Continuous Fractions

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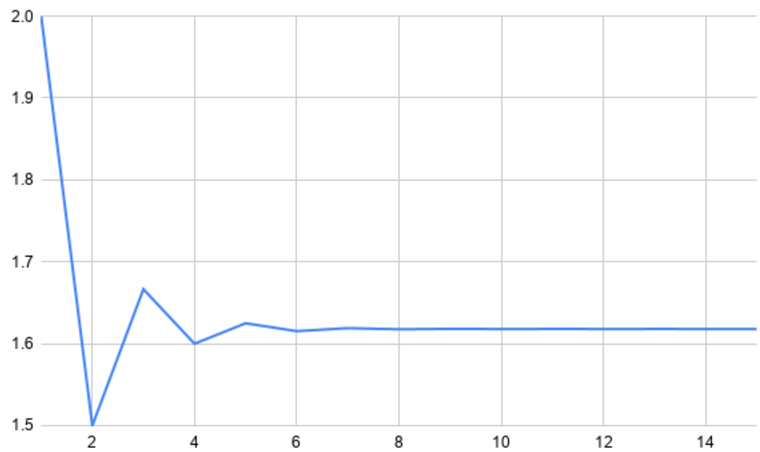
1 Introduction

The continuous fraction $1 + \frac{1}{1+\frac{1}{\dots}}$ which can be generalised as $t_n = k + \frac{1}{t_{n-1}}$ where $k \in \mathbb{R}$. When the first 15 terms t_n where $k = 1$ are plotted t_n against n you get the following chart: Using the graph the observation can be made that the value of t_n converges on a specific value $\approx 1.618033988749895$ therefore it can be determined that $t_{n-1} - t_n$ approaches 0. The problems arisisng for high values of n is that you quickly reach the limit of floating point maths used by computers.

2 Differing values of k

2.1 $k = 2$

The graph of the generalised equation $t_n = k + \frac{1}{t_{n-1}}$ where $k = 2$ is below: From this graph a similar observation can be made as the case where $k = 1$ being that the value of t_n converges on a specific value ≈ 2.4142135623731 however is seems to converge to this value more quickly.



2.2 $k = 0$

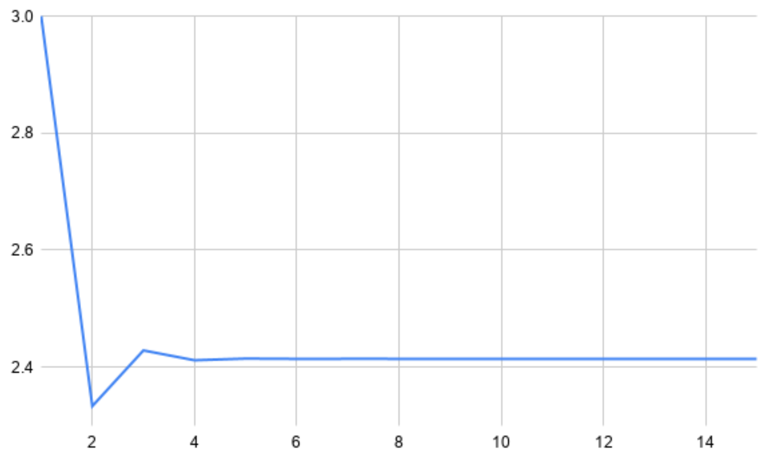
When $k = 0$ the value of t_n stays constant at 1 which would be expected since having a non-zero value of k is what allows the value of t_n to converge .

2.3 $k < -1$

Here since the value of k is given as an uncertainty we can take $k = -1$ and $k = -2$ and extrapolate from there. There is only a value for t_n where $n = 1$ since the value of $t_1 = 1 - 1 = 0$ therefore every value of t_n for higher values of n will be a division by 0. For $k = -2$, the graph displays characteristics to both the graph where $k = 1$ and where $k = 2$ and is shown below: Here the graph appears similar the the graph of $k = 2$ however it's starting value is -1. It converges on a value ≈ -2.4142135623731 which is the negative of the value that $k = 2$ converges on

2.4 $0 < k < -1$

Some of the graphs of values of k between 0 and -1 are somewhat unusual however the graph of $k = 0.5$ is below: This graph is mostly expected as it converges slower than when $k = 1$ however the spike at $n = 4$ is mostly unexpected, the value of $t_4 = 5.5$. The graph where $k = -0.1$ is the most unexpected however t_n seems to diverge as n increases, the graph is below:



3 Determining exact limits

Since some of these continued fractions converge there must be an exact value for t_∞ which we can calculate. Using the continued fraction $1 + \frac{1}{1+\frac{1}{\cdots}}$ the limit is as follows: Consider the quadratic $x^2 - x - 1 = 0$, to find it's solutions we can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solving for x where $a = 1$, $b = -1$, $c = -1$ we get the solutions $x = \frac{1+\sqrt{5}}{2}$ (the positive solution) or $x = \frac{1-\sqrt{5}}{2}$ (the negative solution). If we consider the positive value of x as Φ we can say that:

$$\Phi^2 - \Phi - 1 = 0$$

Rearranging this equation we get:

$$\Phi = 1 + \frac{1}{\Phi}$$

If we substitute Φ into the right side we get:

$$\Phi = 1 + \frac{1}{1 + \frac{1}{\Phi}}$$

If we repeat this substitution "to infinity" we get the continued fraction above. This indicates that the exact value for this continued fraction is $\frac{1+\sqrt{5}}{2}$ which bears a fairly high amount of significance as it equals the **golden ratio** which is often given the capital letter phi (Φ)