Nonlinear Systems

Morse Theory and Lyapunov Stability on Manifolds

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Outline

Introduction

Lyapunov Stability – Introduction

- Introduced by Alexandr Mikhailovich Lyapunov.
- *The general problem of the stability of motion*, 1892.
- Doctoral thesis in Kharkov Mathematical Society.
- The most general theory for analyzing stability of (at least) ordinary differential equations.

Lyapunov Stability – Introduction

- Different notions of stability: input-output stability, periodic orbit stability, etc.
- Stability of equilibrium points usually characterized in the sense of Lyapunov.
  - An equilibrium point is STABLE if all solutions starting at nearby points stay nearby.
  - It is ASYMPTOTICALLY STABLE if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.
- For a linear system $\dot{x} = Ax$, the stability of $x = 0$ can be completely characterized by the eigenvalues of A.
- Stability of a nonlinear system sometimes can be characterized by the same method (through linearization).
- Lyapunov stability theorems give sufficient conditions for stability.

Notations and Definitions

Manifolds and Vector Fields

- $\mathcal{M}$ (state-space) denotes a manifold of finite dimension n.
- $f \in \mathfrak{X}(M)$ is a continuous vector field on $\mathcal{M}$.
- We assume that there exists a unique right maximally defined integral curve of f starting at x.
- We also assume that this integral curve is defined on $[0, \infty]$.

$$\varphi : [0, \infty] \times \mathcal{M} \to \mathcal{M}$$

  with

$$\begin{aligned} \varphi(0, x) &= x, \\ \varphi(t_1, \varphi(t_2, x)) &= \varphi(t_1 + t_2, x). \end{aligned}$$

- The semiflow φ is the evolution function.

Definition

$\Omega \subseteq \mathcal{M}$ is called an INVARIANT SET if for all $x \in \Omega$ and $t \in \mathbb{R}_{\geq 0}$, $\varphi(t, x) \in \Omega$. If $\Omega = \{p\}$ is a singleton, then Ω is called and EQUILIBRIUM POINT of the dynamical system $(\mathcal{M}, \varphi)$.

Definition

$\Omega \subseteq \mathcal{M}$ is STABLE if for *every open neighborhood* $\mathcal{U} \subseteq \mathcal{M}$ of Ω, *there exists a neighborhood* $\mathcal{V} \subseteq \mathcal{M}$ of Ω such that $\varphi(t, \mathcal{V}) \subseteq \mathcal{U}$ for all $t \geq 0$.

An *invariant set* Ω is asymptotically stable if

- Ω is stable,
- Ω is attractive, i.e., for all $x \in \Omega$, there exists an open neighborhood $\mathcal{N} \subseteq \mathcal{M}$ of Ω such that for all $x \in \mathcal{N}$, $\varphi(t, x) \xrightarrow{t \to \infty} \Omega$.

The domain of attraction is denoted by

$$\mathcal{A} = \{x \in \mathcal{M} : \varphi(t, x) \to \Omega \text{ as } t \to \infty\}.$$

Ω is said to be GLOBALLY asymptotically stable if $\mathcal{N} = \mathcal{M}$.

Definition

The LIE DERIVATIVE of $V : \mathcal{M} \to \mathbb{R}$ along $f \in \mathfrak{X}(\mathcal{M})$ is defined by

$$\begin{aligned} \mathcal{L}_f V : \mathcal{M} &\to \mathbb{R}, \\ p &\mapsto \mathrm{d}V_p(f(p)). \end{aligned}$$

Definition

Let $\mathcal{K}$ be an invariant set of the dynamical system $(\mathcal{M}, \varphi)$. A continuous function $V : \mathcal{A} \to \mathbb{R}_{\geq 0}$ is a LYAPUNOV FUNCTION if

- $V(x) > 0$ for all $x \in \mathcal{A} \backslash \mathcal{K}$,
- $V(x) = 0$ for all $x \in \mathcal{K}$,
- V is proper, i.e., $V^{-1}(B)$ is compact for all compact subset $B \subseteq \mathbb{R}_{\geq 0}$,
- V is strictly decreasing along orbits of φ, i.e.,

$$V \circ \varphi(t, x) < V(x),$$

  for all $t > 0$ and $x \in \mathcal{A} \backslash \mathcal{K}$.
  If V is differentiable, this condition may be replaced by

$$\mathcal{L}_f V(x) < 0.$$

(Nondegenerate) Critical Points

Definition

Let $V : \mathcal{M} \to \mathbb{R}$ be a smooth function. A CRITICAL POINT, $p \in \mathcal{M}$, of V is a point where the differential

$$\mathrm{d}V_p : T_p\mathcal{M} \to \mathbb{R}$$

has rank zero, i.e., in any local coordinate system $\{x_i\}_1^n$, one has $\frac{\partial V}{\partial x_i}(p) = 0$ for all $i = 1, \ldots, n$.

Definition

A critical point p is NONDEGENERATE if the Hessian $H_p(V)$ is a nondegenerate bilinear form, i.e., if any coordinate system, the Hessian matrix

$$\left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{1 \leq i,j \leq n}$$

is nondegenerate.

Nondegenerate Critical Points

Definition

The dimension of the subspace of $T_p\mathcal{M}$ on which $H_p(V)$ is negative definite is called the MORSE INDEX of V at p, denoted by $\mathrm{ind}(V, p)$.

Definition

A C^2 function $V : \mathcal{M} \to \mathbb{R}$ is a MORSE FUNCTION if all its critical points are nondegenerate.

Definition

The (SUB)-LEVEL SETS of a function $V : \mathcal{M} \to \mathbb{R}$ are

$$\begin{aligned} \mathcal{M}_a &= V^{-1}\left((-\infty, a]\right), \\ \mathcal{M}_{a,b} &= V^{-1}\left([a, b]\right). \end{aligned}$$

Topological Definitions

- A top. space is an n-CELL if it is homeomorphic to $\mathbb{R}^n$.
- A top. space X is CONTRACTIBLE if it is *homotopy equivalent* to the one-point space.
- A subspace A of X is called a DEFORMATION RETRACT of X if there exists a continuous function $h : [0,1] \times X \to X$ such that for all $x \in X$, $a \in A$,

$$h(0,x) = x,$$
$$h(1,x) \in A,$$
$$h(1,a) = a.$$

- The k^{th} BETTI NUMBER of $\mathcal{M}$, denoted by b_k is the rank of the k^{th} homology group $H^k(\mathcal{M})$.
- The EULER CHARACTERISTIC of $\mathcal{M}$ is defined by

$$\chi(\mathcal{M}) = \sum_{i=1}^{k} (-1)^k b_k.$$

Lyapunov Stability Analysis on Euclidean Spaces

Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{1}$$

where $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz map, with an equilibrium point at $x = 0$.

Definition

The equilibrium point $x = 0$ of the system (1) is

- *stable* if, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that

$$\|x(0)\| < \delta \;\Rightarrow\; \|x(t)\| < \epsilon, \quad \forall t \geq 0.$$

- *unstable* if it is not stable.
- *asymptotically stable* if it is stable and δ can be chosen s.t.

$$\|x(0)\| < \delta \;\Rightarrow\; \lim_{t\to\infty} x(t) = 0.$$

Example – Pendulum

The pendulum equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - b x_2$$

has two equilibrium points at $(x_1 = 0, x_2 = 0)$ and $(x_1 = \pi, x_2 = 0)$.

- If $b = 0$, trajectories in the nbhd. of the first equilibrium are closed orbits.
- By starting sufficiently close to the eq. point, trajectories are guaranteed to stay within any specified ball.
- The point is not asymptotically stable since trajectories don't tend to the eq. point.
- If $b > 0$, the origin becomes asymptotically stable.
- The second eq. point is a saddle point: the $\varepsilon - \delta$ requirement cannot be satisfied (for every $\varepsilon > 0$ there exists a trajectory that will leave the ball B_ε even if $x(0)$ is arbitrarily close to $(\pi, 0)$).

Lyapunov Stability Theorem

Theorem

Let $x = 0 \in D$ be an equilibrium point for (1). Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \;\; and \;\; V(x) > 0 \; in \; D - \{0\},$$
$$\dot{V}(x) \leq 0 \; in \; D.$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \; in \; D - \{0\}$$

then $x = 0$ is asymptotically stable.

Proof of stability.

Given $\varepsilon > 0$, choose $0 < r \leq \varepsilon$ such that $B_r \subseteq D$. Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$. Take $0 < \beta < \alpha$ and consider $\mathcal{M}_\beta = V^{-1}((0, \beta])$.

<u>Claim</u>: $\mathcal{M}_\beta \subseteq \mathring{B}_r$. Argue ad absurdum. Suppose $\mathcal{M}_\beta \cap \mathring{B}_r \neq \mathcal{M}_\beta$. Then $\exists p \in \mathcal{M}_\beta \cap \partial B_r$. Note, $V(p) \geq \alpha > \beta$, but $V(\mathcal{M}_\beta) \subseteq [0, \beta]$.

The set $\mathcal{M}_\beta$ is invariant since

$$\dot{V}(x(t)) \leq 0 \;\Rightarrow\; V(x(t)) \leq V(x(0)) \leq \beta, \; \forall t \geq 0.$$

Because $\mathcal{M}_\beta$ is compact (closed and bounded), we conclude that the ODE (1) has a unique solution $\forall t \geq 0$ whenever $x(0) \in \mathcal{M}_\beta$. Since V is continuous and $V(0) = 0$, $\exists \delta > 0$ such that

$$\|x\| \leq \delta \;\Rightarrow\; V(x) < \beta.$$

$\square$

Proof of stability (cont'd).

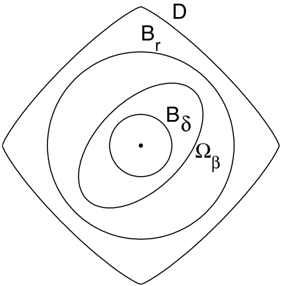
Then,

$$B_\delta \subseteq \mathcal{M}_\beta \subseteq B_r$$

and

$$x(0) \in B_\delta \ \Rightarrow\ x(0) \in \mathcal{M}_\beta \ \Rightarrow\ x(t) \in \mathcal{M}_\beta \ \Rightarrow\ x(t) \in B_r,$$

proving stability. □



Proof of asymptotic stability.

Now assume $\dot{V}(x) < 0$ in $D - \{0\}$. We want to show that $x(t) \xrightarrow{t\to\infty} 0$; i.e., $\forall a > 0$, $\exists T > 0$, s.t. $\|x(t)\| < a, \forall t > T$.

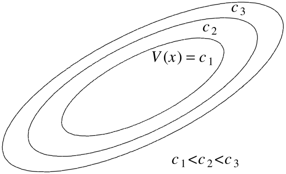
We know that $\forall a > 0$, we can choose $b > 0$ s.t. $\mathcal{M}_b \subseteq B_a$. Therefore, it is sufficient to show that $V(x(t)) \xrightarrow{t\to\infty} 0$. Since V is monotonically decreasing and bounded from below by zero,

$$V(x(t)) \xrightarrow{t\to\infty} c \geq 0.$$

<u>Claim</u>: $c = 0$. Argue ad absurdum. Suppose $c > 0$. By continuity of V, $\exists d > 0$ s.t. $B_d \subseteq \mathcal{M}_c$. The limit $V(x(t)) \to c > 0$ implies that $x(t) \notin B_d, \forall t \geq 0$. Define $\max_{d \leq \|x\| \leq r} \dot{V}(x) =: -\gamma < 0$. It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, \mathrm{d}\tau \leq V(x(0)) - \gamma t.$$

The RHS will eventually become negative: contradiction ($c > 0$). $\square$



- A continuously differentiable function V, satisfying the theorem's conditions is called a LYAPUNOV FUNCTION.
- When $\dot{V} < 0$, the trajectory moves from level set $\mathcal{M}_{c_3} = V^{-1}(c_3)$ to an inner level set $M_{c_2} = V^{-1}(c_2)$ with a smaller c.
- $V^{-1}(c) \xrightarrow{c \downarrow 0} 0.$ Hence the trajectory approaches the origin.
- If we only knew that $\dot{V} \leq 0$, we cannot be sure that the trajectory $x(t) \xrightarrow{t \to \infty} 0$,[1]but we can conclude that the origin is stable.

[1]See, however, Krasovskii-LaSalle's theorem.

Example: Undamped pendulum

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -a \sin x_1.$$

<u>Lyapunov function candidate</u>

$$V(x) = a(1 - \cos x_1) + \tfrac{1}{2}x_2^2.$$

Analysis

Clearly, $V(0) = 0$ and $V(x) > 0$ if $x \neq (2k\pi, 0)$. Compute the Lie derivative of V along f:

$$\dot{V}(x) = \mathcal{L}_f V(x) = a x_2 \sin x_1 - a x_2 \sin x_1 = 0.$$

Thus, the origin is stable. Since $\dot{V}(x) \equiv 0$, we conclude that the origin is not asymptotically stable as solutions starting on the level set $\mathcal{M}_c$ remain in that set.

Example: Damped pendulum

Region of Attraction

Chetaev's Instability Theorem

Theorem

Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Let

$$U := \{x \in B_r : V(x) > 0\}$$

and suppose that $\dot{V}(U) > 0$. Then, $x = 0$ is unstable.

Proof.

$x_0 \in \mathring{U}$ and $V(x_0) = a > 0$. The trajectory $x(t)$ starting at $x(0) = x_0$ must leave U. Indeed, as long as $x(t) \in U$, $V(x(t)) \geq a$, since $\dot{V}(U) > 0$. Let $\min\{\dot{V}(x) : x \in U \text{ and } V(x) \geq a\} := \gamma > 0$. Then,

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))\,\mathrm{d}s \geq a + \int_0^t \gamma\,\mathrm{d}s = a + \gamma t.$$

Hence, $x(t)$ will leave U because $V(x)$ is bounded on U. Now, $x(t)$ cannot leave U through $V(x) = 0$ since $V(x(t)) \geq a$. Hence it must leave U through the sphere $\mathbb{S}_r$. Note: $\|x_0\|$ was arbitrarily small. $\square$

Morse-Lyapunov Functions

Isolated Critical Points

Lemma

Suppose that x_e is an equilibrium points of the dynamical system (M, φ). If $V : \mathcal{M} \to \mathbb{R}$ is a differentiable Lyapunov function then x_e is the only critical point of V.

Proof.

Suppose V has another critical point, x_c, in the domain of attraction. By the definition of a Lyapunov function, we must have $\mathcal{L}_f V(x_c) = 0$. This contradicts the fact that if $x \neq x_e$, $\mathcal{L}_f V(x) < 0$. $\square$

Morse Lemma

Theorem (Morse Lemma)

Let $p \in \mathcal{M}$ be a nondegenerate critical point of a smooth function $V : \mathcal{M} \to \mathbb{R}$. There exists a local coordinate system $\{x_i\}_1^n$ in a nbhd. $\mathcal{N} \subseteq \mathcal{M}$ of p with $x_i(p) = 0$ for all $1 \leq i \leq n$ such that for $x \in \mathcal{N}$,

$$V(x) = V(p) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_n^2$$

where $i = ind(V, p)$.

Corollary

Let $p \in \mathcal{M}$ be an equilibrium point of $(\mathcal{M}, \varphi)$ and $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$ a Morse-Lyapunov function. There exists a local coordinate system $\{x_i\}_1^n$ around p such that V is locally the canonical quadratic Lyapunov function

$$V(x) = \sum_{i=1}^{n} x_i^2$$

with $ind(V, p) = 0$.

Theorem (Deformation Lemma)

Let $V : \mathcal{M} \to \mathbb{R}$ be a smooth function and $a, b \in V(\mathcal{M})$ such that $a < b$. If $\mathcal{M}_{a,b}$ is compact and does not contain critical points of V then $\mathcal{M}_a$ is diffeomorphic to $\mathcal{M}_b$. MOreover, $\mathcal{M}_a$ is a deformation retract of $\mathcal{M}_b$.

Corollary

Let $\mathcal{M}$ be a smooth Riemannian manifold. If $\mathcal{M}$ contains a closed invariant asymptotically stable set, then for all $a, b \in V(\mathcal{M})$, $\mathcal{M}_a$ is diffeomorphic to $\mathcal{M}_b$ and $\mathcal{M}_a$ is a deformation retract of $\mathcal{M}_b$ where V is a smooth Lyapunov function.

Systems with Single Critical Points

Domain of Attraction – Revisited

Theorem (Brown-Stallings Lemma)

Let $\mathcal{M}$ be a paracompact manifold such that every compact subset is contained in an open set diffeomorphic to a Euclidean space. Then $\mathcal{M}$ itself is diffeomorphic to a Euclidean space.

Corollary

Let $\mathcal{M}$ be a paracompact manifold. The domain of attraction of an asymptotically stable equilibrium point is diffeomorphic to a Euclidean space.

Morse and Sontag Theorems

Theorem (Morse Theorem)

Let $V : \mathcal{M} \to \mathbb{R}$ be a Morse function, p a critical point such that $\mathrm{ind}(V, p) = i$ and $c = V(p)$. If there exists $\varepsilon > 0$ such that $\mathcal{M}_{c-\varepsilon,c+\varepsilon}$ is compact and does not contain other critical points p, then $\mathcal{M}_{c-\varepsilon} \cup e_i$ is a deformation retract of $\mathcal{M}_{c+\varepsilon}$ where e_i is an i-cell.

Theorem (Sontag Theorem)

Let us consider the dynamical system $(\mathcal{M}, \varphi)$ with an equilibrium point $x_e \in \mathcal{M}$. Suppose that x_e is asymptotically stable. Then the domain of attraction of x_e, given by

$$\mathcal{A} = \left\{ x \in \mathcal{M} : \lim_{t\to\infty} \varphi(t, x) = x_e \right\},$$

is contractible.

Systems with Multiple Critical Points

Theorem (Morse Theorem)

If $V : \mathcal{M} \to \mathbb{R}$ is a Morse function such that $\mathcal{M}_a$ is compact for each $a \in \mathbb{R}$ then $\mathcal{M}$ has the homotopy type of a CW-complex with one i-cell for each critical point of index i.

Corollary

Suppose that the dynamical system $(\mathcal{M}, \varphi)$ has several equilibria $(x_1, \ldots, x_k)$. If there exists a Morse-Lyapunov function $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$ then $\{x_1, \ldots, x_k\}$ is a retract of the domain of attraction.

Proposition (Reeb Theorem)

Suppose that $\mathcal{M}$ is compact without boundary. If $V : \mathcal{M} \to \mathbb{R}$ is a smooth function with only two critical points, then $\mathcal{M}$ is homeomorphic to the n-sphere $\mathbb{S}^n$.

Morse Inequalities

Theorem (Morse Inequalities)

Let m_k be the number of ciritcal points of a Morse function V with index k. Then, we have

$$b_k \leq m_k, \quad \forall k,$$
$$\sum_{i=0}^{j}(-1)^{j-i} b_i \leq \sum_{i=0}^{j}(-1)^{j-i} m_i \quad \forall j,$$
$$\chi(\mathcal{M}) = \sum_{k}(-1)^k b_k = \sum_{k}(-1)^k m_k.$$

The next corollary states a necesary condition for the existence of a Morse-Lyapunov function based on the Euler characteristic, which is a topological invariant.

Existence of Morse-Lyapunov Functions

Corollary

Consider the dynamical system $(\mathcal{M}, \varphi)$ *with several equilibria* $(x_1, \ldots, x_k)$*. If there exists a Morse-Lyapunov function* $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$ *then* $\chi(\mathcal{M}) = k \geq b_0$*.*

Proof.

If there exists a Morse-Lyapunov function V, $(x_1, \ldots, x_k)$ are the only critical points with indices 0. Then, by the Morse inequalities, $\chi(\mathcal{M}) = m_0 = k$ and $b_0 \leq m_0 = k$. □

Remark

If $\chi(\mathcal{M}) \neq k$ *then there is no Morse-Lyapunov function for the dynamical system.*

The Invariance Principle

Intuition