ECE 661: Nonlinear Systems
Spring 2021 | Homework #1 Solutions

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1. Given $A \in \mathbb{R}^{3\times 3}$, $b \in \mathbb{R}^3$ with the pair (A, b) controllable and q the last row of $\mathcal{C}^{-1} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix}^{-1}$, show that

$$T = \begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix}$$

is nonsingular.

Solution: Since the pair (A, b) is controllable, the matrix $\mathcal{C}$ is invertible and hence $\det \mathcal{C} \in \mathbb{R}^\times$, where $\mathbb{R}^\times$ is the multiplicative group of units of $\mathbb{R}$ (i.e., all nonzero scalars). Premultiplying $\mathcal{C}^{-1}$ by $e_3^\top$, the transpose of the third standard basis vector of $\mathbb{R}^3$, picks out the its last row, i.e., $q = e_3^\top \mathcal{C}^{-1}$. This implies $q\mathcal{C} = e_3^\top$, i.e.,

$$qb = qAb = 0, \quad qA^2b = 1.$$

Now, consider the product $T\mathcal{C}$ and use the equalities above to get

$$T\mathcal{C} = \begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix} \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & \bullet \\ 1 & \bullet & \bullet \end{bmatrix} =: M,$$

where each $\bullet \in \mathbb{R}$ represents some scalar. The matrix $M \in \mathbb{R}^{3\times 3}$ is nonsingular as $\det M = -1$. Therefore,

$$\det T = \det\left(M\mathcal{C}^{-1}\right) = -\det \mathcal{C}^{-1} = -\frac{1}{\det \mathcal{C}} \in \mathbb{R}^\times,$$

where the second and third equalities follow because $\det : GL(3, \mathbb{R}) \to \mathbb{R}^\times$ is a group homomorphism. ■

2. Show, by direct computation, that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = TAT^{-1}z + Tbu = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c} z + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b_c} u.$$

Solution: First, notice that the eigenvalues of similar matrices are the same. Indeed, if $A_c v = \lambda v$ for some $\lambda \in \mathbb{C}$, then

$$A_c v = TAT^{-1}v \implies AT^{-1}v = \lambda T^{-1}v$$

so that if v is an eigenvector of A_c with eigenvalue λ, then $T^{-1}v$ is an eigenvector of A with the same eigenvalue.

This implies that the characteristic polynomials of A and A_c are the same because the coefficients of this polynomial are given by the elementary symmetric polynomials in the eigenvalues of either A or A_c, which are the same as proved above.

Now, we invoke Cayley-Hamilton theorem to deduce that

$$A^3 = -a_0 I - a_1 A - a_2 A^2,$$

which implies

$$TA = \begin{bmatrix} qA \\ qA^2 \\ qA^3 \end{bmatrix} = \begin{bmatrix} qA \\ qA^2 \\ q\left(-a_0 I - a_1 A - a_2 A^2\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix} = A_c T.$$

Further, we showed in Exercise 1 that $Tb = (q\mathcal{C})^\top = e_3 = b_c$, yielding the result. ■

3. **Solution:**

4. **Solution:**