Nonlinear Control Systems   Solutions

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Chapter 1 Solutions

1.1 Exercises

Exercise 1 Given $A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3$ with the pair (A, b) controllable and q the last row of $\mathcal{C}^{-1} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix}^{-1}$, show that

$$T = \begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix}$$

is nonsingular.

The calculation

$$T\mathcal{C} = \begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix} \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} qb & qAb & qA^2b \\ qAb & qA^2b & qA^3b \\ qA^2b & qA^3b & qA^4b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & qA^3b \\ 1 & qA^3b & qA^4b \end{bmatrix}$$

shows that $T\mathcal{C}$ is invertible. So $T = (T\mathcal{C})^{-1}\mathcal{C}^{-1}$ showing T is invertible as it is the product of two invertible matrices.

Exercise 2 With $A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3$ and $\det(sI - A) = s^3 + a_2 s^2 + a_1 s + a_0$ show, by direct computation, that

$$\frac{dx^*}{dt} = TAT^{-1}x^* + Tbu = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c} x^* + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b_c} u.$$

Hint: Show $TA = A_cT$ by using the Cayley-Hamilton theorem.

We need to show

$$TAT^{-1} = A_c$$

or

$$\begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix}$$

or

$$\begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix}$$

or

$$\begin{bmatrix} qA \\ qA^2 \\ qA^3 \end{bmatrix} = \begin{bmatrix} qA \\ qA^2 \\ -a_0 q - a_1 qA - a_2 qA^2 \end{bmatrix}$$

In constructing the control canonical form A_c for A the coefficients of its last row are from the characteristic equation of A given by

$$s^3 + a_2 s^2 + a_1 s + a_0.$$

By the Cayley-Hamilton theorem we have

$$A^3 + a_2 A^2 + a_1 A + a_0 I = 0_{3\times 3}$$

which immediately given

$$qA^3 = -a_0 q - a_1 qA - a_2 qA^2.$$

1.2 Problems

Problem 1 *Nonlinear Transformations*

(a) Let a nonlinear system given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ g_2(x_1,x_2) \end{bmatrix} u.$$

Using the statespace transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} T_1(x_1,x_2) \\ T_2(x_1,x_2) \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ \mathcal{L}_f(T_1) \end{bmatrix} = \begin{bmatrix} x_1 \\ \mathcal{L}_f(x_1) \end{bmatrix}$$

and appropriate feedback the system can be put in the linear form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

Solution

$$\frac{dx_1^*}{dt} = \frac{\partial T_1}{\partial x} f + u\frac{\partial T_1}{\partial x} g = \mathcal{L}_f(T_1) + u\mathcal{L}_g(T_1) = \mathcal{L}_f(T_1) = f_1(x_1,x_2)$$

as

$$\mathcal{L}_g(T_1) = \mathcal{L}_g(x_1) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ g_3(x_1,x_2,x_3) \end{bmatrix} = 0.$$

Further

$$\frac{dx_2^*}{dt} = \frac{\partial T_2}{\partial x} f + u\frac{\partial T_2}{\partial x} g = \mathcal{L}_f(f_1(x_1,x_2)) + u\mathcal{L}_g(f_1(x_1,x_2)) = \mathcal{L}_f(f_1) + u\mathcal{L}_g(f_1).$$

With

$$u = -\frac{\mathcal{L}_f(f_1) + w}{\mathcal{L}_g(f_1)}$$

we have

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

(b) Let a nonlinear system be given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_3(x_1,x_2,x_3) \end{bmatrix} u.$$

Using the statespace transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} T_1(x_1,x_2,x_3) \\ T_2(x_1,x_2,x_3) \\ T_3(x_1,x_2,x_3) \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ \mathcal{L}_f(T_1) \\ \mathcal{L}_f^2(T_1) \end{bmatrix} = \begin{bmatrix} x_1 \\ \mathcal{L}_f(x_1) \\ \mathcal{L}_f^2(x_1) \end{bmatrix}$$

and appropriate feedback the system can be put in the linear form

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w.$$

Solution

$$\frac{dx_1^*}{dt} = \frac{\partial T_1}{\partial x}f + u\frac{\partial T_1}{\partial x}g = \mathcal{L}_f(T_1) + u\mathcal{L}_g(T_1) = \mathcal{L}_f(T_1) = f_1(x_1, x_2)$$

as

$$\mathcal{L}_g(T_1) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ g_3(x_1, x_2, x_3) \end{bmatrix} = 0.$$

$$\frac{dx_2^*}{dt} = \frac{\partial T_2}{\partial x}f + u\frac{\partial T_2}{\partial x}g = \mathcal{L}_f(T_2) + u\mathcal{L}_g(T_2) = \mathcal{L}_f(T_1)$$

as

$$\mathcal{L}_g(T_2) = \mathcal{L}_g(f_1(x_1, x_2)) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ g_3(x_1, x_2, x_3) \end{bmatrix} = 0.$$

Finally

$$\frac{dx_3^*}{dt} = \mathcal{L}_f^2(T_1) + u\mathcal{L}_g\mathcal{L}_f^2(T_1).$$

With

$$u = -\frac{\mathcal{L}_f^2(T_1) + w}{\mathcal{L}_g\mathcal{L}_f^2(T_1)}$$

the equations in the x^* coordinate system are

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w.$$

Problem 2 *Tank Reactor* [1]

Problem 3 *Nonlinear Regulator for a Synchronous Generator* [2]

Problem 4 *Observer for a Predator-Prey Model* [3]
A nonlinear differential equation model for a predator-prey system is given by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} \alpha x_1 - \beta x_1 x_2 \\ \gamma x_1 x_2 - \lambda x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ -x_2 \end{bmatrix}}_{g(x)} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

$x_1 \geq 0$ is the prey population and $x_2 \geq 0$ is the predator population. The constants, $\alpha > 0$ and $\gamma > 0$ are the birth rates of prey and predator populations, respectively while the constants $\beta > 0$ and $\lambda > 0$ are the death rates of the prey and predator populations, respectively. The input $u \geq 0$ represents the rate at which humans can decimate the predator population (e.g., by hunting). The output y is the predator population while the prey population is considered too big to measure. Consider the nonlinear transformation

$$\begin{aligned} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} &= \begin{bmatrix} T_1(x_1, x_2) \\ T_2(x_1, x_2) \end{bmatrix} \triangleq \begin{bmatrix} \gamma x_1 + \beta x_2 - \alpha\ln(x_2) + c_1 \\ \ln(x_2) + c_2 \end{bmatrix} \\ y^* &\triangleq \ln(y) = \ln(x_2) \end{aligned}$$

where c_1, c_2 can be any arbitrary constants.

(a) Find the system equation in the x^* coordinates with $c_1 = c_2 = 0$.

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} &= \begin{bmatrix} \gamma & \beta - \alpha/x_2 \\ 0 & 1/x_2 \end{bmatrix}\begin{bmatrix} \alpha x_1 - \beta x_1 x_2 \\ \gamma x_1 x_2 - \lambda x_2 \end{bmatrix} + \begin{bmatrix} \gamma & \beta - \alpha/x_2 \\ 0 & 1/x_2 \end{bmatrix}\begin{bmatrix} 0 \\ -x_2 \end{bmatrix} u \\
&= \begin{bmatrix} \lambda(\alpha - \beta x_2) \\ \gamma x_1 - \lambda \end{bmatrix} + \begin{bmatrix} \alpha - \beta x_2 \\ -1 \end{bmatrix} u
\end{aligned}
$$

The inverse transformation is

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^*/\gamma - (\beta/\gamma)e^{x_2^*} - (\alpha/\gamma)x_2^* \\ e^{x_2^*} \end{bmatrix}
$$

so the equations become

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} &= \begin{bmatrix} \lambda\alpha - \lambda\beta e^{x_2^*} \\ x_1^* - \beta e^{x_2^*} - \alpha x_2^* - \lambda \end{bmatrix} + \begin{bmatrix} \alpha - \beta e^{x_2^*} \\ -1 \end{bmatrix} u \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} \lambda\alpha - \lambda\beta e^{x_2^*} \\ -\beta e^{x_2^*} - \alpha x_2^* - \lambda \end{bmatrix} + \begin{bmatrix} \alpha - \beta e^{x_2^*} \\ -1 \end{bmatrix} u
\end{aligned}
$$

with output

$$
y^* = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}.
$$

(b) Design an observer for x_1 with linear error dynamics that places the poles of the error system at $-2, -2$.

The pair

$$
c = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
$$

is observable. Let the observer be

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \hat{x}_1^* \\ \hat{x}_2^* \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_1^* \\ \hat{x}_2^* \end{bmatrix} + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \hat{x}_1^* \\ \hat{x}_2^* \end{bmatrix}\right) \\
&\quad + \begin{bmatrix} \lambda\alpha - \lambda\beta e^{x_2^*} \\ -\beta e^{x_2^*} - \alpha x_2^* - \lambda \end{bmatrix} + \begin{bmatrix} \alpha - \beta e^{x_2^*} \\ -1 \end{bmatrix} u.
\end{aligned}
$$

The error system is given by

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} x_1 - \hat{x}_1^* \\ x_2 - \hat{x}_2^* \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 - \hat{x}_1^* \\ x_2 - \hat{x}_2^* \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \hat{x}_1^* \\ \hat{x}_2^* \end{bmatrix}\right) \\
&= \begin{bmatrix} 0 & -\ell_1 \\ 1 & -\ell_2 \end{bmatrix}\begin{bmatrix} x_1 - \hat{x}_1^* \\ x_2 - \hat{x}_2^* \end{bmatrix}.
\end{aligned}
$$

The characteristic equation for the error system is

$$
\det\begin{bmatrix} s & \ell_1 \\ -1 & s + \ell_2 \end{bmatrix} = s^2 + \ell_2 s + \ell_1.
$$

Choose $\ell_2 = 2 + 2 = 4$ and $\ell_1 = 2 \cdot 2 = 4$ to place the poles of the error system at $-2, -2$.

(c) For any given reference input u_0 find all equilibrium points x_0, that is, the solutions to

$$
0 = f(x_0) + g(x_0)u_0.
$$

Explain why the only physically interesting equilibrium point is

$$
\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} (\lambda + u_0)/\gamma \\ \alpha/\beta \end{bmatrix}.
$$

The equilibrium points are solutions of

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha x_{01} - \beta x_{01}x_{02} \\ \gamma x_{01}x_{02} - \lambda x_{02} \end{bmatrix} + \begin{bmatrix} 0 \\ -x_{02} \end{bmatrix} u_0.$$

The first equation gives $x_{01} = 0$ oe $x_{02} = \alpha/\beta$. $x_{01} = 0$ (prey population) requires $x_{02} = 0$ or $u_0 = \lambda$. The equilibrium point $(x_{01}, x_{02}) = (0, 0)$ is not of interest as both popolutions are zero. Next we look at $x_{01}, u_0 = \lambda$, with x_{02} arbitrary. To consider this case we look at the linearization of the system about equilibrium point which is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix} = \begin{bmatrix} \alpha - \beta x_{02} & -\beta x_{01} \\ \gamma x_{01} & \gamma(x_{01} - \lambda) - u_0 \end{bmatrix}\begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix} + \begin{bmatrix} 0 \\ -x_{02} \end{bmatrix}(u - u_0).$$

Then substituting $x_{01} = 0, u_0 = \lambda$ with x_{02} arbitrary this becomes

$$\frac{d}{dt}\begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix} = \begin{bmatrix} \alpha - \beta x_{02} & 0 \\ 0 & -(\gamma + 1)\lambda \end{bmatrix}\begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix} + \begin{bmatrix} 0 \\ -x_{02} \end{bmatrix}(u - u_0).$$

The prey population $x_{01} = 0$ and based on reality cannot increase or decrease. As a consequence it must be that $\alpha - \beta x_{02} = 0$ or $x_{02} = \alpha/\beta$ and x_{02} must *stay* at this value. This is not physically reasonable. We reject this case as well.

With $x_{02} = \alpha/\beta$ it is required that $0 = \gamma x_{01} - \lambda - u_0$ or $x_{01} = (\lambda + u_0)/\gamma$.

(d) With

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \triangleq \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix}, \quad w \triangleq u - u_0, \quad \text{and} \quad \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} (\lambda + u_0)/\gamma \\ \alpha/\beta \end{bmatrix}$$

find the system equations in terms of z_1, z_2, and w. That is, show the equations can be written in the form

$$\frac{dz}{dt} = f^*(z) + g^*(z)u.$$

Explicitly give $f^*(z)$ and $g^*(z)$.

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} \alpha(z_1 + x_{01}) - \beta(z_1 + x_{01})(z_2 + x_{02}) \\ \gamma(z_1 + x_{01})(z_2 + x_{02}) - \lambda(z_2 + x_{02}) \end{bmatrix} + \begin{bmatrix} 0 \\ -(z_2 + x_{02}) \end{bmatrix}(w + u_0) \\ &= \begin{bmatrix} \alpha z_1 - \beta z_1 z_2 - \beta x_{01} z_2 - \beta x_{02} z_1 \\ \gamma z_1 z_2 + \gamma x_{02} z_1 + \gamma x_{01} z_2 - \lambda z_2 - u_0 z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -(z_2 + x_{02}) \end{bmatrix} w \end{aligned}$$

(e) With

$$\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} \triangleq \begin{bmatrix} z_1 \\ \mathcal{L}_{f'}(z_1) \end{bmatrix}$$

find the statespace representation in the z^* coordinates. Choose feedback of the form $w = \mu(z) + \beta(z)u$ so that the system dynamics in z^* are linear given by

$$\frac{d}{dt}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w.$$

With the state feedback

$$w = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix}$$

find the values of the gains k_1, k_2 such that the closed-loop poles of the z^* system are $-1, -1$.

$$\begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} \triangleq \begin{bmatrix} z_1 \\ \mathcal{L}_{f'}(z_1) \end{bmatrix} = \begin{bmatrix} z_1 \\ \alpha z_1 - \beta z_1 z_2 - \beta x_{01} z_2 - \beta x_{02} z_1 \end{bmatrix}$$

$$\begin{aligned}
\frac{dz_2^*}{dt} &= \frac{\partial}{\partial z_1}\left(\alpha z_1 - \beta z_1 z_2 - \beta x_{01} z_2 - \beta x_{02} z_1\right)\frac{dz_1}{dt} + \frac{\partial}{\partial z_2}\left(\alpha z_1 - \beta z_1 z_2 - \beta x_{01} z_2 - \beta x_{02} z_1\right)\frac{dz_2}{dt} \\
&= \left(\alpha - \beta z_2 - \beta x_{02}\right)\left(\alpha z_1 - \beta z_1 z_2 - \beta x_{01} z_2 - \beta x_{02} z_1\right) + \\
&\quad \left(-\beta z_1 - \beta x_{01}\right)\left(\gamma z_1 z_2 + \gamma x_{02} z_1 + \gamma x_{01} z_2 - \lambda z_2 - u_0 z_2 - (z_2 + x_{02})w\right) \\
&= \left(\alpha - \beta z_2 - \beta x_{02}\right)\left(\alpha z_1 - \beta z_1 z_2 - \beta x_{01} z_2 - \beta x_{02} z_1\right) + \\
&\quad \left(-\beta z_1 - \beta x_{01}\right)\left(\gamma z_1 z_2 + \gamma x_{02} z_1 + \gamma x_{01} z_2 - \lambda z_2 - u_0 z_2\right) + \left(\beta z_1 + \beta x_{01}\right)\left(z_2 + x_{02}\right)w \\
&= f^*(z) + g^*(z)u.
\end{aligned}$$

(f) Draw a block diagram illustrating the interconnection of the observer, controller, and predator-prey model.

1.3 References

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[2] R. Marino, "An example of a nonlinear regulator," *IEEE Transactions on Automatic Control*, vol. AC-29, no. 3, pp. 276–279, March 1989.

[3] H. Keller, "Nonlinear observer design by transformation into a generalized observer canonical form," *International Journal of Control*, vol. 46, no. 6, pp. 1915–1930, June 1987.