

# Nonlinear Systems

## Lyapunov Stability

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# Introduction

# Lyapunov Stability – Introduction

- ▶ Introduced by Alexandr Mikhailovich Lyapunov.
- ▶ *The general problem of the stability of motion*, 1892.
- ▶ Doctoral thesis in Kharkov Mathematical Society.
- ▶ The most general theory for analyzing stability of (at least) ordinary differential equations.

# Lyapunov Stability – Introduction

- ▶ Different notions of stability: input-output stability, periodic orbit stability, etc.
- ▶ Stability of equilibrium points usually characterized in the sense of Lyapunov.
  - ▶ An equilibrium point is **STABLE** if all solutions starting at nearby points stay nearby.
  - ▶ It is **ASYMPTOTICALLY STABLE** if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.
- ▶ For a linear system  $\dot{x} = Ax$ , the stability of  $x = 0$  can be completely characterized by the eigenvalues of  $A$ .
- ▶ Stability of a nonlinear system sometimes can be characterized by the same method (through linearization).
- ▶ Lyapunov stability theorems give sufficient conditions for stability.

# Notations and Definitions

# Manifolds and Vector Fields

- ▶  $\mathcal{M}$  (state-space) denotes a manifold of finite dimension  $n$ .
- ▶  $f \in \mathfrak{X}(M)$  is a continuous vector field on  $\mathcal{M}$ .
- ▶ We assume that there exists a unique right maximally defined integral curve of  $f$  starting at  $x$ .
- ▶ We also assume that this integral curve is defined on  $[0, \infty]$ .

$$\varphi : [0, \infty] \times \mathcal{M} \rightarrow \mathcal{M}$$

with

$$\begin{aligned}\varphi(0, x) &= x, \\ \varphi(t_1, \varphi(t_2, x)) &= \varphi(t_1 + t_2, x).\end{aligned}$$

- ▶ The semiflow  $\varphi$  is the evolution function.

# Invariant and Stable Sets

## Definition

$\Omega \subseteq \mathcal{M}$  is called an INVARIANT SET if for all  $x \in \Omega$  and  $t \in \mathbb{R}_{\geq 0}$ ,  $\varphi(t, x) \in \Omega$ . If  $\Omega = \{p\}$  is a singleton, then  $\Omega$  is called an EQUILIBRIUM POINT of the dynamical system  $(\mathcal{M}, \varphi)$ .

## Definition

$\Omega \subseteq \mathcal{M}$  is STABLE if for every open neighborhood  $\mathcal{U} \subseteq \mathcal{M}$  of  $\Omega$ , there exists a neighborhood  $\mathcal{V} \subseteq \mathcal{M}$  of  $\Omega$  such that  $\varphi(t, \mathcal{V}) \subseteq \mathcal{U}$  for all  $t \geq 0$ .

An invariant set  $\Omega$  is asymptotically stable if

- ▶  $\Omega$  is stable,
- ▶  $\Omega$  is attractive, i.e., for all  $x \in \Omega$ , there exists an open neighborhood  $\mathcal{N} \subseteq \mathcal{M}$  of  $\Omega$  such that for all  $x \in \mathcal{N}$ ,  $\varphi(t, x) \xrightarrow{t \rightarrow \infty} \Omega$ .



# Domain (Region) of Attraction

The domain of attraction is denoted by

$$\mathcal{A} = \{x \in \mathcal{M} : \varphi(t, x) \rightarrow \Omega \text{ as } t \rightarrow \infty\}.$$

$\Omega$  is said to be GLOBALLY asymptotically stable if  $\mathcal{N} = \mathcal{M}$ .

## Definition (Lie derivative)

The LIE DERIVATIVE of  $V : \mathcal{M} \rightarrow \mathbb{R}$  along  $f \in \mathfrak{X}(\mathcal{M})$  is defined by

$$\begin{aligned}\mathcal{L}_f V : \mathcal{M} &\rightarrow \mathbb{R}, \\ p &\mapsto dV_p(f(p)).\end{aligned}$$

# Lyapunov Function

## Definition

Let  $\mathcal{K}$  be an invariant set of the dynamical system  $(\mathcal{M}, \varphi)$ . A continuous function  $V : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$  is a LYAPUNOV FUNCTION if

- ▶  $V(x) > 0$  for all  $x \in \mathcal{A} \setminus \mathcal{K}$ ,
- ▶  $V(x) = 0$  for all  $x \in \mathcal{K}$ ,
- ▶  $V$  is proper, i.e.,  $V^{-1}(B)$  is compact for all compact subsets  $B \subseteq \mathbb{R}_{\geq 0}$ ,
- ▶  $V$  is strictly decreasing along orbits of  $\varphi$ , i.e.,

$$V \circ \varphi(t, x) < V(x),$$

for all  $t > 0$  and  $x \in \mathcal{A} \setminus \mathcal{K}$ .

If  $V$  is differentiable, this condition may be replaced by

$$\mathcal{L}_f V(x) < 0.$$

# (Nondegenerate) Critical Points

## Definition

Let  $V : \mathcal{M} \rightarrow \mathbb{R}$  be a smooth function. A CRITICAL POINT,  $p \in \mathcal{M}$ , of  $V$  is a point where the differential

$$dV_p : T_p\mathcal{M} \rightarrow \mathbb{R}$$

has rank zero, i.e., in any local coordinate system  $\{x_i\}_1^n$ , one has  $\frac{\partial V}{\partial x_i}(p) = 0$  for all  $i = 1, \dots, n$ .

## Definition

A critical point  $p$  is NONDEGENERATE if the Hessian  $H_p(V)$  is a nondegenerate bilinear form, i.e., if any coordinate system, the Hessian matrix

$$\left( \frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq n}$$

is nondegenerate.

# Nondegenerate Critical Points

## Definition

The dimension of the subspace of  $T_p\mathcal{M}$  on which  $H_p(V)$  is negative definite is called the MORSE INDEX of  $V$  at  $p$ , denoted by  $\text{ind}(V, p)$ .

## Definition

A  $C^2$  function  $V : \mathcal{M} \rightarrow \mathbb{R}$  is a MORSE FUNCTION if all its critical points are nondegenerate.

## Definition

The (SUB)-LEVEL SETS of a function  $V : \mathcal{M} \rightarrow \mathbb{R}$  are

$$\begin{aligned}\mathcal{M}_a &= V^{-1}((-\infty, a]), \\ \mathcal{M}_{a,b} &= V^{-1}([a, b]).\end{aligned}$$

# Topological Definitions

- ▶ A top. space is an  $n$ -CELL if it is homeomorphic to  $\mathbb{R}^n$ .
- ▶ A top. space  $X$  is CONTRACTIBLE if it is *homotopy equivalent* to the one-point space.
- ▶ A subspace  $A$  of  $X$  is called a DEFORMATION RETRACT of  $X$  if there exists a continuous function  $h : [0, 1] \times X \rightarrow X$  such that for all  $x \in X, a \in A$ ,

$$h(0, x) = x,$$

$$h(1, x) \in A,$$

$$h(1, a) = a.$$

- ▶ The  $k^{\text{th}}$  BETTI NUMBER of  $\mathcal{M}$ , denoted by  $b_k$  is the rank of the  $k^{\text{th}}$  homology group  $H^k(\mathcal{M})$ .
- ▶ The EULER CHARACTERISTIC of  $\mathcal{M}$  is defined by

$$\chi(\mathcal{M}) = \sum_{i=1}^k (-1)^i b_i.$$

# Lyapunov Stability Analysis on Euclidean Spaces

# Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{1}$$

where  $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a locally Lipschitz map, with an equilibrium point at  $x = 0$ .

## Definition

The equilibrium point  $x = 0$  of the system (1) is

- ▶ *stable* if,  $\forall \epsilon > 0$ ,  $\exists \delta = \delta(\epsilon) > 0$  such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0.$$

- ▶ *unstable* if it is not stable.
- ▶ *asymptotically stable* if it is stable and  $\delta$  can be chosen s.t.

$$\|x(0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0.$$

## Example – Pendulum

The pendulum equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a \sin x_1 - bx_2$$

has two equilibrium points at  $(x_1 = 0, x_2 = 0)$  and  $(x_1 = \pi, x_2 = 0)$ .

- ▶ If  $b = 0$ , trajectories in the nbhd. of the first equilibrium are closed orbits.
- ▶ By starting sufficiently close to the eq. point, trajectories are guaranteed to stay within any specified ball.
- ▶ The point is not asymptotically stable since trajectories don't tend to the eq. point.
- ▶ If  $b > 0$ , the origin becomes asymptotically stable.
- ▶ The second eq. point is a saddle point: the  $\varepsilon - \delta$  requirement cannot be satisfied (for every  $\varepsilon > 0$  there exists a trajectory that will leave the ball  $B_\varepsilon$  even if  $x(0)$  is arbitrarily close to  $(\pi, 0)$ ).



# Lyapunov Stability Theorem

## Theorem

*Let  $x = 0 \in D$  be an equilibrium point for (1). Let  $V : D \rightarrow \mathbb{R}$  be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\},$$
$$\dot{V}(x) \leq 0 \text{ in } D.$$

*Then,  $x = 0$  is stable. Moreover, if*

$$\dot{V}(x) < 0 \text{ in } D - \{0\}$$

*then  $x = 0$  is asymptotically stable.*

# Lyapunov Stability Theorem

*Proof of stability.* Given  $\varepsilon > 0$ , choose  $0 < r \leq \varepsilon$  such that  $B_r \subseteq D$ . Let  $\alpha = \min_{\|x\|=r} V(x)$ . Then,  $\alpha > 0$ . Take  $0 < \beta < \alpha$  and consider  $\mathcal{M}_\beta = V^{-1}((0, \beta])$ .

Claim:  $\mathcal{M}_\beta \subseteq \mathring{B}_r$ . Argue ad absurdum. Suppose  $\mathcal{M}_\beta \cap \mathring{B}_r \neq \mathcal{M}_\beta$ . Then  $\exists p \in \mathcal{M}_\beta \cap \partial B_r$ . Note,  $V(p) \geq \alpha > \beta$ , but  $V(\mathcal{M}_\beta) \subseteq [0, \beta]$ .

The set  $\mathcal{M}_\beta$  is invariant since

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta, \forall t \geq 0.$$

Because  $\mathcal{M}_\beta$  is compact (closed and bounded), we conclude that the ODE (1) has a unique solution  $\forall t \geq 0$  whenever  $x(0) \in \mathcal{M}_\beta$ . Since  $V$  is continuous and  $V(0) = 0$ ,  $\exists \delta > 0$  such that

$$\|x\| \leq \delta \Rightarrow V(x) < \beta.$$

# Lyapunov Stability Theorem

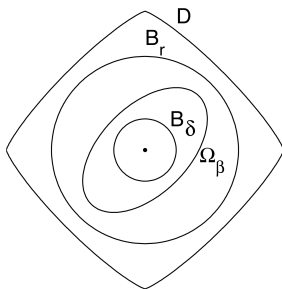
*Proof of stability (cont'd).* Then,

$$B_\delta \subseteq \mathcal{M}_\beta \subseteq B_r$$

and

$$x(0) \in B_\delta \Rightarrow x(0) \in \mathcal{M}_\beta \Rightarrow x(t) \in \mathcal{M}_\beta \Rightarrow x(t) \in B_r,$$

proving stability. □



**Figure:** Geometric representation of Lyapunov stability.

# Lyapunov Stability Theorem

*Proof of asymptotic stability.* Now assume  $\dot{V}(x) < 0$  in  $D - \{0\}$ . We want to show that  $x(t) \xrightarrow{t \rightarrow \infty} 0$ ; i.e.,  $\forall a > 0, \exists T > 0$ , s.t.  $\|x(t)\| < a, \forall t > T$ .

We know that  $\forall a > 0$ , we can choose  $b > 0$  s.t.  $\mathcal{M}_b \subseteq B_a$ . Therefore, it is sufficient to show that  $V(x(t)) \xrightarrow{t \rightarrow \infty} 0$ . Since  $V$  is monotonically decreasing and bounded from below by zero,

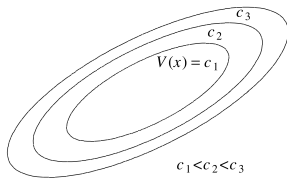
$$V(x(t)) \xrightarrow{t \rightarrow \infty} c \geq 0.$$

Claim:  $c = 0$ . Argue ad absurdum. Suppose  $c > 0$ . By continuity of  $V$ ,  $\exists d > 0$  s.t.  $B_d \subseteq \mathcal{M}_c$ . The limit  $V(x(t)) \rightarrow c > 0$  implies that  $x(t) \notin B_d, \forall t \geq 0$ . Define  $\max_{d \leq \|x\| \leq r} \dot{V}(x) =: -\gamma < 0$ . It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau)) d\tau \leq V(x(0)) - \gamma t.$$

The RHS will eventually become negative: contradiction ( $c > 0$ ). □

# Lyapunov Stability: Intuition



- ▶ A continuously differentiable function  $V$ , satisfying the theorem's conditions is called a LYAPUNOV FUNCTION.
- ▶ When  $\dot{V} < 0$ , the trajectory moves from level set  $\mathcal{M}_{c_3} = V^{-1}(c_3)$  to an inner level set  $\mathcal{M}_{c_2} = V^{-1}(c_2)$  with a smaller  $c$ .
- ▶  $V^{-1}(c) \xrightarrow{c \downarrow 0} 0$ . Hence the trajectory approaches the origin.
- ▶ If we only knew that  $\dot{V} \leq 0$ , we cannot be sure that the trajectory  $x(t) \xrightarrow{t \rightarrow \infty} 0$ ,<sup>1</sup> but we can conclude that the origin is stable.

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<sup>1</sup>See, however, Krasovskii-LaSalle's theorem.

## Example: Undamped pendulum

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -a \sin x_1.$$

Lyapunov function candidate

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2.$$

### Analysis

Clearly,  $V(0) = 0$  and  $V(x) > 0$  if  $x \neq (2k\pi, 0)$ . Compute the Lie derivative of  $V$  along  $f$ :

$$\dot{V}(x) = \mathcal{L}_f V(x) = ax_2 \sin x_1 - ax_2 \sin x_1 = 0.$$

Thus, the origin is stable. Since  $\dot{V}(x) \equiv 0$ , we conclude that the origin is not asymptotically stable as solutions starting on the level set  $\mathcal{M}_c$  remain in that set.

## Example: Damped pendulum

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -a \sin x_1 - bx_2.$$

Lyapunov function candidate

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x^\top P x,$$

$$P = P^\top > 0.$$

The Lie derivative  $\dot{V}(x)$  is given by

$$\dot{V}(x) = a(1 - p_{22})x_2 \sin x_1 - ap_{12}x_1 \sin x_1 + (p_{11} - p_{12}b)x_1x_2 + (p_{12} - p_{22}b)x_2^2.$$

- ▶ Take  $p_{22} = 1$  and  $p_{11} = bp_{12}$ .
- ▶ We must choose  $0 < p_{12} < b$  for  $V$  to be positive definite.
- ▶ Choose  $p_{12} = \frac{b}{2}$ .

$$\dot{V}(x) = -\frac{1}{2}abx_1 \sin x_1 - \frac{1}{2}bx_2^2.$$

This is negative definite for any  $0 < |x_1| < \pi$ .

## Example: Rotational Motion of a Rigid Body in 3D-space

With respect to a coordinate system frame, which is rigidly attached to the body and whose axes are chosen to be the principal axes of the body, define:

- ▶  $\omega$ : angular velocity of the body,
- ▶  $I \in \mathbb{S}_{++}^3$ : inertia matrix of the body.

In the absence of external torques, the motion is described by

$$I\dot{\omega} + \omega \times I\omega = 0.$$
$$\begin{aligned}I_x\dot{\omega}_x &= -(I_z - I_y)\omega_y\omega_z, \\I_y\dot{\omega}_y &= -(I_x - I_z)\omega_x\omega_z, \\I_z\dot{\omega}_z &= -(I_y - I_x)\omega_x\omega_y.\end{aligned}$$



## Example: Rotational Motion of a Rigid Body in 3D-space

Suppose w.l.o.g., that  $I_x \geq I_y \geq I_z > 0$ . For notational simplicity, define

$$\begin{aligned}\omega_x &\mapsto X \\ \omega_y &\mapsto Y \\ \omega_z &\mapsto Z\end{aligned}\qquad \begin{aligned}a &= \frac{I_y - I_z}{I_x}, \\ b &= \frac{I_x - I_z}{I_y}, \\ c &= \frac{I_x - I_y}{I_z}.\end{aligned}$$

Note that  $a, b, c, \geq 0$ . The equations of motion assumes the form

$$\dot{x} = ayz, \quad \dot{y} = -bxz, \quad \dot{z} = cxy.$$

From here on out, assume that the principal axes are unique; this is equivalent to assuming that  $I_x > I_y > I_z$ , or that  $a, b, c > 0$ .

## Example: Rotational Motion of a Rigid Body in 3D-space

The set of equilibria is

$$(\mathbb{R} \times \{0\} \times \{0\}) \cup (\{0\} \times \mathbb{R} \times \{0\}) \cup (\{0\} \times \{0\} \times \mathbb{R}).$$

### Remark

Physically this corresponds to rotation around one of the principal axes at a constant angular velocity. Note that none of the equilibria is isolated.

Consider first, the equilibrium at the origin and try

$$V(x, y, z) = px^2 + qy^2 + rz^2,$$

where  $p, q, r > 0$ . Then  $V$  is a lpdf. Computing  $\dot{V}$ :

$$\dot{V} = 2(px\dot{x} + qy\dot{y} + rz\dot{z}) = 2xyz(ap - bq + cr).$$

Clearly, it is possible to choose  $p, q, r > 0$  such that

$$ap - bq + cr = 0.$$

For such a choice,  $\dot{V} \equiv 0$  and the origin is STABLE.

## Example: Rotational Motion of a Rigid Body in 3D-space

Next, consider the equilibrium of the form  $(x_0, 0, 0)$  where  $x_0 \neq 0$ .

Consider the Lyapunov function candidate  $W$ , such that  $W(x_0, 0, 0) = 0$ , and  $W(x, y, z) > 0$ ,  $\forall (x, y, z) \neq (x_0, 0, 0)$  and sufficiently near  $(x_0, 0, 0)$ :

$$W(x, y, z) = cy^2 + bz^2 + [2acy^2 + abz^2 + bc(x^2 - x_0^2)]^2$$

$W$  is an lpdf w.r.t. the equilibrium  $(x_0, 0, 0)$  and routine computations show that  $\dot{W} \equiv 0$ . Hence  $(x_0, 0, 0)$  is a stable equilibrium.

### Discussion

- ▶ We could also translate the coordinates such that  $(x_0, 0, 0)$  becomes the origin of the new coordinate system and apply the Lyapunov stability theorem directly.
- ▶ Is  $(0, 0, z_0)$ ,  $z_0 \neq 0$  stable?
- ▶ Is  $(0, y_0, 0)$ ,  $y_0 \neq 0$  (w.l.o.g., assume  $y_0 > 0$ ) stable?

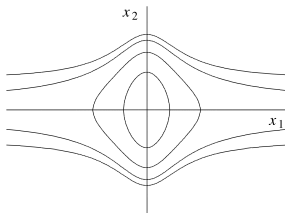
# Region of Attraction

## Definition (Region of Attraction)

The REGION OF ATTRACTION is defined as the set of all points  $x$  such that  $\phi(t; x)$  is defined for all  $t \geq 0$  and  $\lim_{t \rightarrow \infty} \phi(t; x) = 0$ .

- ▶ Finding the exact RoA is usually difficult.
- ▶ Lyapunov fcns. can be used to estimate (inner approx.) the RoA.
- ▶ From the proof of the Lyapunov stability theorem, if there is a Lyapunov fcn. that satisfies asymptotic stability and if  $\mathcal{M}_c$  is bounded and contained in  $D$ , then  $\mathcal{M}_c$  is (positively) invariant.
- ▶ The estimate  $\mathcal{M}_c$  of the RoA may be conservative (inner approximation).
- ▶ QUESTION: Under what conditions is the RoA the whole space?
  - ▶ If so, the origin is said to be *globally asymptotically stable*.
  - ▶ The conditions of the Lyapunov theorem must clearly hold for  $D = \mathbb{R}^n$ . But is this sufficient?

# Region of Attraction



**Figure:** Level sets of  $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$ .

For  $\mathcal{M}_c$  to be bounded ( $\mathcal{M}_c \subseteq \mathring{B}_r$ , for some  $r \geq 0$ ),  $c < \inf_{\|x\| \geq r} V(x)$ . If

$$l = \lim_{r \rightarrow \infty} \inf_{\|x\| \geq r} V(x) < \infty$$

then  $\mathcal{M}_c$  will be bounded only if  $c < l$ . Consider (see figure)

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2.$$

In this example,

$$l = \lim_{r \rightarrow \infty} \min_{\|x\|=r} V(x) = 1.$$

# Region of Attraction

For  $\mathcal{M}_c$  to be bounded ( $\mathcal{M}_c \subseteq \mathring{B}_r$ , for some  $r \geq 0$ ),  $c < \inf_{\|x\| \geq r} V(x)$ . If

$$l = \lim_{r \rightarrow \infty} \inf_{\|x\| \geq r} V(x) < \infty$$

then  $\mathcal{M}_c$  will be bounded only if  $c < l$ . Consider (see figure)

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2.$$

In this example,

$$l = \lim_{r \rightarrow \infty} \min_{\|x\|=r} V(x) = 1.$$

An extra condition that ensures that  $\mathcal{M}_c$  is bounded for all  $c > 0$  is

$$V(x) \rightarrow \infty \text{ as } \|x\| \rightarrow \infty.$$

## Homework

Show that a continuously differentiable map  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is radially unbounded if and only if it is proper (inverse images of compact sets under  $V$  are compact).

## Theorem (Global Asymptotic Stability)

*Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuously differentiable function and the conditions of the Lyapunov stability theorem hold (asymptotic). If, in addition,*

$$\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$$

*then  $x = 0$  is globally asymptotically stable.*

## Remark

*For  $x = 0$  to be GAS, it must be the unique equilibrium point of the system (why?).*

# Chetaev's Instability Theorem

## Theorem

Let  $V : D \rightarrow \mathbb{R}$  be a continuously differentiable function such that  $V(0) = 0$  and  $V(x_0) > 0$  for some  $x_0$  with arbitrarily small  $\|x_0\|$ . Let

$$U := \{x \in B_r : V(x) > 0\}$$

and suppose that  $\dot{V}(U) > 0$ . Then,  $x = 0$  is unstable.

*Proof.*  $x_0 \in \overset{\circ}{U}$  and  $V(x_0) = a > 0$ . The trajectory  $x(t)$  starting at  $x(0) = x_0$  must leave  $U$ . Indeed, as long as  $x(t) \in U$ ,  $V(x(t)) \geq a$ , since  $\dot{V}(U) > 0$ . Let  $\min\{\dot{V}(x) : x \in U \text{ and } V(x) \geq a\} := \gamma > 0$ . Then,

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s)) \, ds \geq a + \int_0^t \gamma \, ds = a + \gamma t.$$

Hence,  $x(t)$  will leave  $U$  because  $V(x)$  is bounded on  $U$ . Now,  $x(t)$  cannot leave  $U$  through  $V(x) = 0$  since  $V(x(t)) \geq a$ . Hence it must leave  $U$  through the sphere  $\mathbb{S}_r$ . Note:  $\|x_0\|$  was arbitrarily small.  $\square$



## Example: Rotational Motion of a Rigid Body

Consider an equilibrium of the form  $(0, y_0, 0)$ ,  $y_0 > 0$  and translate the coordinates so that the equilibrium under study becomes the origin. Setting  $y_s = y - y_0$ , the equations of motion are

$$\dot{x} = ay_s z + ay_0 z, \quad \dot{y}_s = -bxz, \quad \dot{z} = cxy_s + cxy_0.$$

Now, apply Chetaev's theorem with

$$V(x, y, z) = xz,$$

$$B_r = \{(x, y_s, z) : x^2 + y_s^2 + z^2 < r^2\},$$

$$U = \{(x, y_s, z) \in B_{\frac{r}{2}} : x > 0 \text{ and } z > 0\}.$$

Then  $U$  is open and

$$\dot{V} = x\dot{z} + \dot{x}z = 2(y_s + y_0)(cx^2 + az^2).$$

If  $(x, y_s, z) \in U$ , then  $y_s + y_0 > 0$ , so Chetaev's theorem yields that the origin (in the new coordinate system) is UNSTABLE.

# The Invariance Principle

# Intuition: Damped Pendulum

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -a \sin x_1 - bx_2^2.$$

Lyapunov function candidate

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2.$$

$$\dot{V}(x) = -bx_2^2 \leq 0.$$

- ▶  $\dot{V}(x) < 0$  if and only if  $x_2 \neq 0$ .
- ▶ For the system to maintain  $\dot{V}(x) = 0$ , it has to stay on  $x_2 = 0$ .
- ▶ Unless  $x_1 = 0$ , this is impossible:

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow \sin x_1(t) \equiv 0.$$

- ▶ Hence, on the segment  $-\pi < x_1 < \pi$  of the  $x_2 = 0$  line, the system can maintain  $\dot{V}(x) = 0$  only at the origin  $x = 0$ .
- ▶ Therefore,  $V(x(t))$  must decrease towards 0 and, consequently,

$$x(t) \xrightarrow{t \rightarrow \infty} 0.$$

# Limit and Invariant Sets

## Definition (Limit points and limit sets)

A point  $p$  is said to be a *positive limit point* of  $x(t)$  if there is a sequence  $\{t_n\}$ , with  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$ , such that  $x(t_n) \rightarrow p$  as  $n \rightarrow \infty$ .

The set of all positive limit points of  $x(t)$  is called the *positive limit set* of  $x(t)$ .

## Definition (Positively Invariant Set)

A set  $M$  is said to be an *invariant set* w.r.t. (1) if

$$x(0) \in M \Rightarrow x(t) \in M, \forall t \in \mathbb{R}.$$

That is, if a solution belongs to  $M$  at some time instant, then it belongs to  $M$  for all future and past time.

A set  $M$  is said to be a *positively invariant set* if

$$x(0) \in M \Rightarrow x(t) \in M, \forall t \geq 0.$$

# Distance to an (Invariant) Set

## Definition (Distance and Convergence to a Set)

We say that  $x(t)$  approaches a set  $M$  as  $t \rightarrow \infty$ , if for each  $\varepsilon > 0$ ,  $\exists T > 0$  such that

$$\inf_{x \in M} \|p - x\| =: \text{dist}(x(t), M) < \varepsilon, \quad \forall t > T.$$

- ▶ An asymptotically stable equilibrium point is the positive limit set of every solution starting sufficiently near the equilibrium point.
- ▶ A stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle.
- ▶ The solution approaches the limit cycle as  $t \rightarrow \infty$ . Notice: the solution does not approach any specific point on the limit cycle.
- ▶ The statement  $x(t)$  approaches  $M$  as  $t \rightarrow \infty$  does not imply that  $\lim_{t \rightarrow \infty} x(t)$  exists.
- ▶ The set  $\mathcal{M}_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$  with  $\dot{V}(x) \leq 0$  for all  $x \in \mathcal{M}_c$  is a positively invariant set.

# Limit Sets and Krasovskii-LaSalle Theorem

## Lemma

*If a solution  $x(t)$  is bounded and belongs to  $D$  for  $t \geq 0$ , then its positive limit set  $L^+$  is a nonempty, compact, invariant set. Moreover,  $x(t)$  approaches  $L^+$  as  $t \rightarrow \infty$ .*

## Theorem (Krasovskii-LaSalle Theorem)

*Let  $\Omega \subseteq D$  be a compact set that is positively invariant w.r.t. (1). Let  $V : D \rightarrow \mathbb{R}$  be a continuously differentiable function such that  $\dot{V}(x) \leq 0$  in  $\Omega$ . Let  $E$  be the set of all points in  $\Omega$  where  $\dot{V}(x) = 0$ . Let  $M$  be the largest invariant set in  $E$ . Then every solution starting in  $\Omega$  approaches  $M$  as  $t \rightarrow \infty$ .*

# Krasovskii-LaSalle Theorem

*Proof.* Let  $x(t)$  be a solution of (1) starting in  $\Omega$ . Since  $\dot{V}(x) \leq 0$  in  $\Omega$ ,  $V(x(t))$  is a decreasing function of  $t$ . Since  $V(x)$  is continuous on the compact set  $\Omega$ , it is bounded from below on  $\Omega$ . Therefore,  $V(x(t))$  has a limit  $a$  as  $t \rightarrow \infty$ . Note that the positive limit set  $L^+$  is in  $\Omega$  because  $\Omega$  is a closed set. For any  $p \in L^+$ , there is a sequence  $t_n$  with  $t_n \rightarrow \infty$  and  $x(t_n) \rightarrow p$  as  $n \rightarrow \infty$ . By the continuity of  $V(x)$ ,  $V(p) = \lim_{n \rightarrow \infty} V(x(t_n)) = a$ . Hence,  $V(x) = a$  on  $L^+$ . Since  $L^+$  is an invariant set,  $\dot{V}(x) = 0$  on  $L^+$ . Thus,

$$L^+ \subseteq M \subseteq E \subseteq \Omega$$

Since  $x(t)$  is bounded,  $x(t)$  approaches  $L^+$  as  $t \rightarrow \infty$ . Hence,  $x(t)$  approaches  $M$  as  $t \rightarrow \infty$ . □

# Krasovskii-LaSalle Theorem

- ▶ Notice that, this theorem does not require the function  $V(x)$  to be positive definite.
- ▶ The set  $\Omega$  does not have to be tied in with the construction of the function  $V(x)$ .
- ▶ However, in many applications, the construction of  $V(x)$  will itself guarantee the existence of a set  $\Omega$ . In particular, if  $\mathcal{M}_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$  is bounded and  $\dot{V}(x) \leq 0$  in  $\mathcal{M}_c$ , then we can take  $\Omega = \mathcal{M}_c$ .
- ▶ When  $V$  is positive definite,  $\mathcal{M}_c$  is bounded for sufficiently small  $c > 0$ . This is not necessarily true when  $V$  is not positive definite.
- ▶ If  $V$  is radially unbounded (or proper), the set  $\mathcal{M}_c$  is bounded for all values of  $c$ . This is true whether or not  $V$  is positive definite.



# Corollaries of Krasovskii-LaSalle Theorem

## Corollary

*Let  $V : D \rightarrow \mathbb{R}$  be a continuously differentiable positive definite function on a domain  $D$  containing the equilibrium point  $x = 0$ , such that  $\dot{V}(x) \leq 0$  in  $D$ . Let  $S = \{x \in D : \dot{V}(x) = 0\}$  and suppose that no solution can stay identically in  $S$  other than the trivial solution  $x(t) \equiv 0$ . Then, the origin is asymptotically stable.*

## Corollary

*Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuously differentiable, radially unbounded, positive definite function such that  $\dot{V}(x) \leq 0$  for all  $x \in \mathbb{R}^n$ . Let  $S = \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$  and suppose that no solution can stay identically in  $S$  other than the trivial solution  $x(t) \equiv 0$ . Then, the origin is globally asymptotically stable.*

Notice that when  $\dot{V}(x)$  is negative definite, then  $S = \{0\}$ .

## Remarks on Krasovskii-LaSalle Theorem

- ▶ The theorem relaxes the negative definiteness requirement of Lyapunov's theorem.
- ▶ It further extends Lyapunov's theorem in three different directions.
  - ▶ It gives an estimate of the RoA, which is not necessarily of the form  $\mathcal{M}_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ . The set  $\Omega$  of the theorem can be ANY compact positively invariant set.
  - ▶ The theorem can be used in cases where the system has an equilibrium set, rather than an isolated equilibrium point.
  - ▶ The function  $V$  does not have to be positive definite.

## Example: Stabilization of a Rigid Robot without Gravity

### Setup

Let  $q = (q_1, \dots, q_n)$  denote the vector of generalized coordinates of the robot and  $u = (u_1, \dots, u_n)$  denote the vector of generalized forces. The dynamics are given by the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u,$$

where  $L$  is the Lagrangian of the system. Since there is no gravity, the potential energy  $\mathcal{P} = 0$  can be taken. Thus,

$$L = K = \frac{1}{2} \dot{q}^\top M(q) \dot{q}.$$

$M(q) \in \mathbb{S}_{++}^n$  is called the **inertia matrix**. There exist positive constants  $\alpha$  and  $\beta$  such that

$$0 < \alpha \leq \lambda_{\min} [M(q)] \leq \lambda_{\max} [M(q)] \leq \beta, \quad \forall q.$$

# Example: Stabilization of a Rigid Robot without Gravity

## The Euler-Lagrange equations

With  $L = K$ , we have

$$\sum_{j=1}^n m_{ij}(q) \ddot{q}_j + \sum_{j=1}^n \sum_{k=1}^n c_{ijk}(q) \dot{q}_j \dot{q}_k = u_i, \quad i = 1, \dots, n,$$

where

$$c_{ijk} = \frac{1}{2} \left( \frac{\partial m_{ik}}{\partial q_j} + \frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{jk}}{\partial q_i} \right)$$

are called the *Christoffel symbols*. Compactly, we have

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} = u,$$

where the  $(i, j)^{\text{th}}$  element of  $C(q, \dot{q})$  is

$$c_{ij}(q, \dot{q}) = \sum_{k=1}^n c_{ijk}(q) \dot{q}_k.$$

# Example: Stabilization of a Rigid Robot without Gravity

## State equations and naïve control

Introduce the state variables  $x = q$ ,  $y = \dot{q}$  so that

$$\dot{x} = y, \quad \dot{y} = [M(x)]^{-1} [u - C(x, y)y].$$

Suppose we want to asymptotically stabilize the state  $(x, y)$  to a desired value  $(x_d, 0)$ . Let us try the naïve control law

$$u = -K_p(x - x_d) - K_d y,$$

where  $K_p, K_d \in \mathbb{S}_{++}^n$ . The closed-loop dynamics become

$$\dot{x} = y, \quad \dot{y} = -[M(x)]^{-1} [K_p(x - x_d) + K_d y + C(x, y)y].$$

# Example: Stabilization of a Rigid Robot without Gravity

## Lyapunov analysis

Consider the Lyapunov function candidate

$$V = \frac{1}{2} [y^\top M(x)y + (x - x_d)^\top K_p(x - x_d)] .$$

The first term is the kinetic energy, while the second term is the potential energy due to proportional feedback. Note that

$$\frac{d}{dt} [m_{ij}(x)] = \sum_{k=1}^n \frac{\partial m_{ij}(x)}{\partial x_k} y_k .$$

Define the  $(i, j)^{\text{th}}$  element of  $\dot{M}(x, y) \in \mathbb{R}^{n \times n}$  by the RHS above. Now,

$$\begin{aligned} \dot{V} &= y^\top M(x) \dot{y} + \frac{1}{2} y^\top \dot{M}(x, y) y + \dot{x}^\top K_p(x - x_d) \\ &= -y^\top [K_p(x - x_d) + K_d y + C(x, y) y] + \frac{1}{2} y^\top \dot{M}(x, y) y + y^\top K_p(x - x_d) \\ &= -y^\top K_d y + \frac{1}{2} y^\top [\dot{M}(x, y) - 2C(x, y)] y = -y^\top K_d y + \frac{1}{2} y^\top D(x, y) y. \end{aligned}$$

## Example: Stabilization of a Rigid Robot without Gravity

Skew-symmetry of  $D(x, y) := \dot{M}(x, y) - 2C(x, y)$

We perform the computations in coordinates

$$d_{ij} = \dot{m}_{ij} - 2c_{ij} = \left[ \sum_{k=1}^n \frac{\partial m_{ij}}{\partial x_k} - \left( \frac{\partial m_{ik}}{\partial x_j} + \frac{\partial m_{ij}}{\partial x_k} - \frac{\partial m_{jk}}{\partial x_i} \right) \right] y_k$$

## Example: Stabilization of a Rigid Robot without Gravity

Skew-symmetry of  $D(x, y) := \dot{M}(x, y) - 2C(x, y)$

$$\begin{aligned} d_{ij} = \dot{m}_{ij} - 2c_{ij} &= \left[ \sum_{k=1}^n \cancel{\frac{\partial m_{ij}}{\partial x_k}} - \left( \frac{\partial m_{ik}}{\partial x_j} + \cancel{\frac{\partial m_{ij}}{\partial x_k}} - \frac{\partial m_{jk}}{\partial x_i} \right) \right] y_k \\ &= \sum_{k=1}^n \left( \frac{\partial m_{jk}}{\partial x_i} - \frac{\partial m_{ik}}{\partial x_j} \right) y_k. \end{aligned}$$

Interchanging  $i$  and  $j$  gives

$$d_{ji} = \sum_{k=1}^n \left( \frac{\partial m_{ik}}{\partial x_j} - \frac{\partial m_{jk}}{\partial x_i} \right) y_k = -d_{ij}$$



## Example: Stabilization of a Rigid Robot without Gravity

### Lyapunov analysis – resumed

Hence  $D$  is skew-symmetric and hence  $y^\top D y = 0$ , so that

$$\dot{V} = -y^\top K_d y \leq 0.$$

The set  $E$  of Krasovskii-LaSalle theorem is given by

$$E = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \dot{V} \equiv 0\} = \mathbb{R}^n \times \{0\}.$$

Suppose  $(x(t), y(t))$  is a trajectory that lies entirely in  $E$ . Then

$$y \equiv 0 \implies \dot{y} \equiv 0 \implies K_p(x - x_d) \equiv 0 \implies x \equiv x_d, \quad \forall t \geq 0.$$

Hence  $E$  contains no trajectories of the system other than the equilibrium  $(x_d, 0)$ . It follows from the Krasovskii-LaSalle theorem that this equilibrium is GLOBALLY ASYMPTOTICALLY STABLE.

# Stability of Linear Systems

# Autonomous Linear Systems

We restrict our attention to linear *autonomous* systems of the form

$$\dot{x}(t) = Ax(t). \quad (2)$$

## Theorem

*The equilibrium 0 of (2) is (globally) exponentially stable if and only if all eigenvalues of A have negative real parts. The equilibrium is stable if and only if all eigenvalues of A have nonpositive real parts, and in addition, every eigenvalues of A having a zero real part is a simple zero of the minimal polynomial of A.*

# Lyapunov Function

Given the system (2), we choose a Lyapunov function candidate:

$$V(x) = x^\top P x \implies \dot{V} = \dot{x}^\top P x + x^\top P \dot{x} = -x^\top Q x,$$

where  $P = P^\top$  and

$$A^\top P + P A = -Q. \quad (3)$$

Equation (3) is commonly known as the **Lyapunov Matrix Equation**.

## Remark (Stability)

*If a pair of matrices  $(P, Q)$  satisfying (3) can be found such that both  $P$  and  $Q$  are positive definite, then both  $V$  and  $-\dot{V}$  are positive definite functions and  $V$  is radially unbounded. Hence, the equilibrium  $0$  is globally exponentially stable.*

*If a pair  $(P, Q)$  can be found s.t.  $Q > 0$  and  $P$  has at least one nonpositive eigenvalue, then  $-\dot{V} > 0$  and  $V$  assumes nonpositive values arbitrarily close to the origin. Hence  $0$  is unstable.*

# Lyapunov Matrix Equation

## Lemma

*Let  $\{\lambda_i\}_1^n$  denote the eigenvalues of  $A$ . Then equation (3) has a unique solution for  $P$  corresponding to each  $Q \in \mathbb{R}^{n \times n}$  iff*

$$\lambda_i + \lambda_j \neq 0, \quad \forall i, j.$$

## Corollary

*If for some  $Q \in \mathbb{R}^{n \times n}$  does not have a unique solution for  $P$ , then the origin is not an asymptotically stable equilibrium.*

*Proof.* If all eigenvalues of  $A$  has negative real parts, then the equation above is satisfied. □

# Main Result

## Theorem

*Given a matrix  $A \in \mathbb{R}^{n \times n}$ , the following are equivalent:*

- ▶ *A is a Hurwitz matrix (all its e.vals have negative real parts).*
- ▶ *There exists SOME  $Q \in \mathbb{S}_{++}^n$  such that equation (3) has a corresponding unique solution for  $P \in \mathbb{S}_{++}^n$ .*
- ▶ *For EVERY  $Q \in \mathbb{S}_{++}^n$ , equation (3) has a unique solution for  $P \in \mathbb{S}_{++}^n$ .*

*Proof. “(3)  $\implies$  (2)” Obvious.*

*“(2)  $\implies$  (1)” Suppose (2) is true for some particular matrix Q.*

*Consider the candidate  $V(x) = x^\top P x$ . Then  $\dot{V}(x) = -x^\top Q x$ , and one can conclude that 0 is asymptotically stable. Hence A is Hurwitz.*

*“(1)  $\implies$  (3)” Omitted (see Section 5.4, Theorem (42) in Vidyasagar, “Nonlinear Systems Analysis”, 1993.)*

# Control-Lyapunov Functions

# Control-Lyapunov Functions <sup>1</sup>

Consider the control system with state  $x \in \mathbb{R}^n$  and control  $u \in \mathbb{R}^m, \forall t$ :

$$\dot{x}(t) = f(x(t)) + u_1(t)g_1(x(t)) + \cdots + u_m(t)g_m(x(t)), \quad f(0) = 0. \quad (4)$$

## Definition (Control-Lyapunov Function (clf))

A clf is a smooth, proper, and positive definite function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  so that

$$\inf_{u \in \mathbb{R}^m} \{ \mathcal{L}_f V(x) + u_1 \mathcal{L}_{g_1} V(x) + \cdots + u_m \mathcal{L}_{g_m} V(x) \} < 0, \quad \forall x \neq 0.$$

- $V$  is such that for each  $x \neq 0$ , one *can* diminish its value by applying *some* open-loop control.
- Existence of a clf implies that the system is asymp. controllable:

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<sup>1</sup>As discussed in Sontag, “A ‘universal’ construction of Artstein’s theorem on nonlinear stabilization”, 1989.



# Control-Lyapunov Functions: Single input

There exists a feedback law which is smooth on  $\mathbb{R}_0^n := \mathbb{R}^n - 0$

$$u = k(x), \quad k(0) = 0,$$

and which globally stabilizes the system.

Assume  $V$  is a clf for the system

$$\dot{x} = f(x) + ug(x).$$

Denote

$$a(x) := \nabla V(x) \cdot f(x),$$

$$b(x) := \nabla V(x) \cdot g(x).$$

The condition that  $V$  is a clf is precisely the statement that

$$b(x) = 0 \implies a(x) < 0, \quad \forall x \neq 0.$$

On the other hand,  $V$  is a Lyapunov function if

$$\nabla V(x) \cdot (f(x) + k(x)g(x)) < 0,$$

that is

$$a(x) + k(x)b(x) < 0, \quad \forall x \neq 0.$$

## Control-Lyapunov Functions: Single input

In this simple case where the family  $(a(x), b(x))$ , interpreted as a *family of linear systems parametrized by  $x$*  the following works:

$$k := -\frac{1}{b} \left( a + \sqrt{a^2 + b^2} \right).$$

Along trajectories of the closed-loop system, one has

$$\frac{dV}{dt} = -\sqrt{a^2 + b^2} < 0.$$

This feedback law may fail to be continuous, but with the slight modification

$$k := -\frac{1}{b} \left( a + \sqrt{a^2 + b^4} \right),$$

then it does become continuous.

# Control-Lyapunov Functions: Multi input

Now, consider the system back in equation (4).

- ▶ A sufficient conditions for a given  $k$  to be smooth feedback stabilizer is that there exist a Lyapunov function  $V$  so that

$$\nabla V(x) \cdot [f(x) + k_1(x)g_1(x) + \cdots + k_m(x)g_m(x)] < 0, \quad \forall x \neq 0.$$

- ▶ Such a Lyapunov function is automatically a clf.
- ▶ If  $k$  happens to be continuous at the origin, then the following property (**small control property**) holds (with  $u := k(x)$ )  
*For each  $\varepsilon > 0$ , there is  $\delta > 0$  s.t., if  $x \neq 0$  satisfies  $\|x\| < \delta$ , then there is some  $u$  with  $\|u\| < \varepsilon$  s.t.*

$$\nabla V(x) \cdot [f(x) + u_1g_1(x) + \cdots + u_mg_m(x)] < 0.$$

# Control-Lyapunov Functions: Multi input

## Theorem

*If  $\exists$  a smooth clf  $V$  then  $\exists$  a smooth feedback stabilizer  $k$ . If  $V$  satisfies the small control property, then  $k$  can be chosen to be also continuous at 0.*

*Proof. (Sketch).* The proof involves constructing a fixed function  $\phi$  of two variables, and then designing a feedback law in closed-form, from the evaluation of this function at a point determined by  $\nabla V(x) \cdot f(x)$  and the  $\nabla V(x) \cdot g_i(x)$ 's.

Define the following function (and then show that it is analytic.)

$$\phi(a, 0) := 0, \quad \forall a < 0$$

and

$$\phi(a, b) := \frac{1}{b} (a^2 + bq(b)), \quad q(0) = 0 \text{ and } bq(b) > 0.$$

For example, we can choose  $q(b) = b$  or  $q(b) = b^3$ , etc.

## Control-Lyapunov Functions: Multi input

*Proof. (Cont'd).* Assume that  $V$  is a clf and let

$$a(x) := \nabla V(x) \cdot f(x),$$

$$b_i(x) := \nabla V(x) \cdot g_i(x), \quad i = 1, \dots, m.$$

Further, let

$$B(x) := (b_1(x), \dots, b_m(x)),$$

$$\beta(x) := \|B(x)\|^2 = \sum_{i=1}^m b_i^2(x).$$

The condition that  $V$  is a clf is equivalent to  $\beta(x) = 0 \implies a(x) < 0$ .

Now, define the smooth feedback law  $k = (k_1, \dots, k_m)$ :

$$k_i(x) := -b_i(x)\phi(a(x), \beta(x)), \quad x \neq 0,$$

and  $k(0) := 0$ .

## Control-Lyapunov Functions: Multi input

*Proof. (Cont'd).* At a nonzero  $x$  we have that

$$\begin{aligned}\nabla V(x) \cdot \left[ f(x) + \sum_{i=1}^m k_i(x) g_i(x) \right] &= a(x) - \phi(a(x), \beta(x)) \beta(x) \\ &= -\sqrt{a(x)^2 + \beta(x) q(\beta(x))} < 0.\end{aligned}$$

so the original  $V$  decreases along trajectories of the closed-loop system.

We have still yet to show that  $V$  satisfies the small control property. The audience is invited to see the paper for the detailed proof of this. □

# Morse-Lyapunov Functions

# Isolated Critical Points

## Lemma

*Suppose that  $x_e$  is an equilibrium points of the dynamical system  $(M, \varphi)$ . If  $V : \mathcal{M} \rightarrow \mathbb{R}$  is a differentiable Lyapunov function then  $x_e$  is the only critical point of  $V$ .*

*Proof.* Suppose  $V$  has another critical point,  $x_c$ , in the domain of attraction. By the definition of a Lyapunov function, we must have  $\mathcal{L}_f V(x_c) = 0$ . This contradicts the fact that if  $x \neq x_e$ ,  $\mathcal{L}_f V(x) < 0$ .



# Morse Lemma

## Theorem (Morse Lemma)

Let  $p \in \mathcal{M}$  be a nondegenerate critical point of a smooth function  $V : \mathcal{M} \rightarrow \mathbb{R}$ . There exists a local coordinate system  $\{x_i\}_1^n$  in a nbhd.  $\mathcal{N} \subseteq \mathcal{M}$  of  $p$  with  $x_i(p) = 0$  for all  $1 \leq i \leq n$  such that for  $x \in \mathcal{N}$ ,

$$V(x) = V(p) - x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2$$

where  $i = \text{ind}(V, p)$ .

## Corollary

Let  $p \in \mathcal{M}$  be an equilibrium point of  $(\mathcal{M}, \varphi)$  and  $V : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$  a Morse-Lyapunov function. There exists a local coordinate system  $\{x_i\}_1^n$  around  $p$  such that  $V$  is locally the canonical quadratic Lyapunov function

$$V(x) = \sum_{i=1}^n x_i^2$$

with  $\text{ind}(V, p) = 0$ .

# Level Sets of a Lyapunov Function

## Theorem (Deformation Lemma)

Let  $V : \mathcal{M} \rightarrow \mathbb{R}$  be a smooth function and  $a, b \in V(\mathcal{M})$  such that  $a < b$ . If  $\mathcal{M}_{a,b}$  is compact and does not contain critical points of  $V$  then  $\mathcal{M}_a$  is diffeomorphic to  $\mathcal{M}_b$ . Moreover,  $\mathcal{M}_a$  is a deformation retract of  $\mathcal{M}_b$ .

## Corollary

Let  $\mathcal{M}$  be a smooth Riemannian manifold. If  $\mathcal{M}$  contains a closed invariant asymptotically stable set, then for all  $a, b \in V(\mathcal{M})$ ,  $\mathcal{M}_a$  is diffeomorphic to  $\mathcal{M}_b$  and  $\mathcal{M}_a$  is a deformation retract of  $\mathcal{M}_b$  where  $V$  is a smooth Lyapunov function.

## Systems with Single Critical Points

## Theorem (Brown-Stallings Lemma)

*Let  $\mathcal{M}$  be a paracompact manifold such that every compact subset is contained in an open set diffeomorphic to a Euclidean space. Then  $\mathcal{M}$  itself is diffeomorphic to a Euclidean space.*

## Corollary

*Let  $\mathcal{M}$  be a paracompact manifold. The domain of attraction of an asymptotically stable equilibrium point is diffeomorphic to a Euclidean space.*

# Morse and Sontag Theorems

## Theorem (Morse Theorem)

*Let  $V : \mathcal{M} \rightarrow \mathbb{R}$  be a Morse function,  $p$  a critical point such that  $\text{ind}(V, p) = i$  and  $c = V(p)$ . If there exists  $\varepsilon > 0$  such that  $\mathcal{M}_{c-\varepsilon, c+\varepsilon}$  is compact and does not contain other critical points  $p$ , then  $\mathcal{M}_{c-\varepsilon} \cup e_i$  is a deformation retract of  $\mathcal{M}_{c+\varepsilon}$  where  $e_i$  is an  $i$ -cell.*

## Theorem (Sontag Theorem)

*Let us consider the dynamical system  $(\mathcal{M}, \varphi)$  with an equilibrium point  $x_e \in \mathcal{M}$ . Suppose that  $x_e$  is asymptotically stable. Then the domain of attraction of  $x_e$ , given by*

$$\mathcal{A} = \left\{ x \in \mathcal{M} : \lim_{t \rightarrow \infty} \varphi(t, x) = x_e \right\},$$

*is contractible.*

# Systems with Multiple Critical Points

# Morse Theorem – (Third Version)

## Theorem (Morse Theorem)

*If  $V : \mathcal{M} \rightarrow \mathbb{R}$  is a Morse function such that  $\mathcal{M}_a$  is compact for each  $a \in \mathbb{R}$  then  $\mathcal{M}$  has the homotopy type of a CW-complex with one  $i$ -cell for each critical point of index  $i$ .*

## Corollary

*Suppose that the dynamical system  $(\mathcal{M}, \varphi)$  has several equilibria  $(x_1, \dots, x_k)$ . If there exists a Morse-Lyapunov function  $V : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$  then  $\{x_1, \dots, x_k\}$  is a retract of the domain of attraction.*

## Proposition (Reeb Theorem)

*Suppose that  $\mathcal{M}$  is compact without boundary. If  $V : \mathcal{M} \rightarrow \mathbb{R}$  is a smooth function with only two critical points, then  $\mathcal{M}$  is homeomorphic to the  $n$ -sphere  $\mathbb{S}^n$ .*

# Morse Inequalities

## Theorem (Morse Inequalities)

Let  $m_k$  be the number of critical points of a Morse function  $V$  with index  $k$ . Then, we have

$$\begin{aligned} b_k &\leq m_k, \quad \forall k, \\ \sum_{i=0}^j (-1)^{j-i} b_i &\leq \sum_{i=0}^j (-1)^{j-i} m_i \quad \forall j, \\ \chi(\mathcal{M}) &= \sum_k (-1)^k b_k = \sum_k (-1)^k m_k. \end{aligned}$$

The next corollary states a necessary condition for the existence of a Morse-Lyapunov function based on the Euler characteristic, which is a topological invariant.



# Existence of Morse-Lyapunov Functions

## Corollary

*Consider the dynamical system  $(\mathcal{M}, \varphi)$  with several equilibria  $(x_1, \dots, x_k)$ . If there exists a Morse-Lyapunov function  $V : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$  then  $\chi(\mathcal{M}) = k \geq b_0$ .*

*Proof.* If there exists a Morse-Lyapunov function  $V$ ,  $(x_1, \dots, x_k)$  are the only critical points with indices 0. Then, by the Morse inequalities,  $\chi(\mathcal{M}) = m_0 = k$  and  $b_0 \leq m_0 = k$ .

## Remark

*If  $\chi(\mathcal{M}) \neq k$  then there is no Morse-Lyapunov function for the dynamical system.*

