# Simple Geometry Problem

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***Abstract*—This simple geometry problem is my excuse to teach myself some Ti*k*Z.**

***Index Terms*—Geometry, Ti*k*Z**

## I. Introduction

I saw this problem in P. Talwalkar's YouTube channel [1].

## II. Problem Statement

The original problem is stated as in the caption of Figure 1. However, we will extend the statement slightly to make it a tiny bit more interesting. Let $r$ be the radius of the circles and pose the following questions:

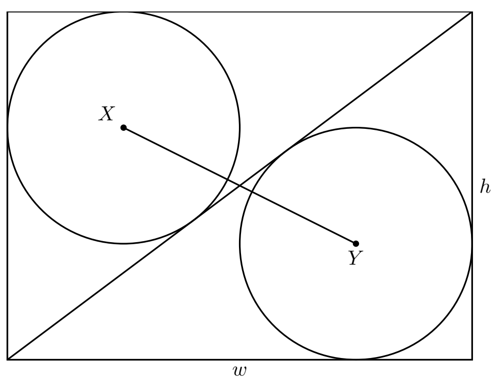


Fig. 1: Given the width $w$ and height $h$ of the rectangle, find the length of the line segment $XY$ between the centres of the inscribed circles, which have the two adjacent sides of the rectangle and its diagonal as tangents.

1) Find the map $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ that takes the side lengths of the rectangle $(w, h)$ to $(r, |XY|)$, the radius of one of the circles and the length of the line segment $XY$.
2) Find the inverse $f^{-1}$ of $f$, that takes $(r, |XY|)$ to $(w, h)$.

## III. Problem Solution

We will make use of Figure 2 in the following derivation. The diagonal length $d \triangleq |AC|$ of the rectangle is given by

$$d^2 = w^2 + h^2. \tag{1}$$

The diagonal length $d$ may also be related to $(w, h)$ and $r$ by noting that $|AE| = h-r$ and $|EC| = w-r$ due to the tangency to the circles. On the other hand, $d = |AC| = |AE| + |EC|$ so that

$$d + 2r = w + h. \tag{2}$$

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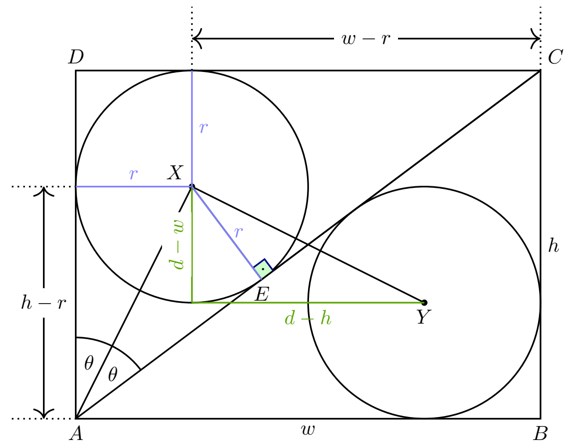


Fig. 2: Auxiliary drawing, identifying key lengths and angles.

### A. Computing $|XY|$, given $w$ and $h$

In order to compute $|XY|$, form the right triangle whose horizontal and vertical sides are drawn in green in Figure 2. The length of the vertical side of this triangle is $h - 2r$ but by equation (2) this is equal to $d - w$. Similarly, the length of the horizontal side of this triangle is $w - 2r$ but, by the same token, this is equal to $d - h$. Hence, we have

$$|XY|^2 = (d-w)^2 + (d-h)^2 = 3d^2 - 2d(w+h), \tag{3}$$

where the second equality may be derived by substituting from equation (1).

### B. Computing $r$, given $w$ and $h$

The right triangle $\triangle CAD$ provides the equality

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{w}{h},$$

which implies the quadratic equation

$$\tan^2\theta + 2{}^{h}\!/\!_{w}\tan\theta - 1 = 0.$$

Solving for $\tan\theta$ such that $0 < \theta < {}^{\pi}\!/_{2}$, we obtain

$$\frac{r}{h-r} = \tan\theta = {}^{1}\!/\!_{w}(d-h),$$

where the first equality follows from the right triangle $\triangle EAX$. We can solve this equation for the radius, $r$, which provides the solution

$$r = \frac{h(d-h)}{w+d-h}. \tag{4}$$

### C. *Computing* f *and its inverse* $f^{-1}$

The development of Sections III-B and III-A readily yields $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$.

$$\begin{aligned} d &= \sqrt{w^2+h^2}, \\ f(w,h) &= \left(\frac{h(d-h)}{w+d-h}, \sqrt{(d-w)^2+(d-h)^2}\right). \end{aligned} \tag{5}$$

In order to look for its inverse, let us recall equation (3) and substitute from (2) for $w+h$, yielding the relationship

$$d^2 - 4rd - |XY|^2 = 0, \tag{6}$$

which implies

$$d = 2r + \sqrt{|XY|^2 + 4r^2}, \tag{7}$$

since $d > 0$. Let us solve for $h$ from equation (2) and substitute into equation (1), which yields

$$w^2 - (2r+d)w + 2r(r+d) = 0.$$

Now, if we notice that equations (1) and (2) are both symmetric in $w$ and $h$, then we can deduce that the two solutions of this quadratic equation correspond to $w$ and $h$. Assuming, without loss of generality, $w > h$, and substituting from (6) and (7), we obtain

$$\begin{aligned} w(r,|XY|) &= 1/2\left(2r+d+\sqrt{d^2-4rd-4r^2}\right) \\ &= 2r + 1/2\left(\sqrt{|XY|^2+4r^2} + \sqrt{|XY|^2-4r^2}\right), \end{aligned}$$

and

$$\begin{aligned} h(r,|XY|) &= 1/2\left(2r+d+\sqrt{d^2-4rd-4r^2}\right) \\ &= 2r + 1/2\left(\sqrt{|XY|^2+4r^2} - \sqrt{|XY|^2-4r^2}\right), \end{aligned}$$

which yields the inverse function $f^{-1}(r,|XY|) = (w,h)$.

## IV. Conclusion

This has been a nice excuse to practice some Ti*k*Z.

## References

[1] P. Talwalker, “Mind your decisions.” https://youtu.be/YBLPzBuEaPc. Accessed: 2022-07-13.