

Nonlinear Systems

Morse Theory and Lyapunov Stability on Manifolds

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Outline

Introduction

Notations and Definitions

Lyapunov Stability Analysis on Euclidean Spaces

Morse-Lyapunov Functions

Systems with Single Critical Points

Systems with Multiple Critical Points

Introduction

Lyapunov Stability – Introduction

- ▶ Introduced by Alexandr Mikhailovich Lyapunov.
- ▶ *The general problem of the stability of motion*, 1892.
- ▶ Doctoral thesis in Kharkov Mathematical Society.
- ▶ The most general theory for analyzing stability of (at least) ordinary differential equations.

Lyapunov Stability – Introduction

- ▶ Different notions of stability: input-output stability, periodic orbit stability, etc.
- ▶ Stability of equilibrium points usually characterized in the sense of Lyapunov.
 - ▶ An equilibrium point is **STABLE** if all solutions starting at nearby points stay nearby.
 - ▶ It is **ASYMPTOTICALLY STABLE** if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.
- ▶ For a linear system $\dot{x} = Ax$, the stability of $x = 0$ can be completely characterized by the eigenvalues of A .
- ▶ Stability of a nonlinear system sometimes can be characterized by the same method (through linearization).
- ▶ Lyapunov stability theorems give sufficient conditions for stability.

Notations and Definitions

Manifolds and Vector Fields

- ▶ \mathcal{M} (state-space) denotes a manifold of finite dimension n .
- ▶ $f \in \mathfrak{X}(M)$ is a continuous vector field on \mathcal{M} .
- ▶ We assume that there exists a unique right maximally defined integral curve of f starting at x .
- ▶ We also assume that this integral curve is defined on $[0, \infty]$.

$$\varphi : [0, \infty] \times \mathcal{M} \rightarrow \mathcal{M}$$

with

$$\begin{aligned}\varphi(0, x) &= x, \\ \varphi(t_1, \varphi(t_2, x)) &= \varphi(t_1 + t_2, x).\end{aligned}$$

- ▶ The semiflow φ is the evolution function.

Invariant and Stable Sets

Definition

$\Omega \subseteq \mathcal{M}$ is called an INVARIANT SET if for all $x \in \Omega$ and $t \in \mathbb{R}_{\geq 0}$, $\varphi(t, x) \in \Omega$. If $\Omega = \{p\}$ is a singleton, then Ω is called an EQUILIBRIUM POINT of the dynamical system (\mathcal{M}, φ) .

Definition

$\Omega \subseteq \mathcal{M}$ is STABLE if for every open neighborhood $\mathcal{U} \subseteq \mathcal{M}$ of Ω , there exists a neighborhood $\mathcal{V} \subseteq \mathcal{M}$ of Ω such that $\varphi(t, \mathcal{V}) \subseteq \mathcal{U}$ for all $t \geq 0$.

An invariant set Ω is asymptotically stable if

- ▶ Ω is stable,
- ▶ Ω is attractive, i.e., for all $x \in \Omega$, there exists an open neighborhood $\mathcal{N} \subseteq \mathcal{M}$ of Ω such that for all $x \in \mathcal{N}$,
$$\varphi(t, x) \xrightarrow{t \rightarrow \infty} \Omega.$$

Domain (Region) of Attraction

The domain of attraction is denoted by

$$\mathcal{A} = \{x \in \mathcal{M} : \varphi(t, x) \rightarrow \Omega \text{ as } t \rightarrow \infty\}.$$

Ω is said to be GLOBALLY asymptotically stable if $\mathcal{N} = \mathcal{M}$.

Definition

The LIE DERIVATIVE of $V : \mathcal{M} \rightarrow \mathbb{R}$ along $f \in \mathfrak{X}(\mathcal{M})$ is defined by

$$\begin{aligned}\mathcal{L}_f V : \mathcal{M} &\rightarrow \mathbb{R}, \\ p &\mapsto dV_p(f(p)).\end{aligned}$$

Lyapunov Function

Definition

Let \mathcal{K} be an invariant set of the dynamical system (\mathcal{M}, φ) . A continuous function $V : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$ is a LYAPUNOV FUNCTION if

- ▶ $V(x) > 0$ for all $x \in \mathcal{A} \setminus \mathcal{K}$,
- ▶ $V(x) = 0$ for all $x \in \mathcal{K}$,
- ▶ V is proper, i.e., $V^{-1}(B)$ is compact for all compact subset $B \subseteq \mathbb{R}_{\geq 0}$,
- ▶ V is strictly decreasing along orbits of φ , i.e.,

$$V \circ \varphi(t, x) < V(x),$$

for all $t > 0$ and $x \in \mathcal{A} \setminus \mathcal{K}$.

If V is differentiable, this condition may be replaced by

$$\mathcal{L}_f V(x) < 0.$$

(Nondegenerate) Critical Points

Definition

Let $V : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth function. A CRITICAL POINT, $p \in \mathcal{M}$, of V is a point where the differential

$$dV_p : T_p\mathcal{M} \rightarrow \mathbb{R}$$

has rank zero, i.e., in any local coordinate system $\{x_i\}_1^n$, one has $\frac{\partial V}{\partial x_i}(p) = 0$ for all $i = 1, \dots, n$.

Definition

A critical point p is NONDEGENERATE if the Hessian $H_p(V)$ is a nondegenerate bilinear form, i.e., if any coordinate system, the Hessian matrix

$$\left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq n}$$

is nondegenerate.

Nondegenerate Critical Points

Definition

The dimension of the subspace of $T_p\mathcal{M}$ on which $H_p(V)$ is negative definite is called the MORSE INDEX of V at p , denoted by $\text{ind}(V, p)$.

Definition

A C^2 function $V : \mathcal{M} \rightarrow \mathbb{R}$ is a MORSE FUNCTION if all its critical points are nondegenerate.

Definition

The (SUB)-LEVEL SETS of a function $V : \mathcal{M} \rightarrow \mathbb{R}$ are

$$\begin{aligned}\mathcal{M}_a &= V^{-1}((-\infty, a]), \\ \mathcal{M}_{a,b} &= V^{-1}([a, b]).\end{aligned}$$

Topological Definitions

- ▶ A top. space is an n -CELL if it is homeomorphic to \mathbb{R}^n .
- ▶ A top. space X is CONTRACTIBLE if it is *homotopy equivalent* to the one-point space.
- ▶ A subspace A of X is called a DEFORMATION RETRACT of X if there exists a continuous function $h : [0, 1] \times X \rightarrow X$ such that for all $x \in X, a \in A$,

$$h(0, x) = x,$$

$$h(1, x) \in A,$$

$$h(1, a) = a.$$

- ▶ The k^{th} BETTI NUMBER of \mathcal{M} , denoted by b_k is the rank of the k^{th} homology group $H^k(\mathcal{M})$.
- ▶ The EULER CHARACTERISTIC of \mathcal{M} is defined by

$$\chi(\mathcal{M}) = \sum_{i=1}^k (-1)^i b_i.$$

Lyapunov Stability Analysis on Euclidean Spaces

Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{1}$$

where $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a locally Lipschitz map, with an equilibrium point at $x = 0$.

Definition

The equilibrium point $x = 0$ of the system (1) is

- ▶ *stable* if, $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0.$$

- ▶ *unstable* if it is not stable.
- ▶ *asymptotically stable* if it is stable and δ can be chosen s.t.

$$\|x(0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0.$$

Example – Pendulum

The pendulum equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a \sin x_1 - bx_2$$

has two equilibrium points at $(x_1 = 0, x_2 = 0)$ and $(x_1 = \pi, x_2 = 0)$.

- ▶ If $b = 0$, trajectories in the nbhd. of the first equilibrium are closed orbits.
- ▶ By starting sufficiently close to the eq. point, trajectories are guaranteed to stay within any specified ball.
- ▶ The point is not asymptotically stable since trajectories don't tend to the eq. point.
- ▶ If $b > 0$, the origin becomes asymptotically stable.
- ▶ The second eq. point is a saddle point: the $\varepsilon - \delta$ requirement cannot be satisfied (for every $\varepsilon > 0$ there exists a trajectory that will leave the ball B_ε even if $x(0)$ is arbitrarily close to $(\pi, 0)$).

Lyapunov Stability Theorem

Theorem

Let $x = 0 \in D$ be an equilibrium point for (1). Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\},$$
$$\dot{V}(x) \leq 0 \text{ in } D.$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\}$$

then $x = 0$ is asymptotically stable.

Lyapunov Stability Theorem

Proof of stability.

Given $\varepsilon > 0$, choose $0 < r \leq \varepsilon$ such that $B_r \subseteq D$. Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$. Take $0 < \beta < \alpha$ and consider $\mathcal{M}_\beta = V^{-1}((0, \beta])$.

Claim: $\mathcal{M}_\beta \subseteq \mathring{B}_r$. Argue ad absurdum. Suppose $\mathcal{M}_\beta \cap \mathring{B}_r \neq \mathcal{M}_\beta$. Then $\exists p \in \mathcal{M}_\beta \cap \partial B_r$. Note, $V(p) \geq \alpha > \beta$, but $V(\mathcal{M}_\beta) \subseteq [0, \beta]$.

The set \mathcal{M}_β is invariant since

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta, \forall t \geq 0.$$

Because \mathcal{M}_β is compact (closed and bounded), we conclude that the ODE (1) has a unique solution $\forall t \geq 0$ whenever $x(0) \in \mathcal{M}_\beta$. Since V is continuous and $V(0) = 0$, $\exists \delta > 0$ such that

$$\|x\| \leq \delta \Rightarrow V(x) < \beta.$$



Lyapunov Stability Theorem

Proof of stability (cont'd).

Then,

$$B_\delta \subseteq \mathcal{M}_\beta \subseteq B_r$$

and

$$x(0) \in B_\delta \Rightarrow x(0) \in \mathcal{M}_\beta \Rightarrow x(t) \in \mathcal{M}_\beta \Rightarrow x(t) \in B_r,$$

proving stability.



Lyapunov Stability Theorem

Proof of asymptotic stability.

Now assume $\dot{V}(x) < 0$ in $D - \{0\}$. We want to show that $x(t) \xrightarrow{t \rightarrow \infty} 0$; i.e., $\forall a > 0, \exists T > 0$, s.t. $\|x(t)\| < a, \forall t > T$.

We know that $\forall a > 0$, we can choose $b > 0$ s.t. $\mathcal{M}_b \subseteq B_a$. Therefore, it is sufficient to show that $V(x(t)) \xrightarrow{t \rightarrow \infty} 0$. Since V is monotonically decreasing and bounded from below by zero,

$$V(x(t)) \xrightarrow{t \rightarrow \infty} c \geq 0.$$

Claim: $c = 0$. Argue ad absurdum. Suppose $c > 0$. By continuity of V , $\exists d > 0$ s.t. $B_d \subseteq \mathcal{M}_c$. The limit $V(x(t)) \rightarrow c > 0$ implies that $x(t) \notin B_d, \forall t \geq 0$. Define $\max_{d \leq \|x\| \leq r} \dot{V}(x) =: -\gamma < 0$. It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau)) d\tau \leq V(x(0)) - \gamma t.$$

The RHS will eventually become negative: contradiction ($c > 0$). \square

Morse-Lyapunov Functions

Isolated Critical Points

Lemma

Suppose that x_e is an equilibrium points of the dynamical system (M, φ) . If $V : \mathcal{M} \rightarrow \mathbb{R}$ is a differentiable Lyapunov function then x_e is the only critical point of V .

Proof.

Suppose V has another critical point, x_c , in the domain of attraction. By the definition of a Lyapunov function, we must have $\mathcal{L}_f V(x_c) = 0$. This contradicts the fact that if $x \neq x_e$, $\mathcal{L}_f V(x) < 0$. \square

Morse Lemma

Theorem (Morse Lemma)

Let $p \in \mathcal{M}$ be a nondegenerate critical point of a smooth function $V : \mathcal{M} \rightarrow \mathbb{R}$. There exists a local coordinate system $\{x_i\}_1^n$ in a nbhd. $\mathcal{N} \subseteq \mathcal{M}$ of p with $x_i(p) = 0$ for all $1 \leq i \leq n$ such that for $x \in \mathcal{N}$,

$$V(x) = V(p) - x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2$$

where $i = \text{ind}(V, p)$.

Corollary

Let $p \in \mathcal{M}$ be an equilibrium point of (\mathcal{M}, φ) and $V : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$ a Morse-Lyapunov function. There exists a local coordinate system $\{x_i\}_1^n$ around p such that V is locally the canonical quadratic Lyapunov function

$$V(x) = \sum_{i=1}^n x_i^2$$

with $\text{ind}(V, p) = 0$.

Level Sets of a Lyapunov Function

Theorem (Deformation Lemma)

Let $V : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth function and $a, b \in V(\mathcal{M})$ such that $a < b$. If $\mathcal{M}_{a,b}$ is compact and does not contain critical points of V then \mathcal{M}_a is diffeomorphic to \mathcal{M}_b . Moreover, \mathcal{M}_a is a deformation retract of \mathcal{M}_b .

Corollary

Let \mathcal{M} be a smooth Riemannian manifold. If \mathcal{M} contains a closed invariant asymptotically stable set, then for all $a, b \in V(\mathcal{M})$, \mathcal{M}_a is diffeomorphic to \mathcal{M}_b and \mathcal{M}_a is a deformation retract of \mathcal{M}_b where V is a smooth Lyapunov function.

Systems with Single Critical Points

Domain of Attraction – Revisited

Theorem (Brown-Stallings Lemma)

Let \mathcal{M} be a paracompact manifold such that every compact subset is contained in an open set diffeomorphic to a Euclidean space. Then \mathcal{M} itself is diffeomorphic to a Euclidean space.

Corollary

Let \mathcal{M} be a paracompact manifold. The domain of attraction of an asymptotically stable equilibrium point is diffeomorphic to a Euclidean space.

Morse and Sontag Theorems

Theorem (Morse Theorem)

Let $V : \mathcal{M} \rightarrow \mathbb{R}$ be a Morse function, p a critical point such that $\text{ind}(V, p) = i$ and $c = V(p)$. If there exists $\varepsilon > 0$ such that $\mathcal{M}_{c-\varepsilon, c+\varepsilon}$ is compact and does not contain other critical points p , then $\mathcal{M}_{c-\varepsilon} \cup e_i$ is a deformation retract of $\mathcal{M}_{c+\varepsilon}$ where e_i is an i -cell.

Theorem (Sontag Theorem)

Let us consider the dynamical system (\mathcal{M}, φ) with an equilibrium point $x_e \in \mathcal{M}$. Suppose that x_e is asymptotically stable. Then the domain of attraction of x_e , given by

$$\mathcal{A} = \left\{ x \in \mathcal{M} : \lim_{t \rightarrow \infty} \varphi(t, x) = x_e \right\},$$

is contractible.

Systems with Multiple Critical Points

Morse Theorem – (Third Version)

Theorem (Morse Theorem)

If $V : \mathcal{M} \rightarrow \mathbb{R}$ is a Morse function such that \mathcal{M}_a is compact for each $a \in \mathbb{R}$ then \mathcal{M} has the homotopy type of a CW-complex with one i -cell for each critical point of index i .

Corollary

Suppose that the dynamical system (\mathcal{M}, φ) has several equilibria (x_1, \dots, x_k) . If there exists a Morse-Lyapunov function $V : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$ then $\{x_1, \dots, x_k\}$ is a retract of the domain of attraction.

Proposition (Reeb Theorem)

Suppose that \mathcal{M} is compact without boundary. If $V : \mathcal{M} \rightarrow \mathbb{R}$ is a smooth function with only two critical points, then \mathcal{M} is homeomorphic to the n -sphere \mathbb{S}^n .

Morse Inequalities

Theorem (Morse Inequalities)

Let m_k be the number of critical points of a Morse function V with index k . Then, we have

$$\begin{aligned} b_k &\leq m_k, \quad \forall k, \\ \sum_{i=0}^j (-1)^{j-i} b_i &\leq \sum_{i=0}^j (-1)^{j-i} m_i \quad \forall j, \\ \chi(\mathcal{M}) &= \sum_k (-1)^k b_k = \sum_k (-1)^k m_k. \end{aligned}$$

The next corollary states a necessary condition for the existence of a Morse-Lyapunov function based on the Euler characteristic, which is a topological invariant.

Existence of Morse-Lyapunov Functions

Corollary

Consider the dynamical system (\mathcal{M}, φ) with several equilibria (x_1, \dots, x_k) . If there exists a Morse-Lyapunov function $V : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$ then $\chi(\mathcal{M}) = k \geq b_0$.

Proof.

If there exists a Morse-Lyapunov function V , (x_1, \dots, x_k) are the only critical points with indices 0. Then, by the Morse inequalities, $\chi(\mathcal{M}) = m_0 = k$ and $b_0 \leq m_0 = k$. □

Remark

If $\chi(\mathcal{M}) \neq k$ then there is no Morse-Lyapunov function for the dynamical system.

