Nonlinear Systems

Morse Theory and Lyapunov Stability on Manifolds

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Outline

Lyapunov stability theory

- Introduced by Alexandr Mikhailovich Lyapunov.
- *The general problem of the stability of motion*, 1892.
- Doctoral thesis in Kharkov Mathematical Society.
- The most general theory for analyzing stability of (at least) ordinary differential equations.

Notations and Definitions

Manifolds and Vector Fields

- $\mathcal{M}$ (state-space) denotes a manifold of finite dimension n.
- $f \in \mathfrak{X}(M)$ is a continuous vector field on $\mathcal{M}$.
- We assume that there exists a unique right maximally defined integral curve of f starting at x.
- We also assume that this integral curve is defined on $[0, \infty]$.

$$\varphi : [0, \infty] \times \mathcal{M} \to \mathcal{M}$$

with

$$\varphi(0, x) = x,$$
$$\varphi(t_1, \varphi(t_2, x)) = \varphi(t_1 + t_2, x).$$

- The semiflow φ is the evolution function.

Invariant and Stable Sets

Definition

$\Omega \subseteq \mathcal{M}$ is called an INVARIANT SET if for all $x \in \Omega$ and $t \in \mathbb{R}_{\geq 0}$, $\varphi(t, x) \in \Omega$. If $\Omega = \{p\}$ is a singleton, then Ω is called and EQUILIBRIUM POINT of the dynamical system $(\mathcal{M}, \varphi)$.

Definition

$\Omega \subseteq \mathcal{M}$ is STABLE if for *every open neighborhood* $\mathcal{U} \subseteq \mathcal{M}$ of Ω, *there exists a neighborhood* $\mathcal{V} \subseteq \mathcal{M}$ of Ω such that $\varphi(t, \mathcal{V}) \subseteq \mathcal{U}$ for all $t \geq 0$.

An *invariant set* Ω is asymptotically stable if

- Ω is stable,
- Ω is attractive, i.e., for all $x \in \Omega$, there exists an open neighborhood $\mathcal{N} \subseteq \mathcal{M}$ of Ω such that for all $x \in \mathcal{N}$, $\varphi(t, x) \xrightarrow{t \to \infty} \Omega$.

Domain (Region) of Attraction

The domain of attraction is denoted by

$$\mathcal{A} = \{x \in \mathcal{M} : \varphi(t, x) \to \Omega \text{ as } t \to \infty\}.$$

Ω is said to be GLOBALLY asymptotically stable if $\mathcal{N} = \mathcal{M}$.

Definition

The LIE DERIVATIVE of $V : \mathcal{M} \to \mathbb{R}$ along $f \in \mathfrak{X}(\mathcal{M})$ is defined by

$$\begin{aligned} \mathcal{L}_f V : \mathcal{M} &\to \mathbb{R}, \\ p &\mapsto \mathrm{d}V_p(f(p)). \end{aligned}$$

Lyapunov Function

Definition

Let $\mathcal{K}$ be an invariant set of the dynamical system $(\mathcal{M}, \varphi)$. A continuous function $V : \mathcal{A} \to \mathbb{R}_{\geq 0}$ is a Lyapunov function if

- $V(x) > 0$ for all $x \in \mathcal{A} \backslash \mathcal{K}$,
- $V(x) = 0$ for all $x \in \mathcal{K}$,
- V is proper, i.e., $V^{-1}(B)$ is compact for all compact subset $B \subseteq \mathbb{R}_{\geq 0}$,
- V is strictly decreasing along orbits of φ, i.e.,

$$V \circ \varphi(t, x) < V(x),$$

for all $t > 0$ and $x \in \mathcal{A} \backslash \mathcal{K}$.
If V is differentiable, this condition may be replaced by

$$\mathcal{L}_f V(x) < 0.$$

(Nondegenerate) Critical Points

Definition

Let $V : \mathcal{M} \to \mathbb{R}$ be a smooth function. A CRITICAL POINT, $p \in \mathcal{M}$, of V is a point where the differential

$$\mathrm{d}V_p : T_p\mathcal{M} \to \mathbb{R}$$

has rank zero, i.e., in any local coordinate system $\{x_i\}_1^n$, one has $\frac{\partial V}{\partial x_i}(p) = 0$ for all $i = 1, \ldots, n$.

Definition

A critical point p is NONDEGENERATE if the Hessian $H_p(V)$ is a nondegenerate bilinear form, i.e., if any coordinate system, the Hessian matrix

$$\left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{1 \le i,j \le n}$$

is nondegenerate.

Nondegenerate Critical Points

Definition

The dimension of the subspace of $T_p\mathcal{M}$ on which $H_p(V)$ is negative definite is called the MORSE INDEX of V at p, denoted by $\operatorname{ind}(V, p)$.

Definition

A C^2 function $V : \mathcal{M} \to \mathbb{R}$ is a MORSE FUNCTION if all its critical points are nondegenerate.

Definition

The (SUB)-LEVEL SETS of a function $V : \mathcal{M} \to \mathbb{R}$ are

$$\begin{aligned} \mathcal{M}_a &= V^{-1}\left((-\infty, a]\right), \\ \mathcal{M}_{a,b} &= V^{-1}\left([a, b]\right). \end{aligned}$$

Topological Definitions

- A top. space is an n-CELL if it is homeomorphic to $\mathbb{R}^n$.
- A top. space X is CONTRACTIBLE if it is *homotopy equivalent* to the one-point space.
- A subspace A of X is called a DEFORMATION RETRACT of X if there exists a continuous function $h : [0,1] \times X \to X$ such that for all $x \in X$, $a \in A$,
$$h(0,x) = x,$$
$$h(1,x) \in A,$$
$$h(1,a) = a.$$
- The k^{th} BETTI NUMBER of $\mathcal{M}$, denoted by b_k is the rank of the k^{th} homology group $H^k(\mathcal{M})$.
- The EULER CHARACTERISTIC of $\mathcal{M}$ is defined by
$$\chi(\mathcal{M}) = \sum_{i=1}^{k} (-1)^k b_k.$$

Morse-Lyapunov Functions

Isolated Critical Points

Lemma

Suppose that x_e is an equilibrium points of the dynamical system (M, φ). If $V : \mathcal{M} \to \mathbb{R}$ is a differentiable Lyapunov function then x_e is the only critical point of V.

Proof.

Suppose V has another critical point, x_c, in the domain of attraction. By the definition of a Lyapunov function, we must have $\mathcal{L}_f V(x_c) = 0$. This contradicts the fact that if $x \neq x_e$, $\mathcal{L}_f V(x) < 0$. $\square$

Morse Lemma

Theorem (Morse Lemma)

Let $p \in \mathcal{M}$ be a nondegenerate critical point of a smooth function $V : \mathcal{M} \to \mathbb{R}$. There exists a local coordinate system $\{x_i\}_1^n$ in a nbhd. $\mathcal{N} \subseteq \mathcal{M}$ of p with $x_i(p) = 0$ for all $1 \leq i \leq n$ such that for $x \in \mathcal{N}$,

$$V(x) = V(p) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_n^2$$

where $i = ind(V, p)$.

Corollary

Let $p \in \mathcal{M}$ be an equilibrium point of $(\mathcal{M}, \varphi)$ and $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$ a Morse-Lyapunov function. There exists a local coordinate system $\{x_i\}_1^n$ around p such that V is locally the canonical quadratic Lyapunov function

$$V(x) = \sum_{i=1}^{n} x_i^2$$

with $ind(V, p) = 0$.

Level Sets of a Lyapunov Function

Theorem (Deformation Lemma)

Let $V : \mathcal{M} \to \mathbb{R}$ be a smooth function and $a, b \in V(\mathcal{M})$ such that $a < b$. If $\mathcal{M}_{a,b}$ is compact and does not contain critical points of V then $\mathcal{M}_a$ is diffeomorphic to $\mathcal{M}_b$. MOreover, $\mathcal{M}_a$ is a deformation retract of $\mathcal{M}_b$.

Corollary

Let $\mathcal{M}$ be a smooth Riemannian manifold. If $\mathcal{M}$ contains a closed invariant asymptotically stable set, then for all $a, b \in V(\mathcal{M})$, $\mathcal{M}_a$ is diffeomorphic to $\mathcal{M}_b$ and $\mathcal{M}_a$ is a deformation retract of $\mathcal{M}_b$ where V is a smooth Lyapunov function.

Systems with Single Critical Points

Domain of Attraction – Revisited

Theorem (Brown-Stallings Lemma)

Let $\mathcal{M}$ be a paracompact manifold such that every compact subset is contained in an open set diffeomorphic to a Euclidean space. Then $\mathcal{M}$ itself is diffeomorphic to a Euclidean space.

Corollary

Let $\mathcal{M}$ be a paracompact manifold. The domain of attraction of an asymptotically stable equilibrium point is diffeomorphic to a Euclidean space.

Morse and Sontag Theorems

Theorem (Morse Theorem)

Let $V : \mathcal{M} \to \mathbb{R}$ be a Morse function, p a critical point such that $\mathrm{ind}(V, p) = i$ and $c = V(p)$. If there exists $\varepsilon > 0$ such that $\mathcal{M}_{c-\varepsilon,c+\varepsilon}$ is compact and does not contain other critical points p, then $\mathcal{M}_{c-\varepsilon} \cup e_i$ is a deformation retract of $\mathcal{M}_{c+\varepsilon}$ where e_i is an i-cell.

Theorem (Sontag Theorem)

Let us consider the dynamical system $(\mathcal{M}, \varphi)$ with an equilibrium point $x_e \in \mathcal{M}$. Suppose that x_e is asymptotically stable. Then the domain of attraction of x_e, given by

$$\mathcal{A} = \left\{ x \in \mathcal{M} : \lim_{t \to \infty} \varphi(t, x) = x_e \right\},$$

is contractible.

Systems with Multiple Critical Points

Morse Theorem – (Third Version)

Theorem (Morse Theorem)

If $V : \mathcal{M} \to \mathbb{R}$ is a Morse function such that $\mathcal{M}_a$ is compact for each $a \in \mathbb{R}$ then $\mathcal{M}$ has the homotopy type of a CW-complex with one i-cell for each critical point of index i.

Corollary

Suppose that the dynamical system $(\mathcal{M}, \varphi)$ has several equilibria $(x_1, \ldots, x_k)$. If there exists a Morse-Lyapunov function $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$ then $\{x_1, \ldots, x_k\}$ is a retract of the domain of attraction.

Proposition (Reeb Theorem)

Suppose that $\mathcal{M}$ is compact without boundary. If $V : \mathcal{M} \to \mathbb{R}$ is a smooth function with only two critical points, then $\mathcal{M}$ is homeomorphic to the n-sphere $\mathbb{S}^n$.

Morse Inequalities

Theorem (Morse Inequalities)

Let m_k be the number of ciritcal points of a Morse function V with index k. Then, we have

$$
\begin{gathered}
b_k \leq m_k, \quad \forall k, \\
\sum_{i=0}^{j}(-1)^{j-i} b_i \leq \sum_{i=0}^{j}(-1)^{j-i} m_i \quad \forall j, \\
\chi(\mathcal{M}) = \sum_{k}(-1)^k b_k = \sum_{k}(-1)^k m_k.
\end{gathered}
$$

The next corollary states a necesary condition for the existence of a Morse-Lyapunov function based on the Euler characteristic, which is a topological invariant.

Existence of Morse-Lyapunov Functions

Corollary

Consider the dynamical system $(\mathcal{M}, \varphi)$ with several equilibria $(x_1, \dots, x_k)$. If there exists a Morse-Lyapunov function $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$ then $\chi(\mathcal{M}) = k \geq b_0$.

Proof.

If there exists a Morse-Lyapunov function V, $(x_1, \dots, x_k)$ are the only critical points with indices 0. Then, by the Morse inequalities, $\chi(\mathcal{M}) = m_0 = k$ and $b_0 \leq m_0 = k$. $\square$

Remark

If $\chi(\mathcal{M}) \neq k$ then there is no Morse-Lyapunov function for the dynamical system.