Chapter 2

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Manifolds and Tangent Vectors

This chapter introduces manifolds and their tangent vectors through the use of examples. This chapter is not intended as a formal treatment of differential geometry; rather the intent is that the presentation here will provide the motivation for the abstract mathematical definitions of manifolds and tangent vectors given in formal treatments of the subject. For example, what is the difference between points in a manifold and their coordinates? Why is a tangent vector defined to be a derivative operator? What is the distinction between a vector tangent and its components?

This and the next chapter are written to give the reader enough background to tackle the more sophisticated treatments such as given in [1][2][3][4]. In particular, see Appendix A of [1], Chapter 2 of [2], Appendix A of [3] and Chapter 3 of [4]. A very nice introduction to abstract manifold theory is given in Chapter 7 of [5]. Further, a very readable introduction to Differential Geometry and General Relativity theory is the book [6].

Before getting into manifolds, we first look at the distinction between $\mathbb{R}^n$ and the Euclidean Space $\mathbf{E}^n$. $\mathbb{R}^n \triangleq \{(x_1, x_2, ..., x_n) |\ x_i \in \mathbb{R}\}$ is just the set of all ordered n-tuples of real numbers. On the other hand, $\mathbf{E}^n$ represents Euclidean n-space. That is, with $\mathbf{e}_1 \triangleq \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \mathbf{e}_2 \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T, ..., \mathbf{e}_n \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T$ the vector

$$p = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n \in \mathbf{E}^n$$

represents a point in space. This distinction is seemly quite fuzzy as the coordinates $(x_1, x_2, ..., x_n)$ of the point p and the point p itself given by

$$p = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

seem to be the same thing. In fact, because of this similarity the point $p = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_3\mathbf{e}_n \in \mathbf{E}^n$ is identified its coordinates $(x_1, x_2, ..., x_n)$. This identification of the coordinates in $\mathbb{R}^n$ with a point in Euclidean space $\mathbf{E}^n$ is called the *Cartesian* coordinate system. That is, n-tuples in $\mathbb{R}^n$ directly correspond to points in $\mathbf{E}^n$. This is not true in other coordinate systems. For example the cylindrical coordinates (ρ, θ, z) of a point in $\mathbf{E}^n$ are a 3-tuple $\mathbb{R}^3$; however, they do *not* represent the point $\rho\mathbf{e}_1 + \theta_2\mathbf{e}_2 + z\mathbf{e}_1$ with $\mathbf{e}_1 \triangleq \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \mathbf{e}_2 \triangleq \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, and $\mathbf{e}_3 \triangleq \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

As pointed out by Boothby [7] (p. 4), Euclidean space should be thought of as studied in high school geometry where definitions are made, theorems are proved, and so forth without the use of coordinates. This is what Euclid did. It was not until the invention of analytic geometry by Fermat and Descartes that coordinate systems were used. The point here is that $\mathbf{E}^n$ represents physical points (locations) in space while an element of $\mathbb{R}^n$ is just an n-tuple of real numbers. They only correspond (seem the same) when the Cartesian coordinate system is used.

$\mathbf{E}^n$ is a vector space with an inner product (e.g., see [8]). For example, with $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$ and $\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + \cdots + y_n\mathbf{e}_n$ their inner product is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \sum_{i=1}^{n} x_i y_i.$$

Using the inner product the norm of a vector is defined by

$$||\mathbf{x}|| \triangleq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

The distance between two points $d(\mathbf{x}, \mathbf{y})$ is defined by

$$d(\mathbf{x}, \mathbf{y}) \triangleq ||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Exercise 1 *Norm*

Show that $||\mathbf{x}|| = \sqrt{\sum_{i=1}^{n} x_i^2}$ has the properties of a norm. That is, (1) $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = 0$, (2) $||c\mathbf{x}|| = |c| \cdot ||\mathbf{x}||$, and (3) $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

1.1 Linear Manifolds

The basic idea of a manifold is that we have a subset of points (often a subset of $\mathbf{E}^n$) with various coordinate systems attached to it. Let's start with a concrete example.

Let $\mathcal{M} \subset \mathbf{E}^3$ be given by

$$\mathcal{M} \triangleq \left\{ \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T \in \mathbf{E}^3 | \ \mathbf{z} = \mathbf{a} + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \right\}$$

where $\mathbf{e}_1 \triangleq \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T, \mathbf{e}_2 \triangleq \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

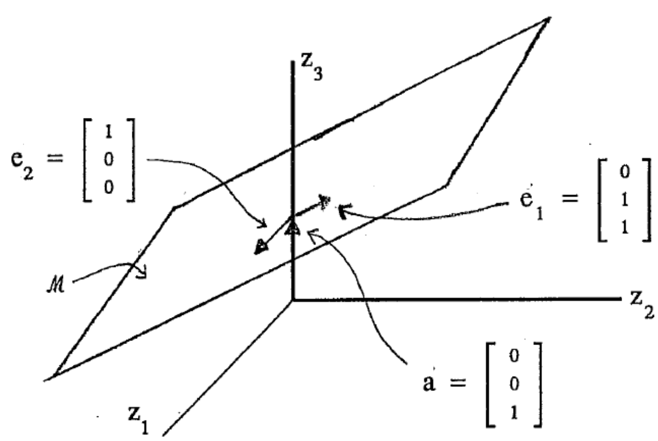

FIGURE 1.1. Linear manifold.

Exercise 2 *Implicit Definition of* $\mathcal{M}$

Show that $\mathcal{M}$ may also be defined by

$$\begin{aligned} \mathcal{M} &\triangleq \left\{ \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T \in \mathbf{E}^3 | \ \mathbf{n} \cdot \mathbf{z} = 1 \text{ where } \mathbf{n} = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T \right\} \\ &= \left\{ \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T \in \mathbf{E}^3 | \ -z_2 + z_3 = 0 \right\} \end{aligned}$$

Note that $\mathbf{n} \cdot \mathbf{e}_1 = 0, \mathbf{n} \cdot \mathbf{e}_2 = 0$ showing that $\mathbf{n}$ is orthogonal to $\mathbf{e}_1$ and $\mathbf{e}_2$.

The set $\mathcal{M}$ is an example of a *linear manifold.*

Definition 1 *Linear Manifold in* $\mathbf{E}^3$
A *linear manifold* in $\mathbf{E}^3$ is the set of vectors in $\mathbf{a}+\mathcal{S}$ where $\mathcal{S}$ is a subspace (sub vector space) of $\mathbf{E}^3$.

In the above example $\mathcal{S} \triangleq \left\{\mathbf{z} \in \mathbf{E}^3 | \ \mathbf{z} = x_1\mathbf{e}_2 + x_2\mathbf{e}_2 \text{ with } (x_1, x_2) \in \mathbb{R}^2\right\}$. For each $p \in \mathcal{M}$ there is a unique pair $(x_1, x_2) \in \mathbb{R}^2$ corresponding to p. The pair $(x_1, x_2) \in \mathbb{R}^2$ are the coordinates of the point $p \in \mathcal{M}$. The coordinates are in $\mathbb{R}^2$ while the points are in $\mathbf{E}^3$. We can explicitly write down the relationship between the coordinates of p and the point p itself. This is $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathcal{M} \subset \mathbf{E}^3$ given by

$$\mathbf{z}(x_1, x_2) = \mathbf{a} + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + 1 \end{bmatrix}.$$

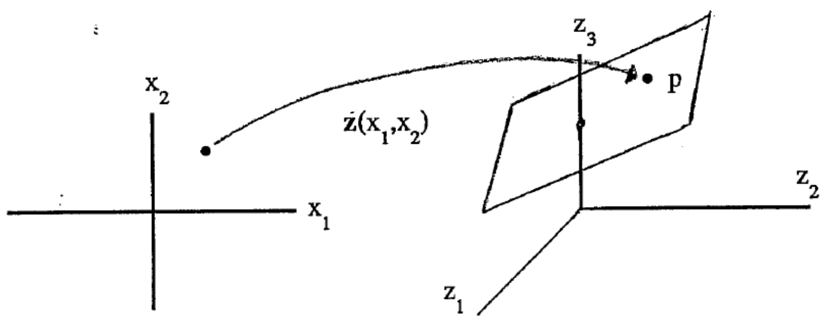

FIGURE 1.2. Coordinate map for a linear manifold.

Of course, (x_1, x_2) is not the only set of coordinates. Consider a different set of basis vectors for the subspace $\mathcal{S}$ given by

$$\mathbf{e}_1^* = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{e}_2^* = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}.$$

Then the above linear manifold is also given by

$$\mathcal{M} \triangleq \{\ \mathbf{z} \in \mathbf{E}^3 | \ \mathbf{z} = \mathbf{a} + x_1^*\mathbf{e}_1^* + x_2^*\mathbf{e}_2^* \text{ with } (x_1^*, x_2^*) \in \mathbb{R}^2$$

This is clear because

$$\begin{aligned} \mathbf{e}_1^* &= \mathbf{e}_1 + 2\mathbf{e}_2 \\ \mathbf{e}_2^* &= 3\mathbf{e}_1 + 4\mathbf{e}_2 \end{aligned}$$

so the span$\{\mathbf{e}_1^*, \mathbf{e}_2^*\}$ =span$\{\mathbf{e}_1, \mathbf{e}_2\}$.

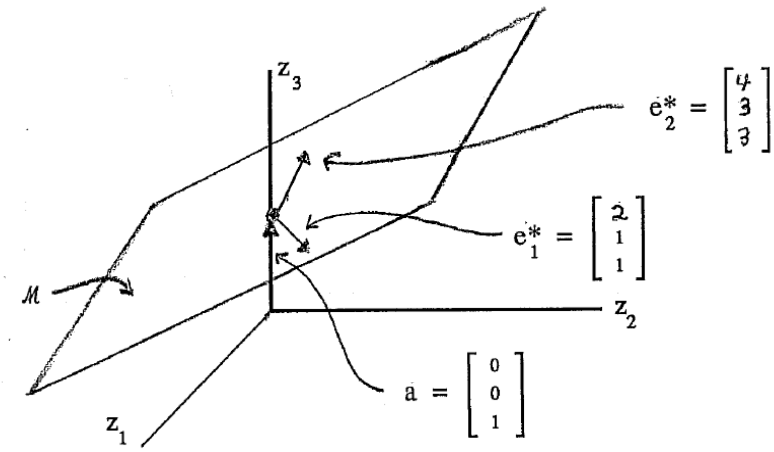

FIGURE 1.3. A second basis for the subspace $\mathcal{S}$.

For each point $p \in \mathcal{M}$ there is a unique pair of coordinates $(x_1^*, x_2^*) \in \mathbb{R}^2$ corresponding to this point. The function relating the coordinates (x_1^*, x_2^*) to the point $p \in \mathcal{M}$ is given by $\mathbf{z}(x_1^*, x_2^*) \mid \mathbb{R}^2 \to \mathcal{M} \subset \mathbf{E}^3$

$$\mathbf{z}(x_1^*, x_2^*) = \mathbf{a} + x_1^* \mathbf{e}_1^* + x_2^* \mathbf{e}_2^* = \begin{bmatrix} 2x_1^* + 4x_2^* \\ x_1^* + 2x_2^* \\ x_1^* + 3x_2^* + 1 \end{bmatrix}.$$

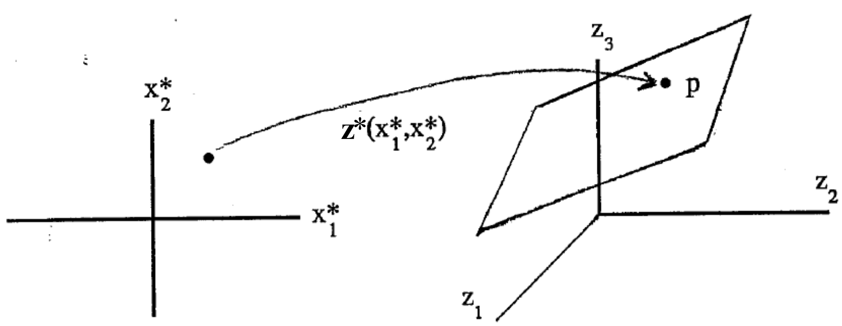

FIGURE 1.4. A second coordinates system for the linear manifold $\mathcal{M}$.

Change of Coordinates

What is the relationship between the two sets of coordinates for the above linear manifold? We have

$$\begin{aligned} \mathbf{e}_1^* &= \mathbf{e}_1 + 2\mathbf{e}_2 \\ \mathbf{e}_2^* &= 3\mathbf{e}_1 + 4\mathbf{e}_2 \end{aligned}$$

with inverse

$$\begin{aligned} \mathbf{e}_1 &= -2\mathbf{e}_1^* + \mathbf{e}_2^* \\ \mathbf{e}_2 &= \frac{3}{2}\mathbf{e}_1^* - \frac{1}{2}\mathbf{e}_2^*. \end{aligned}$$

Then

$$x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 = x_1(-2\mathbf{e}_1^* + \mathbf{e}_2^*) + x_2(\frac{3}{2}\mathbf{e}_1^* - \frac{1}{2}\mathbf{e}_2^*) = (-2x_1 + \frac{3}{2}x_2)\mathbf{e}_1^* + (x_1 - \frac{1}{2}x_2)\mathbf{e}_2^*$$

or

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

More generally let two bases for the subspace $\mathcal{S}$ be related by

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^* \\ \mathbf{e}_2^* \end{bmatrix}. \tag{1.1}$$

The matrix in (1.1) is the *change of basis* matrix. Writing this vertically we have

$$\begin{array}{cc} \mathbf{e}_1 & \mathbf{e}_2 \\ \| & \| \\ a_{11}\mathbf{e}_1^* & a_{21}\mathbf{e}_1^* \\ + & + \\ a_{12}\mathbf{e}_2^* & a_{22}\mathbf{e}_2^* \end{array}$$

so that

$$\begin{array}{cc} x_1\mathbf{e}_1 & x_2\mathbf{e}_2 \\ \| & \| \\ a_{11}x_1\mathbf{e}_1^* & a_{21}x_2\mathbf{e}_1^* \\ + & + \\ a_{12}x_1\mathbf{e}_2^* & a_{22}x_2\mathbf{e}_2^* \end{array}$$

By inspection of this last diagram it follows that

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1.2}$$

The matrix in (1.2) is the *change of coordinates* matrix. Notice that the change of coordinates matrix is the transpose of the change of basis matrix.

$\mathbf{E}^2$ as a manifold

$\mathbf{E}^2$ is itself a linear manifold. Let

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

be an orthogonal basis for $\mathbf{E}^2$. Any point $p \in \mathbf{E}^2$ may be represented by coordinates (x_1, x_2) corresponding the point $p = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. The Cartesian coordinate mapping[1] $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathbf{E}^2$ is given by

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{E}^2.$$

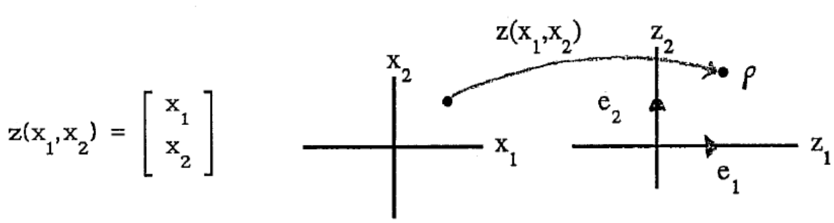

FIGURE 1.5. The manifold $\mathbf{E}^2$.

[1] The terminology *mapping* and *function* are used interchangably throughout this text.

$\mathbf{E}^2$ is a linear manifold because $\mathbf{E}^2 = \mathbf{0} + \mathcal{S}$ where

$$\mathcal{S} \triangleq \{\mathbf{z} \in \mathbf{E}^2 |\ \mathbf{z} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \text{ with } (x_1, x_2) \in \mathbb{R}^2\}.$$

Consider another set of basis vectors

$$\mathbf{e}_1^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{e}_2^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where any point $p \in \mathcal{M}$ is represented by the coordinates (x_1^*, x_2^*) corresponding to $x_1^*\mathbf{e}_1^* + x_2^*\mathbf{e}_2^*$. The coordinate mapping $\mathbf{z}^*(x_1^*, x_2^*) : \mathbb{R}^2 \to \mathbf{E}^2$ is then

$$\mathbf{z}^*(x_1^*, x_2^*) = \begin{bmatrix} x_1^* \\ x_1^* + x_2^* \end{bmatrix}.$$

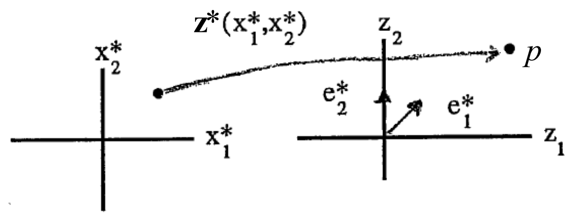

FIGURE 1.6. Another coordinate system for the manifold $\mathbf{E}^3$.

The two bases are related by

$$\begin{bmatrix} \mathbf{e}_1^* \\ \mathbf{e}_2^* \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} \text{ or } \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^* \\ \mathbf{e}_2^* \end{bmatrix}.$$

Next we find the relationship between the (x_1^*, x_2^*) and (x_1, x_2) coordinates. We compute

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = x_1(\mathbf{e}_1^* - \mathbf{e}_2^*) + x_2\mathbf{e}_2^* = x_1\mathbf{e}_1^* + (x_2 - x_1)\mathbf{e}_2^* = x_1^*\mathbf{e}_1^* + x_2^*\mathbf{e}_2^*$$

or in matrix form

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Exercise 3 *Linear Manifold*

Consider the linear manifold defined by

$$\mathcal{M} \triangleq \left\{ \mathbf{z} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T \in \mathbf{E}^3 |\ \mathbf{n} \cdot \mathbf{z} = -1 \text{ with } \mathbf{n} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T \right\}.$$

(a) Sketch this manifold in $\mathbf{E}^3$ to show it is a planar subset in $\mathbf{E}^3$.

(b) Show that $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathcal{M}$ given by

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + 1 \end{bmatrix}$$

is a coordinate mapping, i.e., $\mathbf{z}(x_1, x_2)$ is a $1 - 1$ mapping from $\mathbb{R}^2 \to \mathcal{M}$.

(c) Show that $\mathbf{z}'(x_1', x_2') : \mathbb{R}^2 \to \mathcal{M}$ given by

$$\mathbf{z}'(x_1', x_2') = \begin{bmatrix} x_1' + x_2' \\ x_1' \\ x_1' + 1 \end{bmatrix}$$

is a coordinate mapping, i.e., $\mathbf{z}'(x_1', x_2')$ is a $1-1$ mapping from $\mathbb{R}^2 \to \mathcal{M}$.

(d) Find the change of coordinates from (x_1, x_2) to (x_1', x_2') and its inverse.

Exercise 4 *Linear Manifold*
Consider the linear manifold defined by

$$\mathcal{M} \triangleq \mathbf{E}^3.$$

(a) Show that $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathbf{E}^3$ given by

$$\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_3 \end{bmatrix} = x_2\mathbf{e}_1 + x_1\mathbf{e}_2 + (x_1 + x_3)\vec{e}_3 = x_2\mathbf{e}_1 + x_1(\mathbf{e}_2 + \vec{e}_3) + e_3.$$

is a coordinate mapping, i.e., $\mathbf{z}(x_1, x_2)$ is a $1-1$ mapping from $\mathbb{R}^2 \to \mathbf{E}^3$.

(b) Show that $\mathbf{z}'(x_1', x_2', x_2') : \mathbb{R}^2 \to \mathbf{E}^3$ given by

$$\mathbf{z}'(x_1', x_2', x_3') = \begin{bmatrix} x_1' + x_2' \\ x_1' \\ x_1' + x_3' \end{bmatrix}$$

is a coordinate mapping, i.e., $\mathbf{z}'(x_1', x_2', x_3')$ is a $1-1$ mapping from $\mathbb{R}^3 \to \mathbf{E}^3$.

(d) Find the change of coordinates from (x_1, x_2, x_3) to (x_1', x_2', x_3') and its inverse.

1.2 Functions and Their Inverses

Before going onto general manifolds (not necessarily linear) we need to review some basic notions about functions.

Definition 2 *One-to-One (1-1) and Onto Functions*
Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^{n\mathbb{R}\mathbb{R}\mathbb{R}}$ be two sets and $f : X \to Y$ a function with domain X and range Y. Then f is $1-1$ if for $x_1 \in X$ and $x_2 \in X$ with $f(x_1) = f(x_2)$ then $x_1 = x_2$.
f is onto if for any $y \in Y$ there is at least one $x \in X$ such that $f(x) = y$.

Example 1 *Cartesian and Polar Coordinates*
Let

$$\begin{aligned} X &\triangleq \left\{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 > 0, 0 < x_2 < 2\pi\right\} \\ Y &\triangleq \left\{(y_1, y_2) \in \mathbb{R}^2 |\ \text{If } y_1 > 0 \text{ then } y_2 \neq 0\right\} \end{aligned}$$

and define the mapping from X to Y given by

$$f(x_1, x_2) \triangleq (x_1 \cos(x_2), x_1 \sin(x_2)) = (y_1, y_2).$$

This is just the coordinate transformation from polar coordinates to Cartesian coordinates.

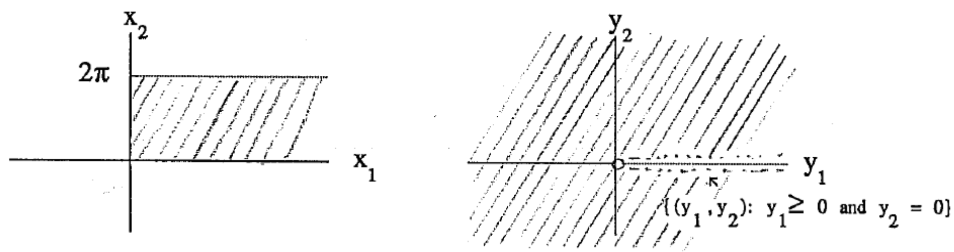

FIGURE 1.7. $X \triangleq \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 > 0, 0 < x_2 < 2\pi\}$, $Y \triangleq \{(y_1, y_2) \in \mathbb{R}^2 |$ If $y_1 > 0$ the $y_2 \neq 0\}$.

The inverse transformation is

$$f^{-1}(y_1, y_2) \triangleq \left(\sqrt{y_1^2 + y_2^2}, \tan^{-1}(y_1, y_2)\right) = (x_1, x_2)$$

where $\tan^{-1}(y_1, y_2) \triangleq \texttt{atan2}(y_2, y_1)$. With this particular domain X and range Y the function f is $1-1$ and onto. Note that both f and f^{-1} are continuous.

Definition 3 *Homeomorphism*
Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ be two sets and $f : X \to Y$ a function with domain X and range Y. If f is $1-1$ on X and onto Y with both f and f^{-1} continuous, then f and f^{-1} are *homeomorphisms.*

Definition 4 *Diffeomorphism*
Let $f : X \subset \mathbb{R}^n \to Y \subset \mathbb{R}^n$ be a homemorphic function. If both f and f^{-1} have continuous derivatives then f and f^{-1} are *diffeomorphisms.*

Example 2 *Cartesian and Polar Coordinates*
Let

$$\begin{aligned} X^* &\triangleq \left\{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 > 0, 0 \leq x_2 < 2\pi\right\} \\ Y^* &\triangleq \left\{(y_1, y_2) \in \mathbb{R}^2 |\ (y_1, y_2) \neq (0, 0)\right\} \end{aligned}$$

and define the mapping from X^* to Y^* given by

$$f(x_1, x_2) \triangleq (x_1 \cos(x_2), x_1 \sin(x_2)) = (y_1, y_2).$$

This is the coordinate transformation from polar coordinates to Cartesian coordinates.

f is $1-1$ and onto from X^* to Y^* so its inverse exists, i.e., f^{-1} is $1-1$ and onto from Y^* to X^*.

f is also continuous from X^* to Y^*. (What does continuity mean on the particular set of points (x_1, x_2) with $x_1 > 0$ and $x_2 = 0$?)

The inverse f^{-1} is *not* continuous from Y^* to X^*. Specifically, f^{-1} is *not* continuous on the set of points (y_1, y_2) with $y_1 > 0, y_2 = 0$. To see this consider the two sets of points in Y^* given by

$$\begin{aligned} y_n &= (y_{n1}, y_{n2}) \triangleq (1, 1/n) \text{ for } n = 1, 2, 3, ... \\ \bar{y}_n &= (\bar{y}_{n1}, \bar{y}_{n2}) \triangleq (1, -1/n) \text{ for } n = 1, 2, 3, ... \end{aligned}$$

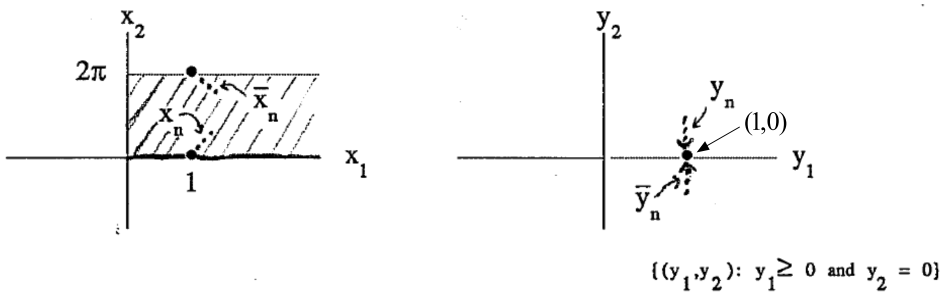

FIGURE 1.8. f^{-1} is not continous on the half-line $(y_1, 0)$, $y_1 \geq 0$.

Both of the sequences y_n and $\bar{y}_n$ converge to $(1, 0)$. However

$$\begin{aligned} x_n &= (x_{n1}, x_{n2}) = f^{-1}(y_n) = \left(\sqrt{1 + (1/n)^2}, \tan^{-1}(1, 1/n)\right) \to (1, 0) \\ \bar{x}_n &= (\bar{x}_{n1}, \bar{x}_{n2}) = f^{-1}(\bar{y}_n) = \left(\sqrt{1 + (-1/n)^2}, \tan^{-1}(1, -1/n)\right) \to (1, 2\pi). \end{aligned}$$

This shows that f^{-1} is not continuous at $(y_1, y_2) = (1, 0)$. A similar argument shows that f^{-1} is not continuous at the any point in the set $\left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_1 > 0, y_2 = 0\right\}$.

Exercise 5 *Cartesian and Polar Coordinates*
Let

$$\begin{aligned} X^{**} &\triangleq \left\{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 > 0, 0 < x_2 < 4\pi\right\} \\ Y^{**} &\triangleq \left\{(y_1, y_2) \in \mathbb{R}^2 |\ (y_1, y_2) \neq (0, 0)\right\} \end{aligned}$$

and define the mapping from X^{**} to Y^{**} given by

$$f((x_1, x_2)) \triangleq (x_1 \cos(x_2), x_1 \sin(x_2)) = (y_1, y_2).$$

Is this function $1 - 1$ from X^{**} to Y^{**}?
Is this function onto Y^{**}?
Does the function f have an inverse?

Single Variable Inverse Function Theorem

Given a function form $f(x) : \mathcal{D} \subset \mathbb{R}$ onto $\mathcal{U} \subset \mathbb{R}$ we would like to know when it has an inverse. I.e., is there a function $g(y) : \mathcal{U} \to \mathcal{D}$ such that

$$\begin{aligned} f(g(y)) &= y \\ g(f(x)) &= x? \end{aligned}$$

Let's consider an example to see what issues are involved.

Example 3 $y = x^2$
Let $\mathcal{D} \triangleq \mathbb{R}$ and $\mathcal{U} \triangleq \{y |\ y \geq 0\}$ with $f : \mathcal{D} \to \mathcal{U}$ given by

$$y = f(x) \triangleq x^2.$$

The function $f(x)$ is graphed on the left side of the figure below.

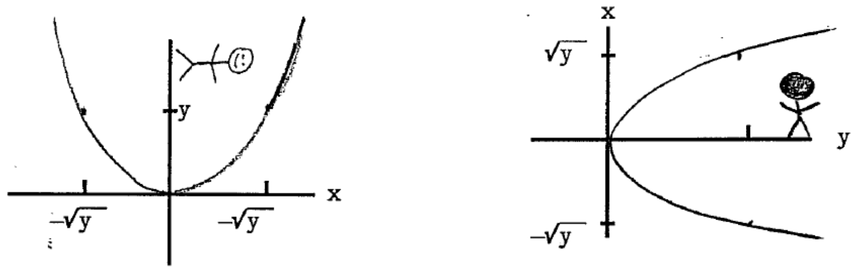

FIGURE 1.9. Left: $y = x^2$. Right: $x = \pm\sqrt{y}$.

The stick man on the left sees a relation (not function!) from $\mathcal{U} \triangleq \{y|\ y \geq 0\}$ to $\mathcal{D} \triangleq \mathbb{R}$ given by $g(y) = \pm\sqrt{y}$. That is, from the point of view of the stick man, the number y is taken is taken to both $\sqrt{y}$ and $-\sqrt{y}$. Thus g is not a function and so f does not have an inverse.

Example 4 $y = x^2$
Let $\mathcal{D} \triangleq \{x|\ x \geq 0\}$ and $\mathcal{U} \triangleq \{y|\ y \geq 0\}$ with $f : \mathcal{D} \to \mathcal{U}$ given by

$$y = f(x) \triangleq x^2.$$

The function $f(x)$ is graphed on the left side of the figure below.

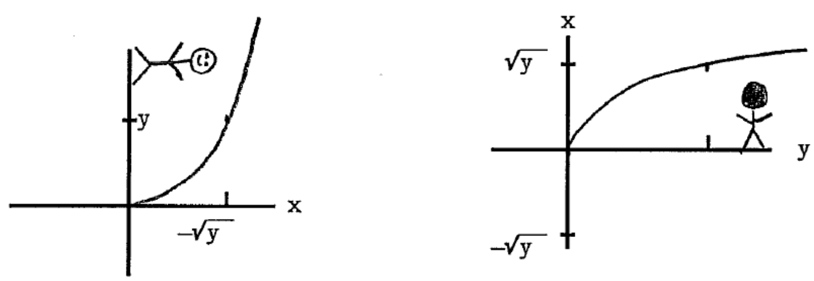

FIGURE 1.10. Left: $f : \mathcal{D} \to \mathcal{U}$ given by $f(x) = x^2$. $g : \mathcal{U} \to \mathcal{D}$ given by $g(y) = \sqrt{y}$.

The stick man on the left sees a function from $\mathcal{U} \triangleq \{y|\ y \geq 0\}$ to $\mathcal{D} \triangleq \{x|\ x \geq 0\}$ given by $g(y) = \sqrt{y}$. That is, from the point of view of the stick man, the number y is taken is to the unique point $\sqrt{y}$. Thus g is a function.

Theorem 1 *Inverse Function Theorem*
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Suppose that the derivative $f'(x) \neq 0$ on the interval $[a, b]$. Then $f(x)$ maps the interval $[a, b]$ in a one-to-one fashion to some interval $[\alpha, \beta]$ and there is a function g defined on $[\alpha, \beta]$ such that $g(f(x)) = x$ for all $x \in [a, b]$ and $f(g(y)) = y$ for all $y \in [\alpha, \beta]$.

Example 5 $y = \sin(x)$
Let $f(x) = \sin(x)$. Restrict f to the interval $[\pi/2, 3\pi/2]$ indicated by the solid line on the left hand side of the figure below. Let $g(y)$ defined on $[-1, 1]$ be the function graphed (solid line) on the right hand side of the figure below. Then $g(f(x)) = x$ for all $x \in [\pi/2, 3\pi/2]$ and $f(g(y)) = y$ for all $y \in [-1, 1]$

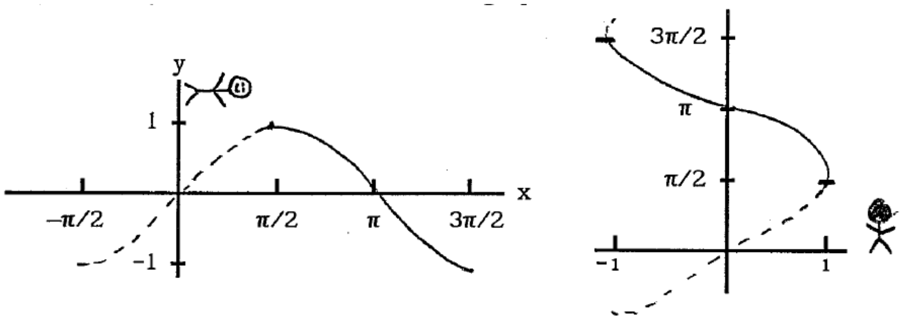

FIGURE 1.11. Left. $\sin(x) : [\pi/2, 3\pi/2] \to [-1, 1]$. Right. Inverse

Remark The conditions of the theorem are sufficient, but not necessary. For example, let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$ which is graphed in the figure below. Though $f'(0) = 0$ this function has an inverse $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) = \sqrt[3]{y}$.

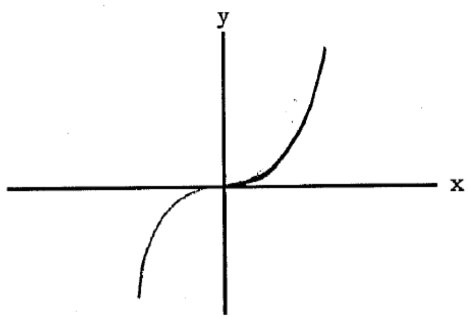

FIGURE 1.12. $f(x) = x^3$.

Multivariable Inverse Function Theorem

Let's start with a simple example.

Example 6 *Polar and Cartesian Coordinates*
Let

$$\begin{aligned}
\mathcal{D} &\triangleq \left\{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 > 0, 0 < x_2 < 2\pi\right\} \\
\mathcal{U} &\triangleq \left\{(y_1, y_2) \in \mathbb{R}^2 |\ \text{If } y_1 \geq 0 \text{ then } y_2 \neq 0\right\}
\end{aligned}$$

and define the function mapping $f : \mathcal{D} \to \mathcal{U}$ by

$$\begin{aligned}
y_1 &= f_1(x_1, x_2) = x_1 \cos(x_2) \\
y_2 &= f_2(x_1, x_2) = x_1 \sin(x_2).
\end{aligned}$$

This function has an inverse $g : \mathcal{U} \to \mathcal{D}$ given by

$$\begin{aligned}
x_1 &= g_1(y_1, y_2) = \sqrt{y_1^2 + y_2^2} \\
x_2 &= g_2(y_1, y_2) = \tan^{-1}(y_1, y_2).
\end{aligned}$$

We would like to know in general when does a function $f : \mathcal{D} \subset \mathbb{R}^2 \rightarrow \mathcal{U} \subset \mathbb{R}^2$ have an inverse? The next theorem provides conditions for such a function to have an inverse.

Theorem 2 *Inverse Function Theorem*
Let $f : X \subset \mathbb{R}^2 \rightarrow Y \subset \mathbb{R}^2$ given by

$$\begin{aligned} y_1 &= f_1(x_1, x_2) \\ y_2 &= f_2(x_1, x_2) \end{aligned}$$

or, more compactly,

$$y = f(x).$$

Let $x_0 = (x_{01}, x_{02}) \in \mathbb{R}^2$ and suppose

$$J_f(x_0) \triangleq \begin{bmatrix} \partial f_1(x_1,x_2)/\partial x_1 & \partial f_1(x_1,x_2)/\partial x_2 \\ \partial f_2(x_1,x_2)/\partial x_1 & \partial f_2(x_1,x_2)/\partial x_2 \end{bmatrix}_{|x_0=(x_{01},x_{02})}$$

is nonsingular, that is,

$$\det J_f(x_0) = \det \begin{bmatrix} \partial f_1(x_1,x_2)/\partial x_1 & \partial f_1(x_1,x_2)/\partial x_2 \\ \partial f_2(x_1,x_2)/\partial x_1 & \partial f_2(x_1,x_2)/\partial x_2 \end{bmatrix}_{|x_0=(x_{01},x_{02})} \neq 0.$$

Let $y_0 = (y_{01}, y_{02})$ be given by

$$\begin{aligned} y_{01} &= f_1(x_{01}, x_{02}) \\ y_{02} &= f_2(x_{01}, x_{02}). \end{aligned}$$

Then in a neighborhood of y_0 the function $y = f(x)$ has an inverse. Specifically, there exists an open neighborhood of $\mathcal{D}$ of x_0 and an open neighborhood $\mathcal{U}$ of y_0 such that f is an 1-1 and onto function from $\mathcal{D}$ to $\mathcal{U}$. Further, there exists functions $g_1(y_1, y_2), g_2(y_1, y_2)$ such that

$$\begin{aligned} x_{01} &= g_1(y_{01}, y_{02}) \\ x_{02} &= g_2(y_{01}, y_{02}) \end{aligned}$$

and for $x \in \mathcal{D}$

$$x_1 = g_1(f_1(x_1, x_2), f_2(x_1, x_2)) \tag{1.3}$$
$$x_2 = g_2(f_1(x_1, x_2), f_2(x_1, x_2)) \tag{1.4}$$

is a 1-1 and onto function from $\mathcal{D}$ to $\mathcal{U}$.
It also follows for all $y \in \mathcal{U}$ that

$$y_1 = f_1(g_1(y_1, y_2), g_2(y_1, y_2)) \tag{1.5}$$
$$y_2 = f_2(g_1(y_1, y_2), g_2(y_1, y_2)). \tag{1.6}$$

Exercise 6 The conditions of this theorem are sufficient, but not necessary. To see this consider the example

$$\begin{aligned} y_1 &= x_1^3 \\ y_2 &= x_2^3. \end{aligned}$$

Show that this has a global inverse, but its Jacobian is singular at $(0, 0)$.

Exercise 7 Show using (1.3) and (1.3) that for all $x \in \mathcal{D}$

$$\underbrace{\left[\begin{array}{cc} \partial g_1(y_1,y_2)/\partial y_1 & \partial g_1(y_1,y_2)/\partial y_2 \\ \partial g_2(y_1,y_2)/\partial y_1 & \partial g_2(y_1,y_2)/\partial y_2 \end{array}\right]_{y=f(x)}}_{J_g(y)_{y=f(x)}} \underbrace{\left[\begin{array}{cc} \partial f_1(x_1,x_2)/\partial x_1 & \partial f_1(x_1,x_2)/\partial x_2 \\ \partial f_2(x_1,x_2)/\partial x_1 & \partial f_2(x_1,x_2)/\partial x_2 \end{array}\right]}_{J_f(x)} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$$

Similarly, using (1.5) and (1.6) that for all $y \in \mathcal{U}$

$$\left[\begin{array}{cc} \partial f_1(x_1,x_2)/\partial x_1 & \partial f_1(x_1,x_2)/\partial x_2 \\ \partial f_2(x_1,x_2)/\partial x_1 & \partial f_2(x_1,x_2)/\partial x_2 \end{array}\right]_{x=g(y)} \left[\begin{array}{cc} \partial g_1(y_1,y_2)/\partial y_1 & \partial g_1(y_1,y_2)/\partial y_2 \\ \partial g_2(y_1,y_2)/\partial y_1 & \partial g_2(y_1,y_2)/\partial y_2 \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$$

Example 7 *Polar and Cartesian Coordinates*
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\begin{aligned} y_1 &= f_1(x_1,x_2) = x_1\cos(x_2) \\ y_2 &= f_2(x_1,x_2) = x_1\sin(x_2). \end{aligned}$$

The Jacobian matrix of f is

$$J_f(x_1,x_2) = \left[\begin{array}{cc} \partial f_1(x_1,x_2)/\partial x_1 & \partial f_1(x_1,x_2)/\partial x_2 \\ \partial f_2(x_1,x_2)/\partial x_1 & \partial f_2(x_1,x_2)/\partial x_2 \end{array}\right] = \left[\begin{array}{cc} \cos(x_2) & -x_1\sin(x_2) \\ \sin(x_2) & x_1\cos(x_2) \end{array}\right]$$

and

$$\det J_f(x_1,x_2) = x_1\cos^2(x_2) + x_1\sin^2(x_2) = x_1.$$

Around any point $x_0 = (x_{01}, x_{02})$ with $x_{01} \neq 0$ this function has an inverse. For example, take $x_0 = (1, 3\pi)$ which is mapped to $(-1, 0)$ where in some neighborhood of $(-1, 0)$ the inverse is given by[2]

$$\begin{aligned} x_1 &= g_1(y_1,y_2) = \sqrt{y_1^2 + y_2^2} \\ x_2 &= g_2(y_1,y_2) = \tan^{-1}(y_1,y_2) + 2\pi. \end{aligned}$$

In this *particular* example it turns out that f is invertible for all $x \in \mathcal{D}$ where

$$\mathcal{D} \triangleq \left\{(x_1,x_2) \in \mathbb{R}^2 |\ x_1 > 0, 2\pi < x_2 < 4\pi\right\}.$$

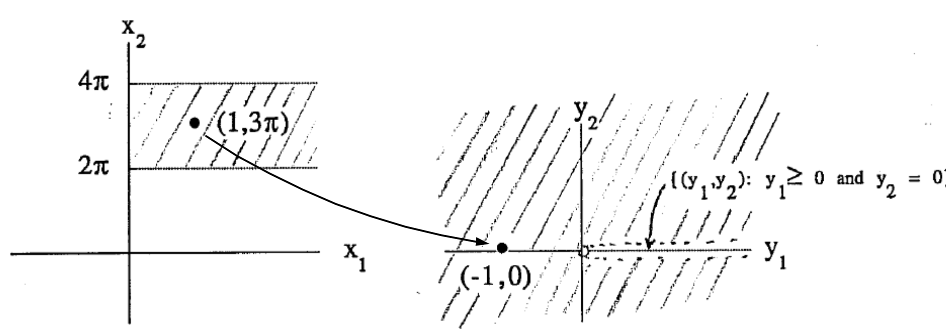

FIGURE 1.13. $\mathcal{D} \triangleq \left\{(x_1,x_2) \in \mathbb{R}^2 |\ x_1 > 0, 2\pi < x_2 < 3\pi\right\}$, $\mathcal{U} \triangleq \left\{(y_1,y_2) \in \mathbb{R}^2 |\ \text{If } y_1 > 0 \text{ the } y_2 \neq 0\right\}$

[2] Recall that we are taking $\theta = \tan^{-1}(y_1,y_2)$ to be in the interval $0 \leq \theta < 2\pi$.

1.3 Manifolds and Coordinate Systems

As far as the examples in this book, a manifold is an n dimensional subset of $\mathbf{E}^N$ $(n \le N)$ with a collection of coordinate systems attache to (defined on) it. Hopefully the following examples clarify what this means.

Example 8 *Spherical Coordinates for* $\mathbf{E}^3$
The standard spherical coordinate system is shown in Figure 1.14.

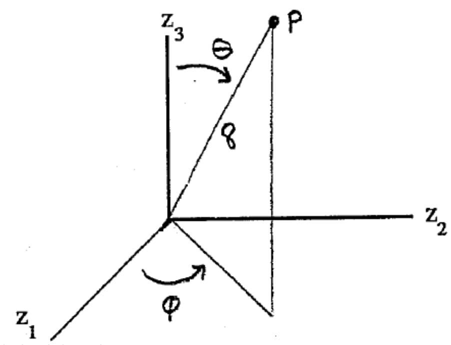

FIGURE 1.14. Spherical coordinate system.

With $x_1 = \rho, x_2 = \theta, x_3 = \varphi$, define the sets

$$\begin{aligned} \mathcal{D} &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\} \\ \mathcal{U} &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\}. \end{aligned}$$

The spherical coordinate system $\mathbf{z}(x_1, x_2, x_3) : \mathcal{D} \to \mathcal{U}$ is

$$\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3.$$

and is illustrated in Figure 1.15. $\mathbf{z}(x_1, x_2, x_3)$ is a one-to-one and onto function from $\mathcal{D}$ to $\mathcal{U}$. Further it is continuously differentiable on $\mathcal{D}$ and it has a continuously differentiable inverse $\mathbf{z}^{-1}(x_1, x_2, x_3) : \mathcal{U} \to \mathcal{D}$.

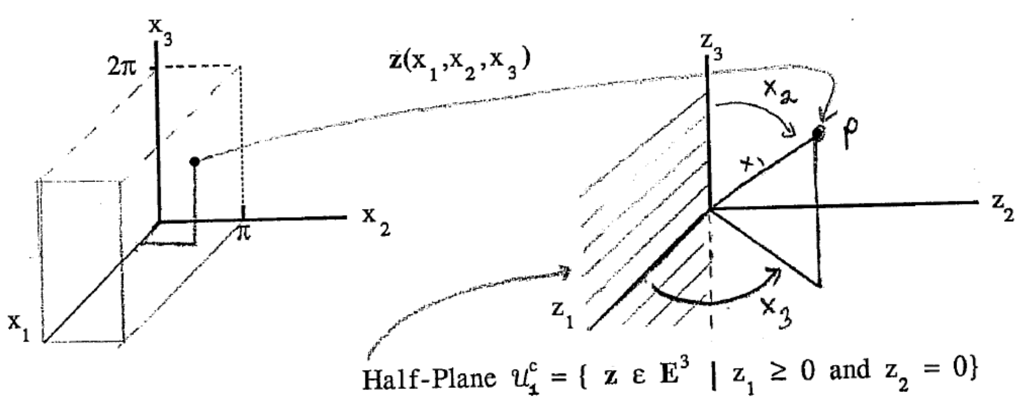

FIGURE 1.15. Spheical coordinate system as mapping from $\mathcal{D}$ to $\mathcal{U}$.

Note that this coordinate system mapping as defined in this example does not cover all of $\mathbf{E}^3$. In particular the points of $\mathbf{E}^3$ in $\mathcal{U}^c \triangleq \{\mathbf{z} \in \mathbf{E}^3 | \text{ If } z_1 \geq 0 \text{ the } z_2 = 0\}$ do not have spherical coordinates.[3] The reason for leaving this part of $\mathbf{E}^3$ out is so that $\mathbf{z}^{-1}(x_1, x_2, x_3)$ will be a continuously differentiable function from $\mathcal{U} \to \mathcal{D}$. If the points $\mathcal{U}^c$ are included then in any neighborhood of $\mathcal{U}^c \triangleq \{\mathbf{z} \in \mathbf{E}^3 | \text{ If } z_1 \geq 0 \text{ the } z_2 = 0\}$ there are points whose $x_3 = \varphi$ coordinate is close to 0 and points whose $x_3 = \varphi$ coordinate are close 2π and thus $\mathbf{z}^{-1}(x_1, x_2, x_3)$ would not be continuous (let alone differentiable).

Example 9 *Cylindrical Coordinates*

The standard cylindrical coordinates (ρ, φ, z) are illustrated in Figure 1.16.

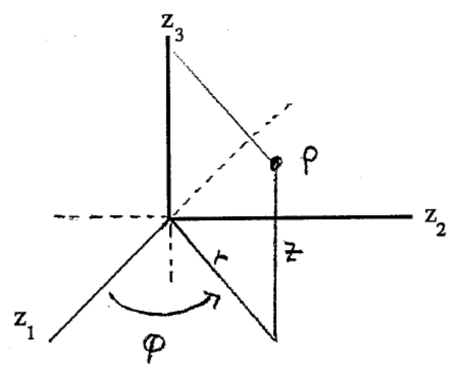

FIGURE 1.16. Cylindrical coordinates.

With $\bar{x}_1 = r, \bar{x}_2 = \varphi, \bar{x}_3 = z$, define the sets

$$\begin{aligned}
\mathcal{D} &= \left\{(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 | \ \bar{x}_1 > 0, 0 < \bar{x}_2 < 2\pi, -\infty < \bar{x}_3 < \infty\right\} \\
\mathcal{U} &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 | \text{ If } z_1 > 0 \text{ the } z_2 \neq 0\right\}.
\end{aligned}$$

The cylindrical coordinate system $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) : \mathcal{D} \to \mathcal{U}$ is

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \begin{bmatrix} \bar{x}_1 \cos(\bar{x}_2) \\ \bar{x}_1 \sin(\bar{x}_2) \\ \bar{x}_3 \end{bmatrix} \in \mathbf{E}^3.$$

and is illustrated in Figure 1.17. $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is a one-to-one and onto function from $\mathcal{D}$ to $\mathcal{U}$. Further it is continuously differentiable on $\mathcal{D}$ and it has a continuously differentiable inverse $\bar{\mathbf{z}}^{-1}(\bar{x}_1, \bar{x}_2, \bar{x}_3) : \mathcal{U} \to \mathcal{D}$.

[3] $\mathcal{U}^c$ denotes the complement of the set $\mathcal{U}$.

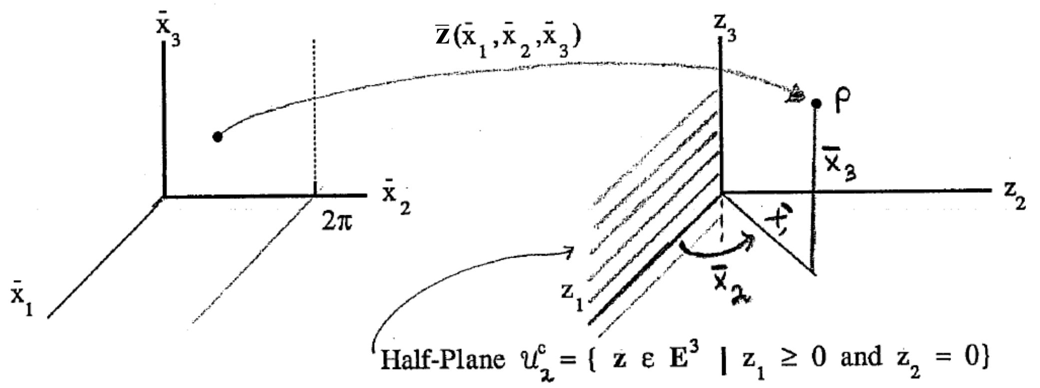

FIGURE 1.17. Cylindrical coordinate system as a mapping from $\mathcal{D}$ to $\mathcal{U}$.

Example 10 *Cartesian Coordinates*
The Cartesian coordinate system is shown in Figure 1.18.

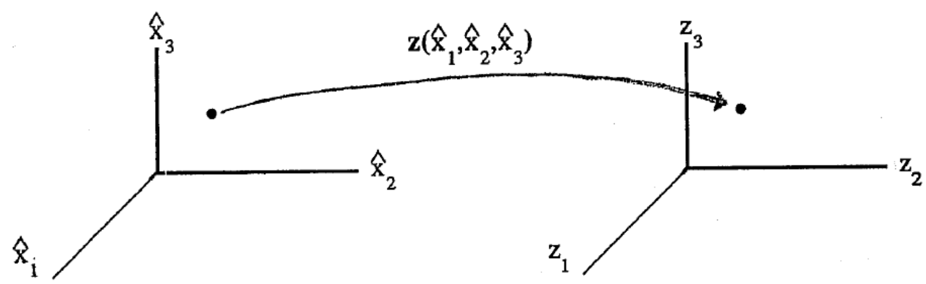

FIGURE 1.18. Cartesian coordinate system as mapping from $\mathcal{D}$ to $\mathcal{U}$.

The Cartesian coordinate system $\hat{\mathbf{z}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) : \mathcal{D} = \mathbb{R}^3 \to \mathcal{U} = \mathbb{R}^3$ is

$$\hat{\mathbf{z}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} \in \mathbf{E}^3.$$

These three examples above are meant to illustrate that a coordinate system for the manifold $\mathbf{E}^3$ consists of a set of coordinates $(x_1, x_2, x_3), (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, or $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ which lie in an open set of $\mathbb{R}^3$ with a mapping (function) that takes each 3-tuple of coordinates to a unique point p of $\mathbf{E}^3$.

The Manifold $\mathbf{S}^2$

The manifold $\mathbf{S}^2$ is a two dimensional subset of $\mathbf{E}^3$ defined by

$$\mathbf{S}^2 \triangleq \left\{ \mathbf{z} \in \mathbf{E^3} \middle| \ z_1^2 + z_2^2 + z_3^2 = 1 \right\}.$$

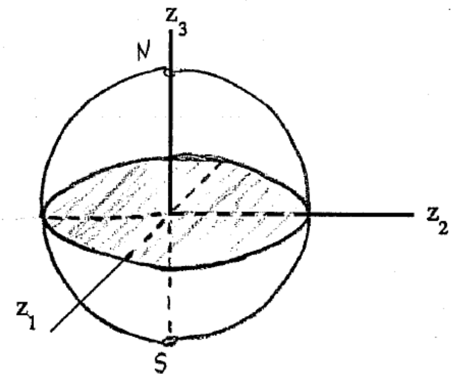

FIGURE 1.19.

Let's now look at some coordinate systems for $\mathbf{S}^2$.

Example 11 *Spherical Coordinate System for* $\mathbf{S}^2$
The spherical coordinates for $\mathbf{S}^2$ are illustrated in Figure 1.20.

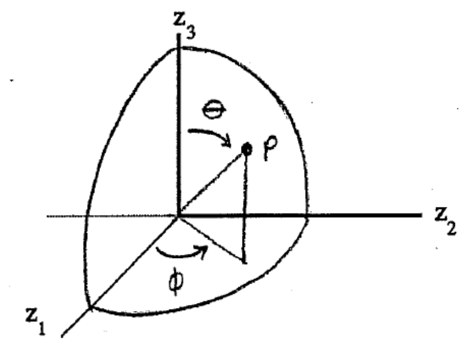

FIGURE 1.20. Spherical coordinates for $\mathbf{S}^2$.

With $x_1 = \theta$ and $x_2 = \varphi$, define the sets

$$\begin{aligned} \mathcal{R}_1 &= \left\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 < x_1 < \pi, 0 < x_2 < 2\pi\right\} \\ \mathcal{V}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ \text{If } z_1 > 0 \text{ then } z_2 \neq 0\right\}. \end{aligned}$$

The spherical coordinate system $\mathbf{z}(x_1, x_2) : \mathcal{R}_1 \to \mathcal{V}_1$ for $\mathbf{S}^2$ is

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} \sin(x_1)\cos(x_2) \\ \sin(x_1)\sin(x_2) \\ \cos(x_1) \end{bmatrix} \in \mathbf{S}^2.$$

and is illustrated in Figure 1.21. $\mathbf{z}(x_1, x_2)$ is a one-to-one and onto function from $\mathcal{R}_1$ to $\mathcal{V}_1$. Further it is continuously differentiable on $\mathcal{V}_1$ and it has a continuously differentiable inverse $\mathbf{z}^{-1}(z_1, z_2, z_3) : \mathcal{V}_1 \to \mathcal{R}_1$.

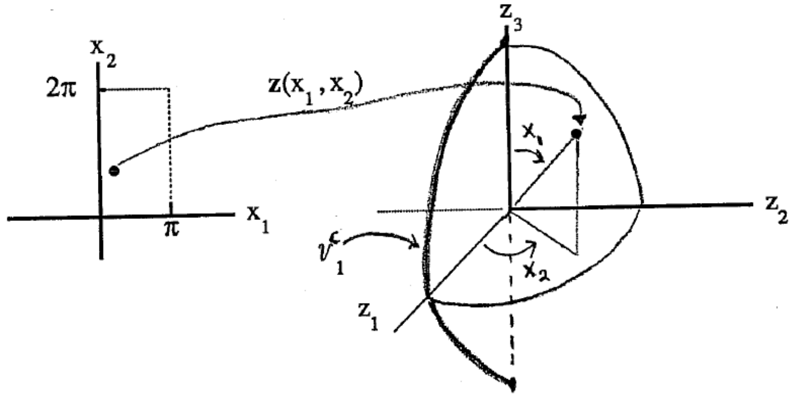

FIGURE 1.21. Spherical coordinate system for S^2.

This particular coordinate system leaves out the points $\mathcal{V}_1^c \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | \text{ If } z_1 > 0 \text{ then } z_2 = 0\right\}$. If this set is included then $\mathbf{z}^{-1}(x_1, x_2)$ would not be continuous.

Example 12 *A Second Spherical Coordinate System for* $\mathbf{S}^2$
Another spherical coordinate system for $\mathbf{S}^2$ is illustrated in Figure 1.22

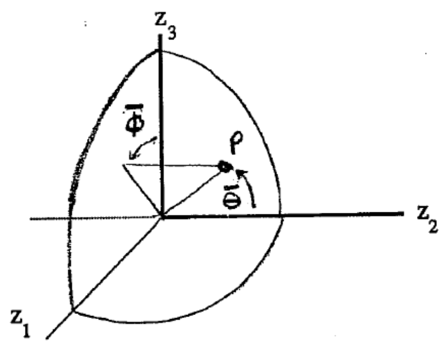

FIGURE 1.22. A second spherical coordinate system for $\mathbf{S}^2$.

With $\bar{x}_1 = \bar{\theta}$ and $\bar{x}_2 = \bar{\varphi}$ define the sets

$$\begin{aligned} \mathcal{R}_2 &= \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 | \ 0 < \bar{x}_1 < \pi, 0 < \bar{x} < 2\pi\right\} \\ \mathcal{V}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | \text{ If } z_1 > 0 \text{ the } z_2 \neq 0\right\}. \end{aligned}$$

This spherical coordinate system $\mathbf{z}(\bar{x}_1, \bar{x}_2) : \mathcal{R}_2 \to \mathcal{V}_2$ for $\mathbf{S}^2$ is

$$\mathbf{z}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \sin(\bar{x}_1)\sin(\bar{x}_2) \\ \cos(\bar{x}_1) \\ \sin(\bar{x}_1)\cos(\bar{x}_2) \end{bmatrix} \in \mathbf{S}^2.$$

and is illustrated in Figure 1.23. $\mathbf{z}(\bar{x}_1, \bar{x}_2)$ is a one-to-one and onto function from $\mathcal{R}_2$ to $\mathcal{V}_2$.

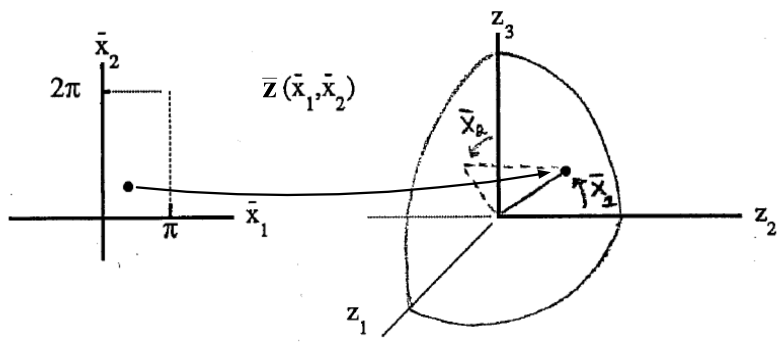

FIGURE 1.23. $\mathcal{R}_2 = \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 |\ 0 < \bar{x}_1 < \pi, 0 < \bar{x} < 2\pi\right\}$, $\mathcal{V}_2 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\}$.

Except for the north pole of $\mathbf{S}^2$, i.e., $\mathbf{z} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ all other points of $\mathbf{S}^2$ are either in $\mathcal{V}_2$ or in $\mathcal{V}_1$ from the previous example.

We now give some Cartesian coordinate systems for $\mathbf{S}^2$.

Example 13 *Northern Hemisphere Coordinate Patch*
Let

$$\begin{aligned} \mathcal{D}_1 &= \left\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 \leq x_1^2 + x_2^2 < 1\right\} \\ \mathcal{U}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}. \end{aligned}$$

Define the northern hemisphere coordinate map $\mathbf{z}(x_1, x_2) : \mathcal{D}_1 \rightarrow \mathcal{U}_1$ to be

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} \in \mathbf{S}^2. \tag{1.7}$$

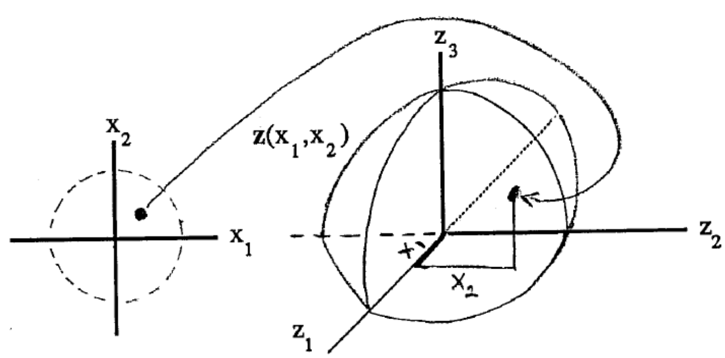

FIGURE 1.24. Northern hemisphere coordinates for $\mathbf{S}^2$.

This is a $1 - 1$ and onto mapping from $\mathcal{D}_1 \rightarrow \mathcal{U}_1$. The sets $\mathcal{D}_1$ and $\mathcal{U}_1$ along with the mapping (1.7) is called a *coordinate patch* or *coordinate chart.* All points of $\mathbf{S}^2$ with $z_3 > 0$ have a unique pair of northern hemisphere coordinates associated with them.

Example 14 *Southern Hemisphere Coordinate Patch*
Let

$$\begin{aligned}\mathcal{D}_2 &= \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 |\ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1\right\} \\ \mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}.\end{aligned}$$

Define the southern hemisphere coordinate map $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) : \mathcal{D}_2 \to \mathcal{U}_2$ to be

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ -\sqrt{1 - (\bar{x}_1^2 + \bar{x}_2^2)} \end{bmatrix} \in \mathbf{S}^2. \tag{1.8}$$

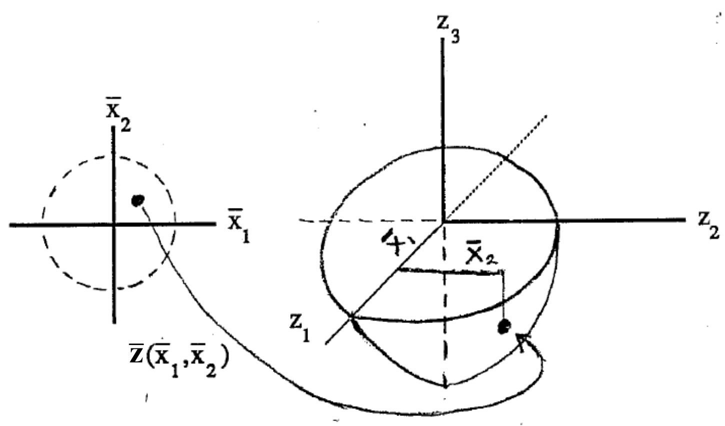

FIGURE 1.25. Southern hemisphere coordinates for $\mathbf{S}^2$.

This is a $1-1$ and onto mapping from $\mathcal{D}_2 \to \mathcal{U}_2$. The sets $\mathcal{D}_2$ and $\mathcal{U}_2$ along with the mapping (1.8) is called a *coordinate patch* or *coordinate chart*. All points of $\mathbf{S}^2$ with $z_3 < 0$ have a unique pair of southern hemisphere coordinates associated with them.

Example 15 *Eastern Hemisphere Coordinate Patch*
Let

$$\begin{aligned}\mathcal{D}_3 &= \left\{(x_1^*, x_2^*) \in \mathbb{R}^2 |\ 0 \le (x_1^*)^2 + (x_2^*)^2 < 1\right\} \\ \mathcal{U}_3 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}.\end{aligned}$$

Define the eastern hemisphere coordinate map $\mathbf{z}^*(x_1^*, x_2^*) : \mathcal{D}_3 \to \mathcal{U}_3$ to be

$$\mathbf{z}^*(x_1^*, x_2^*) = \begin{bmatrix} x_2^* \\ \sqrt{1 - ((x_1^*)^2 + (x_2^*)^2)} \\ x_1^* \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \tag{1.9}$$

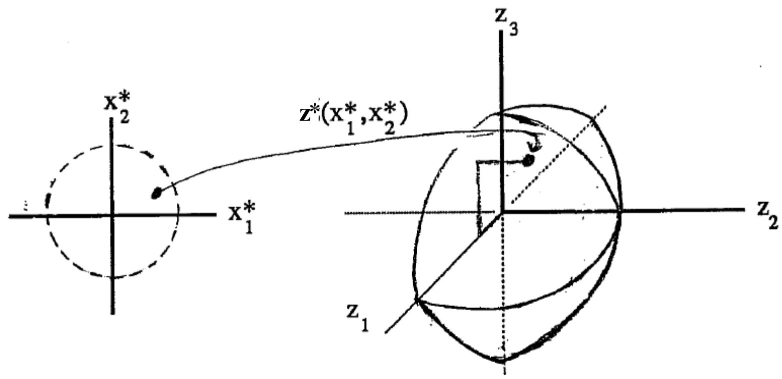

FIGURE 1.26. Eastern hemisphere coordinates for $\mathbf{S}^2$.

This is a $1-1$ and onto mapping from $\mathcal{D}_3 \to \mathcal{U}_3$. The sets $\mathcal{D}_3$ and $\mathcal{U}_3$ along with the mapping (1.9) is called a *coordinate patch* or *coordinate chart*. All points of $\mathbf{S}^2$ with $z_2 > 0$ have a unique pair of eastern hemisphere coordinates associated with them.

We can make similar coordinate patches (charts) for the remaining three hemispheres: $\mathcal{U}_4 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_2 < 0\right\}, \mathcal{U}_5 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_1 > 0\right\}$, and $\mathcal{U}_5 \triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_1 < 0\right\}$. Note that the six patches $\mathcal{U}_1$ through $\mathcal{U}_6$ cover the whole manifold $\mathbf{S}^2$. That is, each point of $\mathbf{S}^2$ is in at least one of the six patches (charts) and therefore each point has at least one set of coordinates associated with it.

How Mathematicians Describe Coordinate Systems

Mathematicians take a different perspective on defining coordinate systems. This is best explained by examples.

Example 16 *Spherical Coordinates for* $\mathbf{S}^2$
Recall with $x_1 = \theta, x_2 = \varphi$ and

$$\begin{aligned}
\mathcal{R}_1 &= \left\{(x_1, x_2) \in \mathbb{R}^3 |\ 0 < x_1 < \pi, 0 < x_2 < 2\pi\right\} \\
\mathcal{V}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\},
\end{aligned}$$

the spherical coordinate map $\mathbf{z}(x_1, x_2) : \mathcal{R}_1 \to \mathcal{V}_1$ for $\mathbf{S}^2$ given by

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} \sin(x_1)\cos(x_2) \\ \sin(x_1)\sin(x_2) \\ \cos(x_1) \end{bmatrix} \in \mathbf{S}^2.$$

This is illustrated in Figure 1.27. The inverse of $\mathbf{z}(x_1, x_2)$ is

$$\mathbf{z}^{-1}(z_1, z_2, z_3) = \left(\tan^{-1}(\sqrt{z_1^2 + z_2^2}/z_3),\ \tan^{-1}(z_1, z_2)\right) = (x_1, x_2).$$

This is one-to-one and onto function from $\mathcal{V}_1$ to $\mathcal{R}_1$.

However, mathematicians view the coordinate patch as a mapping from the manifold to the coordinate system and write it as

$$\varphi_1(p) = \varphi_1\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \left(\tan^{-1}(z_3, \sqrt{z_1^2 + z_2^2}),\ \tan^{-1}(z_1, z_2)\right) = (x_1, x_2).$$

This is just $\mathbf{z}^{-1}(z_1, z_2, z_3)$.

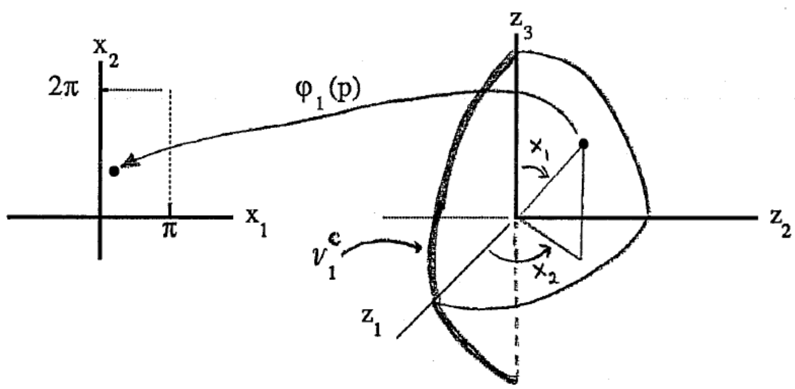

FIGURE 1.27. Spherical Coordinates for $\mathbf{S}^2$.

This makes a lot of sense because the points of the manifold are the fundamental objects of interest while the coordinates are used as a means to access the points conveniently. The mapping $\varphi_1{:}\mathcal{V}_1 \to \varphi_1(\mathcal{V}_1)$ takes any point $p \in \mathcal{V}_1 \subset S^2$ to its coordinates $\varphi_1(p)$ where $\mathcal{R}_1$ is denoted as $\varphi_1(\mathcal{V}_1)$.

Example 17 *Northern Hemisphere Coordinates for* $\mathbf{S}^2$
Recall with

$$\begin{aligned}
\mathcal{D}_2 &= \left\{(x_1, x_2) \in \mathbb{R}^3 |\ 0 \leq x_1^2 + x_2^2 < 1\right\} \\
\mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\},
\end{aligned}$$

the northern hemisphere coordinate map $\mathbf{z}(x_1, x_2) : \mathcal{D}_2 \to \mathcal{U}_2$ is

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} \in \mathbf{S}^2.$$

The inverse of this map $\varphi_2 : \mathcal{U}_2 \to \varphi_2(\mathcal{U}_1)$ is

$$\varphi_2(p) = \varphi_2\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (z_1, z_2) = (x_1, x_2)$$

where $\varphi_2(\mathcal{U}_2) = \mathcal{D}_2$.

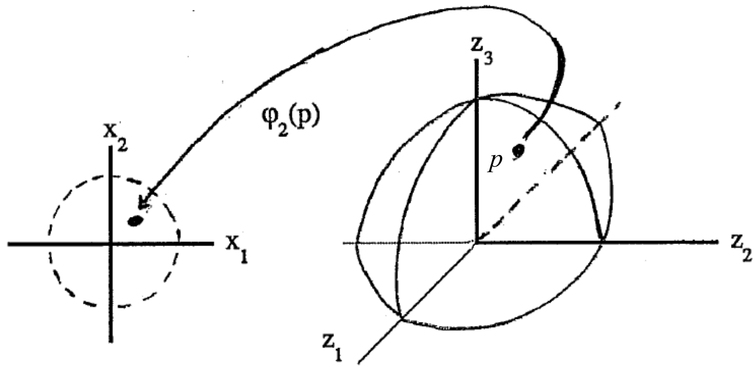

FIGURE 1.28. Northern hemisphere coordinates for $\mathbf{S}^2$.

Example 18 *The Manifold* $\mathbf{E}^3$ *with Spherical Coordinates*

As a last illustration we again look at $\mathbf{E}^3$ with the spherical coordinate system. With $x_1 = \rho, x_2 = \theta, x_3 = \varphi$, and

$$\begin{aligned}
\mathcal{R} &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\} \\
\mathcal{U} &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\},
\end{aligned}$$

the spherical coordinate system $\mathbf{z}(x_1, x_2, x_3) : \mathcal{R} \to \mathcal{U}$ is

$$\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3.$$

The inverse of this is $\varphi(z_1, z_2, z_3) : \mathcal{U} \to \varphi(\mathcal{U})$ is

$$\varphi(p) = \varphi\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \left(\sqrt{z_1^2 + z_2^2 + z_3^2}, \tan^{-1}\left(\sqrt{z_1^2 + z_2^2}/z_3\right), \tan^{-1}(z_1, z_2)\right) = (x_1, x_2, x_3)$$

where $\varphi(\mathcal{U}) = \mathcal{R}$.

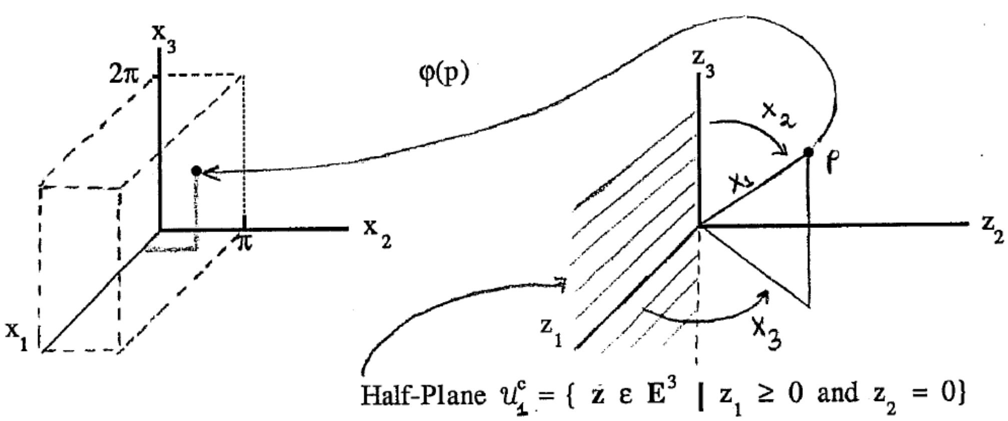

FIGURE 1.29. The manifold $\mathbf{E}^3$ with spherical coordinates.

Coordinate Transformations on Manifolds

We have seen any point p of a given manifold can have many different sets of coordinates attached to it. We now look at the relationship between different coordinates for the same point p of a manifold. Again, this is easiest to understand using examplse. A word about terminology: the particular coordinate patch (chart) being used is also referred to as the *local coordinates*.

Example 19 *Northern and Eastern Coordinate Patches on* $\mathbf{S}^2$
Patch 1 The northern hemisphere coordinate patch for $\mathbf{S}^2$ is $\varphi_1 : \mathcal{U}_1 \to \varphi_1(\mathcal{U}_1) \subset \mathbb{R}^2$ is

$$\varphi_1(p) = \varphi_1\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (z_1, z_2) = (x_1, x_2)$$

where

$$\begin{aligned} \mathcal{U}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\} \\ \varphi_1(\mathcal{U}_1) &= \left\{(x_1, x_2) \in \mathbb{R}^3 | \ 0 \le x_1^2 + x_2^2 < 1\right\}. \end{aligned}$$

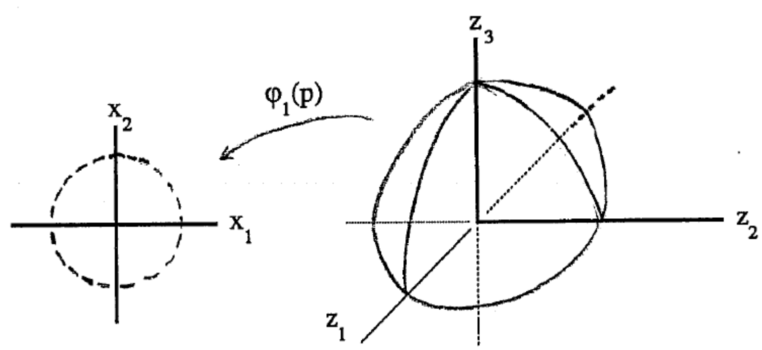

FIGURE 1.30. Northern coordinate patches on $\mathbf{S}^2$.

The inverse $\varphi_1^{-1} : \varphi_1(\mathcal{U}_1) \to \mathcal{U}_1$ is

$$\varphi_1^{-1}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

Patch 2 The eastern hemisphere coordinate patch for $\mathbf{S}^2$ is $\varphi_2 : \mathcal{U}_2 \to \varphi_2(\mathcal{U}_2) \subset \mathbb{R}^2$ is

$$\varphi_2(p) = \varphi_2\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (z_1, z_3) = (\bar{x}_1, \bar{x}_2)$$

where

$$\begin{aligned} \mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 | \ z_2 > 0\right\} \\ \varphi_2(\mathcal{U}_2) &= \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 | \ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1\right\}. \end{aligned}$$

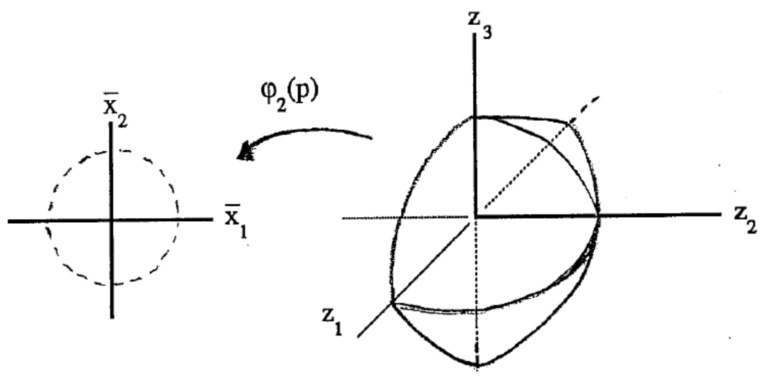

FIGURE 1.31. Eastern coordinate patches on $\mathbf{S}^2$.

The inverse $\varphi_2^{-1} : \varphi_2(\mathcal{U}_2) \to \mathcal{U}_2$ is

$$\varphi_2^{-1}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_1 \\ \sqrt{1-(\bar{x}_1^2+\bar{x}_2^2)} \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

The coordinate patch 1 covers the northern hemisphere $\mathcal{U}_1$ while coordinated patch 2 covers the eastern hemisphere $\mathcal{U}_2$. The intersection of these two sets is

$$\mathcal{U}_1 \cap \mathcal{U}_2 = \left\{ \mathbf{z} \in \mathbf{S}^2 \middle| \; z_2 > 0, z_3 > 0 \right\}$$

and is illustrated in Figure 1.32.

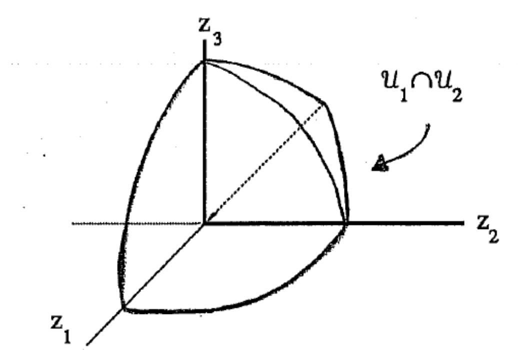

FIGURE 1.32. Intersection of the northern and eastern hemispheres.

Let $p \in \mathcal{U}_1 \cap \mathcal{U}_2$ and suppose we know the coordinates (x_1, x_2) of p in $\mathcal{U}_1$ (northern hemisphere) are known. What are the corresponding coordinates $(\bar{x}_1, \bar{x}_2)$ of p in $\mathcal{U}_2$ (eastern hemisphere)? Well the northern hemisphere coordinates (x_1, x_2) of p in $\mathcal{U}_1$ correspond to the point

$$\varphi_1^{-1}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1-(x_1^2+x_2^2)} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

This point has eastern hemisphere coordinates

$$\varphi_2(\varphi_1^{-1}(x_1,x_2)) = \varphi_2\left(\begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1-(x_1^2+x_2^2)} \end{bmatrix}\right) = (x_1, \sqrt{1-(x_1^2+x_2^2)}) = (\bar{x}_1, \bar{x}_2).$$

That is, the change of coordinates from (x_1, x_2) to $(\bar{x}_1, \bar{x}_2)$ is

$$\bar{x}_1 = x_1 \tag{1.10}$$

$$\bar{x}_2 = \sqrt{1-(x_1^2+x_2^2)}. \tag{1.11}$$

Exercise 8 *Coordinate Transformation from the Eastern Hemisphere to the Northern Hemisphere*
For $p \in \mathcal{U}_1 \cap \mathcal{U}_2$ show that the coordinate transformation from $\mathcal{U}_2$ to $\mathcal{U}_1$ is the inverse of the transformation given by (1.10) and (1.11).

How Mathematicians View Coordinate Transformations

As already pointed out, mathematicians view the coordinate system mapping φ to be from the manifold $\mathcal{M}$ to the coordinates in $\mathbb{R}^n$. We illustrate this point of view using the previous example of the change of coordinates between the northern and eastern coordinate systems.

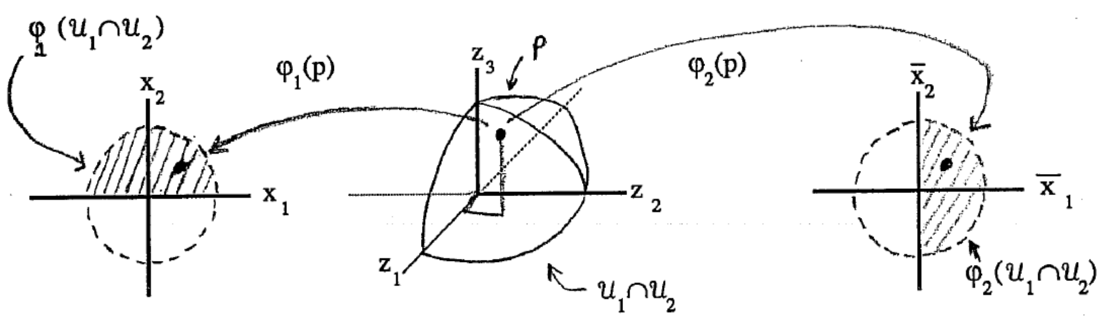

FIGURE 1.33. Change of coordinates between the northern hemisphere patch and the eastern hemisphere patch.

Now $\varphi_1(\mathcal{U}_1) = \left\{(x_1,x_2) \in \mathbb{R}^2 |\ 0 \le x_1^2 + x_2^2 < 1\right\}$ and

$$\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) = \left\{(x_1,x_2) |\ 0 \le x_1^2 + x_2^2 < 1, x_2 > 0\right\}$$

are all the (x_1, x_2) northern hemisphere coordinates of points $p \in \mathcal{U}_1 \cap \mathcal{U}_2$.
Similarly, $\varphi_2(\mathcal{U}_2) = \left\{(\bar{x}_1,\bar{x}_2) \in \mathbb{R}^2 |\ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1\right\}$ and

$$\varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2) = \left\{(\bar{x}_1,\bar{x}_2) |\ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1, \bar{x}_1 > 0\right\}$$

are all the $(\bar{x}_1, \bar{x}_2)$ eastern hemisphere coordinates of points $p \in \mathcal{U}_1 \cap \mathcal{U}_2$. Then $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$ is given by

$$\varphi_2 \circ \varphi_1^{-1}(x_1,x_2) = \varphi_2(\varphi_1^{-1}(x_1,x_2)) = \varphi_2\left(\begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1-(x_1^2+x_2^2)} \end{bmatrix}\right) = (x_1, \sqrt{1-(x_1^2+x_2^2)}) = (\bar{x}_1, \bar{x}_2).$$

This is of course the same expression we got before.
Keep in mind that $\varphi_2 \circ \varphi_1^{-1}(x_1, x_2)$ is a one-to-one map from $\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \subset \mathbb{R}^2$ onto $\varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2) \subset \mathbb{R}^2$ and is nothing more than fancy notation for the change of coordinates between the northern hemisphere and the eastern hemisphere.

Example 20 *Spherical to Cylindrical Coordinates on* $\mathbf{E}^3$

Patch 1 Spherical Coordinates on $\mathbf{E}^3$

With

$$\begin{aligned} \mathcal{U}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 \middle| \text{ If } z_1 > 0 \text{ then } z_2 \neq 0\right\} \\ \varphi_1(\mathcal{U}_1) &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \middle| \; x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\} \end{aligned}$$

the spherical coordinate patch $\varphi_1 : \mathcal{U}_1 \to \varphi_1(\mathcal{U}_1)$ is given by

$$\varphi_1(p) = \varphi_1\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \left(\sqrt{z_1^2 + z_2^2 + z_3^2}, \tan^{-1}\left(z_3, \sqrt{z_1^2 + z_2^2}\right), \tan^{-1}(z_1, z_2)\right) \in \mathbb{R}^3.$$

The inverse $\varphi_1^{-1} : \varphi_1(\mathcal{U}_1) \to \mathcal{U}_1$ is

$$\varphi_1^{-1}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3.$$

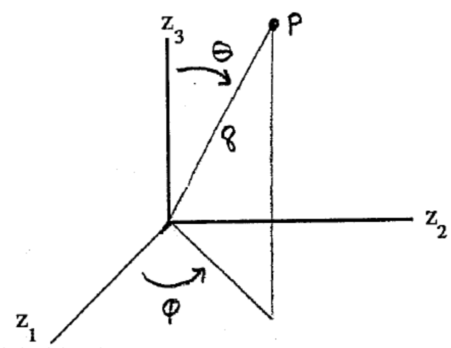

FIGURE 1.34. Spherical coordinates.

Patch 2 Cylindrical Coordinates on $\mathbf{E}^3$

With

$$\begin{aligned} \mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 \middle| \text{ If } z_1 > 0 \text{ the } z_2 \neq 0\right\} \\ \varphi_2(\mathcal{U}_2) &= \left\{(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 \middle| \; \bar{x}_1 > 0, 0 < \bar{x}_2 < 2\pi, -\infty < \bar{x}_3 < \infty\right\}, \end{aligned}$$

the cylindrical coordinate patch $\varphi_2 : \mathcal{U}_2 \to \varphi_1(\mathcal{U}_2)$ is given by

$$\varphi_2(p) = \varphi_2\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (\sqrt{z_1^2 + z_2^2}, \tan^{-1}(z_1, z_2), z_3) \in \mathbb{R}^3.$$

The inverse $\varphi_2^{-1} : \varphi_2(\mathcal{U}_2) \to \mathcal{U}_2$ is

$$\varphi_2^{-1}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \begin{bmatrix} \bar{x}_1 \sin(\bar{x}_2)\cos(\bar{x}_3) \\ \bar{x}_1 \sin(\bar{x}_2)\sin(\bar{x}_3) \\ \bar{x}_1 \cos(\bar{x}_2) \end{bmatrix} \in \mathbf{E}^3.$$

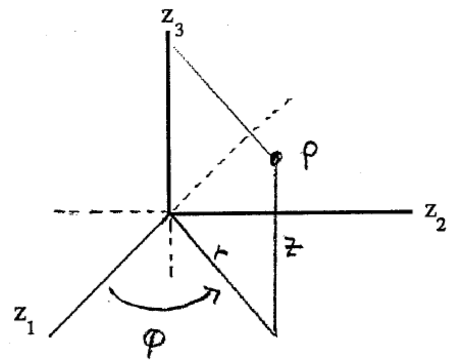

FIGURE 1.35. Cylindrical coordinates.

The change of coordinates from spherical to cylindrical are

$$
\begin{aligned}
&\varphi_2 \circ \varphi_1^{-1}((x_1, x_2, x_3)) \\
&= \varphi_2\left(\left[\begin{array}{l} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1\cos(x_2) \end{array}\right]\right) \\
&= \left(\sqrt{(x_1\sin(x_2)\cos(x_3))^2 + (x_1\sin(x_2)\sin(x_3))^2}, \tan^{-1}\left(x_1\sin(x_2)\cos(x_3),\ x_1\sin(x_2)\sin(x_3)\right), x_1\cos(x_2)\right) \\
&= (x_1\sin(x_2), x_3, x_1\cos(x_2)) \\
&= (\bar{x}_1, \bar{x}_2, \bar{x}_3).
\end{aligned}
$$

That is, the change of coordinates from spherical to cylindrical are

$$
\begin{aligned}
\bar{x}_1 &= x_1\sin(x_2) \\
\bar{x}_2 &= x_3 \\
\bar{x}_3 &= x_1\cos(x_2).
\end{aligned}
$$

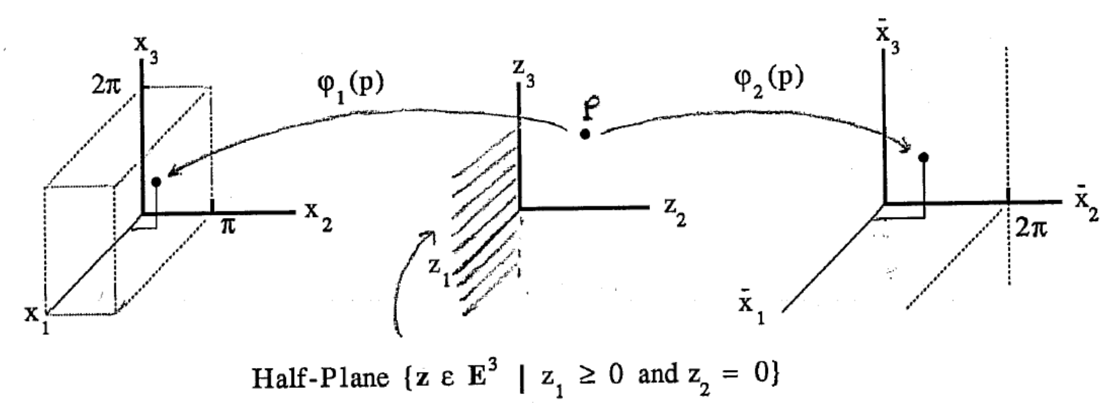

FIGURE 1.36. Change of coordinates from spherical to cylindrical on $\mathbf{E}^3$.

1.4 Tangent Vectors

We now look at the notion of a tangent vector to a manifold. Intuitively, the tangent vectors at a point p of a manifold give the possible directions one can move on the manifold at that point. We introduce tangent vectors using examples.

Tangent vectors on the manifold $\mathbf{S}^2$

Let the manifold be $\mathcal{M} = \mathbf{S}^2$, i.e., the unit sphere given by

$$\mathbf{S}^2 \triangleq \left\{ \mathbf{z} \in \mathbf{E^3} | \ z_1^2 + z_2^2 + z_3^2 = 1 \right\}.$$

The next three examples show how three different coordinate charts can be used to formulate a tangent vector.

Example 21 *Tangent Vectors on* $\mathbf{S}^2$ *using the Northern Hemisphere Coordinate Patch*
As previously shown with

$$\begin{aligned} \mathcal{D}_1 &= \left\{ (x_1, x_2) \in \mathbb{R}^2 | \ 0 \leq x_1^2 + x_2^2 < 1 \right\} \\ \mathcal{U}_1 &\triangleq \left\{ \mathbf{z} \in \mathbf{S}^2 | z_3 > 0 \right\}, \end{aligned}$$

the northern hemisphere coordinate map $\mathbf{z}(x_1, x_2) : \mathcal{D}_1 \to \mathcal{U}_1$ is

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ \sqrt{1 - (x_1^2 + x_2^2)} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

Let $(x_1(t), x_2(t))$ be a curve in $\mathcal{D}_1$. Then

$$\mathbf{z}(x_1(t), x_2(t)) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \sqrt{1 - (x_1^2(t) + x_2^2(t))} \end{bmatrix}$$

is a curve on $\mathcal{U}_1 \subset \mathbf{S}^2$.

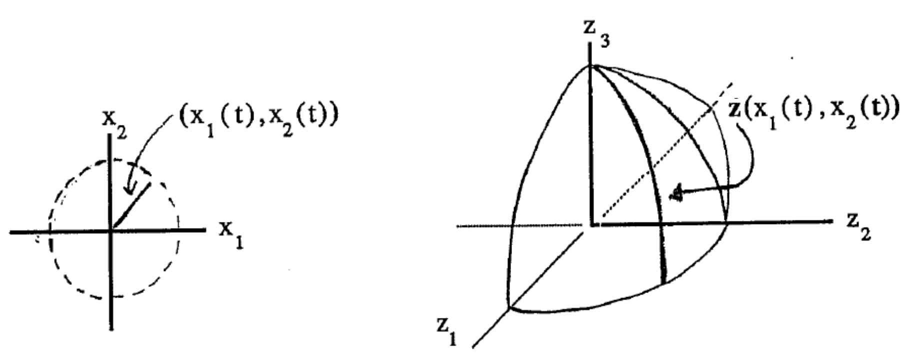

FIGURE 1.37. A curve on $\mathbf{S}^2$ usimg the northern hemisphere coordinate patch.

The tangent to the curve $(x_1(t), x_2(t))$ in $\mathcal{D}_1 \subset \mathbb{R}^2$ is $(dx_1(t)/dt, dx_2(t)/dt)$. The tangent to the curve

$\mathbf{z}(x_1(t), x_2(t))$ on $\mathcal{U}_1 \subset \mathbf{S}^2$ is

$$\begin{aligned}
\frac{d}{dt}\mathbf{z}(x_1(t), x_2(t)) &= \frac{\partial \mathbf{z}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{z}}{\partial x_2}\frac{dx_2}{dt} \\
&= \begin{bmatrix} 1 \\ 0 \\ \dfrac{-x_1(t)}{\sqrt{1-(x_1^2(t)+x_2^2(t))}} \end{bmatrix}\frac{dx_1}{dt} + \begin{bmatrix} 0 \\ 1 \\ \dfrac{-x_2(t)}{\sqrt{1-(x_1^2(t)+x_2^2(t))}} \end{bmatrix}\frac{dx_2}{dt} \\
&= \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x2}\frac{dx_2}{dt}.
\end{aligned}$$

The tangent vectors $\mathbf{z}_{x_1}$ and $\mathbf{z}_{x2}$ are defined on the northern hemisphere $\mathcal{U}_1$ or, in terms of local coordinates (x_1, x_2), $\mathbf{z}_{x_1}$ and $\mathbf{z}_{x2}$ are defined on $\mathcal{D}_1$. In particular, let

$$(x_1(t), x_2(t)) = (x_{01} + t, x_{02})$$

so that

$$\mathbf{z}_{x_1}(x_{01}, x_{02})$$

is the tangent vector to the curve $\mathbf{z}_{x_1}(x_{01} + t, x_{02})$ at $t = 0$. This is illustrated in Figure 1.38.

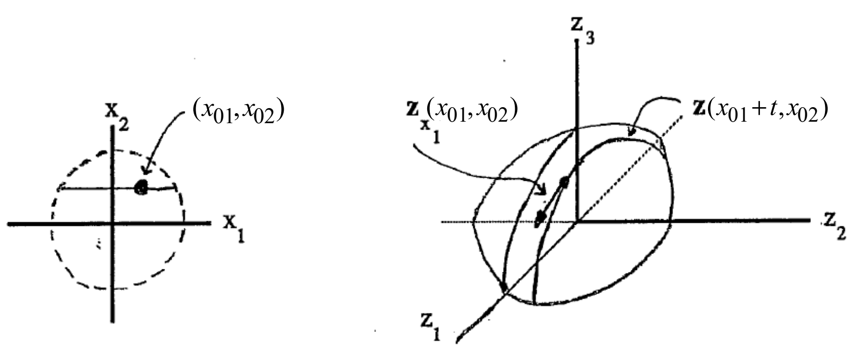

FIGURE 1.38. Tangent vector to the curve $\vec{z}(x_{01} + t, x_{02})$ at $t = 0$.

As similar interpretation holds for $\mathbf{z}_{x_2}(x_{01}, x_{02})$. Let $(x_1(t), x_2(t))$ in $\mathcal{D}_1$ be any curve goes through the coordinates (x_{01}, x_{02}) at $t = 0$, i.e., $(x_1(0), x_2(0)) = (x_{01}, x_{02})$. Then the tangent vector at $\mathbf{z}(x_{01}, x_{02})$ on $\mathbf{S}^2$ is

$$\left.\frac{d}{dt}\mathbf{z}(x_1(t), x_2(t))\right|_{t=0} = \mathbf{z}_{x_1}(x_{01}, x_{02}) \left.\frac{dx_1}{dt}\right|_{t=0} + \mathbf{z}_{x2}(x_{01}, x_{02}) \left.\frac{dx_2}{dt}\right|_{t=0}.$$

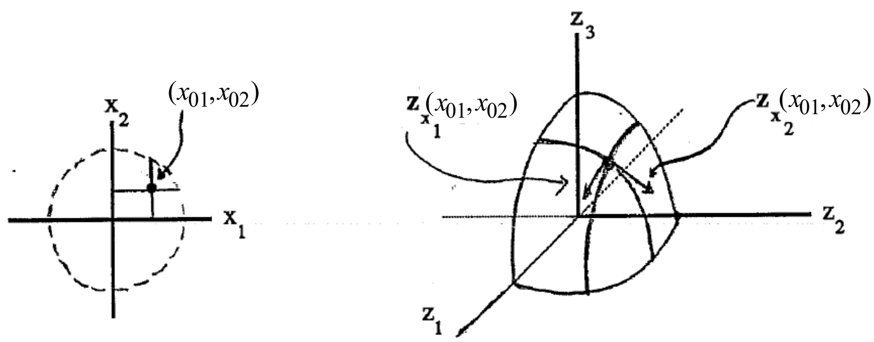

FIGURE 1.39. Basis vectors of the tangent space to $\mathbf{S}^2$ at $\vec{z}(x_{01}, x_{02})$

Any tangent vector to $\mathbf{S}^2$ at $\mathbf{z}(x_{01}, x_{02})$ is a linear combination of $\mathbf{z}_{x_1}(x_{01}, x_{02})$ and $\mathbf{z}_{x2}(x_{01}, x_{02})$. The *tangent space* at a point $p \in \mathbf{S}^2$ is the set of all tangent vectors at that point. In this example the tangent space at $\mathbf{z}(x_{01}, x_{02})$ is

$$\mathbf{T}_p(\mathbf{S}^2) = \left\{a_1\mathbf{z}_{x_1}(x_{01}, x_{02}) + a_2\mathbf{z}_{x_2}(x_{01}, x_{02})|\ (a_1, a_2) \in \mathbb{R}^2\right\}.$$

$\mathbf{T}_p(\mathbf{S}^2)$ is a two dimensional *vector space* with basis vectors $\mathbf{z}_{x_1}(x_{01}, x_{02}), \mathbf{z}_{x_2}(x_{01}, x_{02})$. The pair (a_1, a_2) are referred to as the *components* of the tangent vector $a_1\mathbf{z}_{x_1}(x_{01}, x_{02}) + a_2\mathbf{z}_{x_2}(x_{01}, x_{02})$.

Example 22 *Tangent Vectors on* $\mathbf{S}^2$ *using the Eastern Hemisphere Coordinate Patch*
As shown previously, with

$$\begin{aligned}\mathcal{D}_2 &= \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2|\ 0 \le \bar{x}_1^2 + \bar{x}_2^2 < 1\right\}\\ \mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2|z_3 > 0\right\},\end{aligned}$$

the eastern hemisphere coordinate map $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) : \mathcal{D}_2 \to \mathcal{U}_2$ is

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_2 \\ \sqrt{1 - (\bar{x}_1^2 + \bar{x}_2^2)} \\ \bar{x}_1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

The left side of Figure 1.40 shows a curve in $\mathcal{D}_2$ which maps to a curve on $\mathbf{S}^2$ shown on the right side of Figure 1.40.

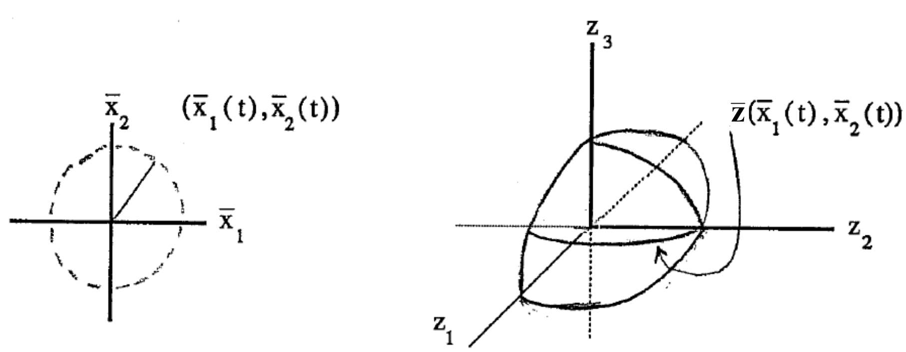

FIGURE 1.40. A curve on $\mathbf{S}^2$ using the eastern hemisphere coordinate patch.

More generally, for any curve $(\bar{x}_1(t), \bar{x}_2(t))$ in $\mathcal{D}_2$ the curve $\mathbf{z}(\bar{x}_1(t), \bar{x}_2(t))$ lies on the manifold $\mathbf{S}^2$ as illustrated in Figure 1.40. The tangent to this curve at $\bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t))$ is given by

$$\begin{aligned}
\frac{d}{dt}\bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t)) &= \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_1}\frac{d\bar{x}_1}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_2}\frac{d\bar{x}_2}{dt} \\
&= \begin{bmatrix} 0 \\ \dfrac{-\bar{x}_1(t)}{\sqrt{1-(\bar{x}_1^2(t)+\bar{x}_2^2(t))}} \\ 1 \end{bmatrix}\frac{d\bar{x}_1}{dt} + \begin{bmatrix} 1 \\ \dfrac{-\bar{x}_2(t)}{\sqrt{1-(\bar{x}_1^2(t)+\bar{x}_2^2(t))}} \\ 0 \end{bmatrix}\frac{d\bar{x}_2}{dt} \\
&= \bar{\mathbf{z}}_{\bar{x}_1}\frac{d\bar{x}_1}{dt} + \bar{\mathbf{z}}_{\bar{x}_2}\frac{d\bar{x}_2}{dt}.
\end{aligned}$$

The left side of Figure 1.41 shows two straight lines going through $(\bar{x}_{01}, \bar{x}_{02})$ and their corresponding curves on $\mathbf{S}^2$.

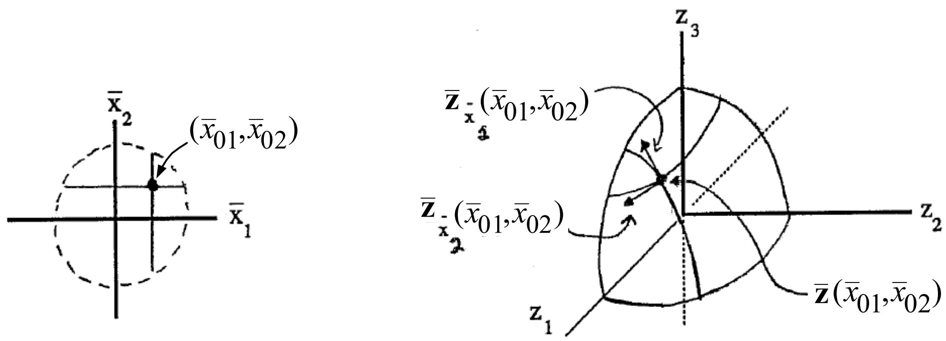

FIGURE 1.41. Tangent space using the eastern hemisphere coordinate chart.

Note that $\bar{\mathbf{z}}_{\bar{x}_1}, \bar{\mathbf{z}}_{\bar{x}_2}$ are defined for all points $p \in \mathcal{U}_2$ (eastern hemisphere) or, in terms of the local coordinates, $\bar{\mathbf{z}}_{\bar{x}_1}$ and $\bar{\mathbf{z}}_{\bar{x}_2}$ are defined for all $(\bar{x}_1, \bar{x}_2) \in \mathcal{D}_2$. Any curve going through the point p of $\mathbf{S}^2$ which has local coordinates $(\bar{x}_{01}, \bar{x}_{02})$ has a tangent vector that is a linear combination of $\bar{\mathbf{z}}_{\bar{x}_1}$ and $\bar{\mathbf{z}}_{\bar{x}_2}$. As in the previous example, the set of all possible tangent vectors at $p \in \mathcal{U}_2 \subset \mathbf{S}^2$ is the tangent space given by

$$\mathbf{T}_p(\mathbf{S}^2) = \left\{\bar{a}_1\bar{\mathbf{z}}_{\bar{x}_1}(\bar{x}_{01}, \bar{x}_{02}) + \bar{a}_2\bar{\mathbf{z}}_{\bar{x}_2}(\bar{x}_{01}, \bar{x}_{02}) \middle| \, (\bar{a}_1, \bar{a}_2) \in \mathbb{R}^2\right\}.$$

$\bar{\mathbf{z}}_{\bar{x}_1}$ and $\bar{\mathbf{z}}_{\bar{x}_2}$ are the basis vectors of the two dimensional vector space $\mathbf{T}_p(\mathbf{S}^2)$ and $(\bar{a}_1, \bar{a}_2)$ are the components of the tangent vector $\bar{a}_1\bar{\mathbf{z}}_{\bar{x}_1}(\bar{x}_{01}, \bar{x}_{02}) + \bar{a}_2\bar{\mathbf{z}}_{\bar{x}_2}(\bar{x}_{01}, \bar{x}_{02})$.

Example 23 *Tangent Vectors on* $\mathbf{S}^2$ *using the Spherical Coordinate Patch*
With

$$\begin{aligned}
\mathcal{D}_3 &= \left\{(x_1^*, x_2^*) \in \mathbb{R}^2 \middle| \, 0 < x_1^* < \pi, 0 < x_2^* < 2\pi\right\} \\
\mathcal{U}_3 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 \middle| \text{ If } z_1 > 0 \text{ then } z_2 \neq 0\right\}.
\end{aligned}$$

The spherical coordinate patch $\mathbf{z}^*(x_1^*, x_2) : \mathcal{D}_3 \to \mathcal{U}_3$ for $\mathbf{S}^2$ is

$$\mathbf{z}^*(x_1^*, x_2^*) = \begin{bmatrix} \sin(x_1^*)\cos(x_2^*) \\ \sin(x_1^*)\sin(x_2^*) \\ \cos(x_1^*) \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbf{S}^2.$$

The left side of Figure 1.42 shows a curve in $\mathcal{D}_3$ which maps to a curve on $\mathbf{S}^2$ shown on the right side of Figure 1.42.

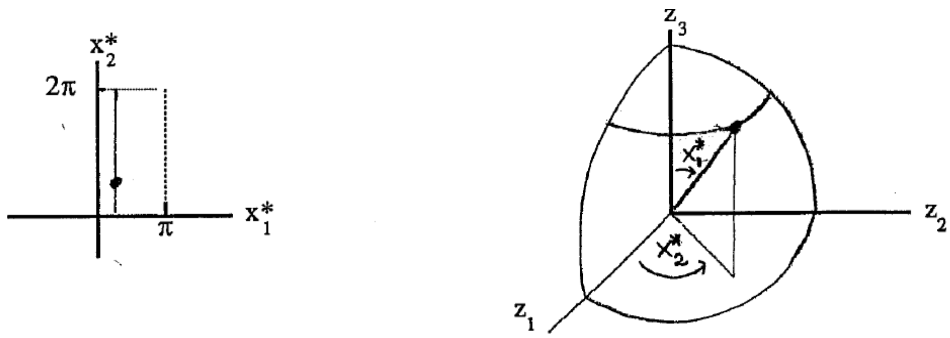

FIGURE 1.42. Spherical coordinate patch.

In general, for any curve $(x_1^*(t), x_2^*(t))$ in $\mathcal{D}_2$ the curve $\mathbf{z}^*(x_1^*(t), x_2^*(t))$ lies on the manifold $\mathbf{S}^2$. The tangent to this curve at $\mathbf{z}^*(x_1^*(t), x_2^*(t))$ is given by

$$\begin{aligned}
\frac{d}{dt}\mathbf{z}^*(x_1^*(t), x_2^*(t)) &= \frac{\partial \mathbf{z}^*}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_2^*}\frac{dx_2^*}{dt} \\
&= \begin{bmatrix} \cos(x_1^*)\cos(x_2^*) \\ \cos(x_1^*)\sin(x_2^*) \\ -\sin(x_1^*) \end{bmatrix} \frac{dx_1^*}{dt} + \begin{bmatrix} -\sin(x_1^*)\sin(x_2^*) \\ \sin(x_1^*)\cos(x_2^*) \\ \cos(x_1^*) \end{bmatrix} \frac{dx_2^*}{dt} \\
&= \mathbf{z}^*_{x_1^*}\frac{dx_1^*}{dt} + \mathbf{z}^*_{x_2^*}\frac{dx_2^*}{dt}.
\end{aligned}$$

Figure 1.43 illustrates the tangents to the curve $\mathbf{z}^*(x_{01}^*, t)$ for $0 < t < 2\pi$ and the curve $\mathbf{z}^*(t, x_{02}^*)$ for $0 < t < \pi$.

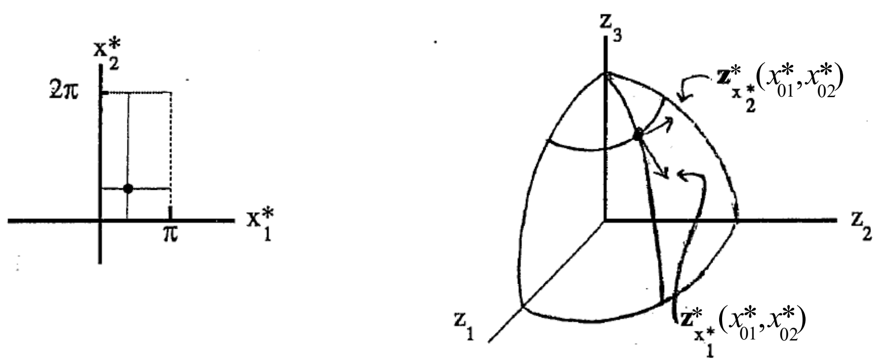

FIGURE 1.43. Basis vectors of the tangent space to $\mathbf{S}^2$ at $\mathbf{z}^*(x_{01}^*, x_{02}^*)$.

Again the set of all possible tangent vectors at $p = \mathbf{z}^*(x_1^*, x_2^*) \in \mathcal{U}_1 \subset \mathbf{S}^2$ is the tangent space given by

$$\mathbf{T}_p(\mathbf{S}^2) = \left\{ a_1^*\mathbf{z}^*_{x_1^*}(x_1^*, x_2^*) + a_2^*\mathbf{z}^*_{x_2^*}(x_1^*, x_2^*) | \ (a_1^*, a_2^*) \in \mathbb{R}^2 \right\}$$

where $\mathbf{z}^*_{x_1^*}(x_1^*, x_2^*)$ and $\mathbf{z}^*_{x_2^*}(x_1^*, x_2^*)$ are the basis vectors of the two dimensional vector space $\mathbf{T}_p(\mathbf{S}^2)$. For any vector $a_1^*\mathbf{z}^*_{x_1^*}(x_1^*, x_2^*) + a_2^*\mathbf{z}^*_{x_2^*}(x_1^*, x_2^*)$ in the tangent space, the pair $(a_1^*, a_2^*) \in \mathbb{R}^2$ are the components of this vector.

The three previous examples illustrated the idea of a manifold and its tangent vectors. We considered three different sets of local coordinates (patches/charts) and there are many points of $\mathbf{S}^2$ that are in all three coordinate charts. Specifically (see Figure 1.44)

$$\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 = \left\{ \mathbf{z} = \begin{bmatrix} z_1 & z_2 & z_2 \end{bmatrix}^T \in \mathbf{S}^2 | \ z_2 > 0, z_3 > 0 \right\}.$$

For $0 < t < 1/2$ consider the curve $c(t)$ in $\mathbf{S}^2$ given by

$$c(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} \sin(\pi t) \\ \frac{1}{\sqrt{2}} \sin(\pi t) \\ \cos(\pi t) \end{bmatrix} \in \mathbf{S}^2.$$

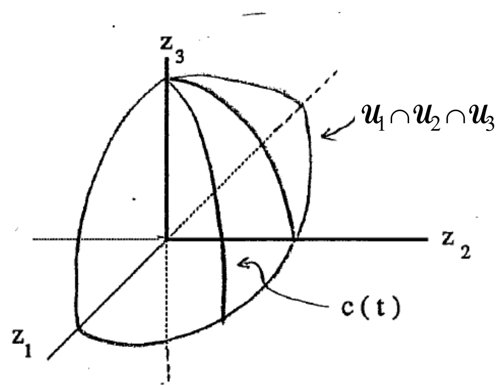

FIGURE 1.44. $c(t)$ on $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 = \left\{ \vec{z} \in \mathbf{S}^2 | \ z_2 > 0, z_3 > 0 \right\}$.

The tangent vector along the curve c is

$$\frac{d}{dt} c(t) = \begin{bmatrix} \frac{\pi}{\sqrt{2}} \cos(\pi t) \\ \frac{\pi}{\sqrt{2}} \cos(\pi t) \\ -\pi \sin(\pi t) \end{bmatrix} \in \mathbf{S}^2$$

and is illustrated in Figure 1.45.

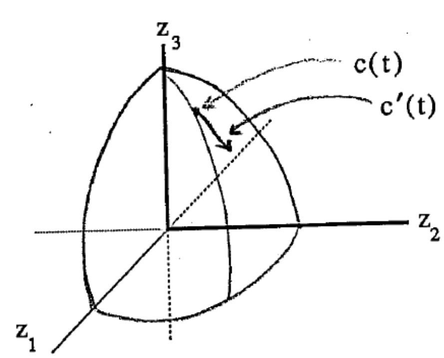

FIGURE 1.45. $c'(t)$ on $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 = \left\{ \vec{z} \in \mathbf{S}^2 | \ z_2 > 0, z_3 > 0 \right\}$.

In the three coordinate systems the curve $c(t)$ is represented by

$$c(t) = \mathbf{z}(x_1(t), x_2(t)) = \bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t)) = \mathbf{z}^*(x_1^*(t), x_2^*(t))$$

where $(x_1(t), x_2(t)), (\bar{x}_1(t), \bar{x}_2(t))$, and $(x_1^*(t), x_2^*(t))$ are, respectively, the local coordinates for the curve $c(t)$ in the northern, eastern, and spherical coordinate systems. By the chain rule for partial differentiation the tangent vector represented in these three sets of local coordinates are

$$\begin{aligned}
\frac{dc}{dt} &= \frac{\partial \mathbf{z}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{z}}{\partial x_2}\frac{dx_2}{dt} = \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x_2}\frac{dx_2}{dt} \\
&= \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_1}\frac{d\bar{x}_1}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_2}\frac{d\bar{x}_2}{dt} = \bar{\mathbf{z}}_{\bar{x}_1}\frac{d\bar{x}_1}{dt} + \bar{\mathbf{z}}_{\bar{x}_2}\frac{d\bar{x}_2}{dt} \\
&= \frac{\partial \mathbf{z}^*}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_2^*}\frac{dx_2^*}{dt} = \mathbf{z}^*_{x_1^*}\frac{dx_1^*}{dt} + \mathbf{z}^*_{x_2^*}\frac{dx_2^*}{dt}.
\end{aligned}$$

Transformation of the Components of a Tangent Vector in $\mathbf{S}^2$

We just saw how the same curve on a manifold can represented in three different coordinate charts. Let's focus on the northern hemisphere and spherical coordinate charts where

$$c(t) = \mathbf{z}(x_1(t), x_2(t)) = \mathbf{z}^*(x_1^*(t), x_2^*(t))$$

with tangent vector

$$\frac{dc}{dt} = \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x_2}\frac{dx_2}{dt} = \mathbf{z}^*_{x_1^*}\frac{dx_1^*}{dt} + \mathbf{z}^*_{x_2^*}\frac{dx_2^*}{dt}.$$

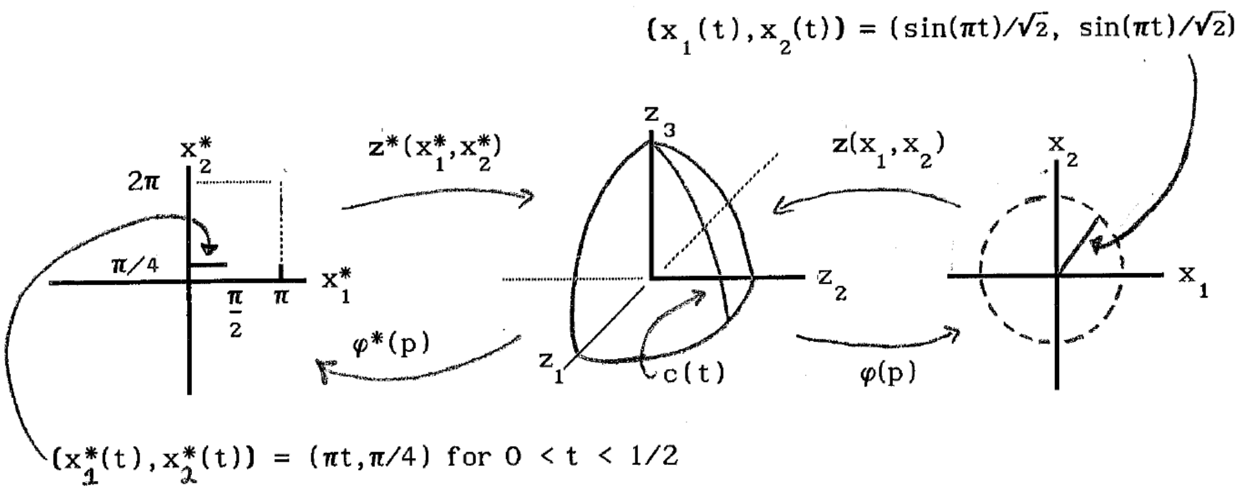

FIGURE 1.46. $c(t)$ in spherical and northern hemisphere coordinates.

We want to now find the relationship between the components of the tangent vector $\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}\right)$ in the northern hemisphere coordinates and the components of the tangent vector $\left(\frac{dx_1^*}{dt}, \frac{dx_2^*}{dt}\right)$ in spherical coordinates.

Recall with $x_1^* = \theta, x_2^* = \varphi$ and

$$\begin{aligned}
\mathcal{D}_1 &= \left\{(x_1^*, x_2^*) \in \mathbb{R}^3 |\ 0 < x_1^* < \pi, 0 < x_2^* < 2\pi\right\} \\
\mathcal{U}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\},
\end{aligned}$$

the spherical coordinate map $\varphi^{*-1} : \mathcal{D}_1 \to \mathcal{U}_1$ for $\mathbf{S}^2$ given by

$$\varphi^{*-1}(x_1^*, x_2^*) = \begin{bmatrix} \sin(x_1^*)\cos(x_2^*) \\ \sin(x_1^*)\sin(x_2^*) \\ \cos(x_1^*) \end{bmatrix} \in \mathbf{S}^2.$$

Further with

$$\begin{aligned} \mathcal{U}_2 &\triangleq \left\{\mathbf{z} \in \mathbf{S}^2 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\} \\ \mathcal{D}_2 &= \left\{(x_1, x_2) \in \mathbb{R}^3 |\ 0 < x_1 < \pi, 0 < x_2 < 2\pi\right\}. \end{aligned}$$

the northern hemisphere coordinate map $\varphi : \mathcal{U}_2 \to \mathcal{D}_2$ for $\mathbf{S}^2$ given by

$$\varphi\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (z_1, z_2) = (x_1, x_2).$$

Then $\varphi \circ \varphi^{*-1} : \varphi^*(\mathcal{U}_1 \cap \mathcal{U}_2) \to \varphi(\mathcal{U}_1 \cap \mathcal{U}_2)$ given by

$$\varphi \circ \varphi^{*-1}(x_1^*, x_2^*) = \varphi\left(\varphi^{*-1}(x_1^*, x_2^*)\right) = (\sin(x_1^*)\cos(x_2^*), \sin(x_1^*)\sin(x_2^*)) = (x_1, x_2)$$

is the coordinate transformation from the spherical to the northern hemisphere coordinates. We may also write this as

$$\begin{aligned} x_1 &= \sin(x_1^*)\cos(x_2^*) \\ x_2 &= \sin(x_1^*)\sin(x_2^*). \end{aligned}$$

A curve $(x_1^*(t), x_2^*(t))$ in spherical coordinates gives the curve

$$\varphi^{*-1}(x_1^*(t), x_2^*(t)) = \begin{bmatrix} \sin(x_1^*(t))\cos(x_2^*(t)) \\ \sin(x_1^*(t))\sin(x_2^*(t)) \\ \cos(x_1^*(t)) \end{bmatrix} \in \mathbf{S}^2.$$

This is represented in the northern hemisphere coordinates by

$$(x_1(t), x_2(t)) = (\sin(x_1^*(t))\cos(x_2^*(t)), \sin(x_1^*(t))\cos(x_2^*(t))).$$

By the chain rule for partial differentiation we have

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{\partial x_1}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial x_1}{\partial x_2^*}\frac{dx_2^*}{dt} = \cos(x_1^*(t))\cos(x_2^*(t))\frac{dx_1^*}{dt} - \sin(x_1^*(t))\sin(x_2^*(t))\frac{dx_2^*}{dt} \\ \frac{dx_2}{dt} &= \frac{\partial x_2}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial x_2}{\partial x_2^*}\frac{dx_2^*}{dt} = \cos(x_1^*(t))\sin(x_2^*(t))\frac{dx_1^*}{dt} + \sin(x_1^*(t))\cos(x_2^*(t))\frac{dx_2^*}{dt} \end{aligned}$$

or in matrix notation

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \cos(x_1^*(t))\cos(x_2^*(t)) & -\sin(x_1^*(t))\sin(x_2^*(t)) \\ \\ \cos(x_1^*(t))\sin(x_2^*(t)) & \sin(x_1^*(t))\cos(x_2^*(t)) \end{bmatrix} \begin{bmatrix} \dfrac{dx_1^*}{dt} \\ \\ \dfrac{dx_2^*}{dt} \end{bmatrix}.$$

Exercise 9 *Northern Hemisphere Coordinates to Spherical Coordinates*

Show that the coordinate transformation for $\mathbf{S}^2$ from the northern hemisphere coordinates (x_1, x_2) to the spherical coordinates (x_1^*, x_2^*) is

$$\varphi^* \circ \varphi^{-1}(x_1, x_2) = \left(\sin^{-1}\left(\sqrt{x_1^2 + x_2^2}\right), \tan^{-1}(x_1, x_2)\right)$$

or

$$
\begin{aligned}
x_1^* &= \sin^{-1}\left(\sqrt{x_1^2 + x_2^2}\right) \\
x_2^* &= \tan^{-1}(x_1, x_2).
\end{aligned}
$$

Show that

$$
\begin{bmatrix} \dfrac{\partial x_1^*}{\partial x_1} & \dfrac{\partial x_1^*}{\partial x_2} \\ \dfrac{\partial x_2^*}{\partial x_1} & \dfrac{\partial x_2^*}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial x_1^*} & \dfrac{\partial x_1}{\partial x_2^*} \\ \dfrac{\partial x_2}{\partial x_1^*} & \dfrac{\partial x_2}{\partial x_2^*} \end{bmatrix}^{-1}_{x^* = \varphi^*(\varphi(x))}
$$

This is equivalent to

$$
\sum_{j=1}^{2} \frac{\partial x_\ell}{\partial x_j^*} \frac{\partial x_j^*}{\partial x_k} = \delta_k^\ell.
$$

Exercise 10 *Northern Hemisphere Coordinates to Eastern Hemisphere Coordinates*
On $\mathbf{S}^2$ let $\mathbf{z}(x_1, x_2) : \mathcal{D}_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 \leq x_1^2 + x_2^2 < 1\right\} \to \mathcal{U}_1 \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}$ be the northern hemisphere coordinate map and $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) : \mathcal{D}_2 = \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 |\ 0 \leq \bar{x}_1^2 + \bar{x}_2^2 < 1\right\} \to \mathcal{U}_2 = \left\{\mathbf{z} \in \mathbf{S}^2 | z_3 > 0\right\}$ be the eastern hemisphere coordinate map. Using the change of coordinates from (x_1, x_2) to $(\bar{x}_1, \bar{x}_2)$ find the change of coordinates for the components (a_1, a_2) of a tangent vector represented in the northern hemisphere patch to the components $(\bar{a}_1, \bar{a}_2)$ of the same tangent vector represented in an eastern hemisphere coordinate patch.

Tangent vectors on $\mathbf{E}^3$

We continue to look at tangent vectors, but now on the manifold $\mathbf{E}^3$.

Example 24 *Spherical Coordinates on* $\mathbf{E}^3$
With

$$
\begin{aligned}
\mathcal{D}_1 &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\} \\
\mathcal{U}_1 &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\},
\end{aligned}
$$

recall the spherical coordinate system $\mathbf{z}(x_1, x_2, x_3) : \mathcal{D}_1 \to \mathcal{U}_1$ given by

$$
\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2) \cos(x_3) \\ x_1 \sin(x_2) \sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3.
$$

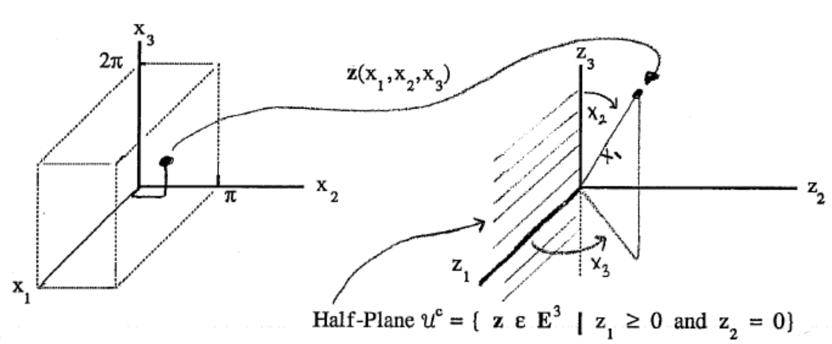

FIGURE 1.47. Spherical coordinates on $\mathbf{E}^3$

Let $(x_1(t), x_2(t), x_3(t))$ be a curve in $\mathcal{D}_1$ corresponding to the curve $\mathbf{z}(x_1(t), x_2(t), x_3(t))$ on $\mathcal{U}_1$ in $\mathbf{E}^3$. The tangent vector at time t is

$$\begin{aligned}
\frac{d}{dt}\mathbf{z}(x_1(t), x_2(t), x_3(t)) &= \frac{\partial \mathbf{z}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{z}}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \mathbf{z}}{\partial x_3}\frac{dx_3}{dt} \\
&= \begin{bmatrix} \sin(x_2)\cos(x_3) \\ \sin(x_2)\sin(x_3) \\ \cos(x_2) \end{bmatrix}\frac{dx_1}{dt} + \begin{bmatrix} x_1\cos(x_2)\cos(x_3) \\ x_1\cos(x_2)\sin(x_3) \\ -x_1\sin(x_2) \end{bmatrix}\frac{dx_2}{dt} + \begin{bmatrix} -x_1\sin(x_2)\sin(x_3) \\ x_1\sin(x_2)\cos(x_3) \\ -x_1\sin(x_2) \end{bmatrix}\frac{dx_3}{dt} \\
&= \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x_2}\frac{dx_2}{dt} + \mathbf{z}_{x_3}\frac{dx_3}{dt}
\end{aligned}$$

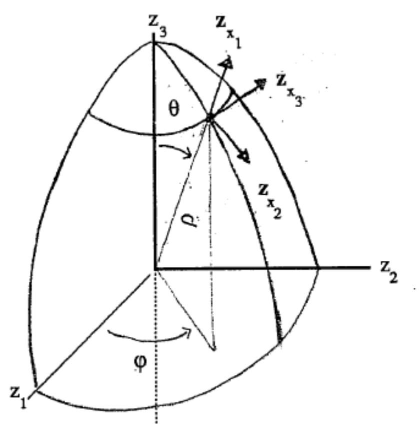

FIGURE 1.48. Tangent vectors on $\mathbf{E}^3$ using spherical coordinates

Any curve through a point $p \in \mathcal{U}_1 \subset \mathbf{E}^3$ with local spherical coordinates $(x_1(t), x_2(t), x_3(t))$ has a tangent vector which is a linear combination of $\mathbf{z}_{x_1}(x_1, x_2, x_3)$, $\mathbf{z}_{x_2}(x_1, x_2, x_3)$, and $\mathbf{z}_{x_3}(x_1, x_2, x_3)$. The set of all possible tangent vectors at a point p of the manifold of $\mathbf{E}^3$ is the tangent space $\mathbf{T}_p(\mathbf{E}^3)$ given by

$$\mathbf{T}_p(\mathbf{E}^3) = \left\{ a_1\mathbf{z}_{x_1} + a_2\mathbf{z}_{x_2} + a_3\mathbf{z}_{x_3} \middle| \, (a_1, a_2, a_3) \in \mathbb{R}^3 \right\}.$$

The tangent vectors $\mathbf{z}_{x_1}, \mathbf{z}_{x_2}$, and $\mathbf{z}_{x3}$ form a basis for the three dimensional vector space $\mathbf{T}_p(\mathbf{E}^3)$ with (a_1, a_2, a_3) the components of the vector $a_1\mathbf{z}_{x_1} + a_2\mathbf{z}_{x_2} + a_3\mathbf{z}_{x_3}$.

Example 25 *Cylindrical Coordinates on* $\mathbf{E}^3$

With

$$\begin{aligned}
\mathcal{D}_2 &= \left\{ (x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3 \middle| \, x_1^* > 0, 0 < x_2^* < 2\pi, -\infty < x_3^* < \infty \right\} \\
\mathcal{U}_2 &\triangleq \left\{ \mathbf{z} \in \mathbf{E}^3 \middle| \text{ If } z_1 > 0 \text{ the } z_2 \neq 0 \right\},
\end{aligned}$$

the cylindrical coordinate system $\mathbf{z}(x_1^*, x_2^*, x_3^*) : \mathcal{D}_2 \to \mathcal{U}_2$ is

$$\mathbf{z}^*(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} x_1^*\cos(x_2^*) \\ x_1^*\sin(x_2^*) \\ x_3^* \end{bmatrix} \in \mathbf{E}^3.$$

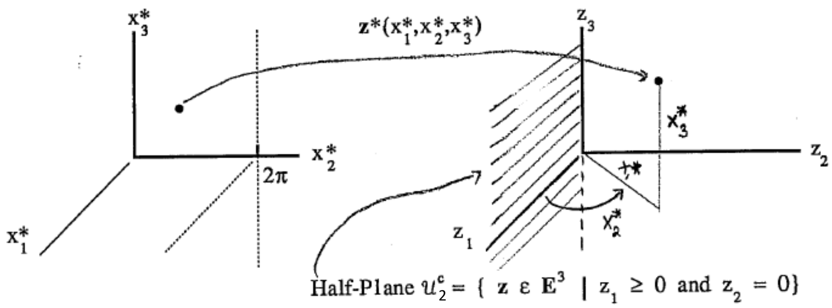

FIGURE 1.49. $\mathcal{D}_2 = \left\{(x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3 | \; x_1^* > 0, 0 < x_2^* < 2\pi, -\infty < x_3^* < \infty\right\}, \mathcal{U}_2 \triangleq \left\{\vec{z} \in \mathbf{E}^3 | \text{ If } z_1 > 0 \text{ the } z_2 \neq 0\right\}$

Let $(x_1^*(t), x_2^*(t), x_3^*(t))$ be a curve in $\mathcal{D}_2$ with corresponding curve $\mathbf{z}^*(x_1^*(t), x_2^*(t), x_3^*(t))$ on $\mathcal{U}_2$ in $\mathbf{E}$. The tangent vector to this curve at time t is

$$\begin{aligned}
\frac{d}{dt}\mathbf{z}(x_1^*(t), x_2^*(t), x_3^*(t)) &= \frac{\partial \mathbf{z}^*}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_2^*}\frac{dx_2^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_3^*}\frac{dx_3^*}{dt} \\
&= \begin{bmatrix} \sin(x_2^*) \\ \sin(x_2^*) \\ 0 \end{bmatrix} \frac{dx_1^*}{dt} + \begin{bmatrix} -x_1^* \sin(x_2^*) \\ x_1^* \cos(x_2^*) \\ 0 \end{bmatrix} \frac{dx_2^*}{dt} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{dx_3^*}{dt} \\
&= \mathbf{z}^*_{x_1^*}\frac{dx_1^*}{dt} + \mathbf{z}^*_{x_2^*}\frac{dx_2^*}{dt} + \mathbf{z}^*_{x_3^*}\frac{dx_3^*}{dt}
\end{aligned}$$

The curve $(x_1^*(t), x_2^*(t), x_3^*(t)) = (x_{01}^*, t, x_{03}^*)$ for $0 < t < 2\pi$ maps to $c(t) = \mathbf{z}^*(x_{01}^*, t, x_{03}^*)$ which is a circle in $\mathbf{E}^3$ in the $z_1 - z_2$ plane as shown in Figure 1.50. The tangent vector to the curve is

$$\frac{d}{dt}c(t) = \frac{d}{dt}\mathbf{z}^*(x_{01}^*, t, x_{03}^*) = \mathbf{z}^*_{x_2^*}(x_{01}^*, t, x_{03}^*).$$

.

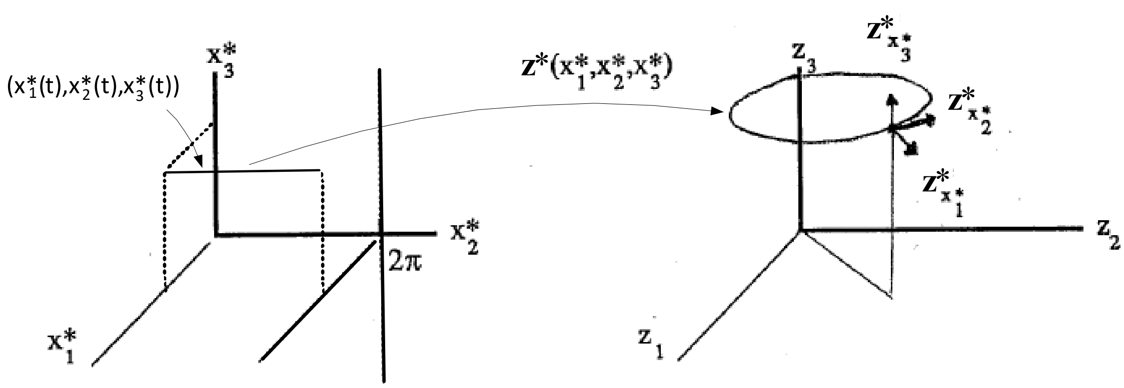

FIGURE 1.50. Curve in cylindrical coordinates.

Any curve through a point $p \in \mathcal{U}_2 \subset \mathbf{E}^3$ with local cylindrical coordinates $(x_1(t), x_2(t), x_3(t))$ has a tangent vector which is a linear combination of $\mathbf{z}^*_{x_1^*}(x_1^*, x_2^*, x_3^*), \mathbf{z}^*_{x_2^*}(x_1^*, x_2^*, x_3^*)$, and $\mathbf{z}^*_{x_3^*}(x_1^*, x_2^*, x_3^*)$. The set of all possible tangent vectors at a point $p = \mathbf{z}^*(x_1^*, x_2^*, x_3^*)$ of the manifold of $\mathbf{E}^3$ is the tangent space $\mathbf{T}_p(\mathbf{E}^3)$

given by $\mathbf{T}_p(\mathbf{E}^3) = \left\{a_1^*\mathbf{z}^*_{x_1^*} + a_2^*\mathbf{z}^*_{x_2^*} + a_3^*\mathbf{z}^*_{x_3^*} |\ (a_1^*, a_2^*, a_3^*) \in \mathbb{R}^3\right\}$. The tangent vector $a_1^*\mathbf{z}^*_{x_1^*} + a_2^*\mathbf{z}^*_{x_2^*} + a_3^*\mathbf{z}^*_{x_3^*}$ at $p = \mathbf{z}^*(x_1^*, x_2^*, x_3^*)$ has components a_1^*, a_2^*, and a_3^*.

Example 26 *Cartesian Coordinates on* $\mathbf{E}^3$
With

$$\begin{aligned}
\mathcal{D}_3 &= \left\{(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 |\ -\infty < \bar{x}_1 < \infty, -\infty < \bar{x}_2 < \infty, -\infty < \bar{x}_3 < \infty\right\} \\
\mathcal{U}_3 &\triangleq \left\{\mathbf{z} \in \mathbf{E}^3 |\ \text{If } z_1 > 0 \text{ the } z_2 \neq 0\right\},
\end{aligned}$$

the Cartesian coordinates $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) : \mathcal{D}_3 \to \mathcal{U}_3$ is

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} \in \mathbf{E}^3.$$

Let $(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$ be a curve in $\mathcal{D}_3$ corresponding to the curve $\bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$ on $\mathcal{U}_3$ in $\mathbf{E}^3$. The tangent vector at time t is

$$\begin{aligned}
\frac{d}{dt}\bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t)) &= \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_1}\frac{d\bar{x}_1}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_2}\frac{d\bar{x}_2}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_3}\frac{d\bar{x}_3}{dt} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\frac{d\bar{x}_1}{dt} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\frac{d\bar{x}_2}{dt} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\frac{d\bar{x}_3}{dt} \\
&= \bar{\mathbf{z}}_{\bar{x}_1}\frac{d\bar{x}_1}{dt} + \bar{\mathbf{z}}_{\bar{x}_2}\frac{d\bar{x}_2}{dt} + \bar{\mathbf{z}}_{\bar{x}_3}\frac{d\bar{x}_3}{dt}.
\end{aligned}$$

Any curve through a point $p \in \mathcal{U}_3 = \mathbf{E}^3$ with local Cartesian coordinates $(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$ has a tangent vector which is a linear combination of $\bar{\mathbf{z}}_{\bar{x}_1}, \bar{\mathbf{z}}_{\bar{x}_2}$, and $\bar{\mathbf{z}}_{\bar{x}_3}$. The set of all possible tangent vectors at a point $p = \bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ of the manifold of $\mathbf{E}^3$ is the tangent space $\mathbf{T}_p(\mathbf{E}^3)$ given by

$$\mathbf{T}_p(\mathbf{E}^3) = \left\{\bar{a}_1\bar{\mathbf{z}}_{\bar{x}_1} + \bar{a}_2\bar{\mathbf{z}}_{\bar{x}_2} + \bar{a}_3\bar{\mathbf{z}}_{\bar{x}_3} |\ (\bar{a}_1, \bar{a}_2, \bar{a}_3) \in \mathbb{R}^3\right\}.$$

The tangent vector $\bar{a}_1\bar{\mathbf{z}}_{\bar{x}_1} + \bar{a}_2\bar{\mathbf{z}}_{\bar{x}_2} + \bar{a}_3\bar{\mathbf{z}}_{\bar{x}_3}$ at $p = \bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ has components $\bar{a}_1, \bar{a}_2$, and $\bar{a}_3$.

Let's now consider the representation of a helical curve in $\mathbf{E}^3$ using each of the three coordinate systems above.

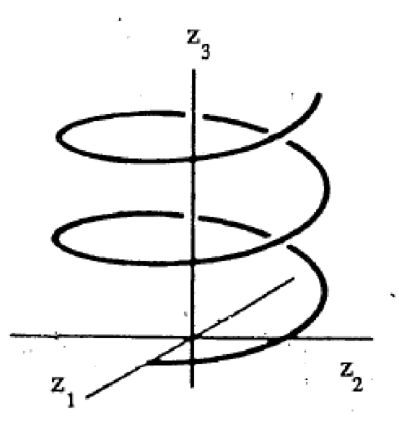

FIGURE 1.51. A helix in $\mathbf{E}^3$.

In *spherical coordinates* with $t > 0$ the helix is

$$(x_1(t), x_2(t), x_3(t)) = \left(\sqrt{1+t^2}, \tan^{-1}(1/t), 2\pi t\right)$$

as $\mathbf{z}(x_1(t), x_2(t), x_3(t))$ is given by

$$c(t) = \mathbf{z}(\sqrt{1+t^2}, \tan(1/t), 2\pi t) = \begin{bmatrix} \sqrt{1+t^2}\sin(\tan^{-1}(1/t))\cos(2\pi t) \\ \sqrt{1+t^2}\sin(\tan^{-1}(1/t))\sin(2\pi t) \\ \sqrt{1+t^2}\cos(\tan^{-1}(1/t)) \end{bmatrix} = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ t \end{bmatrix} \in \mathbf{E}^3$$

where $\sin(\tan^{-1}(1/t)) = 1/\sqrt{1+t^2}$ and $\cos(\tan^{-1}(1/t)) = t/\sqrt{1+t^2}$ was used. The components of the tangent vector are

$$\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\right) = \left(\frac{d}{dt}\sqrt{1+t^2}, \frac{d}{dt}\tan^{-1}(1/t), \frac{d}{dt}2\pi t\right) = \left(\frac{t}{\sqrt{1+t^2}}, -\frac{1}{1+t^2}, 2\pi\right).$$

In *cylindrical coordinates* with $t > 0$ the helix is

$$(x_1^*(t), x_2^*(t), x_3^*(t)) = (1, 2\pi t, t)$$

as $\mathbf{z}^*(x_1^*(t), x_2^*(t), x_3^*(t))$ is given by

$$c(t) = \mathbf{z}^*(x_1^*(t), x_2^*(t), x_3^*(t)) = \begin{bmatrix} x_1^*\cos(x_2^*) \\ x_1^*\sin(x_2^*) \\ x_3^* \end{bmatrix}_{x^*(t)=(1,2\pi t,t)} = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ t \end{bmatrix} \in \mathbf{E}^3.$$

The components of the tangent vector are

$$\left(\frac{dx_1^*}{dt}, \frac{dx_2^*}{dt}, \frac{dx_3^*}{dt}\right) = \left(\frac{d}{dt}1, \frac{d}{dt}2\pi t, \frac{d}{dt}t\right) = (0, 2\pi, 1).$$

In *Cartesian coordinates* with $t > 0$ the helix is

$$(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t)) = (\cos(2\pi t), \sin(2\pi t), t)$$

as $\bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$ is given by

$$c(t) = \bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t)) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \end{bmatrix}_{x^*(t)=(\cos(2\pi t),\sin(2\pi t),t)} = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ t \end{bmatrix} \in \mathbf{E}^3.$$

$$\left(\frac{d\bar{x}_1}{dt}, \frac{d\bar{x}_2}{dt}, \frac{d\bar{x}_3}{dt}\right) = \left(\frac{d}{dt}\cos(2\pi t), \frac{d}{dt}\sin(2\pi t), \frac{d}{dt}t\right) = (-2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1).$$

In summary, the local coordinate representation of the helical curve $c(t)$ and the components of its tangent vector are quite different in each of the coordinate systems.

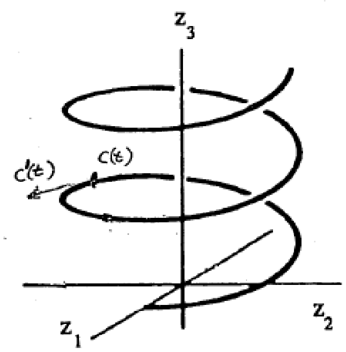

FIGURE 1.52. $\dfrac{dc}{dt} = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ t \end{bmatrix} \in \mathbf{E}^3$.

Representing the curve $c(t) \in \mathbf{E}^3$ in any of the local coordinates will still result in the *same* tangent vector $\dfrac{dc}{dt} \in \mathbf{E}^3$. This is seen explicitly as

$$\mathbf{z}(x_1(t), x_2(t), x_3(t)) = \mathbf{z}^*(x_1^*(t), x_2^*(t), x_3^*(t)) = \bar{\mathbf{z}}(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$$

we have

$$\begin{aligned} \frac{dc}{dt} &= \frac{\partial \mathbf{z}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathbf{z}}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \mathbf{z}}{\partial x_3}\frac{dx_3}{dt} = \mathbf{z}_{x_1}\frac{dx_1}{dt} + \mathbf{z}_{x_2}\frac{dx_2}{dt} + \mathbf{z}_{x_3}\frac{dx_3}{dt} \\ &= \frac{\partial \mathbf{z}^*}{\partial x_1^*}\frac{dx_1^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_2^*}\frac{dx_2^*}{dt} + \frac{\partial \mathbf{z}^*}{\partial x_3^*}\frac{dx_3^*}{dt} = \mathbf{z}_{x_1^*}\frac{dx_1^*}{dt} + \mathbf{z}_{x_2^*}\frac{dx_2^*}{dt} + \mathbf{z}_{x_3^*}\frac{dx_3^*}{dt} \\ &= \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_1}\frac{d\bar{x}_1}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_2}\frac{d\bar{x}_2}{dt} + \frac{\partial \bar{\mathbf{z}}}{\partial \bar{x}_3}\frac{d\bar{x}_3}{dt} = \bar{\mathbf{z}}_{\bar{x}_1}\frac{d\bar{x}_1}{dt} + \bar{\mathbf{z}}_{\bar{x}_2}\frac{d\bar{x}_2}{dt} + \bar{\mathbf{z}}_{\bar{x}_3}\frac{d\bar{x}_3}{dt}. \end{aligned}$$

Transformation of the Components of a Tangent Vector in $\mathbf{E}^3$

We now show how the components of the *same* tangent vector in different coordinate systems of $\mathbf{E}^3$ are related. To see how this is done consider the transformation between spherical and cylindrical coordinates charts as illustrated in Figure 1.53.

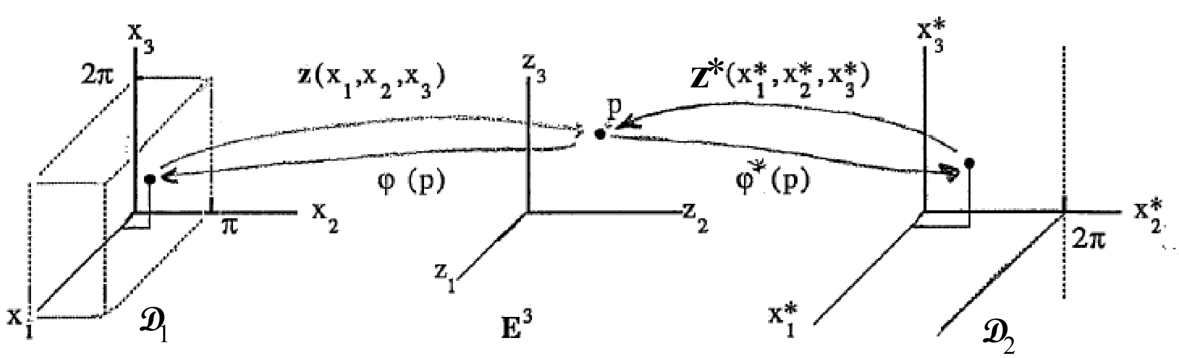

FIGURE 1.53. Transformation between spherical and cylindrical coordinates in $\mathbf{E}^3$.

The mapping from a point $\mathbf{z} \in \mathbf{E}^3$ to its spherical coordinates is

$$\varphi(p) = \varphi\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \left(\sqrt{z_1^2 + z_2^2 + z_3^2}, \tan^{-1}\left(z_3, \sqrt{z_1^2 + z_2^2}\right), \tan^{-1}(z_1, z_2)\right) = (x_1, x_2, x_3)$$

while the mapping from the spherical coordinates (x_1, x_2, x_3) to $\mathbf{E}^3$ is

$$\varphi^{-1}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix} \in \mathbf{E}^3.$$

Further the mapping from a point $\mathbf{z} \in \mathbf{E}^3$ to its cylindrical coordinates is.

$$\varphi^*(p) = \varphi^*\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \left(\sqrt{z_1^2 + z_2^2}, \tan^{-1}(z_1, z_2), z_3\right) = (x_1^*, x_2^*, x_3^*)$$

while the mapping from the cylindrical coordinates (x_1^*, x_2^*, x_3^*) to $\mathbf{E}^3$ is

$$\varphi^{*-1}(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} x_1^* \sin(x_2^*) \\ x_1^* \cos(x_2^*) \\ x_3^* \end{bmatrix}.$$

The coordinate transformation between the spherical coordinates on

$$\mathcal{D}_1 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 > 0, 0 < x_2 < \pi, 0 < x_3 < 2\pi\right\}$$

to the cylindrical coordinates on

$$\mathcal{D}_2 = \left\{(x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3 |\ x_1^* > 0, 0 < x_2^* < 2\pi, -\infty < x_3^* < \infty\right\}$$

is

$$\begin{aligned} (x_1^*, x_2^*, x_3^*) = \varphi^* \circ \varphi^{-1}(x_1, x_2, x_3) &= \varphi^*(\varphi^{-1}(x_1, x_2, x_3)) \\ &= \varphi^*\left(\begin{bmatrix} x_1 \sin(x_2)\cos(x_3) \\ x_1 \sin(x_2)\sin(x_3) \\ x_1 \cos(x_2) \end{bmatrix}\right) \\ &= \left(x_1 \sin(x_2), x_3,\ x_1 \cos(x_2)\right) \end{aligned}$$

or

$$\begin{aligned} x_1^* &= x_1 \sin(x_2) \\ x_2^* &= x_3 \\ x_3^* &= x_1 \cos(x_2). \end{aligned}$$

A curve $(x_1(t), x_2(t), x_3(t))$ in the spherical coordinate system gives the curve $\varphi^{-1}(x_1(t), x_2(t), x_3(t))$ is represented in cylindrical coordinates by

$$(x_1^*(t), x_2^*(t), x_3^*(t)) = \left(x_1(t)\sin(x_2(t)), x_3(t),\ x_1(t)\cos(x_2(t))\right).$$

The components of the tangent vectors in these two coordinate charts are

$$\begin{pmatrix} \frac{dx_1^*}{dt} \\ \frac{dx_2^*}{dt} \\ \frac{dx_3^*}{dt} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x_1^*}{\partial x_1} & \frac{\partial x_1^*}{\partial x_2} & \frac{\partial x_1^*}{\partial x_3} \\ \frac{\partial x_2^*}{\partial x_1} & \frac{\partial x_2^*}{\partial x_2} & \frac{\partial x_2^*}{\partial x_3} \\ \frac{\partial x_3^*}{\partial x_1} & \frac{\partial x_3^*}{\partial x_2} & \frac{\partial x_3^*}{\partial x_3} \end{pmatrix}}_{\left[\frac{\partial x_i^*}{\partial x_j}\right]} \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} \sin(x_2) & -x_1\cos(x_2) & 0 \\ 0 & 0 & 1 \\ \cos(x_2) & -x_1\sin(x_2) & 0 \end{pmatrix} \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix}.$$

Exercise 11 *Cylindrical to Spherical Coordinate Transformation in* $\mathbf{E}^3$

(a) Compute the change of coordinates $\varphi \circ \varphi^{*-1}$ from cylindrical to spherical coordinates.

(b) Use your answer in part (a) to find the transformation matrix $\left[\frac{\partial x_i}{\partial x_j^*}\right]$ of the components of a tangent vector in spherical coordinates in terms of the components of the same tangent vector in cylindrical coordinates.

(c) Show that $\left[\frac{\partial x_i}{\partial x_j^*}\right]^{-1} = \left[\frac{\partial x_i^*}{\partial x_j}\right]$ which is equivalent to $\sum_{\ell=1}^{3} \frac{\partial x_i}{\partial x_\ell^*}\frac{\partial x_\ell^*}{\partial x_j} = \delta_j^i$.

Exercise 12 *Linear Manifold and its Tangent Vectors*
Consider the linear manifold defined by

$$\mathcal{M} = \left\{\mathbf{z} \in \mathbf{E}^3 |\ z_2 - z_3 = -1\right\}$$

with the two coordinate charts given by $\mathbf{z}(x_1, x_2) : \mathbb{R}^2 \to \mathcal{M}$ given by

$$\mathbf{z}(x_1, x_2) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + 1 \end{bmatrix}$$

and $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) : \mathbb{R}^2 \to \mathcal{M}$

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_1 + \bar{x}_2 \\ \bar{x}_1 \\ \bar{x}_1 + 1 \end{bmatrix}.$$

(a) Find a basis for the tangent space $\mathbf{T}_p(\mathcal{M})$ using the coordinate chart $\mathbf{z}(x_1, x_2)$.

(b) Find a basis for the tangent space $\mathbf{T}_p(\mathcal{M})$ using the coordinate chart $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2)$.

(c) Consider the curve $c(t) : \mathbb{R} \to \mathcal{M}$ given by

$$c(t) = \begin{bmatrix} t^3 \\ t^2 \\ t^2 - 1 \end{bmatrix}.$$

Show $c(t) \in \mathcal{M}$ for all t. Show that $(x_1(t), x_2(t)) = (t^2, t^3)$ and $(\bar{x}_1(t), \bar{x}_2(t)) = (t^2, t^3 - t^2)$ are the coordinate representations of $c(t)$ in the two coordinate system respectively.

(d) Compute the tangent vector to $c(t)$ in terms of $\mathbf{z}_{x_1}, \mathbf{z}_{x_2}$ as well as in terms of $\bar{\mathbf{z}}_{\bar{x}_1}, \bar{\mathbf{z}}_{\bar{x}_2}$.

(e) Compute the transformation from the components of the tangent vector in terms of $\mathbf{z}_{x_1}, \mathbf{z}_{x_2}$ to the components of the same tangent vector in terms of $\bar{\mathbf{z}}_{\bar{x}_1}, \bar{\mathbf{z}}_{\bar{x}_2}$.

Exercise 13 *Linear Manifold and its Tangent Vector*

Consider the linear manifold defined by

$$\mathcal{M} = \mathbf{E}^3$$

with the two coordinate charts given by $\mathbf{z}(x_1, x_2, x_3) : \mathbb{R}^3 \rightarrow \mathbf{E}^3$

$$\mathbf{z}(x_1, x_2, x_3) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_3 \end{bmatrix}$$

and $\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3)| \ \mathbb{R}^2 \rightarrow \mathcal{M}$

$$\bar{\mathbf{z}}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \begin{bmatrix} \bar{x}_1 + \bar{x}_2 \\ \bar{x}_1 \\ \bar{x}_1 + \bar{x}_3 \end{bmatrix}.$$

Answer $(a), (b), (c), (d)$, and(e) of the previous exercise.

1.5 Problems

Problem 1 *Inverse Function Theorem*
Define a nonlinear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned} y_1 &= e^{x_1}\cos(x_2) \\ y_2 &= e^{x_2}\sin(x_2). \end{aligned}$$

(a) Compute the Jacobian matrix and its determinant.

(b) Is this transformation one-to-one?

(c) If the domain is restricted to $\{(x_1, x_2) \in \mathbb{R}^2 |\ 0 < x_2 < 2\pi\}$ show that transformation is one-to-one and find its inverse.

Problem 2 *Inverse Function Theorem*
Define a nonlinear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned} y_1 &= x_1^2 \\ y_2 &= x_2/x_1 \end{aligned}$$

(a) Compute the Jacobian matrix and its determinant.

(b) With $x_1 > 0$ for what region in the (x_1, x_2) plane is this transformation invertible. For this region find the corresponding range (image) of this transformation.

Problem 3 *Field Controlled DC Motor* [1]
The equations describing a separately excited DC motor are

$$\begin{aligned} J\frac{d\omega}{dt} &= K_T L_f i_f i_a - \tau_L \\ L\frac{di_a}{dt} &= -Ri_a - K_b L_f i_f \omega + V_{a0} \\ L_f\frac{di_f}{dt} &= -R_f i_f + V_f. \end{aligned}$$

Here ω is the rotor angular speed, V_{a0} is the (constant) armature voltage, i_a is the armature current, V_f is the field voltage, i_f is the field current, τ_L is the load torque, K_T is the torque constant, and K_b is the back-emf constant. The armature resistance and armature inductance are denoted by R and L,respectively, and the field resistance and field inductance are R_f and L_f, respectively.

Historically field controlled DC motors were used in mills for rolling out sheets of steel. This application required armature currents of $1000 - 2000$ Amperes to obtain the torques need to roll out the steel. However, there were not *variable* voltage sources available that could handle that amount of current. So a *constant* voltage source for the armature was used which could supply the large current. The field current i_f was on the order of only 25 Amperes and there were voltage sources that could provide a *varying* voltage while supplying that amount of current. By varying the field voltage V_f the current i_f and speed ω could then be controlled.

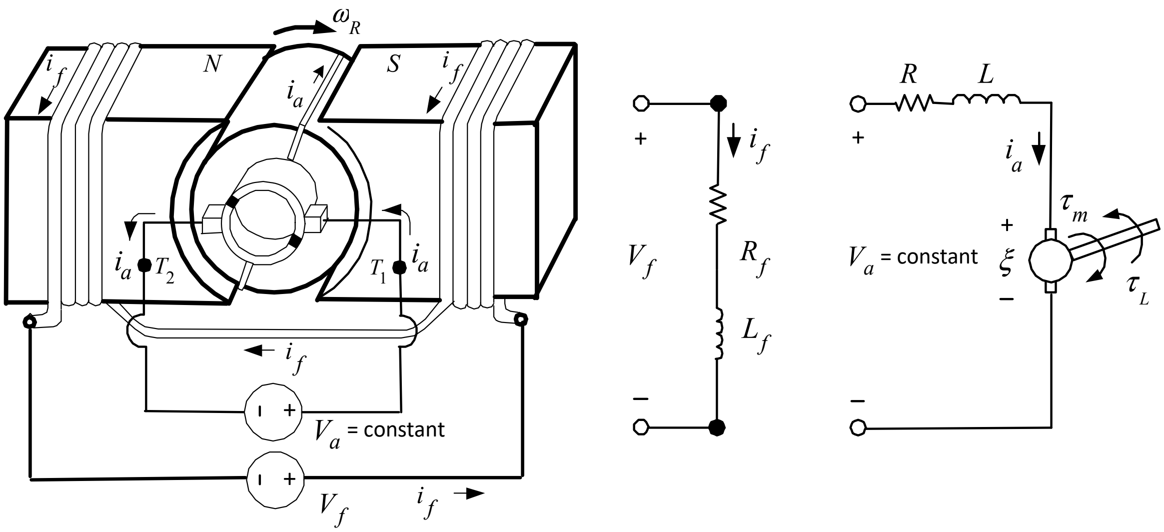

FIGURE 1.54. Field controlled DC motor. $\xi = K_b L_f i_f$ and $\tau_m = K_T L_f i_f i_a$.

Let $x_1 = \omega, x_2 = i_a, x_3 = i_f, u = V_f/L_f$, and define the constants $c_0 = V_{a0}/L, c_1 = R_f/L_f, c_2 = R/L, c_3 = K_b L_f/L, c_4 \triangleq K_T L_f/J, c_5 = 1/L$. The equations describing the field controlled DC motor are then

$$\begin{aligned}
\frac{dx_1}{dt} &= -c_1 x_1 + u \\
\frac{dx_2}{dt} &= -c_2 x_2 - c_3 x_1 x_3 + c_0 \\
\frac{dx_3}{dt} &= c_4 x_1 x_2 - \tau_L/J
\end{aligned} \tag{1.12}$$

or

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -c_1 x_1 \\ -c_2 x_2 - c_3 x_1 x_3 + c_0 \\ c_4 x_1 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{g(x)} u + \begin{bmatrix} 0 \\ 0 \\ -1/J \end{bmatrix} \tau_L.$$

(a) Define $T_1(x) = \dfrac{c_4}{c_3} x_2^2 + x_3^2 = (L i_a^2 + J\omega^2)/J$. Compute the transformation

$$\begin{aligned}
x_1^* &= T_1(x) \\
x_2^* &= \mathcal{L}_f(T_1) \\
x_3^* &= \mathcal{L}_f^2(T_1).
\end{aligned}$$

(b) Find the equations of the field controlled DC motor in the x^* coordinates.

(c) Use feedback linearization so that in the x^* coordinates the system is linear. What conditions on the state variables x_1, x_2, x_3 are needed to use this feedback?

(d) The model (1.12) is taken to be in Cartesian coordinates for $\mathbf{E}^3$, that is,

$$\phi\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = (x_1, x_2, x_3) \text{ and } \mathbf{z}(x_1, x_2, x_3) = \varphi^{-1}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{E}^3.$$

The basis vectors for the tangent space are

$$\mathbf{z}_{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{z}_{x_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{z}_{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with corresponding components $(-c_1x_1+u, -c_2x_2-c_3x_1x_3+c_0, c_4x_1x_2-\tau_L/J)$ for the tangent vector of any solution of (1.12). In the x^* coordinate system give both the basis vectors $\mathbf{z}^*_{x^*_1}, \mathbf{z}^*_{x^*_2}, \mathbf{z}^*_{x^*_3}$ of the tangent space and the components of the tangent vector of any solution of (1.12).

1.6 References

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