# Nonlinear Systems

## Lyapunov Stability

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# Outline

# Introduction

- Introduced by Alexandr Mikhailovich Lyapunov.
- *The general problem of the stability of motion*, 1892.
- Doctoral thesis in Kharkov Mathematical Society.
- The most general theory for analyzing stability of (at least) ordinary differential equations.

# Lyapunov Stability – Introduction

- Different notions of stability: input-output stability, periodic orbit stability, etc.
- Stability of equilibrium points usually characterized in the sense of Lyapunov.
  - An equilibrium point is STABLE if all solutions starting at nearby points stay nearby.
  - It is ASYMPTOTICALLY STABLE if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.
- For a linear system $\dot{x} = Ax$, the stability of $x = 0$ can be completely characterized by the eigenvalues of $A$.
- Stability of a nonlinear system sometimes can be characterized by the same method (through linearization).
- Lyapunov stability theorems give sufficient conditions for stability.

# Notations and Definitions

# Manifolds and Vector Fields

- $\mathcal{M}$ (state-space) denotes a manifold of finite dimension $n$.
- $f \in \mathfrak{X}(M)$ is a continuous vector field on $\mathcal{M}$.
- We assume that there exists a unique right maximally defined integral curve of $f$ starting at $x$.
- We also assume that this integral curve is defined on $[0, \infty]$.

$$\varphi : [0, \infty] \times \mathcal{M} \to \mathcal{M}$$

  with

$$\begin{aligned} \varphi(0, x) &= x, \\ \varphi(t_1, \varphi(t_2, x)) &= \varphi(t_1 + t_2, x). \end{aligned}$$

- The semiflow $\varphi$ is the evolution function.

# Invariant and Stable Sets

## Definition

$\Omega \subseteq \mathcal{M}$ is called an INVARIANT SET if for all $x \in \Omega$ and $t \in \mathbb{R}_{\geq 0}$, $\varphi(t,x) \in \Omega$. If $\Omega = \{p\}$ is a singleton, then $\Omega$ is called and EQUILIBRIUM POINT of the dynamical system $(\mathcal{M}, \varphi)$.

## Definition

$\Omega \subseteq \mathcal{M}$ is STABLE if for *every open neighborhood* $\mathcal{U} \subseteq \mathcal{M}$ of $\Omega$, *there exists a neighborhood* $\mathcal{V} \subseteq \mathcal{M}$ of $\Omega$ such that $\varphi(t, \mathcal{V}) \subseteq \mathcal{U}$ for all $t \geq 0$.

An *invariant set* $\Omega$ is asymptotically stable if

- $\Omega$ is stable,
- $\Omega$ is attractive, i.e., for all $x \in \Omega$, there exists an open neighborhood $\mathcal{N} \subseteq \mathcal{M}$ of $\Omega$ such that for all $x \in \mathcal{N}$, $\varphi(t,x) \xrightarrow{t\to\infty} \Omega$.

# Domain (Region) of Attraction

*The* domain of attraction is denoted by

$$\mathcal{A} = \{x \in \mathcal{M} : \varphi(t, x) \to \Omega \text{ as } t \to \infty\}.$$

$\Omega$ is said to be GLOBALLY asymptotically stable if $\mathcal{N} = \mathcal{M}$.

**Definition (Lie derivative)**

The LIE DERIVATIVE of $V : \mathcal{M} \to \mathbb{R}$ along $f \in \mathfrak{X}(\mathcal{M})$ is defined by

$$\begin{aligned} \mathcal{L}_f V : \mathcal{M} &\to \mathbb{R}, \\ p &\mapsto \mathrm{d}V_p(f(p)). \end{aligned}$$

# Lyapunov Function

## Definition

Let $\mathcal{K}$ be an invariant set of the dynamical system $(\mathcal{M}, \varphi)$. A continuous function $V : \mathcal{A} \to \mathbb{R}_{\geq 0}$ is a LYAPUNOV FUNCTION if

- $V(x) > 0$ for all $x \in \mathcal{A} \setminus \mathcal{K}$,
- $V(x) = 0$ for all $x \in \mathcal{K}$,
- $V$ is proper, i.e., $V^{-1}(B)$ is compact for all compact subsets $B \subseteq \mathbb{R}_{\geq 0}$,
- $V$ is strictly decreasing along orbits of $\varphi$, i.e.,

  $$V \circ \varphi(t, x) < V(x),$$

  for all $t > 0$ and $x \in \mathcal{A} \setminus \mathcal{K}$.
  If $V$ is differentiable, this condition may be replaced by

  $$\mathcal{L}_f V(x) < 0.$$

# (Nondegenerate) Critical Points

## Definition

Let $V : \mathcal{M} \to \mathbb{R}$ be a smooth function. A CRITICAL POINT, $p \in \mathcal{M}$, of $V$ is a point where the differential

$$\mathrm{d}V_p : T_p\mathcal{M} \to \mathbb{R}$$

has rank zero, i.e., in any local coordinate system $\{x_i\}_1^n$, one has $\frac{\partial V}{\partial x_i}(p) = 0$ for all $i = 1, \ldots, n$.

## Definition

A critical point $p$ is NONDEGENERATE if the Hessian $H_p(V)$ is a nondegenerate bilinear form, i.e., if any coordinate system, the Hessian matrix

$$\left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{1 \le i,j \le n}$$

is nondegenerate.

# Nondegenerate Critical Points

## Definition

The dimension of the subspace of $T_p\mathcal{M}$ on which $H_p(V)$ is negative definite is called the MORSE INDEX of $V$ at $p$, denoted by $\mathrm{ind}(V, p)$.

## Definition

A $C^2$ function $V : \mathcal{M} \to \mathbb{R}$ is a MORSE FUNCTION if all its critical points are nondegenerate.

## Definition

The (SUB)-LEVEL SETS of a function $V : \mathcal{M} \to \mathbb{R}$ are

$$\begin{aligned}\mathcal{M}_a &= V^{-1}\left((-\infty, a]\right),\\ \mathcal{M}_{a,b} &= V^{-1}\left([a, b]\right).\end{aligned}$$

# Topological Definitions

- A top. space is an $n$-CELL if it is homeomorphic to $\mathbb{R}^n$.
- A top. space $X$ is CONTRACTIBLE if it is *homotopy equivalent* to the one-point space.
- A subspace $A$ of $X$ is called a DEFORMATION RETRACT of $X$ if there exists a continuous function $h : [0,1] \times X \to X$ such that for all $x \in X$, $a \in A$,

$$h(0,x) = x,$$
$$h(1,x) \in A,$$
$$h(1,a) = a.$$

- The $k^{\text{th}}$ BETTI NUMBER of $\mathcal{M}$, denoted by $b_k$ is the rank of the $k^{\text{th}}$ homology group $H^k(\mathcal{M})$.
- The EULER CHARACTERISTIC of $\mathcal{M}$ is defined by

$$\chi(\mathcal{M}) = \sum_{i=1}^{k} (-1)^k b_k.$$

# Lyapunov Stability Analysis on Euclidean Spaces

# Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{1}$$

where $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz map, with an equilibrium point at $x = 0$.

## Definition

The equilibrium point $x = 0$ of the system (1) is

- *stable* if, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that

$$\|x(0)\| < \delta \ \Rightarrow \ \|x(t)\| < \epsilon, \quad \forall t \geq 0.$$

- *unstable* if it is not stable.
- *asymptotically stable* if it is stable and $\delta$ can be chosen s.t.

$$\|x(0)\| < \delta \ \Rightarrow \ \lim_{t \to \infty} x(t) = 0.$$

# Example – Pendulum

The pendulum equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - b x_2$$

has two equilibrium points at $(x_1 = 0, x_2 = 0)$ and $(x_1 = \pi, x_2 = 0)$.

- If $b = 0$, trajectories in the nbhd. of the first equilibrium are closed orbits.
- By starting sufficiently close to the eq. point, trajectories are guaranteed to stay within any specified ball.
- The point is not asymptotically stable since trajectories don't tend to the eq. point.
- If $b > 0$, the origin becomes asymptotically stable.
- The second eq. point is a saddle point: the $\varepsilon - \delta$ requirement cannot be satisfied (for every $\varepsilon > 0$ there exists a trajectory that will leave the ball $B_\varepsilon$ even if $x(0)$ is arbitrarily close to $(\pi, 0)$).

# Lyapunov Stability Theorem

**Theorem**

*Let $x = 0 \in D$ be an equilibrium point for* (1)*. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \ \ \textit{and} \ \ V(x) > 0 \ \textit{in} \ D - \{0\},$$

$$\dot{V}(x) \left[ = \frac{\partial V}{\partial x}(x) \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{d}V(x) \cdot f(x) = \mathcal{L}_f V(x) \right] \leq 0 \ \textit{in} \ D.$$

*Then, $x = 0$ is stable. Moreover, if*

$$\dot{V}(x) < 0 \ \textit{in} \ D - \{0\}$$

*then $x = 0$ is asymptotically stable.*

# Lyapunov Function Candidates

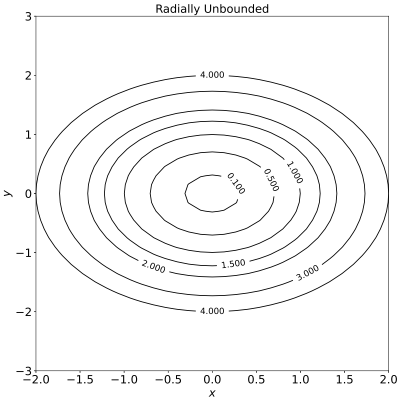



Figure: $V(x) = x_1^2 + x_2^2$.

$$\lim_{r \to \infty} \min_{\|x\|=r} V(x) = \infty.$$

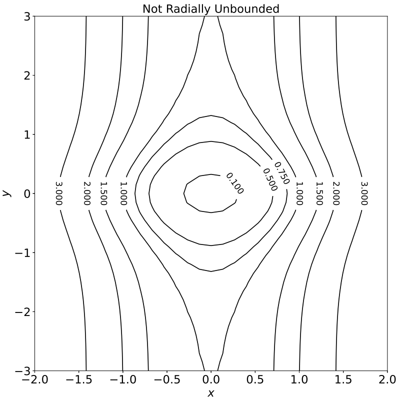



Figure: $V(x) = x_1^2 + \tanh{(x_2)}^2$.

$$\lim_{r \to \infty} \min_{\|x\|=r} V(x) = 1.$$

# Lyapunov Stability Theorem

*Proof of stability.* Given $\varepsilon > 0$, choose $0 < r \leq \varepsilon$ such that $B_r \subseteq D$. Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$. Take $0 < \beta < \alpha$ and consider $\mathcal{M}_\beta = V^{-1}((0, \beta])$.

Claim: $\mathcal{M}_\beta \subseteq \mathring{B}_r$. Argue ad absurdum. Suppose $\mathcal{M}_\beta \cap \mathring{B}_r \subsetneq \mathcal{M}_\beta$. Then $\exists p \in \mathcal{M}_\beta \cap \partial B_r$. Note, $V(p) \geq \alpha > \beta$, but $V(\mathcal{M}_\beta) \subseteq [0, \beta]$. ♦

The set $\mathcal{M}_\beta$ is invariant since

$$\dot{V}(x(t)) \leq 0 \;\Rightarrow\; V(x(t)) \leq V(x(0)) \leq \beta, \;\forall t \geq 0.$$

Because $\mathcal{M}_\beta$ is compact (closed and bounded), we conclude that the ODE (1) has a unique solution $\forall t \geq 0$ whenever $x(0) \in \mathcal{M}_\beta$. Since $V$ is continuous and $V(0) = 0$, $\exists \delta > 0$ such that

$$\|x\| \leq \delta \;\Rightarrow\; V(x) < \beta.$$

*Proof of stability (cont'd).* Then,

$$B_\delta \subseteq \mathcal{M}_\beta \subseteq B_r$$

and

$$x(0) \in B_\delta \;\Rightarrow\; x(0) \in \mathcal{M}_\beta \;\Rightarrow\; x(t) \in \mathcal{M}_\beta \;\Rightarrow\; x(t) \in B_r,$$

proving stability. □

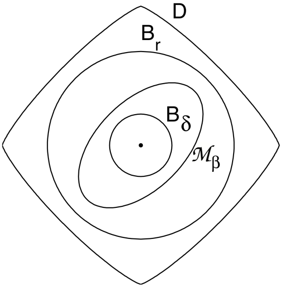



**Figure:** Geometric representation of Lyapunov stability.

*Proof of asymptotic stability.* Now assume $\dot{V}(x) < 0$ in $D - \{0\}$. We want to show that $x(t) \xrightarrow{t\to\infty} 0$; i.e., $\forall a > 0$, $\exists T > 0$, s.t. $\|x(t)\| < a, \forall t > T$.

We know that $\forall a > 0$, we can choose $b > 0$ s.t. $\mathcal{M}_b \subseteq B_a$. Therefore, it is sufficient to show that $V(x(t)) \xrightarrow{t\to\infty} 0$. Since $V$ is monotonically decreasing and bounded from below by zero,

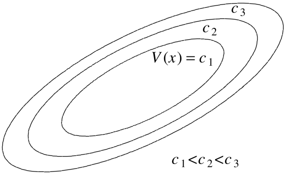
$$V(x(t)) \xrightarrow{t\to\infty} c \geq 0.$$

Claim: $c = 0$. Argue ad absurdum. Suppose $c > 0$. By continuity of $V$, $\exists d > 0$ s.t. $B_d \subseteq \mathcal{M}_c$. The limit $V(x(t)) \to c > 0$ implies that $x(t) \notin B_d, \forall t \geq 0$. Define $\max_{d \leq \|x\| \leq r} \dot{V}(x) =: -\gamma < 0$. It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, \mathrm{d}\tau \leq V(x(0)) - \gamma t.$$

The RHS will eventually become negative: contradiction ($c > 0$). $\square$

# Lyapunov Stability: Intuition




- A continuously differentiable function $V$, satisfying the theorem's conditions is called a Lyapunov function.
- When $\dot{V} < 0$, the trajectory moves from level set $\mathcal{M}_{c_3} = V^{-1}(c_3)$ to an inner level set $M_{c_2} = V^{-1}(c_2)$ with a smaller $c$.
- $V^{-1}(c) \xrightarrow{c \downarrow 0} 0.$ Hence the trajectory approaches the origin.
- If we only knew that $\dot{V} \leq 0$, we cannot be sure that the trajectory $x(t) \xrightarrow{t \to \infty} 0$,[1] but we can conclude that the origin is stable.

[1] See, however, Krasovskii-LaSalle's theorem.

# Example: Undamped pendulum

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -a \sin x_1.$$

Lyapunov function candidate

$$V(x) = a(1 - \cos x_1) + \tfrac{1}{2}x_2^2.$$

## Analysis

Clearly, $V(0) = 0$ and $V(x) > 0$ if $x \neq (2k\pi, 0)$. Compute the Lie derivative of $V$ along $f$:

$$\dot{V}(x) = \mathcal{L}_f V(x) = ax_2 \sin x_1 - ax_2 \sin x_1 = 0.$$

Thus, the origin is stable. Since $\dot{V}(x) \equiv 0$, we conclude that the origin is not asymptotically stable as solutions starting on the level set $\mathcal{M}_c$ remain in that set.

# Example: Damped pendulum

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -a \sin x_1 - b x_2.$$

<u>Lyapunov function candidate</u>

$$V(x) = a(1 - \cos x_1) + \frac{1}{2} x^\top P x,$$
$$P = P^\top > 0.$$

The Lie derivative $\dot{V}(x)$ is given by

$$\dot{V}(x) = a(1 - p_{22}) x_2 \sin x_1 - a p_{12} x_1 \sin x_1 + (p_{11} - p_{12} b) x_1 x_2 + (p_{12} - p_{22} b) x_2^2.$$

- Take $p_{22} = 1$ and $p_{11} = b p_{12}$.
- We must choose $0 < p_{12} < b$ for $V$ to be positive definite.
- Choose $p_{12} = \frac{b}{2}$.

$$\dot{V}(x) = -\frac{1}{2} a b x_1 \sin x_1 - \frac{1}{2} b x_2^2.$$

This is negative definite for any $0 < |x_1| < \pi$.

# Example: Rotational Motion of a Rigid Body in 3D-space

With respect to a coordinate system frame, which is rigidly attached to the body and whose axes are chosen to be the principal axes of the body, define:

- $\omega$: angular velocity of the body,
- $I \in \mathbb{S}^3_{++}$: inertia matrix of the body.

In the absence of external torques, the motion is described by

$$I\dot{\omega} + \omega \times I\omega = 0.$$

$$\begin{aligned} I_x\dot{\omega}_x &= -(I_z - I_y)\omega_y\omega_z, \\ I_y\dot{\omega}_y &= -(I_x - I_z)\omega_x\omega_z, \\ I_z\dot{\omega}_z &= -(I_y - I_x)\omega_x\omega_y. \end{aligned}$$

# Example: Rotational Motion of a Rigid Body in 3D-space

Suppose w.l.o.g., that $I_x \geq I_y \geq I_z > 0$. For notational simplicity, define

$$\omega_x \mapsto x \qquad a = \frac{I_y - I_z}{I_x},$$

$$\omega_y \mapsto y \qquad b = \frac{I_x - I_z}{I_y},$$

$$\omega_z \mapsto z \qquad c = \frac{I_x - I_y}{I_z}.$$

Note that $a, b, c, \geq 0$. The equations of motion assumes the form

$$\dot{x} = ayz, \quad \dot{y} = -bxz, \quad \dot{z} = cxy.$$

From here on out, assume that the principal axes are unique; this is equivalent to assuming that $I_x > I_y > I_z$, or that $a, b, c > 0$.

# Example: Rotational Motion of a Rigid Body in $3$D-space

The set of equilibria is

$$(\mathbb{R} \times \{0\} \times \{0\}) \cup (\{0\} \times \mathbb{R} \times \{0\}) \cup (\{0\} \times \{0\} \times \mathbb{R}) \,.$$

## Remark

Physically this corresponds to rotation around one of the principal axes at a constant angular velocity. Note that none of the equilibria is isolated.

Consider first, the equilibrium at the origin and try

$$V(x, y, z) = px^2 + qy^2 + rz^2,$$

where $p, q, r > 0$. Then $V$ is a lpdf. Computing $\dot{V}$:

$$\dot{V} = 2(px\dot{x} + qy\dot{y} + rz\dot{z}) = 2xyz(ap - bq + cr).$$

Clearly, it is possible to choose $p, q, r > 0$ such that

$$ap - bq + cr = 0.$$

For such a choice, $\dot{V} \equiv 0$ and the origin is STABLE.

# Example: Rotational Motion of a Rigid Body in 3D-space

Next, consider the equilibrium of the form $(x_0, 0, 0)$ where $x_0 \neq 0$.

Consider the Lyapunov function candidate $W$, such that $W(x_0, 0, 0) = 0$, and $W(x, y, z) > 0, \ \forall (x, y, z) \neq (x_0, 0, 0)$ and sufficiently near $(x_0, 0, 0)$:

$$W(x, y, z) = cy^2 + bz^2 + \left[2acy^2 + abz^2 + bc(x^2 - x_0^2)\right]^2$$

$W$ is an lpdf w.r.t. the equilibrium $(x_0, 0, 0)$ and routine computations show that $\dot{W} \equiv 0$. Hence $(x_0, 0, 0)$ is a stable equilibrium.

## Discussion

- We could also translate the coordinates such that $(x_0, 0, 0)$ becomes the origin of the new coordinate system and apply the Lyapunov stability theorem directly.
- Is $(0, 0, z_0)$, $z_0 \neq 0$ stable?
- Is $(0, y_0, 0)$, $y_0 \neq 0$ (w.l.o.g., assume $y_0 > 0$) stable?

# Region of Attraction

**Definition (Region of Attraction)**

The REGION OF ATTRACTION is defined as the set of all points $x$ such that $\phi(t; x)$ is defined for all $t \geq 0$ and $\lim_{t\to\infty} \phi(t; x) = 0$.

- Finding the exact RoA is usually difficult.
- Lyapunov fcns. can be used to estimate (inner approx.) the RoA.
- From the proof of the Lyapunov stability theorem, if there is a Lyapunov fcn. that satisfies asymptotic stability and if $\mathcal{M}_c$ is bounded and contained in $D$, then $\mathcal{M}_c$ is (positively) invariant.
- The estimate $\mathcal{M}_c$ of the RoA may be conservative (inner approximation).
- QUESTION: Under what conditions is the RoA the whole space?
  - If so, the origin is said to be *globally asymptotically stable*.
  - The conditions of the Lyapunov theorem must clearly hold for $D = \mathbb{R}^n$. But is this sufficient?

# Region of Attraction

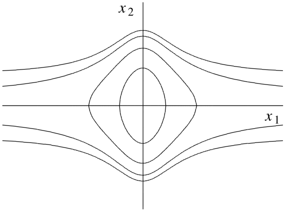



**Figure:** Level sets of $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$.

For $\mathcal{M}_c$ to be bounded ($\mathcal{M}_c \subseteq \mathring{B}_r$, for some $r \geq 0$), $c < \inf_{\|x\| \geq r} V(x)$. If

$$l = \lim_{r\to\infty} \inf_{\|x\| \geq r} V(x) < \infty$$

then $\mathcal{M}_c$ will be bounded only if $c < l$. Consider (see figure)

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2.$$

In this example,

$$l = \lim_{r\to\infty} \min_{\|x\| = r} V(x) = 1.$$

# Region of Attraction

For $\mathcal{M}_c$ to be bounded ($\mathcal{M}_c \subseteq \mathring{B}_r$, for some $r \geq 0$), $c < \inf_{\|x\| \geq r} V(x)$. If

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$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2.$$

In this example,

$$l = \lim_{r \to \infty} \min_{\|x\| = r} V(x) = 1.$$

An extra condition that ensures that $\mathcal{M}_c$ is bounded for all $c > 0$ is

$$V(x) \to \infty \ \text{ as } \ \|x\| \to \infty.$$

**Homework**

Show that a continuously differentiable map $V : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded if and only if it is proper (inverse images of compact sets under $V$ are compact).

# Region of Attraction

### Theorem (Global Asymptotic Stability)

*Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and the conditions of the Lyapunov stability theorem hold (asymptotic). If, in addition,*

$$\|x\| \to \infty \;\Rightarrow\; V(x) \to \infty$$

*then $x = 0$ is globally asymptotically stable.*

### Remark

*For $x = 0$ to be GAS, it must be the unique equilibrium point of the system (why?).*

# Chetaev's Instability Theorem

## Theorem

*Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some $x_0$ with arbitrarily small $\|x_0\|$. Let*

$$U := \{x \in B_r : V(x) > 0\}$$

*and suppose that $\dot{V}(U) > 0$. Then, $x = 0$ is unstable.*

*Proof.* $x_0 \in \mathring{U}$ and $V(x_0) = a > 0$. The trajectory $x(t)$ starting at $x(0) = x_0$ must leave $U$. Indeed, as long as $x(t) \in U$, $V(x(t)) \geq a$, since $\dot{V}(U) > 0$. Let $\min\{\dot{V}(x) : x \in U \text{ and } V(x) \geq a\} := \gamma > 0$. Then,

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))\, \mathrm{d}s \geq a + \int_0^t \gamma\, \mathrm{d}s = a + \gamma t.$$

Hence, $x(t)$ will leave $U$ because $V(x)$ is bounded on $U$. Now, $x(t)$ cannot leave $U$ through $V(x) = 0$ since $V(x(t)) \geq a$. Hence it must leave $U$ through the sphere $\mathbb{S}_r$. Note: $\|x_0\|$ was arbitrarily small. $\square$

# Example: Rotational Motion of a Rigid Body

Consider an equilibrium of the form $(0, y_0, 0)$, $y_0 > 0$ and translate the coordinates so that the equilibrium under study becomes the origin. Setting $y_s = y - y_0$, the equations of motion are

$$\dot{x} = ay_s z + ay_0 z, \quad \dot{y}_s = -bxz, \quad \dot{z} = cxy_s + cxy_0.$$

Now, apply Chetaev's theorem with

$$\begin{aligned} V(x, y, z) &= xz, \\ B_r &= \{(x, y_s, z) : x^2 + y_s^2 + z^2 < r^2\}, \\ U &= \{(x, y_s, z) \in B_{\frac{r}{2}} : x > 0 \text{ and } z > 0\}. \end{aligned}$$

Then $U$ is open and

$$\dot{V} = x\dot{z} + \dot{x}z = 2(y_s + y_0)(cx^2 + az^2).$$

If $(x, y_s, z) \in U$, then $y_s + y_0 > 0$, so Chetaev's theorem yields that the origin (in the new coordinate system) is UNSTABLE.

# The Invariance Principle

# Intuition: Damped Pendulum

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -a \sin x_1 - b x_2^2.$$

Lyapunov function candidate

$$V(x) = a(1 - \cos x_1) + \tfrac{1}{2} x_2^2.$$

$$\dot{V}(x) = -b x_2^2 \le 0.$$

- $\dot{V}(x) < 0$ if and only if $x_2 \neq 0$.
- For the system to maintain $\dot{V}(x) = 0$, it has to stay on $x_2 = 0$.
- Unless $x_1 = 0$, this is impossible:

$$x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_2 \equiv 0 \;\Rightarrow\; \sin x_1(t) \equiv 0.$$

- Hence, on the segment $-\pi < x_1 < \pi$ of the $x_2 = 0$ line, the system can maintain $\dot{V}(x) = 0$ only at the origin $x = 0$.
- Therefore, $V(x(t))$ must decrease towards 0 and, consequently,

$$x(t) \xrightarrow{t \to \infty} 0.$$

## Definition (Limit points and limit sets)

A point $p$ is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$.

The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$.

## Definition (Positively Invariant Set)

A set $M$ is said to be an *invariant set* w.r.t. (1) if

$$x(0) \in M \;\Rightarrow\; x(t) \in M, \;\forall t \in \mathbb{R}.$$

That is, if a solution belongs to $M$ at some time instant, then it belongs to $M$ for all future and past time.

A set $M$ is said to be a *positively invariant set* if

$$x(0) \in M \;\Rightarrow\; x(t) \in M, \;\forall t \geq 0.$$

# Distance to an (Invariant) Set

**Definition (Distance and Convergence to a Set)**

We say that $x(t)$ approaches a set $M$ as $t \to \infty$, if for each $\varepsilon > 0$, $\exists T > 0$ such that

$$\inf_{x \in M} \|p - x\| =: \operatorname{dist}(x(t), M) < \varepsilon, \ \forall t > T.$$

- An asymptotically stable equilibrium point is the positive limit set of every solution starting sufficiently near the equilibrium point.
- A stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle.
- The solution approaches the limit cycle as $t \to \infty$. Notice: the solution does not approach any specific point on the limit cycle.
- The statement $x(t)$ approaches $M$ as $t \to \infty$ does not imply that $\lim_{t \to \infty} x(t)$ exists.
- The set $\mathcal{M}_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ for all $x \in \mathcal{M}_c$ is a positively invariant set.

# Limit Sets and Krasovskii-LaSalle Theorem

### Lemma

*If a solution $x(t)$ is bounded and belongs to $D$ for $t \geq 0$, then its positive limit set $L^+$ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches $L^+$ as $t \to \infty$.*

### Theorem (Krasovskii-LaSalle Theorem)

*Let $\Omega \subseteq D$ be a compact set that is positively invariant w.r.t. (1). Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in $\Omega$. Let $E$ be the set of all points in $\Omega$ where $\dot{V}(x) = 0$. Let $M$ be the largest invariant set in $E$. Then every solution starting in $\Omega$ approaches $M$ as $t \to \infty$.*

*Proof.* Let $x(t)$ be a solution of (1) starting in $\Omega$. Since $\dot{V}(x) \leq 0$ in $\Omega$, $V(x(t))$ is a decreasing function of $t$. Since $V(x)$ is continuous on the compact set $\Omega$, it is bounded from below on $\Omega$. Therefore, $V(x(t))$ has a limit $a$ as $t \to \infty$. Note that the positive limit set $L^+$ is in $\Omega$ because $\Omega$ is a closed set. For any $p \in L^+$, there is a sequence $t_n$ with $t_n \to \infty$ and $x(t_n) \to p$ as $n \to \infty$. By the continuity of $V(x)$, $V(p) = \lim_{n\to\infty} V(x(t_n)) = a$. Hence, $V(x) = a$ on $L^+$. Since $L^+$ is an invariant set, $\dot{V}(x) = 0$ on $L^+$. Thus,

$$L^+ \subseteq M \subseteq E \subseteq \Omega$$

Since $x(t)$ is bounded, $x(t)$ approaches $L^+$ as $t \to \infty$. Hence, $x(t)$ approaches $M$ as $t \to \infty$. $\square$

# Krasovskii-LaSalle Theorem

- Notice that, this theorem does not require the function $V(x)$ to be positive definite.
- The set $\Omega$ does not have to be tied in with the construction of the function $V(x)$.
- However, in many applications, the construction of $V(x)$ will itself guarantee the existence of a set $\Omega$. In particular, if $\mathcal{M}_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ is bounded and $\dot{V}(x) \leq 0$ in $\mathcal{M}_c$, then we can take $\Omega = \mathcal{M}_c$.
- When $V$ is positive definite, $\mathcal{M}_c$ is bounded for sufficiently small $c > 0$. This is not necessarily true when $V$ is not positive definite.
- If $V$ is radially unbounded (or proper), the set $\mathcal{M}_c$ is bounded for all values of $c$. This is true whether or not $V$ is positive definite.

# Corollaries of Krasovskii-LaSalle Theorem

## Corollary

*Let $V : D \to \mathbb{R}$ be a continuously differentiable positive definite function on a domain $D$ containing the equilibrium point $x = 0$, such that $\dot{V}(x) \leq 0$ in $D$. Let $S = \{x \in D : \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in $S$ other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.*

## Corollary

*Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable, radially unbounded, positive defintie function such that $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$. Let $S = \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in $S$ other than the trivial solution $x(t) \equiv 0$. Then, the origin is globally asymptotically stable.*

Notice that when $\dot{V}(x)$ is negative definite, then $S = \{0\}$.

# Remarks on Krasovskii-LaSalle Theorem

- The theorem relaxes the negative definiteness requirement of Lyapunov's theorem.
- It further extends Lyapunov's theorem in three different directions.
  - It gives an estimate of the RoA, which is not necessarily of the form $\mathcal{M}_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$. The set $\Omega$ of the theorem can be ANY compact positively invariant set.
  - The theorem can be used in cases where the system has an equilibrium set, rather than an isolated equilibrium point.
  - The function $V$ does not have to be positive definite.

# Example: Stabilization of a Rigid Robot without Gravity

## Setup

Let $q = (q_1, \dots, q_n)$ denote the vector of generalized coordinates of the robot and $u = (u_1, \dots, u_n)$ denote the vector of generalized forces. The dynamics are given by the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u,$$

where $L$ is the Lagrangian of the system. Since there is no gravity, the potential energy $\mathcal{P} = 0$ can be taken. Thus,

$$L = K = \frac{1}{2}\dot{q}^\top M(q)\dot{q}.$$

$M(q) \in \mathbb{S}^n_{++}$ is called the **inertia matrix**. There exist positive constants $\alpha$ and $\beta$ such that

$$0 < \alpha \leq \lambda_{\min}\left[M(q)\right] \leq \lambda_{\max}\left[M(q)\right] \leq \beta, \quad \forall q.$$

# Example: Stabilization of a Rigid Robot without Gravity

## The Euler-Lagrange equations

With $L = K$, we have

$$\sum_{j=1}^{n} m_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk}(q)\dot{q}_j\dot{q}_k = u_i, \quad i = 1, \ldots, n,$$

where

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial m_{ik}}{\partial q_j} + \frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{jk}}{\partial q_i}\right)$$

are called the *Christoffel symbols*. Compactly, we have

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u,$$

where the $(i,j)^{\text{th}}$ element of $C(q,\dot{q})$ is

$$c_{ij}(q,\dot{q}) = \sum_{k=1}^{n} c_{ijk}(q)\dot{q}_k.$$

# Example: Stabilization of a Rigid Robot without Gravity

## State equations and naïve control

Introduce the state variables $x = q$, $y = \dot{q}$ so that

$$\dot{x} = y, \qquad \dot{y} = [M(x)]^{-1} [u - C(x, y)y] .$$

Suppose we want to asymptotically stabilize the state $(x, y)$ to a desired value $(x_d, 0)$. Let us try the naïve control law

$$u = -K_p(x - x_d) - K_d y,$$

where $K_p, K_d \in \mathbb{S}^n_{++}$. The closed-loop dynamics become

$$\dot{x} = y, \qquad \dot{y} = -[M(x)]^{-1} [K_p(x - x_d) + K_d y + C(x, y)y] .$$

## Lyapunov analysis

Consider the Lyapunov function candidate

$$V = \frac{1}{2}\left[y^\top M(x)y + (x - x_d)^\top K_p(x - x_d)\right].$$

The first term is the kinetic energy, while the second term is the potiential energy due to proportional feedback. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[m_{ij}(x)\right] = \sum_{k=1}^{n} \frac{\partial m_{ij}(x)}{\partial x_k} y_k.$$

Define the $(i,j)^{\text{th}}$ element of $\dot{M}(x,y) \in \mathbb{R}^{n\times n}$ by the RHS above. Now,

$$\begin{aligned}
\dot{V} &= y^\top M(x)\dot{y} + \frac{1}{2}y^\top \dot{M}(x,y)y + \dot{x}^\top K_p(x - x_d) \\
&= -y^\top\left[K_p(x - x_d) + K_d y + C(x,y)y\right] + \frac{1}{2}y^\top \dot{M}(x,y)y + y^\top K_p(x - x_d) \\
&= -y^\top K_d y + \frac{1}{2}y^\top\left[\dot{M}(x,y) - 2C(x,y)\right]y = -y^\top K_d y + \frac{1}{2}y^\top D(x,y)y.
\end{aligned}$$

# Example: Stabilization of a Rigid Robot without Gravity

## Skew-symmetry of $D(x, y) := \dot{M}(x, y) - 2C(x, y)$

We perform the computations in coordinates

$$d_{ij} = \dot{m}_{ij} - 2c_{ij} = \left[\sum_{k=1}^{n} \frac{\partial m_{ij}}{\partial x_k} - \left(\frac{\partial m_{ik}}{\partial x_j} + \frac{\partial m_{ij}}{\partial x_k} - \frac{\partial m_{jk}}{\partial x_i}\right)\right] y_k$$

# Example: Stabilization of a Rigid Robot without Gravity

## Skew-symmetry of $D(x, y) := \dot{M}(x, y) - 2C(x, y)$

$$d_{ij} = \dot{m}_{ij} - 2c_{ij} = \left[\sum_{k=1}^{n} \cancel{\frac{\partial m_{ij}}{\partial x_k}} - \left(\frac{\partial m_{ik}}{\partial x_j} + \cancel{\frac{\partial m_{ij}}{\partial x_k}} - \frac{\partial m_{jk}}{\partial x_i}\right)\right] y_k$$

$$= \sum_{k=1}^{n} \left(\frac{\partial m_{jk}}{\partial x_i} - \frac{\partial m_{ik}}{\partial x_j}\right) y_k.$$

Interchanging $i$ and $j$ gives

$$d_{ji} = \sum_{k=1}^{n} \left(\frac{\partial m_{ik}}{\partial x_j} - \frac{\partial m_{jk}}{\partial x_i}\right) y_k = -d_{ij}.$$

# Example: Stabilization of a Rigid Robot without Gravity

## Lyapunov analysis – resumed

Hence $D$ is skew-symmetric and hence $y^\top D y = 0$, so that

$$\dot{V} = -y^\top K_d y \leq 0.$$

The set $E$ of Krasovskii-LaSalle theorem is given by

$$E = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \dot{V} \equiv 0\} = \mathbb{R}^n \times \{0\}.$$

Suppose $(x(t), y(t))$ is a trajectory that lies entirely in $E$. Then

$$y \equiv 0 \implies \dot{y} \equiv 0 \implies K_p(x - x_d) \equiv 0 \implies x \equiv x_d, \quad \forall t \geq 0.$$

Hence $E$ contains no trajectories of the system other than the equilibrium $(x_d, 0)$. It follows from the Krasovskii-LaSalle theorem that this equilibrium is GLOBALLY ASYMPTOTICALLY STABLE.

# Stability of Linear Systems

# Autonomous Linear Systems

We restrict our attention to linear *autonomous* systems of the form

$$\dot{x}(t) = Ax(t). \tag{2}$$

**Theorem**

*The equilibrium 0 of (2) is (globally) exponentially stable if and only if all eigenvalues of $A$ have negative real parts. The equilibrium is stable if and only if all eigenvalues of $A$ have nonpositive real parts, and in addition, every eigenvalues of $A$ having a zero real part is a simple zero of the minimal polynomial of $A$.*

# Lyapunov Function

Given the system (2), we choose a Lyapunov function candidate:

$$V(x) = x^\top P x \implies \dot{V} = \dot{x}^\top P x + x^\top P \dot{x} = -x^\top Q x,$$

where $P = P^\top$ and

$$A^\top P + PA = -Q. \tag{3}$$

Equation (3) is commonly known as the **Lyapunov Matrix Equation**.

## Remark (Stability)

*If a pair of matrices $(P, Q)$ satisfying (3) can be found such that both $P$ and $Q$ are positive definite, then both $V$ and $-\dot{V}$ are positive definite functions and $V$ is radially unbounded. Hence, the equilibrium $0$ is globally exponentially stable.*

*If a pair $(P, Q)$ can be found s.t. $Q > 0$ and $P$ has at least one nonpositive eigenvalue, then $-\dot{V} > 0$ and $V$ assumes nonpositive values arbitrarily close to the origin. Hence $0$ is unstable.*

**Lemma**

*Let $\{\lambda_i\}_1^n$ denote the eigenvalues of A. Then equation (3) has a unique solution for P corresponding to each $Q \in \mathbb{R}^{n\times n}$ iff*

$$\lambda_i + \lambda_j \neq 0, \quad \forall i, j.$$

**Corollary**

*If for some $Q \in \mathbb{R}^{n\times n}$ does not have a unique solution for P, then the origin is not an asymptotically stable equilibrium.*

*Proof.* If all eigenvalues of $A$ has negative real parts, then the equation above is satisfied. $\square$

# Main Result

## Theorem

*Given a matrix $A \in \mathbb{R}^{n\times n}$, the following are equivalent:*

- *$A$ is a Hurwitz matrix (all its e.vals have negative real parts).*
- *There exists* SOME *$Q \in \mathbb{S}^n_{++}$ such that equation (3) has a corresponding unique solution for $P \in \mathbb{S}^n_{++}$.*
- *For* EVERY *$Q \in \mathbb{S}^n_{++}$, equation (3) has a unique solution for $P \in \mathbb{S}^n_{++}$.*

*Proof.* "(3) $\implies$ (2)" Obvious.
"(2) $\implies$ (1)" Suppose (2) is true for some particular matrix $Q$. Consider the candidate $V(x) = x^\top Px$. Then $\dot{V}(x) = -x^\top Qx$, and one can conclude that 0 is asymptotically stable. Hence $A$ is Hurwitz.
"(1) $\implies$ (3)" Omitted (see Section 5.4, Theorem (42) in Vidyasagar, "*Nonlinear Systems Analysis*", 1993.)

# Control-Lyapunov Functions

# Control-Lyapunov Functions [1]

Consider the control system with state $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$, $\forall t$:

$$\dot{x}(t) = f(x(t)) + u_1(t)g_1(x(t)) + \cdots + u_m(t)g_m(x(t)), \quad f(0) = 0. \tag{4}$$

**Definition (Control-Lyapunov Function (clf))**

A clf is a smooth, proper, and positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ so that

$$\inf_{u \in \mathbb{R}^m} \{\mathcal{L}_f V(x) + u_1 \mathcal{L}_{g_1} V(x) + \cdots + u_m \mathcal{L}_m g_M V(x)\} < 0, \quad \forall x \neq 0.$$

- $V$ is such that for each $x \neq 0$, one *can* diminish its value by applying *some* open-loop control.
- Existence of a clf implies that the system is asymp. controllable:

[1] As discussed in *Sontag, "A 'universal' construction of Artstein's theorem on nonlinear stabilization"*, 1989.

# Control-Lyapunov Functions: Single input

There exists a feedback law which is smooth on $\mathbb{R}^n_0 := \mathbb{R}^n - 0$

$$u = k(x), \quad k(0) = 0,$$

and which globally stabilizes the system.

Assume $V$ is a clf for the system

$$\dot{x} = f(x) + ug(x).$$

Denote

$$a(x) := \nabla V(x) \cdot f(x),$$
$$b(x) := \nabla V(x) \cdot g(x).$$

The condition that $V$ is a clf is precisely the statement that

$$b(x) = 0 \implies a(x) < 0, \quad \forall x \neq 0.$$

On the other hand, $V$ is a Lyapunov function if

$$\nabla V(x) \cdot (f(x) + k(x)g(x)) < 0,$$

that is

$$a(x) + k(x)b(x) < 0, \quad \forall x \neq 0.$$

# Control-Lyapunov Functions: Single input

In this simple case where the family $(a(x), b(x))$, interpreted as a *family of linear systems parametrized by $x$* the following works:

$$k := -\frac{1}{b}\left(a + \sqrt{a^2 + b^2}\right).$$

Along trajectories of the closed-loop system, one has

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\sqrt{a^2 + b^2} < 0.$$

This feedback law may fail to be continuous, but with the slight modification

$$k := -\frac{1}{b}\left(a + \sqrt{a^2 + b^4}\right),$$

then it does become continuous.

# Control-Lyapunov Functions: Multi input

Now, consider the system back in equation (4).

- A sufficient conditions for a given $k$ to be smooth feedback stabilizer is that there exist a Lyapunov function $V$ so that

$$\nabla V(x) \cdot [f(x) + k_1(x)g_1(x) + \cdots + k_m(x)g_m(x)] < 0, \quad \forall x \neq 0.$$

- Such a Lyapunov function is automatically a clf.
- If $k$ happens to be continous at the origin, then the following property (`small control property`) holds (with $u := k(x)$)
  *For each $\varepsilon > 0$, there is $\delta > 0$ s.t., if $x \neq 0$ satisfies $\|x\| < \delta$, then there is some $u$ with $\|u\| < \varepsilon$ s.t.*

$$\nabla V(x) \cdot [f(x) + u_1 g_1(x) + \cdots + u_m g_m(x)] < 0.$$

# Control-Lyapunov Functions: Multi input

## Theorem

*If $\exists$ a smooth clf $V$ then $\exists$ a smooth feedback stabilizer $k$. If $V$ satisfies the small control property, then $k$ can be chosen to be also continuous at* $0$.

*Proof. (Sketch).* The proof involves constructing a fixed function $\phi$ of two variables, and then designing a feedback law in closed-form, from the evaluation of this function at a point determined by $\nabla V(x) \cdot f(x)$ and the $\nabla V(x) \cdot g_i(x)$'s.

Define the following function (and then show that it is analytic.)

$$\phi(a, 0) := 0, \quad \forall a < 0$$

and

$$\phi(a, b) := \frac{1}{b}\left(a^2 + bq(b)\right), \quad q(0) = 0 \ \text{ and } \ bq(b) > 0.$$

For example, we can choose $q(b) = b$ or $q(b) = b^3$, etc.

# Control-Lyapunov Functions: Multi input

*Proof. (Cont'd).* Assume that $V$ is a clf and let

$$a(x) := \nabla V(x) \cdot f(x),$$
$$b_i(x) := \nabla V(x) \cdot g_i(x), \quad i = 1, \ldots, m.$$

Further, let

$$B(x) := (b_1(x), \ldots, b_m(x)),$$
$$\beta(x) := \|B(x)\|^2 = \sum_{i=1}^{m} b_i^2(x).$$

The condition that $V$ is a clf is equivalent to $\beta(x) = 0 \implies a(x) < 0$.
Now, define the smooth feedback law $k = (k_1, \ldots, k_m)$:

$$k_i(x) := -b_i(x)\phi(a(x), \beta(x)), \quad x \neq 0,$$

and $k(0) := 0$.

# Control-Lyapunov Functions: Multi input

*Proof. (Cont'd).* At a nonzero $x$ we have that

$$\begin{aligned}\nabla V(x) \cdot \left[ f(x) + \sum_{i=1}^{m} k_i(x) g_i(x) \right] &= a(x) - \phi\left(a(x), \beta(x)\right) \beta(x) \\ &= -\sqrt{a(x)^2 + \beta(x) q\left(\beta(x)\right)} < 0.\end{aligned}$$

so the original $V$ decreases along trajectories of the closed-loop system.

We have still yet to show that $V$ satisfies the small control property. The audience is invited to see the paper for the detailed proof of this. □

# Morse-Lyapunov Functions

# Isolated Critical Points

**Lemma**

*Suppose that $x_e$ is an equilibrium points of the dynamical system $(M, \varphi)$. If $V : \mathcal{M} \to \mathbb{R}$ is a differentiable Lyapunov function then $x_e$ is the only critical point of $V$.*

*Proof.* Suppose $V$ has another critical point, $x_c$, in the domain of attraction. By the definition of a Lyapunov function, we must have $\mathcal{L}_f V(x_c) = 0$. This contradicts the fact that if $x \neq x_e$, $\mathcal{L}_f V(x) < 0$.

# Morse Lemma

## Theorem (Morse Lemma)

*Let $p \in \mathcal{M}$ be a nondegenerate critical point of a smooth function $V : \mathcal{M} \to \mathbb{R}$. There exists a local coordinate system $\{x_i\}_1^n$ in a nbhd. $\mathcal{N} \subseteq \mathcal{M}$ of $p$ with $x_i(p) = 0$ for all $1 \leq i \leq n$ such that for $x \in \mathcal{N}$,*

$$V(x) = V(p) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_n^2$$

*where $i = ind(V, p)$.*

## Corollary

*Let $p \in \mathcal{M}$ be an equilibrium point of $(\mathcal{M}, \varphi)$ and $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$ a Morse-Lyapunov function. There exists a local coordinate system $\{x_i\}_1^n$ around $p$ such that $V$ is locally the canonical quadratic Lyapunov function*

$$V(x) = \sum_{i=1}^{n} x_i^2$$

*with $ind(V, p) = 0$.*

## Theorem (Deformation Lemma)

*Let $V : \mathcal{M} \to \mathbb{R}$ be a smooth function and $a, b \in V(\mathcal{M})$ such that $a < b$. If $\mathcal{M}_{a,b}$ is compact and does not contain critical points of $V$ then $\mathcal{M}_a$ is diffeomorphic to $\mathcal{M}_b$. MOreover, $\mathcal{M}_a$ is a deformation retract of $\mathcal{M}_b$.*

## Corollary

*Let $\mathcal{M}$ be a smooth Riemannian manifold. If $\mathcal{M}$ contains a closed invariant asymptotically stable set, then for all $a, b \in V(\mathcal{M})$, $\mathcal{M}_a$ is diffeomorphic to $\mathcal{M}_b$ and $\mathcal{M}_a$ is a deformation retract of $\mathcal{M}_b$ where $V$ is a smooth Lyapunov function.*

# Systems with Single Critical Points

# Domain of Attraction – Revisited

**Theorem (Brown-Stallings Lemma)**

*Let $\mathcal{M}$ be a paracompact manifold such that every compact subset is contained in an open set diffeomorphic to a Euclidean space. Then $\mathcal{M}$ itself is diffeomorphic to a Euclidean space.*

**Corollary**

*Let $\mathcal{M}$ be a paracompact manifold. The domain of attraction of an asymptotically stable equilibrium point is diffeomorphic to a Euclidean space.*

# Morse and Sontag Theorems

**Theorem (Morse Theorem)**

*Let $V : \mathcal{M} \to \mathbb{R}$ be a Morse function, $p$ a critical point such that $ind(V, p) = i$ and $c = V(p)$. If there exists $\varepsilon > 0$ such that $\mathcal{M}_{c-\varepsilon,c+\varepsilon}$ is compact and does not contain other critical points $p$, then $\mathcal{M}_{c-\varepsilon} \cup e_i$ is a deformation retract of $\mathcal{M}_{c+\varepsilon}$ where $e_i$ is an i-cell.*

**Theorem (Sontag Theorem)**

*Let us consider the dynamical system $(\mathcal{M}, \varphi)$ with an equilibrium point $x_e \in \mathcal{M}$. Suppose that $x_e$ is asymptotically stable. Then the domain of attraction of $x_e$, given by*

$$\mathcal{A} = \left\{ x \in \mathcal{M} : \lim_{t\to\infty} \varphi(t, x) = x_e \right\},$$

*is contractible.*

# Systems with Multiple Critical Points

# Morse Theorem – (Third Version)

## Theorem (Morse Theorem)

*If $V : \mathcal{M} \to \mathbb{R}$ is a Morse function such that $\mathcal{M}_a$ is compact for each $a \in \mathbb{R}$ then $\mathcal{M}$ has the homotopy type of a CW-complex with one $i$-cell for each critical point of index $i$.*

## Corollary

*Suppose that the dynamical system $(\mathcal{M}, \varphi)$ has several equilibria $(x_1, \ldots, x_k)$. If there exists a Morse-Lyapunov function $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$ then $\{x_1, \ldots, x_k\}$ is a retract of the domain of attraction.*

## Proposition (Reeb Theorem)

*Suppose that $\mathcal{M}$ is compact without boundary. If $V : \mathcal{M} \to \mathbb{R}$ is a smooth function with only two critical points, then $\mathcal{M}$ is homeomorphic to the $n$-sphere $\mathbb{S}^n$.*

# Morse Inequalities

## Theorem (Morse Inequalities)

*Let $m_k$ be the number of ciritcal points of a Morse function $V$ with index $k$. Then, we have*

$$b_k \leq m_k, \quad \forall k,$$

$$\sum_{i=0}^{j}(-1)^{j-i} b_i \leq \sum_{i=0}^{j}(-1)^{j-i} m_i \quad \forall j,$$

$$\chi(\mathcal{M}) = \sum_k (-1)^k b_k = \sum_k (-1)^k m_k.$$

The next corollary states a necesary condition for the existence of a Morse-Lyapunov function based on the Euler characteristic, which is a topological invariant.

# Existence of Morse-Lyapunov Functions

## Corollary

*Consider the dynamical system* $(\mathcal{M}, \varphi)$ *with several equilibria* $(x_1, \ldots, x_k)$*. If there exists a Morse-Lyapunov function* $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$ *then* $\chi(\mathcal{M}) = k \geq b_0$*.*

*Proof.* If there exists a Morse-Lyapunov function $V$, $(x_1, \ldots, x_k)$ are the only critical points with indices 0. Then, by the Morse inequalities, $\chi(\mathcal{M}) = m_0 = k$ and $b_0 \leq m_0 = k$.

## Remark

*If* $\chi(\mathcal{M}) \neq k$ *then there is no Morse-Lyapunov function for the dynamical system.*