Chapter 1

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Linear and Nonlinear Control Systems

In this first chapter we review the use of state feedback and state estimation for linear systems. Specifically the transformation of a controllable linear system into control canonical form and the transformation of an observable linear system are considered. The procedure of linearizing a nonlinear system about an equilibrium point is also reviewed.

Next examples of nonlinear systems are presented. These examples are used to show how the linear methods of state feedback control and state estimation can be generalized without having to linearize the system about an equilibrium points. These examples are used to motivate the differential geometric approach to the control of nonlinear systems.

1.1 The Control Canonical Form for Linear Time Invariant Systems

Recall that a single-input linear time invariant (LTI) statespace model is given as

$$\frac{dx}{dt} = Ax + bu,\ x \in \mathbb{R}^n,\ u \in \mathbb{R},\ A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n. \tag{1.1}$$

With $n = 3$ suppose A and b have special form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.2}$$

Then a simple calculation shows that the characteristic equation is

$$\det(sI - A) = s^3 + a_2 s^2 + a_1 s + a_0.$$

With the state feedback

$$u = -kx + r = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + r \tag{1.3}$$

the closed-loop system is

$$\frac{dx}{dt} = (A - bk)x + br.$$

More explicitly we have

$$\begin{aligned} \frac{dx}{dt} &= \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r. \end{aligned}$$

Choosing the control gains as $k_1 = \alpha_0 - a_1, k_2 = \alpha_2 - k_2, k_3 = \alpha_3 - a_3$ results in

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

with $\det\Big(sI - (A - bk)\Big) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$.

The special form of the pair (A, b) above is called the control canonical form and its usefulness is clear: If (A, b) is in control canonical form then the feedback (row)vector may be straightforwardly chosen (by inspection!) to assign the coefficients of the closed-loop characteristic equation to any desired values. That is, arbitrary pole placement is possible.

We now consider the above in the context of a physical system: the armature controlled DC motor. The figure below is a physical representation of a single-loop DC motor from [3].

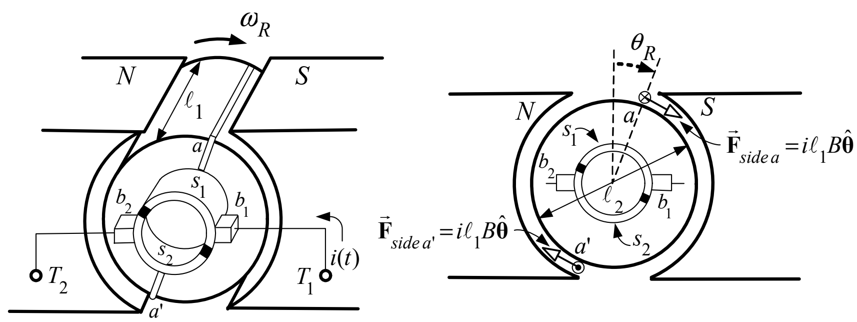

FIGURE 1.1. Physical structure of a single-loop DC motor.

The standard schematic diagram for the DC motor is shown on the right side of Figure 1.2.

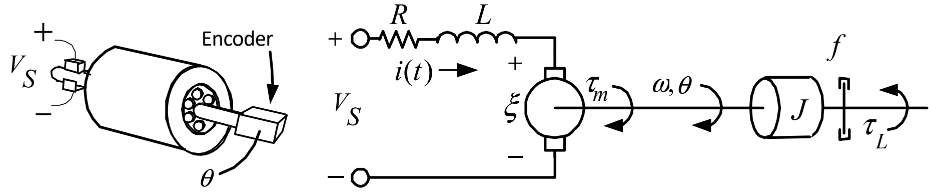

FIGURE 1.2. DC motor drawing and schematic.

Here R is the resistance of the rotor loop, L is the inductance of the rotor loop, i is the current in the rotor loop (armature current), ω is the rotor speed, θ is the rotor position, $\tau_m = K_T i$ is the torque produced by the motor, $\xi = K_b\omega$ is the induced voltage (back emf) in the rotor loop, J is the moment of inertia of the rotor, f is the viscous friction of the rotor (small and due to the ball bearings), and τ_l represents any load torque on the rotor. The differential equation model for the DC motor is then [3]

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega + V_S \\ J\frac{d\omega}{dt} &= K_T i - f\omega - \tau_L \\ \frac{d\theta}{dt} &= \omega. \end{aligned} \tag{1.4}$$

In matrix form this is

$$\frac{d}{dt}\begin{bmatrix} i \\ \omega \\ \theta \end{bmatrix} = \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 \\ K_T/J & -f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} i \\ \omega \\ \theta \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{b} V_S - \begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix}\tau_L. \tag{1.5}$$

For now let $\tau_l = 0$. Note that the pair (A, b) is not in control canonical form and

$$\det(sI - A) = s\Big((s + R/L)(s + f/J) \ + K_T K_b\Big) = s^3 + (R/L + f/J)s^2 + \Big(K_T K_b + \ Rf/(JL)\Big)\, s \tag{1.6}$$

showing the system is unstable as it has a pole at $s = 0$.

Next we look at transforming the system into control canonical form. Set $x_1 = i, x_2 = \omega, x_3 = \theta$, and $u = V_S$ and consider the statespace transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \underbrace{\frac{JL}{K_T}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ K_T/J & -f/J & 0 \end{bmatrix}}_{T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Tx. \tag{1.7}$$

To get into the "spirit" of how we will work with nonlinear transformations we take what may seem to be a rather (unnecessarily) involved approach. Start with rewriting this transformation as

$$\begin{aligned} x_1^* &= T_1(x) \triangleq \frac{JL}{K_T} x_3 \\ x_2^* &= T_2(x) \triangleq \frac{JL}{K_T} x_2 \\ x_3^* &= T_3(x) \triangleq \frac{JL}{K_T}\left(\frac{K_T}{J} x_1 - \frac{f}{J} x_2\right). \end{aligned} \tag{1.8}$$

Then

$$\begin{aligned} \frac{dx_1^*}{dt} = \frac{dT_1}{dt} &= \frac{\partial T_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial T_1}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial T_1}{\partial x_3}\frac{dx_3}{dt} \\ &= \underbrace{\begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \end{bmatrix}}_{dT_1} \underbrace{\begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{bmatrix}}_{Ax+bu}. \end{aligned}$$

Digression on Gradients and Dual Product

The gradient of the scalar function $T_1(x) : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$dT_1 \triangleq \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \end{bmatrix} \in \mathbb{R}^{1\times 3}.$$

We can also write $\dfrac{\partial T_1}{\partial x}$ to denote the gradient, i.e.,

$$\frac{\partial T_1}{\partial x} = \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \end{bmatrix}.$$

The gradient will always be take to be a row vector.

Definition 1 *Dual Product*

With the gradient (row) vector dT_1 and the column vector $\dfrac{dx}{dt}$, their *dual product* $\left\langle dT_1, \dfrac{dx}{dt}\right\rangle$ is defined by

$$\left\langle dT_1, \frac{dx}{dt}\right\rangle \triangleq \begin{bmatrix} \dfrac{\partial T_1}{\partial x_1} & \dfrac{\partial T_1}{\partial x_2} & \dfrac{\partial T_1}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{bmatrix} = \frac{\partial T_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial T_1}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial T_1}{\partial x_3}\frac{dx_3}{dt}.$$

Remark 1 An *inner product* is defined similarly except both must be row vectors or column vectors.

Remark 2 We can also write $\left\langle dT_1, \dfrac{dx}{dt} \right\rangle = dT_1 \dfrac{dx}{dt}$. The gradient (row) vector is a *covariant* vector and the (column) vector $\dfrac{dx}{dt}$ is a *contravarient* vector. This will be explained in the next chapter.

Let's return to the task of finding the system equations in the x^* coordinates. We have

$$\frac{dx_1^*}{dt} = \left\langle dT_1, \frac{dx}{dt} \right\rangle = \langle dT_1, Ax + bu \rangle = \langle dT_1, Ax \rangle + \langle dT_1, b \rangle\, u.$$

Then

$$\begin{aligned}
\langle dT_1, Ax \rangle &= \underbrace{\frac{JL}{K_T} \left[\begin{array}{ccc} 0 & 0 & 1 \end{array}\right]}_{dT_1} \underbrace{\left[\begin{array}{c} -(R/L)x_1 - (K_b/L)x_2 \\ (K_T/J)x_1 - (f/J)x_2 \\ x_2 \end{array}\right]}_{Ax} = \frac{JL}{K_T} x_2 = x_2^* \\
\langle dT_1, b \rangle\, u &= \underbrace{\frac{JL}{K_T} \left[\begin{array}{ccc} 0 & 0 & 1 \end{array}\right]}_{dT_1} \underbrace{\left[\begin{array}{c} 1/L \\ 0 \\ 0 \end{array}\right]}_{b} u = 0.
\end{aligned}$$

Next

$$\frac{dx_2^*}{dt} = \left\langle dT_2, \frac{dx}{dt} \right\rangle = \langle dT_2, Ax + bu \rangle = \langle dT_2, Ax \rangle + \langle dT_2, b \rangle\, u$$

and

$$\begin{aligned}
\langle dT_2, Ax \rangle &= \underbrace{\frac{JL}{K_T} \left[\begin{array}{ccc} 0 & 1 & 0 \end{array}\right]}_{dT2} \underbrace{\left[\begin{array}{c} -(R/L)x_1 - (K_b/L)x_2 \\ (K_T/J)x_1 - (f/J)x_2 \\ x_2 \end{array}\right]}_{Ax} = \frac{JL}{K_T} \left(\frac{K_T}{J} x_1 - \frac{f}{J} x_2 \right) x_2 = x_3^* \\
\langle dT_2, b \rangle\, u &= \underbrace{\frac{JL}{K_T} \left[\begin{array}{ccc} 0 & 1 & 0 \end{array}\right]}_{dT_2} \underbrace{\left[\begin{array}{c} 1/L \\ 0 \\ 0 \end{array}\right]}_{b} u = 0.
\end{aligned}$$

Finally

$$\frac{dx_3^*}{dt} = \left\langle dT_3, \frac{dx}{dt} \right\rangle = \langle dT_3, Ax + bu \rangle = \langle dT_3, Ax \rangle + \langle dT_3, b \rangle\, u$$

and

$$\begin{aligned}
\langle dT_3, Ax \rangle &= \underbrace{\frac{JL}{K_T} \left[\begin{array}{ccc} \frac{K_T}{J} & -\frac{f}{J} & 0 \end{array}\right]}_{dT_3} \underbrace{\left[\begin{array}{c} -(R/L)x_1 - (K_b/L)x_2 \\ (K_T/J)x_1 - (f/J)x_2 \\ x_2 \end{array}\right]}_{Ax} \\
&= \frac{JL}{K_T} \left(-\frac{fK_T}{J^2} - \frac{RK_T}{JL} \right) x_1 + \frac{JL}{K_T} \left(\frac{f^2}{J^2} - \frac{K_T K_b}{JL} \right) x_2 \\
\langle dT_3, b \rangle\, u &= \underbrace{\frac{JL}{K_T} \left[\begin{array}{ccc} \frac{K_T}{J} & -\frac{f}{J} & 0 \end{array}\right]}_{dT_3} \underbrace{\left[\begin{array}{c} 1/L \\ 0 \\ 0 \end{array}\right]}_{b} u = u.
\end{aligned}$$

Summarizing we have shown

$$\begin{aligned}\frac{dx_1^*}{dt} &= x_2^*\\ \frac{dx_2^*}{dt} &= x_3^* \qquad (1.9)\\ \frac{dx_3^*}{dt} &= \underbrace{\frac{JL}{K_T}\left(-\frac{fK_T}{J^2}-\frac{RK_T}{JL}\right)x_1+\frac{JL}{K_T}\left(\frac{f^2}{J^2}-\frac{K_TK_b}{JL}\right)x_2}_{\langle dT_3, Ax\rangle}+u.\end{aligned}$$

However, $\langle dT_3, Ax\rangle = \frac{JL}{K_T}\left(-\frac{fK_T}{J^2}-\frac{RK_T}{JL}\right)x_1+\frac{JL}{K_T}\left(\frac{f^2}{J^2}-\frac{K_TK_b}{JL}\right)x_2$ needs to be rewritten in terms of the x^* coordinates. The inverse of (1.8) is

$$\begin{aligned}x_1 &= \frac{f}{JL}x_2^*+\frac{1}{L}x_3^*\\ x_2 &= \frac{K_T}{JL}x_2^*\\ x_3 &= \frac{K_T}{JL}x_1^*.\end{aligned}$$

Substituting for x_1 and x_2 into the third equation of (1.9) gives

$$\begin{aligned}\frac{dx_1^*}{dt} &= x_2^*\\ \frac{dx_2^*}{dt} &= x_3^* \qquad (1.10)\\ \frac{dx_3^*}{dt} &= -\frac{Rf+K_TK_b}{JL}x_2^*-\frac{fL+RJ}{JL}x_3^*+u.\end{aligned}$$

With $a_0=0, a_1=\frac{Rf+K_TK_b}{JL}, a_2=\frac{fL+RJ}{JL}$ this is written in matrix form as

$$\frac{d}{dt}\begin{bmatrix}x_1^*\\x_2^*\\x_3^*\end{bmatrix}=\underbrace{\begin{bmatrix}0&1&0\\0&0&1\\-a_0&-a_1&-a_2\end{bmatrix}}_{A_c}\begin{bmatrix}x_1^*\\x_2^*\\x_3^*\end{bmatrix}+\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{b_c}u \qquad (1.11)$$

which is in control canonical form.

Note from (1.8) that, except for the factor $\frac{JL}{K_T}$, the transformed coordinates x_1^*, x_2^*, and x_3^* are the position, speed, and acceleration of the motor, respectively. It is usually more convenient to specify the trajectory of a mechanical system by its desired position, speed, and acceleration rather than by position, speed, and current. Let

$$x_d^*=\frac{JL}{K_T}\begin{bmatrix}\theta_{Rd}\\ \omega_{Rd}\\ \alpha_{Rd}\end{bmatrix}$$

be the reference trajectory where $\omega_{Rd}=d\theta_{Rd}/dt, \alpha_{Rd}=d\omega_{Rd}/dt$. With

$$r_d(t)\triangleq\frac{JL}{K_T}\frac{d\alpha_{Rd}}{dt}+a_0x_{d1}^*+a_1x_{d2}^*+a_2x_{d3}^*$$

we have

$$\frac{d}{dt}\begin{bmatrix}x_{d1}^*\\x_{d2}^*\\x_{d3}^*\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-a_0&-a_1&-a_2\end{bmatrix}\begin{bmatrix}x_{d1}^*\\x_{d2}^*\\x_{d3}^*\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}r_d. \qquad (1.12)$$

Subtracting (1.12) from (1.11) gives the error system

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x_1^* - x_{d1}^* \\ x_2^* - x_{d2}^* \\ x_3^* - x_{d3}^* \end{bmatrix}}_{\epsilon^*} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c}\underbrace{\begin{bmatrix} x_1^* - x_{d1}^* \\ x_2^* - x_{d2}^* \\ x_3^* - x_{d3}^* \end{bmatrix}}_{\epsilon^*} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b_c}\underbrace{(u - r_d)}_{w} \tag{1.13}$$

or in more compact form

$$\epsilon^* = A_c^*\epsilon + b_c w.$$

With

$$k = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \triangleq \begin{bmatrix} \alpha_0 - a_0 & \alpha_1 - a_1 & \alpha_2 - a_2 \end{bmatrix}$$

the feedback

$$w = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\begin{bmatrix} x_1^* - x_{d1}^* \\ x_2^* - x_{d2}^* \\ x_3^* - x_{d3}^* \end{bmatrix}$$

results in

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x_1^* - x_{d1}^* \\ x_2^* - x_{d2}^* \\ x_3^* - x_{d3}^* \end{bmatrix}}_{\epsilon^*} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix}}_{A_{CL}}\underbrace{\begin{bmatrix} x_1^* - x_{d1}^* \\ x_2^* - x_{d2}^* \\ x_3^* - x_{d3}^* \end{bmatrix}}_{\epsilon^*} \tag{1.14}$$

With the α_i chosen so that A_{CL} is stable the error $\epsilon^*(t) \triangleq x^*(t) - x_d^*(t)$ is given by

$$\epsilon^*(t) = e^{A_{CL}t}\epsilon^*(0) \to 0_{3\times 1}.$$

The feedback results in the motor tracking the trajectory. The actual input to the motor is

$$\begin{aligned} u = w + r_d &= -k_1(x_1^* - x_{d1}^*) - k_2(x_2^* - x_{d2}^*) - k_3(x_3^* - x_{d3}^*) + r_d \\ &= -k_1(T_1(x) - x_{d1}^*) - k_2(T_2(x) - x_{d2}^*) - k_3(T_3(x) - x_{d3}^*) + r_d. \end{aligned}$$

So if the original coordinates are the signals that are measured, then they must be transformed to the x^* coordinate system to apply this feedback. In this particular example

$$T_3(x) = \frac{JL}{K_T}\left(\frac{K_T}{J}x_1 - \frac{f}{J}x_2\right) = \frac{JL}{K_T}\left(\frac{K_T}{J}i - \frac{f}{J}\omega\right) = \frac{JL}{K_T}\alpha.$$

If the current and speed are measured then the acceleration can be computed. However this requires the motor parameters being known accurately to get an accurate value of $\dfrac{JL}{K_T}\alpha$.

General Procedure To Transform a SISO LTI System to Control Canonical Form

For single-input single-output (SISO) linear time invariant (LTI) there is a simple procedure for transforming it to control canonical form if it is *controllable*. Recall that the LTI system

$$\frac{dx}{dt} = Ax + bu, \quad A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n$$

is controllable if and only if

$$\mathcal{C} \triangleq \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$$

is nonsingular. To continue with the procedure of transforming this LTI system to control canonical form, let $n = 3$ to simplify the presentation. Assume $\mathcal{C} \triangleq \begin{bmatrix} b & Ab & A^2b \end{bmatrix}$ is nonsingular so it has an inverse, i.e.,

$$\mathcal{C}^{-1}\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define $q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ be the last row of $\mathcal{C}^{-1}$ so that

$$q\mathcal{C} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{1.15}$$

That is, $qb = 0, qAb = 0, qA^2b = 1$. Then the change of coordinates to transform this system into control canonical form is given by

$$\begin{aligned} T_1(x) &\triangleq qx = q_1x_1 + q_2x_2 + q_3x_3, \quad && dT_1 = q \in \mathbb{R}^{1\times 3} \\ T_2(x) &\triangleq \langle dT_1, Ax\rangle = qAx, && qA \in \mathbb{R}^{1\times 3} \\ T_3(x) &\triangleq \langle dT_2, Ax\rangle = qA^2x, && qA^2 \in \mathbb{R}^{1\times 3}. \end{aligned}$$

That is, with

$$\begin{aligned} x_1^* &= T_1(x) = qx \\ x_2^* &= T_2(x) = qAx \\ x_3^* &= T_3(x) = qA^2x. \end{aligned} \tag{1.16}$$

and using (1.15) we have

$$\begin{aligned} \frac{dx_1^*}{dt} &= \left\langle dT_1, \frac{dx}{dt} \right\rangle = \langle q, Ax + bu\rangle = qAx + qbu = qAx = T_2(x) \\ \frac{dx_2^*}{dt} &= \left\langle dT_2, \frac{dx}{dt} \right\rangle = \langle qA, Ax + bu\rangle = qA^2x + qAbu = qA^2x = T_3(x) \\ \frac{dx_3^*}{dt} &= \left\langle dT_3, \frac{dx}{dt} \right\rangle = \left\langle qA^2, Ax + bu\right\rangle = qA^3x + qA^2bu = qA^3x + u \end{aligned}$$

or

$$\begin{aligned} \frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= x_3^* \\ \frac{dx_3^*}{dt} &= qA^3x + u. \end{aligned} \tag{1.17}$$

By (1.16) we have

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \underbrace{\begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix}}_{T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

so $x = T^{-1}x^*$ and (1.17) becomes

$$\begin{aligned} \frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= x_3^* \\ \frac{dx_3^*}{dt} &= (qA^3T^{-1})x^* + u = -a_0x_1^* - a_1x_2^* - a_2x_3^* + u \end{aligned}$$

where $\begin{bmatrix} -a_0 & -a_1 & -a_2 \end{bmatrix} \triangleq qA^3T^{-1}$.

Exercise 1 Given $A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3$ with the pair (A, b) controllable and q the last row of $\mathcal{C}^{-1} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix}^{-1}$, show that

$$T = \begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix}$$

is nonsingular.

Exercise 2 With $A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3$ and $\det(sI - A) = s^3 + a_2 s^2 + a_1 s + a_0$ show, by direct computation, that

$$\frac{dx^*}{dt} = TAT^{-1}x^* + Tbu = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c} x^* + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b_c} u.$$

Hint: Show $TA = A_c T$ by using the Cayley-Hamilton theorem.

1.2 Lie Derivatives

We now introduce the Lie derivative which will be used extensively in our dealings with nonlinear systems. For simplicity of exposition we stay with $n = 3$. Let

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} \in \mathbb{R}^3$$

be a (in general nonlinear) function from $\mathbb{R}^3 \to \mathbb{R}^3$ and $h(x) : \mathbb{R}^3 \to \mathbb{R}$ be a scalar function. Then the Lie derivative $\mathcal{L}_f(h)$ of h with respect to f is defined as

$$\mathcal{L}_f(h) \triangleq \langle dh, f \rangle = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}.$$

More precisely, let $\mathcal{C}^\infty$ denote all infinitely differentiable functions on some open subset $U \subset \mathbb{R}^3$. By a function h being infinitely differentiable on U is meant that partial derivatives of h of all orders exist on U.[1] Let $h(x) : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function of x defined on U and the components of $f(x) : \mathbb{R}^3 \to \mathbb{R}^3$ also $\mathcal{C}^\infty$ functions of x defined on U. Then the operator $\mathcal{L}_f(h) : \mathcal{C}^\infty \to \mathcal{C}^\infty$ is defined by

$$\mathcal{L}_f(h) \triangleq \langle dh, f \rangle$$

is the *Lie derivative* of h with respect f (Lie is pronounced as "Lee").

Repeated Lie derivatives are defined recursively as follows.

$$\mathcal{L}_f^2(h) \triangleq \mathcal{L}_f(\mathcal{L}_f(h)), \mathcal{L}_f^3(h) \triangleq \mathcal{L}_f(\mathcal{L}_f^2(h)), \text{ etc.}$$

Now consider the LTI system

$$\frac{dx}{dt} = Ax + bu, \; x \in \mathbb{R}^3, \; u \in \mathbb{R}, \; A \in \mathbb{R}^{3\times 3}, \; b \in \mathbb{R}^3.$$

Define

$$f(x) \triangleq Ax \in \mathbb{R}^3$$

[1] So $\frac{\partial h}{\partial x_1}, \frac{\partial^2 h}{\partial x_1 \partial x_3}, \frac{\partial^3 h}{\partial x_2^3}$, etc. all exist.

so we may write this system as

$$\frac{dx}{dt} = f(x) + bu.$$

With $h(x)$ a differentiable function we define the derivative of h along the trajectory $x(t)$ to be $\frac{dh}{dt}$ where $x(t)$ is the solution to the above system. In the notation of Lie derivatives we write

$$\begin{aligned}
\frac{dh(x(t))}{dt} = \mathcal{L}_{f+bu}(h) = \langle dh, f+bu\rangle &= \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1(x) + b_1 u \\ f_2(x) + b_2 u \\ f_3(x) + b_3 u \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} + \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} b_1 u \\ b_2 u \\ b_3 u \end{bmatrix} \\
&= \mathcal{L}_f(h) + u\mathcal{L}_b(h)
\end{aligned}$$

For example, with

$$h(x) = qx = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad f(x) = Ax$$

we have

$$\frac{dh}{dt} = \frac{dh(x(t))}{dt} = \mathcal{L}_f(h) + u\mathcal{L}_b(h) = qAx + uqb.$$

More generally, with $h(x) = qx$ and $\frac{dx}{dt} = f(x) + bu = Ax + bu$ we have

$$\begin{aligned}
\mathcal{L}_{f+bu}(h) &= \mathcal{L}_f(h) + u\mathcal{L}_b(h) = qAx + uqb \in \mathbb{R} \\
\mathcal{L}^2_{f+bu}(h) &= \mathcal{L}_{f+bu}(\mathcal{L}_{f+bu}) = \mathcal{L}_{f+bu}(qAx + uqb) = \langle qA, Ax + bu\rangle = qA^2x + qAbu \in \mathbb{R} \\
\mathcal{L}^2_{f+bu}(h) &= \mathcal{L}_{f+bu}(\mathcal{L}^2_{f+bu}) = \mathcal{L}_{f+bu}(qA^2x + qAbu) = \left\langle qA^2, Ax + bu\right\rangle = qA^3x + qA^2bu \in \mathbb{R}
\end{aligned}$$

Remark Though $\frac{dh}{dt} = \mathcal{L}_{f+bu}(h)$, $\frac{d^2h}{dt^2} \neq \mathcal{L}^2_{f+bu}(h)$ (why?).

1.3 Linearization about an Equilibrium Point

We now look at the standard approach of dealing with nonlinear control systems which is to find a linear approximate model for it. To explain, consider the nonlinear control system

$$\frac{dx_1}{dt} = f_1(x_1, x_2) + g_1(x_1, x_2)u \tag{1.18}$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2) + g_2(x_1, x_2)u \tag{1.19}$$

written compactly as

$$\frac{dx}{dt} = f(x) + g(x)u.$$

Let $x_0 = (x_{01}, x_{02})$ be a constant equilibrium point with reference input u_0 such that x_0, u_0 satisfy

$$\begin{aligned}
\frac{dx_{01}}{dt} &= 0 = f_1(x_{01}, x_{02}) + g_1(x_{01}, x_{02})u_0 \\
\frac{dx_{02}}{dt} &= 0 = f_2(x_{01}, x_{02}) + g_2(x_{01}, x_{02})u_0
\end{aligned}$$

or

$$0_{2\times 1} = f(x_0) + g(x_0)u_0.$$

Next do a Taylor series expansion of $f_1(x_1,x_2), f_2(x_1,x_2), g_1(x_1,x_2)$, and $g_2(x_1,x_2)$ about $x_0 = (x_{01}, x_{02})$. Specifically, we have

$$\begin{aligned}
f_1(x_1,x_2) &= f_1(x_{01},x_{02}) + \frac{\partial f_1(x_{01},x_{02})}{\partial x_1}(x_1 - x_{01}) + \frac{\partial f_1(x_{01},x_{02})}{\partial x_2}(x_2 - x_{02}) \\
&\quad + \frac{1}{2!}\left(\frac{\partial^2 f_1(x_{01},x_{02})}{\partial x_1^2}(x_1-x_{01})^2 + \frac{\partial^2 f_1(x_{01},x_{02})}{\partial x_1 \partial x_2}(x_1-x_{01})(x_2-x_{02}) + \frac{\partial^2 f_1(x_{01},x_{02})}{\partial x_2^2}(x_2-x_{02})^2\right) \\
&\quad + \cdots
\end{aligned}$$

$$\begin{aligned}
g_1(x_1,x_2) &= g_1(x_{01},x_{02}) + \frac{\partial g_1(x_{01},x_{02})}{\partial x_1}(x_1 - x_{01}) + \frac{\partial g_1(x_{01},x_{02})}{\partial x_2}(x_2 - x_{02}) \\
&\quad + \frac{1}{2!}\left(\frac{\partial^2 g_1(x_{01},x_{02})}{\partial x_1^2}(x_1-x_{01})^2 + \frac{\partial^2 g_1(x_{01},x_{02})}{\partial x_1 \partial x_2}(x_1-x_{01})(x_2-x_{02}) + \frac{\partial^2 g_1(x_{01},x_{02})}{\partial x_2^2}(x_2-x_{02})^2\right) \\
&\quad + \cdots
\end{aligned}$$

and similarly for $f_2(x_1,x_2)$ and $g_2(x_1,x_2)$. With $u = w + u_0$ we may rewrite (1.18) and (1.19) as

$$\begin{aligned}
\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} &= \begin{bmatrix} f_1(x_{01},x_{02}) \\ f_2(x_{01},x_{02}) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1(x_{01},x_{02})}{\partial x_1} & \frac{\partial f_1(x_{01},x_{02})}{\partial x_2} \\ \frac{\partial f_2(x_{01},x_{02})}{\partial x_1} & \frac{\partial f_2(x_{01},x_{02})}{\partial x_2} \end{bmatrix} \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix} + \cdots \\
&\quad + \begin{bmatrix} g_1(x_{01},x_{02}) \\ g_2(x_{01},x_{02}) \end{bmatrix}(w+u_0) + \begin{bmatrix} \frac{\partial g_1(x_{01},x_{02})}{\partial x_1}(x_1-x_{01}) + \frac{\partial g_1(x_{01},x_{02})}{\partial x_2}(x_2-x_{02}) \\ \frac{\partial g_2(x_{01},x_{02})}{\partial x_1}(x_1-x_{01}) + \frac{\partial g_2(x_{01},x_{02})}{\partial x_2}(x_2-x_{02}) \end{bmatrix}(w+u_0) + \cdots
\end{aligned}$$

or

$$\begin{aligned}
\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} &= \underbrace{\begin{bmatrix} f_1(x_{01},x_{02}) \\ f_2(x_{01},x_{02}) \end{bmatrix} + \begin{bmatrix} g_1(x_{01},x_{02}) \\ g_2(x_{01},x_{02}) \end{bmatrix} u_0}_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]} \\
&\quad + \begin{bmatrix} \frac{\partial f_1(x_{01},x_{02})}{\partial x_1} + \frac{\partial g_1(x_{01},x_{02})}{\partial x_1}u_0 & \frac{\partial f_1(x_{01},x_{02})}{\partial x_2} + \frac{\partial g_1(x_{01},x_{02})}{\partial x_2}u_0 \\ \frac{\partial f_2(x_{01},x_{02})}{\partial x_1} + \frac{\partial g_2(x_{01},x_{02})}{\partial x_1}u_0 & \frac{\partial f_2(x_{01},x_{02})}{\partial x_2} + \frac{\partial g_2(x_{01},x_{02})}{\partial x_2}u_0 \end{bmatrix} \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix} \\
&\quad + \begin{bmatrix} g_1(x_{01},x_{02}) \\ g_2(x_{01},x_{02}) \end{bmatrix} w + \cdots
\end{aligned}$$

The terms left out all have factors of the form $w(x_1 - x_{01}), w(x_2 - x_{02}), (x_1 - x_{01})^2, (x_1 - x_{01})(x_2 - x_{02}), (x_2 - x_{02})^2$ or higher and are referred to as higher order terms. The idea is that the state $[x_1, x_2]^T$ starts off close to the equilibrium point $[x_{01}, x_{02}]^T$ and that a feedback controller is designed to keep the state close to the equilibrium point for all time with $w = u - u_0$ also small. In this case these higher order terms are small and we take them to be zero to end up with the linear system

$$\frac{d}{dt}\begin{bmatrix}(x_1-x_{01})\\(x_2-x_{02})\end{bmatrix} = \underbrace{\begin{bmatrix}\dfrac{\partial f_1(x_{01},x_{02})}{\partial x_1}+\dfrac{\partial g_1(x_{01},x_{02})}{\partial x_1}u_0 & \dfrac{\partial f_1(x_{01},x_{02})}{\partial x_2}+\dfrac{\partial g_1(x_{01},x_{02})}{\partial x_2}u_0\\ \dfrac{\partial f_2(x_{01},x_{02})}{\partial x_1}+\dfrac{\partial g_2(x_{01},x_{02})}{\partial x_1}u_0 & \dfrac{\partial f_2(x_{01},x_{02})}{\partial x_2}+\dfrac{\partial g_2(x_{01},x_{02})}{\partial x_2}u_0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1-x_{01}\\x_2-x_{02}\end{bmatrix}}_{z}$$
$$+\underbrace{\begin{bmatrix}g_1(x_{01},x_{02})\\g_2(x_{01},x_{02})\end{bmatrix}}_{b}w. \tag{1.20}$$

This system can then be controlled by linear methods. For example, with $z=x-x_0$ we rewrite (1.20) as

$$\frac{dz}{dt}=Az+bw.$$

If the pair (A,b) is controllable then, as shown previously, we can find $k=\begin{bmatrix}k_1 & k_2\end{bmatrix}$ such that the feedback $w=-kz$ results in the closed-loop system

$$\frac{dz}{dt}=Az-bkz=(A-bk)z$$

being stable. More generally, consider a nonlinear system described by

$$\frac{dx}{dt}=f(x,u),\quad x\in\mathbb{R}^n, u\in\mathbb{R}^m.$$

Let x_0 be an equilibrium point with corresponding input reference u_0 so that

$$\frac{dx_0}{dt}=0_{n\times 1}=f(x_0,u_0).$$

With $z\triangleq x-x_0\in\mathbb{R}^n, w\triangleq u-u_0\in\mathbb{R}^m$ and $||z||=\sqrt{z_1^2+\cdots+z_n^2}, ||w||=\sqrt{w_1^2+\cdots+w_m^2}$ small we have

$$\begin{aligned}\frac{d}{dt}(x-x_0) &= f(x,u)-f(x_0,u_0)\\ &\approx \frac{\partial f(x_0,u_0)}{\partial x}(x-x_0)+\frac{\partial f(x_0,u_0)}{\partial u}(u-u_0)\end{aligned}$$

or

$$\frac{dz}{dt}=Az+Bw$$

where $A=\dfrac{\partial f(x_0,u_0)}{\partial x}\in\mathbb{R}^{n\times n}, B=\dfrac{\partial f(x_0,u_0)}{\partial u}\in\mathbb{R}^{n\times m}$. This is a LTI system which is (hopefully) valid for x close to x_0 and u close to u_0.

Linear Statespace Model of a Magnetically Levitated Steel Ball

Figure 1.3 shows a representation of using an electromagnet to provide a magnetic force F_{mag} to keep a steel ball levitated against the force of gravity. A current command amplifier is used, and, with the gains K_p, K_I chosen appropriately, the current can be considered as the input, that is, $i_r=i$. That is, a PI controller forces $i(t)\to i_r(t)$ so fast (compared to the motion of the steel ball) that the current $i(t)$ can be considered as the input.

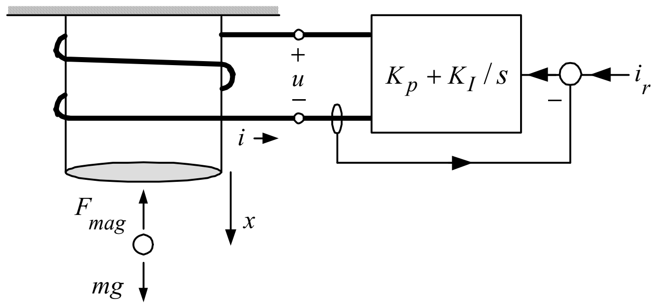

FIGURE 1.3. Current command amplifier for the magnetic levitation system.

With $u = i$ the input and $x_1 = x, x_2 = dx/dt$, the model is given by [4]

$$\frac{dx_1}{dt} = x_2 \tag{1.21}$$
$$\frac{dx_2}{dt} = g - \frac{C}{m}\frac{u^2}{x_1^2}. \tag{1.22}$$

Here C is an empirically determined force constant, m is the mass of the steel ball, and g is the acceleration due to gravity. An equilibrium point for the system is a *constant* solution to the system equations (1.21) and (1.22). We choose an equilibrium point so that the steel ball is at a distance x_{eq} below the electromagnet. Specifically, choose the equilibrium point to be $(x_{01}, x_{02}) = (x_{eq}, 0)$ for some u_0. At the equilibrium point (1.21) and (1.22) are simply

$$\begin{aligned}\frac{dx_{01}}{dt} &= 0 = x_{02}\\ \frac{dx_{02}}{dt} &= 0 = g - \frac{C}{m}\frac{u_0^2}{x_{01}^2}.\end{aligned}$$

For the right side of the second equation to hold it must be that $(x_{01} = x_{eq})$

$$g = \frac{C}{m}\frac{u_0^2}{x_{eq}^2} \quad \text{or} \quad u_0 = x_{eq}\sqrt{mg/C}. \tag{1.23}$$

The model (1.21) and (1.22) is nonlinear due to the $\dfrac{u^2}{x_1^2}$ term. To find a linear approximate model a Taylor series expansion of

$$f(x_1, u) = g - \frac{C}{m}\frac{u^2}{x_1^2}$$

is done about the equilibrium point $(x_{01}, x_{02}) = (x_{eq}, 0)$ with reference input $u_0 = x_0\sqrt{g/C}$. First note that

$$f(x_0, u_0) = g - \frac{C}{m}\frac{u_0^2}{x_{eq}^2} = 0$$

as u_0 was chosen to make this true. The Taylor series expansion of $f(x, u)$ about (x_{eq}, u_0) is then (dropping higher-order terms)

$$
\begin{aligned}
f(x_1,u) &\approx f(x_{eq},u_0) + \frac{\partial f(x_{eq},u_0)}{\partial x}(x_1 - x_{eq}) + \frac{\partial f(x_{eq},u_0)}{\partial u}(u-u_0) \\
&= \underbrace{g - \frac{C}{m}\frac{u_0^2}{x_{eq}^2}}_{0} + 2\frac{C}{m}\frac{u_0^2}{x^3}(x_1 - x_{eq}) - 2\frac{C}{m}\frac{u_0}{x^2}(u-u_0) \\
&= 2\frac{C}{m}\frac{u_0^2}{x^3}(x_1 - x_{eq}) - 2\frac{C}{m}\frac{u_0}{x^2}(u-u_0) \\
&= \frac{2g}{x_0}(x_1 - x_{eq}) - \frac{2g}{u_0}(u-u_0)
\end{aligned}
$$

where in the third line we used (1.23) to obtain

$$\frac{2g}{x_{eq}} = 2\frac{C}{m}\frac{u_0^2}{x_{eq}^3}, \quad \frac{2g}{u_0} = 2\frac{C}{m}\frac{u_0}{x_{eq}^2}.$$

The linear statespace model is then

$$
\begin{aligned}
\frac{d}{dt}(x_1 - x_{eq}) &= x_2 - 0 \\
\frac{d}{dt}(x_2 - 0) &= \frac{2g}{x_0}(x_1 - x_{eq}) - \frac{2g}{u_0}(u-u_0).
\end{aligned}
$$

With $z_1 = x_1 - x_{eq}, z_2 = x_2 - 0, w = u - u_0$ this becomes

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2g/x_{eq} & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2g/u_0 \end{bmatrix} w$$

1.4 Feedback Linearization of Nonlinear Control Systems

The fundamental reason we want to approximate a nonlinear system by a linear system is that we know how to stabilize a linear system.[2] However, for the linear system to be a “good” approximation to the nonlinear system, the state must start off “close” to the equilibrium point and the controller must keep the state “close” to the equilibrium point. We use the quotes “good” and “close” indicate that these are vague terms. In reality one only knows if the controller will work by implementing it. The concept of feedback linearization is to use a nonlinear change of coordinates so that in the new coordinate system the nonlinear system can be made linear by using feedback to cancel out the nonlinearities. We use this section to present several examples of this approach. The remaining chapters develop the theory of how one finds these nonlinear transformations.

Example 1 *Feedback Linearization*
Consider the nonlinear control system given by

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
\frac{dx_3}{dt} &= f(x_1,x_2,x_3) + u.
\end{aligned}
$$

[2] Assuming it is a *controllable* linear system.

With the feedback

$$u = -f(x_1, x_2, x_3) + v$$

the nonlinear control system becomes the *linear* control system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v.$$

Setting

$$v = -\underbrace{\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{bmatrix}}_{k}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

gives the closed-loop system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

with closed-loop characteristic polynomial

$$\det\left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix}\right) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0.$$

Note that the feedback to the system is

$$u = -f(x_1, x_2, x_3) - kx.$$

Feedback linearization is a generalization of the previous example. To explain, consider the nonlinear second-order control system given by

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2) + g_1(x_1, x_2)u \\ \frac{dx_2}{dt} &= f_2(x_1, x_2) + g_2(x_1, x_2)u. \end{aligned}$$

Suppose we can find a nonlinear change of coordinates

$$\begin{aligned} x_1^* &= T_1(x_1, x_2) \\ x_2^* &= T_2(x_1, x_2) \end{aligned}$$

such that in the x^* coordinates

$$\begin{aligned} \frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= f^*(x_1, x_2) + g^*(x_1, x_2)u. \end{aligned}$$

With the feedback

$$u = \frac{-f^*(x_1, x_2) + v}{g^*(x_1, x_2)}$$

this nonlinear system becomes the linear system (double integrator)

$$\begin{aligned} \frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= v. \end{aligned}$$

By means of a nonlinear transformation and state feedback the system is now linear from the input v to the state (x_1^*, x_2^*).

Example 2 *Direct Current to Direct Current (DC-DC) Converter* [5]

A DC-to-DC converter is an switching electronic circuit which converts a source of direct current (DC) from one voltage level to another. A circuit model of a DC-DC converter is shown in Figure 1.4. The variable $d(t)$ indicates if the switch is attached to ground ($d = 1$) or attached to the output ($d = 0$).

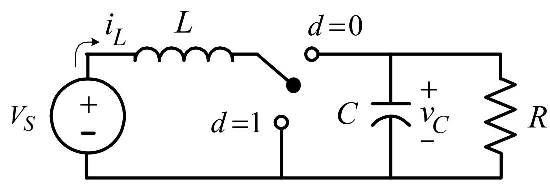

FIGURE 1.4. DC-DC converter.

Given the value of $d \in \{0, 1\}$ the equations of the circuit are

$$\frac{di_L}{dt} = -(1-d)\frac{v_C}{L} + \frac{V_S}{L} \tag{1.24}$$

$$\frac{dv_C}{dt} = (1-d)\frac{i_L}{C} - \frac{v_C}{RC}. \tag{1.25}$$

Let T_s denote the switching period so that $f_s \triangleq 1/T_s$ is the switching rate which is typically $10 - 20$ kHz. The duty ratio D for each period is the fraction of time that $d = 1$. In more detail, for the n^{th} time period starting at nT_s, $d(t)$ is given as

$$d(t) = \begin{cases} 1, & nT_s \le t \le nT_s + DT_s \\ 0, & nT_s + DT_s \le t \le nT_s + T_s. \end{cases}$$

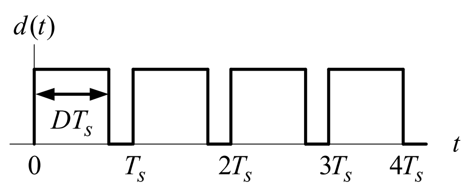

FIGURE 1.5. Illustration of the duty ratio.

The duty ratio is defined as

$$D = < d > \triangleq \frac{1}{T_s}\int_{t-T_s}^{t} d(\tau)d\tau.$$

Define

$$\begin{aligned} V_0 &\triangleq < V_S > = \frac{1}{T_s}\int_{t-T_s}^{t} V_S(\tau)d\tau \\ x_1(t) &\triangleq < i_L > = \frac{1}{T_s}\int_{t-T_s}^{t} i_L(\tau)d\tau \\ x_2(t) &\triangleq < v_C > = \frac{1}{T_s}\int_{t-T_s}^{t} v_C(\tau)d\tau. \end{aligned}$$

Typically V_S is constant so $V_0 \triangleq < V_S > = V_S$. Over a single switching period the duty ratio is constant. Averaging the set of equations (1.24) and (1.24) over one period gives

$$\begin{aligned}\frac{d}{dt} < i_L > &= -(1-D)\frac{< v_C >}{L} + \frac{< V_S >}{L}\\ \frac{d}{dt} < v_C > &= (1-D)\frac{< i_L >}{C} - \frac{< v_C >}{RC}\end{aligned}$$

where approximations

$$\begin{aligned}\langle (1-d)v_C \rangle &\approx \langle 1-d \rangle \langle v_C \rangle = (1-D)\langle v_C \rangle\\ \langle (1-d)i_L \rangle &\approx \langle 1-d \rangle \langle i_L \rangle = (1-D)\langle i_L \rangle .\end{aligned}$$

were used. Summarizing, a mathematical model of a DC-DC converter is given by

$$\begin{aligned}\frac{dx_1}{dt} &= -(1-u)\frac{x_2}{L} + \frac{V_0}{L}\\ \frac{dx_2}{dt} &= (1-u)\frac{x_1}{C} - \frac{x_2}{RC}\end{aligned}$$

where $u = D$ is the input (duty ratio) with $0 \leq u \leq 1$, x_1 is the (average) current in the inductor, and x_2 is the (average) voltage across the capacitor. The nonlinear terms are ux_2 and ux_1 in the two equations. This model of the DC-DC is given in the standard nonlinear form by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -x_2/L + V_0/L \\ x_1/C - x_2/(RC) \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} x_2/L \\ -x_1/C \end{bmatrix}}_{g(x)} u.$$

This way of setting up the nonlinear statespace equations going from the discrete input d to the continuous input $u = D$ is called *statespace averaging* [6]. Consider the nonlinear transformation

$$\begin{aligned}x_1^* &= T_1(x) \triangleq \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2\\ x_2^* &= T_2(x) \triangleq x_1V_0 - x_2^2/R.\end{aligned}$$

Then

$$\begin{aligned}\frac{d}{dt}x_1^* &\triangleq Lx_1\frac{d}{dt}x_1 + Cx_2\frac{d}{dt}x_2\\ &= Lx_1\left(-(1-u)\frac{x_2}{L} + \frac{V_0}{L}\right) + Cx_2\left((1-u)\frac{x_1}{C} - \frac{x_2}{RC}\right)\\ &= x_1V_0 - x_2^2/R\\ &= x_2^*\end{aligned}$$

and

$$\begin{aligned}\frac{dx_2^*}{dt} &= V_0\frac{dx_1}{dt} - \frac{2}{R}x_2\frac{dx_2}{dt}\\ &= V_0\left(-(1-u)\frac{x_2}{L} + \frac{V_0}{L}\right) - \frac{2}{R}x_2\left((1-u)\frac{x_1}{C} - \frac{x_2}{RC}\right)\\ &= \underbrace{V_0\left(-\frac{x_2}{L} + \frac{V_0}{L}\right) - \frac{2}{R}x_2\left(-\frac{x_1}{C} + \frac{x_2}{RC}\right)}_{f_2^*(x_1,x_2)} + u\underbrace{\left(\frac{V_0}{L}x_2 + \frac{2}{RC}x_1x_2\right)}_{g_2^*(x_1,x_2)}.\end{aligned}$$

We now have

$$\begin{aligned}\frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= f^*(x_1,x_2) + ug^*(x_1,x_2).\end{aligned}$$

If we set the input as

$$u = \frac{-f^*(x_1,x_2)+v}{g^*(x_1,x_2)}$$

we obtain

$$\begin{aligned}\frac{dx_1^*}{dt} &= x_2^* \\ \frac{dx_2^*}{dt} &= v\end{aligned}$$

which is a *linear* system.

Example 3 *Series Connected DC Motor* [7][8]
A series connected DC motor is one in which the field circuit is connected in series with the armature (rotor) loop. This is indicated in Figure 1.6 which shows the terminal T_2' of the field winding connected to the terminal T_1 of the armature loop. R_f is the resistance of the field windings and R_a is the resistance of the rotor loops. The total flux (flux linkage) in the field windings due to the current i_f in the field windings is $L_f i_f$ with L_f the self inductance of the field windings. The total flux in the rotor loops is $L_a i_a$ due to the current i_a in the rotor loops with L_a the self inductance of the rotor loop. It turns out (see [3]) that the torque produced by the motor is

$$\tau_m = K_T L_f i_f i_a$$

and the voltage induced in rotor loops as they rotate in the magnetic field of the air gap produced by the field current is

$$\xi = K_b L_f i_f \omega.$$

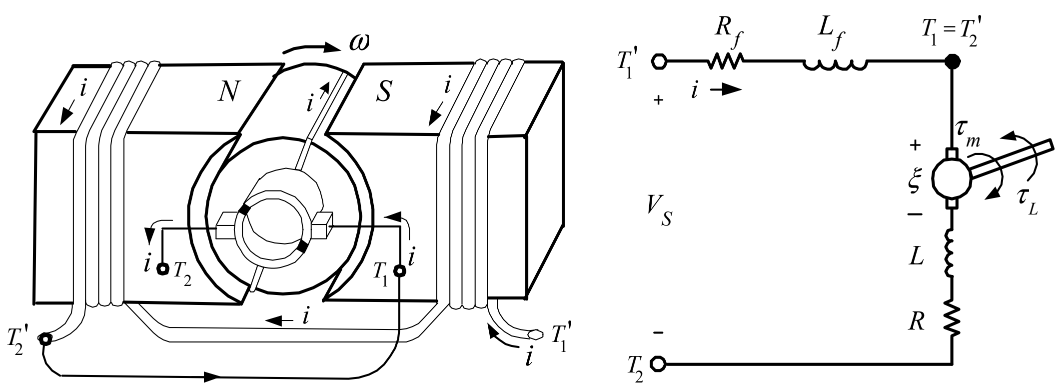

FIGURE 1.6. Series connected DC motor.

As this is a series connected DC motor we have $i = i_f = i_a$. Conservation of energy requires $\tau_m \omega = \xi i$ which implies $K_T = K_b$. Using the equivalent circuit on the right side of Figure 1.6 it is seen that the equations describing this motor are

$$\begin{aligned} V_S &= (R_f + R_a)i + (L_f + L_a)\frac{di}{dt} + \xi = Ri + L\frac{di}{dt} + K_b L_f i\omega \\ J\frac{d\omega}{dt} &= \tau_m - \tau_L = K_T L_f i^2 - \tau_L \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

Here $R \triangleq R_f + R_a, L \triangleq L_f + L_a$, and J is the moment of inertia of the motor shaft. Rearranging these equations into statespace form we have

$$\begin{aligned} \frac{di}{dt} &= -\frac{R}{L}i - \frac{K_b L_f}{L}i\omega + \frac{V_S}{L} \\ \frac{d\omega}{dt} &= \frac{K_T L_f}{J}i^2 - \frac{\tau_L}{J} \\ \frac{d\theta}{dt} &= \omega \end{aligned}$$

Set $x_1 = \theta, x_2 = \omega, x_3 = i$, and $u = V_S/L$ to have

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= \frac{K_T L_f}{J}x_3^2 - \frac{\tau_L}{J} \\ \frac{dx_3}{dt} &= -\frac{R}{L}x_3 - \frac{K_b L_f}{L}x_3 x_2 + u \end{aligned}$$

or

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}}_{p} \tau_L$$

with $c_1 \triangleq K_T L_f/J, c_2 \triangleq R/L$, and $c_3 \triangleq K_b L_f/L$.

In this case of the series connected DC motor the nonlinearity in the dx_3/dt equation can be canceled out by feedback, but we would still have a nonlinearity in the dx_2/dt equation. To get around this consider the nonlinear statespace transformation

$$\begin{aligned} x_1^* &= T_1(x) = x_1 \\ x_2^* &= T_2(x) = \mathcal{L}_f(T_1) = \langle dT_1, f\rangle = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix} = x_2 \\ x_3^* &= T_3(x) = \mathcal{L}_f(T_2) = \langle dT_2, f\rangle = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix} = c_1 x_3^2. \end{aligned}$$

We now show that this nonlinear statespace transformation, i.e.,

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_2(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ c_1 x_3^2 \end{bmatrix}$$

will transform the equations of the series connected DC motor into a form where the nonlinearities can be canceled out by feedback. We proceed as follows.

$$\begin{aligned}
\frac{dx_1^*}{dt} = \frac{d}{dt}T_1(x) &= \langle dT_1, f(x) + g(x)u + p\tau_L \rangle \\
&= \mathcal{L}_f(T_1) + u\mathcal{L}_g(T_1) + \tau_L \mathcal{L}_p(T_1) \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix} + u \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \tau_L \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix} \\
&= x_2
\end{aligned}$$

$$\begin{aligned}
\frac{dx_2^*}{dt} = \frac{d}{dt}T_2(x) &= \langle dT_2, f(x) + g(x)u + p\tau_L \rangle \\
&= \mathcal{L}_f(T_2) + u\mathcal{L}_g(T_2) + \tau_L \mathcal{L}_p(T_2) \\
&= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix} + u \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \tau_L \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix} \\
&= c_1 x_3^2 - \tau_L/J
\end{aligned}$$

$$\begin{aligned}
\frac{dx_3^*}{dt} = \frac{d}{dt}T_3(x) &= \langle dT_3, f(x) + g(x)u + p\tau_L \rangle \\
&= \mathcal{L}_f(T_3) + u\mathcal{L}_g(T_3) + \tau_L \mathcal{L}_p(T_3) \\
&= \begin{bmatrix} 0 & 0 & 2c_1 x_3 \end{bmatrix} \begin{bmatrix} x_2 \\ c_1 x_3^2 \\ -c_2 x_3 - c_3 x_3 x_2 \end{bmatrix} + u \begin{bmatrix} 0 & 0 & 2c_1 x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \tau_L \begin{bmatrix} 0 & 0 & 2c_1 x_3 \end{bmatrix} \begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix} \\
&= -2c_1 c_2 x_3^2 - 2c_1 c_3 x_2 x_3^2 + 2c_1 x_3 u
\end{aligned}$$

Summarizing we have

$$\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^* \\
\frac{dx_2^*}{dt} &= x_3^* - \tau_L/J \\
\frac{dx_3^*}{dt} &= \underbrace{-2c_1 c_2 x_3^2 - 2c_1 c_3 x_2 x_3^2}_{a(x)} + \underbrace{2c_1 x_3}_{b(x)} u.
\end{aligned}$$

For $x_3 = i \neq 0$ we can apply the state feedback

$$u = \frac{-a(x) + w}{b(x)}$$

to obtain the linear statespace system

$$\begin{aligned}
\frac{dx_1^*}{dt} &= x_2^* \\
\frac{dx_2^*}{dt} &= x_3^* - \tau_L/J \\
\frac{dx_3^*}{dt} &= w.
\end{aligned}$$

As $\tau_m = K_T L_f i^2$ a series connected DC motor can only produce torque in one direction so it is used for applications (e.g., locomotives) where speed needs to be controlled, but it is not required to reverse it. By decreasing the current the load torque decreases the speed.

With a constant load torque τ_{L0} consider speed control with ω_0 the desired constant speed. Define ($x_2^* = \omega$)

$$\begin{aligned}
z_0 &\triangleq \int_0^t (x_2^*(t') - \omega_0)dt' = \int_0^t (\omega(t') - \omega_0)dt' \\
z_1 &\triangleq x_2^* - \omega_0 = \omega - \omega_0 \\
z_2 &\triangleq x_3^*
\end{aligned}$$

with corresponding linear model

$$\begin{aligned}
\frac{dz_0}{dt} &= z_1 \\
\frac{dz_1}{dt} &= z_2 - \tau_{L0}/J \\
\frac{dz_2}{dt} &= w.
\end{aligned}$$

In matrix form this is

$$\frac{d}{dt}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix} \tau_{L0}.$$

The feedback

$$w = -k_0 z_0 - k_1 z_1 - k_2 z_2$$

results in

$$\frac{d}{dt}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_0 & -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix} \tau_{L0}.$$

Taking Laplace transform we have

$$\begin{aligned}
\begin{bmatrix} Z_0(s) \\ Z_1(s) \\ Z_2(s) \end{bmatrix} &= \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k_0 & k_1 & s + k_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix} \frac{\tau_{L0}}{s} \\
&= \frac{1}{s^3 + k_2 s^2 + k_1 s + k_0} \begin{bmatrix} s^2 + k_2 s + k_1 & s + k_2 & 1 \\ -k_0 & s^2 + k_2 s & s \\ -s k_0 & -k_0 - s k_1 & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix} \frac{\tau_{L0}}{s}.
\end{aligned}$$

By the final value theorem ($z_1(t) = \omega(t) - \omega_0$) it follows that

$$\begin{bmatrix} z_0(\infty) \\ z_1(\infty) \\ z_2(\infty) \end{bmatrix} = \lim_{s \to 0} s \begin{bmatrix} Z_0(s) \\ Z_1(s) \\ Z_2(s) \end{bmatrix} = \begin{bmatrix} -k_2 \tau_{L0}/J \\ 0 \\ \tau_{L0}/J \end{bmatrix}$$

Though the controller is designed without knowledge of the value of τ_{L0}, it forces $\omega(t) \to \omega_0$.

Remark The series connected DC motor puts out more torque per Ampere than any other DC motor. Historically they were used in applications that require high torque such as diesel electric locomotives. In the case of the locomotive the diesel engine runs a generator with the output of the generator connected to the input of the series connected DC motor. The motor is controlled to bring the locomotive up to some speed $\omega > 0$. As the train moves across the landscape the speed can be increased or decreased, but keeping $\omega > 0$ and $i > 0$. However, in the last thirty years or so series connected DC motors have been replaced by AC induction drives made possible by the technological advancements in power electronics.)

Remark Note that if an AC voltage is applied to the series connected DC motor then the current will also change direction due to this alternating voltage input. However, the torque $\tau_m = K_T i^2$ will remain positive as it depends on the square of the current. For this reason the series connected DC motor is also referred to as a *universal* motor as it can run either of an AC or a DC voltage source yet keeping the torque always positive. For this reason vacuum cleaners, blenders, hair dyers, drills, sanders, jig saws and starter motors in cars typically use series connected DC motors.

1.5 Permanent Magnet Synchronous Motor [1][2]

Figure 1.7 is an illustration of the structure of a two-phase permanent magnet synchronous motor. The rotor is a permanent magnet and the stator has two sets of windings (phases) denoted as $a-a'$ and $b-b'$. The current in phase $a-a'$ is i_{Sa} and the current in phase $b-b'$ is i_{Sb}. L_S denotes the self-inductance of each phase, R_S denote the resistance of each phase, J is the moment of inertia of the rotor, K_m is the torque constant, and τ_L denotes the load torque. u_{Sa} is the voltage applied to phase $a-a'$ and u_{Sb} is the voltage applied to phase $b-b'$. Details on the physical operation of this motor based on elementary electricity and magnetism are given in [3].

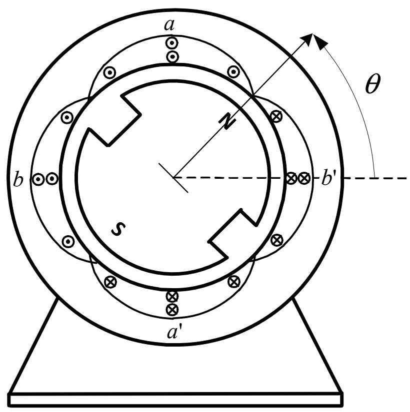

FIGURE 1.7. Two-phase permanent magnet synchronous machine.

The differential equation model of this motor is derived in [3] and given by

$$\begin{aligned}
L_S\frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m\sin(n_p\theta)\omega + u_{Sa}\\
L_S\frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m\cos(n_p\theta)\omega + u_{Sb}\\
J\frac{d\omega}{dt} &= K_m(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)) - \tau_L\\
\frac{d\theta}{dt} &= \omega.
\end{aligned}\tag{1.26}$$

More common in use are three-phase PM synchronous machines. However these machines have an equivalent two-phase model also given by (1.26) so the development below holds for them as well.

The currents and voltages are transformed by the *direct-quadrature* (dq) transformation defined by

$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix}\begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix}\tag{1.27}$$

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix}\begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix}\tag{1.28}$$

where u_d is the direct voltage, u_q is the quadrature voltage, i_d is the direct current, and i_q is the quadrature current. The direct current i_d corresponds to the component of the stator magnetic field along the axis of the rotor magnetic field, while the quadrature current i_q corresponds to the orthogonal component of the stator current that produces torque. The application of the dq transformation to the original system (1.26) yields the system of equations

$$L_S \frac{di_d}{dt} = -R_S i_d + n_p \omega L_S i_q + u_d \tag{1.29}$$

$$L_S \frac{di_q}{dt} = -R_S i_q - n_p \omega L_S i_d - K_m \omega + u_q \tag{1.30}$$

$$J \frac{d\omega}{dt} = K_m i_q - \tau_L \tag{1.31}$$

$$\frac{d\theta}{dt} = \omega \tag{1.32}$$

The resulting dq system model (1.29)−(1.32) is still nonlinear, however the nonlinear terms can now be canceled by state feedback. Specifically, choosing u_d and u_q to be

$$u_d = R_S i_d - n_p \omega L_S i_q + L_S w_d \tag{1.33}$$

$$u_q = R_S i_q + n_p \omega L_S i_d + K_m \omega + L_S w_q \tag{1.34}$$

results in the *feedback linearized* system

$$\frac{di_d}{dt} = w_d \tag{1.35}$$

$$\frac{di_q}{dt} = w_q \tag{1.36}$$

$$\frac{d\omega}{dt} = (K_m/J) i_q - \tau_L/J \tag{1.37}$$

$$\frac{d\theta}{dt} = \omega. \tag{1.38}$$

Note that the original fourth-order system has been transformed into a first-order linear system (1.35) and a third-order linear system (equations (1.36)−(1.38)) which are decoupled from each other.

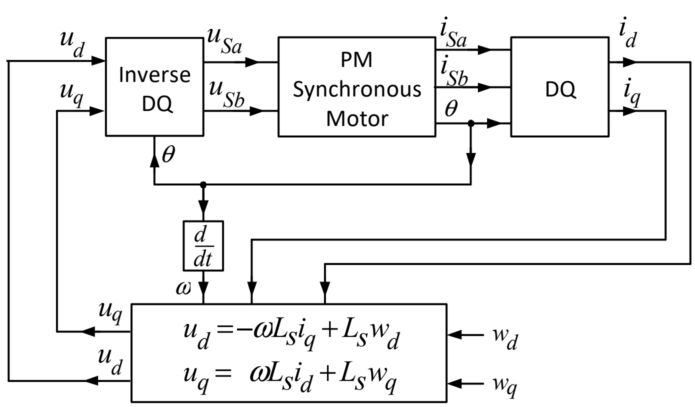

FIGURE 1.8. Block diagram for controlling a PM synchronous machine using the dq coordinates.

It is important to note that feedback linearization is carried out on the error system making easier to avoid actuator saturation. To explain suppose a desired trajectory has been designed for the motor which satisfies the system equations (1.29)−(1.32), that is,

$$L_S\frac{di_{dref}}{dt} = -R_S i_{dref} + n_p\omega_{ref}L_S i_{qref} + u_{dref} \tag{1.39}$$

$$L_S\frac{di_{qref}}{dt} = -R_S i_{qref} - n_p\omega_{ref}L_S i_{dref} - K_m\omega_{ref} + u_{qref} \tag{1.40}$$

$$J\frac{d\omega_{ref}}{dt} = K_m i_{qref} \tag{1.41}$$

$$\frac{d\theta_{ref}}{dt} = \omega_{ref} \tag{1.42}$$

State Feedback Controller

Let τ_L constant and subtract the system model (1.29)−(1.32) from the reference model (1.39)− (1.42) to obtain the error system

$$\begin{aligned} L_S\frac{d}{dt}(i_{dref} - i_d) &= -R_S(i_{dref} - i_d) + n_p\omega_{ref}L_S i_{qref} - n_p\omega L_S i_q \\ &\quad + u_{dref} - u_d \end{aligned} \tag{1.43}$$

$$\begin{aligned} L_S\frac{d}{dt}(i_{qref} - i_q) &= -R_S(i_{qref} - i_q) - n_p\omega_{ref}L_S i_{dref} + n_p\omega L_S i_d \\ &\quad - K_m(\omega_{ref} - \omega) + u_{qref} - u_q \end{aligned} \tag{1.44}$$

$$J\frac{d}{dt}(\omega_{ref} - \omega) = K_m(i_{qref} - i_q) + \tau_L/J \tag{1.45}$$

$$\frac{d}{dt}(\theta_{ref} - \theta) = \omega_{ref} - \omega. \tag{1.46}$$

The first step in specifying the state feedback control law is to use a *feedback linearizing* controller given by

$$u_d = -n_p\omega L_S i_q + u_{dref} + n_p\omega_{ref}L_S i_{qref} - L_S v_d \tag{1.47}$$

$$u_q = +n_p\omega L_S i_d + u_{qref} - n_p\omega_{ref}L_S i_{dref} - L_S v_q \tag{1.48}$$

where v_d and v_q are new inputs yet to be defined. If the controller keeps i_q close to i_{qref} then the two terms $-n_p\omega L_S i_q$ and $\omega_{ref}L_S i_{qref}$ in (1.47) approximately cancel each other so that a lot the direct voltage u_d is not used to cancel them. Similarly, if the controller keeps i_d close to i_{dref} then the two terms $n_p\omega L_S i_d$ and $-n_p\omega_{ref}L_S i_{dref}$ in (1.48) approximately cancel each other so that a lot of the quadrature voltage u_q is not used to cancel them.

Substituting equations (1.47) and (1.48) for u_d and u_q into (1.43)−(1.44) results in the *linear* system of equations

$$L_S\frac{d}{dt}(i_{dref} - i_d) = -R_S(i_{dref} - i_d) + v_d \tag{1.49}$$

$$L_S\frac{d}{dt}(i_{qref} - i_q) = -R_S(i_{qref} - i_q) - K_m(\omega_{ref} - \omega) + v_q \tag{1.50}$$

$$J\frac{d}{dt}(\omega_{ref} - \omega) = K_m(i_{qref} - i_q) + \tau_L \tag{1.51}$$

$$\frac{d}{dt}(\theta_{ref} - \theta) = \omega_{ref} - \omega. \tag{1.52}$$

Define the tracking error as

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} \triangleq \begin{bmatrix} i_{dref} - i_d \\ i_{qref} - i_q \\ \omega_{ref} - \omega \\ \theta_{ref} - \theta \\ \int_0^t (\theta_{ref}(\tau) - \theta(\tau))d\tau \end{bmatrix} \tag{1.53}$$

where the integrator is added to the controller to eliminate any steady-state error due to τ_L. Using (1.49)−(1.52), the error (1.53) satisfies

$$\frac{de}{dt} = \begin{bmatrix} -R_S/L_S & 0 & 0 & 0 & 0 \\ 0 & -R_S/L_S & -K_m/L_S & 0 & 0 \\ 0 & K_m/J & -f/J & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} e + \begin{bmatrix} 1/L_S & 0 \\ 0 & 1/L_S \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1/J \\ 0 \\ 0 \end{bmatrix} \tau_L. \tag{1.54}$$

where $v \triangleq \begin{bmatrix} v_d & v_q \end{bmatrix}^T$. More compactly,

$$\frac{de}{dt} = Ae + Bv + p\tau_L$$

with the obvious definitions for A, B, and p. Through the use of a nonlinear state transformation, input transformation, and nonlinear feedback, a *linear time-invariant* error system has been obtained. The input v is chosen as the linear state feedback

$$v = -Ke \tag{1.55}$$

where K is taken to be of the form

$$K = \begin{bmatrix} k_{11} & 0 & 0 & 0 & 0 \\ 0 & k_{22} & k_{23} & k_{24} & k_{25} \end{bmatrix}. \tag{1.56}$$

The closed-loop error system is then

$$\frac{de}{dt} = (A - BK)\, e + p\tau_L.$$

Using linear statespace techniques one can choose K to place the closed-loop poles of $A - BK$ in the open left-half plane at any desired location there. It is straightforward to show that $e_4(t) = \theta_{ref}(t) - \theta(t) \to 0$ and $e_5(t) = \omega_{ref}(t) - \omega(t) \to 0$ despite any constant load torque τ_L.

Remarks The details of using this approach for controlling actual PM synchronous machines is given in [9]. The dq transformation is the feedback linearizing transformation for the model (1.26). If one identifies the direct current i_d with the field current i_f of a wound field DC motor and the quadrature current i_q with the armature current i_a of a wound field DC motor then the model (1.35)−(1.38) is essentially the model of a wound field DC motor. For this reason, a PM synchronous machine with this type of control is referred to as a *brushless* DC motor.

1.6 Magnetic Levitation - Again

Let's go back to the levitation of a steel ball using an electromagnet shown in Figure 1.9. Now we take the voltage applied to coil of the electromagnet as the input and the state variables to be the current i in the coil, the position x of the steel ball below the electromagnet, and the velocity $v = dx/dt$ of the steel ball.

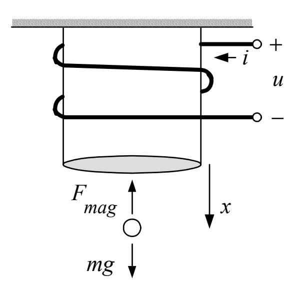

FIGURE 1.9. Magnetic levitation system.

With $L(x) = L_0 + rL_1/x$ denoting the inductance of the coil with the ball at distance x below the magnet, the nonlinear statespace model of this magnetic levitation system is [4]

$$\frac{di}{dt} = rL_1 \frac{i}{x^2 L(x)} v - \frac{R}{L(x)} i + \frac{1}{L(x)} u \tag{1.57}$$

$$\frac{dx}{dt} = v \tag{1.58}$$

$$\frac{dv}{dt} = g - \frac{rL_1}{2m} \frac{i^2}{x^2}. \tag{1.59}$$

Define a nonlinear statespace transformation by

$$x_1 \triangleq x \tag{1.60}$$

$$x_2 \triangleq v \tag{1.61}$$

$$x_3 \triangleq g - \frac{rL_1}{2m} \frac{i^2}{x^2}. \tag{1.62}$$

We then have

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
\frac{dx_3}{dt} &= \frac{d}{dt}\left(g - \frac{rL_1}{2m} \frac{i^2}{x^2}\right).
\end{aligned}$$

x_1 is the position of the steel ball, x_2 is its velocity, x_3 is its acceleration. We calculate $\frac{dx_3}{dt}$ as follows.

$$\begin{aligned}
\frac{d}{dt}\left(g - \frac{rL_1}{2m} \frac{i^2}{x^2}\right) &= -\frac{rL_1}{m} \frac{i}{x^2} \frac{di}{dt} + \frac{rL_1}{m} \frac{i^2}{x^3} \frac{dx}{dt} \\
&= -\frac{rL_1}{m} \frac{i}{x^2} \left(rL_1 \frac{i}{x^2 L(x)} v - \frac{R}{L(x)} i + \frac{1}{L(x)} u\right) + \frac{rL_1}{m} \frac{i^2}{x^3} v \\
&= -\frac{rL_1}{m} \frac{i^2}{x^4} \frac{rL_1}{L(x)} v + \frac{rL_1}{m} \frac{i^2}{x^2} \frac{R}{L(x)} - \frac{rL_1}{m} \frac{i}{x^2} \frac{u}{L(x)} + \frac{rL_1}{m} \frac{i^2}{x^3} v \\
&= \underbrace{-\frac{rL_1}{m} \frac{i^2}{x^4} \frac{rL_1}{L(x)} v + \frac{rL_1}{m} \frac{i^2}{x^2} \frac{R}{L(x)} + \frac{rL_1}{m} \frac{i^2}{x^3} v}_{f(i,x,v)} + \underbrace{\left(-\frac{rL_1}{m} \frac{i}{x^2 L(x)}\right)}_{g(i,x)} u.
\end{aligned}$$

In terms of the state variables x_1, x_2, and x_3, the nonlinear statespace model is

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
\frac{dx_3}{dt} &= f(i,x,v) + g(i,x)u.
\end{aligned}$$

The point of this transformation is that the nonlinear terms are in the *same* equation as the input. Setting

$$u = \frac{-f(i,x,v) + w}{g(i,x)}$$

we obtain the *linear* system

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= x_3\\ \frac{dx_3}{dt} &= w.\end{aligned}$$

Let's first design a trajectory for the system. Let x_{d1} be the position reference, $x_{d2} \triangleq dx_{d1}/dt$ be the speed reference, $x_{d3} \triangleq dx_{d2}/dt$ be the acceleration reference, and $j_d \triangleq dx_{d3}/dt$ be the jerk reference so that

$$\begin{aligned}\frac{dx_{d1}}{dt} &= x_{d2}\\ \frac{dx_{d2}}{dt} &= x_{d3}\\ \frac{dx_{d3}}{dt} &= j_d.\end{aligned}$$

Using (1.60), (1.61), and (1.62) we have

$$x_d \triangleq x_{d1}, v_d \triangleq x_{d2}, \text{ and } i_d \triangleq \sqrt{\frac{2m}{rL_1}(g - x_{d3})\, x_{d1}^2}.$$

We want

$$\frac{dx_{d3}}{dt} = f(i_d, x_d, v_d) + g(i_d, x_d)u_d = j_d,$$

so define the reference input voltage u_d to be

$$u_d \triangleq \frac{j_d - f(i_d, x_d, v_d)}{g(i_d, x_d)}.$$

With $\epsilon_1 = x_{d1} - x_1, \epsilon_2 = x_{d2} - x_2, \epsilon_3 = x_{d3} - x_3$ the error system is

$$\begin{aligned}\frac{d\epsilon_1}{dt} &= \epsilon_2\\ \frac{d\epsilon_2}{dt} &= \epsilon_3\\ \frac{d\epsilon_3}{dt} &= f(i_d, x_d, v_d) + g(i_d, x_d)u_d - f(i, x, v) - g(i, x)u\\ &= f(i_d, x_d, v_d) - f(i, x, v) + g(i_d, x_d)u_d - g(i, x)u.\end{aligned}$$

Then choose the input u to satisfy

$$f(i_d, x_d, v_d) - f(i, x, v) + g(i_d, x_d)u_d - g(i, x)u = -\underbrace{\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}}_{k}\underbrace{\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}}_{\epsilon}.$$

That is, let the input voltage be given by (see Figure 1.10)

$$u = \frac{f(i_d, x_d, v_d) - f(i, x, v) + g(i_d, x_d)u_d + k\epsilon}{g(i, x)} = \frac{-f(i, x, v) + j_d + k\epsilon}{g(i, x)}. \tag{1.63}$$

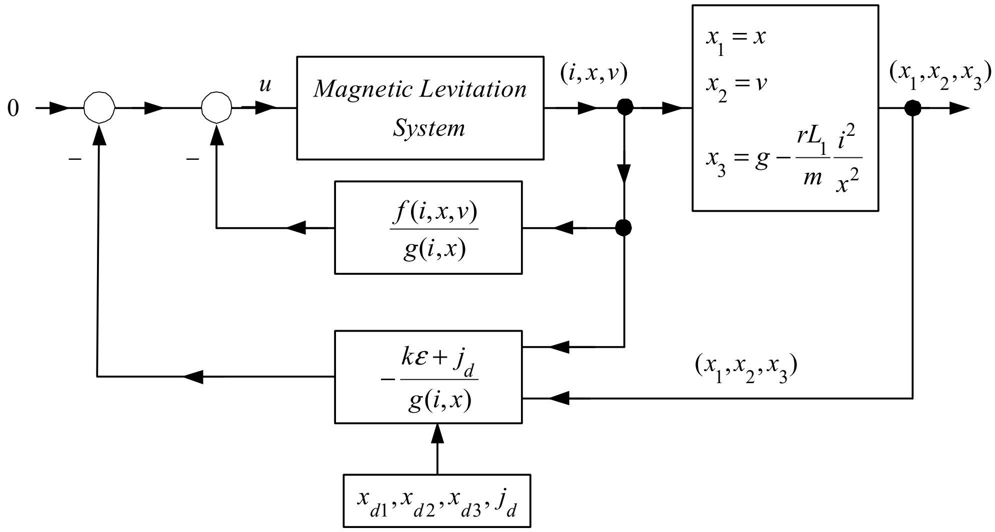

FIGURE 1.10. Block diagram for the magnetic levitation system.

The error system is then

$$\begin{aligned}
\frac{d\epsilon_1}{dt} &= \epsilon_2 \\
\frac{d\epsilon_2}{dt} &= \epsilon_3 \\
\frac{d\epsilon_3}{dt} &= -k_1\epsilon_1 - k_2\epsilon_2 - k_3\epsilon_3.
\end{aligned}$$

In matrix form we have

$$\frac{d}{dt}\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

with

$$\det(sI - (A_c - b_c k)) = s^3 + k_3 s^2 + k_2 s + k_1.$$

The feedback gains k_1, k_2, k_3 can be chosen to place the poles of $A_c - b_c k$ at any desired location in the open left half-plane. The feedback voltage u to the amplifier is then

$$u = \frac{-f(i,x,v) + j_d + k\epsilon}{g(i,x)}.$$

The state variables i, x, v are sampled in real-time and, along with the stored variables $i_d, x_d, v_d, x_{d1}, x_{d2}, x_{d3}, u_d$, the input u computed in real-time according to (1.63) as the voltage commanded to the amplifier. This nonlinear controller allows for trajectory tracking and does *not* require (i, x, v) to be close to an equilibrium state. However, it must be ensured that the controller does not violate the voltage limit of the amplifier, that is, the feedback gains k_1, k_2, k_3 must be chosen so that

$$|u| = \left| \frac{-f(i,x,v) + j_d + k\epsilon}{g(i,x)} \right| \leq V_{\max}.$$

1.7 State Observers for Linear Systems

We first review the concept of observers for state estimation in LTI systems. Consider the single-input single-output (SISO) linear time invariant system given by

$$\begin{aligned}\frac{dx}{dt} &= Ax + bu, \quad A \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^n \\ y &= cx, \quad c \in \mathbb{R}^{1\times n}.\end{aligned}$$

Define an observer for this system by

$$\begin{aligned}\frac{d\hat{x}}{dt} &= A\hat{x} + bu + \ell(y - \hat{y}), \quad \ell \in \mathbb{R}^n \\ \hat{y} &= c\hat{x}.\end{aligned}$$

Note that with $\ell = 0_{n\times 1}$ the observer is simply a simulation of the original system. The addition of the error term $\ell(y-\hat{y})$ provides a way to force $\hat{x}(t) \to x(t)$ despite $\hat{x}(0)$ not being known. The idea here is that the observer puts out the value $\hat{x}(t)$ and uses it to predict the value of $\hat{y}(t) = c\hat{x}(t)$. Then, as $y(t) = cx(t)$ is measured, the observer uses the difference (error) $y(t) - \hat{y}(t) = cx(t) - c\hat{x}(t)$ to adjust the state estimate $\hat{x}(t)$ through $\ell(y(t) - \hat{y}(t))$. To show that this actually works, let

$$\epsilon(t) \triangleq x(t) - \hat{x}(t)$$

which has the dynamics

$$\frac{d\epsilon}{dt} = Ax + bu - (A\hat{x} + bu + \ell(y - \hat{y})) = A(x - \hat{x}) + \ell c(x - \hat{x}) = (A - \ell c)\epsilon(t).$$

If ℓ can be chosen so that $A - \ell c$ is stable, then $\epsilon(t) \to 0$. That is, the state estimate $\hat{x}(t)$ goes to $x(t)$ for any initial condition $x(0)$. To see how ℓ can be chosen to make $A - \ell c$ stable let's consider an example where A and c have a special form.

Example 4 *The Pair* (c, A) *in Observer Canonical Form*
Let the matrices c and A have the special form

$$c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}.$$

A straightforward calculation shows $\det(sI - A) = s^3 + a_2 s^2 + a_1 s + a_0$. With $\ell = \begin{bmatrix} \ell_0 & \ell_1 & \ell_2 \end{bmatrix}^T$ we have

$$A - \ell c = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} - \begin{bmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\ell_0 - a_0 \\ 1 & 0 & -\ell_1 - a_1 \\ 0 & 1 & -\ell_2 - a_2 \end{bmatrix}.$$

Setting $\ell = \begin{bmatrix} a_0 - \alpha_0 & a_1 - \alpha_1 & a_2 - \alpha_2 \end{bmatrix}^T$ we then have

$$A - \ell c = \begin{bmatrix} 0 & 0 & -\alpha_0 \\ 1 & 0 & -\alpha_1 \\ 0 & 1 & -\alpha_2 \end{bmatrix}$$

with $\det(sI - (A - \ell c)) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$. That is, the coefficients of the characteristic polynomial of $A - \ell c$ can be arbitrarily assigned. This special form of the pair (c, A) is called the *observer canonical* form.

For arbitrary $c \in \mathbb{R}^{1\times 3}$ and $A \in \mathbb{R}^{3\times 3}$ the pair (c, A) is called observable if matrix $\mathcal{O}$ defined by

$$\mathcal{O} \triangleq \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix}$$

is nonsingular. If this is true then the eigenvalues of $A - \ell c$ can be assigned arbitrarily. Problem 5 shows how to transform an observable linear system into observer form.

Example 5 *An Observer for Speed and Current in a DC motor*
Recall the model of the DC motor given as

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega + V_S \\ J\frac{d\omega}{dt} &= K_T i - f\omega - \tau_L \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

With $x_1 = i, x_2 = \omega, x_3 = \theta, u = V_S$, and $\tau_L = 0$ we may write

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 \\ K_T/J & -f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{b} u \\ y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

where the rotor angle is taken as the output, i.e., it is measured. The observability matrix is

$$\mathcal{O} \triangleq \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ K_T/J & -f/J & 0 \end{bmatrix}$$

which is nonsingular. Define a coordinate transformation

$$x^* = Tx = \begin{bmatrix} K_T/J & R/L & (Rf + K_T K_b)/(JL) \\ 0 & 1 & R/L + f/J \\ 0 & 0 & 1 \end{bmatrix} x$$

which has inverse

$$x = T^{-1}x^* = \begin{bmatrix} J/K_T & -RJ/(LK_T) & R^2J/(L^2K_T) - K_b/L \\ 0 & 1 & -\left(RJ + Lf\right)/(JL) \\ 0 & 0 & 1 \end{bmatrix} x^*.$$

In the new coordinate system we have

$$\begin{aligned} \frac{dx^*}{dt} &= TAT^{-1}x^* + Tbu \\ y &= cT^{-1}x^* \end{aligned}$$

where

$$\begin{aligned} A_o &\triangleq TAT^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & (Rf + K_T K_b)/(JL) \\ 0 & 1 & R/L + f/J \end{bmatrix}, b_o \triangleq Tb = \begin{bmatrix} K_T/(JL) \\ 0 \\ 0 \end{bmatrix} \\ c_o &\triangleq cT^{-1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

With $a_0 = 0, a_1 = (Rf + K_T K_b)/(JL), a_2 = R/L + f/J$ we see that the pair

$$c_o = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},\ A_o \begin{bmatrix} 0 & 0 & a_0 \\ 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{bmatrix}$$

is in observer canonical form. Thus the procedure given in the previous example can be used to estimate x^* and thus the estimate of $\hat{x}$ is given by

$$\hat{x} = T^{-1}\hat{x}^*$$

Problem 9 shows how to estimate the load torque τ_L in addition to the speed ω and the current i.

1.8 State Observers for Nonlinear Systems

Consider a nonlinear system in the following special form.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}}_{A_o}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \varphi_1(y) \\ \varphi_2(y) \\ \varphi_3(y) \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b_o} u \tag{1.64}$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{c_o}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{1.65}$$

where the $\varphi_i(y) = \varphi_i(x_3)$ are arbitrary functions of the output $y = x_3$. Define an observer by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} \varphi_1(y) \\ \varphi_2(y) \\ \varphi_3(y) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u + \begin{bmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \end{bmatrix}(y - \hat{y}) \\ \hat{y} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}. \end{aligned}$$

Then the error $\epsilon \triangleq x - \hat{x}$ satisfies

$$\frac{d}{dt}\epsilon = (A_o - \ell c_o)\epsilon.$$

This is a linear system with the pair (c_o, A_o) in observer canonical form so $\ell \in \mathbb{R}^3$ can used to place the poles of $A_o - \ell c_o$ in the open left-half plane so that $\epsilon(t) \triangleq x(t) - \hat{x}(t) \to 0_{3\times 1}$.

The differential-geometric approach to the design of observers for nonlinear systems is to find a statespace transformation so that in the new coordinates the system has the form of (1.64) and (1.65).

Example 6 *Series Connected DC Motor*

Recall the nonlinear equations of the series connected DC motor.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ c_2 x_3 - c_3 x_3 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}}_{p} \tau_L$$

where $x_1 = \theta, x_2 = \omega, x_3 = i = i_a = i_f, u = V_S/L, c_1 = K_T L_f/J, c_2 = -R/L$, and $c_3 = K_b L_f/L$. As discussed above, the series connected DC motor is used in speed control applications so let's remove $x_1 = \theta$

from the model. The load torque is taken to be constant, but is unknown so it will need to be estimated in order to estimate the motor speed ω. To do this τ_L/J is added to the model as a *state variable* with $\dfrac{d}{dt}(\tau_L/J) = 0$. With $z_1 = x_2 = \omega, z_2 = x_3 = i$, $z_3 = \tau_L/J$, and assuming only the current is measured, the system equations are now

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 z_2^2 - z_3 \\ -c_2 z_2 - c_3 z_1 z_2 \\ 0 \end{bmatrix}}_{f(z)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{g(z)} u \tag{1.66}$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{c} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \tag{1.67}$$

More compactly this is written as

$$\begin{aligned} \frac{dz}{dt} &= f(z) + g(z)u \\ y &= cz. \end{aligned}$$

$f(z)$ in (1.66) is *not* linear in the unmeasured state variable $z_1 = \omega$. We transform the equations into a new set of coordinates for which the statespace model is linear in i and ω. Specifically, consider the nonlinear transformation given by

$$\begin{aligned} z_1^* &= T_1(z) = c_3 z_3 \\ z_2^* &= T_2(z) = -c_3 z_1 \\ z_3^* &= T_3(z) = \ln(z_2) \end{aligned}$$

with inverse

$$\begin{aligned} z_1 &= -z_2^*/c_3 \\ z_2 &= e^{z_3^*} \\ z_3 &= z_1^*/c_3. \end{aligned}$$

The system equations in the z^* coordinate system are then

$$\begin{aligned} \frac{dz_1^*}{dt} = \mathcal{L}_{f+gu}(T_1(z)) = \langle dT_1, f + gu \rangle &= \begin{bmatrix} 0 & 0 & c_3 \end{bmatrix} \left(\begin{bmatrix} c_1 z_2^2 - z_3 \\ -c_2 z_2 - c_3 z_1 z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \right) \\ &= 0 \\ \frac{dz_2^*}{dt} = \mathcal{L}_{f+gu}(T_2(z)) = \langle dT_2, f + gu \rangle &= \begin{bmatrix} -c_3 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} c_1 z_2^2 - z_3 \\ -c_2 z_2 - c_3 z_1 z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \right) \\ &= -c_1 c_3 z_2^2 + c_3 z_3 \\ &= -c_1 c_3 y^2 + c_3 z_3 \\ \frac{dz_3^*}{dt} = \mathcal{L}_{f+gu}(T_3(z)) = \langle dT_3, f + gu \rangle &= \begin{bmatrix} 0 & 1/z_2 & 0 \end{bmatrix} \left(\begin{bmatrix} c_1 z_2^2 - z_3 \\ -c_2 z_2 - c_3 z_1 z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \right) \\ &= -c_2 - c_3 z_1 + u/z_2. \end{aligned}$$

In summary the system in the z^* coordinates is given by

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A_o}\begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ -c_1c_3y^2 \\ -c_2 + u/z_2 \end{bmatrix} \\ y^* &= \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{c_o}\begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix}.\end{aligned}$$

Note that the pair (c_o, A_o) is in observer canonical form. The output is taken to be $y^* \triangleq z_3^* = \ln(z_2) = \ln(i)$. As the current is measured the output y^* is known.

The observer for $z_1^* = c_3 z_3 = \dfrac{K_b L_f}{L}\dfrac{\tau_L}{J}$ and $z_2^* = -c_3 z_1 = -\dfrac{K_b L_f}{L}\omega$ is

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} \hat{z}_1^* \\ \hat{z}_2^* \\ \hat{z}_3^* \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{z}_1^* \\ \hat{z}_2^* \\ \hat{z}_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ -c_1c_3y^2 \\ -c_2 + u/z_2 \end{bmatrix} + \begin{bmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \end{bmatrix}(y^* - \hat{y}^*) \\ \hat{y}^* &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{z}_1^* \\ \hat{z}_2^* \\ \hat{z}_3^* \end{bmatrix}.\end{aligned}$$

The error system is

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} z_1^* - \hat{z}_1^* \\ z_2^* - \hat{z}_2^* \\ z_3^* - \hat{z}_3^* \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} z_1^* - \hat{z}_1^* \\ z_2^* - \hat{z}_2^* \\ z_3^* - \hat{z}_3^* \end{bmatrix} - \begin{bmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} z_1^* - \hat{z}_1^* \\ z_2^* - \hat{z}_2^* \\ z_3^* - \hat{z}_3^* \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -\ell_0 \\ 1 & 0 & -\ell_1 \\ 0 & 1 & -\ell_2 \end{bmatrix}\begin{bmatrix} z_1^* - \hat{z}_1^* \\ z_2^* - \hat{z}_2^* \\ z_3^* - \hat{z}_3^* \end{bmatrix}\end{aligned}$$

The gains ℓ_0, ℓ_1, ℓ_2 can be chosen to place the eigenvalues of this error system arbitrarily so $\hat{z}^*(t) \to z^*(t)$. As $z_1^* = \dfrac{K_b L_f}{L}\dfrac{\tau_L}{J}$ and $z_2^* = -\dfrac{K_b L_f}{L}\omega$ the parameters $K_b, L_f, L = L_f + L_a$ must be known accurately to obtain accurate estimates for τ_L/J and ω.

Remark An implementation of this observer is given in [10].

1.9 Lie Brackets, Lie Derivatives, and Differential Equations

Let's look at a general nonlinear control system given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{bmatrix}}_{g(x)} u. \tag{1.68}$$

Examples of nonlinear control systems have been presented showing that it is possible to find a nonlinear change of coordinates such that the system can be made linear using feedback. The resulting linear system can then be straightforwardly controlled. The first goal is determine the conditions on a given nonlinear system that determine if such a transformation exists. If such a transformation exists, the next goal is to find the transformation. This will done in the following chapters. Here we develop some tools that are needed to achieve these goals.

Consider an invertible nonlinear change of coordinates $x^* = T(x) : \mathbb{R}^3 \to \mathbb{R}^3$ for (1.68) given by

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} \triangleq \begin{bmatrix} T_1(x_1,x_2,x_3) \\ T_2(x_1,x_2,x_3) \\ T_3(x_1,x_2,x_3) \end{bmatrix}.$$

In the x^* coordinate system the system (1.68) has the form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \underbrace{\begin{bmatrix} f_1^*(x^*) \\ f_2^*(x^*) \\ f_3^*(x^*) \end{bmatrix}}_{f^*(x^*)} + \underbrace{\begin{bmatrix} g_1^*(x^*) \\ g_2^*(x^*) \\ g_3^*(x^*) \end{bmatrix}}_{g^*(x^*)} u$$

where

$$f^*(x^*) = \frac{\partial T}{\partial x} f(x)\Big|_{x=T^{-1}(x^*)}, \quad g^*(x^*) = \frac{\partial T}{\partial x} g(x)\Big|_{x=T^{-1}(x^*)} \tag{1.69}$$

and

$$\frac{\partial T}{\partial x} \triangleq \begin{bmatrix} \partial T_1/\partial x_1 & \partial T_1/\partial x_2 & \partial T_1/\partial x_3 \\ \partial T_2/\partial x_1 & \partial T_2/\partial x_2 & \partial T_2/\partial x_3 \\ \partial T_3/\partial x_1 & \partial T_3/\partial x_2 & \partial T_3/\partial x_3 \end{bmatrix}. \tag{1.70}$$

The matrix $\dfrac{\partial T}{\partial x} \in \mathbb{R}^{3\times 3}$ is referred to as the *Jacobian* of the transformation $T \in \mathbb{R}^3$.

Example 7 *Linear Systems*

Let $f(x) = Ax, g(x) = b$ so that $\dfrac{dx}{dt} = f(x) + g(x)u = Ax + bu$ is a linear system. Further, let $T(x) \triangleq \mathbf{T}x$ be a linear transformation with $\mathbf{T} \in \mathbb{R}^{3\times 3}$ and $\det(\mathbf{T}) \neq 0$.

Then

$$\begin{aligned} f^*(x) &\triangleq \frac{\partial T}{\partial x} f(x) = \mathbf{T}Ax \\ g^*(x) &\triangleq \mathbf{T}b \end{aligned}$$

and

$$\begin{aligned} f^*(x^*) &= \frac{\partial T}{\partial x} f(x)\Big|_{x=T^{-1}(x^*)} = \mathbf{T}Ax_{x=T^{-1}(x^*)} = \mathbf{T}A\mathbf{T}^{-1}x^* \\ g^*(x^*) &= \frac{\partial T}{\partial x} g(x)\Big|_{x=T^{-1}(x^*)} = \mathbf{T}b\big|_{x=T^{-1}(x^*)} = \mathbf{T}b. \end{aligned}$$

We have the following definition of the *Lie bracket* of $f(x)$ and $g(x)$.

Definition 2 *Lie Bracket*

Let $f(x)$ and $g(x)$ be as in (1.68). Define the *Lie bracket* of the pair $f(x), g(x)$ as

$$[f,g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \triangleq \begin{bmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \partial g_1/\partial x_3 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 & \partial g_2/\partial x_3 \\ \partial g_3/\partial x_1 & \partial g_3/\partial x_2 & \partial g_3/\partial x_3 \end{bmatrix}\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} - \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \end{bmatrix}\begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{bmatrix}.$$

We will find out in Chapter 3 that the Lie bracket is a tool used in determining whether or not a feedback linearizing transformation exists for a system of the form $dx/dt = f(x) + g(x)u$.

Lie Brackets Under a Change of Coordinates $[f^*, g^*] = \dfrac{\partial T}{\partial x}[f, g]$

An important fact of the Lie bracket is that under a change of coordinates the Lie bracket $[f, g]$ transforms the *same* way as f, g transform as given in (1.69). That is, with $f^*(x^*), g^*(x^*)$ defined as in (1.69) it is now shown that

$$[f^*, g^*] = \frac{\partial g^*(x^*)}{\partial x^*}f^*(x^*) - \frac{\partial f^*(x^*)}{\partial x^*}g^*(x^*) = \frac{\partial T}{\partial x}\left(\frac{\partial g(x)}{\partial x}f(x) - \frac{\partial f(x)}{\partial x}g(x)\right)\bigg|_{x=T^{-1}(x^*)} = \frac{\partial T}{\partial x}[f, g]\bigg|_{|x=T^{-1}(x^*)}. \tag{1.71}$$

To begin let $S = T^{-1}$ so that $x = T^{-1}(x^*) = S(x^*)$. By the chain rule we have

$$\begin{aligned}
\frac{\partial f^*(x^*)}{\partial x^*} = \frac{\partial}{\partial x^*}\left(\frac{\partial T}{\partial x}f(x)\bigg|_{x=T^{-1}(x^*)}\right) &= \frac{\partial}{\partial x^*}\left(\begin{bmatrix} \partial T_1/\partial x_1 & \partial T_1/\partial x_2 & \partial T_1/\partial x_3 \\ \partial T_2/\partial x_1 & \partial T_2/\partial x_2 & \partial T_2/\partial x_3 \\ \partial T_3/\partial x_1 & \partial T_3/\partial x_2 & \partial T_3/\partial x_3 \end{bmatrix}\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}\Bigg|_{x=T^{-1}(x^*)}\right) \\
&= \underbrace{\frac{\partial}{\partial x}\begin{bmatrix} d_1(x) \\ d_2(x) \\ d_3(x) \end{bmatrix}}_{\left(\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}f(x)\right)\right)}\underbrace{\begin{bmatrix} \partial x_1/\partial x_1^* & \partial x_1/\partial x_2^* & \partial x_1/\partial x_3^* \\ \partial x_2/\partial x_1^* & \partial x_2/\partial x_2^* & \partial x_2/\partial x_3^* \\ \partial x_3/\partial x_1^* & \partial x_3/\partial x_2^* & \partial x_3/\partial x_3^* \end{bmatrix}}_{\frac{\partial x}{\partial x^*} = \frac{\partial S(x^*)}{\partial x^*}} \\
&= \underbrace{\begin{bmatrix} \partial d_1/\partial x_1 & \partial d_1/\partial x_2 & \partial d_1/\partial x_3 \\ \partial d_2/\partial x_1 & \partial d_2/\partial x_2 & \partial d_2/\partial x_3 \\ \partial d_3/\partial x_1 & \partial d_3/\partial x_2 & \partial d_3/\partial x_3 \end{bmatrix}}_{\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}f(x)\right)}\underbrace{\begin{bmatrix} \partial x_1/\partial x_1^* & \partial x_1/\partial x_2^* & \partial x_1/\partial x_3^* \\ \partial x_2/\partial x_1^* & \partial x_2/\partial x_2^* & \partial x_2/\partial x_3^* \\ \partial x_3/\partial x_1^* & \partial x_3/\partial x_2^* & \partial x_3/\partial x_3^* \end{bmatrix}}_{\frac{\partial x}{\partial x^*} = \frac{\partial S(x^*)}{\partial x^*}} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}f(x)\right)\bigg|_{x=T^{-1}(x^*)}\frac{\partial S(x^*)}{\partial x^*}.
\end{aligned} \tag{1.72}$$

Then

$$\begin{aligned}
\frac{\partial f^*(x^*)}{\partial x^*}g^*(x^*) &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}f(x)\right)\bigg|_{x=T^{-1}(x^*)}\frac{\partial S(x^*)}{\partial x^*}\frac{\partial T}{\partial x}g(x)\bigg|_{x=T^{-1}(x^*)} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}f(x)\right)\bigg|_{x=T^{-1}(x^*)}\underbrace{\frac{\partial S(x^*)}{\partial x^*}\frac{\partial T}{\partial x}_{|x=T^{-1}(x^*)}}_{I_{3\times 3}}g(x)_{|x=T^{-1}(x^*)} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}f(x)\right)\bigg|_{x=T^{-1}(x^*)}g(x)_{|x=T^{-1}(x^*)}.
\end{aligned}$$

The last step used the fact that $x = S(x^*) = T^{-1}(x^*)$ so $S(x^*) = S(T(x)) = x$ and therefore

$$\frac{\partial}{\partial x}S(T(x)) = \frac{\partial S(x^*)}{\partial x^*}\frac{\partial T(x)}{\partial x} = \frac{\partial x}{\partial x} = I_{3\times 3}.$$

Next we find a more convenient expression for $\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}f(x)\right)$. To do this note that $\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}f(x)\right)$ is more explicitly written as

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}f(x)\right)=\begin{bmatrix}\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right)\\ \\ \frac{\partial}{\partial x}\left(\frac{\partial T_2}{\partial x}f(x)\right)\\ \\ \frac{\partial}{\partial x}\left(\frac{\partial T_3}{\partial x}f(x)\right)\end{bmatrix}.$$

Expanding $\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right)$ we have

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right)\\
&= \begin{bmatrix}\frac{\partial}{\partial x_1}\left(\frac{\partial T_1}{\partial x}f(x)\right) & \frac{\partial}{\partial x_2}\left(\frac{\partial T_1}{\partial x}f(x)\right) & \frac{\partial}{\partial x_3}\left(\frac{\partial T_1}{\partial x}f(x)\right)\end{bmatrix}\\
&= \begin{bmatrix}\frac{\partial}{\partial x_1}\left(\sum_{i=1}^{3}\frac{\partial T_1}{\partial x_i}f_i(x)\right) & \frac{\partial}{\partial x_2}\left(\sum_{i=1}^{3}\frac{\partial T_1}{\partial x_i}f_i(x)\right) & \frac{\partial}{\partial x_3}\left(\sum_{i=1}^{3}\frac{\partial T_1}{\partial x_i}f_i(x)\right)\end{bmatrix}\\
&= \begin{bmatrix}\sum_{i=1}^{3}\frac{\partial^2 T_1}{\partial x_1\partial x_i}f_i(x)+\frac{\partial T_1}{\partial x_i}\frac{\partial f_i(x)}{\partial x_1} & \sum_{i=1}^{3}\frac{\partial^2 T_1}{\partial x_2\partial x_i}f_i(x)+\frac{\partial T_1}{\partial x_i}\frac{\partial f_i(x)}{\partial x_2} & \sum_{i=1}^{3}\frac{\partial^2 T_1}{\partial x_3\partial x_i}f_i(x)+\frac{\partial T_1}{\partial x_i}\frac{\partial f_i(x)}{\partial x_3}\end{bmatrix}\\
&= \begin{bmatrix}\sum_{i=1}^{3}\left(\frac{\partial^2 T_1}{\partial x_1\partial x_i}f_i(x)\right) & \sum_{i=1}^{3}\left(\frac{\partial^2 T_1}{\partial x_2\partial x_i}f_i(x)\right) & \sum_{i=1}^{3}\left(\frac{\partial^2 T_1}{\partial x_3\partial x_i}f_i(x)\right)\end{bmatrix}+\\
&\quad\begin{bmatrix}\sum_{i=1}^{3}\left(\frac{\partial T_1}{\partial x_i}\frac{\partial f_i(x)}{\partial x_1}\right) & \sum_{i=1}^{3}\left(\frac{\partial T_1}{\partial x_i}\frac{\partial f_i(x)}{\partial x_2}\right) & \sum_{i=1}^{3}\left(\frac{\partial T_1}{\partial x_i}\frac{\partial f_i(x)}{\partial x_3}\right)\end{bmatrix}.
\end{aligned}$$

Continuing the expansion of $\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right)$ gives

$$\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right)=$$

$$\left[\begin{bmatrix}\frac{\partial^2 T_1}{\partial x_1\partial x_1} & \frac{\partial^2 T_1}{\partial x_1\partial x_2} & \frac{\partial^2 T_1}{\partial x_1\partial x_3}\end{bmatrix}\begin{bmatrix}f_1(x)\\ f_2(x)\\ f_3(x)\end{bmatrix}\ \begin{bmatrix}\frac{\partial^2 T_1}{\partial x_2\partial x_1} & \frac{\partial^2 T_1}{\partial x_2\partial x_2} & \frac{\partial^2 T_1}{\partial x_2\partial x_3}\end{bmatrix}\begin{bmatrix}f_1(x)\\ f_2(x)\\ f_3(x)\end{bmatrix}\ \begin{bmatrix}\frac{\partial^2 T_1}{\partial x_3\partial x_1} & \frac{\partial^2 T_1}{\partial x_3\partial x_2} & \frac{\partial^2 T_1}{\partial x_3\partial x_3}\end{bmatrix}\begin{bmatrix}f_1(x)\\ f_2(x)\\ f_3(x)\end{bmatrix}\right]$$

$$+\left[\begin{bmatrix}\frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3}\end{bmatrix}\begin{bmatrix}\frac{\partial f_1(x)}{\partial x_1}\\ \frac{\partial f_2(x)}{\partial x_1}\\ \frac{\partial f_3(x)}{\partial x_1}\end{bmatrix}\ \begin{bmatrix}\frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3}\end{bmatrix}\begin{bmatrix}\frac{\partial f_1(x)}{\partial x_2}\\ \frac{\partial f_2(x)}{\partial x_2}\\ \frac{\partial f_3(x)}{\partial x_2}\end{bmatrix}\ \begin{bmatrix}\frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3}\end{bmatrix}\begin{bmatrix}\frac{\partial f_1(x)}{\partial x_2}\\ \frac{\partial f_2(x)}{\partial x_2}\\ \frac{\partial f_3(x)}{\partial x_2}\end{bmatrix}\right]$$

Using $\dfrac{\partial^2 T_1}{\partial x_i \partial x_j} = \dfrac{\partial^2 T_1}{\partial x_j \partial x_i}$ we rewrite this last expression as

$$\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f(x)\right) =$$

$$\begin{bmatrix} \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 T_1}{\partial x_1 \partial x_1} \\ \frac{\partial^2 T_1}{\partial x_2 \partial x_1} \\ \frac{\partial^2 T_1}{\partial x_3 \partial x_1} \end{bmatrix} & \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 T_1}{\partial x_1 \partial x_2} \\ \frac{\partial^2 T_1}{\partial x_2 \partial x_2} \\ \frac{\partial^2 T_1}{\partial x_3 \partial x_2} \end{bmatrix} & \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 T_1}{\partial x_1 \partial x_3} \\ \frac{\partial^2 T_1}{\partial x_2 \partial x_3} \\ \frac{\partial^2 T_1}{\partial x_3 \partial x_3} \end{bmatrix} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \end{bmatrix} \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \end{bmatrix}$$

Finally we have the expression we were looking for given by

$$\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f(x)\right) =$$

$$\underbrace{\begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \end{bmatrix}}_{f^T} \underbrace{\begin{bmatrix} \frac{\partial^2 T_1}{\partial x_1 \partial x_1} & \frac{\partial^2 T_1}{\partial x_1 \partial x_2} & \frac{\partial^2 T_1}{\partial x_1 \partial x_3} \\ \frac{\partial^2 T_1}{\partial x_2 \partial x_1} & \frac{\partial^2 T_1}{\partial x_2 \partial x_2} & \frac{\partial^2 T_1}{\partial x_2 \partial x_3} \\ \frac{\partial^2 T_1}{\partial x_3 \partial x_1} & \frac{\partial^2 T_1}{\partial x_3 \partial x_2} & \frac{\partial^2 T_1}{\partial x_3 \partial x_3} \end{bmatrix}}_{\frac{\partial^2 T_1}{\partial x^2}} + \underbrace{\begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \end{bmatrix}}_{\frac{\partial T_1}{\partial x}} \underbrace{\begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \end{bmatrix}}_{\frac{\partial f}{\partial x}}$$

or

$$\frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x} f(x)\right) = f^T \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial T_1}{\partial x}\frac{\partial f}{\partial x}. \tag{1.73}$$

The *symmetric* matrix $\dfrac{\partial^2 T_1}{\partial x^2}$ is referred to as the *Hessian* of T_1. Now we can write

$$\begin{aligned} \frac{\partial f^*(x^*)}{\partial x^*} g^*(x^*) = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} f(x)\right) g(x)_{|x=T^{-1}(x^*)} &= \left. \begin{bmatrix} f^T \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial T_1}{\partial x}\frac{\partial f}{\partial x} \\ f^T \frac{\partial^2 T_2}{\partial x^2} + \frac{\partial T_2}{\partial x}\frac{\partial f}{\partial x} \\ f^T \frac{\partial^2 T_3}{\partial x^2} + \frac{\partial T_3}{\partial x}\frac{\partial f}{\partial x} \end{bmatrix} \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{bmatrix} \right|_{x=T^{-1}(x^*)} \\ &= \begin{bmatrix} f^T \frac{\partial^2 T_1}{\partial x^2} g(x) + \frac{\partial T_1}{\partial x}\frac{\partial f}{\partial x} g(x) \\ f^T \frac{\partial^2 T_2}{\partial x^2} g(x) + \frac{\partial T_2}{\partial x}\frac{\partial f}{\partial x} g(x) \\ f^T \frac{\partial^2 T_3}{\partial x^2} g(x) + \frac{\partial T_3}{\partial x}\frac{\partial f}{\partial x} g(x) \end{bmatrix}_{|x=T^{-1}(x^*)} \end{aligned} \tag{1.74}$$

Thus

$$\begin{aligned}
\frac{\partial g^*(x^*)}{\partial x^*}f^*(x^*) - \frac{\partial f^*(x^*)}{\partial x^*}g^*(x^*) &= \begin{bmatrix} g^T\dfrac{\partial^2 T_1}{\partial x^2}f(x) + \dfrac{\partial T_1}{\partial x}\dfrac{\partial g}{\partial x}f(x) \\ \\ g^T\dfrac{\partial^2 T_2}{\partial x^2}f(x) + \dfrac{\partial T_2}{\partial x}\dfrac{\partial g}{\partial x}f(x) \\ \\ g^T\dfrac{\partial^2 T_3}{\partial x^2}f(x) + \dfrac{\partial T_3}{\partial x}\dfrac{\partial g}{\partial x}f(x) \end{bmatrix} - \begin{bmatrix} f^T\dfrac{\partial^2 T_1}{\partial x^2}g(x) + \dfrac{\partial T_1}{\partial x}\dfrac{\partial f}{\partial x}g(x) \\ \\ f^T\dfrac{\partial^2 T_2}{\partial x^2}g(x) + \dfrac{\partial T_2}{\partial x}\dfrac{\partial f}{\partial x}g(x) \\ \\ f^T\dfrac{\partial^2 T_3}{\partial x^2}g(x) + \dfrac{\partial T_3}{\partial x}\dfrac{\partial f}{\partial x}g(x) \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial T_1}{\partial x}\dfrac{\partial g}{\partial x}f(x) \\ \\ \dfrac{\partial T_2}{\partial x}\dfrac{\partial g}{\partial x}f(x) \\ \\ \dfrac{\partial T_3}{\partial x}\dfrac{\partial g}{\partial x}f(x) \end{bmatrix} - \begin{bmatrix} \dfrac{\partial T_1}{\partial x}\dfrac{\partial f}{\partial x}g(x) \\ \\ \dfrac{\partial T_2}{\partial x}\dfrac{\partial f}{\partial x}g(x) \\ \\ \dfrac{\partial T_3}{\partial x}\dfrac{\partial f}{\partial x}g(x) \end{bmatrix} \\
&= \frac{\partial T}{\partial x}\left(\frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g\right)_{|x=T^{-1}(x^*)}
\end{aligned}$$

We have shown

$$[f^*, g^*] = \frac{\partial T}{\partial x}\,[f,g]\Big|_{|x=T^{-1}(x^*)}$$

Lie brackets and Lie derivatives $\mathcal{L}_{[f,g]}(T_1) = \mathcal{L}_f(\mathcal{L}_g(T_1)) - \mathcal{L}_g(\mathcal{L}_f(T_1))$

There is a connection between Lie brackets and Lie derivatives. To show this connection recall by definition that

$$\mathcal{L}_f(T_1) = \frac{\partial T_1}{\partial x}f(x) \in \mathbb{R}$$

so

$$\mathcal{L}_g(\mathcal{L}_f(T_1)) = \frac{\partial(\mathcal{L}_f(T_1))}{\partial x}g(x) = \frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right)g(x).$$

By (1.74) we have

$$\mathcal{L}_g(\mathcal{L}_f(T_1)) = \frac{\partial}{\partial x}\left(\frac{\partial T_1}{\partial x}f(x)\right)g(x) = f^T\frac{\partial^2 T_1}{\partial x^2}g(x) + \frac{\partial T_1}{\partial x}\frac{\partial f}{\partial x}g(x).$$

It then follows that

$$\begin{aligned}
\mathcal{L}_f(\mathcal{L}_g(T_1)) - \mathcal{L}_g(\mathcal{L}_f(T_1)) &= \left(g^T\frac{\partial^2 T_1}{\partial x^2}f(x) + \frac{\partial T_1}{\partial x}\frac{\partial g}{\partial x}f(x)\right) - \left(f^T\frac{\partial^2 T_1}{\partial x^2}g(x) + \frac{\partial T_1}{\partial x}\frac{\partial f}{\partial x}g(x)\right) \\
&= \frac{\partial T_1}{\partial x}\left(\frac{\partial g}{\partial x}f(x) - \frac{\partial f}{\partial x}g(x)\right) \\
&= \mathcal{L}_{[f,g]}(T_1).
\end{aligned}$$

That is

$$\mathcal{L}_{[f,g]}(T_1) = \mathcal{L}_f(\mathcal{L}_g(T_1)) - \mathcal{L}_g(\mathcal{L}_f(T_1)).$$

Repeated Lie Brackets

Definition 3 *Repeated Lie Brackets*
Define

$$
\begin{aligned}
ad_f^0 g &\triangleq g \\
ad_f^1 g &\triangleq [f, g] \\
ad_f^2 g &= [f, [f, g]] \\
&\vdots \\
ad_f^k g &= [f, ad_f^{k-1} g].
\end{aligned}
$$

Example 8 *Linear Systems*
Let $f(x) = Ax, g(x) = b$ so that $\dfrac{dx}{dt} = f(x) + g(x)u = Ax + bu$ is a linear system. Then

$$
\begin{aligned}
ad_f^0 g &= b, \\
ad_f^1 g &\triangleq [Ax, b] = \frac{\partial b}{\partial x} Ax - \frac{\partial (Ax)}{\partial x} b = -Ab \\
ad_f^2 g &\triangleq [Ax, ad_f^1 g] = \frac{\partial ad_f^1 g}{\partial x} Ax - \frac{\partial (Ax)}{\partial x} ad_f^1 g = A^2 b
\end{aligned}
$$

or, in general,

$$ad_f^k g = (-1)^k A^k b.$$

1.10 Problems

Problem 1 *Nonlinear Transformations*

(a) Let a nonlinear system be given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ g_2(x_1,x_2) \end{bmatrix} u.$$

Using the statespace transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} T_1(x_1,x_2) \\ T_2(x_1,x_2) \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ \mathcal{L}_f(T_1) \end{bmatrix} = \begin{bmatrix} x_1 \\ \mathcal{L}_f(x_1) \end{bmatrix}$$

and appropriate feedback the system can be put in the linear form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(b) Let a nonlinear system be given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_3(x_1,x_2,x_3) \end{bmatrix} u.$$

Using the statespace transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} T_1(x_1,x_2,x_3) \\ T_2(x_1,x_2,x_3) \\ T_3(x_1,x_2,x_3) \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ \mathcal{L}_f(T_1) \\ \mathcal{L}_f^2(T_1) \end{bmatrix} = \begin{bmatrix} x_1 \\ \mathcal{L}_f(x_1) \\ \mathcal{L}_f^2(x_1) \end{bmatrix}$$

and appropriate feedback the system can be put in the linear form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

(c) Let a nonlinear system be given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3,x_4) \\ f_3(x_1,x_2,x_3,x_4) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_4(x_1,x_2,x_3,x_4) \end{bmatrix} u.$$

Using the statespace transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} T_1(x_1,x_2,x_3,x_4) \\ T_2(x_1,x_2,x_3,x_4) \\ T_3(x_1,x_2,x_3,x_4) \\ T_4(x_1,x_2,x_3,x_4) \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ \mathcal{L}_f(T_1) \\ \mathcal{L}_f^2(T_1) \\ \mathcal{L}_f^3(T_1) \end{bmatrix} = \begin{bmatrix} x_1 \\ \mathcal{L}_f(x_1) \\ \mathcal{L}_f^2(x_1) \\ \mathcal{L}_f^3(x_1) \end{bmatrix}$$

and appropriate feedback the system can be put in the linear form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Problem 2 *Tank Reactor* [11]

A dimensionless pair of equations for the model of a constant volume, non-adiabatic (occurring with loss or gain of heat), stirred tank reactor with first irreversible kinetics is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + c_1(1-x_1)e^{\frac{x_2}{1+x_2/\gamma}} \\ -x_2 + c_1c_2(1-x_1)e^{\frac{x_2}{1+x_2/\gamma}} \end{bmatrix} + \begin{bmatrix} 0 \\ -(x_2 - x_c) \end{bmatrix} u$$

with $c_1 = 0.05, c_2 = 8$, $x_c = 0$, and γ is a constant.

(a) Show that the equilibrium point

$$x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 3 \end{bmatrix}$$

implies $e^{\frac{x_2}{1+x_2/\gamma}} = 20$ and $u_0 = 1/3$.

(b) Rewrite the model equations in terms of

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix} \quad \text{and} \quad w = -u_0.$$

(c) Use part (a) of Problem 1 to find a transformation

$$z^* = T(z)$$

so that in the new coordinates along with feedback of the form

$$u = f^*(z) + g^*(z)w$$

the model in part (b) becomes

$$\frac{d}{dt}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Explicitly given $a(z)$ and $\beta(z)$. Also, with the state feedback

$$w = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix}$$

find the values of the gains k_1, k_2 such that the closed-loop poles of the z^* system are $-3, -3$.

Problem 3 *Nonlinear Regulator for a Synchronous Generator* [12]

A nonlinear statespace model of a synchronous generator connected to an infinite bus is given in [12] as

$$\frac{d}{dt}\begin{bmatrix} \delta \\ \omega \\ \psi_f \\ \psi_A \\ \psi_B \end{bmatrix} = \underbrace{\begin{bmatrix} \omega - \omega_s \\ c_{mo} - K_2\omega\psi_f \sin(\delta) - K_3\omega\psi_A \sin(\delta) - K_4\omega\psi_B \sin(\delta) + K_5 \sin(\delta)\cos(\delta) \\ \nu_{f0} - K_8\omega\psi_f + K_9\omega\psi_A + K_{10}\cos(\delta) \\ K_{11}\omega\psi_f - K_{12}\omega\psi_A + K_{13}\cos(\delta) \\ -K_{14}\omega\psi_B - K_{15}\sin(\delta) \end{bmatrix}}_{f(\delta,\omega,\psi_f,\psi_A,\psi_B)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{g_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{g_2} u_2$$

where δ is the rotor angle referred to the infinite bus, ω is the rotor angular speed, ψ_f is the field flux linkage, ψ_A is the direct axis flux linkage, ψ_B is the quadrature axis flux linkage, ω_s is the constant synchronous angular velocity, c_m is the rotor angular acceleration produced by the input torque, c_{mo} is the reference input angular acceleration, ν_f is the field excitation voltage, ν_{f0} is the constant reference field excitation voltage, $u_1 = c_m - c_{mo}$, and $u_2 = \nu_f - \nu_{f0}$.

Let $\begin{bmatrix} \delta_0 & \omega_0 & \psi_{f0} & \psi_{A0} & \psi_{B0} \end{bmatrix}^T$ denote a constant stable operating point with $u_1 = 0, u_2 = 0$. All the parameters $K_1, ..., K_{15}$ are positive constants. With

$$\begin{aligned} x &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T \triangleq \begin{bmatrix} \delta & \omega & \psi_f & \psi_A & \psi_B \end{bmatrix}^T \\ u &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} c_m - c_{mo} & \nu_f - \nu_{f0} \end{bmatrix} \end{aligned}$$

the model of the synchronous generator has the compact form

$$\frac{d}{dt}x = f(x) + g(x)u.$$

(a) Define a nonlinear transformation given by

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \\ T_4(x) \\ T_5(x) \end{bmatrix} = \begin{bmatrix} \ln(x_5)/K_{14} + x_1 + c_1 \\ -\omega_s - (K_{15}/K_{14})\sin(x_1)/x_5 \\ -(K_{15}/K_{14})(x_2 - \omega_s)\cos(x_1)/x_5 - K_{15}x_2\sin(x_1)/x_5 - (K_{15}^2/K_{14})\sin^2(x_1)/x_5^2 \\ x_4 + c_2 \\ K_{11}x_2x_3 - K_{12}x_2x_4 + K_{13}\cos(x_1) \end{bmatrix}.$$

The system equations in this new coordinate system have the form

$$\frac{d}{dt}x^* = f^*(x)_{|x=T^{-t}(x^*)} + g_1^*(x)_{|x=T^{-t}(x^*)}u_1 + g_2^*(x)_{|x=T^{-t}(x^*)}u_2.$$

Compute $f^*(x), g_1^*(x)$, and $g_2^*(x)$.

(b) Show that the nonlinear feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{g_1^*}(T_3) & \mathcal{L}_{g_2^*}(T_3) \\ \mathcal{L}_{g_1^*}(T_5) & \mathcal{L}_{g_2^*}(T_5) \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathcal{L}_{f^*}(T_3) \\ \mathcal{L}_{f^*}(T_5) \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right)$$

results in the linear model

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w_2.$$

(c) What condition is needed for the feedback in part (b) to be valid. When does the condition hold?

Problem 4 *Observer for a Predator-Prey Model* [13]
A nonlinear differential equation model for a predator-prey system is given by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} \alpha x_1 - \beta x_1 x_2 \\ \gamma x_1 x_2 - \lambda x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ -x_2 \end{bmatrix}}_{g(x)} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

$x_1 \geq 0$ is the prey population and $x_2 \geq 0$ is the predator population. The constants, $\alpha > 0$ and $\gamma > 0$ are the birth rates of prey and predator populations, respectively while the constants $\beta > 0$ and $\lambda > 0$ are the death rates of the prey and predator populations, respectively. The input $u \geq 0$ represents the rate at which humans can decimate the predator population (e.g., by hunting). The output y is the predator population while the prey population is considered too big to measure. Consider the nonlinear transformation

$$\begin{aligned} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} &= \begin{bmatrix} T_1(x_1, x_2) \\ T_2(x_1, x_2) \end{bmatrix} \triangleq \begin{bmatrix} \gamma x_1 + \beta x_2 - \alpha \ln(x_2) + c_1 \\ \ln(x_2) + c_2 \end{bmatrix} \\ y^* &\triangleq \ln(y) = \ln(x_2) \end{aligned}$$

where c_1, c_2 can be any arbitrary constants.

(a) Find the system equation in the x^* coordinates with $c_1 = c_2 = 0$.

(b) Design an observer for x_1 with linear error dynamics that places the poles of the error system at $-2, -2$.

(c) For any given reference input u_0 find all equilibrium points x_0, that is, the solutions to

$$0 = f(x_0) + g(x_0)u_0.$$

Explain why the only physically interesting equilibrium point is

$$\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} (\lambda + u_0)/\gamma \\ \alpha/\beta \end{bmatrix}.$$

(d) With

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \triangleq \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix}, \quad w \triangleq u - u_0, \quad \text{and} \quad \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} (\lambda + u_0)/\gamma \\ \alpha/\beta \end{bmatrix}$$

find the system equations in terms of z_1, z_2, and w. That is, show the equations can be written in the form

$$\frac{dz}{dt} = f'(z) + g'(z)w.$$

Explicitly give $f'(z)$ and $g'(z)$.

(e) With

$$\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} \triangleq \begin{bmatrix} z_1 \\ \mathcal{L}_{f'}(z_1) \end{bmatrix}$$

find the statespace representation in the z^* coordinates. Choose feedback of the form $w = f^*(z) + g^*(z)u$ so that the system dynamics in z^* are linear given by

$$\frac{d}{dt}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w.$$

With the state feedback

$$w = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix}$$

find the values of the gains k_1, k_2 such that the closed-loop poles of the z^* system are $-1, -1$.

(f) Draw a block diagram illustrating the interconnection of the observer, controller, and predator-prey model.

Problem 5 *Linear System Not in Observer Canonical Form*

Let

$$\begin{aligned} \frac{dx}{dt} &= Ax + bu, \quad A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3 \\ y &= cx, \quad c \in \mathbb{R}^{1\times 3} \end{aligned}$$

with $\det(sI - A) = s^3 + a_2 s^2 + a_1 s + a_0$ the characteristic polynomial of A and the observability matrix given by

$$\mathcal{O} \triangleq \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix}.$$

Show that if this system is observable, i.e., $\mathcal{O}$ has full rank, then it can be transformed to observer canonical form.

Hint. Define q as the vector that satisfies

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathcal{O}q.$$

That is, q is the third column of $\mathcal{O}^{-1}$. Show that the transformation $x^* = Tx$ with $T \triangleq \begin{bmatrix} q & Aq & A^2q \end{bmatrix}^{-1}$ takes the system to observer canonical form. You need to show

$$TAT^{-1} = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}, cT^{-1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

or

$$\begin{aligned} A\begin{bmatrix} q & Aq & A^2q \end{bmatrix} &= \begin{bmatrix} q & Aq & A^2q \end{bmatrix}\begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \\ c\begin{bmatrix} q & Aq & A^2q \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Problem 6 *PM Synchronous Motor with a Salient Rotor* [5]

Some PM synchronous machines have rotors constructed so that the air gap is not uniform and are referred to as a *salient* rotors. In this case the flux linkages (total flux) in the stator phases due to the stator currents depends on the rotor position. Specifically, the stator flux linkages are given by

$$\begin{aligned} \lambda_{Sa} &= L_S i_{Sa} + K_m \cos(n_p\theta) + L_g\left(i_{Sa}\cos(2n_p\theta) + i_{Sb}\sin(2n_p\theta)\right) \\ \lambda_{Sb} &= L_S i_{Sb} + K_m \sin(n_p\theta) + L_g\left(i_{Sa}\sin(2n_p\theta) - i_{Sb}\cos(2n_p\theta)\right) \end{aligned}$$

where the third term in each expression is due to the nonuniform air gap (salient rotor). Notice that these terms have a period of π/n_p while the flux linkage due to the permanent magnet rotor has a period of $2\pi/n_p$.

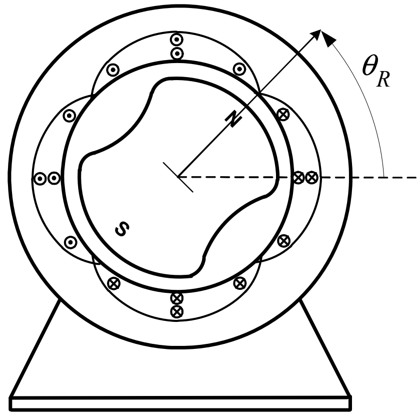

FIGURE 1.11. PM sychronous motor with a salient rotor ($n_p = 1$).

By Faraday's law we have

$$\begin{aligned}\frac{d\lambda_{Sa}}{dt}+Ri_{Sa} &= u_{Sa}\\ \frac{d\lambda_{Sb}}{dt}+Ri_{Sb} &= u_{Sb}.\end{aligned}$$

It follows from these equations that the electrical power $i_{Sa}u_{Sa}+i_{Sb}u_{Sb}$ into the motor is equal to

$$i_{Sa}u_{Sa}+i_{Sb}u_{Sb}=i_{Sa}\frac{d\lambda_{Sa}}{dt}+i_{Sb}\frac{d\lambda_{Sb}}{dt}+R_S(i_{Sa}^2+i_{Sb}^2).$$

First compute

$$\begin{aligned}\lambda_{Sa} &= L_S i_{Sa}+K_m\cos(n_p\theta)+L_g\left(i_{Sa}\cos(2n_p\theta)+i_{Sb}\sin(2n_p\theta)\right)\\ \lambda_{Sb} &= L_S i_{Sb}+K_m\sin(n_p\theta)+L_g\left(i_{Sa}\sin(2n_p\theta)-i_{Sb}\cos(2n_p\theta)\right)\end{aligned}$$

$$\begin{aligned}&i_{Sa}\frac{d\lambda_{Sa}}{dt}+i_{Sb}\frac{d\lambda_{Sb}}{dt}\\ =\;&\frac{d}{dt}\frac{1}{2}L_S(i_{Sa}^2+i_{Sb}^2)+i_{Sa}K_m\frac{d}{dt}\cos(n_p\theta)+i_{Sb}K_m\frac{d}{dt}\sin(n_p\theta)+\\ &i_{Sa}L_g\left(\frac{di_{Sa}}{dt}\cos(2n_p\theta)+\frac{di_{Sb}}{dt}\sin(2n_p\theta)\right)+i_{Sb}L_g\left(\frac{di_{Sa}}{dt}\sin(2n_p\theta)-\frac{di_{Sb}}{dt}\cos(2n_p\theta)\right)+\\ &i_{Sa}\left(L_g\left(i_{Sa}\frac{d}{dt}\cos(2n_p\theta)+\;i_{Sb}\frac{d}{dt}\sin(2n_p\theta)\right)\right)+i_{Sb}\left(L_g\left(i_{Sa}\frac{d}{dt}\sin(2n_p\theta)\;-i_{Sb}\frac{d}{dt}\cos(2n_p\theta)\right)\right)\end{aligned}$$

Continuing this becomes

$$\begin{aligned}&i_{Sa}\frac{d\lambda_{Sa}}{dt}+i_{Sb}\frac{d\lambda_{Sb}}{dt}\\ =\;&\frac{d}{dt}\frac{1}{2}L_S(i_{Sa}^2+i_{Sb}^2)-i_{Sa}K_m n_p\omega\sin(n_p\theta)+i_{Sb}K_m n_p\omega\cos(n_p\theta)\\ &i_{Sa}L_g\left(\frac{di_{Sa}}{dt}\cos(2n_p\theta)+\frac{di_{Sb}}{dt}\sin(2n_p\theta)\right)+i_{Sb}L_g\left(\frac{di_{Sa}}{dt}\sin(2n_p\theta)-\frac{di_{Sb}}{dt}\cos(2n_p\theta)\right)\\ &-L_g i_{Sa}^2 2n_p\omega\sin(2n_p\theta)+L_g i_{Sa}i_{Sb}2n_p\omega\cos(2n_p\theta)+L_g i_{Sb}i_{Sa}2n_p\omega\cos(2n_p\theta)+L_g i_{Sb}^2 2n_p\omega\sin(2n_p\theta)\end{aligned}$$

Combining the above expression the power into the motor may now be written as

$$\begin{aligned}i_{Sa}u_{Sa}+i_{Sb}u_{Sb} &= R_S(i_{Sa}^2+i_{Sb}^2)+i_{Sa}\frac{d\lambda_{Sa}}{dt}+i_{Sb}\frac{d\lambda_{Sb}}{dt}\\ &= R_S(i_{Sa}^2+i_{Sb}^2)+\frac{d}{dt}\left(\frac{1}{2}L_S(i_{Sa}^2+i_{Sb}^2)\right)+\omega\left(n_pK_m(-i_{Sa}\sin(n_p\theta)\;+i_{Sb}\cos(n_p\theta))\right)+\\ &\quad i_{Sa}L_g\left(\frac{di_{Sa}}{dt}\cos(2n_p\theta)+\frac{di_{Sb}}{dt}\sin(2n_p\theta)\right)+i_{Sb}L_g\left(\frac{di_{Sa}}{dt}\sin(2n_p\theta)-\frac{di_{Sb}}{dt}\cos(2n_p\theta)\right)\\ &\quad \omega 2n_pL_g\left((i_{Sb}^2-i_{Sa}^2)\sin(2n_p\theta)+\;2i_{Sa}i_{Sb}\cos(2n_p\theta)\right).\end{aligned}$$

Again, $i_{Sa}u_{Sa}+i_{Sb}u_{Sb}$ is the electrical power supplied to the motor. Part of this power changes the stored magnetic energy according to

$$\frac{d}{dt}\left(\frac{1}{2}L_S(i_{Sa}^2+i_{Sb}^2)\right)+i_{Sa}L_g\left(\frac{di_{Sa}}{dt}\cos(2n_p\theta)+\frac{di_{Sb}}{dt}\sin(2n_p\theta)\right)+i_{Sb}L_g\left(\frac{di_{Sa}}{dt}\sin(2n_p\theta)-\frac{di_{Sb}}{dt}\cos(2n_p\theta)\right)$$

and part of this power goes into Ohmic losses according to

$$R_S(i_{Sa}^2 + i_{Sb}^2).$$

The rest of the power given by

$$\omega\left(n_pK_m(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta))\right) + \omega 2n_pL_g\left((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) + 2i_{Sa}i_{Sb}\cos(2n_p\theta)\right)$$

is the power absorbed by the windings. By conservation of energy, this power reappears as the mechanical power $\tau\omega$. That is, conservation of energy requires

$$\omega\left(n_pK_m(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta))\right) + \omega 2n_pL_g\left((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) + 2i_{Sa}i_{Sb}\cos(2n_p\theta)\right) = \tau\omega$$

or

$$\tau = n_pK_m\left(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)\right) + 2n_pL_g\left((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) + 2i_{Sa}i_{Sb}\cos(2n_p\theta)\right).$$

The complete set of equations for this PM salient machine are then

$$\begin{aligned}
\frac{d\lambda_{Sa}}{dt} + Ri_{Sa} &= u_{Sa}\\
\frac{d\lambda_{Sb}}{dt} + Ri_{Sb} &= u_{Sb}\\
J\frac{d\omega}{dt} &= n_pK_m\left(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)\right) +\\
&\quad 2n_pL_g\left((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) + 2i_{Sa}i_{Sb}\cos(2n_p\theta)\right) - \tau_L
\end{aligned}$$

with

$$\begin{aligned}
\lambda_{Sa} &= L_Si_{Sa} + K_m\cos(n_p\theta) + L_g\left(i_{Sa}\cos(2n_p\theta) + i_{Sb}\sin(2n_p\theta)\right)\\
\lambda_{Sb} &= L_Si_{Sb} + K_m\sin(n_p\theta) + L_g\left(i_{Sa}\sin(2n_p\theta) - i_{Sb}\cos(2n_p\theta)\right).
\end{aligned}$$

Transforming these equations in the dq frame greatly simplifies them from which a feedback controller can be designed.

(a) *Fluxes in the dq coordinate system.*

With

$$\begin{bmatrix}\lambda_d\\ \lambda_q\end{bmatrix} \triangleq \begin{bmatrix}\cos(n_p\theta) & \sin(n_p\theta)\\ -\sin(n_p\theta) & \cos(n_p\theta)\end{bmatrix}\begin{bmatrix}\lambda_{Sa}\\ \lambda_{Sb}\end{bmatrix}, \begin{bmatrix}i_d\\ i_q\end{bmatrix} \triangleq \begin{bmatrix}\cos(n_p\theta) & \sin(n_p\theta)\\ -\sin(n_p\theta) & \cos(n_p\theta)\end{bmatrix}\begin{bmatrix}i_{Sa}\\ i_{Sb}\end{bmatrix}$$

show that

$$\begin{bmatrix}\lambda_{Sa}\\ \lambda_{Sb}\end{bmatrix} = \begin{bmatrix}L_S & 0\\ 0 & L_S\end{bmatrix}\begin{bmatrix}i_{Sa}\\ i_{Sb}\end{bmatrix} + K_m\begin{bmatrix}\cos(n_p\theta)\\ \sin(n_p\theta)\end{bmatrix} + L_g\begin{bmatrix}\cos(2n_p\theta) & \sin(2n_p\theta)\\ \sin(2n_p\theta) & -\cos(2n_p\theta)\end{bmatrix}\begin{bmatrix}i_{Sa}\\ i_{Sb}\end{bmatrix}$$

are represented in the dq frame by

$$\begin{bmatrix}\lambda_d\\ \lambda_q\end{bmatrix} = \begin{bmatrix}L_S & 0\\ 0 & L_S\end{bmatrix}\begin{bmatrix}i_d\\ i_q\end{bmatrix} + \begin{bmatrix}K_m\\ 0\end{bmatrix} + L_g\begin{bmatrix}i_d\\ -i_q\end{bmatrix} = \begin{bmatrix}L_di_d + K_m\\ L_qi_q\end{bmatrix}$$

where $L_d \triangleq L_S + L_g, L_q \triangleq L_S - L_g$.

(b) *Flux equations in the dq coordinate system*

Show that the flux equations

$$\begin{aligned}\frac{d\lambda_{Sa}}{dt} + Ri_{Sa} &= u_{Sa}\\ \frac{d\lambda_{Sb}}{dt} + Ri_{Sb} &= u_{Sb}\end{aligned}$$

in the dq system are given by

$$\frac{d}{dt}\begin{bmatrix} L_d i_d \\ L_q i_q \end{bmatrix} - n_p\omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} L_d i_d + K_m \\ L_q i_q \end{bmatrix} = -R\begin{bmatrix} i_d \\ i_q \end{bmatrix} + \begin{bmatrix} u_d \\ u_q \end{bmatrix}$$

where

$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix}\begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix}.$$

(c) *Torque Equation*

Show that the motor torque

$$\tau = n_p K_m\Big(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)\Big) + 2n_p L_g\Big((i_{Sb}^2 - i_{Sa}^2)\sin(2n_p\theta) + 2i_{Sa}i_{Sb}\cos(2n_p\theta)\Big)$$

is given in the dq system by

$$\tau = 2n_p(L_d - L_q)i_d i_q + n_p K_m i_q.$$

(d) Equations of the salient PM synchronous machine in the dq system.

Show that the equations of the salient PM synchronous motor are given by

$$\begin{aligned}L_d\frac{di_d}{dt} &= -Ri_d + n_p\omega L_q i_q + u_d\\ L_q\frac{di_q}{dt} &= -Ri_q - n_p\omega L_d i_d - n_p\omega K_m + u_q\\ J\frac{d\omega}{dt} &= n_p\left(K_m i_q + 2(L_d - L_q)i_q i_d\right)\\ \frac{d\theta}{dt} &= \omega\end{aligned}$$

where

$$\begin{aligned}L_d &\triangleq L_S + L_g\\ L_q &\triangleq L_S - L_g.\end{aligned}$$

(e) Consider the change of coordinates

$$\begin{aligned}\theta &= \theta\\ \omega &= \omega\\ \alpha &= \frac{n_p}{J}\left(K_m i_q + 2(L_d - L_q)i_q i_d\right)\\ i_d &= i_d.\end{aligned}$$

Show that the motor model is then given by

$$\begin{aligned}\frac{d\theta}{dt} &= \omega\\ \frac{d\omega}{dt} &= \alpha\\ \frac{d\alpha}{dt} &= f_1(\omega, i_d, i_q) + 2L_g n_p u_d/L_d + n_p\left(K_m + 2L_g i_d\right)u_q/L_q\\ \frac{di_d}{dt} &= f_2(\omega, i_d, i_q) + u_d/L_d.\end{aligned}$$

Give the explicit expressions for f_1 and f_2. Show that the nonlinear feedback

$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} = \begin{bmatrix} \dfrac{2L_g n_p}{L_d} & \dfrac{n_p\left(K_m + 2L_g i_d\right)}{L_q} \\ \dfrac{n_p\left(K_m + 2L_g i_d\right)}{L_q} & 0 \end{bmatrix}^{-1} \begin{bmatrix} -f_1 + v_d \\ -f_2 + v_q \end{bmatrix}$$

results in a *linear* system. Under what conditions is the nonlinear feedback of part (e) singular? Is it a practical problem? Explain.

Problem 7 *Series Connected DC Motor*
With $x_1 = \theta, x_2 = \omega$, $x_3 = i$, and $u = V_S/L$ the equations describing the series connected DC motor are

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ c_1 x_3^2 \\ c_2 x_3 - c_3 x_3 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ -1/J \\ 0 \end{bmatrix}}_{p} \tau_L.$$

Using the transformation

$$\begin{aligned} x_1^* &= T_1(x) = x_1 \\ x_2^* &= T_2(x) = 2c_2 x_2 + c_3 x_2^2 + c_1 x_3^2 \\ x_3^* &= T_3(x) = x_2 \end{aligned}$$

give the equations of this system in the x^* coordinate system.
If $x_1 = \theta$ and $x_2 = \omega$ are measured, can an observer for $x_3 = i$ be constructed using the equations in the x^* coordinate system? If so, do so.

Problem 8 *Magnetic Levitation*
Recall the equations of the current command magnetic levitation system given by

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= g - \frac{C}{m}\frac{u^2}{x_1^2}. \end{aligned}$$

Note that this nonlinear system is not of the form $dx/dt = f(x) + g(x)u$ as it is not linear in the input u.

(a) Show that this system can be made linear by the appropriate choice of u. Design a state feedback controller to keep the steel ball at $x_1 = x_0$.

(b) Given that the position $x = x_1$ and coil current $u = i$ are measured, design an observer to estimate the velocity $v = x_2$.

Problem 9 *An Observer for Speed, Current, and Load Torque of a DC Motor*
Recall the model of the DC motor given as

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega + V_S \\ J\frac{d\omega}{dt} &= K_T i - f\omega - \tau_L \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

With $x_1 = i, x_2 = \omega, x_3 = \theta$, and $u = V_S$ we may write

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 \\ K_T/J & -f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{b} u + \begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix}\tau_L$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where the rotor angle is taken as the output, i.e., it is measured. The load torque affects the speed and so it must be included in the observer. With the load torque taken to be constant and setting $x_4 = \tau_L/J$ the system is now modeled by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -R/L & -K_b/L & 0 & 0 \\ K_T/J & -f/J & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_a}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_a} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}}_{c_a}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Design an observer that estimates $x_1 = i, x_2 = \omega$, and $x_4 = \tau/J$ based on the measurement $y = \theta$.

Problem 10 *Shunt Connected DC Motor*

A shunt connected DC motor has the field circuit and the armature circuit connected in parallel as illustrated in Figure 1.12. By connected in parallel is meant that the T_1 terminal of the armature is connected to the T_1' terminal of the field circuit and similarly for the T_2 and T_2'.

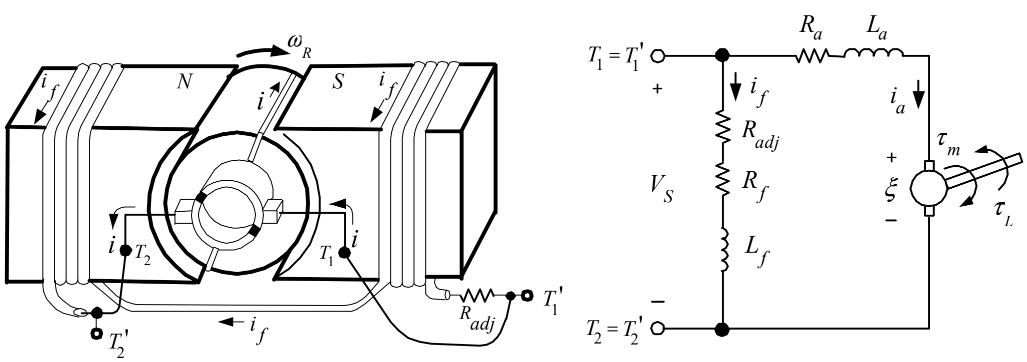

FIGURE 1.12. Shunt connected DC motor.

The equations describing the shunt motor are [3]

$$\begin{aligned} J\frac{d\omega}{dt} &= K_T L_f i_f i_a - \tau_L \\ L_a\frac{di_a}{dt} &= -R_a i_a - K_b L_f i_f \omega + V_S \\ L_f\frac{di_f}{dt} &= -(R_{adj} + R_f) i_f + V_S. \end{aligned}$$

Here ω is the rotor angular speed, V_S is the terminal (source) voltage, i_a is the armature current, i_f is the field current, τ_L is the load torque, K_T is the torque constant, and K_b is the back-emf constant. The armature resistance and armature inductance are denoted by R_a and L_a,respectively, and the field resistance and field inductance are R_f and L_f, respectively. R_{adj} is an adjustable resistor so that the total field resistance $R_{adj} + R_f$ can be varied.

Let $x_1 = \omega, x_2 = i_a, x_3 = i_f, u = V_S$, and define the constants $c_1 \triangleq \dfrac{K_T L_f}{J}, c_2 = \dfrac{R_a}{L_a}, c_3 = \dfrac{K_b L_f}{L_a}, c_4 = \dfrac{1}{L_a}, c_5 = \dfrac{R_{adj} + R_f}{L_f}, c_6 = \dfrac{1}{L_f}$ so that the statespace model becomes

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 x_2 x_3 \\ -c_2 x_2 - c_3 x_1 x_3 \\ -c_f x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ c_4 \\ c_6 \end{bmatrix} u + \begin{bmatrix} -1/J \\ 0 \\ 0 \end{bmatrix} \tau_L.$$

The constant load torque is not known. Assuming that $x_1 = \omega, x_2 = i_a$, and $x_3 = i_f$ are measured this problem shows how to design an observer to estimate τ_L/J.

(a) Let $x_4 \triangleq \tau_L/J$ with $dx_4/dt = 0$ and suppose $x_1 = \omega, x_2 = i_a$, and $x_3 = i_f$ are all measured. Show the model of the shunt connected DC motor is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} c_1 y_2 y_3 \\ -c_2 y_2 - c_3 y_1 y_3 \\ -c_f y_3 \\ 0 \end{bmatrix}}_{\varphi(y)} + \underbrace{\begin{bmatrix} 0 \\ c_4 \\ c_6 \\ 0 \end{bmatrix}}_{b} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(b) Is the pair (C, A) observable? Explain.

(c) Let

$$T_o \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, T_o^{-1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and define the linear transformation $x^* = T_o x$. Transform the model given in part (a) into the x^* coordinate system.

(d) With $A_o \triangleq T_o A T_o^{-1}$ and $C_o = C T_o^{-1}$ let

$$L_o = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \\ \ell_{41} & \ell_{42} & \ell_{43} \end{bmatrix}$$

give the equations for an observer to estimate the state x^*.

(e) Give the equations for the estimate error $x^* - \hat{x}^*$ and show that the components of L_o can be chosen so that poles of the estimation error system can be put at $-r_1, -r_2, -r_3, -r_4$.

Problem 11 *Speed and Load Torque Estimation for the PM Synchronous Motor*
The differential equation model of PM synchronous motor is

$$\begin{aligned}
L_S\frac{di_{Sa}}{dt} &= -R_S i_{Sa} + K_m \sin(n_p\theta)\omega + u_{Sa} \\
L_S\frac{di_{Sb}}{dt} &= -R_S i_{Sb} - K_m \cos(n_p\theta)\omega + u_{Sb} \\
J\frac{d\omega}{dt} &= K_m(-i_{Sa}\sin(n_p\theta) + i_{Sb}\cos(n_p\theta)) - \tau_L \\
\frac{d\theta}{dt} &= \omega.
\end{aligned}$$

With currents and voltages transformed by *direct-quadrature* (dq) transformation

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} i_{Sa} \\ i_{Sb} \end{bmatrix}, \quad \begin{bmatrix} u_d \\ u_q \end{bmatrix} \triangleq \begin{bmatrix} \cos(n_p\theta) & \sin(n_p\theta) \\ -\sin(n_p\theta) & \cos(n_p\theta) \end{bmatrix} \begin{bmatrix} u_{Sa} \\ u_{Sb} \end{bmatrix}$$

the model is given by

$$\begin{aligned}
L_S\frac{di_d}{dt} &= -R_S i_d + n_p\omega L_S i_q + u_d \\
L_S\frac{di_q}{dt} &= -R_S i_q - n_p\omega L_S i_d - K_m\omega + u_q \\
J\frac{d\omega}{dt} &= K_m i_q - \tau_L \\
\frac{d\theta}{dt} &= \omega.
\end{aligned}$$

With the currents i_{Sa}, i_{Sb} measured along with θ design an observer for ω and τ_L/J.

1.11 References

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