ECE 661: Nonlinear Systems
Spring 2021 | Homework #5

Question 1 .... 20 (main) + 0 (bonus) points

For the rotational motion of a rigid body in 3D-space example, it is desired to analyze the stability of an equilibrium of the form $(0, 0, z_0)$ where $z_0 \neq 0$. Set up a new set of coordinates such that the equilibrium under study is the origin of the new set. Define a suitable Lyapunov function such that the stability of the equilibrium can be established by applying Lyapunov stability theorem.

Solution:

Consider the change of coordinates

$$x \mapsto x, \quad y \mapsto y, \quad z \mapsto z_s := z - z_0,$$

under which the system dynamics becomes $(\dot{x}, \dot{y}, \dot{z}_s) = f(x, y, z_s)$ where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$f(x, y, z) = (ay(z + z_0), -bx(z + z_0), cxy)\,.$$

Now, consider the Lyapunov function candidate $V : \mathbb{R}^3 \to \mathbb{R}$

$$V(x, y, z_s) = bx^2 + ay^2 + \big(bcx^2 + 2acy^2 + abz_s(z_s + 2z_0)\big)^2$$

This is a sum-of-squares polynomial function of (x, y, z) and hence is positive definite. Further, we have $V(0, 0, 0) = 0$ and rote computation shows that

$$\mathcal{L}_f V(x, y, z_s) = 0.$$

Hence, the equilibrium $(x, y, z) = (0, 0, z_s)$ is stable.

Question 2 .... 20 (main) + 0 (bonus) points

Suppose a particle of mass m is moving in a smooth potential field. To simplify the problem, suppose the motion is one-dimensional. Let x denote the position coordinate of the particle, and let $\phi(x)$ denote the potential energy at x. If the only force acting on the particle is due to the potential, then the motion of the particle is described by

$$m\ddot{x} = -\phi'(x) =: f(x),$$

where the prime denotes differentiation with respect to x. Show that every local minimum of the function ϕ is a stable equilibrium.

Solution:

Let x_0 be a local minimum of the function ϕ. Notice that this is an equilibrium point since $\phi'(x_0) = 0$. Consider the Lyapunov function candidate

$$V(x) = \frac{1}{m}\left(\phi(x) - \phi(x_0)\right) + \frac{1}{2}\dot{x}^2.$$

Computing the time derivative of V along the trajectories of the system gives

$$\dot{V}(x) = \frac{1}{m}\phi'(x)\dot{x} + \dot{x}\ddot{x} = \frac{1}{m}\phi'(x)\dot{x} + \dot{x}\left(-\frac{\phi'(x)}{m}\right) = 0.$$

Hence, the equilibrium point x_0 is stable.

Question 3 . . . . 20 (main) + 0 (bonus) points

Consider the autonomous differential equation

$$\dot{x}(t) = f(x(t)),$$

and suppose f is a C^1 function such that $f(0) = 0$. Then there exists a C^1 matrix-valued function A such that

$$f(x) = A(x)x, \quad \forall x \in \mathbb{R}^n.$$

(a) [15 points] Show that if the matrix $A^\top(0)+A(0)$ is negative definite, then the origin is an exponentially stable equilibrium. More generally, show that if there exists a positive definite matrix P such that $A^\top(0)P + PA(0)$ is negative definite, then the origin is an exponentially stable equilibrium. (Hint: Consider the Lyapunov function candidate $V(x) = \|x\|^2$.)

Solution:
Let $V(x) = x^\top x$ so that

$$\dot{V}(x) = x^\top\left(A(x)^\top + A(x)\right)x.$$

Since $\dot{V}$ is a continuous function $\dot{V}(x) < 0$ in a suitable neighborhood of the origin whenever $A(0)^\top + A(0)$ is negative definite.
More generally, let $V(x) = x^\top Px$ so that

$$\dot{V}(x) = x^\top\left(A(x)^\top P + PA(x)\right)x.$$

By the same reasoning above, $\dot{V}$ is negative in a neighborhood of the origin whenever $A^\top(0)P + PA(0)$ is negative definite.

(b) [5 points] Extend the results in (a) to global stability.

Solution: If the matrix $A(x)^\top P + PA(x)$ is negative definite for all x, then $\dot{V}$ is negative over all of $\mathbb{R}^n$ and hence the origin is globally asymptotically stable.

Question 4 .... 20 (main) + 0 (bonus) points

Consider the system

$$\dot{x}_1 = x_1 + 2x_2^2, \quad \dot{x}_2 = 2x_1x_2 + x_2^2.$$

Using the Lyapunov function candidate

$$V(x) = x_1^2 - x_2^2,$$

show that 0 is an unstable equilibrium.

Solution:

Let $r = 1$ and $U = \{x \in B_r(0) : x_1^2 - x_2^2 \geq 0\}$. Notice that $V(y) > 0$ for arbitrarily small y and hence $U \neq \emptyset$. Indeed, U contains all points of the form (x_1, x_2) with $|x_1| \geq |x_2|$. Computing the Lie derivative of V along trajectories gives

$$\dot{V} = 2x_1(x_1 + 2x_2^2) - 2x_2(2x_1x_2 + x_2^2) = 2(x_1^2 - x_2^3) > 0, \quad \forall (x_1, x_2) \in U.$$

Question 5 .... 20 (main) + 0 (bonus) points

Given a finite collection of $n \times n$ matrices $A_1, \ldots, A_k$, define their *convex hull* S as

$$S = \{A = \sum_{i=1}^{k} \lambda_i A_i : \lambda_i \geq 0, \ \forall i, \sum_{i=1}^{k} \lambda_i = 1\}.$$

(a) Suppose there exists a positive definite matrix P such that $A_i^\top P + PA_i$ is negative definite for each i between 1 and k. Show that every matrix in the set S is Hurwitz.

Solution:

Claim: A convex combination of positive definite matrices is positive definite.
Proof. Let $\{Q_i\}_{i=1}^k$ be a set of positive definite matrices and let $\{\lambda_i\}_{i=1}^k$ be nonnegative with $\sum_{i=1}^k \lambda_i = 1$. Let $x \in \mathbb{R}^n$ be arbitrary and consider the quadratic form

$$x^\top \left(\sum_{i=1}^{k} \lambda_i Q_i \right) x = \sum_{i=1}^{k} \lambda_i \left(x^\top Q_i x \right) > 0, \quad \forall x \neq 0.$$

Let $A = \sum_{i=1}^k \lambda_i A_i$ be a convex combination of A_i's. We have

$$A^\top P + PA = \sum_{i=1}^{k} \lambda_i \left(A_i^\top P + PA_i \right) =: -\sum_{i=1}^{k} \lambda_i Q_i =: Q < 0.$$

(b) Consider the differential equation

$$\dot{x}(t) = A(t)x(t), \quad \text{where } A(t) \in S, \quad \forall t \geq 0.$$

Show that 0 is an exponentially stable equilibrium of this system.

Solution:

Let $V(x) = x^\top Px$. We have

$$\dot{V} = x^\top (A(t)^\top P + PA(t)) \leq -x^\top Qx < 0,$$

where

$$Q = \arg\inf_t \left\{ \lambda_{\min}(Q(t)) : A(t)^\top P + PA(t) = -Q(t) \right\}.$$

Question 6 .... 20 (bonus) points

Show that a continuously differentiable map $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) \geq 0$ is radially unbounded if and only if it is proper (inverse images of compact sets under f are compact).

Solution:

"($\Rightarrow$)" Assume that $f(x) \xrightarrow{\|x\|\to\infty} \infty$. We want to show that for all compact $K \subseteq \mathbb{R}$, $f^{-1}(K) \subseteq \mathbb{R}^n$ is compact. Since f is continuous, the inverse images of closed sets under f are closed sets. It remains to show that if K is (closed and) bounded in $\mathbb{R}$ then $f^{-1}(K)$ is bounded on $\mathbb{R}^n$. Let $c = \max\{x \in K \subseteq \mathbb{R}\}$ and consider the set $f^{-1}([0, c])$. Notice that $f^{-1}(K) \subseteq f^{-1}([0, c]) = \{x \in \mathbb{R}^n : f(x) \leq c\}$. But since f is radially unbounded, there exists a $\delta \in \mathbb{R}$ such that whenever $x \in B_\delta(0)$, $f(x) > c$. That is to say, $f^{-1}(K) \subseteq f^{-1}([0, c]) \subseteq B_\delta(0)$. Hence $f^{-1}(K)$ is closed and bounded, i.e., compact, showing that f is proper.

"($\Leftarrow$)" Assume that f is proper so that $f^{-1}(K)$ is compact whenever $K \subseteq \mathbb{R}$ is. We want to show that $f(x) \xrightarrow{\|x\|\to\infty} \infty$. Argue ad absurdum. Suppose $f(x)$ does not approach ∞ as $\|x\| \to \infty$. This means that there exists $c \in \mathbb{R}$ such that $f(x) < c$ for all $x \in \mathbb{R}^n$. But then $f^{-1}([0, c]) = \mathbb{R}^n$, which is not compact. This contradicts the fact that f is proper. $\square$