7 Sequences and series

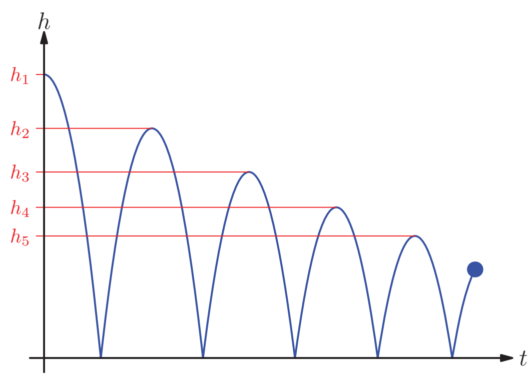
In this chapter you will learn:

- how to describe sequences mathematically
- a way to describe sums of sequences
- about sequences with a constant difference between terms
- about finite sums of sequences with a constant difference between terms
- about sequences with a constant ratio between terms
- about finite sums of sequences with a constant ratio between terms
- about infinite sums of sequences with a constant ratio between terms
- how to apply sequences to real life problems.

Introductory problem

A mortgage of \$100 000 is fixed at 5% compound interest. It needs to be paid off over 25 years by annual instalments. Interest is added at the end of each year, just before the payment is made. How much should be paid each year?

If you drop a tennis ball, it will bounce a little lower each time it hits the ground. The heights to which the ball bounces form a **sequence**. Although the study of sequences may just seem to be the maths of number patterns, it also has a remarkable number of applications in the real world, from calculating mortgages to estimating the harvests on farms.



7A General sequences

A sequence is a list of numbers in a specified order. You may recognise a pattern in each of the following examples:

1, 3, 5, 7, 9, 11, ...

1, 4, 9, 16, 25, ...

100, 50, 25, 12.5, ...

To study sequences further, it is useful to have a notation to describe them.

KEY POINT 7.1

u_n is the value of the nth term of a sequence.

So in the sequence 1, 3, 5, 7, 9, 11, ... above, we could say that $u_1 = 1$, $u_2 = 3$, $u_5 = 9$.

The whole of a sequence $u_1, u_2, u_3, \ldots$ is sometimes written $\{u_n\}$.

We are mainly interested in sequences with well-defined mathematical rules. There are two types: **recursive definitions** and **deductive** rules.

Recursive definitions link new terms to previous terms in the sequence. For example, if each term is three times the previous term we would write $u_{n+1} = 3u_n$.

EXAM HINT

u_n is a conventional symbol for a sequence, but there is nothing special about the letters used. We could also have a sequence t_x or a_h. The important thing is that the letter with a subscript represents a value and the subscript represents where the term is in the sequence.

Worked example 7.1

A sequence is defined by $u_{n+1} = u_n + u_{n-1}$ with $u_1 = 1$ and $u_2 = 1$. What is the fifth term of this sequence?

The sequence is defined recursively, so we have to work our way up to u_5

To find u_3 we set $n = 2$

$$\begin{aligned} u_3 &= u_2 + u_1 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

To find u_4 we set $n = 3$

$$\begin{aligned} u_4 &= u_3 + u_2 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

To find u_5 we set $n = 4$

$$\begin{aligned} u_5 &= u_4 + u_3 \\ &= 3 + 2 \\ &= 5 \end{aligned}$$

You may recognise this as the famous Fibonacci Sequence, based on a model Leonardo Fibonacci made for the breeding of rabbits. This has many applications from the arrangement of seeds in pine cones to a proof of the infinity of prime numbers.

There is also a beautiful link to the golden ratio: $\dfrac{1 \pm \sqrt{5}}{2}$

Deductive rules link the value of the term to where it is in the sequence. For example, if each term is the square of its position in the sequence then we would write $u_n = n^2$.

Worked example 7.2

A sequence is defined by $u_n = 2n - 1$. List the first four terms of this sequence.

With a deductive rule, we can find the first four terms by setting $n = 1, 2, 3, 4$

$u_1 = 2 \times 1 - 1 = 1$
$u_2 = 2 \times 2 - 1 = 3$
$u_3 = 2 \times 3 - 1 = 5$
$u_4 = 2 \times 4 - 1 = 7$

EXAM HINT

There are several alternative names used for deductive and recursive definitions.

An recursive definition may also be referred to as 'term-to-term rule', 'recurrence relation' or 'recursive definition'.

A deductive rule may be referred to as 'position-to-term rule', 'nth term rule' or simply 'the formula' of the sequence.

Exercise 7A

1. Write out the first five terms of the following sequences, using the recursive definitions.

(a) (i) $u_{n+1} = u_n + 5,\ u_1 = 3.1$ (ii) $u_{n+1} = u_n - 3.8,\ u_1 = 10$

(b) (i) $u_{n+1} = 3u_n + 1,\ u_1 = 0$ (ii) $u_{n+1} = 9u_n - 10,\ u_1 = 1$

(c) (i) $u_{n+2} = u_{n+1} \times u_n,\ u_1 = 2, u_2 = 3$

(ii) $u_{n+2} = u_{n+1} \div u_n,\ u_1 = 2u_2 = 1$

(d) (i) $u_{n+2} = u_n + 5,\ u_1 = 3u_2 = 4$

(ii) $u_{n+2} = 2u_n + 1,\ u_1 = -3u_2 = 3$

(e) (i) $u_{n+1} = u_n + 4,\ u_4 = 12$ (ii) $u_{n+1} = u_n - 2,\ u_6 = 3$

2. Write out the first five terms of the following sequences, using the deductive definitions.

(a) (i) $u_n = 3n + 2$ (ii) $u_n = 1.5n - 6$

(b) (i) $u_n = n^3 - 1$ (ii) $u_n = 5n^2$

(c) (i) $u_n = 3^n$ (ii) $u_n = 8 \times (0.5)^n$

(d) (i) $u_n = n^n$ (ii) $u_n = \sin(90n°)$

3. Give a recursive definition for each of these sequences.

(a) (i) 7, 10, 13, 16, … (ii) 1, 0.2, −0.6, −1.4, …

(b) (i) 3, 6, 12, 24, … (ii) 12, 18, 27, 40.5, …

(c) (i) 1, 3, 6, 10, … (ii) 1, 2, 6, 24, …

4. Give a deductive definition for each of the following sequences.

(a) (i) 2, 4, 6, 8, … (ii) 1, 3, 5, 7, …

(b) (i) 2, 4, 8, 16, … (ii) 5, 25, 125, 625, …

(c) (i) 1, 4, 9, 16, … (ii) 1, 8, 27, 64, …

(d) (i) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots$ (ii) $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16} \ldots$

5. A sequence $\{u_n\}$ is defined by $u_0 = 1$, $u_1 = 2$, $u_{n+1} = 3u_n - 2u_{n-1} - 1$ where $n \in \mathbb{Z}$.

(a) Find u_2, u_3 and u_4.

(b) (i) Based on your answer to (a), suggest a formula for u_n in terms of n.

(ii) Verify that your answer to part (b)(i) satisfies the equation $u_{n+1} = 3u_n - 2u_{n-1}$. *[6 marks]*

7B General series and sigma notation

If 10% interest is paid on money in a bank account each year, the amounts paid form a sequence. While it is good to know how much is paid in each year, you may be even more interested to know how much will be paid in altogether.

This is an example of a situation where we may want to sum a sequence. The sum of a sequence up to a certain point is called a **series**, and we often use the symbol S_n to denote the sum of the first n terms of a sequence.

Worked example 7.3

Adding up consecutive odd numbers starting at 1 forms a series.

Let S_n denote the sum of the first n terms. List the first five terms of the sequence S_n and suggest a rule for it.

Start by examining the first few terms

$S_1 = 1$
$S_2 = 1 + 3 = 4$
$S_3 = 1 + 3 + 5 = 9$
$S_4 = 1 + 3 + 5 + 7 = 16$
$S_5 = 1 + 3 + 5 + 7 + 9 = 25$

Do we recognise these numbers?

It seems that $S_n = n^2$

Defining such sums by saying 'Add up a defined sequence from a given start point to a given end point' is too wordy and imprecise for mathematicians.

Exactly the same thing is written in a shorter (although not necessarily simpler) way using **sigma notation**:

KEY POINT 7.2

This is the last value taken by r, where counting ends

r is a placeholder; it shows what changes with each new term

Greek capital sigma means 'add up'

$$\sum_{r=1}^{r=n} f(r) = f(1) + f(2) + \cdots + f(n)$$

This is the first value taken by r; where counting starts

EXAM HINT

Do not be intimidated by this complicated-looking notation.

If you struggle with it, try writing out the first few terms.

If there is only one variable in the expression being summed, it is acceptable to miss out the '$r =$' above and below the sigma.

In the example we use both the letters n and r as unknowns – but they are not the same type of unknown.

If we replaced r by any other letter (apart from f or n) then the expression on the right would be unchanged. r is called a dummy variable. If we replaced n by any other letter then the expression would change.

Worked example 7.4

$T_n = \sum_{2}^{n} r^2$ Find the value of T_4.

Put the starting value, $r = 2$ into the expression to be summed

$T_4 = 2^2 + \ldots$

We've not reached the end value, so put in $r = 3$

$T_4 = 2^2 + 3^2 + \ldots$

We've not reached the end value, so put in $r = 4$

$T_4 = 2^2 + 3^2 + 4^2$

We've reached the end value, so evaluate

$T_4 = 4 + 9 + 16 = 29$

Worked example 7.5

Write the series $T_4 = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$ using sigma notation.

We must write in terms of the dummy variable r what eachterm of the sequence looks like

General term $= \frac{1}{r}$

What is the first value of r?

Starts when $r = 2$

What is the final value of r?

Ends when $r = 6$

Summarise in sigma notation

Series $= \sum_{2}^{6} \frac{1}{r}$

Exercise 7B

1. Evaluate the following expressions.

(a) (i) $\sum_{2}^{4} 3r$ (ii) $\sum_{5}^{7} (2r+1)$

(b) (i) $\sum_{3}^{6} (2^r - 1)$ (ii) $\sum_{-1}^{4} 1.5^r$

(c) (i) $\sum_{a=1}^{a=4} b\ (a+1)$ (ii) $\sum_{q=-3}^{q=2} pq^2$

2. Write the following expressions in sigma notation. Be aware that there is more than one correct answer.

(a) (i) $2+3+4 \ldots + 43$ (ii) $6+8+10 \ldots + 60$

(b) (i) $\frac{1}{4}+\frac{1}{8}+\frac{1}{16} \ldots + \frac{1}{128}$ (ii) $2+\frac{2}{3}+\frac{2}{9} \ldots + \frac{2}{243}$

(c) (i) $14a+21a+28a \ldots + 70a$ (ii) $0+1+2^b+3^b \ldots + 19^b,\ (b \neq 0)$

7C Arithmetic sequences

We will now focus on one particular type of sequence: one where there is a constant difference, known as the common difference, between consecutive terms.

This is called an **arithmetic sequence** (or an **arithmetic progression**). The standard notation for the difference between terms is d, so arithmetic sequences obey the recursive definition $u_{n+1} = u_n + d$.

Knowing the common difference is not enough to fully define the sequence. There are many different sequences with common difference 2, for example:

$$1, 3, 5, 7, 9, 11, \ldots \text{ and } 106, 108, 110, 112, 114, \ldots$$

To fully define the sequence we also need the first term. Conventionally this is given the symbol u_1.

So the sequence 106, 108, 110, 112, 114, … is defined by:

$$u_1 = 106, d = 2.$$

Worked example 7.6

What is the fourth term of an arithmetic sequence with $u_1 = 300$, $d = -5$?

Use the recursive definition to find the first four terms $u_{n+1} = u_n + d$

$u_1 = 300$
$u_2 = u_1 - 5 = 295$
$u_3 = u_2 - 5 = 290$
$u_4 = u_3 - 5 = 285$

In the above example it did not take long to find the first four terms. But what if you had been asked to find the hundredth term? To do this efficiently we must move from the inductive definition of arithmetic sequences to the deductive definition.

We need to think about how arithmetic sequences are built up. To get to the nth term we start at the first term and add on the common difference $n - 1$ times. This suggests a formula:

KEY POINT 7.3

$$u_n = u_1 + (n - 1)d$$

Worked example 7.7

The fifth term of an arithmetic sequence is 7 and the eighth term is 16. What is the 100th term?

Write down the information given and relate it to u_1 and d to give an expression for the fifth term in terms of u_1 and d

$u_5 = u_1 + 4d$

But we are told that $u_5 = 7$

$7 = u_1 + 4d \quad (1)$

Repeat for the eighth term

$16 = u_1 + 7d \quad (2)$

continued . . .

Solve simultaneously (2) – (1)

$$9 = 3d$$
$$\Leftrightarrow d = 3$$
$$\therefore u_1 = -5$$

Write down the general term and use it to answer the question

$$u_n = -5 + (n-1) \times 3$$
$$\therefore u_{100} = -5 + 99 \times 3 = 292$$

EXAM HINT

Many exam-style questions on sequences and series involve writing the given information in the form of simultaneous equations and then solving them.

Worked example 7.8

An arithmetic progression has first term 5 and common difference 7. What is the term number corresponding to the value 355?

The question is asking for n when $u_n = 355$. Write this as an equation

$$355 = u_1 + (n-1)d = 5 + 7(n-1)$$

Solve this equation

$$350 = 7(n-1)$$
$$\Leftrightarrow 50 = n - 1$$
$$\Leftrightarrow n = 51$$

So 355 is the 51st term.

EXAM HINT

'Arithmetic progression' is just another way of saying 'arithmetic sequence'.

Make sure you know all the alternative expressions for the same thing.

Exercise 7C

1. Using Key point 7.3, find the general formula for each arithmetic sequence given the following information.

(a) (i) First term 9, common difference 3

(ii) First term 57, common difference 0.2

(b) (i) First term 12, common difference −1

(ii) First term 18, common difference $\frac{1}{2}$

(c) (i) First term 1, second term 4

(ii) First term 9, second term 19

(d) (i) First term 4, second term 0

(ii) First term 27, second term 20

(e) (i) Third term 5, eighth term 60

(ii) Fifth term 8, eighth term 38

2. How many terms are there in the following sequences?

(a) (i) 1, 3, 5, …, 65

(ii) 18, 13, 8, …, −122

(b) (i) First term 8, common difference 9, last term 899

(ii) First term 0, ninth term 16, last term 450

3. An arithmetic sequence has 5 and 13 as its first two terms.

(a) Write down, in terms of n, an expression for the nth term, u_n.

(b) Find the number of terms of the sequence which are less than 400. *[8 marks]*

4. The 10th term of an arithmetic sequence is 61 and the 13th term is 79. Find the value of the 20th term. *[4 marks]*

5. The 8th term of an arithmetic sequence is 74 and the 15th term is 137. Which term has the value 227? *[4 marks]*

6. The heights of the rungs in a ladder form an arithmetic sequence. The third rung is 70 cm above the ground and the tenth rung is 210 cm above the ground. If the top rung is 350 cm above the ground, how many rungs does the ladder have? *[5 marks]*

7. The first four terms of an arithmetic sequence are 2, $a-b$, $2a+b+7$ and $a-3b$, where a and b are constants. Find a and b. *[5 marks]*

8. A book starts at page 1 and is numbered on every page.

(a) Show that the first eleven pages contain thirteen digits.

(b) If the total number of digits used is 1260, how many pages are in the book? *[8 marks]*

7D Arithmetic series

When you add up terms of an arithmetic sequence you get an arithmetic series. There is a formula for the sum of an arithmetic series, the proof of which is not required in the IB. See Fill-in proof 4 'Arithmetic series and the story of Gauss' on the CD-ROM if you are interested.

There are two different forms for the formula.

KEY POINT 7.4

If you know the first and last terms:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

If you know the first term and the common difference:

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

Worked example 7.9

Find the sum of the first 30 terms of an arithmetic progression with first term 8 and common difference 0.5.

We have all the information we need to use the second formula

$$S_{30} = \frac{30}{2}(2 \times 8 + (30-1) \times 0.5) = 457.5$$

Sometimes you have to interpret the question carefully to be sure that it is about an arithmetic sequence.

Worked example 7.10

Find the sum of all the multiples of 3 between 100 and 1000.

Write out the first few terms to see what is happening

Sum $= 102 + 105 + 108 + \ldots + 999$

This is an arithmetic series with $u_1 = 102$ and $d = 3$

To use either sum formula, we also need to know how many terms are in this sequence

We do this by setting $u_n = 999$

$$999 = 102 + 3(n-1)$$

$$\Leftrightarrow 897 = 3(n-1)$$

$$\Leftrightarrow \quad n = 300$$

Use the first sum formula

$$S_{300} = \frac{300}{2}(102 + 999) = 165\,150$$

You must be able to work backwards too; given information which includes the sum of the series, you may be asked to find out how many terms are in the series. Remember that the number of terms can only be a positive integer.

Worked example 7.11

An arithmetic sequence has first term 5 and common difference 10.

If the sum of all the terms is 720, how many terms are in the sequence?

We need to find n and it is the only unknown in the second sum formula

$$720 = \frac{n}{2}(2\times 5 + (n-1)\times 10)$$
$$= \frac{n}{2}(10 + 10n - 10)$$
$$= 5n^2$$

Solve this equation

$$n^2 = 144$$
$$n = \pm 12$$

But n must be a positive integer, so $n = 12$

Exercise 7D

1. Find the sum of the following arithmetic sequences:
   (a) (i) 12, 33, 54, … (17 terms)
   (ii) −100, −85, −70, … (23 terms)
   (b) (i) 3, 15, …, 459
   (ii) 2, 11, …, 650
   (c) (i) 28, 23, …, −52
   (ii) 100, 97, …,40
   (d) (i) 15, 15.5, …, 29.5
   (ii) $\frac{1}{12}, \frac{1}{6}, \ldots, 1.5$
2. An arithmetic sequence has first term 4 and common difference 8.

   How many terms are required to get a sum of:
   (a) (i) 676 (ii) 4096 (iii) 11236
   (b) $x^2, x > 0$
3. The second term of an arithmetic sequence is 7. The sum of the first four terms of the sequence is 12. Find the first term, a, and the common difference, d, of the sequence. *[5 marks]*

4. Consider the arithmetic series $2 + 5 + 8 + \ldots$:

   (a) Find an expression for S_n, the sum of the first n terms.

   (b) Find the value of n for which $S_n = 1365$. *[5 marks]*

5. Find the sum of the positive terms of the arithmetic sequence 85, 78, 71, … *[6 marks]*

6. The second term of an arithmetic sequence is 6. The sum of the first four terms of the arithmetic sequence is 8. Find the first term, a, and the common difference, d, of the sequence. *[6 marks]*

7. Consider the arithmetic series $-6 + 1 + 8 + 15 + \ldots$

   Find the least number of terms so that the sum of the series is greater than 10 000. *[6 marks]*

8. The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 - 2n$. Find the nth term u_n. *[6 marks]*

9. A circular disc is cut into twelve sectors whose angles are in an arithmetic sequence.

   The angle of the largest sector is twice the angle of the smallest sector. Find the size of the angle of the smallest sector. *[6 marks]*

10. The ratio of the fifth term to the twelfth term of a sequence in an arithmetic progression is $\frac{6}{13}$.

    If each term of this sequence is positive, and the product of the first term and the third term is 32, find the sum of the first 100 terms of this sequence. *[7 marks]*

11. What is the sum of all three-digit numbers which are multiples of 14 but not 21? *[8 marks]*

7E Geometric sequences

Geometric sequences have a constant ratio, called the *common ratio, r*, between terms:

$$u_{n+1} = r \times u_n$$

So examples of geometric sequences might be:

1, 2, 4, 8, 16, … $(r = 2)$

100, 50, 25, 12.5, 6.25, … $(r = \frac{1}{2})$

1, −3, 9, −27, 81, … $(r = -3)$

As with arithmetic sequences, we also need to know the first term to fully define a geometric sequence. Again this is normally given the symbol u_1.

To get immediately to the deductive rule, we can see that to get to the nth term you start at the first term and multiply by the common ratio $n - 1$ times.

KEY POINT 7.5

$$u_n = u_1 r^{n-1}$$

Worked example 7.12

The seventh term of a geometric sequence is 13. The ninth term is 52.

What values could the common ratio take?

Write an expression for the seventh term in terms of u_1 and r

$u_7 = u_1 r^6$

But $u_7 = 13$

$13 = u_1 r^6 \quad (1)$

Repeat the same process for the ninth term

$52 = u_1 r^8 \quad (2)$

Solve the two equations simultaneously. Divide to eliminate u_1

$(2) \div (1)$

$4 = r^2$

$\Leftrightarrow r = \pm 2$

EXAM HINT

Notice that the question asked for values rather than value. This is a big hint that there is more than one solution.

When questions on geometric sequences ask what term satisfies a particular condition, you will usually use logarithms to solve an equation. Be careful when dealing with logarithms and inequalities; if you divide by the logarithm of a number less than 1 then you must flip the inequality.

Worked example 7.13

A geometric sequence has first term 10 and common ratio $\frac{1}{3}$. What is the first term that is less than 10^{-6}?

Express the condition as an inequality

$$10 \times \left(\frac{1}{3}\right)^{n-1} < 10^{-6}$$

$$\Leftrightarrow \left(\frac{1}{3}\right)^{n-1} < 10^{-7}$$

The unknown is in the power so we solve it using logarithms

$$\Leftrightarrow \log\left(\frac{1}{3}\right)^{n-1} < \log 10^{-7}$$

continued . . .

Use the logarithm law $\log_a x^p = p\log_a x$

$\log\left(\frac{1}{3}\right) < 0$ so reverse the inequality when dividing

$\Leftrightarrow \ (n-1)\log\left(\frac{1}{3}\right) < \log 10^{-7}$

$\Leftrightarrow \ (n-1) > \dfrac{\log 10^{-7}}{\log\left(\frac{1}{3}\right)}$

$\Leftrightarrow \ n > 15.7$ (3SF)

But n is a whole number so the first term satisfying the condition is 16

A particular problem is when the common ratio is negative, as we cannot take the log of a negative number. However, we can get around this problem using the fact that a negative number raised to an even power is always positive.

Inequalities are covered in Prior learning Section L.

Worked example 7.14

A geometric sequence has first term 2 and common ratio –3. What term has the value –4374?

Write the information given as an equation

$-4374 = 2\times(-3)^{n-1}$

Multiply both sides by –3 to make the left hand side positive

$\Leftrightarrow 13\,122 = 2\times(-3)^n$

Since the LHS is positive the RHS must also be positive, so n must be even and we can replace $(-3)^n$ with $(3)^n$

Since both sides must be positive:

$13\,122 = 2\times(3)^n$

Solve the equation using logarithms

$6561 = 3^n$

$\log 6561 = n\log 3$

$n = \dfrac{\log 6561}{\log 3} = 8$

In Worked example 7.14, you might have tried taking logarithms at the first line. Although this would have meant logs of negative numbers, using the rules of logs leads to the correct answer. This suggests that there may be some interpretation of logs of negative numbers, and when we meet complex numbers there will indeed be an interpretation. So does the logarithm of a negative number 'exist'? To some extent in mathematics if a concept is useful, that is enough to justify its introduction.

Exercise 7E

1. Find an expression for the nth term of these geometric sequences:

   (a) (i) 6, 12, 24, … (ii) 12, 18, 27, …

   (b) (i) 20, 5, 1.25, … (ii) $1, \frac{1}{2}, \frac{1}{4}, \ldots$

   (c) (i) 1, −2, 4, … (ii) 5, −5, 5, …

   (d) (i) $a, ax, ax^2, \ldots$ (ii) $3, 6x, 12x^2, \ldots$

2. How many terms are there in the following geometric sequences?

   (a) (i) 6, 12, 24, …, 24576 (ii) 20, 50, …, 4882.8125

   (b) (i) $1, -3, \ldots, -19683$ (ii) 2, −4, 8, …, −1024

   (c) (i) $\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{1024}$ (ii) $3, 2, \frac{4}{3}, \ldots, \frac{128}{729}$

3. How many terms are needed in the following geometric sequences to get within 10^{-9} of zero?

   (a) (i) $5, 1, \frac{1}{5}, \ldots$ (ii) 0.6, 0.3, 0.15, …

   (b) (i) 4, −2, 1, … (ii) −125, 25, −5, …

4. The second term of a geometric sequence is 6 and the fifth term is 162. Find the tenth term. *[5 marks]*

5. The third term of a geometric sequence is 112 and the sixth term is 7168.

   Which term takes the value 1 835 008? *[5 marks]*

6. Which is the first term of the sequence $\frac{2}{5}, \frac{4}{25}, \ldots, \frac{2^n}{5^n}$ that is less than 10^{-6}? *[6 marks]*

7. The difference between the fourth and the third term of a geometric sequence is $\frac{75}{8}$ times the first term. Find all possible values of the common ratio. *[6 marks]*

8. The third term of a geometric progression is 12 and the fifth term is 48. Find the two possible values of the eighth term. *[6 marks]*

9. The first three terms of a geometric sequence are a, $a + 14$ and $9a$. Find all possible values of a. *[6 marks]*

10. The three terms a, 1, b are in arithmetic progression.
The three terms 1, a, b are in geometric progression.
Find the value of a and b given that $a \neq b$. *[7 marks]*

11. The sum of the first n terms of an arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m. *[7 marks]*

7F Geometric series

As with arithmetic series there is a complicated formula for the sum of geometric sequences. See Fill-in proof 5 'Self-similarity and geometric series', on the CD-ROM for the derivation.

KEY POINT 7.6

$$S_n = \frac{u_1(1-r^n)}{1-r} \qquad (r \neq 1)$$

or equivalently

$$S_n = \frac{u_1(r^n-1)}{r-1} \qquad (r \neq 1)$$

We generally use the first of these formulae when the common ratio is less than one and the second when the common ratio is greater than one. This avoids working with negative numbers.

Worked example 7.15

Find the exact value of the sum of the first 6 terms of the geometric sequence with first term 8 and common ratio $\frac{1}{2}$.

$r < 1$, so use the first sum formula

$$S_6 = \frac{8\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}}$$

$$= \frac{8\left(1-\frac{1}{64}\right)}{\frac{1}{2}}$$

$$= 16\left(\frac{63}{64}\right)$$

$$= \frac{63}{4}$$

We may be given information about the sum and have to deduce other information.

Worked example 7.16

How many terms are needed for the sum of the geometric series 3 + 6 + 12 + 24 + … to exceed 10 000?

State the values of u_1 and r

$u_1 = 3$
$r = 2$

$r > 1$, so use the second sum formula and express the condition as an inequality

$$S_n = \frac{3(2^n - 1)}{2 - 1} > 10\,000$$

The unknown n is in the power, so use logarithms to solve the inequality

$$3(2^n - 1) > 10\,000$$

$$2^n > \frac{10\,003}{3}$$

$$\Leftrightarrow \log 2^n > \log\left(\frac{10\,003}{3}\right)$$

$$\Leftrightarrow n\log 2 > \log\left(\frac{10\,003}{3}\right)$$

$$\Leftrightarrow \quad n > \log\left(\frac{10\,003}{3}\right) \div \log 2$$

$$n > 11.7 \text{ (3SF)}$$

But n must be a whole number so 12 terms are needed.

Exercise 7F

Geometric series get very large very quickly. A mathematical legend involving the supposed inventor of chess, Sissa Ibn Dahir, illustrates how poor our intuition is with large numbers. It is explored on Supplementary sheet 6 'The chess legend and extreme numbers'.

1. Find the sums of the following geometric series. (There may be more than one possible answer!)

(a) (i) 7, 35, 175, … (10 terms)

(ii) 1152, 576, 288, … (12 terms)

(b) (i) 16, 24, 36, …, 182.25

(ii) 1, 1.1, 1.21, …, 1.771651

(c) (i) First term 8, common ratio −3, last term 52 488

(ii) First term −6, common ratio −3, last term 13 122

(d) (i) Third term 24, fifth term 6, 12 terms

(ii) Ninth term 50, thirteenth term 0.08, last term 0.0032

2. Find the possible values of the common ratio if the:

   (a) (i) first term is 11 and the sum of the first 12 terms is 2 922 920

   (ii) first term is 1 and the sum of the first 6 terms is 1.249 94

   (b) (i) first term 12 and the sum of the first 6 terms is −79 980

   (ii) first term is 10 and the sum of the first 4 terms is 1.

3. The nth term, u_n, of a geometric sequence is given by $u_n = 3\times 5^{n+2}, n\in\mathbb{Z}^+$.

   (a) Find the common ratio, r.

   (b) Hence or otherwise, find S_n, the sum of the first n terms of this sequence. *[5 marks]*

4. The sum of the first three terms of a geometric sequence is $23\frac{3}{4}$ and the sum of the first four terms is $40\frac{5}{8}$. Find the first term and the common ratio. *[6 marks]*

5. (a) A geometric sequence has first term 1 and common ratio x. Express the sum of the first four terms as a polynomial in x.

   (b) Factorise $x^6 - 1$ into a linear factor and a polynomial of order 5. *[6 marks]*

7G Infinite geometric series

If we keep adding together terms of any arithmetic sequence the answer grows (or decreases) without limit. The series is **divergent**.

Sometimes this happens with geometric series too, but there are cases where the sum gets closer and closer to a finite number. The series is **convergent**.

Not all geometric series converge. To decide which ones do, we need to use the formula for a geometric series from Key point 7.6.

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

and look at the r^n term. We are interested in what happens to this as n gets very large.

When you raise most numbers to a large power the result gets bigger and bigger, except when r is a number between −1 and 1. In this case, r^n gets smaller as n increases – in fact it tends to 0.

Many other sequences and series show interesting long-term behaviour. For example if the series $\frac{4}{1}-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}$ continues for ever, the result is π. Series like these are investigated in Supplementary sheet 5 'Long-term behaviour of sequences and series' on the CD-ROM.

EXAM HINT

The condition that $|r|<1$ is just as important as the formula.

We can summarise:

KEY POINT 7.7

As n increases the sum of a geometric series converges to:

$$S_\infty = \frac{u_1}{1-r} \text{ if } |r|<1$$

This is called the **sum to infinity** of the series.

When $r = 1$ the geometric sequence definitely diverges. When $r = -1$ it is uncertain whether the sequence converges or not. It might converge to 0, to u_1 or to $\frac{u_1}{2}$ depending upon how the terms are grouped. This is an example of a situation where mathematics is open to interpretation.

Worked example 7.17

The sum to infinity of a geometric series is 5. The second term is $-\frac{6}{5}$. Find the common ratio.

Express the information given as equations in terms of u_1 and r

$$S_\infty = \frac{u_1}{1-r} = 5 \qquad (1)$$

$$u_2 = u_1 r = -\frac{6}{5} \qquad (2)$$

Solve the equations simultaneously

From (2) $u_1 = -\frac{6}{5r}$

Substituting into (1)

$$-\frac{6}{5r(1-r)} = 5$$

$$\Leftrightarrow \quad -6 = 25(r - r^2)$$

$$\Leftrightarrow \quad 0 = 25r^2 - 25r - 6 = (5r-6)(5r+1)$$

Therefore $r = \frac{6}{5}$ or $r = -\frac{1}{5}$

Watch out for a trick! Check that the series converges

But since the sum to infinity exists, $|r|<1$ so

$$r = -\frac{1}{5}$$

Remember that some questions may focus on the condition for the sequence to converge as well as the value that it converges to.

Worked example 7.18

The geometric series $(2-x)+(2-x)^2+(2-x)^3+\dots$ converges. What values can x take?

Identify r	$r=(2-x)$
Use the fact that the series converges	Since the series converges $\lvert 2-x\rvert<1$
Solve the inequality	$-1<2-x<1$ $\Leftrightarrow -3<-x<-1$ Therefore $1<x<3$

Exercise 7G

1. Find the value of these infinite geometric series, or state that they are divergent.
   (a) (i) $9+3+1+\frac{1}{3}+\dots$
   (ii) $56+8+1\frac{1}{7}+\dots$
   (b) (i) $0.3+0.03+0.003+\dots$
   (ii) $0.78+0.0078+0.000078+\dots$
   (c) (i) $0.01+0.02+0.04+\dots$
   (ii) $\frac{19}{10000}+\frac{19}{1000}+\frac{19}{100}+\dots$
   (d) (i) $10-2+0.4-\dots$
   (ii) $6-4+\frac{8}{3}-\dots$
   (e) (i) $10-40+160-\dots$
   (ii) $4.2-3.36+2.688-\dots$

2. Find the values of x which allow these geometric series to converge.
   (a) (i) $9+9x+9x^2+\dots$
   (ii) $-2-2x-2x^2-\dots$
   (b) (i) $1+3x+9x^2+\dots$
   (ii) $1+10x+100x^2+\dots$
   (c) (i) $-2-10x-50x^2-\dots$
   (ii) $8+24x+72x^2+\dots$

(d) (i) $40+10x+2.5x^2+\dots$

(ii) $144+12x+x^2+\dots$

(e) (i) $243-81x+27x^2-\dots$

(ii) $1-\frac{5}{4}x+\frac{25}{16}x^2-\dots$

(f) (i) $3-\frac{6}{x}+\frac{12}{x^2}-.$

(ii) $18-\frac{9}{x}+\frac{1}{x^2}-\dots$

(g) (i) $5+5(3-2x)+5(3-2x)^2+\dots$

(ii) $7+\frac{7(2-x)}{2}+\frac{7(2-x)^2}{4}+\dots$

(h) (i) $1+\left(3-\frac{2}{x}\right)+\left(3-\frac{2}{x}\right)^2+\dots$

(ii) $1+\frac{1+x}{x}+\frac{(1+x)^2}{x^2}+\dots$

(i) (i) $7+7x^2+7x^4+\dots$

(ii) $12-48x^3+192x^6-\dots$

3. Find the sum to infinity of the geometric sequence
$-18, 12, -8, \dots$ *[4 marks]*

4. The first and fourth terms of a geometric series are 18 and $-\frac{2}{3}$ respectively. Find:

(a) the sum of the first n terms of the series

(b) the sum to infinity of the series. *[5 marks]*

5. A geometric sequence has all positive terms. The sum of the first two terms is 15 and the sum to infinity is 27. Find the value of

(a) the common ratio

(b) the first term. *[5 marks]*

6. The sum to infinity of a geometric sequence is 32. The sum of the first four terms is 30 and all the terms are positive.

Find the difference between the sum to infinity and the sum of the first eight terms. *[5 marks]*

7. Consider the infinite geometric series:

$$1+\left(\frac{2x}{3}\right)+\left(\frac{2x}{3}\right)^2+\dots$$

(a) For what values of x does the series converge?

(b) Find the sum of the series if $x=1.2$. *[6 marks]*

8. The sum of an infinite geometric sequence is 13.5, and the sum of the first three terms is 13. Find the first term. *[6 marks]*

9. An infinite geometric series is given by $\sum_{k=1}^{\infty} 2(4-3x)^k$.

(a) Find the values of x for which the series has a finite sum.

(b) When $x = 1.2$, find the minimum number of terms needed to give a sum which is greater than 1.328. *[7 marks]*

10. The common ratio of the terms in a geometric series is 2^x.

(a) State the set of values of x for which the sum to infinity of the series exists.

(b) If the first term of the series is 35, find the value of x for which the sum to infinity is 40. *[6 marks]*

11. $f(x) = 1 + 2x + 4x^2 + 8x^3 \ldots$ is an infinitely long expression. Evaluate:

(a) $f\left(\frac{1}{3}\right)$ (b) $f\left(\frac{2}{3}\right)$ *[6 marks]*

7H Mixed questions

Be very careful when dealing with sequences and series questions.

It is vital that you

- identify whether it is a geometric or an arithmetic sequence
- identify whether it is asking for a term in the sequence or the sum of terms in the sequence
- interpret the information given in the question into equations.

One frequently examined topic is **compound interest**. This is about savings or loans, where the interest added is a percentage of the current amount. As long as no other money is added or removed, the value of the investment will follow a geometric sequence.

If the compound interest rate is p% then this is equivalent to a ratio of $r = 1 + \frac{p}{100}$.

Exercise 7H

1. Philippa invests £1000 at 3% compound interest for 6 years.

(a) How much interest does she get paid in the 6th year?

(b) How much does she get back after 6 years? *[6 marks]*

2. Lars starts a job on an annual salary of \$32 000 and is promised an annual increase of \$1500.

(a) How much will his 20th year's salary be?

(b) After how many complete years will he have earned a total of \$1 million? *[6 marks]*

3. A sum of \$5000 is invested at a compound interest rate of 6.3% per annum.

(a) Write down an expression for the value of the investment after n full years.

(b) What will be the value of the investment at the end of 5 years?

(c) The value of the investment will exceed \$10 000 after n full years.

(i) Write an inequality to represent this information.

(ii) Calculate the minimum value of n. *[8 marks]*

4. Each consecutive row of seats in a theatre has 200 more seats than the previous row. There are 50 seats in the front row and the designer wants the theatre capacity to be at least 8000.

(a) How many rows are required?

(b) Assuming the spacings between rows are equal, what percentage of people are seated in the front half of the theatre? *[7 marks]*

5. A sum of \$100 is invested.

(a) If the interest is compounded annually at a rate of 5% per year, find the total value V of the investment after 20 years.

(b) If the interest is compounded monthly at a rate of $\frac{5}{12}$ % per month, find the minimum number of months for the value of the investment to exceed V. *[6 marks]*

6. A marathon is a 26 mile race. In his training program, a marathon a runner runs 1 mile on his first day of training and each day increases his distance by $\frac{1}{4}$ of a mile.

(a) After how many days has he run for a total of 26 miles?

(b) On which day does he first run over 26 miles? [6 *marks*]

7. A ball is dropped vertically from 2 m in the air. Each time it bounces up to a height of 80% of its previous height.

(a) How high does it bounce on the fourth bounce?

(b) How far has it travelled when it hits the ground for the ninth time? *[7 marks]*

8. Samantha puts \$1000 into a bank account at the beginning of each year, starting in 2010.

At the end of each year 4% interest is added to the account.

(a) Show that at the beginning of 2012 there is

$\$1000 + \$1000 \times 1.04 + \$1000 \times (1.04)^2$ in the account.

(b) Find an expression for the amount in the account at the beginning of year n.

(c) When Samantha has a total of at least \$50 000 in her account at the beginning of a year she will buy a house. In which year will this happen? *[7 marks]*

Summary

- Sequences can be described using either **recursive definitions**– from term-to-term – or **deductive rules** – finding the nth term.
- A **series** is a sum of terms in a sequence and it can be described neatly using sigma notation:

$$\sum_{r=1}^{r=n} f(r) = f(1) + f(2) + \ldots + f(n)$$

- One very important type of sequence is an **arithmetic sequence** which has a constant difference, d, between terms. The relevant formulae are given in the Formula booklet:
  - If you know the first term, u_1, the general term is:

$$u_n = u_1 + (n-1)d$$

  - If you know the first and last terms the sum of all n terms in the sequence is:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

  - If you know the first term and the common difference:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

- Another very important type of sequence is a **geometric sequence** which has a constant ratio, r, between terms. The following are also given in the Formula booklet:
  - If you know the first term, u_1, the general term is:

$$u_n = u_1 r^{n-1}$$

  - The sum of the first n terms is:

$$S_n = \frac{u_1(1-r^n)}{1-r} \text{ or } \frac{u_1(r^n-1)}{r-1} (r \neq 1)$$

- A series can be **convergent** (the sum gets closer and closer to a single value) or **divergent** (keeps increasing or decreasing without limit)
  - If $|r|<1$ the sum to infinity of a geometric sequence is given by:

$$S_\infty = \frac{u_1}{1-r}$$

Introductory problem revisited

A mortgage of \$100 000 is fixed at 5% compound interest. It needs to be paid off over 25 years by annual instalments. Interest is added at the end of each year, just before the payment is made. How much should be paid each year?

Imagine that you have two separate bank accounts. One is overdrawn and interest is added annually to the debt. You make regular payments to the second account, and this account earns interest each year at the same rate.

The first payment you make will have interest paid on it 24 times. The second payment will have 23 interest payments and so on.

The amount in the debt account after 25 years will be $100\,000 \times 1.05^{25}$.

If the annual payment is \$$x$, the amount in the credit account will be:

$$x \times 1.05^{24} + x \times 1.05^{23} + x \times 1.05^{22} + \ldots + x \times 1.05^{1} + x$$

But this is a finite geometric series with 25 terms, first term x and common ratio 1.05 so it can be simplified to:

$$\frac{x(1.05^{25}-1)}{1.05-1}$$

If the debt is to be paid off, the amount in the credit account must equal the amount in the debit account, so:

$$100\,000 \times 1.05^{25} = \frac{x(1.05^{25}-1)}{1.05-1}$$

Solving this gives $x = 7095.25$ and the annual payments must be \$7095.25.

Mixed examination practice 7

Short questions

1. The fourth term of an arithmetic sequence is 9.6 and the ninth term is 15.6.
Find the sum of the first nine terms. *[5 marks]*

2. The sum of the first n terms of a series is given by:

$$S_n = 2n^2 - n, \text{ where } n \in \mathbb{Z}^+.$$

(a) Find the first three terms of the series.

(b) Find an expression for the nth term of the series, giving your answer in terms of n. *[6 marks]*

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3. Which is the first term of this sequence which is less than 10^{-6}?

$$\frac{1}{3}, \frac{1}{9} \ldots \frac{1}{3^n}$$ *[5 marks]*

4. The fifth term of an arithmetic sequence is three times larger than the second term. Find the ratio: $\dfrac{\text{common difference}}{\text{first term}}$ *[6 marks]*

5. A geometric sequence and an arithmetic sequence both start with a first term of 1. The third term of the arithmetic sequence is the same as the second term of the geometric sequence. The fourth term of the arithmetic sequence is the same as the third term of the geometric sequence. Find the possible values of the common difference of the arithmetic sequence. *[7 marks]*

6. Evaluate $\displaystyle\sum_{i=0}^{i=\infty} \frac{(2^i + 4^i)}{6^i}$. *[6 marks]*

7. Find the sum of all the integers between 300 and 600 which are divisible by 7. *[7 marks]*

8. Find an expression for the sum of the first 23 terms of the series

$$\ln\frac{a^3}{\sqrt{b}} + \ln\frac{a^3}{b} + \ln\frac{a^3}{b\sqrt{b}} + \ln\frac{a^3}{b^2} + \ldots$$

giving your answer in the form $\ln\dfrac{a^m}{b^n}$, where $m, n \in \mathbb{Z}$. *[7 marks]*

Long questions

1. Kenny is offered two investment plans, each requiring an initial investment of \$10 000:

Plan A offers a fixed return of \$800 per year.

Plan B offers a return of 5% each year, reinvested in the plan.

(a) Find an expression for the amount in plan A after n years.

(b) Find an expression for the amount in plan B after n years.

(c) Over what period of time is plan A better than plan B? *[10 marks]*

2. Ben builds a pyramid out of toy bricks. The top row contains one brick, the second row contains three bricks and each row after that contains two more bricks than the previous row.

(a) How many bricks are in the nth row?

(b) If a total of 36 bricks are used how many rows are there?

(c) In Ben's largest ever pyramid he noticed that the total number of bricks was four more than four times the number of bricks in the bottom row. What is the total number of bricks? *[10 marks]*

3. A pupil writes '1' on the first line of a page, then the next two integers '2, 3' on the second line of the page then the next three integers '4, 5, 6' on the third line. He continues this pattern.

(a) How many integers are on the nth line?

(b) What is the last integer on the nth line?

(c) What is the first integer on the nth line?

(d) Show that the sum of all the integers on the nth line is $\frac{n}{2}\left(n^2+1\right)$.

(e) The sum of all the integers on the last line of the page is 16 400. How many lines are on the page? *[10 marks]*

4. Selma has a mortgage of £150 000. At the end of each year 6% interest is added before Selma pays £10 000.

(a) Show that at the end of the third year the amount owing is

$$£150000\times(1.06)^3-10000\times(1.06)^2-10000\times1.06-10000$$

(b) Find an expression for how much is owed at the end of the nth year.

(c) After how many years will the mortgage be paid off? *[10 marks]*