8 Binomial expansion

In this chapter you will learn:

- how to expand an expression of the form $(a+b)^n$ for positive integer n
- how to find individual terms in the expansion of $(a+b)^n$ for positive integer n
- how to use partial expansions of $(a+bx)^n$ to find approximate values.

Introductory problem

Without using a calculator, find the value of $(1.002)^{10}$ correct to 8 decimal places.

A **binomial** expression is one which contains two terms, for example, $a+b$.

Expanding a power of such a binomial expression could be performed by expanding brackets; for example $(a+b)^7$ could be found by calculating, at length,

$$(a+b)(a+b)(a+b)(a+b)(a+b)(a+b)(a+b).$$

This is time-consuming, but happily there is a much faster approach.

The Ancient Babylonians made an unexpected use of expanding brackets – it helped them multiply numbers in their base 60 number system. This is explored in Supplementary sheet 7 'Babylonian multiplication' on the CD-ROM.

8A Introducing the binomial theorem

To see how to expand an expression of the form $(a+b)^n$ for integer n rapidly, consider the following expansions of $(a+b)^n$, done using the slow method of repeatedly multiplying out brackets.

$$\begin{aligned}
(a+b)^0 &= 1\\
&= 1a^0b^0\\
(a+b)^1 &= a+b\\
&= 1a^1b^0 + 1a^0b^1\\
(a+b)^2 &= a^2+2ab+b^2\\
&= 1a^2b^0 + 2a^1b^1 + 1a^0b^2\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\
&= 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\
&= 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4
\end{aligned}$$

The red numbers form a famous mathematical construction called Pascal's triangle. Each value not on the edge is formed by adding up the value directly above and the value to its left. There are many amazing patterns in Pascal's triangle – for example, if you highlight all the even numbers you generate an ever-repeating pattern called a fractal.

See chapter 1, Section D for more combinations $\binom{n}{r}$.

We can see several patterns in this structure.

- The powers of a and b always total n (in the 3rd row, $3+0=2+1=1+2=0+3=3$).
- Each power of a from 0 up to n is present in one of the terms, with the corresponding complementary power of b.
- Each **term** has a **coefficient** (given in red), and the pattern of coefficients in each line is symmetrical.

In this section we shall focus on the coefficients. The pattern of numbers may seem familiar as they are all numbers which were found when calculating combinations $\binom{n}{r}$. In this context, $\binom{n}{r}$ is called a **binomial coefficient**.

See Calculator skills sheet 3 on the CD-ROM for a reminder of how to use your calculator to find $\binom{n}{r}$.

KEY POINT 8.1

Binomial coefficient

The binomial coefficient, $\binom{n}{r}$, is the coefficient of the term containing $a^{n-r}b^r$ in the expansion of $(a+b)^n$.

Worked example 8.1

Find the coefficient of x^5y^3 in the expansion of $(x+y)^8$.

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a=x$, $b=y$, $n=8$, $r=3$

The relevant term is $\binom{8}{3}(x)^5(y)^3$

Calculate the coefficient and apply the powers to the bracketed terms

$\binom{8}{3}=56$

$(x)^5=x^5$

$(y)^3=y^3$

Combine the elements to calculate the coefficient

The term is $56x^5y^3$

The coefficient is 56

EXAM HINT

Questions might ask you to give either the whole term or just the coefficient. Make sure that you answer the question!

Exercise 8A

1. (a) Find the coefficient of xy^3 in the expansion of $(x+y)^4$.
   (b) Find the coefficient of x^3y^4 in the expansion of $(x+y)^7$.
   (c) Find the coefficient of ab^6 in the expansion of $(a+b)^7$.
   (d) Find the coefficient of a^5b^3 in the expansion of $(a+b)^8$.

2. (a) Find the term in x^5y^7 in the expansion of $(x+y)^{12}$.
   (b) Find the term in a^7b^9 in the expansion of $(a+b)^{16}$.
   (c) Find the term in c^3d^2 in the expansion of $(c+d)^5$.
   (d) Find the term in a^2b^7 in the expansion of $(a+b)^9$.
   (e) Find the term in x^2y^4 in the expansion of $(x+y)^6$.

8B Applying the binomial theorem

In Section 8A, you saw the general pattern for expanding powers of a binomial expression $(a+b)$. When expanding powers of more complicated expressions, you will still use this method, but may substitute more complicated expressions for a and b.

Worked example 8.2

Find the term in x^6y^4 in the expansion of $(x+3y^2)^8$.

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a=x$, $b=3y^2$, $n=8$, $r=2$

The relevant term is $\binom{8}{2}(x)^6(3y^2)^2$

Calculate the coefficient and apply the powers to the bracketed terms

$\binom{8}{2}=28$

$(x)^6=x^6$

$(3y^2)^2=9y^4$

The term is $28\times x^6\times 9y^4=252x^6y^4$

Many examination questions ask you to focus on just one term, but you should also be able to find the entire expansion. To do this you repeat the process for each term for every possible value of r (from 0 up to n) and add together the results. This result is quoted in the Formula booklet.

EXAM HINT

Take care to apply the power not only to the algebraic part but also to its coefficient. In Worked example 8.2, $(3y^2)^2 = 9y^4$, not $3y^4$.

KEY POINT 8.2

Binomial theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

Worked example 8.3

Use the Binomial theorem to expand and simplify $(2x-3y)^5$.

Write down each term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a = 2x$, $b = -3y$, $n = 5$

Coefficients: 1, 5, 10, 10, 5, 1

The expansion is

$1(2x)^5 + 5(2x)^4(-3y)^1 + 10(2x)^3(-3y)^2$

$+ 10(2x)^2(-3y)^3 + 5(2x)^1(-3y)^4 + 1(-3y)^5$

Apply the powers to the bracketed terms and multiply through

$= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$

EXAM HINT

A common mistake is to assume that the powers of each variable correspond to the value of r in the expansion.

A question may ask for a binomial expansion where both of the terms in the binomial expression contain the same variable. You can use the rules of exponents to determine which term of the expansion is needed.

Worked example 8.4

Find the coefficient of x^5 in the expansion of $\left(2x^2 - \frac{1}{x}\right)^7$.

Start with the form of a general term and simplify using the rules of exponents

Each term will be of the form

$\binom{7}{r}(2x^2)^{7-r}(-x^{-1})^r$

$= \binom{7}{r}(2)^{7-r}x^{14-2r}(-1)^r x^{-r}$

$= \binom{7}{r}(2)^{7-r}(-1)^r x^{14-3r}$

continued . . .

We need the term in x^5, so equate that to the power of x in the general term

Require that $14 - 3r = 5$
$\Leftrightarrow \quad 3r = 9$
$\Leftrightarrow \quad r = 3$

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a = 2x^2$, $b = x^{-1}$, $n = 7$, $r = 3$

The relevant term is $\binom{7}{3}(2x^2)^4(-x^{-1})^3$

Calculate the coefficient and apply the powers to the bracketed terms

$\binom{7}{3} = 35$

$(2x^2)^4 = 16x^8$

$(-x^{-1})^3 = -x^{-3}$

Combine the elements to calculate the coefficient

The term is $35 \times 16x^8 \times (-x^{-3}) = -560x^5$
The coefficient is -560

EXAM HINT

Don't forget that if there is a negative sign it must stay part of the coefficient of the term it is acting upon. Lots of people fall into this trap!

Applying this process in reverse is always quite tricky, since in general you cannot 'undo' the $\binom{n}{r}$ operation. So, you must use the formula for $\binom{n}{r}$, seen in chapter 1, to rewrite it as a polynomial $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Remind yourself of Key point 1.6 in chapter 1.

Worked example 8.5

The coefficient of x^2 in the expansion of $(1+3x)^n$ is 189. Find n if $n > 0$.

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a = 1$, $b = 3x$, $r = 2$

Required term is
$\binom{n}{2}(1)^{n-2}(3x)^2$

continued . . .

Simplify the coefficient algebraically and apply the powers to the bracketed terms	$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$ $(1)^{n-2} = 1$ $(3x)^2 = 9x^2$
Combine these and equate to the given information	$\frac{9n(n-1)}{2}x^2 \equiv 189x^2$
Compare coefficients	$\frac{9n(n-1)}{2} = 189$ $9n^2 - 9n = 378$ $n^2 - n - 42 = 0$ $(n-7)(n+6) = 0$ $n = 7$ or $n = -6$ but $n > 0$ $\therefore n = 7$

The binomial expansion can be generalised to situations where n is negative or a fraction. This general binomial expansion results in an infinite polynomial.

Exercise 8B

EXAM HINT

The constant coefficient is the term in x^0. It may also be described as the term independent of x.

1. (a) Find the coefficient of xy^3 in:

(i) $(2x+3y)^4$ (ii) $(5x+y)^4$

(b) Find the term in x^3y^4 in:

(i) $(x-2y)^7$ (ii) $(y-2x)^7$

(c) Find the coefficient of a^2b^3 in:

(i) $\left(2a - \frac{1}{2}b\right)^5$ (ii) $(17a+3b)^5$

2. (a) Find the coefficient of x^2 in the expansion of:

(i) $\left(x + \frac{1}{x}\right)^8$ (ii) $\left(2x + \frac{1}{\sqrt{x}}\right)^5$

(b) Find the constant coefficient in the expansion of:

(i) $(x-2x^{-2})^9$ (ii) $(x^3-2x^{-1})^4$

3. (a) Fully expand and simplify:

(i) $(2-x)^5$ (ii) $(3+x)^6$

(b) (i) Find the first three terms in descending powers of x of $(3x+y)^5$.

(ii) Find the first three terms in ascending powers of d of $(2c-d)^4$.

(c) Fully expand and simplify:

(i) $(2x^2-3x)^3$ (ii) $(2x^{-1}+5y)^3$

(d) Fully expand and simplify:

(i) $\left(2z^2+\dfrac{3}{z}\right)^4$ (ii) $\left(3xy+\dfrac{5x}{y}\right)^3$

4. Write in polynomial form:

(a) $\begin{pmatrix} n \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} n \\ 2 \end{pmatrix}$ (c) $\begin{pmatrix} n \\ 3 \end{pmatrix}$

5. Which term in the expansion of $(x-2y)^5$ has coefficient:

(a) 80 (b) −80 *[6 marks]*

6. Find the coefficient of x^2y^6 in $(3x+2y^2)^5$. *[5 marks]*

7. Find the term in x^5 in $\left(x^2-\dfrac{3}{x}\right)^7$. *[6 marks]*

8. Find the term that is independent of x in the expansion of $\left(2x-\dfrac{5}{x^2}\right)^{12}$. *[6 marks]*

9. The expansion of $(1+3x)^n$ starts with $1+42x\ldots$ Find the value of n. *[4 marks]*

10. The coefficient of x^2 in $(1+2x)^n$ is 264. Find the value of n given that $n\in\mathbb{N}$. *[6 marks]*

11. The coefficient of x^3 in $(1-5x)^n$ is −10 500. Find the value of n given that $n\in\mathbb{N}$. *[5 marks]*

12. The coefficient of x^2 in $(3+2x)^n$ is 20 412. Find the value of n given that $n\in\mathbb{N}$. *[6 marks]*

8C Products of binomial expansions

We may need to work with a product of a binomial and another expression. It is possible to do this by working with the entire expansion.

Worked example 8.6

Use the Binomial theorem to expand and simplify $(5-3x)(2-x)^4$.

Expand $(2-x)^4$. Coefficients are: 1, 4, 6, 4, 1

The expansion is

$(5-3x)\left[1(2)^4+4(2)^3(-x)+6(2)^2(-x)^2+4(2)^1(-x)^3+1(-x)^4\right]$

Multiply each term in the second bracket by 5 and then by $-3x$

$=5\left[16-32x+24x^2-8x^3+x^4\right]$
$\quad -3x\left[16-32x+24x^2-8x^3+x^4\right]$
$=80-208x+216x^2-112x^3+29x^4-3x^5$

This method is quite long-winded. In examinations you will usually only be asked to find a small number of terms from such an expression.

Worked example 8.7

Find the coefficient of a^4b^3 in the expansion of $(a-5b)(a+b)^6$.

Split the product into two parts, and treat each separately

$a(a+b)^6-5b(a+b)^6$

Decide which term from each expansion is needed to make a^4b^3

For a^4b^3, we need a^3b^3 from the first bracket $(a\times a^3b^3=a^4b^3)$ and a^4b^2 from the second $(b\times a^4b^2=a^4b^3)$

Write down the required terms in the form $\binom{n}{r}(a)^{n-r}(b)^r$

a^4b^3 will arise from

$a\times\binom{6}{3}(a)^3(b)^3-5b\times\binom{6}{2}(a)^4(b)^2$

Calculate the coefficient and apply the powers to the bracketed terms

$\binom{6}{3}=20$ $\qquad$ $\binom{6}{2}=15$

$(a)^3=a^3$ $\qquad$ $(a)^4=a^4$

$(b)^3=b^3$ $\qquad$ $(b)^2=b^2$

Combine the elements to calculate the coefficient

The term is $20a^4b^3-75a^4b^3=-55a^4b^3$
The coefficient is -55

As you get more practised, you may not need to split the problem into several parts explicitly; always consider all possible ways that you can multiply to get the required term.

Worked example 8.8

Find the coefficient of x^4 in the expansion of $(1+3x-x^2)(2+x)^5$.

Split the product into three parts, and treat each separately	$1(2+x)^5+3x(2+x)^5-x^2(2+x)^5$
Decide which term from each expansion is needed to make x^4	
Write down the required terms in the form $\binom{n}{r}(a)^{n-r}(b)^r$	x^4 will arise from $1\times\binom{5}{4}(2)^1x^4+3x\times\binom{5}{3}(2)^2x^3-x^2\times\binom{5}{2}(2)^3x^2$
Calculate the coefficient and apply the powers to the bracketed terms	$\binom{5}{4}=5$ $\quad\binom{5}{3}=10$ $\quad\binom{5}{2}=10$ $(2)^1=2$ $\quad(2)^2=4$ $\quad(2)^3=8$
Combine the elements to calculate the coefficient	The term is $1\times5\times2x^4+3x\times10\times4x^3-x^2\times10\times8x^2$ $=10x^4+120x^4-80x^4$ $=50x^4$ The coefficient is 50

Exercise 8C

1. (a) (i) Find the coefficient of x^2y^5 in the expansion of $(x-y)(x+y)^6$.

   (ii) Find the coefficient of x^5 in the expansion of $(1+3x)(1+x)^7$.

   (b) (i) Find the coefficient of x^6 in the expansion of $(1-x^2)(1+x)^5$.

   (ii) Find the coefficient of x^6 in the expansion of $(1-x^2)(1+x)^7$.

2. (a) (i) Find the coefficient of x^2y in the expansion of $(1+x)^3(1+y)^5$.

(ii) Find the coefficient of xy^3 in the expansion of $(1+x)^4(1+y)^5$.

3. (i) Find the coefficient of c^4d^{11} in the expansion of $(2c+5d)(c+d)^{14}$.

(ii) Find the coefficient of a^3b^{15} in the expansion of $(3a-b)(a+b)^{17}$.

4. (a) (i) Find the first three terms in descending powers of x of $(3x+7)(x^2-2x)^4$.

(ii) Find the first three terms in descending powers of x of $(x-x^2)(x-3)^5$.

(b) (i) Find the first three terms in ascending powers of x of $(x-1)^4(x+1)^5$.

(ii) Find the first three terms in ascending powers of x of $(x+2)^4(2-x)^3$.

5. Find the first 4 terms in the expansion of $(y+3y^2)^6$ in ascending powers of y. *[6 marks]*

6. Find the first three non-zero terms of the expansion of $(1-x)^{10}(1+x)^{10}$ in ascending powers of x. *[6 marks]*

7. Find the first 4 terms in the expansion of $(1-2x+x^2)^{10}$ in ascending powers of x. *[6 marks]*

8. Given that $(1+x)^3(1+mx)^4 \equiv 1+nx+93x^2+\ldots+m^4x^7$, find the possible values of m and n. *[8 marks]*

9. Given that $(1+kx)^4(1+x)^n \equiv 1+13x+74x^2+\ldots+k^4x^{n+4}$, find the possible positive integer values of k and n. *[8 marks]*

8D Binomial expansions as approximations

Calculators and computers use binomial expansions to work out powers and roots.

One of the main applications of the binomial expansion is in calculating approximate values of powers and roots. When x is a very small value, high powers of x become increasingly small, and so they have little impact on the value of the total sum, even when multiplied by the binomial coefficient. For this reason, using only the first few terms gives a good approximation to the total value of the sum.

KEY POINT 8.3

If the value of x is much less than one, large powers of x will be extremely small.

Worked example 8.9

Find the first 3 terms in ascending powers of x of the expansion of $(2-x)^5$.
By setting $x = 0.01$, use your answer to find an approximate value of 1.99^5.

Write down each term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a = 2, b = -x, n = 5$

The first 3 terms are
$1(2)^5 + 5(2)^4(-x)^1 + 10(2)^3(-x)^2$

Apply the powers to the bracketed terms and multiply through

$= 32 - 80x + 80x^2$

Calculate the powers of the appropriate value of x and thus the value of each term

$x^0 = 1 \qquad 32x^0 = 32$

$x^1 = 0.01 \qquad -80x^1 = -0.8$

$x^2 = 0.0001 \qquad 80x^2 = 0.008$

Total the values of the terms

Hence $1.99^5 \simeq 31.208$

Exercise 8D

1. (a) Find the first 4 terms in the expansion of $(1+5x)^7$.
By setting $x = 0.01$, find an approximation for 1.05^7, leaving your answer correct to 6 significant figures.

(b) Find the first 3 terms in the expansion of $(2+3x)^6$.
By setting $x = 0.001$, find an approximation for 2.003^6, leaving your answer correct to 6 significant figures.

2. (a) Find the first 3 terms in the expansion of $(3-5x)^4$.
(b) Using a suitable value of x, use your answer to find a 6 significant figure approximation for 2.995^4. *[7 marks]*

3. (a) Find the first 3 terms in the expansion of $(2+5x)^7$.
(b) Using a suitable value of x, use your answer to find a 6 significant figure approximation for 2.005^7. *[7 marks]*

4. (a) Find the first 3 terms in the expansion of $(2+3x)^7$.

(b) Hence find an approximation to:

(i) 2.3^7 (ii) 2.03^7

(c) Which of your answers in part (b) provides a more accurate approximation? Justify your answer. *[6 marks]*

Summary

- A **binomial** expression is one that contains two terms, e.g., $a + b$.
- The **binomial coefficient**, $\binom{n}{r}$, is the coefficient of the term containing $a^{n-r}b^r$ in the expansion of $(a+b)^n$.
- The expansion of $(a+b)^n$ can be accomplished directly using the **binomial theorem**:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

- The coefficient of individual terms of a binomial expansion can be found by considering the powers of the algebraic components and using the formula for the rth term:

$$\binom{n}{r}(a)^{n-r}(b)^r$$

- By reversing this process and using the polynomial form of $\binom{n}{r}\frac{n!}{r!(n-r)!}$, you can also find the value of n if you have a term of the binomial expansion.
- Approximations for powers and roots can be made using the first few terms of a binomial expansion $(a+bx)^n$, valid when bx is very much less than one, meaning that terms with higher powers are increasingly small.

Introductory problem revisited

Without using a calculator, find the value of $(1.002)^{10}$ correct to 8 decimal places.

We recognise that $(1.002)^{10}$ can be calculated by evaluation of the binomial expansion $(1 + 2x)^{10}$ with $x = 0.001$.

To ensure accuracy to 8 decimal places, we need to include terms at least to x^3, but can safely disregard terms in x^4 and greater powers, since the magnitudes of the coefficients mean these are too small to affect the first 8 decimal places.

Write down each term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a = 1, b = 2x, n = 10$

The first 4 terms are
$1(1)^{10} + 10(1)^9(2x)^1 + 45(1)^8(2x)^2 + 120(1)^7(2x)^3$

Apply the powers to the bracketed terms and multiply through

The first 4 terms are
$1 + 20x + 180x^2 + 960x^3$

Calculate the powers of the appropriate value of x and thus the value of each term

$x^0 = 1$	$1 = 1$
$x^1 = 0.001$	$20x^1 = 0.02$
$x^2 = 0.000\,001$	$180x^2 = 0.000\,18$
$x^3 = 0.000\,000\,001$	$960x^3 = 0.000\,000\,96$

Total the values of the terms

Hence $1.002^{10} \simeq 1.020\,180\,96$

From calculator, $1.002^{10} = 1.020\,180\,963\,368\,08\ldots$
So approximation error is $3.30 \times 10^{-9} = 0.000\,000\,33\%$.

Mixed examination practice 8

Short questions

1. Find the coefficient of x^5 in the expansion of $(2-x)^{12}$. *[5 marks]*

2. $a = 2-\sqrt{2}$. Using the binomial theorem or otherwise, express a^5 in the form $m + n\sqrt{2}$. *[5 marks]*

3. (a) Find the expansion of $(2+x)^5$, giving your answer in ascending powers of x.

(b) By letting $x = 0.01$ or otherwise, find the exact value of 2.01^5. *[7 marks]*

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4. Determine the first 3 terms in the expansion of $(1-2x)^3(3+4x)^5$. *[7 marks]*

5. Fully expand and simplify $\left(x^2 - \dfrac{2}{x}\right)^4$. *[6 marks]*

6. The coefficient of x in the expansion of $\left(x + \dfrac{1}{ax^2}\right)^7$ is $\dfrac{7}{3}$. Find the possible values of a. *[3 marks]*

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7. Given that $(1+x)^6(1+mx)^5 \equiv 1 + nx + 415x^2 + \ldots + m^5x^{11}$, find the possible values of m and n. *[8 marks]*

Long questions

1. (a) Sketch the graph of $y = (x+2)^3$.

(b) Find the binomial expansion of $(x+2)^3$.

(c) Find the exact value of 2.001^3.

(d) Solve the equation $x^3 + 6x^2 + 12x + 16 = 0$. *[12 marks]*

2. $f(x) = (1+x)^5$ and $g(x) = (2+x)^4$.

(a) Write down the vertical asymptote and axes intercepts of the graph $y = \dfrac{f(x)}{g(x)}$.

(b) Write down binomial expansions for $f(x)$ and $g(x)$.

(c) (i) Show that $\dfrac{f(x)}{g(x)} = x - k + \dfrac{ax^3 + 50x^2 + 85x + 49}{g(x)}$, where k and a are constants to be found.

(ii) Hence explain why the graph of $y = \dfrac{f(x)}{g(x)}$ approaches a straight line when x is large, and write down the equation of this straight line.

(d) Sketch the curve $y = \dfrac{f(x)}{g(x)}$ for $-10 \le x \le 10$. *[12 marks]*

3. (a) Write $\left(1+\sqrt{2}\right)^3$ in the form $p+q\sqrt{2}$ where $p,\ q \in \mathbb{Z}$.

(b) Write down the general term in the binomial expansion of $\left(1+\sqrt{2}\right)^n$.

(c) Hence show that $\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$ is always an integer.

(d) What is the smallest value of n such that $\left(1+\sqrt{2}\right)^n$ is within 10^{-9} of a whole number? *[12 marks]*

4. The expansion of $(a+x)^n$ where $n \in \mathbb{N}$ has the form:

$$a^n + \ldots + \alpha x^r + \beta x^{r+1} + \gamma x^{r+2} + \ldots + x^n$$

(a) Show that the ratio of $\dfrac{\alpha}{\beta}$ is $\dfrac{r+1}{n-r}a$.

(b) If $a=1$ show that the expansion will contain two consecutive terms with the same coefficient as long as n is odd.

(c) Using the result of part (a) deduce an expression for $\dfrac{\beta}{\gamma}$

(d) Prove that there are no values for a such that in the expansion of $(a+x)^n$, $n \in \mathbb{N}$, three consecutive terms have the same coefficient. *[16 marks]*