HW 3 Solutions

John Chiasson
Boise State University

March 16, 2021

Contents

1

Chapter3 Solutions

1.1 Problems Chapter 3

Problem 1 *Implicit Function Theorem*
Let $F_1(x_1, x_2, x_3, x_4, x_5)$ and $F_2(x_1, x_2, x_3, x_4, x_5)$ be continuously differentiable functions from an open set $\mathcal{U} \subset \mathbb{R}^5 \to \mathbb{R}$. Further suppose $x_0 = (x_{01}, x_{02}, x_{03}, x_{04}, x_{05}) \in \mathcal{U}$ satisfying $F_1(x_0) = F_2(x_0) = 0$ and

$$\left.\frac{\partial(F_1, F_2)}{\partial x_4 \partial x_5}\right|_{x_0} \triangleq \det \left.\begin{bmatrix} \frac{\partial F_1}{\partial x_4} & \frac{\partial F_1}{\partial x_5} \\ \frac{\partial F_2}{\partial x_4} & \frac{\partial F_2}{\partial x_5} \end{bmatrix}\right|_{x_0} \neq 0.$$

(a) Using the inverse function theorem show there exists functions $s_1(x_1, x_2, x_3)$ and $s_2(x_1, x_2, x_3)$ defined on a neighborhood $\mathcal{D} \subset \mathbb{R}^3$ containing (x_{01}, x_{02}, x_{03}) such that

$$\begin{aligned} x_{04} &= s_1(x_{01}, x_{02}, x_{03}) \\ x_{05} &= s_2(x_{01}, x_{02}, x_{03}) \end{aligned}$$

and for all $(x_1, x_2, x_3) \in \mathcal{D}$

$$\begin{aligned} F_1(x_1, x_2, x_3, s_1(x_1, x_2, x_3), s_2(x_1, x_2, x_3)) &\equiv 0 \\ F_2(x_1, x_2, x_3, s_1(x_1, x_2, x_3), s_2(x_1, x_2, x_3)) &\equiv 0. \end{aligned}$$

About x_0 define transformation

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \\ x'_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ F_1(x_1, x_2, x_3, x_4, x_5) \\ F_2(x_1, x_2, x_3, x_4, x_5) \end{bmatrix}.$$

The Jacobian of this transformation is

$$\frac{\partial x'}{\partial x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} & \frac{\partial F_1}{\partial x_5} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} & \frac{\partial F_2}{\partial x_5} \end{bmatrix}$$

and

$$\det \frac{\partial x'}{\partial x}_{|x_0} = \det \left.\begin{bmatrix} \frac{\partial F_1}{\partial x_4} & \frac{\partial F_1}{\partial x_5} \\ \frac{\partial F_2}{\partial x_4} & \frac{\partial F_2}{\partial x_5} \end{bmatrix}\right|_{x_0} \neq 0$$

By the inverse function theorem we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ h_4(x'_1, x'_2, x'_3, x'_4, x'_5) \\ h_5(x'_1, x'_2, x'_3, x'_4, x'_5) \end{bmatrix}.$$

With $x_4' \equiv 0, x_5' \equiv 0$ we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ h_4(x_1', x_2', x_3', 0, 0) \\ h_5(x_1', x_2', x_3', 0, 0) \end{bmatrix}.$$

Setting $s_1(x_1, x_2, x_3) \triangleq h_4(x_1, x_2, x_3, 0, 0)$ and $s_2(x_1, x_2, x_3) \triangleq h_5(x_1, x_2, x_3, 0, 0)$ gives

$$\begin{aligned} F_1(x_1, x_2, x_3, s_1(x_1, x_2, x_3), s_2(x_1, x_2, x_3)) &= F_1(x_1, x_2, x_3, h_4(x_1, x_2, x_3, 0, 0), h_5(x_1, x_2, x_3, 0, 0)) \equiv 0 \\ F_2(x_1, x_2, x_3, s_1(x_1, x_2, x_3), s_2(x_1, x_2, x_3)) &= F_1(x_1, x_2, x_3, h_4(x_1, x_2, x_3, 0, 0), h_5(x_1, x_2, x_3, 0, 0)) \equiv 0. \end{aligned}$$

(b) Use part (a) to construct a coordinate chart for the manifold defined by

$$\mathcal{M} \triangleq \big\{x \in \mathbb{R}^5 |\ F_1(x_1, x_2, x_3, x_4, x_5) = 0,\ F_2(x_1, x_2, x_3, x_4, x_5) = 0\big\}$$

which contains x_0.

The coordinate chart is

$$\varphi^{-1}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ h_4(x_1, x_2, x_3, 0, 0) \\ h_5(x_1, x_2, x_3, 0, 0) \end{bmatrix}.$$

(c) Consider the system of equations

$$\begin{aligned} F_1(x) &= x_1^2 - x_2 x_4 = 0 \\ F_2(x) &= x_1 x_2 + x_4 x_5 = 0. \end{aligned}$$

Show that $x_0 = (-1, 1, 0, 1, 1)$ satisfies this system of equations. Show that $\left.\dfrac{\partial(F_1, F_2)}{\partial x_4 \partial x_5}\right|_{x_0} \neq 0$. Explicitly find $x_4 = s_1(x_1, x_2, x_3), x_5 = s_2(x_1, x_2, x_3)$ which are valid in a neighborhood of $(-1, 1, 0)$. Use s_1, s_2 to define a coordinate chart for $\mathcal{M}$.

Simply compute

$$\begin{aligned} x_4 &= x_1^2 / x_2 \\ x_5 &= -x_1 x_2 / x_4 = -x_2^2 / x_1 \end{aligned}$$

to have the coordinate chart

$$\varphi^{-1}(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1^2/x_2 \\ -x_2^2/x_1 \end{bmatrix}.$$

which is defined in a neighborhood of

$$(x_{01}, x_{02}, x_{03}) = (-1, 1, 0).$$

2

Chapter 4 Solutions

2.1 Problems Chapter 4

Problem 1 *Tangent Vectors as Derivations*
Let $D \subset \mathbb{R}^n$ be an open subset and denote by $\mathcal{C}^\infty(\mathcal{D})$ the infinitely differentiable functions on $\mathcal{D}$. A *derivation* is a map D from $\mathcal{C}^\infty(\mathcal{D})$ to $\mathbb{R}$ such that for any $h_1, h_2 \in \mathcal{C}^\infty(\mathcal{D})$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ it satisfies
(i) Linearity
$D(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 D(h_1) + \alpha_2 D(h_2)$
(ii) Product Rule
$D(h_1 h_2) = h_1 D(h_2) + h_2 D(h_1)$

(a) Let $\mathcal{U} \subset \mathbf{S}^2$ and $\varphi : \mathcal{D} \triangleq \varphi(\mathcal{U})$ be a coordinate chart on $\mathbf{S}^2$. Let h be a function defined on $\mathcal{U}$ so that $\mathfrak{h} = h \circ \varphi^{-1}(x_1, x_2)$ is defined on $\mathcal{D} = \varphi(\mathcal{U})$. Define $\dfrac{\partial}{\partial x_1} : \mathcal{C}^\infty(\mathbf{S}^2) \to \mathbb{R}$ by

$$\frac{\partial}{\partial x_1} : h \to \frac{\partial h \circ \varphi^{-1}(x_1, x_2)}{\partial x_1} = \frac{\partial \mathfrak{h}}{\partial x_1}.$$

Show that this satisfies (i) and (ii), i.e., it is a derivation.

(b) Recall the modern definition of a tangent vector as a mapping $\mathbf{z}_p : \mathcal{C}^\infty(\mathbf{S}^2) \to \mathbb{R}$ given by

$$\mathbf{z}_p : h \to \mathbf{z}_p(h) \triangleq f_1(x_1, x_2)\frac{\partial h \circ \varphi^{-1}(x_1, x_2)}{\partial x_1} + f_2(x_1, x_2)\frac{\partial h \circ \mathbf{z}(x_1, x_2)}{\partial x_2}$$

where $f_1(x_1, x_2), f_2(x_1, x_2)$ are the *components* of the vector. Show that $\mathbf{z}_p$ is a derivation.

Also recall that we might "prefer" to look at this mapping as

$$\begin{aligned}
\mathbf{z}_p(h) &= \begin{bmatrix} \dfrac{\partial h(z)}{\partial z_1} & \dfrac{\partial h(z)}{\partial z_2} & \dfrac{\partial h(z)}{\partial z_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \\ \dfrac{\partial z_3}{\partial x_1} & \dfrac{\partial z_3}{\partial x_2} \end{bmatrix}_{(x_{01}, x_{02})} \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \\
&= dh \begin{bmatrix} \mathbf{z}_{x_1} & \mathbf{z}_{x2} \end{bmatrix} \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}
\end{aligned}$$

with $f_1(x_1, x_2)\mathbf{z}_{x_1} + f_2(x_1, x_2)\mathbf{z}_{x_2}$ the "tangent vector". However, this does not make sense as h is defined only on $\mathbf{S}^2$ (so $\partial h(z)/\partial z_1$, etc. are not defined) and $\mathbf{z}_{x_1}, \mathbf{z}_{x2}$ stick off of $\mathbf{S}^2$.

(c) Let $f = (f_1(x_1, x_2), f_2(x_1, x_2))$ and $g = (g_1(x_1, x_2), g_2(x_1, x_2))$ be the components of two vector fields defined on $\mathcal{D}$. With $\mathfrak{h} = h \circ \varphi^{-1}(x_1, x_2)$ we have shown that

$$\mathcal{L}_f(\mathcal{L}_g(\mathfrak{h})) = g^T \frac{\partial^2 \mathfrak{h}}{\partial x^2} f + \frac{\partial \mathfrak{h}}{\partial x}\frac{\partial g}{\partial x} f.$$

Is this a derivation, i..e., does it satisfy (i) and (ii) above? Explain.

(d) Show that $D \triangleq \mathcal{L}_f\mathcal{L}_g - \mathcal{L}_g\mathcal{L}_f : \mathcal{C}^\infty(\mathbf{S}^2) \to \mathbb{R}$ given by

$$\mathfrak{h} \to \mathcal{L}_f(\mathcal{L}_g(\mathfrak{h})) - \mathcal{L}_g(\mathcal{L}_f(\mathfrak{h}))$$

is a derivation. Equivalently, $\mathcal{L}_{[f,g]}(\mathfrak{h}) = \mathcal{L}_f(\mathcal{L}_g(\mathfrak{h})) - \mathcal{L}_g(\mathcal{L}_f(\mathfrak{h}))$ is a derivation.

Problem 2 *The Frobenius Theorem* [**?**]
Consider the following system of partial differential equations

$$\begin{aligned} \frac{\partial S}{\partial u_1} &= f^{(1)}(u_1, u_2, s_1(u_1,u_2), s_2(u_1,u_2), s_3(u_1,u_2)) \qquad (2.1) \\ \frac{\partial S}{\partial u_2} &= f^{(2)}(u_1, u_2, s_1(u_1,u_2), s_2(u_1,u_2), s_3(u_1,u_2)) \qquad (2.2) \end{aligned}$$

where

$$S(u_1,u_2) = \begin{bmatrix} s_1(u_1,u_2) \\ s_2(u_1,u_2) \\ s_3(u_1,u_2) \end{bmatrix}$$

and

$$f^{(1)}(u_1,u_2,x_1,x_2,x_3) = \begin{bmatrix} f_1^{(1)}(u_1,u_2,x_1,x_2,x_3) \\ f_2^{(1)}(u_1,u_2,x_1,x_2,x_3) \\ f_3^{(1)}(u_1,u_2,x_1,x_2,x_3) \end{bmatrix}, \quad f^{(2)}(u_1,u_2,x_1,x_2,x_3) = \begin{bmatrix} f_1^{(2)}(u_1,u_2,x_1,x_2,x_3) \\ f_2^{(2)}(u_1,u_2,x_1,x_2,x_3) \\ f_3^{(2)}(u_1,u_2,x_1,x_2,x_3) \end{bmatrix}.$$

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^3$, $\mathcal{D}$ an open subset of $\mathbb{R}^2$. Suppose, given any point $x_0 \in \mathcal{U}$ and any $u_0 \in (u_{01}, u_{02}) \in \mathcal{D}$, there is a surface $S(u_1,u_2)$ satisfying (2.1) and (2.2) in some neighborhood of (u_{01}, u_{02}) with $S(u_{01},u_{02}) = x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$.

(a) Show that

$$\frac{\partial f^{(1)}(u,x)}{\partial u_2} - \frac{\partial f^{(2)}(u,x)}{\partial u_1} + \frac{\partial f^{(1)}(u,x)}{\partial x} f^{(2)} - \frac{\partial f^{(2)}(u,x)}{\partial x} f^{(1)} \equiv 0 \qquad (2.3)$$

for all $(u_1,u_2) \in \mathcal{D}$ and $x = (x_1,x_2,x_3) \in \mathcal{U}$.

Let

$$S(u_1,u_2) = \begin{bmatrix} s_1(u_1,u_2) \\ s_2(u_1,u_2) \\ s_3(u_1,u_2) \end{bmatrix} \quad \text{with} \quad S(u_{01},u_{02}) = x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

satisfying (2.1) and (2.2). Take the partial derivative of (2.1) with respect to u_2 to obtain

$$\begin{aligned} \frac{\partial^2 S}{\partial u_2 \partial u_1} &= \frac{\partial}{\partial u_2} f^{(1)}(u_1,u_2,s_1(u_1,u_2),s_2(u_1,u_2),s_3(u_1,u_2)) \\ &= \frac{\partial f^{(1)}(u,x)}{\partial u_2}_{|x=S(u_1,u_2)} + \frac{\partial f^{(1)}(u,x)}{\partial x}_{|x=S(u_1,u_2)} \frac{\partial S}{\partial u_2} \\ &= \frac{\partial f^{(1)}(u,x)}{\partial u_2}_{|x=S(u_1,u_2)} + \frac{\partial f^{(1)}(u,x)}{\partial x}_{|x=S(u_1,u_2)} f^{(2)}(u,x)_{|x=S(u_1,u_2)} \\ &= \frac{\partial f^{(1)}(u,S(u_1,u_2))}{\partial u_2} + \frac{\partial f^{(1)}(u,S(u_1,u_2))}{\partial x} f^{(2)}(u,S(u_1,u_2)). \end{aligned}$$

Similarly, taking the partial derivative of (2.2) with respect to u_1 gives

$$\begin{aligned}
\frac{\partial^2 S}{\partial u_1 \partial u_2} &= \frac{\partial}{\partial u_1} f^{(2)}(u_1, u_2, s_1(u_1,u_2), s_2(u_1,u_2), s_3(u_1,u_2)) \\
&= \frac{\partial f^{(2)}(u,x)}{\partial u_1}_{|x=S(u_1,u_2)} + \frac{\partial f^{(2)}(u,x)}{\partial x}_{|x=S(u_1,u_2)} \frac{\partial S}{\partial u_1} \\
&= \frac{\partial f^{(2)}(u,x)}{\partial u_1}_{|x=S(u_1,u_2)} + \frac{\partial f^{(2)}(u,x)}{\partial x}_{|x=S(u_1,u_2)} f^{(1)}(u,x)_{|x=S(u_1,u_2)} \\
&= \frac{\partial f^{(2)}(u,S(u_1,u_2))}{\partial u_1} + \frac{\partial f^{(2)}(u,S(u_1,u_2))}{\partial x} f^{(1)}(u,S(u_1,u_2)).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial^2 S}{\partial u_2 \partial u_1} - \frac{\partial^2 S}{\partial u_1 \partial u_2} &= \frac{\partial f^{(1)}(u,S(u_1,u_2))}{\partial u_2} - \frac{\partial f^{(2)}(u,S(u_1,u_2))}{\partial u_1} + \\
&\quad \frac{\partial f^{(1)}(u,S(u_1,u_2))}{\partial x} f^{(2)}(u,S(u_1,u_2)) - \frac{\partial f^{(2)}(u,S(u_1,u_2))}{\partial x} f^{(1)}(u,S(u_1,u_2)) \\
&= 0
\end{aligned}$$

for all (u_1, u_2) in a neighborhood of (u_{01}, u_{02}). In particular at $u_0, x_0 = S(u_0)$ this becomes

$$\frac{\partial f^{(1)}(u_0,x_0)}{\partial u_2} - \frac{\partial f^{(2)}(u_0,x_0)}{\partial u_1} + \frac{\partial f^{(1)}(u_0,x_0)}{\partial x} f^{(2)}(u_0,x_0) - \frac{\partial f^{(2)}(u_0,x_0)}{\partial x} f^{(1)}(u_0,x_0) = 0 \tag{2.4}$$

The statement of the problem says we can find a surface such that for any point $x_0 \in \mathcal{U}$ and any $u_0 \in (u_{01}, u_{02}) \in \mathcal{D}$ that Equation (2.4) holds.

(b) Let

$$f^{(1)}(u_1,u_2,x_1,x_2,x_3) = \begin{bmatrix} x_1 - u_1 - u_2 \\ -x_2 + u_1 + u_2 \\ 0 \end{bmatrix}, \quad f^{(2)}(u_1,u_2,x_1,x_2,x_3) = \begin{bmatrix} x_1^2 - u_1^2 - u_2^2 - 2u_1 - 2u_2 - 2u_1u_2 \\ 1 \\ 0 \end{bmatrix}. \tag{2.5}$$

Are the integrability conditions (2.3) satisfied for these vector fields.

The integrability conditions are not satisfied as

$$\begin{aligned}
&\frac{\partial f^{(1)}(u,x)}{\partial u_2} - \frac{\partial f^{(2)}(u,x)}{\partial u_1} + \frac{\partial f^{(1)}(u,x)}{\partial x} f^{(2)}(u,x) - \frac{\partial f^{(2)}(u,x)}{\partial x} f^{(1)}(u,x) \\
&= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -2u_1 - 2 - 2u_2 \\ 0 \\ 0 \end{bmatrix} + \\
&\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^2 - u_1^2 - u_2^2 - 2u_1 - 2u_2 - 2u_1u_2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - u_1 - u_2 \\ -x_2 + u_1 + u_2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2x_1(u_1 + u_2 - x_1) - u_1^2 - u_2^2 + x_1^2 - 2u_1u_2 + 1 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

(c) Let

$$S(u_1,u_2) = \begin{bmatrix} u_1 + u_2 + 1 \\ u_1 + u_2 - 1 \\ x_{03} \end{bmatrix}. \tag{2.6}$$

Does this satisfy the partial differential equations (2.1) (2.2) with $f^{(1)}, f^{(2)}$ given by (2.5)? Does (2.6) satisfy

$$\left(\frac{\partial f^{(1)}(u,x)}{\partial u_2} - \frac{\partial f^{(2)}(u,x)}{\partial u_1} + \frac{\partial f^{(1)}(u,x)}{\partial x} f^{(2)} - \frac{\partial f^{(2)}(u,x)}{\partial x} f^{(1)}\right)_{x=s(u_1,u_2)} \equiv 0 \ ?$$

Is there any contradiction with your answer to part (b)? Explain.

The surface (2.6) does satisfy (2.1) and (2.2) as

$$\begin{aligned}
\frac{\partial}{\partial u_1} S(u_1,u_2) &= \frac{\partial}{\partial u_1}\begin{bmatrix} u_1+u_2+1 \\ u_1+u_2-1 \\ x_{03}\end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
f^{(1)}(u_1,u_2,x_1,x_2,x_3)_{|x=S(u_1,u_2)} &= \begin{bmatrix} x_1-u_1-u_2 \\ -x_2+u_1+u_2 \\ 0 \end{bmatrix}_{|x=\begin{bmatrix} u_1+u_2+1 \\ u_1+u_2-1 \\ x_{03}\end{bmatrix}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial u_2} S(u_1,u_2) &= \frac{\partial}{\partial u_1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
f^{(2)}(u_1,u_2,x_1,x_2,x_3)_{|x=S(u_1,u_2)} &= \begin{bmatrix} x_1^2-u_1^2-u_2^2-2u_1-2u_2-2u_1u_2 \\ 1 \\ 0 \end{bmatrix}_{|x=\begin{bmatrix} u_1+u_2+1 \\ u_1+u_2-1 \\ x_{03}\end{bmatrix}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.
\end{aligned}$$

We have

$$\begin{aligned}
&\left(\frac{\partial f^{(1)}(u,x)}{\partial u_2} - \frac{\partial f^{(2)}(u,x)}{\partial u_1} + \frac{\partial f^{(1)}(u,x)}{\partial x} f^{(2)} - \frac{\partial f^{(2)}(u,x)}{\partial x} f^{(1)}\right)_{x=s(u_1,u_2)} \\
&= \begin{bmatrix} 2x_1\left(u_1+u_2-x_1\right)-u_1^2-u_2^2+x_1^2-2u_1u_2+1 \\ 0 \\ 0 \end{bmatrix}_{|x=\begin{bmatrix} u_1+u_2+1 \\ u_1+u_2-1 \\ x_{03}\end{bmatrix}} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

There is no contradiction with part (b) as the surface (2.6) has

$$S(0,0) = \begin{bmatrix} 1 \\ -1 \\ x_{03} \end{bmatrix}$$

and the theorem requires there be a neighborhood of a point x_0 for which a surface satisfying (2.6) exists and contains any given point in that neighborhood.

Problem 3 *The Frobenius Theorem*

Problem 4 *The Frobenius Theorem*

Problem 5 *The Lie Derivative of a Vector $\mathcal{L}_f g$*

Consider two vector fields f and g in $\mathcal{U} \subset \mathbf{E}^3$. Let $\phi^f(t,x) = \phi_t^f(x)$ and $\phi^g(t,x) = \phi_t^g(x)$ be the flows of f and g, respectively. That is,

$$\begin{aligned}\frac{d}{dt}\phi^f(t,x) &= f(\phi^f(t,x)) \text{ with } \phi^f(0,x) = x\\ \frac{d}{dt}\phi^g(t,x) &= g(\phi^g(t,x)) \text{ with } \phi^g(0,x) = x.\end{aligned}$$

For each fixed t, $\phi^f(t,x)$ represents starting at x and moving in the direction specified by f for the time t to reach the point $x' \triangleq \phi^f(t,x)$. That is, for each fixed t, $\phi^f(t,\cdot) : \mathbf{E}^3 \to \mathbf{E}^3$ that takes x to $x' = \phi^f(t,x)$. Further, starting at $x' = \phi^f(t,x)$ and following the vector field f for a time $-t$ results in coming back to x, that is,

$$x = \phi^f(-t,x') = \phi^f(-t,\phi^f(t,x)).$$

Then

$$\frac{\partial}{\partial x}x = \frac{\partial}{\partial x}\phi^f(-t,\phi^f(t,x)) = \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\frac{\partial}{\partial x}\phi^f(t,x) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

(a) Use this relationship to show that

$$\left(\frac{d}{dt}\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\right)_{t=0} = -\frac{\partial f(x)}{\partial x}.$$

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} &= \frac{d}{dt}\left(\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\frac{\partial}{\partial x}\phi^f(t,x)\right)\\ &= \left(\frac{d}{dt}\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\right)\frac{\partial}{\partial x}\phi^f(t,x) + \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\left(\frac{d}{dt}\frac{\partial}{\partial x}\phi^f(t,x)\right)\end{aligned} \tag{2.7}$$

or

$$\begin{aligned}\left(\frac{d}{dt}\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\right)\frac{\partial}{\partial x}\phi^f(t,x) &= -\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\left(\frac{d}{dt}\frac{\partial}{\partial x}\phi^f(t,x)\right)\\ &= -\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\frac{\partial}{\partial x}f(\phi^f(t,x)).\end{aligned}$$

At $t=0$ this becomes

$$\left(\frac{d}{dt}\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\right)_{t=0} = -I_{3\times3}\frac{\partial}{\partial x}f(x) = -\frac{\partial f}{\partial x}. \tag{2.8}$$

(b) With

$$\left(\phi_{-t}^f\right)_* \triangleq \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}$$

define

$$\mathcal{L}_f g \triangleq \left(\frac{d}{dt}\left(\left(\phi_{-t}^f\right)_* g(\phi^f(t,x))\right)\right)_{t=0}$$

and show that

$$\mathcal{L}_f g = [f, g].$$

We have

$$\begin{aligned}
\mathcal{L}_f g &\triangleq \left(\frac{d}{dt}\left(\left(\phi^f_{-t}\right)_* g(\phi^f(t,x))\right)\right)_{t=0} \\
&= \left(\frac{d}{dt}\left(\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)} g(\phi^f(t,x))\right)\right)_{t=0} \\
&= \left(\frac{d}{dt}\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\right)_{t=0} g(\phi^f(t,x))_{|t=0} + \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)} \left(\frac{d}{dt} g(\phi^f(t,x))\right)_{|t=0} \\
&= -\frac{\partial f}{\partial x} g + f\frac{\partial g}{\partial x} \\
&= [f, g].
\end{aligned}$$

Remark A more enlightening way to evaluate the definition of $\mathcal{L}_f g$ is shown in [?] (page 61) as follows. Using Equation (2.8) a two term Taylor series expansion of

$$\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}$$

about $t = 0$ is

$$\begin{aligned}
\left(\phi^f_{-t}\right)_* &\triangleq \left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)} \\
&\approx \left(\frac{\partial}{\partial x'}\phi^f(-0,x')\right)_{|x'=\phi^f(0,x)} + \left(\frac{d}{dt}\left(\frac{\partial}{\partial x'}\phi^f(-t,x')\right)_{|x'=\phi^f(t,x)}\right)_{t=0} t \\
&= I_{3\times 3} - \frac{\partial f(x)}{\partial x} t.
\end{aligned}$$

A two term Taylor series expansion of $g(\phi^f(t,x))$ about $t = 0$ is

$$\begin{aligned}
g(\phi^f(t,x)) &\approx g(\phi^f(0,x)) + \left(\frac{d}{dt} g(\phi^f(t,x))\Big|_{t=0}\right) t \\
&= g(\phi^f(0,x)) + \left(\frac{\partial g}{\partial x}_{|\phi^f(0,x)}\right)\left(\frac{d\phi^f(t,x))}{dt}\Big|_{t=0}\right) t \\
&= g(\phi^f(0,x)) + \frac{\partial g}{\partial x}\left(f(\phi^f(t,x))_{|t=0}\right) t \\
&= g(x)) + \left(\frac{\partial g}{\partial x} f(x)\right) t
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{L}_f g &\triangleq \left(\frac{d}{dt} \left(\left(\phi^f_{-t} \right)_* g(\phi^f(t,x)) \right) \right)_{t=0} \\
&= \lim_{t \to 0} \frac{\left(I_{3\times 3} - \frac{\partial f(x)}{\partial x} t \right) \left(g(x) + \left(\frac{\partial g}{\partial x} f(x) \right) t \right) - g(x)}{t} \\
&= \lim_{t \to 0} \frac{g(x) + \left(\frac{\partial g}{\partial x} f(x) \right) t - \frac{\partial f(x)}{\partial x} t \left(g(x) + \left(\frac{\partial g}{\partial x} f(x) \right) t \right) - g(x)}{t} \\
&= \lim_{t \to 0} \left(\frac{\partial g}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} \left(g(x) + \left(\frac{\partial g}{\partial x} f(x) \right) t \right) \right) \\
&= \frac{\partial g}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x).
\end{aligned}$$