Simple Geometry Problem

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***Abstract*— This simple geometry problem is my excuse to teach myself some Ti*k*Z.**

***Index Terms*— Geometry, Ti*k*Z**

I. Introduction

I saw this problem in P. Talwalkar's YouTube channel [1].

II. Problem Statement

The original problem is stated as in the caption of Figure 1. However, we will extend the statement slightly to make it a tiny bit more interesting. Let r be the radius of the circles and pose the following questions:

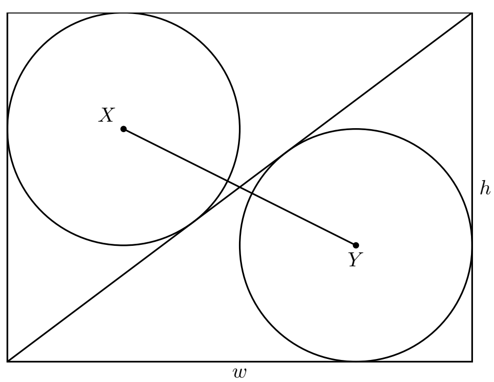

Fig. 1: Given the width w and height h of the rectangle, find the length of the line segment XY between the centres of the inscribed circles, which have the two adjacent sides of the rectangle and its diagonal as tangents.

1) Find the map $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ that takes the side lengths of the rectangle (w, h) to $(r, |XY|)$, the radius of one of the circles and the length of the line segment XY.
2) Find the inverse f^{-1} of f, that takes $(r, |XY|)$ to (w, h).

III. Problem Solution

We will make use of Figure 2 in the following derivation. The diagonal length $d \triangleq |AC|$ of the rectangle is given by

$$d^2 = w^2 + h^2. \tag{1}$$

The diagonal length d may also be related to (w, h) and r by noting that $|AE| = h-r$ and $|EC| = w-r$ due to the tangency to the circles. On the other hand, $d = |AC| = |AE| + |EC|$ so that

$$d + 2r = w + h. \tag{2}$$

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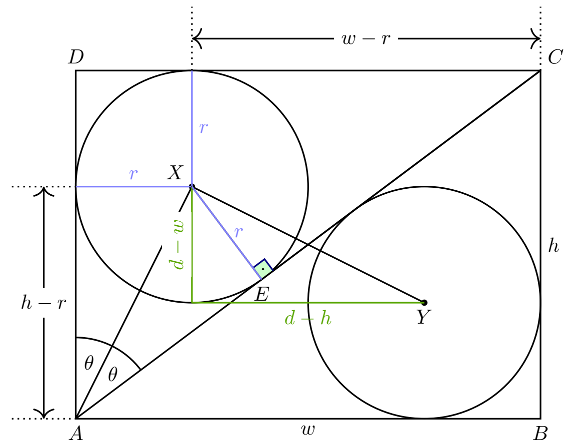

Fig. 2: Auxiliary drawing, identifying key lengths and angles.

A. Computing |XY| and r, given w and h

In order to compute $|XY|$, form the right triangle whose horizontal and vertical sides are drawn in green in Figure 2. The length of the vertical side of this triangle is $h - 2r$ but by equation (2) this is equal to $d - w$. Similarly, the length of the horizontal side of this triangle is $w - 2r$ but, by the same token, this is equal to $d - h$. Hence, we have

$$|XY|^2 = (d - w)^2 + (d - h)^2 = 3d^2 - 2d(w + h), \tag{3}$$

where the second equality may be derived by substituting from equation (1). The radius r can be read off from equation (2).

$$r = {}^1/_2(w + h - d) \tag{4}$$

B. Computing f and its inverse f^{-1}

The computation in the previous section determines $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$.

$$\begin{aligned} d &= \sqrt{w^2 + h^2}, \\ f(w, h) &= \left({}^1/_2(w + h - d), \sqrt{(d - w)^2 + (d - h)^2}\right). \end{aligned} \tag{5}$$

In order to look for its inverse, let us recall equation (3) and substitute from (2) for $w + h$, yielding the relationship

$$d^2 - 4rd - |XY|^2 = 0, \tag{6}$$

which implies

$$d = 2r + \sqrt{|XY|^2 + 4r^2}, \tag{7}$$

since $d > 0$. Let us solve for h from equation (2) and substitute into equation (1), which yields

$$w^2 - (2r + d)w + 2r(r + d) = 0.$$

Now, if we notice that equations (1) and (2) are both symmetric in w and h, then we can deduce that the two solutions of this quadratic equation correspond to w and h. Assuming, without loss of generality, $w > h$, and substituting from (6) and (7), we obtain

$$\begin{aligned} w(r, |XY|) &= 1/2\left(2r + d + \sqrt{d^2 - 4rd - 4r^2}\right) \\ &= 2r + 1/2\left(\sqrt{|XY|^2 + 4r^2} + \sqrt{|XY|^2 - 4r^2}\right), \end{aligned}$$

and

$$\begin{aligned} h(r, |XY|) &= 1/2\left(2r + d - \sqrt{d^2 - 4rd - 4r^2}\right) \\ &= 2r + 1/2\left(\sqrt{|XY|^2 + 4r^2} - \sqrt{|XY|^2 - 4r^2}\right), \end{aligned}$$

which yields the inverse function $f^{-1}(r, |XY|) = (w, h)$.

IV. Conclusion

This has been a nice excuse to practice some Ti*k*Z.

References

[1] P. Talwalker, "Mind your decisions." https://youtu.be/YBLPzBuEaPc. Accessed: 2022-07-13.

Appendix

The right triangle $\triangle CAD$ provides the equality

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{w}{h},$$

which implies the quadratic equation

$$\tan^2\theta + 2{}^h\!/\!{}_w \tan\theta - 1 = 0.$$

Solving for $\tan\theta$ such that $0 < \theta < {}^\pi\!/\!{}_2$, we obtain

$$\frac{r}{h - r} = \tan\theta = {}^1\!/\!{}_w(d - h),$$

where the first equality follows from the right triangle $\triangle EAX$. We can solve this equation for the radius, r, which provides the solution

$$r = \frac{h(d - h)}{w + d - h}. \tag{8}$$

Simple computation shows that

$$\frac{h(d - h)}{w + d - h} = 1/2(w + h - d).$$