ra} - \hat{\psi}_{Ra},\ \varepsilon_{Rb} \triangleq \psi_{Rb} - \hat{\psi}_{Rb}$ are the errors in the estimates.

Consider a (Lyapunov) function defined by

$$V(t) \triangleq \left(\psi_{Ra}(t) - \hat{\psi}_{Ra}(t)\right)^2 + \left(\psi_{Rb}(t) - \hat{\psi}_{Rb}(t)\right)^2 = \varepsilon_{Ra}^2(t) + \varepsilon_{Rb}^2(t).$$

If it can be shown $V(t) \to 0$ as $t \to \infty$, then $\hat{\psi}_{Ra}(t) \to \psi_{Ra}(t),\ \hat{\psi}_{Rb}(t) \to \psi_{Rb}(t)$ as $t \to \infty$. To do this, compute

$$dV/dt = 2\varepsilon_{Ra}\dot{\varepsilon}_{Ra} + 2\varepsilon_{Rb}\dot{\varepsilon}_{Rb} = 2\varepsilon_{Ra}\left(-\eta\varepsilon_{Ra} - n_p\omega\varepsilon_{Rb}\right) + 2\varepsilon_{Rb}\left(-\eta\varepsilon_{Rb} + n_p\omega\varepsilon_{Ra}\right) = -2\eta\left(\varepsilon_{Ra}^2 + \varepsilon_{Rb}^2\right) = -2\eta V.$$

That is, $dV/dt = -2\eta V$ with solution $V(t) = V(0)e^{-2\eta t}$. Now, $V(0) = \left(\psi_{Ra}(0) - \hat{\psi}_{Ra}(0)\right)^2 + \left(\psi_{Rb}(0) - \hat{\psi}_{Rb}(0)\right)^2$ is unknown, but $V(t) \to 0$ regardless of the value of $V(0)$ and thus $\hat{\psi}_{Ra}(t) \to \psi_{Ra}(t),\ \hat{\psi}_{Rb}(t) \to \psi_{Rb}(t)$ as $t \to \infty$ independent of the initial conditions used for (1.83).

1.6 Nonlinear State Observers with Linear Error Dynamics

Recall the approach for state estimation in linear time-invariant systems. With a single-input single-output (SISO) linear time invariant system given by

$$\begin{aligned}\frac{dx}{dt} &= Ax + bu, \quad A \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^n \\ y &= cx, \quad c \in \mathbb{R}^{1\times n}\end{aligned}$$

an observer is defined by

$$\begin{aligned}\frac{d\hat{x}}{dt} &= A\hat{x} + bu + \ell(y - \hat{y}), \quad \ell \in \mathbb{R}^n \\ \hat{y} &= c\hat{x}.\end{aligned}$$

The idea here is that the observer puts out the value $\hat{x}(t)$ and uses it to predict the value of $\hat{y}(t) = c\hat{x}(t)$. Then, as $y(t) = cx(t)$ is measured, the observer uses the difference (error) $y(t) - \hat{y}(t) = cx(t) - c\hat{x}(t)$ to adjust the state estimate $\hat{x}(t)$ through $\ell(y(t) - \hat{y}(t))$. Specifically, let

$$\epsilon(t) \triangleq x(t) - \hat{x}(t)$$

which has the dynamics

$$\frac{d\epsilon}{dt} = Ax + bu - (A\hat{x} + bu + \ell(y - \hat{y})) = A(x - \hat{x}) + \ell c(x - \hat{x}) = (A - \ell c)\epsilon(t).$$

If ℓ can be chosen so that $A - \ell c$ is stable, then $\epsilon(t) \to 0$. That is, the state estimate $\hat{x}(t)$ goes to $x(t)$ for any initial condition $x(0)$. To see how ℓ can be chosen to make $A - \ell c$ stable let's consider an example where $A \in \mathbb{R}^{4\times 4}$ and $c \in \mathbb{R}^{1\times 4}$ have the special form

$$c = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix}.$$

A calculation shows $\det(sI - A) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$. Set $\ell = \begin{bmatrix} \ell_0 & \ell_1 & \ell_2 & \ell_4 \end{bmatrix}^T$ so

$$A - \ell c = \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix} - \begin{bmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \\ \ell_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\ell_0 - a_0 \\ 1 & 0 & 0 & -\ell_1 - a_1 \\ 0 & 1 & 0 & -\ell_2 - a_2 \\ 0 & 0 & 1 & -\ell_2 - a_3 \end{bmatrix}.$$

Let $\ell = \begin{bmatrix} \alpha_0 - a_0 & \alpha_1 - a_1 & \alpha_2 - a_2 & \alpha_3 - a_3 \end{bmatrix}^T$ to obtain

$$A - \ell c = \begin{bmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{bmatrix}$$

with $\det(sI - (A - \ell c)) = s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$. That is, the coefficients of $\det(sI - (A - \ell c))$ can be arbitrarily assigned. This special form of the pair (c, A) is called the *observer canonical* form.

Let's now show the general procedure for choosing the observer gain ℓ for arbitrary $c \in \mathbb{R}^{1\times 4}$ and $A \in \mathbb{R}^{4\times 4}$. (This procedure will be used to motivate the approach for nonlinear systems.) In order to assign the eigenvalues of $A - \ell c$ arbitrarily we must assume the pair (c, A) is observable, that is, the matrix

$$\mathcal{O} \triangleq \begin{bmatrix} c \\ cA \\ cA^2 \\ cA^3 \end{bmatrix} \tag{1.85}$$

is nonsingular. Let $\det(sI - A) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$ be the characteristic polynomial of A. Choose $q \in \mathbb{R}^4$ to be

$$q \triangleq \mathcal{O}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.86}$$

That is, q is the last column of $\mathcal{O}^{-1}$. Define a change of coordinates

$$x^* \triangleq \underbrace{\begin{bmatrix} q & Aq & A^2q & A^3q \end{bmatrix}^{-1}}_{T} x \tag{1.87}$$

or

$$x = \underbrace{\begin{bmatrix} q & Aq & A^2q & A^3q \end{bmatrix}}_{T^{-1}} x^*. \tag{1.88}$$

That is, x^* is a representation of x with respect to the basis vectors $\{q, Aq, A^2q, A^3q\}$.

The system of equations

$$\begin{aligned} \frac{dx}{dt} &= Ax + bu, \quad A \in \mathbb{R}^{4\times 4}, \quad b \in \mathbb{R}^4 \\ y &= cx, \quad c \in \mathbb{R}^{1\times 4} \end{aligned}$$

become

$$\begin{aligned} \frac{dx^*}{dt} &= \underbrace{TAT^{-1}}_{A_o} x^* + \underbrace{Tb}_{b_o} u, \\ y &= \underbrace{cT^{-1}}_{c_o} x^*. \end{aligned}$$

Note that A_o must satisfy

$$T^{-1}A_o = AT^{-1}$$

or

$$\left[\begin{array}{cccc} q & Aq & A^2q & A^3q \end{array}\right] A_o = A\left[\begin{array}{cccc} q & Aq & A^2q & A^3q \end{array}\right] = \left[\begin{array}{cccc} Aq & A^2q & A^3q & A^4q \end{array}\right].$$

By inspection A_o has the form

$$A_o = \begin{bmatrix} 0 & 0 & 0 & a_{o14} \\ 1 & 0 & 0 & a_{o24} \\ 0 & 1 & 0 & a_{o34} \\ 0 & 0 & 1 & a_{o44} \end{bmatrix}$$

where $a_{o14}, a_{o24}, a_{o34}, a_{o44}$ must be found to satisfy

$$a_{o14}q + a_{o24}Aq + a_{o34}A^2q + a_{o44}A^3q = A^4q.$$

By the Cayley-Hamilton theorem A satisfies $A^4 + a_3A^3 + a_2A^2 + a_1A + a_0I = 0_{4\times 4}$ from which it follows that

$$A^4q + a_3A^3q + a_2A^2q + a_1Aq + a_0q = 0_4.$$

That is, $a_{o14} = -a_0, a_{o24} = -a_1, a_{o34} = -a_2, a_{o44} = -a_3$ so that

$$\left[\begin{array}{cccc} q & Aq & A^2q & A^3q \end{array}\right] \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix} = A\left[\begin{array}{cccc} q & Aq & A^2q & A^3q \end{array}\right].$$

Further as

$$\mathcal{O}q \triangleq \begin{bmatrix} c \\ cA \\ cA^2 \\ cA^3 \end{bmatrix} q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

we also have

$$cT^{-1} = c\left[\begin{array}{cccc} q & Aq & A^2q & A^3q \end{array}\right] = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right].$$

In the x^* coordinates the model is

$$\begin{aligned} \frac{dx^*}{dt} &= \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix} x^* + b_o u \\ y &= \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right] x^*. \end{aligned}$$

We rewrite this as

$$\begin{aligned} \frac{dx^*}{dt} &= \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_o} x^* + \underbrace{\begin{bmatrix} -a_0y \\ -a_1y \\ -a_2y \\ -a_3y \end{bmatrix}}_{\varphi(y)} + b_o u \\ y &= \underbrace{\left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right]}_{c_o} x^*. \end{aligned}$$

Remark Another way to look at the transformation (1.88) being generated by the basis $\{q, Aq, A^2q, A^3q\}$ is as follows. We have $\phi^{(1)}_{t_1}(x_0) = qt_1 + x_0$ is the solution to $dx/dt_1 = q, x(0) = x_0$, $\phi^{(2)}_{t_2}(x'_0) = Aqt_2 + x'_0$ is the solution to $dx/dt_2 = Aq, x(0) = x_0''$, $\phi_{t_3}^{(3)}(x_0'') = A^2qt_3 + x_0''$ is the solution to $dx/dt_3 = A^2q, x(0) = x_0''$, and $\phi_{t_4}^{(4)}(x_0''') = A^3qt_4 + x_0'''$ is the solution to $dx/dt_4 = A^3q, x(0) = x_0'''$. Then define

$$x(t_1, t_2, t_3, t_4) \triangleq \phi_{t_4}^{(4)}(\phi_{t_3}^{(3)}(\phi_{t_2}^{(2)}(\phi_{t_1}^{(1)}(x_0)))), \quad x(0,0,0,0) = x_0$$

where it is easy to see that

$$\phi_{t_4}^{(4)}(\phi_{t_3}^{(3)}(\phi_{t_2}^{(2)}(\phi_{t_1}^{(1)}(x_0)))) = qt_1 + qA^2t_2 + qA^3t_3 + qA^4t_4 = \begin{bmatrix} q & Aq & A^2q & A^3q \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}.$$

Replacing $t = (t_1, t_2, t_3, t_4)$ by $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ we obtain (1.88).

Nonlinear Observers

Consider the nonlinear control system defined in a neighborhood $\mathcal{U}$ containing $x = 0$ given by[5]

$$\begin{aligned} \frac{dx}{dt} &= f(x) + g(x)u, \quad f(x), g(x) \in \mathbb{R}^n \\ y &= h(x) \in \mathbb{R}, \quad h(0) = 0. \end{aligned}$$

We now develop conditions for which an observer with linear error dynamics can be designed for this system. We follow the presentation given in [13].

Definition 3 *Local Observability*
The system (without a control input) given by

$$\begin{aligned} \frac{dx}{dt} &= f(x), \ x \in \mathbb{R}^n \\ y &= h(x) \in \mathbb{R}, \quad h(0) = 0 \end{aligned} \tag{1.89}$$

is *locally observable* in a neighborhood $\mathcal{U}$ containing $x = 0$ if

$$\mathcal{O} \triangleq \begin{bmatrix} dh \\ d\mathcal{L}_f h \\ \vdots \\ d\mathcal{L}_f^{n-1} h \end{bmatrix} = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial \mathcal{L}_f h}{\partial x_1} & \frac{\partial \mathcal{L}_f h}{\partial x_2} & \cdots & \frac{\partial \mathcal{L}_f h}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \mathcal{L}_f^{n-1} h}{\partial x_1} & \frac{\partial \mathcal{L}_f^{n-1} h}{\partial x_2} & \cdots & \frac{\partial \mathcal{L}_f^{n-1} h}{\partial x_n} \end{bmatrix} \tag{1.90}$$

has rank n for all $x \in \mathcal{U}$.

[5] As usual, it is always assumed the vector fields $f(x), g(x)$ and the output function $h(x)$ are smooth (their derivatives of all orders exist in the neighborhood $\mathcal{U}$).

Example 12 *State Estimator*

Let the system (1.89) be locally observable in a neighborhood of $x = 0$. Then, as (1.90) has full rank in a neighborhood of $x = 0$, by the inverse function theorem the transformation

$$
\begin{aligned}
x_1^* &= h(x) \\
x_2^* &= \mathcal{L}_f h(x) \\
x_2^* &= \mathcal{L}_f^2 h(x) \\
\vdots &= \ \vdots \\
x_n^* &= \mathcal{L}_f^{n-1} h(x)
\end{aligned}
\tag{1.91}
$$

is invertible in a neighborhood of $x = 0$. This transformation and its inverse depend only the known functions $h(x), f(x)$ and therefore the full state is (theoretically) computable from the output. The idea here is to measure $y(t)$ and calculate $\dot{y}, \ddot{y}, ...y^{(n-1)}$ so that inverting

$$
\begin{aligned}
y &= h(x(t)) \\
\dot{y} &= \mathcal{L}_f h(x(t)) \\
\ddot{y} &= \mathcal{L}_f^2 h(x(t)) \\
\vdots &= \ \vdots \\
y^{(n-1)} &= \mathcal{L}_f^{n-1} h(x(t))
\end{aligned}
\tag{1.92}
$$

would give $x(t)$. Even if the inverse of (1.91) was found, the calculation of the derivatives $\dot{y}, \ddot{y}, ...y^{(n-1)}$ can result in them being too noisy for this approach to work..

Theorem 4 *Nonlinear Observers with Linear Error Dynamics*

In a neighborhood $\mathcal{U}$ of $\mathbb{R}^4$ containing $x = 0$ consider the nonlinear control system[6]

$$
\begin{aligned}
\frac{dx}{dt} &= f(x) + g(x)u, \quad f(x), g(x) \in \mathbb{R}^4 \\
y &= h(x) \in \mathbb{R}, \quad h(0) = 0.
\end{aligned}
\tag{1.93}
$$

Then there exists an invertible transformation

$$
x^* = T(x) \in \mathbb{R}^4, \quad T(0) = 0 \tag{1.94}
$$

such that

$$
\frac{dx^*}{dt} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_o} x^* + \underbrace{\begin{bmatrix} \varphi_1(y) \\ \varphi_2(y) \\ \varphi_3(y) \\ \varphi_4(y) \end{bmatrix}}_{\varphi(y)} + g^*(y)u \tag{1.95}
$$

$$
y = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}}_{c_o} x^* \tag{1.96}
$$

if and only if for all x in a neighborhood $\mathcal{U}$ containing $x = 0$ the following three conditions hold.

[6] For expository reasons we take $n = 4$.

(1) With $u = 0$ the system is locally observable, that is,

$$rank \underbrace{\begin{bmatrix} dh \\ d\mathcal{L}_f h \\ d\mathcal{L}_f^2 h \\ d\mathcal{L}_f^3 h \end{bmatrix}}_{\mathcal{O}} = 4. \tag{1.97}$$

(2) With

$$q \triangleq \begin{bmatrix} dh \\ d\mathcal{L}_f h \\ d\mathcal{L}_f^2 h \\ d\mathcal{L}_f^3 h \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{so} \quad \underbrace{\begin{bmatrix} dh \\ d\mathcal{L}_f h \\ d\mathcal{L}_f^2 h \\ d\mathcal{L}_f^3 h \end{bmatrix}}_{\mathcal{O}} q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.98}$$

the set of vector fields

$$\left\{q, ad_f q, ad_f^2 q, ad_f^3 q\right\}$$

have Lie brackets equal to 0, i.e.,

$$[q, ad_f q] = 0, [q, ad_f^2 q] = 0, [q, ad_f^3 q] = 0, [ad_f q, ad_f^2 q] = 0, [ad_f q, ad_f^3 q] = 0, [ad_f^2 q, ad_f^3 q] = 0. \tag{1.99}$$

(3) And

$$[g, ad_f^i q] = 0 \ \text{ for } \ i = 0, 1, 2. \tag{1.100}$$

Proof. First we show the set of vectors $\left\{q, ad_f q, ad_f^2 q, ad_f^3 q\right\}$ are linearly independent. Recall from Exercise **??** of Chapter **??** that

$$\begin{aligned}
\mathcal{L}_{ad_f q}(h) &= \mathcal{L}_f(\mathcal{L}_q(h)) - \mathcal{L}_q(\mathcal{L}_f(h)) \\
\mathcal{L}_{ad_f^2 q}(h) &= \mathcal{L}_f^2 \mathcal{L}_q(h)) - 2\mathcal{L}_f \mathcal{L}_q \mathcal{L}_f(h) + \mathcal{L}_q \mathcal{L}_f^2(h) \\
\mathcal{L}_{ad_f^3 q}(h) &= \mathcal{L}_f^3 \mathcal{L}_q(h)) - 3\mathcal{L}_f^2 \mathcal{L}_q \mathcal{L}_f(h) + 3\mathcal{L}_f \mathcal{L}_q \mathcal{L}_f^2(h) - \mathcal{L}_q \mathcal{L}_f^3(h).
\end{aligned}$$

Using these expressions along with the right hand side of 1.98 we have

$$\begin{aligned}
\begin{bmatrix} dh \\ d\mathcal{L}_f h \\ d\mathcal{L}_f^2 h \\ d\mathcal{L}_f^3 h \end{bmatrix} \begin{bmatrix} q & ad_f q & ad_f^2 q & ad_f^3 q \end{bmatrix} &= \begin{bmatrix} \mathcal{L}_q h & \mathcal{L}_{ad_f q} h & \mathcal{L}_{ad_f^2 q} h & \mathcal{L}_{ad_f^3 q} h \\ \mathcal{L}_q \mathcal{L}_f h & \mathcal{L}_{ad_f q} \mathcal{L}_f h & \mathcal{L}_{ad_f^2 q} \mathcal{L}_f h & \mathcal{L}_{ad_f^3 q} \mathcal{L}_f h \\ \mathcal{L}_q \mathcal{L}_f^2 h & \mathcal{L}_{ad_f q} \mathcal{L}_f^2 h & \mathcal{L}_{ad_f^2 q} \mathcal{L}_f^2 h & \mathcal{L}_{ad_f^3 q} \mathcal{L}_f^2 h \\ \mathcal{L}_q \mathcal{L}_f^3 h & \mathcal{L}_{ad_f q} \mathcal{L}_f^3 h & \mathcal{L}_{ad_f^2 q} \mathcal{L}_f^3 h & \mathcal{L}_{ad_f^3 q} \mathcal{L}_f^3 h \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & \mathcal{L}_{ad_f^3 q} \mathcal{L}_f h \\ 0 & -1 & \mathcal{L}_{ad_f^2 q} \mathcal{L}_f^2 h & \mathcal{L}_{ad_f^3 q} \mathcal{L}_f^2 h \\ 1 & \mathcal{L}_{ad_f q} \mathcal{L}_f^3 h & \mathcal{L}_{ad_f^2 q} \mathcal{L}_f^3 h & \mathcal{L}_{ad_f^3 q} \mathcal{L}_f^3 h \end{bmatrix}.
\end{aligned} \tag{1.101}$$

As $\mathcal{O}$ is invertible and the matrix (1.101) is invertible it follows that $\begin{bmatrix} q & ad_f q & ad_f^2 q & ad_f^3 q \end{bmatrix}$ is invertible, equivalently, the set of vectors $\left\{q, ad_f q, ad_f^2 q, ad_f^3 q\right\}$ are linearly independent. For later reference note that a similar computation using the vector fields $\left\{q, ad_{(-f)}q, ad^2_{(-f)}q, ad^3_{(-f)}q\right\}$ shows that

$$\begin{bmatrix} dh \\ d\mathcal{L}_f h \\ d\mathcal{L}_f^2 h \\ d\mathcal{L}_f^3 h \end{bmatrix} \begin{bmatrix} q & ad_{(-f)}q & ad^2_{(-f)}q & ad^3_{(-f)}q \end{bmatrix} = \begin{bmatrix} \mathcal{L}_q h & \mathcal{L}_{ad_{(-f)}q} h & \mathcal{L}_{ad^2_{(-f)}q} h & \mathcal{L}_{ad^3_{(-f)}q} h \\ \mathcal{L}_q \mathcal{L}_f h & \mathcal{L}_{ad_{(-f)}q} \mathcal{L}_f h & \mathcal{L}_{ad^2_{(-f)}q} \mathcal{L}_f h & \mathcal{L}_{ad^3_{(-f)}q} \mathcal{L}_f h \\ \mathcal{L}_q \mathcal{L}_f^2 h & \mathcal{L}_{ad_{(-f)}q} \mathcal{L}_f^2 h & \mathcal{L}_{ad^2_{(-f)}q} \mathcal{L}_f^2 h & \mathcal{L}_{ad^3_{(-f)}q} \mathcal{L}_f^2 h \\ \mathcal{L}_q \mathcal{L}_f^3 h & \mathcal{L}_{ad_{(-f)}q} \mathcal{L}_f^3 h & \mathcal{L}_{ad^2_{(-f)}q} \mathcal{L}_f^3 h & \mathcal{L}_{ad^3_{(-f)}q} \mathcal{L}_f^3 h \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \mathcal{L}_{ad^3_{(-f)}q} \mathcal{L}_f h \\ 0 & 1 & \mathcal{L}_{ad^2_{(-f)}q} \mathcal{L}_f^2 h & \mathcal{L}_{ad^3_{(-f)}q} \mathcal{L}_f^2 h \\ 1 & \mathcal{L}_{ad_{(-f)}q} \mathcal{L}_f^3 h & \mathcal{L}_{ad^2_{(-f)}q} \mathcal{L}_f^3 h & \mathcal{L}_{ad^3_{(-f)}q} \mathcal{L}_f^3 h \end{bmatrix}. \tag{1.102}$$

In the linear case given in (1.88) with $f = Ax$ and the constant vector q, the vector fields $\begin{bmatrix} q & ad_f q & ad_f^2 q & ad_f^3 q \end{bmatrix}$ would be $\begin{bmatrix} q & -Aq & A^2 q & -A^3 q \end{bmatrix}$ due to the way the Lie bracket is defined. However we want $\begin{bmatrix} q & Aq & A^2 q & A^3 q \end{bmatrix}$ so we now start with the vector fields $\begin{bmatrix} q & ad_{(-f)}q & ad^2_{(-f)}q & ad^3_{(-f)}q \end{bmatrix}$, i.e., f is replaced by $-f$. It should be clear that if the conditions (1), (2), and (3) above hold for f then they also hold $-f$ and vice-versa. As in Chapter **??** on page **??** we want to create a special coordinate system using the vector fields $\begin{bmatrix} q & ad_{(-f)}q & ad^2_{(-f)}q & ad^3_{(-f)}q \end{bmatrix}$. To do so let $\phi^{(1)}_{t_1}(x_0)$ be the solution to $dx/dt_1 = q, x(0) = x_0$, $\phi^{(2)}_{t_2}(x_0')$ be the solution to $dx/dt_2 = ad_{(-f)}q, x(0) = x_0'$, $\phi^{(3)}_{t_3}(x_0'')$ be the solution to $dx/dt_3 = ad^2_{(-f)}q, x(0) = x_0''$, and $\phi^{(4)}_{t_4}(x_0''')$ be the solution to $dx/dt_4 = ad^3_{(-f)}q, x(0) = x_0'''$. Then define the nonlinear transformation[7]

$$x(t_1, t_2, t_3, t_4) \triangleq \phi^{(4)}_{t_4}(\phi^{(3)}_{t_3}(\phi^{(2)}_{t_2}(\phi^{(1)}_{t_1}(x_0)))), \quad x(0,0,0,0) = x_0. \tag{1.103}$$

Its Jacobian at $t = (0,0,0,0)$ is given by

$$\frac{\partial x}{\partial t}_{|t=0} = \begin{bmatrix} q & ad_{(-f)}q & ad^2_{(-f)}q & ad^3_{(-f)}q \end{bmatrix}_{|x_0}$$

which is full rank. Further, by the conditions (1.99), it follows that

$$\frac{\partial x}{\partial t} = \begin{bmatrix} \frac{\partial x}{\partial t_1} & \frac{\partial x}{\partial t_2} & \frac{\partial x}{\partial t_3} & \frac{\partial x}{\partial t_4} \end{bmatrix} = \begin{bmatrix} q & ad_{(-f)}q & ad^2_{(-f)}q & ad^3_{(-f)}q \end{bmatrix}$$

for *all* $t = (t_1, t_2, t_3, t_4)$ in a neighborhood of $(0,0,0,0)$. Let

$$\begin{aligned} x_1^* &\triangleq t_1 = T_1(x) \\ x_2^* &\triangleq t_2 = T_2(x) \\ x_3^* &\triangleq t_3 = T_3(x) \\ x_4^* &\triangleq t_4 = T_4(x) \end{aligned}$$

denote the inverse transformation of (1.103) which is valid in an open neighborhood of x_0. We now take

[7] This transformation is a generalization of the transformation in (1.88) for an observable linear system where $x^* = \begin{bmatrix} x_1^* & x_2^* & x_3^* & x_4^* \end{bmatrix}^T$ in (1.88) corresponds to the coordinates $\begin{bmatrix} t_1 & t_2 & t_3 & t_4 \end{bmatrix}^T$ in (1.103).

$x_0 = 0$ so that in a neighborhood of $x_0 = 0$ we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} & \frac{\partial T_1}{\partial x_4} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} & \frac{\partial T_2}{\partial x_3} & \frac{\partial T_2}{\partial x_4} \\ \frac{\partial T_3}{\partial x_1} & \frac{\partial T_3}{\partial x_2} & \frac{\partial T_3}{\partial x_3} & \frac{\partial T_3}{\partial x_4} \\ \frac{\partial T_4}{\partial x_1} & \frac{\partial T_4}{\partial x_2} & \frac{\partial T_4}{\partial x_3} & \frac{\partial T_4}{\partial x_4} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \frac{\partial x_1}{\partial t_3} & \frac{\partial x_1}{\partial t_4} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & \frac{\partial x_2}{\partial t_3} & \frac{\partial x_2}{\partial t_4} \\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} & \frac{\partial x_3}{\partial t_3} & \frac{\partial x_3}{\partial t_4} \\ \frac{\partial x_4}{\partial t_1} & \frac{\partial x_4}{\partial t_2} & \frac{\partial x_4}{\partial t_3} & \frac{\partial x_4}{\partial t_4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} & \frac{\partial T_1}{\partial x_4} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} & \frac{\partial T_2}{\partial x_3} & \frac{\partial T_2}{\partial x_4} \\ \frac{\partial T_3}{\partial x_1} & \frac{\partial T_3}{\partial x_2} & \frac{\partial T_3}{\partial x_3} & \frac{\partial T_3}{\partial x_4} \\ \frac{\partial T_4}{\partial x_1} & \frac{\partial T_4}{\partial x_2} & \frac{\partial T_4}{\partial x_3} & \frac{\partial T_4}{\partial x_4} \end{bmatrix} \begin{bmatrix} q & ad_{(-f)}q & ad^2_{(-f)}q & ad^3_{(-f)}q \end{bmatrix}.$$

Recalling Equation (??) of Chapter ?? (page ??) we see that in the x^* coordinates q is given by $q^* = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$ad_{(-f)}q$ by $ad_{(-f^*)}q^* = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $ad^2_{(-f)}q$ by $ad^2_{(-f^*)}q^* = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $ad^3_{(-f)}q$ by $ad^3_{(-f^*)}q^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

That is, the transformation of the basis vector field in the x coordinates to the basis vector field in the x^* coordinates is given by[8]

$$\frac{\partial x^*}{\partial x} \begin{bmatrix} q & ad_{(-f)}q & ad^2_{(-f)}q & ad^3_{(-f)}q \end{bmatrix} = \begin{bmatrix} q^* & ad_{(-f^*)}q^* & ad^2_{(-f^*)}q^* & ad^3_{(-f^*)}q^* \end{bmatrix}.$$

We now compute the components of f in the x^* coordinate system, that is, $f^* = \frac{\partial T}{\partial x} f = \frac{\partial x^*}{\partial x} f$ and show f^* has the form given in (1.95). Following [13] we compute

$$ad_{(-f^*)}q^* = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = [-f^*, q^*] = -\frac{\partial(-f^*)}{\partial x^*} q^* = \begin{bmatrix} \frac{\partial f_1^*}{\partial x_1^*} \\ \frac{\partial f_2^*}{\partial x_1^*} \\ \frac{\partial f_3^*}{\partial x_1^*} \\ \frac{\partial f_4^*}{\partial x_1^*} \end{bmatrix}$$

$$ad^2_{(-f^*)}q^* = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \left[-f^*, ad_{(-f^*)}q^*\right] = -\frac{\partial(-f^*)}{\partial x^*} ad_{(-f^*)}q^* = \begin{bmatrix} \frac{\partial f_1^*}{\partial x_2^*} \\ \frac{\partial f_2^*}{\partial x_2^*} \\ \frac{\partial f_3^*}{\partial x_2^*} \\ \frac{\partial f_4^*}{\partial x_2^*} \end{bmatrix}$$

[8] Recall the notation $\frac{\partial x^*}{\partial x} \triangleq \frac{\partial T}{\partial x}$

$$ad^3_{(-f^*)}q^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \left[-f^*, ad^2_{(-f^*)}q^*\right] = -\frac{\partial(-f^*)}{\partial x^*}ad^2_{(-f^*)}q^* = \begin{bmatrix} \dfrac{\partial f_1^*}{\partial x_3^*} \\ \dfrac{\partial f_2^*}{\partial x_3^*} \\ \dfrac{\partial f_3^*}{\partial x_3^*} \\ \dfrac{\partial f_4^*}{\partial x_3^*} \end{bmatrix}$$

Putting this together we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1^*}{\partial x_1^*} & \dfrac{\partial f_1^*}{\partial x_2^*} & \dfrac{\partial f_1^*}{\partial x_3^*} \\ \dfrac{\partial f_2^*}{\partial x_1^*} & \dfrac{\partial f_2^*}{\partial x_2^*} & \dfrac{\partial f_2^*}{\partial x_3^*} \\ \dfrac{\partial f_3^*}{\partial x_1^*} & \dfrac{\partial f_3^*}{\partial x_2^*} & \dfrac{\partial f_3^*}{\partial x_3^*} \\ \dfrac{\partial f_4^*}{\partial x_1^*} & \dfrac{\partial f_4^*}{\partial x_2^*} & \dfrac{\partial f_4^*}{\partial x_3^*} \end{bmatrix}$$

or

$$\begin{bmatrix} f_1^* \\ f_2^* \\ f_3^* \\ f_4^* \end{bmatrix} = \begin{bmatrix} \varphi_1(x_4^*) \\ x_1^* + \varphi_2(x_4^*) \\ x_2^* + \varphi_3(x_4^*) \\ x_3^* + \varphi_4(x_4^*) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x^* + \begin{bmatrix} \varphi_1(x_4^*) \\ \varphi_2(x_4^*) \\ \varphi_3(x_4^*) \\ \varphi_4(x_4^*) \end{bmatrix}$$

We next transform the input vector field $g(x)$ to the x^* coordinate system, that is, we find $g^* = \dfrac{\partial T}{\partial x} g = \dfrac{\partial x^*}{\partial x} g$. First note that

$$\begin{aligned} \frac{\partial x^*}{\partial x}\begin{bmatrix} [g,q] & [g, ad_{(-f)}q] & [g, ad^2_{(-f)}q] & [g, ad^3_{(-f)}q] \end{bmatrix} &= \begin{bmatrix} [g^*, q^*] & [g^*, ad_{(-f^*)}q^*] & [g^*, ad^2_{(-f^*)}q^*] & [g^*, ad^3_{(-f^*)}q^*] \end{bmatrix} \\ &= \begin{bmatrix} 0_{n\times 1} & 0_{n\times 1} & 0_{n\times 1} & [g^*, ad^3_{(-f^*)}q^*] \end{bmatrix} \end{aligned}$$

where the last step follows by condition (3) of the theorem. Again following [13] we compute

$$[g^*, q^*] = -\frac{\partial g^*}{\partial x^*}q^* = -\begin{bmatrix} \dfrac{\partial g_1^*}{\partial x_1^*} \\ \dfrac{\partial g_2^*}{\partial x_1^*} \\ \dfrac{\partial g_3^*}{\partial x_1^*} \\ \dfrac{\partial g_4^*}{\partial x_1^*} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [g^*, ad_{(-f^*)}q^*] = -\frac{\partial g^*}{\partial x^*}ad_{(-f^*)}q^* = -\begin{bmatrix} \dfrac{\partial g_1^*}{\partial x_2^*} \\ \dfrac{\partial g_2^*}{\partial x_2^*} \\ \dfrac{\partial g_3^*}{\partial x_2^*} \\ \dfrac{\partial g_4^*}{\partial x_2^*} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$[g^*, ad^2_{(-f^*)}q^*] = -\frac{\partial g^*}{\partial x^*}ad^2_{(-f^*)}q^* = -\begin{bmatrix} \dfrac{\partial g_1^*}{\partial x_3^*} \\ \dfrac{\partial g_2^*}{\partial x_3^*} \\ \dfrac{\partial g_3^*}{\partial x_3^*} \\ \dfrac{\partial g_4^*}{\partial x_3^*} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

These computations show that g^* is only a function of x_4^* so we may write it as $g^*(x_4^*)$.

To find the output equation $h^*(x^*) = h(x)_{|x=T^{-1}(x^*)}$ we use (1.102) which shows that

$$dh \begin{bmatrix} q & ad_{(-f)}q & ad^2_{(-f)}q & ad^3_{(-f)}q \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Lie derivative is invariant under a change of coordinates so[9]

$$dh^* \begin{bmatrix} q^* & ad_{(-f^*)}q^* & ad^2_{(-f^*)}q^* & ad^3_{(-f^*)}q^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} \dfrac{\partial h^*}{\partial x_1^*} & \dfrac{\partial h^*}{\partial x_2^*} & \dfrac{\partial h^*}{\partial x_3^*} & \dfrac{\partial h^*}{\partial x_4^*} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

This last expression shows that

$$h^*(x^*) = x_4^*.$$

Putting it altogether we have (1.95) and (1.96).

The proof of necessity is left to the reader. ■

Example 13 *Series Connected DC Motor*

Recall the nonlinear equations of the series connected DC motor.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ c_2 x_3 - c_3 x_3 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}}_{p} \tau_L$$

where $x_1 = \theta, x_2 = \omega, x_3 = i = i_a = i_f, u = V_S/L, c_1 = K_T L_f/J, c_2 = -R/L$, and $c_3 = K_b L_f/L$. As discussed in Chapter **??**, the series connected DC motor is used in speed control applications so let's remove $x_1 = \theta$ from the model. The load torque is taken to be constant, but is unknown so it will need to be estimated in order to estimate the motor speed ω. To do this τ_L/J is added to the model as a *state variable* with $\dfrac{d}{dt}(\tau_L/J) = 0$. With $z_1 = x_2 = \omega, z_2 = x_3 = i$, $z_3 = \tau_L/J$, and assuming only the current is measured, the system equations are now

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 z_2^2 - z_3 \\ -c_2 z_2 - c_3 z_1 z_2 \\ 0 \end{bmatrix}}_{f(z)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{g(z)} u \tag{1.104}$$

$$y = \ln(z_2). \tag{1.105}$$

More compactly this is written as

$$\frac{dz}{dt} = f(z) + g(z)u$$
$$y = h(z).$$

Remark In Theorem 4 it is assumed that the conditions of theorem hold in a neighborhood of $z = 0$ and $h(0) = 0$. However, the output equation is singular at $z_2 = i = 0$. We could get around this by defining new state variables $z_1' = z_1, z_2' = z_2 - 1, z_3' = z_3$ and a new output equation $y = h'(z') = \ln(z_2' + 1)$ so that $z' = 0$

[9] $dh^* = dh\dfrac{\partial x}{\partial x^*}, q^* = \dfrac{\partial x^*}{\partial x}q$ so $\mathcal{L}_{q^*}h^* = \langle dh^*, q^*\rangle = \left\langle dh\dfrac{\partial x}{\partial x^*}, \dfrac{\partial x^*}{\partial x}q \right\rangle = < dh, q >= \mathcal{L}_q h$. See Chapter **??** page **??** where this was explained.

corresponds to $h'(0) = 0$ and $z = (0, 1, 0)$. However, this is not essential for the theorem to hold. As shown below it is only essential that $z_2 > 0$.

Remark It seems strange to take the output to be $\ln(z_2)$ rather than z_2, but in Theorem 4 the output is assumed to be given and the proof is for that *particular* output. If the output is taken to be $y = z_2$ then it turns out the conditions of the theorem do not hold.

Notice that $f(z)$ in (1.104) is *not* linear in the unmeasured state variable $z_1 = \omega$. We now check the conditions of Theorem 4 to see if a statespace transformation exists such that in the new coordinates a nonlinear observer with linear error dynamics can be designed.

To check condition (1) we compute

$$\mathcal{O} = \begin{bmatrix} dh \\ d\mathcal{L}_f h \\ d\mathcal{L}_f^2 h \end{bmatrix} = \begin{bmatrix} 0 & 1/z_2 & 0 \\ -c_3 & 0 & 0 \\ 0 & -2c_1c_3z_2 & c_3 \end{bmatrix}$$

and a further computation shows $\det \mathcal{O} = c_3^2/z_2$.

To check condition (2) we have

$$q \triangleq \begin{bmatrix} dh \\ d\mathcal{L}_f h \\ d\mathcal{L}_f^2 h \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/z_2 & 0 \\ -c_3 & 0 & 0 \\ 0 & -2c_1c_3z_2 & c_3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/c_3 \end{bmatrix}$$

and find that

$$\begin{aligned} ad_f q &= - \begin{bmatrix} 0 & 2z_2 & -1 \\ -c_3z_2 & -c_2 - c_3z_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1/c_3 \end{bmatrix} = \begin{bmatrix} 1/c_3 \\ 0 \\ 0 \end{bmatrix} \\ ad_f^2 q &= - \begin{bmatrix} 0 & 2z_2 & -1 \\ -c_3z_2 & -c_2 - c_3z_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/c_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ z_2 \\ 0 \end{bmatrix}. \end{aligned}$$

It is easy to check that the Lie bracket of any two vector fields in $\left\{q, ad_f q, ad_f^2 q\right\}$ is zero so condition (2) is satisfied.

To check condition (3) is also easy see that the Lie bracket of the vector field $g = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ with any vector field in $\{q, ad_f q\}$ is zero so condition (3) is satisfied.

In this particular example we can construct the transformation as shown in the proof of Theorem 4. Using the above computations we have

$$\begin{bmatrix} q & ad_{(-f)}q & ad_{(-f)}^2 q \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1/c_3 \end{bmatrix} & \begin{bmatrix} -1/c_3 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ z_2 \\ 0 \end{bmatrix} \end{bmatrix}$$

and it is straightforward to verify that

$$\phi_{t_1}^{(1)}(z_0) = \begin{bmatrix} z_{01} \\ z_{02} \\ t_1/c_3 + z_{03} \end{bmatrix}, \phi_{t_2}^{(2)}(z_0') = \begin{bmatrix} -t_2/c_3 + z_{01}' \\ z_{02}' \\ z_{03}' \end{bmatrix}, \phi_{t_3}^{(3)}(z_0'') = \begin{bmatrix} z_{01}'' \\ e^{t_3} z_{02}'' \\ z_{03}'' \end{bmatrix}.$$

First computing

$$\phi_{t_2}^{(2)}(\phi_{t_1}^{(1)}(z_0)) = \begin{bmatrix} -t_2/c_3 + z_{01}' \\ z_{02}' \\ z_{03}' \end{bmatrix}_{z_0' = \begin{bmatrix} z_{01} \\ z_{02} \\ t_1/c_3 + z_{03} \end{bmatrix}} = \begin{bmatrix} -t_2/c_3 + z_{01} \\ z_{02} \\ t_1/c_3 + z_{03} \end{bmatrix}$$

we can then obtain

$$z(t_1,t_2,t_3) = \phi^{(3)}_{t_3}(\phi^{(2)}_{t_2}(\phi^{(1)}_{t_1}(z_0))) = \begin{bmatrix} z''_{01} \\ e^{t_3} z''_{02} \\ z''_{03} \end{bmatrix}_{|z''_0 = \begin{bmatrix} -t_2/c_3 + z_{01} \\ z_{02} \\ t_1/c_3 + z_{03} \end{bmatrix}} = \begin{bmatrix} -t_2/c_3 + z_{01} \\ e^{t_3} z_{02} \\ t_1/c_3 + z_{03} \end{bmatrix}.$$

That is, the transformation is

$$\begin{aligned} z_1 &= -t_2/c_3 + z_{01} \\ z_2 &= e^{t_3} z_{02} \\ z_3 &= t_1/c_3 + z_{03} \end{aligned}$$

with inverse

$$\begin{aligned} z_1^* &= t_1 = c_3(z_3 - z_{03}) \\ z_2^* &= t_2 = -c_3(z_1 - z_{01}) \\ z_3^* &= t_3 = \ln(z_2/z_{02}) \end{aligned}$$

Taking $z_{01} = 0, z_{02} = 1$, and $z_{03} = 0$ this transformation reduces to

$$\begin{aligned} z_1^* &= T_1(z) = c_3 z_3 \\ z_2^* &= T_2(z) = -c_3 z_1 \\ z_3^* &= T_3(z) = \ln(z_2) \end{aligned}$$

which was given in Example **??** of Chapter **??**.

1.7 Problems

Problem 1 *Controllability Matrix*
Consider the control system given by

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u.$$

(a) Compute the controllability matrix

$$\mathcal{C} \triangleq \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 & A^3b_1 & A^3b_2 & A^4b_1 & A^4b_2 \end{bmatrix} \in \mathbb{R}^{5\times 25}.$$

Search $\mathcal{C}$ from left to right to find the first 5 linearly independent columns of $\mathcal{C}$.

(b) Compute the controllability indices κ_i. In this problem $\kappa_1 < \kappa_2$. Compute

$$C \triangleq \begin{bmatrix} b_1 & Ab_1 & \cdots & A^{\kappa_1 - 1}b_1 & b_2 & Ab_2 & \cdots & A^{\kappa_2 - 1}b_2 \end{bmatrix} \in \mathbb{R}^{5\times 5}.$$

(c) Let q_{κ_1} be the κ_1 row of C^{-1} and $q_{\kappa_1+\kappa_2}$ be the $\kappa_1 + \kappa_2$ row of C^{-1}. Use $q_{\kappa_1}, q_{\kappa_1+\kappa_2}$ to find the transformation T that is used to put the system into control canonical form.

Problem 2 *Controllability Matrix*
Consider the control system given by

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u.$$

(a) Compute the controllability matrix

$$\mathcal{C} \triangleq \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 & A^3b_1 & A^3b_2 & A^4b_1 & A^4b_2 \end{bmatrix} \in \mathbb{R}^{5\times 25}.$$

(b) Searching $\mathcal{C}$ from left to right find the first 5 linearly independent columns of $\mathcal{C}$. Compute the controllability indices κ_i and

$$C \triangleq \begin{bmatrix} b_1 & Ab_1 & \cdots & A^{\kappa_1 - 1}b_1 & b_2 & Ab_2 & \cdots & A^{\kappa_2 - 1}b_2 \end{bmatrix} \in \mathbb{R}^{5\times 5}.$$

(c) Let q_{κ_1} be the κ_1 row of C^{-1} and $q_{\kappa_1+\kappa_2}$ be the $\kappa_1 + \kappa_2$ row of C^{-1}. Use $q_{\kappa_1}, q_{\kappa_1+\kappa_2}$ to find the transformation T that is used to put the system into control canonical form.

Problem 3 *Multi-Input Control Canonical Form*
Consider the controllable linear time-invariant control system

$$\frac{dx}{dt} = Ax + Bu, \quad x \in \mathbb{R}^4, u \in \mathbb{R}^2, A \in \mathbb{R}^{4\times 4}, B \in \mathbb{R}^{4\times 2}$$

with $rank[B] = 2$. Suppose $\kappa_1 = 2$ and $\kappa_2 = 2$ so that by Theorem 2 it follows that

$$C \triangleq \begin{bmatrix} b_1 & Ab_1 & b_2 & Ab_2 \end{bmatrix}$$

is invertible. Transform this control system into the form

$$\frac{d}{dt}x^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^* + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} v^*$$

by constructing the appropriate state and input transformations and the appropriate state feedback matrix.

Problem 4 *Exact Differentials*

Let $d\omega = \begin{bmatrix} \omega_1(x) & \omega_2(x) & \omega_3(x) \end{bmatrix} \in \mathbb{R}^{1\times 3}$ for all x in an open set $\mathcal{U} \subset \mathbb{R}^3$. Show that the necessary and sufficient condition for there to exist a scalar function $T_1(x)$ such that

$$\frac{\partial T_1}{\partial x_1} = \omega_1(x), \frac{\partial T_1}{\partial x_2} = \omega_2(x), \frac{\partial T_1}{\partial x_3} = \omega_3(x) \tag{1.106}$$

is that

$$\begin{aligned} \frac{\partial \omega_1(x)}{\partial x_2} &= \frac{\partial \omega_2(x)}{\partial x_1} \\ \frac{\partial \omega_1(x)}{\partial x_3} &= \frac{\partial \omega_3(x)}{\partial x_1} \\ \frac{\partial \omega_2(x)}{\partial x_3} &= \frac{\partial \omega_3(x)}{\partial x_2}. \end{aligned} \tag{1.107}$$

If $d\omega$ satisfies (1.107) it is said to be an *exact differential*. Hint: For sufficiency, show that

$$T_1(x) \triangleq \int_0^{x_1} \omega_1(x_1', 0, 0)dx_1' + \int_0^{x_2} \omega_1(x_1, x_2', 0)dx_2' + \int_0^{x_3} \omega_1(x_1, x_2, x_3')dx_3$$

will satisfy (1.106).

Problem 5 *Feedback Linearization*

Consider the following nonlinear control system.

$$\begin{aligned} \frac{dx_1}{dt} &= x_2(1+u) \\ \frac{dx_2}{dt} &= c\sin(u)/x_1 \end{aligned}$$

where $c > 0$ and $x_1 \neq 0$. Clearly this system is not linear in the control.

(a) Suppose the input u is kept small so that $\sin(u) \approx u$. The model for the control system is then

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ c/x_1 \end{bmatrix} u.$$

Is this model feedback linearizable? If so, find the feedback linearizing transformation.

(b) Define a new state variable $x_3 \triangleq u$ and a new input $w \triangleq dx_3/dt$. The extended control system is then

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2(1+x_3) \\ c\sin(x_3)/x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w.$$

Does this system satisfy the necessary and sufficient conditions for feedback linearization? If so, find the transformation.

Problem 6 *Doubly Excited DC Motor*
The equations describing a doubly excited DC motor are

$$
\begin{aligned}
\frac{d\theta}{dt} &= \omega \\
J\frac{d\omega}{dt} &= K_T L_f i_f i_a - \tau_L \\
L\frac{di_a}{dt} &= -Ri_a - K_b L_f i_f \omega + V_a \\
L_f\frac{di_f}{dt} &= -R_f i_f + V_f.
\end{aligned}
$$

Here θ is the rotor angle, ω is the rotor angular speed, V_{a0} is the (constant) armature voltage, i_a is the armature current, V_f is the field voltage, i_f is the field current, τ_L is the load torque, K_T is the torque constant, and K_b is the back-emf constant. The armature resistance and armature inductance are denoted by R and L,respectively, and the field resistance and field inductance are R_f and L_f, respectively.

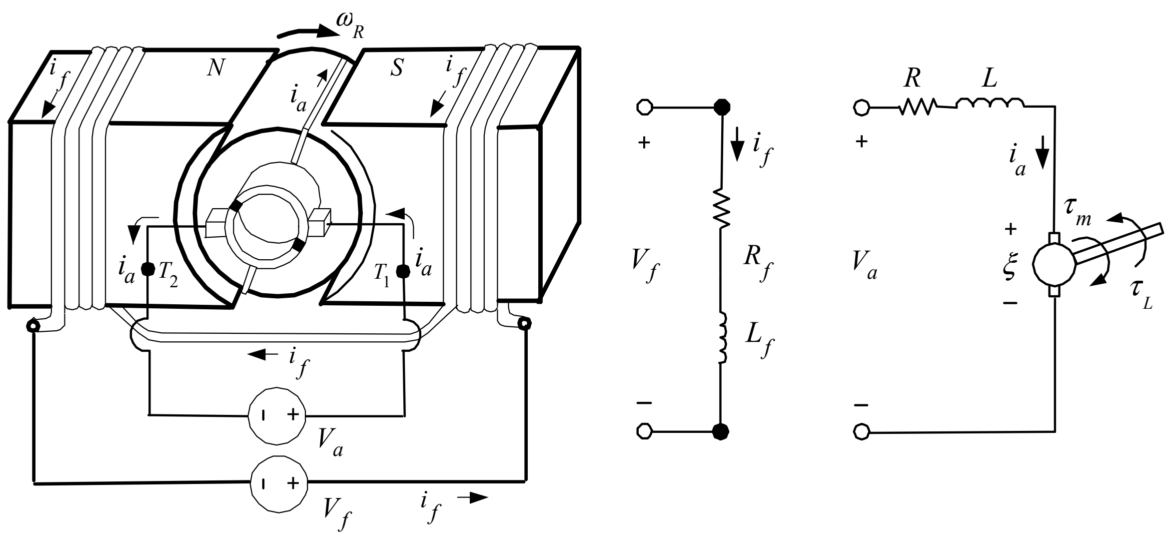

FIGURE 1.3. Field controlled DC motor. $\xi = K_b L_f i_f$ and $\tau_m = K_T L_f i_f i_a$.

Let $x_1 = i_f, x_2 = i_a, x_3 = \omega, x_4 = \theta, u_1 = V_a/L_a, u_2 = V_f/L_f$, and define the constants $c_1 = R_f/L_f, c_2 = R/L, c_3 = K_b L_f/L, c_4 \triangleq K_T L_f/J$. The equations describing the doubly excited DC motor are then

$$
\begin{aligned}
\frac{dx_1}{dt} &= -c_1 x_1 + u_2 \\
\frac{dx_2}{dt} &= -c_2 x_2 - c_3 x_1 x_3 + u_1 \\
\frac{dx_3}{dt} &= c_4 x_1 x_2 - \tau_L/J \\
\frac{dx_4}{dt} &= x_3
\end{aligned}
$$

or

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -c_1x_1 \\ -c_2x_2 - c_3x_1x_3 \\ c_4x_1x_2 \\ x_3 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{g_1(x)} u_1 + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{g_2(x)} u_2 + \begin{bmatrix} 0 \\ 0 \\ -1/J \\ 0 \end{bmatrix} \tau_L.$$

(a) Compute the controllability indices of this nonlinear system with $x_1 = i_f \neq 0$ and $i_a = x_2 \neq 0$.

(b) Can you find a feedback linearizing transformation? If so, do so. What conditions on the state variables x_1, x_2, x_3 are needed to use this feedback?

Problem 7 *Multi-Input Feedback Linearization* [14]
Consider the nonlinear control system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 \\ (1-\ln(x_3))x_2 \\ -px_1x_3 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix}}_{g_1(x)} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g_2(x)} u_2$$

with $x_1 > 0, x_2 > 0, x_3 > 0$.

(a) Compute the controllability indices. Why must they be $\kappa_1 = 2, \kappa_2 = 1$?

(b) Show this model satisfies the necessary and sufficient conditions for a feedback linearzing transformation to exist. What region of $\mathbb{R}^3$ are they satisfied?

(c) Explicitly compute the coordinate transformation that allows this control system to be feedback linearized.

(c) Compute the system equations in the new coordinates.

Problem 8 *Nonlinear Regulator for a Synchronous Generator* [15]
In problem **??** of Chapter **??** a nonlinear statespace model of a synchronous generator connected to an infinite bus was given as

$$\frac{d}{dt}\begin{bmatrix} \delta \\ \omega \\ \psi_f \\ \psi_A \\ \psi_B \end{bmatrix} = \underbrace{\begin{bmatrix} \omega - \omega_s \\ c_{mo} - K_2\omega\psi_f\sin(\delta) - K_3\omega\psi_A\sin(\delta) - K_4\omega\psi_B\sin(\delta) + K_5\sin(\delta)\cos(\delta) \\ \nu_{f0} - K_8\omega\psi_f + K_9\omega\psi_A + K_{10}\cos(\delta) \\ K_{11}\omega\psi_f - K_{12}\omega\psi_A + K_{13}\cos(\delta) \\ -K_{14}\omega\psi_B - K_{15}\sin(\delta) \end{bmatrix}}_{f(\delta,\omega,\psi_f,\psi_A,\psi_B)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{g_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{g_2} u_2$$

where δ is the rotor angle referred to the infinite bus, ω is the rotor angular speed, ψ_f is the field flux linkage, ψ_A is the direct axis flux linkage, ψ_B is the quadrature axis flux linkage, ω_s is the constant synchronous angular velocity, c_m is the rotor angular acceleration produced by the input torque, c_{mo} is the reference input angular acceleration, ν_f is the field excitation voltage, ν_{f0} is the constant reference field excitation voltage, $u_1 = c_m - c_{mo}$, and $u_2 = \nu_f - \nu_{f0}$.

Let $\begin{bmatrix} \delta_0 & \omega_0 & \psi_{f0} & \psi_{A0} & \psi_{B0} \end{bmatrix}^T$ denote a constant stable operating point with $u_1 = 0, u_2 = 0$. All the parameters $K_1, ..., K_{15}$ are positive constants. With

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T \triangleq \begin{bmatrix} \delta & \omega & \psi_f & \psi_A & \psi_B \end{bmatrix}^T$$
$$u = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} c_m - c_{mo} & \nu_f - \nu_{f0} \end{bmatrix}$$

the model of the synchronous generator has the compact form

$$\frac{d}{dt}x = f(x) + g(x)u.$$

Show that this model satisfies the necessary and sufficient conditions for feedback linearization.

Problem 9 *Nonlinear Observer with Linear Error Dynamics* [14]
Consider the nonlinear control system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 \\ (1-\ln(x_3))x_2 \\ -px_1x_3 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix}}_{g_1(x)} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g_2(x)} u_2$$

$$y = \ln(x_2)$$

with $x_1 > 0, x_2 > 0, x_3 > 0$.

(a) With $u_1 = u_2 = 0$ show that this control system satisfies the necessary and sufficient conditions for there to exist a transformation $x^* = T(x) \in \mathbb{R}^3$ such that in the x^* coordinates the system is given by

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} + \begin{bmatrix} \varphi_1(y) \\ \varphi_2(y) \\ \varphi_3(y) \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix}.$$

(b) Find the transformation.

(c) Would this observer work if $u_1 \neq 0$, but $u_2 = 0$. If not, suppose in addition x_1 is also measurable?

(c) Would this observer work if $u_1 = 0$, but $u_2 \neq 0$. If not, suppose in addition x_3 is also measurable?

Problem 10 *Dynamic Feedback Linearization - Aircraft Control* [16]
A statespace model for an aircraft is given by

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \\ \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} u\cos(\psi)\cos(\theta) + v(\cos(\psi)\sin(\theta)\sin(\phi) - \sin(\psi)\cos(\phi)) + w(\cos(\psi)\sin(\theta)\cos(\phi) + \sin(\psi)\sin(\phi) \\ u\sin(\psi)\cos(\theta) + v(\sin(\psi)\sin(\theta)\sin(\phi) + \cos(\psi)\cos(\phi)) + w(\sin(\psi)\sin(\theta)\cos(\phi) - \cos(\psi)\sin(\phi) \\ u\sin(\theta) - v\cos(\theta)\sin(\phi) - w\cos(\theta)\cos(\phi) \\ -g\sin(\theta) + X/m \\ g\cos(\theta)\sin(\phi) + Y/m \\ g\cos(\theta)\cos(\phi) + Z/m \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ w \\ -v \\ 1 \\ 0 \\ 0 \end{bmatrix} p + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -w \\ 0 \\ u \\ \tan(\theta)\sin(\phi) \\ \sin(\phi) \\ \sin(\phi)/\cos(\theta) \end{bmatrix} q + \begin{bmatrix} 0 \\ 0 \\ 0 \\ v \\ -u \\ 0 \\ \tan(\theta)\cos(\phi) \\ -\sin(\phi) \\ r\cos(\phi)/\cos(\theta) \end{bmatrix} r + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rho$$

Here x, y, z are the coordinates of the center of mass with respect to earth (inertial frame) with the $z - axis$ oriented down, u, v, w are the components of the velocity in the body frame, and ϕ, θ, ψ are the Euler angles which locate the body frame with respect to the earth frame. In a little more detail let $\hat{\boldsymbol{x}}_e, \hat{\boldsymbol{y}}_e, \hat{\boldsymbol{z}}_e$ be the inertial (earth) basis vectors and note that $\hat{\boldsymbol{z}}_e$ points down into the earth with $\hat{\boldsymbol{z}}_e = \hat{\boldsymbol{x}}_e \times \hat{\boldsymbol{y}}_e$. To go from the earth axis to the body fixed axis, the aircraft undergoes three rotations. Figure 1.4 show the aircraft is first rotated by the *yaw* angle ψ about the $\hat{\boldsymbol{z}}_e$ axis. As shown in Figure 1.5, the aircraft is next *pitched* up

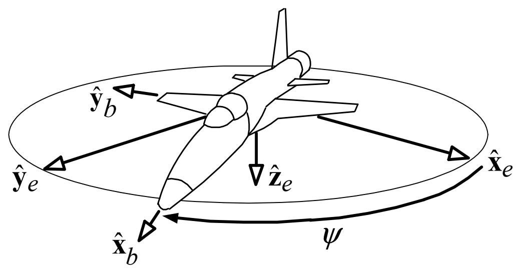

FIGURE 1.4. Yaw angle ψ.

by the angle θ about the $\hat{\boldsymbol{y}}_b$ axis. Finally, as shown in Figure 1.6, the aircraft is rotated about the $\hat{\boldsymbol{x}}_b$ axis

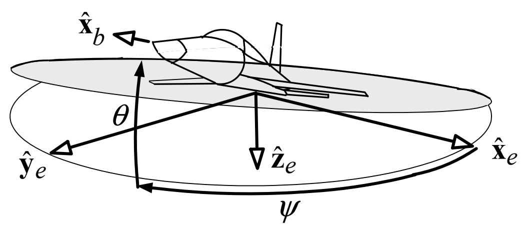

FIGURE 1.5. The pitch angle θ.

by the roll angle ϕ. The velocity of the aircraft in the earth frame in terms of the velocity in the body frame

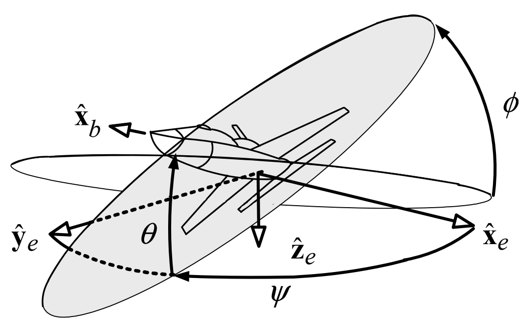

FIGURE 1.6. Roll angle ϕ.

is given by

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = R(\phi, \theta, \psi) \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

where

$$R(\phi,\theta,\psi)=\begin{bmatrix}\cos(\psi) & -\sin(\psi) & 0\\ \sin(\psi) & \cos(\psi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\theta) & 0 & \sin(\theta)\\ 0 & 1 & 0\\ -\sin(\theta) & 0 & \cos(\theta)\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\phi) & -\sin(\phi)\\ 0 & \sin(\phi) & \cos(\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi-\cos\phi\sin\psi & \sin\phi\sin\psi+\cos\phi\cos\psi\sin\theta\\ \cos\theta\sin\psi & \cos\phi\cos\psi+\sin\theta\sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi-\cos\psi\sin\phi\\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi\end{bmatrix}.$$

The mass of the aircraft is denoted by m and g denotes the acceleration due to gravity. The aerodynamic forces in the body frame (drag, lift, etc.) are denoted by $X=X(x,y,z,u,v,w,\phi,\theta,\psi), Y=Y(x,y,z,u,v,w,\phi,\theta,\psi)$ and $Z=Z(x,y,z,u,v,w,\phi,\theta,\psi)$. The angular rates in the body frame p,q,r are taken as inputs along with the thrust ρ.

(a) Compute the controllability indices and then show the system is not feedback linearizable.

(b) Define a new state $\xi\triangleq\rho/m$ and a new input $\sigma\triangleq d\xi/dt$ so the state and input are

$$\xi=\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\\ \xi_5\\ \xi_6\\ \xi_7\\ \xi_8\\ \xi_9\\ \xi_{10}\end{bmatrix}=\begin{bmatrix}x\\ y\\ z\\ u\\ v\\ w\\ \phi\\ \theta\\ \psi\\ \rho\end{bmatrix},\quad u=\begin{bmatrix}p\\ q\\ r\\ \sigma\end{bmatrix}$$

so the system has the form

$$\frac{d\xi}{dt}=f(\xi)+g_1(\xi)u_1+g_2(\xi)u_2+g_3(\xi)u_3+g_4\sigma\in\mathbb{R}^{10}.$$

Compute the controllability indices and show this system is feedback linearizable. Can you find the transformation?

(c) If part (b) is too messy, try just doing it for the longitudinal equations. That is, set $r=p=\psi=\phi=0$ so the equations become

$$\frac{d}{dt}\begin{bmatrix}x\\ z\\ u\\ w\\ \theta\\ \rho/m\end{bmatrix}=\begin{bmatrix}u\cos(\theta)+w\sin(\theta)\cos(\phi)\\ u\sin(\theta)-w\cos(\theta)\\ -g\sin(\theta)+X/m+\rho/m\\ g\cos(\theta)+Z/m\\ 0\\ \sigma\end{bmatrix}+\begin{bmatrix}0\\ 0\\ -w\\ u\\ 1\\ 0\end{bmatrix}q+\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 1\end{bmatrix}\sigma.$$

Problem 11 *Nonlinear Observer with Linear Error Dynamics*

Consider the nonlinear control system

$$\frac{d}{dt}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\underbrace{\begin{bmatrix}x_3\\ x_1x_2\\ x_3\end{bmatrix}}_{f(x)}+\underbrace{\begin{bmatrix}0\\ 0\\ e^{x_2}\end{bmatrix}}_{g(x)}u_1$$

$$y=\ln(x_2/x_{02})$$

with $x_2>0$.

(a) Show that the necessary and sufficient conditions are satisfied for a nonlinear observer with linear error dynamics to exist.

(b) Find the transformation and the system equations in the new coordinates.

Problem 12 *Observer for a Predator-Prey Model* [17]

In Problem **??** of Chapter **??** a nonlinear differential equation model for a predator-prey system was given as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha x_1 - \beta x_1 x_2 \\ \gamma x_1 x_2 - \lambda x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ -x_2 \end{bmatrix}}_{g(x)} u$$

$$y = \ln(y) = \ln(x_2).$$

$x_1 \geq 0$ is the prey population and $x_2 > 0$ is the predator population. The constants, $\alpha > 0$ and $\gamma > 0$ are the birth rates of prey and predator populations, respectively while the constants $\beta > 0$ and $\lambda > 0$ are the death rates of the prey and predator populations, respectively. The input $u \geq 0$ represents the rate at which humans can decimate the predator population (e.g., by hunting). The output y is the predator population while the prey population is considered too big to measure.

Can a nonlinear observer with linear error dynamics be designed? Specifically, check if the necessary and sufficient conditions for the existence of such an observer are satisfied.

Problem 13 *Addition of an Integrator to the q-axis Input* [6][7]

In this problem an integrator is added to the q-axis of the system model (1.57). With $x_1 = \omega$, $x_2 = \psi_d$, $x_3 = \rho$ and $x_4 = u_2 = i_{qr}$, $dx_4/dt = v_2$, and $v_1 = u_1 = i_{dr}$ the extended system is

$$\frac{dx}{dt} = f(x) + g_1 v_1 + g_1 v_2$$

with

$$f(x) = \begin{bmatrix} \mu x_2 x_4 - \tau_L/J \\ -\eta x_2 \\ n_p x_1 + \eta M x_4/x_2 \\ 0 \end{bmatrix}, \; g_1 = \begin{bmatrix} 0 \\ \eta M \\ 0 \\ 0 \end{bmatrix}, \; g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^4.$$

(a) Show that the transformation

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= \mu x_2 x_4 - \tau_L/J \\ z_3 &= x_3 \\ z_4 &= n_p x_1 + \eta M x_4/x_2. \end{aligned}$$

results in

$$\begin{aligned} \frac{dz_1}{dt} &= z_2 \\ \frac{dz_2}{dt} &= a_1(x) + v_1 b_{11}(x) + v_2 b_{12}(x) \\ \frac{dz_3}{dt} &= z_4 \\ \frac{dz_4}{dt} &= a_2(x) + v_1 b_{21}(x) + v_2 b_{22}(x). \end{aligned}$$

Explicitly compute $a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x)$, and $b_{22}(x)$.

$$
\begin{aligned}
a_1(x) &= -\eta\mu x_2 x_4 \\
a_2(x) &= \mu n_p x_2 x_4 + \eta^2 M \frac{x_4}{x_2} - \frac{n_p \tau_L}{J} \\
b_{11}(x) &= \eta M \mu x_4, b_{12}(x) = \mu x_2, b_{21}(x) = -\eta^2 M^2 x_4/x_2^2, b_{22}(x) = \eta M/x_2.
\end{aligned}
$$

(b) Use feedback and an input transformation to show the system can be put in Brunovsky canonical form. What are the values of the controllability indices?

(c) Compute the conditions under which the controller in part (b) is singular, i.e., the input transformation is not invertible. Do you seen any practical problems with this controller? Hint: The torque of the motor is $\tau = J\mu\psi_d i_q = J\mu x_2 x_4 \neq 0$. Does the system go through the singularity to change the sign of the torque?

Problem 14 *Flux and Speed Observer*

Consider the following approach to estimating the flux linkages and speed simultaneously. The stator currents i_{Sa} and i_{Sb} and flux linkages ψ_{Ra} and ψ_{Rb} are rotated by the angle $n_p\theta$ to obtain

$$
\begin{bmatrix} i_{Sx} \\ i_{Sy} \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix}
$$

$$
\begin{bmatrix} \psi_{Rx} \\ \psi_{Ry} \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} \psi_{Ra} \\ \psi_{Rb} \end{bmatrix}.
$$

The currents i_{Sx} and i_{Sy} are *known* as the angle θ is measured. In order to estimate the load torque, it is modeled as a constant so that its dynamic equation is taken to be $d(\tau_L/J)/dt = 0$.

(a) In terms of the new state variables $i_{Sx}, i_{Sy}, \psi_{Rx}$, and ψ_{Ry} show that

$$
\begin{aligned}
\frac{d\psi_{Rx}}{dt} &= -\frac{R_R}{L_R}\psi_{Rx} + \frac{MR_R}{L_R} i_{Sx} \\
\frac{d\psi_{Ry}}{dt} &= -\frac{R_R}{L_R}\psi_{Ry} + \frac{MR_R}{L_R} i_{Sy} \\
\frac{d\theta}{dt} &= \omega \\
\frac{d\omega}{dt} &= \mu(i_{Sy}\psi_{Rx} - i_{Sx}\psi_{Ry}) - \tau_L/J \\
\frac{d(\tau_L/J)}{dt} &= 0.
\end{aligned}
$$

The key point here is that in this state-space representation, the (unmeasured) speed ω has been *eliminated* from the flux equations.

(b) Define an estimator for the flux linkages, speed, and load torque by

$$
\begin{aligned}
\frac{d\hat{\psi}_{Rx}}{dt} &= -\frac{R_R}{L_R}\hat{\psi}_{Rx} + \frac{MR_R}{L_R} i_{Sx} \\
\frac{d\hat{\psi}_{Rx}}{dt} &= -\frac{R_R}{L_R}\hat{\psi}_{Ry} + \frac{MR_R}{L_R} i_{Sy} \\
\frac{d\theta}{dt} &= \hat{\omega} + \ell_1(\theta - \hat{\theta}) \\
\frac{d\omega}{dt} &= \mu(i_{Sy}\hat{\psi}_{Rx} - i_{Sx}\hat{\psi}_{Ry}) - \frac{f}{J}\omega - \frac{\hat{\tau}_L}{J} + \ell_2(\theta - \hat{\theta}) \\
\frac{d(\hat{\tau}_L/J)}{dt} &= 0 + \ell_3(\theta - \hat{\theta}).
\end{aligned}
$$

With $e_{Rx} \triangleq \psi_{Rx} - \hat{\psi}_{Rx}, e_{Ry} \triangleq \psi_{Ry} - \hat{\psi}_{Ry}$, $e_\theta \triangleq \theta - \hat{\theta}$, $e_\omega \triangleq \omega - \hat{\omega}$, and $e_{\tau_L} \triangleq \tau_L/J - \hat{\tau}_L$, show that the error system is given by

$$\begin{aligned}
\frac{de_{Rx}}{dt} &= -\frac{R_R}{L_R} e_{Rx} \\
\frac{de_{Ry}}{dt} &= -\frac{R_R}{L_R} e_{Ry} \\
\frac{de_\theta}{dt} &= e_\omega - \ell_1(\theta - \hat{\theta}) \\
\frac{de_\omega}{dt} &= \mu(i_{Sy} e_{Rx} - i_{Sx} e_{Ry}) - \frac{f}{J} e_\omega - e_{\tau_L} - \ell_2(\theta - \hat{\theta}) \\
\frac{de_{\tau_L}}{dt} &= 0 - \ell_3(\theta - \hat{\theta}).
\end{aligned}$$

(c) As long as the currents i_{Sx} and i_{Sy} are bounded (consistent with the assumed current-command operation), show that the error system is stable. Note that the rate of convergence of the error dynamics of this observer is still limited by the rotor time constant $T_R = L_R/R_R = 1/\eta$ [18].

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