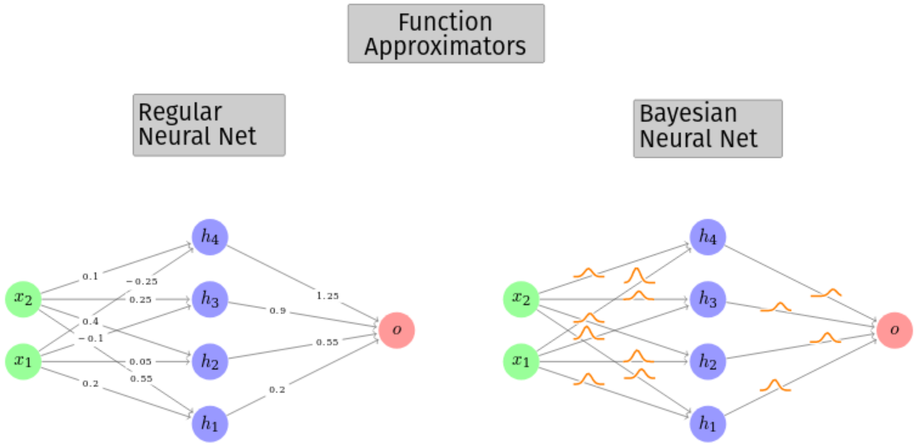
Robust Control Under Uncertainties

Optimal Control under System Parameter and Measurement Uncertainties

$$
\begin{aligned}
\underset{\theta}{\text{minimize}} \qquad J(\theta, x_0) &= \int_0^T \ell\left(\phi, u^\theta, \theta\right) \mathrm{d}t \\
\text{subject to} \qquad dx &= f\left(x, u^\theta\right) dt + \nabla_x u(x) dW_t \\
p &\sim \mathcal{N}\left(\widehat{p}, \sigma_p\right)
\end{aligned}
$$



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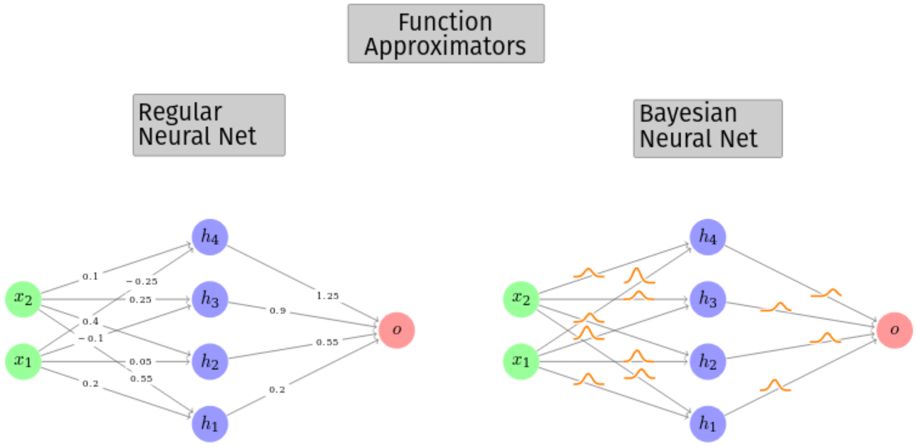

- BNNs capture variance in controller
- Akin to learning ensemble of controllers that each minimize J

Bayesian Learning

Bayesian Passivity-based Control

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& p && \sim \mathcal{N}\left(\widehat{p}, \sigma_p\right) \\
& u^\theta && = -G^\dagger \nabla_q H_d^\theta - K_D G^\top \nabla_p H_d^\theta \\
& \theta && \sim q(\theta; z)
\end{aligned}$$

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p(\theta \mid \mathcal{D}) = \frac{\overbrace{p(\mathcal{D} \mid \theta)}^{\text{likelihood}}\overbrace{p(\theta)}^{\text{prior}}}{\underbrace{\int_\theta p(\mathcal{D} \mid \theta')p(\theta')d\theta'}_{\text{evidence}}} \underbrace{\approx q(\theta; z)}_{\text{VI}}.
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KL-divergence and ELBO

$$
\begin{aligned}
D_{\text{KL}} &= \mathbb{E}_{\theta \sim q}\left[\log \tfrac{q(\theta; z)}{p(\theta \mid \mathcal{D})} \right] \\
&= \log p(\mathcal{D}) - \mathbb{E}_{\theta \sim q}\left[\log \tfrac{p(\mathcal{D} \mid \theta) p(\theta)}{q(\theta; z)} \right] \\
\mathcal{L}(\mathcal{D}; z) &= \mathbb{E}_{\theta \sim q}[\log p(\mathcal{D} \mid \theta) p(\theta) - \log q(\theta; z)]
\end{aligned}
$$

Bayesian Solution

Computing ELBO

$$\mathcal{L}(\mathcal{D}; z) = \mathbb{E}_{\theta \sim q}[\ \log p(\mathcal{D} \mid \theta)p(\theta) - \ \log q(\theta; z)]$$

requires:

Bayesian Solution

Computing ELBO

$$\mathcal{L}(\mathcal{D}; z) = \mathbb{E}_{\theta \sim q}[\,\log p(\mathcal{D} \mid \theta)p(\theta) - \,\log q(\theta; z)]$$

requires:

- *Likelihood*:

$$p(J(\theta, x_0)) = \mathcal{N}(0, s).$$

Bayesian Solution

Computing ELBO

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- *Prior*:
  - Uninformed
  - Deterministic

Bayesian Solution

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- *Prior*:
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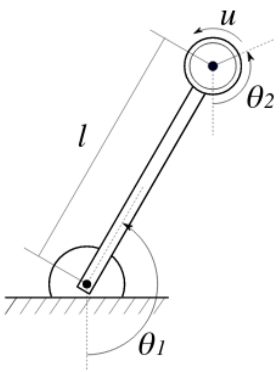
Prediction through maximum aposteriori

$$u(x) = u\left(x, \operatorname*{argmax}_{\theta} p(\theta; z)\right).$$

Prediction through marginalization

$$u(x) = \frac{1}{N} \sum_{\theta \sim q} u(x, \theta).$$

Case Study: Inertia Wheel Pendulum



The control input u is torque applied to the wheel. The equations of motion are

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -mgl \sin q_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} u.$$

<u>NeuralPbc training setup</u>

	Deterministic	**Bayesian**
H_d neural net size	(6, 12, 3, 1)	(6, 5, 3, 1)
Learned parameters	133	128
Optimizer	ADAM	DecayedAdaGrad
Initial learning rate	0.001	0.01
Replay buffer size	400	50

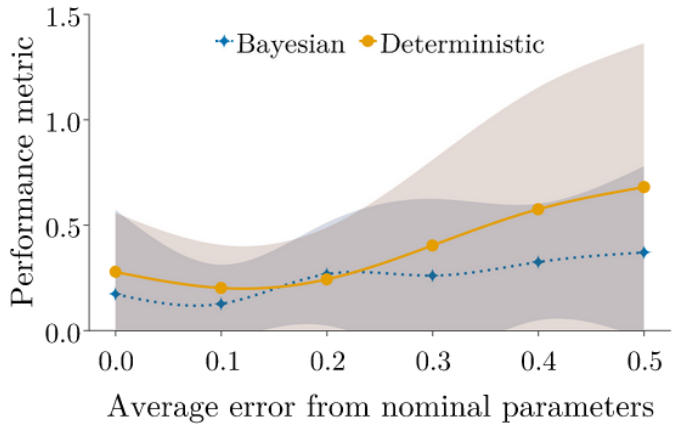
Deterministic vs Bayesian in Simulation

- Tested on swingup-task of the inertia wheel pendulum
- Parameter uncertainities and measurement noise are modelled as

$$dx = \left(\begin{bmatrix} \nabla_p H \\ -\nabla_q H \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u^\theta(x) \right) dt + \nabla_x u(x) dW_t.$$

- Measurement noise given by Wiener process with state uncertainties of 0.05 rad and 0.0005 rad/s
- Performance metric given by

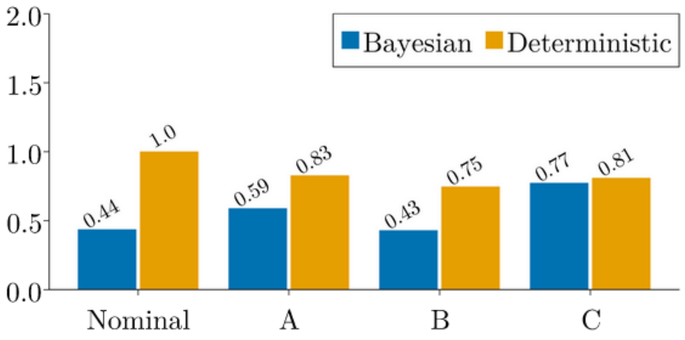
$$J_T = \frac{1}{2} \int_0^T (x^\top Q x + u^\top R u) dt$$

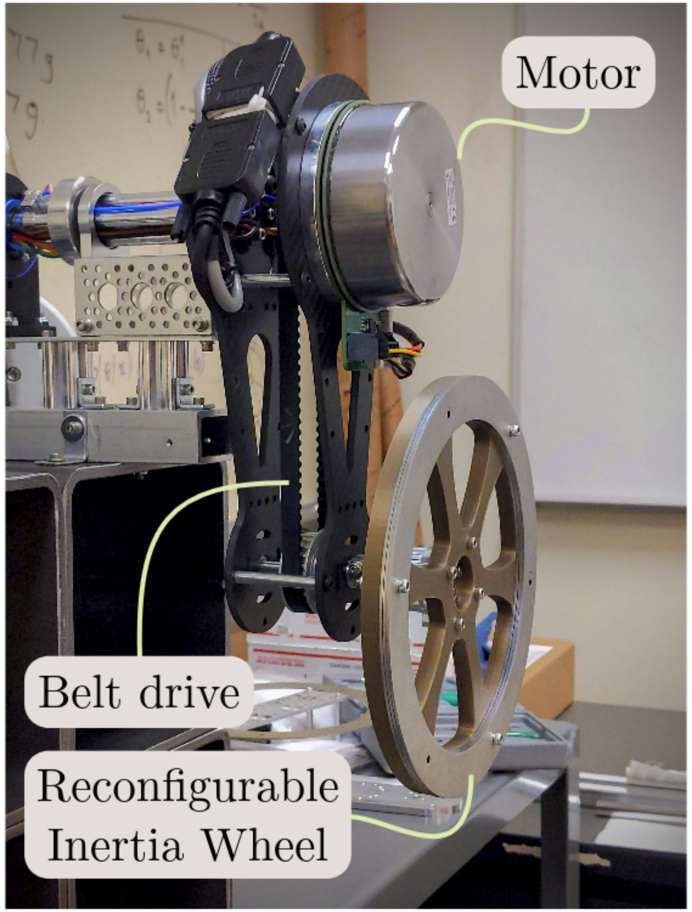


Deterministic vs Bayesian in Experiment

System parameters used in real-world experiments. The errors in the last column are $\|p_s - p_s^{\mathrm{nom}}\| / \|p_s^{\mathrm{nom}}\|$.

Parameter set p_s	I_1	I_2	mgl
Nominal	0.0455	0.00425	1.795
A	0.0417	0.00330	1.577
B	0.0378	0.00235	1.358
C	0.0340	0.00141	1.140





Closing Thoughts and Future Directions

PBC + machine learning techniques ✨

- We uncovered the engineering foundations for combining them
- Extensive experimental results in simulation and on hardware

Future directions—applications in:

- Dynamical models with uncertainity
- Hybrid dynamical systems (walking machines)