3 Polynomials

In this chapter you will learn:

- the formal definition of a polynomial
- operations with polynomials
- a trick for factorising polynomials and finding remainders
- how to sketch the graphs of polynomials
- how to identify the number of solutions of a quadratic equation.

Introductory problem

Without using your calculator, solve the cubic equation

$$x^3 - 13x^2 + 47x - 35 = 0$$

You may think that your calculator can 'magically' do any calculation, such as $\sin 45°$ or e^{-2} or $\ln 2$. However, like any computer, it is built from a device called a logic circuit which can only really do addition. If you repeatedly do addition you get multiplication and if you repeatedly do multiplication you raise to a power. Any expression which only uses these operations is called a polynomial. What is particularly surprising is that all the other operations your calculator does can be approximated very accurately using these polynomials.

3A Working with polynomials

EXAM HINT

The order of a polynomial is sometimes also called its degree.

The **polynomial functions** of x make up a family of functions, each of which can be written as a sum of non-negative integer powers of x. Polynomial functions are classified according to the highest power of x occurring in the function, called the **order of the polynomial**.

General form of the polynomial	Order	Classification	Example
a	0	Constant polynomial	$y = 5$
$ax + b$	1	Linear polynomial	$y = x + 7$
$ax^2 + bx + c$	2	Quadratic polynomial	$y = -3x^2 + 4x - 1$
$ax^3 + bx^2 + cx + d$	3	Cubic polynomial	$y = 2x^3 + 7x$
$ax^4 + bx^3 + cx^2 + dx + e$	4	Quartic polynomial	$y = x^4 - x^3 + 2x + \frac{1}{2}$

The letters a, b, c, … are called the **coefficients** of the powers of x, and the coefficient of the highest power of x in the function (a in the table above) is called the **lead coefficient** and the term containing it is the **leading order term**.

Coefficients can take any value, with the restriction that the lead coefficient cannot equal zero; a polynomial of order n which had a lead coefficient 0 could be more simply written as a polynomial of order $(n-1)$. The sign of the lead coefficient dictates whether the polynomial is a **positive polynomial** $(a > 0)$ or a **negative polynomial** $(a < 0)$.

Adding and subtracting two polynomials is straightforward – it is just collecting like terms. For example:

$$(x^4+3x^2-1)-(2x^3-x^2+2)=x^4-2x^3+4x^2-3.$$

Multiplying is a little more difficult. Below is one suggested way of setting out polynomial multiplication to ensure that you include all of the terms.

The Greeks knew how to solve quadratic equations, and general cubics and quartics were first solved in 14th Century Italy. For over three hundred years nobody was able to find a general solution to the quintic equation, until in 1821 Niels Abel used a branch of mathematics called group theory to prove that there could never be a 'quintic formula'.

Worked example 3.1

Expand $(x^3+3x^2-2)(x^2-5x+4)$.

Multiply each term in the 1st bracket by the whole of the 2nd bracket

$$\begin{aligned}
&(x^3+3x^2-2)(x^2-5x+4)\\
&= x^3(x^2-5x+4)+3x^2(x^2-5x+4)-2(x^2-5x+4)\\
&= x^5-5x^4+4x^3\\
&\quad +3x^4-15x^3+12x^2\\
&\quad\quad -2x^2+10x-8\\
&= x^5-2x^4-11x^3+10x^2+10x-8
\end{aligned}$$

It is important to be able to decide when two polynomials are 'equal'. x^2 and $3x-2$ may take the same value when $x=1$ or 2 but when $x=3$ they do not. However, $3x-2$ and $4x-(x+2)$ are always equal, no matter what value of x you choose.

This leads to what seems like a very obvious definition of equality of polynomials:

You will find that this type of equality is called an identity and is looked at in more detail in Section 4H.

KEY POINT 3.1

Two polynomials being equal means that they have the same order and all of their coefficients are equal.

Ideas similar to comparing coefficients will be applied to vectors (chapter 13) and complex numbers (chapter 15).

However, this definition can be used to solve some quite tricky problems. We tend to use it in questions which start from telling us that two polynomials are equal; we can then **compare coefficients**.

Worked example 3.2

If $(ax)^2 + bx + 3(ax^2 + x) = 4x - 2x^2$ find the values of a and b.

Rearrange each side to make the coefficients clear

LHS: $a^2x^2 + bx + 3ax^2 + 3x$
$= a^2x^2 + 3ax^2 + bx + 3x$
$= x^2(a^2 + 3a) + x(b + 3)$

Compare coefficients and solve the resulting equations

x^2: $(a^2 + 3a) = -2$
$\Leftrightarrow a^2 + 3a + 2 = 0$
$\Leftrightarrow (a+1)(a+2) = 0$
$\Leftrightarrow a = -1$ or $a = -2$
x: $b + 3 = 4$
$\Leftrightarrow b = 1$

A very important use of this technique is factorising a polynomial if one factor is already known. To do this we need to know what the remaining factor looks like. For example, if a cubic has one linear factor, the other factor must be quadratic.

Worked example 3.3

$(x-1)$ is a factor of $x^4 + 3x^2 + 2x - 6$. Find the remaining factor.

The remaining factor must be a cubic. We can write the original function as a product of $(x - 1)$ and a general cubic

$x^4 + 3x^2 + 2x - 6 = (x-1)(ax^3 + bx^2 + cx + d)$

Multiplying out the right hand side and grouping like terms allows us to compare coefficients

$$= ax^4 + bx^3 + cx^2 + dx$$
$$-ax^3 - bx^2 - cx - d$$
$$= ax^4 + x^3(b-a) + x^2(c-b) + x(d-c) - d$$

continued . . .

Remember that the coefficient of x^3 in the original expression is zero!

Compare x^4:

$a = 1$

Compare x^3:

$b - a = 0$

$b = 1$

Compare x^2:

$c - b = 3$

$c = 4$

Compare x:

$d - c = 2$

$d = 6$

Answer the question

The remaining factor is $x^3 + x^2 + 4x + 6$

'Finding the remaining factor' is another way of asking you to divide. We can conclude from this example that:

$$\frac{x^4 + 3x^2 + 2x - 6}{x - 1} = x^3 + x^2 + 4x + 6$$

EXAM HINT

Notice in the previous example we could check using the constant term that $d = 6$.

Exercise 3A

1. Decide whether each of the following expressions are polynomials. For those that are, give the order and the lead coefficient.

(a) $3x^3 - 3x^2 + 2x$ (b) $1 - 3x - x^5$

(c) $5x^2 - x^{-3}$ (d) $9x^4 - \frac{5}{x}$

(e) $4e^x + 3e^{2x}$ (f) $x^4 + 5x^2 - 3\sqrt{x}$

(g) $4x^5 - 3x^3 + 2x^7 - 4$ (h) 1

2. Expand the brackets for the following expressions:

(a) (i) $(3x - 2)(2x^2 + 4x - 7)$

(ii) $(3x + 1)(x^2 + 5x + 6)$

(b) (i) $(2x + 1)(x^3 - 8x^2 + 6x - 1)$

(ii) $(2x + 5)(x^3 - 6x^2 + 3)$

(c) (i) $(b^2+3b-1)(b^2-2b+4)$

(ii) $(r^2-3r+7)(r^2-8r+2)$

(d) (i) $(5-x^2)(x^4-2x^3+1)$

(ii) $(x-x^3)(x^3-x-1)$

3. Find the remaining factor if

(a) (i) $x^3+3x^2-11x+2$ has a factor of $x-2$

(ii) $x^3+4x^2-3x-18$ has a factor of $x+3$

(b) (i) $6x^3+13x^2+2x-6$ has a factor of $2x+3$

(ii) $25x^3+5x^2-10x-2$ has a factor of $5x+1$

(c) (i) $x^4-5x^3+9x^2-2x-21$ has a factor of $x-3$

(ii) $x^4-5x^3+5x^2+3x-28$ has a factor of $x-4$

(d) (i) $x^4-3x^3+12x^2-15x+35$ has a factor of x^2-3x+7

(ii) x^3-2x^2+3x-6 has a factor of x^2+3

4. Given that the result of each division is a polynomial, simplify each expression.

(a) (i) $\dfrac{x^4-x^3-2x^2+3x-6}{x-2}$ (ii) $\dfrac{x^4+3x^3+2x^2+2x+4}{x+2}$

(b) (i) $\dfrac{2x^4+27x^2+36}{x^2+12}$ (ii) $\dfrac{2x^3-3x^2-27}{2x^2+3x+9}$

5. Find the unknown constants a and b in these identities.

(a) (i) $ax^2+bx=4x^2-6x$ (ii) $ax^2+4=3x^2+4b$

(b) (i) $ax^2+bx=4x+bx^2-ax$ (ii) $ax^2+2bx+6x=0$

(c) (i) $(ax+1)^2+3bx=2ax^2-2x+1$

(ii) $(x+a)^2+b=x^2+4x+9$

(d) (i) $ax^2+bx-2ax=2x-4x^2$

(ii) $ax^2-3bx^2+bx+4x=x^2+7x$

(e) (i) $(ax)^2-(bx)^2+bx=2x$

(ii) $(ax+b)^2=4x^2-20x+25$

6. In what circumstances might you want to expand brackets? In what circumstances is the factorised form better?

7. (a) Is it always true that the sum of a polynomial of order n and a polynomial of order $n-1$ has order n?

(b) Is it always true that the sum of a polynomial of order n and a polynomial of order n has order n?

3B Remainder and factor theorems

We saw in the last section that we can factorise polynomials by comparing coefficients. For example, if we know that $(x+2)$ is one factor of x^3+2x^2+x+2, we can write $x^3+2x^2+x+2=(x+2)(ax^2+bx+c)$ and compare coefficients to find that the other factor is (x^2+1).

If we try to factorise x^3+2x^2+x+5 using $(x+2)$ as one factor, we find that it is not possible; $(x+2)$ is not a factor of x^3+2x^2+x+5. However, using the factorisation of x^3+2x^2+x+2 we can write:

$$x^3+2x^2+x+5=(x+2)(x^2+1)+3$$

The number 3 is the **remainder** – it is what is left over when we try to write x^3+2x^2+x+5 as a multiple of $(x+2)$. In the last section we saw that factorising is related to division. In this case, we could say that:

$$\frac{x^3+2x^2+x+5}{x+2}=(x^2+1)\text{ with remainder }3$$

This is similar to the concept of a remainder when dividing numbers: for example, $25=3\times7+4$, so we would say that 4 is the remainder when 25 is divided by 7.

We can find the reminder by including it as another unknown coefficient. For example, to find the reminder when x^3+2x^2+x+5 is divided by $(x+2)$, we could write

$$x^3+2x^2+x+5=(x+2)(ax^2+bx+c)+R$$

then expand and compare coefficients. This is not a quick task. Luckily there is a shortcut which can help us find the reminder without finding all the other coefficients. If we substitute in a value of x that makes the first bracket equal to zero, in this case $x=-2$, into the above equation, it becomes

$$3=(0)(ax^2+bx+c)+R$$

so $R=3$. This means that R is the value we get when we substitute $x=-2$ into the polynomial expression on the left. Fill-in proof sheet 6 on the CD-ROM, 'Remainder theorem', shows you that the same reasoning can be applied when dividing any polynomial by a linear factor. This leads us to the **Remainder theorem**.

EXAM HINT

Notice that $x = \frac{b}{a}$ is the value which makes $ax - b = 0$.

KEY POINT 3.2

The remainder theorem

The remainder when a polynomial expression is divided by $(ax - b)$ is the value of the expression when $x = \frac{b}{a}$.

Worked example 3.4

Find the remainder when $x^3 + 2x + 7$ is divided by $x + 2$.

Use the remainder theorem by rewriting the divisor in the form $(ax - b)$…

$(x+2) = (x-(-2))$

… then substitute the value of x (obtained from $x = \frac{b}{a}$) into the expression when $x = \frac{b}{a}$

When $x = -2: (-2)^3 + 2\times(-2) + 7 = -5$

So the remainder is -5

If the remainder is zero then $(ax - b)$ is a factor. This is summarised by the **factor theorem**.

KEY POINT 3.3

The factor theorem

If the value of a polynomial expression is zero when $x = \frac{b}{a}$, then $(ax - b)$ is a factor of the expression.

Worked example 3.5

Show that $2x - 3$ is a factor of $2x^3 - 13x^2 + 19x - 6$.

To use the factor theorem we need to substitute in $x = \frac{3}{2}$

When $x = \frac{3}{2}$:

$$2\times\left(\frac{3}{2}\right)^3 - 13\left(\frac{3}{2}\right)^2 + 19\times\left(\frac{3}{2}\right) - 6$$

$$= \frac{27}{4} - \frac{117}{4} + \frac{57}{2} - 6 = 0$$

continued . . .

Therefore, by the factor theorem, $(2x-3)$ is a factor of $2x^3-13x^2+19x-6$.

We can also use the factor theorem to identify a factor of an expression, by trying several different numbers. Once one factor has been found, then comparing coefficients can be used to find the remaining factors. This is the recommended method for factorising cubic expressions on the non-calculator paper.

Worked example 3.6

Fully factorise $x^3+3x^2-33x-35$.

When factorising a cubic with no obvious factors we must put in some numbers and hope that we can apply the factor theorem

When $x=1$ the expression is -64
When $x=2$ the expression is -81
When $x=-1$ the expression is 0
Therefore $x+1$ is a factor.

We can rewrite the expression as $(x+1)\times$general quadratic and compare coefficients

$$x^3+3x^2-33x-35$$
$$=(x+1)(ax^2+bx+c)$$
$$=ax^3+(a+b)x^2+(b+c)x+c$$
$$a=1, b=2, c=-35$$
$$x^3+3x^2-33x-35=(x+1)(x^2+2x-35)$$

The remaining quadratic also factorises

$$=(x+1)(x+7)(x-5)$$

EXAM HINT

If the expression is going to factorise easily then you only need to try numbers which are factors of the constant term.

A very common type of question asks you to find unknown coefficients in an expression if factors or remainders are given.

Worked example 3.7

x^3+4x^2+ax+b has a factor of $(x-1)$ and leaves a remainder of 17 when divided by $(x-2)$. Find the constants a and b.

Apply factor theorem

when $x=1$:
$1+4+a+b=0$
$\Leftrightarrow a+b=-5 \quad (1)$

Apply remainder theorem

when $x=2$:
$8+16+2a+b=17$
$\Leftrightarrow 2a+b=-7 \quad (2)$

Two equations with two unknowns can be solved simultaneously

(2) − (1)
$a=-2$
$b=-3$

Exercise 3B

1. Use the remainder theorem to find the remainder when:
   (a) (i) x^2+3x+5 is divided by $x+1$
   (ii) x^2+x-4 is divided by $x+2$
   (b) (i) x^3-6x^2+4x+8 is divided by $x-3$
   (ii) x^3-7x^2+11x is divided by $x-1$
   (c) (i) $6x^4+7x^3-5x^2+5x+10$ is divided by $2x+3$
   (ii) $12x^4-10x^3+11x^2-5$ is divided by $3x-1$
   (d) (i) x^3 is divided by $x+2$
   (ii) $3x^4$ is divided by $x-1$

2. Decide whether each of the following expressions are factors of $2x^3-73-3x+2$.
   (a) x
   (b) $x-1$
   (c) $x+1$
   (d) $x-2$
   (e) $x+2$
   (f) $x-\frac{1}{2}$
   (g) $x+\frac{1}{2}$
   (h) $2x-1$
   (i) $2x+1$
   (j) $3x-1$

3. Fully factorise the following expressions:

(a) (i) x^3+2x^2-x-2 (ii) x^3+x^2-4x-4

(b) (i) $x^3-7x^2+16x-12$ (ii) $x^3+6x^2+12x+8$

(c) (i) $x^3-3x^2+12x-10$ (ii) $x^3-2x^2+2x-15$

(d) (i) $6x^3-11x^2+6x-1$ (ii) $12x^3+13x^2-37x-30$

4. $6x^3+ax^2+bx+8$ has a factor $(x+2)$ and leaves a remainder of -3 when divided by $(x-1)$.
Find a and b. *[5 marks]*

5. x^3+8x^2+ax+b has a factor of $(x-2)$ and leaves a remainder of 15 when divided by $(x-3)$.
Find a and b. *[5 marks]*

6. The polynomial $x^2+kx-8k$ has a factor $(x-k)$. Find the possible values of k. *[5 marks]*

7. The polynomial $x^2-(k+1)x-3$ has a factor $(x-k+1)$. Find k. *[6 marks]*

8. $x^3-ax^2-bx+168$ has factors $(x-7)$ and $(x-3)$.

(a) Find a and b.

(b) Find the remaining factor of the expression. *[6 marks]*

9. x^3+ax^2+9x+b has a factor of $(x-11)$ and leaves a remainder of -52 when divided by $(x+2)$.

(a) Find a and b.

(b) Find the remainder when x^3+ax^2+9x+b is divided by $(x-2)$. *[6 marks]*

10. $f(x)=x^3+ax^2+3x+b$.
The remainder when $f(x)$ is divided by $(x+1)$ is 6. Find the remainder when $f(x)$ is divided by $(x-1)$. *[5 marks]*

$f(x)$ is just a name given to the expression. You will learn more about this notation in chapter 5.

11. The polynomial x^2-5x+6 is a factor of $2x^3-15x^2+ax+b$. Find the values of a and b. *[6 marks]*

3C Sketching polynomial functions

The word quadratic indicates that the term with the highest power in the equation is x^2. It comes from the Latin *quadratus*, meaning square.

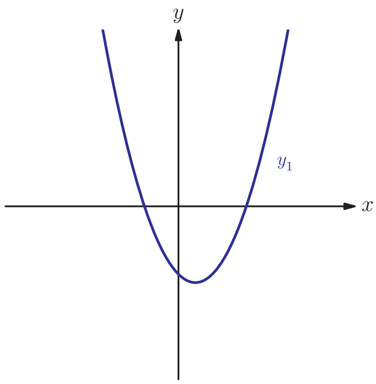
The simplest polynomial with a curved graph is a quadratic.

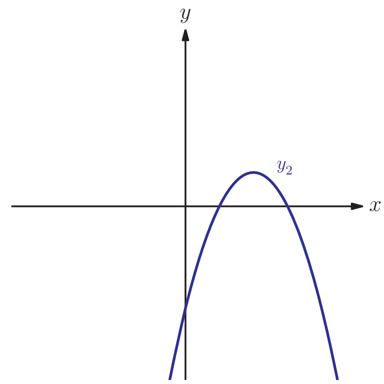
Let us look at two examples of quadratic functions:

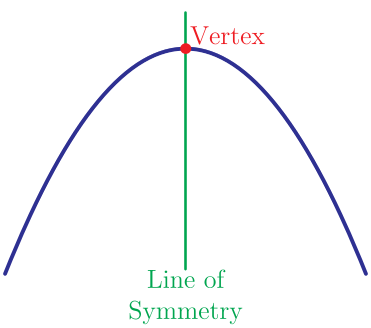
$y_1 = 2x^2 + 2x - 4$ and $y_2 = -x^2 + 4x - 3$ $(x \in \mathbb{R})$

$x \in \mathbb{R}$ means that x can be any number. See Prior learning Section G on the CD-ROM for the meaning of other similar statements.

You can use your calculator to plot the two graphs. See Calculator sheet 2 on the CD-ROM to see how to sketch graphs on your GDC.

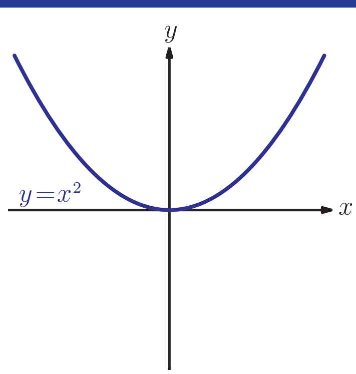
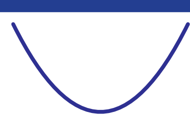
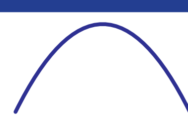


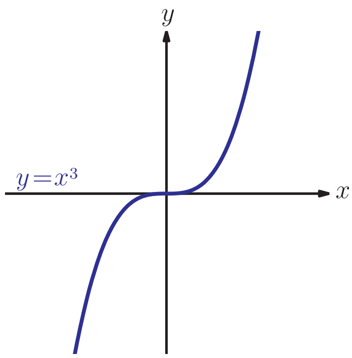
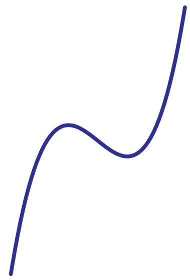
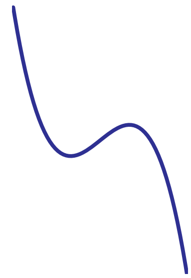
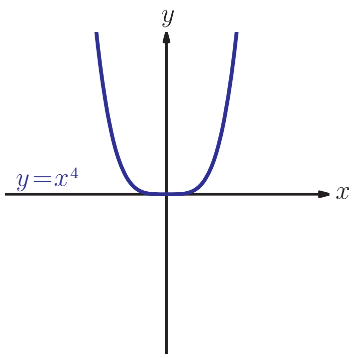
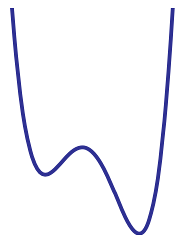
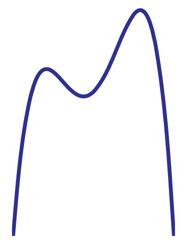




These two graphs have a similar shape, called a **parabola.** A parabola has a single turning point (also called the **vertex**) and a vertical line of symmetry. The most obvious difference is that y_1 has a minimum point, whereas y_2 has a maximum point. This is because of the different signs of the x^2 term.

The graphs of other polynomial functions are also smooth curves. You need to know the shapes to expect for these graphs.

n	$y = x^n$	Positive degree n polynomial	Negative degree n polynomial	Number of times graph can cross or touch the x-axis	Number of turning points
2				0, 1 or 2	1

3				1, 2, or 3	0 or 2
4				0, 1, 2 or 3	1 or 3

The constant term in the polynomial expression gives the position of the y-intercept of the graph (where the curve crosses the y-axis). This is because all other terms contain x, so when $x = 0$ the only non-zero term is the constant term. For example, $2x^3 - 3x + 5 = 5$ when $x = 0$.

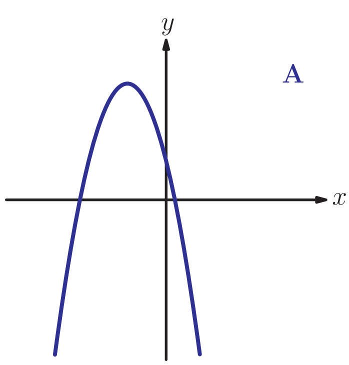
Worked example 3.8

Match each equation to the corresponding graph, explaining your reasons.

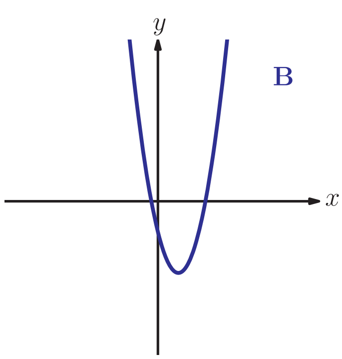
(a) $y = 3x^2 - 4x - 1$

(b) $y = -2x^2 - 4x$

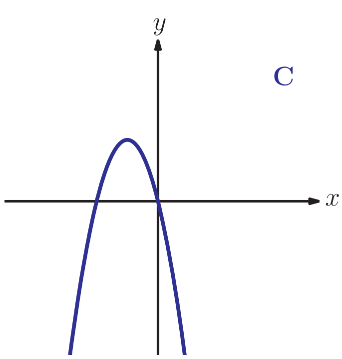
(c) $y = -x^2 - 4x + 2$



A



B



C

continued . . .

Graph B is the only positive quadratic

Graph B shows a positive quadratic, so graph B corresponds to equation (a).

We can distinguish between the other two graphs based on their *y*-intercept

Graph A has a positive y-intercept, so graph A corresponds to equation (c).
Graph C corresponds to equation (b).

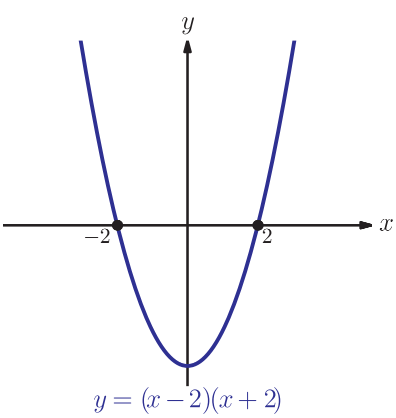
Factorising is covered in Prior learning Section N.

To sketch the graph of a polynomial it is also useful to know its x-intercepts. These are the points where the graph crosses the x-axis, so at those points $y = 0$. For this reason, they are called **zeros** of the polynomial. For example, the quadratic polynomial $x^2 - 5x + 6$ has zeros at $x = 2$ and $x = 3$. They are the **roots** (solutions) of the equation $x^2 - 5x + 6 = 0$ and can be found from the factorised form of the polynomial: $(x-2)(x-3)$.

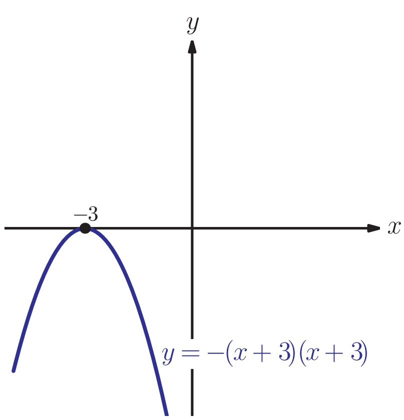
KEY POINT 3.4

The polynomial $a(x-p_1)(x-p_2)(x-p_3)\ldots$ has zeros $p_1, p_2, p_3, \ldots$

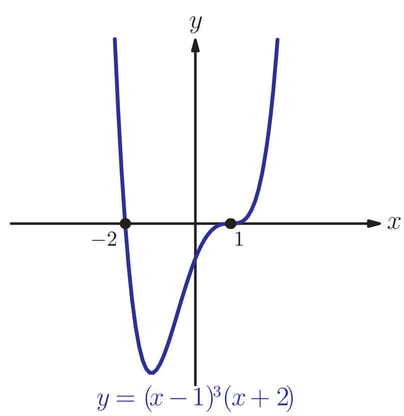
If some of the zeros are the same, we say that the polynomial has a repeated root. For example, the equation $(x-1)^2(x+3)$ has a repeated (double) root $x = 1$ and a single root $x = -3$. Repeated roots tell us how the graph meets the x-axis.



If a polynomial has a factor $(x-a)$ then the curve passes straight through the x-axis at a.



If a polynomial has a double factor $(x-a)^2$ then the curve touches the x-axis at a.



If a polynomial has a triple factor $(x-a)^3$ then the curve passes through the x-axis at a, flattening as it does so.

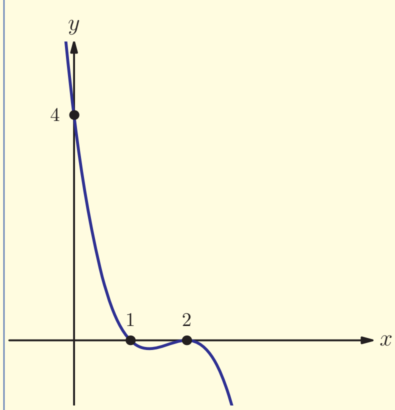
KEY POINT 3.5

The stages of sketching graphs of polynomials are:

- classify the order of the polynomial and whether it is positive or negative to deduce the basic shape
- set $x = 0$ to find the y-intercept
- write in factorised form
- find x-intercepts
- decide on how the curve meets the x-axis at each intercept
- connect all this information to make a smooth curve.

Worked example 3.9

Sketch the graph of $y = (1-x)(x-2)^2$.

Classify the basic shape	This is a negative cubic
Find y-intercept	When $x = 0$ $\quad y = 1 \times (-2)^2 = 4$
Find x-intercepts	When $y = 0$, $x = 1$ or $x = 2$
Decide on shape of curve at x-intercepts	At $x = 1$ curve passes through the axis At $x = 2$ curve just touches the axis
Sketch the curve	

EXAM HINT

A sketch does not have to be accurate or to scale. It must have approximately the correct shape and all important points, such as axis intercepts, must be clearly labelled.

Sometimes we need to deduce possible equations from a given curve.

We will examine the Fundamental theorem of algebra in more detail in Section 15D.

The first thing we should ask is what order polynomial we need to use. There is a result called the Fundamental Theorem of Algebra, which states that a polynomial of order n can have at most n real roots. So, for example, if the given curve has three x-intercepts, we know that the corresponding polynomial must have degree at least 3. We can then use those intercepts to write down the factors of the polynomial.

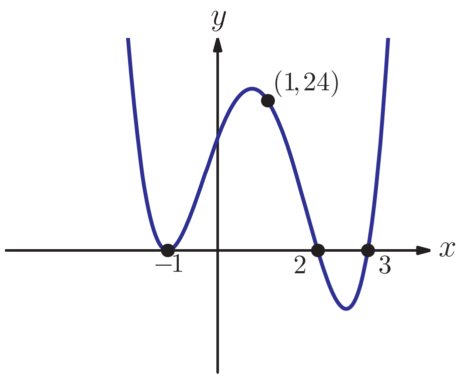
KEY POINT 3.6

To find the equation of a polynomial from its graph:

- use the shape and position of the x-intercepts to write down the factors of the polynomial
- use any other point to find the lead coefficient.

Worked example 3.10

Find a possible equation for this graph.



Describe x-intercepts

Single root at $x = 2$ and $x = 3$
Double root at $x = -1$

Convert this to a factorised form

$$\therefore\ y = k(x+1)^2(x-2)(x-3)$$

Use the fact that when $x = 1$, $y = 24$

$$24 = k \times (2)^2 \times (-1) \times (-2)$$
$$\Leftrightarrow 24 = 8k$$
$$\Leftrightarrow k = 3$$

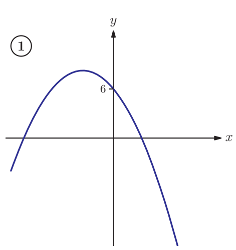
So the equation is $y = 3(x+1)^2(x-2)(x-3)$

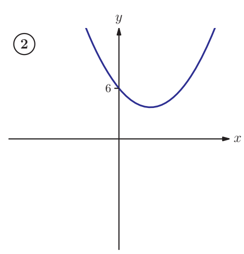
Exercise 3C

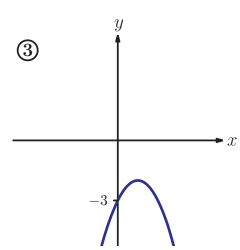
1. Sketch the following graphs, labelling all axis intercepts.
   (a) (i) $y = 2(x-2)(x-3)$
   (ii) $y = 7(x-5)(x+1)$
   (b) (i) $y = 4(5-x)(x-3)(x-3)$
   (ii) $y = 2(x-1)(2-x)(x-3)$
   (c) (i) $y = -(x-4)^2$ (ii) $y = (x-2)^2$
   (d) (i) $y = x(x^2+4)$ (ii) $y = (x+1)(x^2-3x+7)$
   (e) (i) $y = (1-x)^2(1+x)$ (ii) $y = (2-x)(3-x)^2$

2. Sketch the following graphs, labelling all axis intercepts.
   (a) (i) $y = x(x-1)(x-2)(2x-3)$
   (ii) $y = (x+2)(x+3)(x-2)(x-3)$
   (b) (i) $y = -4(x-3)(x-2)(x+1)(x+3)$
   (ii) $y = -5x(x+2)(x-3)(x-4)$
   (c) (i) $y = (x-3)^2(x-2)(x-4)$
   (ii) $y = -x^2(x-1)(x+2)$
   (d) (i) $y = 2(x+1)^3(x-3)$
   (ii) $y = -x^3(x-4)$
   (e) (i) $y = (x^2+3x+12)(x+1)(3x-1)$
   (ii) $y = (x+2)^2(x^2+4)$

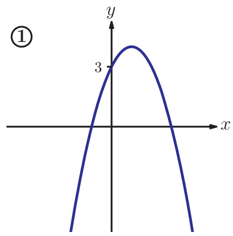
3. Match equations and corresponding graphs.
   (i) A: $y = -x^2-3x+6$ B: $y = 2x^2-3x+3$ C: $y = x^2-3x+6$







(ii) A: $y = -x^2 + 2x - 3$ B: $y = -x^2 + 2x + 3$ C: $y = x^2 + 2x + 3$

①


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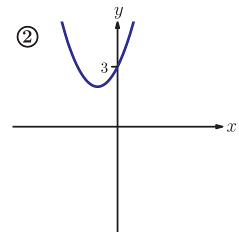

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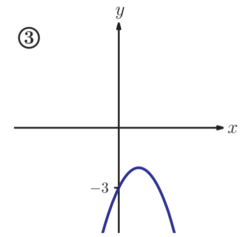
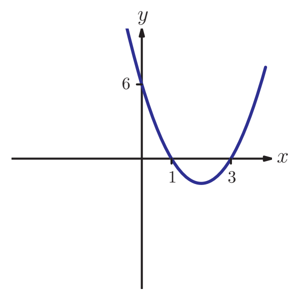
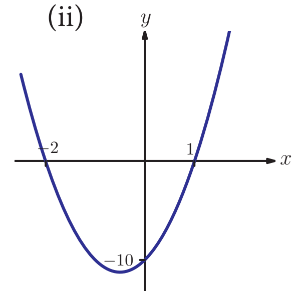
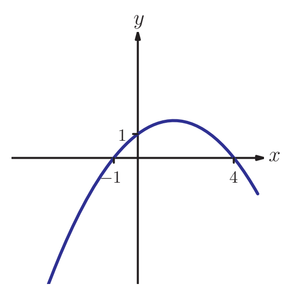
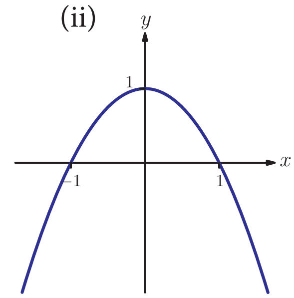


4. The diagrams show graphs of quadratic functions of the form $y = ax^2 + bx + c$. Write down the value of c, and then find the values of a and b.

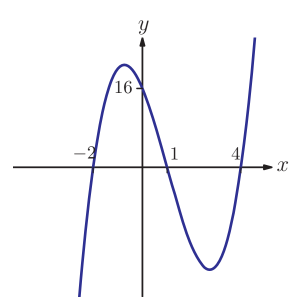
(a) (i)


(ii)


(b) (i)


(ii)


5. Find lowest order polynomial equation for each of these graphs.

(a) (i)


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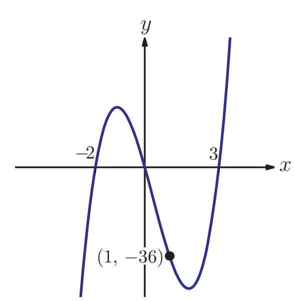
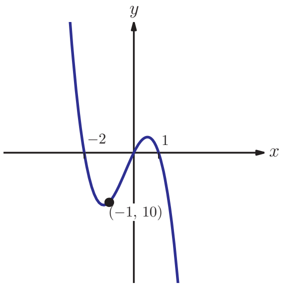


(b) (i)



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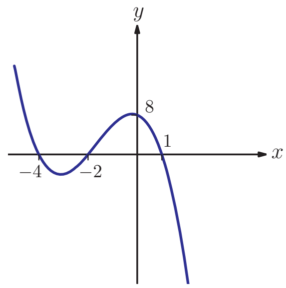
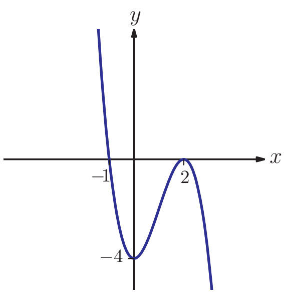
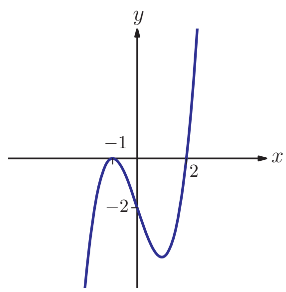


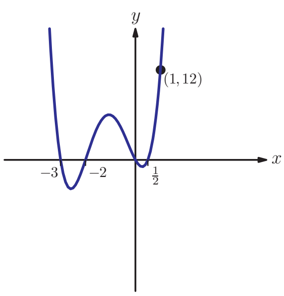
(c) (i)



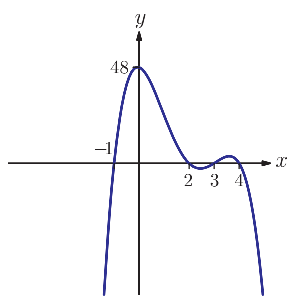
(ii)



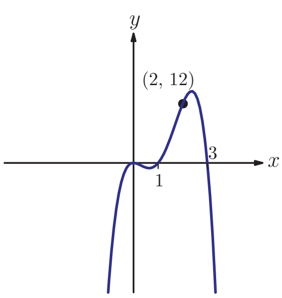
(d) (i)



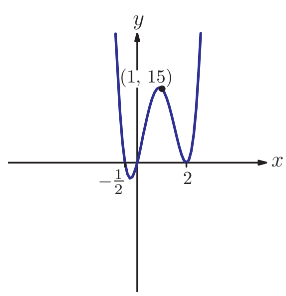
(ii)



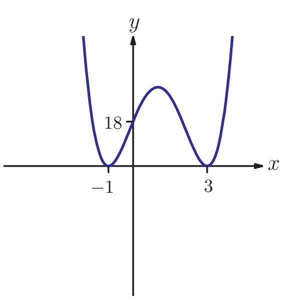
(e) (i)



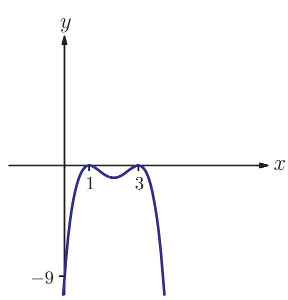
(ii)



(f) (i)



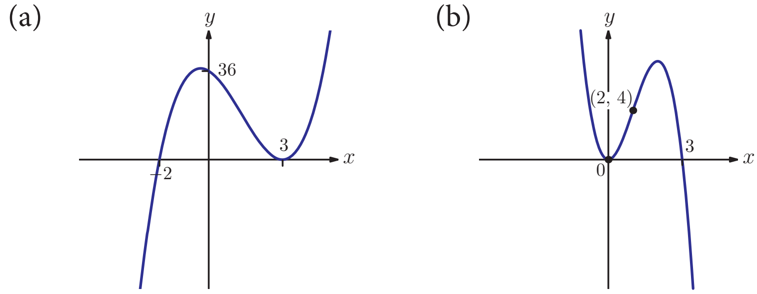
(ii)



6. (a) Show that $(x-2)$ is a factor of $f(x) = 2x^3 - 5x^2 + x + 2$.
(b) Factorise $f(x)$.
(c) Sketch the graph $y = f(x)$.

7. Sketch the graph of $y = 2(x+2)^2(3-x)$, labelling clearly any axes intercepts. *[5 marks]*

8. The two graphs below each have equations of the form $y = px^3 + qx^2 + rx + s$. Find the values of p, q, r and s for each graph.



[10 marks]

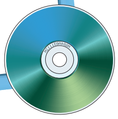
9. (a) Factorise fully $x^4 - q^4$ where q is a positive constant.
(b) Hence or otherwise sketch the graph $y = x^4 - q^4$, labelling any points where the graph meets an axis. *[5 marks]*

10. (a) Sketch the graph of $y = (x-p)^2(x-q)$ where $p < q$.
(b) How many solutions does the equation $(x-p)^2(x-q) = k$ have when $k > 0$? *[5 marks]*

EXAM HINT

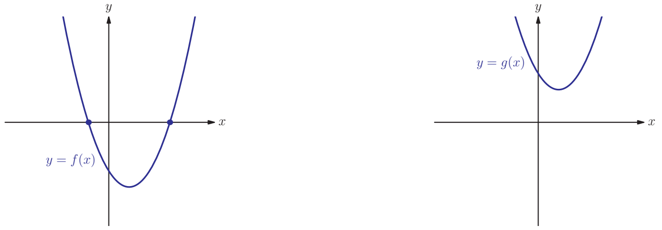
Don't spend too long trying to factorise a quadratic – use the formula in Key point 3.7 if you are asked to find exact solutions and use a calculator (graph or equation solver) otherwise.

See Calculator sheets 4 and 6 on the CD-ROM.



3D The quadratic formula and the discriminant

It is not always possible to find zeros of a polynomial using factorising. Try factorising the two quadratic expressions $f(x) = x^2 - 3x - 3$ and $g(x) = x^2 - 3x + 3$. It appears that neither of the expressions can be factorised, but sketching the graphs reveals that y_1 has two zeros, while y_2 has no zeros.



In the case where the polynomial is a quadratic we have another option – using the quadratic formula to find the zeros (roots):

KEY POINT 3.7

The zeros of $ax^2 + bx + c$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You can find the proof of this formula in Fill-in proof 3 'Proving quadratic formula'.

The solution to quadratic equations was known to the Greeks, although they did everything without algebra, using geometry instead. They would have described the quadratic equation $x^2 + 3x = 10$ as 'the area of a square plus three times its length measures ten units'.

Worked example 3.11

Use the quadratic formula to find the zeros of $x^2 - 5x - 3$.

It is not obvious how to factorise the quadratic expression, so use the quadratic formula

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 1 \times (-3)}}{2}$$

$$= \frac{5 \pm \sqrt{37}}{2}$$

The roots are

$$\frac{5+\sqrt{37}}{2} = 5.54 \text{ and } \frac{5-\sqrt{37}}{2} = -0.541 \text{ (3SF)}$$

EXAM HINT

In International Baccalaureate® exams you should either give exact answers (such as $\frac{5+\sqrt{37}}{2}$) or round your answers to 3 significant figures, unless you are told otherwise. See Prior learning Section B on the CD-ROM for rules of rounding.

Let us examine what happens if we try to apply the quadratic formula to find the zeros of $x^2 - 3x + 3$:

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times 3}}{2} = \frac{3 \pm \sqrt{-3}}{2}$$

In chapter 15 you will meet imaginary numbers, a new type of number which makes it possible to find zeros of functions like this.

As the square root of a negative number is not a real number, it follows that the expression has no real zeros.

Looking more closely at the quadratic formula, we see that it can be separated into two parts:

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

The line of symmetry of the parabola lies halfway between the two roots:

KEY POINT 3.8

The line of symmetry of $y = ax^2 + bx + c$ is

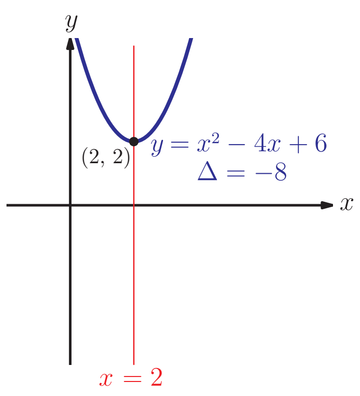
$$x = -\frac{b}{2a}$$

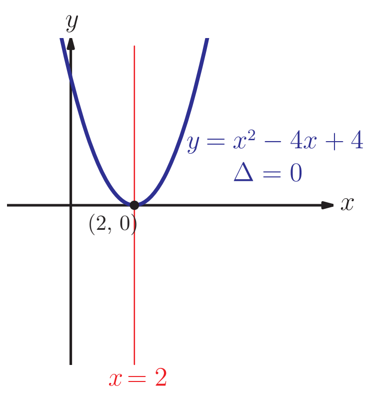
The second part of the formula involves a root expression: $\sqrt{b^2 - 4ac}$.

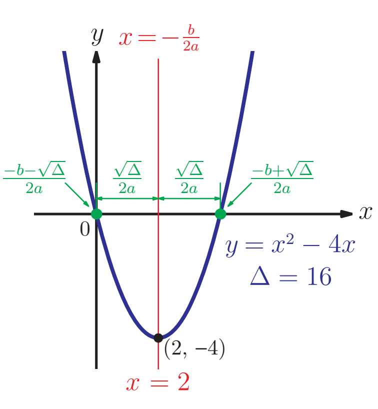
The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic (often symbolised by the Greek letter Δ) and $\frac{\sqrt{b^2 - 4ac}}{2a}$ is the distance of the zeros from the line of symmetry $x = -\frac{b}{2a}$.

The square root of a negative number is not a real number, so if the discriminant is negative, there can be no real zeros and the graph will not cross the x-axis. If the discriminant is zero, the graph is tangent to the x-axis at a point which lies on the line of symmetry (it touches the x-axis rather than crossing it).

The graphs below are examples of the three possible situations:







KEY POINT 3.9

For a quadratic expression ax^2+bx+c, the discriminant is: $\Delta = b^2-4ac$.

- If $\Delta < 0$ the expression has no real zeros
- If $\Delta = 0$ the expression has one (repeated) zero
- If $\Delta > 0$ the expression has two distinct real zeros

There is also a formula for solving cubic equations called 'Cardano's Formula'. It too has a discriminant which can be used to decide how many solutions there will be:

$$b^2c^2-4ac^3-4b^3d-27a^2d^2+18abcd$$

This gets a lot more complicated than the quadratic version!

Worked example 3.12

Find the exact values of k for which the quadratic equation $kx^2-(k+2)x+3=0$ has a repeated root.

Repeated root means that $\Delta = b^2-4ac=0$

$$b^2-4ac=0$$

We can form an equation in k using $a=k$ and $b=-(k+2)$

$$(k+2)^2-4(k)(3)=0$$

This is a quadratic equation

$$k^2+4k+4-12k=0$$
$$k^2-8k+4=0$$

It doesn't appear to factorise, so use the quadratic formula to find k

$$k=\frac{8\pm\sqrt{8^2-4\times4}}{2}$$
$$=\frac{8\pm\sqrt{48}}{2}$$
$$=\frac{(8\pm4\sqrt{3})}{2}$$
$$=4\pm2\sqrt{3}$$

When $\Delta < 0$, the graph does not intersect the x-axis, so it is either entirely above or entirely below it. The two cases are distinguished by the value of a.

Questions involving discriminants often lead to quadratic inequalities, which are covered in Prior learning Section Z.

KEY POINT 3.10

For a quadratic function with $\Delta < 0$:

if $a > 0$ then $y > 0$ for all x

if $a < 0$ then $y < 0$ for all x

Worked example 3.13

Let $y = -3x^2 + kx - 12$.
Find the values of k for which $y < 0$ for all x.

y is a negative quadratic. $y < 0$ means that the graph is entirely below the x-axis. This will happen when $y = 0$ has no real roots

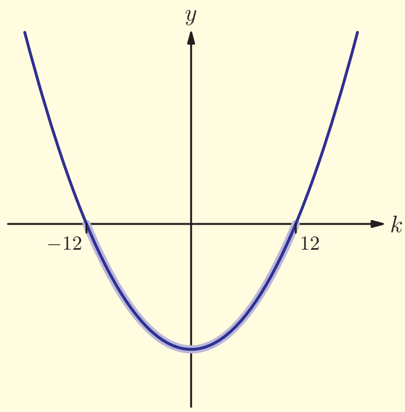
No real roots, $\therefore \Delta < 0$

$k^2 - 4(-3)(-12) < 0$

$k^2 < 144$

This is a quadratic inequality. A sketch of the graph will help

$\Rightarrow k^2 - 144 < 0$



$\therefore -12 < k < 12$

Exercise 3D

1. Evaluate the discriminant of these quadratic expressions.

(a) (i) $x^2 + 4x - 5$	(ii) $x^2 - 6x - 8$
(b) (i) $2x^2 + x + 6$	(ii) $3x^2 - x + 10$
(c) (i) $3x^2 - 6x + 3$	(ii) $9x^2 - 6x + 1$
(d) (i) $12 - x - x^2$	(ii) $-x^2 - 3x + 10$

2. State the number of zeros for each expression in Question 1.

3. Find the exact solutions of these equations.

(a) (i) $x^2-3x+1=0$ (ii) $x^2-x-1=0$

(b) (i) $3x^2+x-2=0$ (ii) $2x^2-6x+1=0$

(c) (i) $4+x-3x^2=0$ (ii) $1-x-2x^2=0$

(d) (i) $x^2-3=4x$ (ii) $3-x=2x^2$

4. Find the values of k for which:

(a) (i) the equation $x^2-x+k=$ has two distinct real roots

(ii) the equation $3x^2-5x+k=0$ has two distinct real roots

(b) (i) the equation $5x^2-2x+(2k-1)=0$ has equal roots

(ii) the equation $2x^2+3x-(3k+1)=0$ has equal roots

(c) (i) the equation $-x^2+3x+(k-1)=0$ has real roots

(ii) the equation $-2x^2+3x-(2k+1)=0$ has real roots

(d) (i) the equation $3kx^2-3x+2=0$ has no real solutions

(ii) the equation $kx^2+5x+3=0$ has no real solutions

(e) (i) the quadratic expression $(k-2)x^2+3x+1$ has a repeated zero

(ii) the quadratic expression $-4x^2+5x+(2k-5)$ has a repeated zero

(f) (i) the graph of $y=x^2-4x+(3k+1)$ is tangent to the x-axis

(ii) the graph of $y=-2kx^2+x-4$ is tangent to the x-axis

(g) (i) the expression $-3x^2+5k$ has no real zeros

(ii) the expression $2kx^2-3$ has no real zeros

Some of the many applications of quadratic equations are explored in Supplementary sheet 1 'The many applications of quadratic equations'.

5. Find the values of parameter m for which the quadratic equation $mx^2-4x+2m=0$ has equal roots. *[5 marks]*

6. Find the exact values of k such that the equation $-3x^2+(2k+1)x-4k=0$ has a repeated root. *[6 marks]*

7. Find the range of values of the parameter c such that $2x^2-3x+(2c-1)\geq 0$ for all x. *[6 marks]*

8. Find the set of values of k for which the equation $x^2-2kx+6k=0$ has no real solutions. *[6 marks]*

9. Find the range of value of k for which the quadratic equation $kx^2-(k+3)x-1=0$ has no real roots. *[6 marks]*

10. Find the range of values of m for which the equation $mx^2 + mx - 2 = 0$ has one or two real roots. *[6 marks]*

11. Find the possible values of m such that $mx^2 + 3x - 4 < 0$ for all x. *[6 marks]*

12. The positive difference between the zeros of the quadratic expression $x^2 + kx + 3$ is $\sqrt{69}$. Find the possible values of k. *[5 marks]*

Summary

- **Polynomials** are expressions involving only addition, multiplication and raising to a power.
- If two polynomials are equal we can **compare coefficients**. This can be used to divide two polynomials.
- The **remainder theorem** says that the remainder when a polynomial is divided by $ax - b$ is the value of the polynomial expression when $x = \frac{b}{a}$.
- The **factor theorem** says that if a polynomial expression is zero when $x = \frac{b}{a}$ then $ax - b$ is a factor of the expression.
- The graphs of polynomial are best sketched using their factorised form. Look for repeated factors and find where the graph crosses the axes.
- You can use the shape and position of the x-intercepts to find the equation of a polynomial graph.
- The number of solutions of a quadratic equation depends on the value of the **discriminant**, $\Delta = b^2 - 4ac$:
  - If $\Delta > 0$ there are two distinct solutions.
  - If $\Delta = 0$ there is one solution.
  - If $\Delta < 0$ there is no solution.

Introductory problem revisited

Without using your calculator, solve the cubic equation

$$x^3 - 13x^2 + 47x - 35 = 0$$

If we put $x = 1$ into this expression we get 0, so we can conclude that $x - 1$ is a factor. The remaining quadratic factor can be found by comparing coefficients; it is $x^2 - 12x + 35$, which factorises to give $(x - 5)(x - 7)$.

The equation therefore becomes $(x - 1)(x - 5)(x - 7) = 0$ which has solutions $x = 1, 5,$ or 7.

Mixed examination practice 3

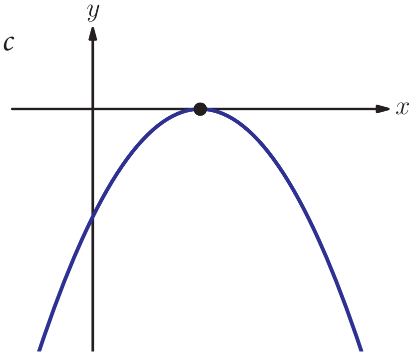
Short questions

1. A quadratic graph passes through the points $(k, 0)$ and $(k+4, 0)$. Find in terms of k the x-coordinates of the turning point. *[4 marks]*

2. The diagram shows the graph of $y = ax^2 + bx + c$

Complete the table to show whether each expression is positive, negative or zero.

expression	positive	negative	zero
a			
c			
$b^2 - 4ac$			
b			



[6 marks]
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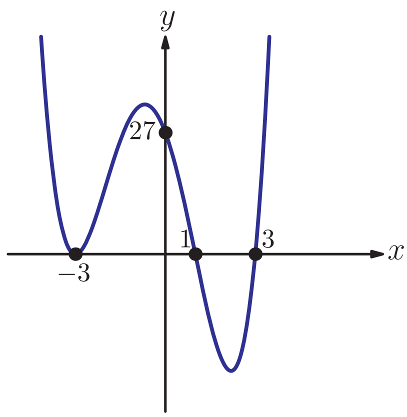
3. The diagram shows the graph with equation $y = ax^4 + bx^3 + cx^2 + dx + e$.

Find the values of a, b, c, d and e. *[6 marks]*



4. The remainder when $(ax+b)^3$ is divided by $(x-2)$ is 8 and the remainder when it is divided by $(x+3)$ is -27. Find the values of a and b. *[5 marks]*

5. (a) Show that $(x-2)$ is a factor of $f(x) = x^3 - 4x^2 + x + 6$.

(b) Factorise $f(x)$.

(c) Sketch the graph of $y = f(x)$. *[7 marks]*

6. The remainder when $(ax+b)^4$ is divided by $(x-2)$ is 16 and the remainder when it is divided by $(x+1)$ is 81. Find the possible values of a and b. *[6 marks]*

7. Sketch the graph of $y=(x-a)^2(x-b)(x-c)$ where $b<0<a<c$. *[5 marks]*

8. Find the exact values of k for which the equation $2kx^2+(k+1)x+1=0$ has equal roots. *[5 marks]*

9. Find the set of values of k for which the equation $2x^2+kx+6=0$ has no real roots. *[6 marks]*

10. Find the range of values of k for which the quadratic function $x^2-(2k+1)x+5$ has at least one real zero. *[6 marks]*

11. The polynomial x^2-4x+3 is a factor of the polynomial $x^3+ax^2+27x+b$. Find the values of a and b. *[6 marks]*

12. Let α and β denote the roots of the quadratic equation

$x^2-kx+(k-1)=0$.

(a) Express α and β in terms of the real parameter k.

(b) Given that $\alpha^2+\beta^2=17$, find the possible values of k. *[7 marks]*

13. Let $q(x)=kx^2+(k-2)x-2$. Show that the equation $q(x)=0$ has real roots for all values of k. *[7 marks]*

14. Find the range of values of k such that for all x, $kx-2\le x^2$. *[7 marks]*

Long questions

1. (a) Find the coordinates of the point where the curve $y=x^2+bx-a$ crosses the y-axis, giving your answer in terms of a and/or b.

(b) State the equation of the axis of symmetry of x^2+bx-a, giving your answer in terms of a and/or b.

(c) Show that the remainder when x^2+bx-a is divided by $x-\frac{a}{b}$ is always positive.

(d) The remainder when x^2+bx-a is divided by $x-a$ is -9. Find the possible values that b can take.

[14 marks]

2. (a) Show that for all values of p, $(x-2)$ is a factor of

$$f(x)=x^3+(p-2)x^2+(5-2p)x-10.$$

(b) By factorising $f(x)$, or otherwise, find the exact values of p for which the equation

$$x^3+(p-2)x^2+(5-2p)x-10=0$$

has exactly two real roots.

(c) For the smaller of the two values of p found above, sketch the graph of $y=f(x)$.

[10 marks]

3. (a) On the graph of $y=\dfrac{x^2+4x+5}{x+2}$ prove that there is no value of x for which $y=0$.

(b) Find the equation of the vertical asymptote of the graph.

(c) Rearrange the equation to find x in terms of y.

(d) Hence show that y cannot take values between -2 and 2.

(e) Sketch the graph of $y=\dfrac{x^2+4x+5}{x+2}$.

[18 marks]

4. Let $f(x)=x^4+x^3+x^2+x+1$.

(a) Evaluate $f(1)$.

(b) Show that $(x-1)f(x)\equiv x^5-1$.

(c) Sketch $y=x^5-1$.

(d) Hence show that $f(x)$ has no real roots.

[10 marks]