# ECE 661: Nonlinear Systems
# Spring 2021 | Homework #5

Question 1 ........................................ 20 (main) + 0 (bonus) points

For the rotational motion of a rigid body in 3D-space example, it is desired to analyze the stability of an equilibrium of the form $(0, 0, z_0)$ where $z_0 \neq 0$. Set up a new set of coordinates such that the equilibrium under study is the origin of the new set. Define a suitable Lyapunov function such that the stability of the equilibrium can be established by applying Lyapunov stability theorem.

Question 2 ........................................ 20 (main) + 0 (bonus) points

Suppose a particle of mass $m$ is moving in a smooth potential field. To simplify the problem, suppose the motion is one-dimensional. Let $x$ denote the position coordinate of the particle, and let $\phi(x)$ denote the potential energy at $x$. If the only force acting on the particle is due to the potential, then the motion of the particle is described by

$$m\ddot{x} = -\phi'(x) =: f(x),$$

where the prime denotes differentiation with respect to $x$. Show that every local minimum of the function $\phi$ is a stable equilibrium.

Question 3 ........................................ 20 (main) + 0 (bonus) points

Consider the autonomous differential equation

$$\dot{x}(t) = f(x(t)),$$

and suppose $f$ is a $C^1$ function such that $f(0) = 0$. Then there exists a $C^1$ matrix-valued function $A$ such that

$$f(x) = A(x)x, \quad \forall x \in \mathbb{R}^n.$$

(a) [15 points] Show that if the matrix $A^\top(0) + A(0)$ is negative definite, then the origin is an exponentially stable equilibrium. More generally, show that if there exists a positive definite matrix $P$ such that $A^\top(0)P + PA(0)$ is negative definite, then the origin is an exponentially stable equilibrium. (Hint: Consider the Lyapunov function candidate $V(x) = \|x\|^2$.)

(b) [5 points] Extend the results in $(a)$ to global stability.

Question 4 ........................................ 20 (main) + 0 (bonus) points

Consider the system

$$\dot{x}_1 = x_1 + 2x_2^2, \quad \dot{x}_2 = 2x_1x_2 + x_2^2.$$

Using the Lyapunov function candidate

$$V(x) = x_1^2 - x_2^2,$$

show that 0 is an unstable equilibrium.

Question 5 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20 (main) + 0 (bonus) points

Given a finite collection of $n \times n$ matrices $A_1, \dots, A_k$, define their *convex hull* $S$ as

$$S = \{A = \sum_{i=1}^{k} \lambda_i A_i : \lambda_i \geq 0,\ \forall i, \sum_{i=1}^{k} \lambda_i = 1\}.$$

(a) Suppose there exists a positive definite matrix $P$ such that $A_i^\top P + PA_i$ is negative definite for each $i$ between 1 and $k$. Show that every matrix in the set $S$ is Hurwitz.

(b) Consider the differential equation

$$\dot{x}(t) = A(t)x(t), \quad \text{where } A(t) \in S, \quad \forall t \geq 0.$$

Show that 0 is an exponentially stable equilibrium of this system.

Question 6 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10 (bonus) points

Show that a continuously differentiable map $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) \geq 0$ is radially unbounded if and only if it is proper (inverse images of compact sets under $f$ are compact).