2 Exponents and logarithms

In this chapter you will learn:

- some rules for dealing with exponents
- about a function where the unknown is in the exponent
- about the value e and some of its properties
- how to undo exponential functions using an operation called a logarithm
- the rules of logarithms
- about graphs of logarithms
- to use logarithms to find exact solutions to simple exponential equations.

Introductory problem

A radioactive substance has a half-life of 72 years. A 1 kg block of the substance is found to have a radioactivity of 25 million Becquerel (Bq). How long, to the nearest 10 years, would it take for the radioactivity to have fallen below 10 000 Bq?

Many mathematical models (biological, physical and financial in particular) involve the concept of continuous growth or decay where the rate of growth/decay of the population is linked to the size of that population. You may have met similar situations already, for example when a bank account earns compound interest: the increase in amount each year depends upon how much is in the account. Any similar situation is governed by an exponential function, which you will learn about in this chapter.

We shall also look at how we can find out how long an exponential process has been occurring, using a function called a logarithm.

2A Laws of exponents

The exponent of a number shows you how many times the number is to be multiplied by itself. You will have already met some of the rules for dealing with exponents before, and in this section we shall revisit and extend these rules. (In other courses, the exponent might have been called the 'index' or 'power'.)

See Prior learning Section C on the CD-ROM which looks at exponents.

A number written in **exponent form** is one which explicitly looks like:

$$a^n$$

n is referred to as the **exponent** or **power**
a is referred to as the **base**

a^n is pronounced 'a to the exponent n' or, more simply, 'a to the n'.

To investigate the rules of exponents let us consider an example:

Mathematics is often considered a subject without ambiguity. However, the value of 0^0 is undetermined; it depends upon how you get there!

Worked example 2.1

Simplify.

(a) $a^3 \times a^4$ (b) $a^3 \div a^4$ (c) $(a^4)^3$ (d) $a^4 + a^3$

(a) $a^3 \times a^4 = (a \times a \times a \times a) \times (a \times a \times a) = a^7$

(b) $a^3 \div a^4 = \dfrac{\cancel{a \times a \times a}}{\cancel{a \times a \times a} \times a} = \dfrac{1}{a} = a^{-1}$

Use the idea from part (a)

(c) $(a^4)^3 = a^4 \times a^4 \times a^4 = a^{12}$

(d) $a^4 + a^3 = a^3(a+1)$

The example above suggests some rules of exponents.

It is questionable whether in part (d) we have actually simplified the expression. Sometimes the way mathematicians choose to simplify expressions is governed by how it looks as well as how it is used.

KEY POINT 2.1

$$a^m \times a^n = a^{m+n}$$

KEY POINT 2.2

$$a^m \div a^n = a^{m-n}$$

KEY POINT 2.3

$$(a^m)^n = a^{m \times n}$$

EXAM HINT

These rules are NOT given in the formula booklet. Make sure that you can use them in both directions, e.g. if you see 2^6 you can rewrite it as $(2^3)^2$ and if you see $(2^3)^2$ you can rewrite it as 2^6. Both ways will be important!

We can use Key point 2.3 to justify the interpretation of $a^{\frac{1}{n}}$ as the nth root of a, since $\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n} \times n} = a$. This is exactly the property we require of the nth root of a. So, we get the rule: $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$.

Worked example 2.2

Evaluate $64^{\frac{2}{3}}$.

Use $a^{m/n} = \left(a^{\frac{1}{n}}\right)^m$ to split the calculation into two steps

$$64^{\frac{2}{3}} = \left(64^{\frac{1}{3}}\right)^2$$
$$= (4)^2$$

So, $64^{\frac{2}{3}} = 16$

You must take care when expressions with *different* bases are to be combined by multiplication or division, for example $2^3 \times 4^2$. The rules such as 'multiplication means add the exponents together' are only true *when the bases are the same*. You cannot use this rule to simplify $2^3 \times 4^2$.

There is however, a rule that works when the bases are different *but* the exponents are the *same*.

Consider the following example:

$$\begin{aligned} 3^2 \times 5^2 &= 3 \times 3 \times 5 \times 5 \\ &= 3 \times 5 \times 3 \times 5 \\ &= 15 \times 15 \\ &= 15^2 \end{aligned}$$

This suggests the following rules:

KEY POINT 2.4

$$a^n \times b^n = (ab)^n$$

KEY POINT 2.5

$$a^n \div b^n = \left(\frac{a}{b}\right)^n$$

Exercise 2A

1. Simplify the following, leaving your answer in exponent form.
   - (a) (i) $6^4 \times 6^3$ (ii) $5^3 \times 5^5$
   - (b) (i) $a^3 \times a^5$ (ii) $x^6 \times x^3$
   - (c) (i) $7^{11} \times 7^{-14}$ (ii) $5^7 \times 5^{-2}$
   - (d) (i) $x^4 \times x^{-2}$ (ii) $x^8 \times x^{-3}$
   - (e) (i) $g^{-3} \times g^{-9}$ (ii) $k^{-2} \times k^{-6}$

2. Simplify the following, leaving your answer in exponent form.
   - (a) (i) $6^4 \div 6^3$ (ii) $5^3 \div 5^5$
   - (b) (i) $a^3 \div a^5$ (ii) $x^6 \div x^3$
   - (c) (i) $5^7 \div 5^{-2}$ (ii) $7^{11} \div 7^{-4}$
   - (d) (i) $x^4 \div x^{-2}$ (ii) $x^8 \div x^{-3}$
   - (e) (i) $2^{-5} \div 2^{-7}$ (ii) $3^{-6} \div 3^8$
   - (f) (i) $g^{-3} \div g^{-9}$ (ii) $k^{-2} \div k^6$

3. Express the following in the form required.
   - (a) (i) $(2^3)^4$ as 2^n (ii) $(3^2)^7$ as 3^n
   - (b) (i) $(5^{-1})^4$ as 5^n (ii) $(7^{-3})^2$ as 7^n
   - (c) (i) $(11^{-2})^{-1}$ as 11^n
     (ii) $(13^{-3})^{-5}$ as 13^n
   - (d) (i) $4 \times (2^5)^3$ as 2^n
     (ii) $3^{-5} \times (9^{-1})^{-4}$ as 3^n
   - (e) (i) $(4^2)^3 \times 3^{12}$ as 6^n
     (ii) $(6^3)^2 \div (2^2)^3$ as 3^n

4. Simplify the following, leaving your answer in exponent form with a prime number as the base.
   - (a) (i) 4^5 (ii) 9^7
   - (b) (i) 8^3 (ii) 16^5
   - (c) (i) $4^2 \times 8^3$ (ii) $9^5 \div 27^2$
   - (d) (i) $4^{-3} \times 8^5$ (ii) $3^7 \div 9^{-2}$
   - (e) (i) $\left(\frac{1}{4}\right)^3$ (ii) $\left(\frac{1}{9}\right)^3$
   - (f) (i) $\left(\frac{1}{8}\right)^2 \div \left(\frac{1}{4}\right)^4$ (ii) $9^7 \times \left(\frac{1}{3}\right)^4$

5. Write the following without brackets or negative exponents:

(a) (i) $(2x^2)^3$ (ii) $(3x^4)^2$

(b) (i) $2(x^2)^3$ (ii) $3(x^4)^2$

(c) (i) $\dfrac{(3a^3)^4}{9a^2}$ (ii) $\dfrac{(4x)^4}{8(2x)^4}$

(d) (i) $(2x)^{-1}$ (ii) $\left(\dfrac{3}{y}\right)^{-2}$

(e) (i) $2x^{-1}$ (ii) $\dfrac{3}{y^{-2}}$

(f) (i) $5\div\left(\dfrac{3}{xy^2}\right)^2$ (ii) $\left(\dfrac{ab}{2}\right)^3\div\left(\dfrac{a}{b}\right)^2$

(g) (i) $\left(\dfrac{2}{q}\right)^2\div\left(\dfrac{p}{2}\right)^{-3}$ (ii) $\left(\dfrac{6}{x}\right)^4\div\left(2\times\dfrac{3^2}{x}\right)^{-3}$

6. Simplify the following:

(a) (i) $(x^6)^{\frac{1}{2}}$ (ii) $(x^9)^{\frac{4}{3}}$

(b) (i) $(4x^{10})^{0.5}$ (ii) $(8x^{12})^{-\frac{1}{3}}$

(c) (i) $\left(\dfrac{27x^9}{64}\right)^{-\frac{1}{3}}$ (ii) $\left(\dfrac{x^4}{y^8}\right)^{-1.5}$

7. Solve for x, giving your answer as a rational value:

(a) (i) $8^x = 32$ (ii) $25^x = \dfrac{1}{125}$

(b) (i) $\dfrac{1}{49^x} = 7$ (ii) $\dfrac{1}{16^x} = 8$

(c) (i) $2\times 3^x = 162$ (ii) $3\div 5^x = 0.12$

(d) (i) $2\times 5^{x-1} = 250$ (ii) $5+3^{x+2} = 14$

(e) (i) $16+2^x = 2^{x+1}$ (ii) $100^{x+5} = 10^{3x-1}$

(f) (i) $6^{x+1} = 162\times 2^x$ (ii) $4^{1.5x} = 2\times 16^{x-1}$

In Section 2G you will see that there is an easier way to solve equations like this when you have a calculator and can use logarithms.

8. Any simple computer program is able to sort n input values in $k\times n^{1.5}$ microseconds. Observations show that it sorts a million values in half a second. Find the value of k. *[3 marks]*

9. A square-ended cuboid has volume xy^2, where x and y are lengths. A cuboid for which $x = 2y$ has volume 128 cm^3. Find x. *[3 marks]*

10. The volume and surface area of a family of regular solid shapes are related by the formula $V = kA^{1.5}$, where V is given in cm^3 and A in cm^2.

(a) For one such shape, $A = 81$ and $V = 243$. Find k.

(b) Hence determine the surface area of a shape with volume $\frac{64}{3}$ cm^3. *[4 marks]*

11. Prove that 2^{350} is larger than 5^{150}. *[5 marks]*

12. Given that there is more than one solution value of x to the equation $4^{ax} = b \times 8^x$, find all possible values of a and b. *[5 marks]*

2B Exponential functions

In most of the equations that you have met so far, the unknown appears as the base, for example $x^3 = 27$.

In an exponential function, the unknown appears in the exponent, leading to a fundamentally different type of function.

The general form of a simple exponential function is $f(x) = a^x$.

We will only consider situations when a is positive, because otherwise some exponents cannot be easily defined (for example, we cannot square root a negative number).

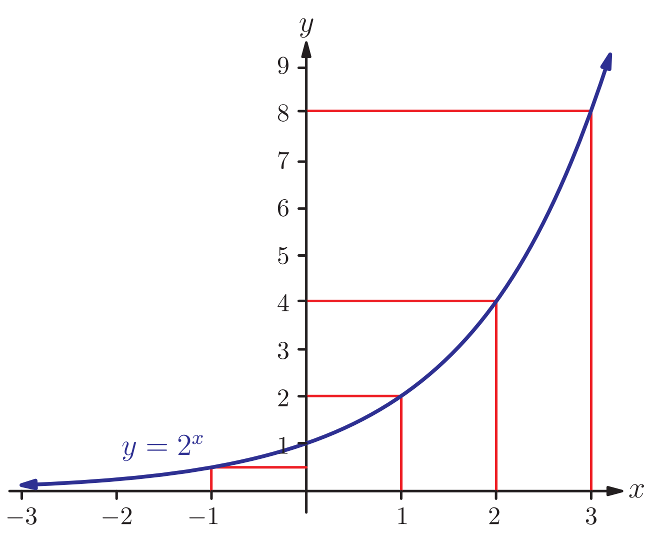
What is $(-1)^{\frac{1}{2}}$?

What about $\left((-1)^2\right)^{\frac{1}{4}}$?

What about $(-1)^{\frac{2}{4}}$?

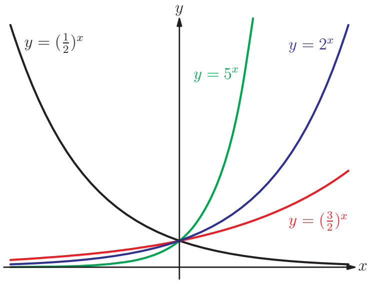
Not all mathematics is unambiguous!

Here is the graph of $y = 2^x$:



For very large positive values of x, the y-value approaches infinity. For very large negative values of x, the y-value approaches (but never reaches) zero.

A line that a function gets increasingly close to but never reaches is called an **asymptote**. In this case we would say that the x-axis is an asymptote to the graph.



If we look at the graphs of other exponential functions with different bases we can begin to make some generalisations.

For all the graphs $y = a^x$:

- The y-intercept is always (0, 1) because $a^0 = 1$.
- The graph of the function lies entirely above the x-axis since $a^x > 0$ for all values of x.
- The x-axis is an asymptote.
- If $a > 1$, then as x increases, so does $f(x)$. This is called a **positive exponential**.
- If $0 < a < 1$, then as x increases, $f(x)$ decreases. This is called a **negative exponential**.

You may observe that the blue line is a reflection of the black line. You will see why this is the case in chapter 6.

Many mathematical models use the characteristics of exponential functions: as time (t) increases by a fixed value, the value we are interested in (N) will change by a fixed *factor*, called the **growth factor**. Exponential functions can therefore be used to model many physical, financial and biological forms of **exponential growth** (positive exponential models) and **exponential decay** (negative exponential models).

To model more complex situations we may need to add more constants to our exponential equation.

We can use a function of the form $N = Ba^{\left(\frac{t}{k}\right)}$.

We can interpret the constants in the following way:

- When $t = 0$, $N = B$ so B is the **initial value** of N.
- When $t = k$, $N = Ba$ so k is the time taken for N to increase by a factor of a.
- If $k = 1$ then a is the growth factor.
- As long as k is positive when $a > 1$ the function models exponential growth.
- When $0 < a < 1$ the function models exponential decay.
- When modelling exponential decay there might sometimes be a **background level**. This means the asymptote is not $N = 0$.

We can change the asymptote to $N = c$ by adding on a constant.

The new function is then $N = Ba^{\left(\frac{t}{k}\right)} + c$.

B represents how much N starts above the background level, so the initial value is $B + c$.

KEY POINT 2.6

If $N = Ba^{\left(\frac{t}{k}\right)} + c$:

- The background level (and the asymptote) is c.
- The initial value is $B + c$.
- k is the time taken for the difference between N and the background level to increase by a factor of a.
- If $a > 1$ this models exponential growth.
- If $0 < a < 1$ this models exponential decay.

EXAM HINT

See Calculator skills sheet 2 on the CD-ROM to find out how to sketch graphs on your calculator. As asymptotes are not a part of the graph, your calculator might not show them, though you can guess approximately where they are by looking at large values of x. This is why it is important to know how to find asymptotes directly from the equation.

In many applications we are given certain information and need to find the constants in the model.

Worked example 2.3

A population of bacteria in a culture medium doubles in size every 15 minutes.

(a) Write down a model for N, the number of bacteria in terms of time, t, in hours.
(b) If there are 1000 bacterial cells at 08:00, how many cells are there at
   (i) 08:15?
   (ii) 09:24?

There is a constant increasing factor so use an exponential growth model

(a) Let N be the number of cells at time t hours

$$N = Ba^{\left(\frac{t}{k}\right)}$$

Every time t increases by 0.25, N doubles

Doubles every quarter hour
$a = 2, k = 0.25$
$\therefore\ N = B \times 2^{4t}$

All details are relative to a start time 08:00, so set $t = 0$ at that time

(b) When $t = 0, N = 1000 = B$
$N = 1000 \times 2^{4t}$

Remember to convert minutes to hours

(i) When $t = 0.25$, $N = 2000$ cells
(ii) When $t = 1.4$, $N = 48\,503$ cells

You may also be given a model and have to interpret it. Unfortunately, it might not be given in exactly the same form as in Key point 2.6 $\left(N = Ba^{\frac{t}{k}}\right)$, so you will have to use the rules of exponents to rewrite it in the correct form.

Worked example 2.4

The temperature in degrees (θ) of a cup of coffee a time t minutes after it was made is modelled using the function:

$$\theta = 70 \times 3^{-\beta t} + 22, \beta > 0$$

(a) Show that $\theta - 22$ follows an exponential decay.
(b) What was the initial temperature of the coffee?
(c) If the coffee is left for a very long time, what temperature does the model predict it will reach?
(d) Find, in terms of β, how long it takes for the temperature difference between the coffee and the room to fall by a factor of 9.
(e) Coffee made at the same time is put into a thermos flask which is much better insulated than the cup. State with a reason whether β would increase or decrease.

Use rules of exponents to rewrite the exponent in the form $N = Ba^{\left(\frac{t}{k}\right)}$

(a) $\theta - 22 = 70 \times 3^{-\beta t}$

$= 70 \times (3^{-1})^{\beta t}$

$= 70 \times \left(\frac{1}{3}\right)^{\frac{t}{\beta^{-1}}}$

which is exponential decay since the base is between 0 and 1 and $\beta > 0$

Initial means $t = 0$

(b) When $t = 0$, $\theta = 70 \times 3^0 + 22$

$= 92°$

The asymptote is given by the background level

(c) 22 degrees

Use algebra to rewrite the exponent in the form $N = Ba^{\frac{t}{k}}$

(d) $\theta - 22 = 70\left(\frac{1}{3}\right)^{\frac{t}{\beta^{-1}}}$

So the time to fall by a factor of 3 is $\frac{1}{\beta}$

To fall by a factor of 9 is a fall by a factor of 3 followed by another fall by a factor of 3. Therefore the time to fall by a factor of 9 is $2 \times \frac{1}{\beta}$ minutes.

continued . . .

(e) The time taken to fall by any given factor will be longer for a better insulated container so $\frac{1}{\beta}$ must be larger so β must be smaller.

When modelling exponential growth or decay, you may be given a percentage increase or decrease. This needs to be converted into a growth factor to be used in the exponential model.

Worked example 2.5

A car costs \$17 500. It then loses value at a rate of 18% each year.

(a) Write a model for the value of the car (V) after n years in the form $V = ka^n$.

(b) Hence or otherwise find the value of the car after 20 years.

Find the growth factor

(a) The growth factor is $1-\frac{18}{100}=0.82$

Use initial value information

When $n=0,\ V=k=17500$

$V=17500\times 0.82^n$

Substitute for n

(b) After 20 years, $V=\$330.61$

Exercise 2B

1. Using your calculator, sketch the following functions for $-5 \le x \le 5$ and $0 \le y \le 10$.

   Show all the axis intercepts and state the equation of the horizontal asymptote.

   (a) (i) $y=1.5^x$ (ii) $y=3^x$

   (b) (i) $y=2\times 3^x$ (ii) $y=6\times 1.4^x$

   (c) (i) $y=\left(\frac{1}{2}\right)^x$ (ii) $y=\left(\frac{2}{3}\right)^x$

   (d) (i) $y=5+2^x$ (ii) $y=8+3^x$

   (e) (i) $y=6-2^x$ (ii) $y=1-5^x$

2. An algal population on the surface of a pond grows by 10% every day, and the area it covers can be modelled by the equation $y=k\times 1.1^t$, where t is measured in days. At 09:00 on Tuesday it covered 10 m². What area will it cover by 09:00 on Friday? *[4 marks]*

3. The air temperature T °C around a light bulb is given by equation

$$T = A + B \times 2^{-\frac{x}{k}}$$

where x is the distance from the surface of the light bulb in millimetres. The background temperature in the room is a constant 25 °C, and the temperature on the surface of the light bulb is 125 °C.

(a) You find that the air temperature 3 mm from the surface of the bulb is only 75 °C. Find the integer values of A, B and k.

(b) Determine the air temperature 2 cm from the surface of the bulb.

(c) Sketch a graph of air temperature against distance.

[10 marks]

4. A tree branch is observed to bend as the fruit growing on it increases in size. By estimating the mass of the developing fruit, and plotting the data over time, a student finds that the height in metres of the branch tip above the ground closely follows the graph of:

$$h = 2 - 0.2 \times 1.6^{0.2m}$$

where m is the estimated mass, in kilograms, of fruit on the branch.

(a) Plot a graph of h against m.

(b) What height above ground level is a branch without fruit?

(c) The total mass of fruit on the branch at harvest was 7.5 kg. Find the height of the branch immediately prior to harvest.

(d) The student wishes to estimate what mass of fruit would cause the branch tip to touch the ground. Why might his model not be suitable to assess this? *[10 marks]*

5. (a) Sketch the graph of $y = 1 + 16^{1-x^2}$.

Label clearly the horizontal asymptote and maximum value.

(b) Find all values of x for which $y = 3$. *[6 marks]*

6. A bowl of soup is served at a temperature of 55 °C in a room with a constant air temperature of 20 °C. Every 5 minutes, the temperature difference between the soup and the room air decreases by 30%. Assuming the room air temperature is constant, at what temperature will the soup be seven minutes after serving? *[7 marks]*

7. The speed (V metres per second) of a parachutist t seconds after jumping from an aeroplane is modelled by the expression:

$$V = 40(1 - 3^{-0.1t})$$

(a) Find his initial speed.
(b) What speed does the model predict that he will eventually reach? *[6 marks]*

2C The value e

In this section we introduce a mathematical constant, e, which will be used extensively in the rest of this chapter and in many other chapters throughout the course.

Consider the following different situations, each of which is typical of early population growth of a cell culture.

(a) There is a 100% increase every 100 seconds.
(b) There is a 50% increase every 50 seconds.
(c) There is a 25% increase every 25 seconds.

Although these may at first appear to be equivalent statements, they are subtly different because of the compounding nature of percentage increases.

If we begin with a population of size P, then after 100 seconds, we have

(a) $P \times (1+1) = 2P$

(b) $P \times \left(1 + \frac{1}{2}\right)^2 = 2.25P$

(c) $P \times \left(1 + \frac{1}{4}\right)^4 = 2.44P$

To generalise the situation, if we considered an increase of $\frac{1}{n}\%$ which occurred n times every 100 seconds, the population after 100 seconds would be given by $P \times \left(1 + \frac{1}{n}\right)^n$.

It may seem from the above that as n increases, the overall increase over 100 seconds will keep on getting larger, and this is indeed the case, but not without limit.

π and e have many similar properties. Both are irrational, meaning that they cannot be written as a ratio of two whole numbers and both are transcendental, meaning that they cannot be written as the solution to a polynomial equation. The proof of these facts is intricate but beautiful.

In chapter 16 you will see that e plays a major role in studying rates of change.

In fact, as can be seen by taking greater and greater values of n, the resultant increase factor $\left(1+\frac{1}{n}\right)^n$ tends towards a value of approximately 2.71828182849 which, much like π, is such an important part of mathematical studies that it has been given its own letter, e.

KEY POINT 2.7

$$e = 2.71828182849\ldots$$

Although e has many important properties in mathematics, it is still just a number, and so all the standard rules of arithmetic and exponents apply. You are not likely to find an exam question just on e, but the number e *is* likely to appear in questions on many other topics.

EXAM HINT

In questions involving the number e you may be asked to either give an exact answer (such as e^2) or to use your calculator, in which case you should usually round the answer to 3 significant figures.

Exercise 2C

1. Find the values of the following to 3 significant figures:

(a) (i) $e+1$ (ii) $e-4$

(b) (i) $3e$ (ii) $\frac{e}{2}$

(c) (i) e^2 (ii) e^{-3}

(d) (i) $5e^{0.5}$ (ii) $\frac{3}{e^7}$

2. Evaluate $\sqrt[6]{(\pi^4+\pi^5)}$. What do you notice about this result?

2D Introducing logarithms

See Self-Discovery Worksheet 1 'An introduction to logarithms' on the CD-ROM if you would like to discover more about logarithms for yourself.

In this section we shall look at a new operation which reverses the effect of exponentiating (that is, raising a number to an exponent).

If you are asked to solve

$$x^2 = 3, x \geq 0$$

you can either find a decimal approximation (using a calculator or trial and improvement) or you can use the square root symbol: $x = \sqrt{3}$.

This restates the problem as 'x is the positive value which when squared gives 3'.

Similarly, if asked to solve:

$$10^x = 50$$

you could use trial and improvement to seek a decimal value:

$$10^1 = 10$$
$$10^2 = 100$$

So x is between 1 and 2:

$$10^{1.5} = 31.6$$
$$10^{1.6} = 39.8$$
$$10^{1.7} = 50.1$$

So the answer is around 1.7.

However, just as with squares and square roots there is also a function to answer the question 'What is the number, which when put as the exponent of 10, gives this value?'

The function is called a base-10 **logarithm**, written $\log_{10}$.

So in the above example: $10^x = 50$ so $x = \log_{10} 50$.

This means that $y = 10^x$ may be re-expressed as $x = \log_{10}(y)$.

In fact, the base involved need not be 10, but could be any positive value other than 1.

KEY POINT 2.8

$$b = a^x \Leftrightarrow x = \log_a b$$

The symbol $\Leftrightarrow$ has a very specific meaning (implies and is implied by) in mathematical logic. It means that if the left-hand side is true then so is the right-hand side, and also if the right-hand side is true then so is the left-hand side.

This is the form written in the Formula booklet; you only need to remember that it means you can switch between them.

It is worth noting that the two most common bases have abbreviations for their logarithms.

Since we use a decimal system of counting, base 10 is the default base for a logarithm, so that $\log_{10} x$ is generally written more simply as just $\log x$, called the **common logarithm**. Also e, encountered in Section 2C, is considered the 'natural' base, and its counterpart the **natural logarithm** is denoted by $\ln x$.

KEY POINT 2.9

$\log_{10} x$ is often written as $\log x$.

$\log_e x$ is often written as $\ln x$.

See Key point 2.18.

Since taking a logarithm reverses the process of exponentiation, it follows that:

KEY POINT 2.10

$$\log_a(a^x) = x$$

KEY POINT 2.11

$$a^{\log_a x} = x$$

These are sometimes referred to as the cancellation principles. This sort of 'cancellation', similar to stating that (for positive x) $\sqrt[n]{x^n} = x = \left(\sqrt[n]{x}\right)^n$, is frequently useful when simplifying logarithm expressions, but you can only apply it when the base of the logarithm and the base of the exponential match.

Worked example 2.6

Evaluate:

(a) $\log_5 625$ (b) $\log_8 16$

Express the argument of the logarithm in exponent form with the same base

(a) $\log_5 625 = \log_5 5^4$

Apply the cancellation principle $\log_a(a^x) = x$

$= 4$

16 is not a power of 8, but they are both powers of 2

(b) $\log_8(16) = \log_8(2^4)$

We need to convert 2^4 to an exponent of $8 = 2^3$

Rewrite 4 as $3 \times \frac{4}{3}$ and use $a^{mn} = (a^m)^n$

$= \log_8\left(2^{3\times\frac{4}{3}}\right) = \log_8\left(8^{\frac{4}{3}}\right)$

Apply the cancellation principle $\log_a(a^x) = x$

$= \frac{4}{3}$

Actually, you can find the logarithm of a negative number, but the answer turns out not to be real; it is a complex number, which is a new type of number you will meet in chapter 15.

Whenever you raise a positive number to a power, positive or negative, the answer is always positive. Therefore we currently have no answer to a question such as $10^x = -3$.

This means:

KEY POINT 2.12

The logarithm of a negative number or zero has no real value.

Exercise 2D

1. Evaluate the following:

(a) (i) $\log_3 27$ (ii) $\log_4 16$

(b) (i) $\log_5 5$ (ii) $\log_3 3$

(c) (i) $\log_{12} 1$ (ii) $\log_{15} 1$

(d) (i) $\log_3 \frac{1}{3}$ (ii) $\log_4 \frac{1}{64}$

(e) (i) $\log_4 2$ (ii) $\log_{27} 3$

(f) (i) $\log_8 \sqrt{8}$ (ii) $\log_2 \sqrt{2}$

(g) (i) $\log_8 4$ (ii) $\log_{81} 27$

(h) (i) $\log_{25} 125$ (ii) $\log_{16} 32$

(i) (i) $\log_4 2\sqrt{2}$ (ii) $\log_9 81\sqrt{3}$

(j) (i) $\log_{25} 0.2$ (ii) $\log_4 0.5$

2. Use a calculator to evaluate each of the following, giving your answer correct to 3 significant figures:

(a) (i) $\log 50$ (ii) $\log\left(\frac{1}{4}\right)$

(b) (i) $\ln 0.1$ (ii) $\ln 10$

EXAM HINT

On most calculators $\log x$ is denoted with a $\boxed{\log}$ button, and $\ln x$ by a $\boxed{\ln}$ button.

3. Simplify the following expressions:

(a) (i) $7\log x - 2\log x$ (ii) $2\log x + 3\log x$

(b) (i) $(\log x - 1)(\log y + 3)$ (ii) $(\log x + 2)^2$

(c) (i) $\dfrac{\log a + \log b}{\log a \log b}$ (ii) $\dfrac{(\log a)^2 - 1}{\log a - 1}$

EXAM HINT

Remember that '$\log x$' is just a number so can be treated the same way as any variable.

4. Make x the subject of the following:

(a) (i) $\log_3 x = y$ (ii) $\log_4 x = 2y$

(b) (i) $\log_a x = 1 + y$ (ii) $\log_a x = y^2$

(c) (i) $\log_x 3y = 3$ (ii) $\log_x y = 2$

5. Find the value of x in each of the following:

(a) (i) $\log_2 x = 32$ (ii) $\log_2 x = 4$

(b) (i) $\log_5 25 = 5x$ (ii) $\log_{49} 7 = 2x$

(c) (i) $\log_x 36 = 2$ (ii) $\log_x 10 = \frac{1}{2}$

6. Solve the equation $\log_{10}(9x+1)=3$. *[4 marks]*

7. Solve the equation $\log_8 \sqrt{1-x}=\frac{1}{3}$. *[4 marks]*

8. Find the exact solution to the equation $\ln(3x-1)=2$. *[5 marks]*

9. Find all values of x which satisfy $(\log_3 x)^2=4$. *[5 marks]*

10. Solve the simultaneous equations:

$$\log_3 x+\log_5 y=6$$
$$\log_3 x-\log_5 y=2$$

[6 marks]

11. Solve the equation $3(1+\log x)=6+\log x$. *[5 marks]*

12. Solve the equation $\log_x 4=9$. *[4 marks]*

13. The Richter scale is a way of measuring the strength of earthquakes. An increase of one unit on the Richter scale corresponds to an increase by a factor of 10 in the strength of the earthquake. What would be the Richter level of an earthquake which is twice as strong as a level 5.2 earthquake? *[5 marks]*

2E Laws of logarithms

Just as there are rules which hold when working with exponents, so there are corresponding rules which apply to logarithms. These are derived from the laws of exponents and can be found on the Fill-in proof sheet 2 ‘Proving log rules’ on the CD-ROM.

- The logarithm of a *product* is the *sum* of the logarithms.

KEY POINT 2.13

$\log_a xy=\log_a x+\log_a y$ for $x, y>0$

- The logarithm of a *quotient* is the *difference* of the logarithms.

KEY POINT 2.14

$$\log_a \frac{x}{y} = \log_a x - \log_a y \text{ for } x, y > 0$$

EXAM HINT

These formulae are *not* in the Formula booklet.

- The logarithm of a *reciprocal* is the *negative* of the logarithm.

KEY POINT 2.15

$$\log_a \frac{1}{x} = -\log_a x \text{ for } x > 0$$

EXAM HINT

It is important to know what you *cannot* do with logarithms. One very common mistake is to try to simplify $\log(x+y)$ into either $\log x + \log y$ or $\log x \log y$.

- The logarithm of a *exponent* is the *multiple* of the logarithm.

KEY POINT 2.16

$$\log_a x^p = p\log_a x \text{ for } x > 0$$

- The logarithm of 1 is always 0, irrespective of the base.

KEY POINT 2.17

$$\log_a 1 = 0$$

The rules of logarithms can be used to manipulate expressions and solve equations involving logarithms.

Worked example 2.7

If $x = \log_{10} a$ and $y = \log_{10} b$, express $\log_{10} \dfrac{100a^2}{b}$ in terms of x, y and integers.

Use laws of logs to isolate $\log_{10} a$ and $\log_{10} b$. First, use $\log \frac{x}{y} = \log x - \log y$	$\log_{10} \dfrac{100a^2}{b} = \log_{10}(100a^2) - \log_{10} b$
...then $\log xy = \log x + \log y$...	$= \log_{10} 100 + \log_{10} a^2 - \log_{10} b$
... then $\log x^p = p \log x$...	$= \log_{10} 100 + 2\log_{10} a - \log_{10} b$
... then calculate $\log_{10} 100$ ... then write in terms of x and y	$= 2 + 2\log_{10} a - \log_{10} b$ $= 2 + 2x - y$

Worked example 2.8

Solve the equation $\log_2 x + \log_2(x+4) = 5$.

Rewrite one side as a single logarithm using $\log x + \log y = \log xy$	$\log_2 x + \log_2(x+4) = 5$ $\Leftrightarrow \log_2(x(x+4)) = 5$
Undo the logarithm by exponentiating both sides with base 2 (to balance, you must also exponentiate 5 by base 2) ...	$\Leftrightarrow 2^{\log_2(x(x+4))} = 2^5$
and use the cancellation principle	$\Leftrightarrow x^2 + 4x = 32$
Use standard methods for quadratic equations	$\Leftrightarrow x^2 + 4x - 32 = 0$ $\Leftrightarrow (x+8)(x-4) = 0$ $\Rightarrow x = -8$ or $x = 4$
Check your solution in the original equation	When $x = -8$: LHS: $\log_2(-8)$ and $\log_2(-4)$ are not real so this solution does not work When $x = 4$: LHS: $\log_2 4 + \log_2 8 = 2 + 3 = 5$ $=$ RHS

EXAM HINT

Checking your solutions is more than just looking for an arithmetic error – as you can see from the example above, false solutions can occur through *correct* algebraic manipulation.

You will notice that although we have discussed logarithms for a general base, *a*, your calculator may only have buttons for the common logarithm and the natural logarithm ($\log x$ and $\ln x$).

To use a calculator to evaluate, for example, $\log_5 20$ you can use the **change of base rule** of logarithms.

KEY POINT 2.18

Change of base rule for logarithms:

$$\log_b a = \frac{\log_c a}{\log_c b}$$

So, we can calculate $\log_5 20$ using the logarithm to the base 10:

$$\log_5 20 = \frac{\log 20}{\log 5} = 1.86 \text{ (3 SF)}$$

The change of base rule is useful for more than just evaluating logarithms.

For an insight into what mathematics was like before calculators, have a look at Supplementary sheet 3, 'A history of logarithms' on the CD-ROM.

Worked example 2.9

Solve the equation $\log_3 x + \log_9 x = 2$.

We want to have logarithms involving just one base, so use the change of base rule to turn logs with base 9 into logs with base 3

$$\log_9 x = \frac{\log_3 x}{\log_3 9} = \frac{\log_3 x}{2}$$

Therefore:

$$\log_3 x + \log_9 x = 2$$

$$\Leftrightarrow \log_3 x + \frac{\log_3 x}{2} = 2$$

Collecting the logs together

$$\Leftrightarrow \frac{3}{2}\log_3 x = 2$$

$$\Leftrightarrow \log_3 x = \frac{4}{3}$$

Exponentiate both sides with base 3

$$\Leftrightarrow x = 3^{\frac{4}{3}} = 4.33 \text{ (3SF)}$$

Equations combining logarithms and quadratics can be found in Section 4B.

Exercise 2E

1. Given $b > 0$, simplify each of the following:

   (a) (i) $\log_b b^4$ (ii) $\log_b \sqrt{b}$

   (b) (i) $\log_{\sqrt{b}} b^3$ (ii) $\log_b b^2 - \log_{b^2} b$

2. If $x = \log a$, $y = \log b$ and $z = \log c$, express the following in terms of x, y and z:

   (a) (i) $\log b^7$ (ii) $\log a^2 b$

   (b) (i) $\log\left(\frac{ab^2}{c}\right)$ (ii) $\log\left(\frac{a^2}{bc^3}\right)$

(c) (i) $\log\left(\frac{100}{bc^5}\right)$ (ii) $\log(5b)+\log(2c^2)$

(d) (i) $\log a^3 - 2\log ab^2$ (ii) $\log(4b)+2\log(5ac)$

(e) (i) $\log_a a^2b$ (ii) $\log_b\left(\frac{a}{bc}\right)$

(f) (i) $\log_{ab}(b^a)$ (ii) $\log_{ab} ac^2$

3. Solve the following for x:

(a) (i) $\log_3\left(\frac{2+x}{2-x}\right)=3$ (ii) $\log_2(7x+4)=5$

(b) (i) $\log_3 x - \log_3(x-6)=1$ (ii) $\log_8 x - 2\log_8\left(\frac{1}{x}\right)=1$

(c) (i) $\log_2 x+1=\log_4 x$ (ii) $\log_8 x+\log_2 x=4$

(d) (i) $\log_4 x+\log_8 x=2$ (ii) $\log_{16} x-\log_{32} x=0.5$

(e) (i) $\log_3(x-7)+\log_3(x+1)=2$

(ii) $2\log(x-2)-\log x=0$

(f) (i) $\log(x^2+1)=1+2\log x$

(ii) $\log(3x+6)=\log 3+1$

4. Find the exact solution to the equation $2\ln x+\ln 9=3$, giving your answer in the form Ae^B where A and B are rational numbers. *[5 marks]*

5. If $a=\ln 2$ and $b=\ln 5$, find in terms of a and b: *[6 marks]*

(a) $\ln 50$ (b) $\ln 0.16$

6. Solve $\log_2 x=\log_x 2$. *[5 marks]*

7. Prove that if $a^x=b^y=(ab)^{xy}$ where $a, b>1$ then $x+y=1$ or $x=y=0$. *[5 marks]*

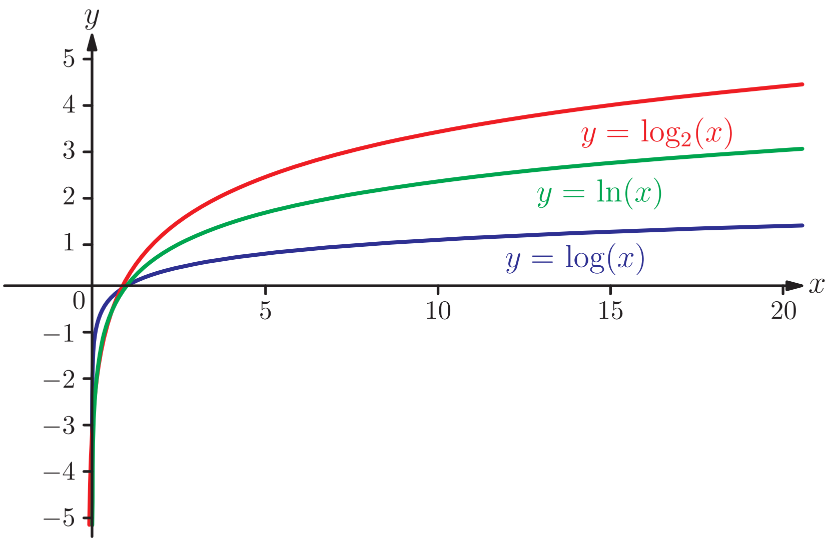
8. Evaluate $\log\frac{1}{2}+\log\frac{2}{3}+\log\frac{3}{4}+\log\frac{4}{5}\ldots+\log\frac{8}{9}+\log\frac{9}{10}$. *[4 marks]*

9. Given that $\log_a b=\log_b a$, and that $a,b\neq 1$ and $a\neq b$, find b in terms of a. *[5 marks]*

2F Graphs of logarithms

It is important also to know the graph of the logarithm function, and the various properties of logarithms which we can deduce from it.

Below are the graphs of $y = \log x, y = \log_2 x$ and $y = \ln x$:



These three curves all have a similar shape.

The change of base rule (Key point 2.18) states that $\log_b a = \dfrac{\log_c a}{\log_b a}$.

So, $\log_2 x = \dfrac{\log x}{\log 2}$ and $\ln x = \dfrac{\log x}{\log \mathrm{e}}$ and all other logarithm functions are simply multiples of the common logarithm function $y = \log x$.

In chapter 6 you will see how this type of change causes a vertical stretch of the graph.

EXAM HINT

Vertical asymptotes are even harder to detect accurately from a calculator display than the horizontal ones. This is why it is important that you know how to find an asymptote of a logarithmic graph.

The graph shows the following important facts about the logarithm function (Key point 2.19).

KEY POINT 2.19

If $y = \log_a x$ then:

- The graph of y against x crosses the x-axis at (1, 0), because $\log_a 1 = 0$ (for any positive value of a).
- Log x is negative for $0 < x < 1$ and positive for $x > 1$.
- The graph of the function lies entirely to the right of the y-axis, since the logarithm of a negative value does not produce a real solution.
- The logarithm graph increases throughout: as x tends to infinity so does y.
- The y-axis is an asymptote to the curve.

You may observe that a logarithm graph is the reflection of an exponential graph. You will see why this is the case in chapter 5.

Exercise 2F

It is unlikely you will find exam questions testing *just* this topic, but you may be required to sketch a graph involving a logarithm as a part of another question.

1. Sketch the following graphs, labelling clearly the vertical asymptote and all axis intercepts.

   (a) (i) $y = \log(x^2)$ (ii) $y = \log(x^3)$

   (b) (i) $y = \log 4x$ (ii) $y = \log 2x$

   (c) (i) $y = \log(x-2)$ (ii) $y = \log(x+1)$

2. Why are the graphs of $y = \log(x^2)$ and $y = 2\log(x)$ different?

2G Solving exponential equations

One of the main uses of logarithms is to solve equations with the unknown in the exponent. By taking logarithms, the unknown becomes a factor, which is easier to deal with.

Worked example 2.10

Solve the equation $3 \times 2^x = 5^{x-1}$, giving your answer in the form $\dfrac{\log p}{\log q}$ where p and q are rational numbers.

Since the unknown is in the exponent, taking logarithms is a good idea

$$\log(3 \times 2^x) = \log(5^{x-1})$$

Use the rules of logarithms to simplify the expression

$$\log 3 + \log 2^x = \log(5^{x-1})$$
$$\log 3 + x\log 2 = (x-1)\log 5$$

Expand the brackets and get all of the x's on one side and everything else on the other side

$$\log 3 + x\log 2 = x\log 5 - \log 5$$
$$\log 3 + \log 5 = x\log 5 - x\log 2$$

Factorise and divide to find x

$$x = \frac{\log 3 + \log 5}{\log 5 - \log 2}$$

Use the rules of logarithms to write it in the correct form

$$x = \frac{\log 15}{\log\left(\frac{5}{2}\right)}$$

EXAM HINT

A common mistake is to say that $\log(3 \times 2^x)$ is $\log 3 \times \log 2^x$ Make sure you learn the rules of logarithms carefully.

Logarithmic scales make it easy to compare values which are very different. Some of these scales are explored in Supplementary sheet 2 on the CD-ROM.

You can use similar ideas to solve inequalities with the unknown in the exponent:

Worked example 2.11

The number of bacteria in a culture medium is given by $N = 1000 \times 2^{4t}$, where t is the number of hours elapsed since 08:00. At what time will the population first exceed one million?

Simplify the equation where possible	$1000 \times 2^{4t} \geq 1\,000\,000$ $\Leftrightarrow 2^{4t} \geq 1000$
Take logarithms of each side	$\Leftrightarrow \log(2^{4t}) \geq \log 1000$
Use the rule $\log_a x^p = p \log_a x$	$\Leftrightarrow 4t \log 2 \geq 3$
Note that $\log 2 > 0$, so the inequality remains in the same orientation when we divide	$\Leftrightarrow t \geq \dfrac{3}{4 \log 2} = 2.49 (3SF)$
To answer the question, convert 2.49 to hours and minutes, then add it to 08:00	10:29

You know that you can always apply the same operation to both sides of an equation, but in this worked example we took logarithms of both sides of an inequality. You might like to investigate for which operations this is valid.

EXAM HINT

Whenever dividing an inequality by a logarithm it is important to remember to check if it is positive or negative.

There is another type of exponential equation which you will meet – a disguised quadratic equation. These are explored in Section 4B.

Exercise 2G

1. Solve for x, giving your answer correct to 3 significant figures.

   (a) (i) $3\times 4^x = 90$ (ii) $1000\times 1.02^x = 10\,000$

   (b) (i) $6\times 7^{3x+1} = 1.2$ (ii) $5\times 2^{2x-5} = 94$

   (c) (i) $3^{2x} = 4^{x-1}$ (ii) $5^x = 6^{1-x}$

   (d) (i) $3\times 2^{3x} = 7\times 3^{3x-2}$ (ii) $4\times 8^{x-1} = 3\times 5^{2x+1}$

2. In a yeast culture, cell numbers are given by $N = 100e^{1.03t}$, where t is measured in hours after the cells are introduced to the culture.

   (a) What is the initial number of cells? *[1 mark]*

   (b) How many cells will be present after 6 hours? *[1 mark]*

   (c) How long will it take for the population to exceed one thousand? *[2 marks]*

3. A rumour spreads exponentially through a school. When school begins (at 9 a.m.) 18 people know it. By 10 a.m. 42 people know it.

   (a) How many people know it at 10.30? *[3 marks]*

   (b) There are 1200 people in the school. According to the exponential model, at what time will everyone know the rumour? *[2 marks]*

4. In an experimental laboratory, a scientist sets up a positive feedback loop for a fission reaction and extracts heat to control the experiment and produce power. When the reaction is established, and while sufficient fuel is present, the power he can siphon off is given by $P = 32\left(e^{0.0012t} - 1\right)$, where P is measured in units of energy per second and t in seconds.

   (a) How much energy is being produced after 2 minutes? *[1 mark]*

   (b) The equipment reaches a dangerous temperature when P exceeds 7×10^5. For how long can the experiment safely be run? *[2 marks]*

5. The weight of a block of salt W in a salt solution after t seconds is given by:

$$W = ke^{-0.01t}$$

   (a) Sketch the graph of W against t. [2 marks]

   (b) How long will it take to reach 25% of its original weight? [2 marks]

6. Solve the equation $5\times 4^{x-1}=\frac{1}{3^{2x}}$, giving your answer in the form $x=\frac{\ln p}{\ln q}$, where p and q are rational numbers. *[5 marks]*

7. Solve the equation $\frac{1}{7^x}=3\times 49^{5-x}$, giving your answer in the form $a+\log_7 b$ where $a, b\in\mathbb{Z}$.

8. A cup of tea is poured at 98 °C. After two minutes it has reached 94 °C. The difference between the temperature of the tea and the room temperature (22 °C) falls exponentially. Find the time it takes for the tea to cool to 78 °C. *[5 marks]*

9. (a) Show that the equation $3^x=3-x$ has only one solution. *[2 marks]*

(b) Find the solution, giving your answer to 3 significant figures. *[4 marks]*

Summary

- In this chapter, we revisited the rules for **exponents**.

$$a^m\times a^n=a^{m+n}$$
$$a^m\div a^n=a^{m-n}$$
$$a^n\times b^n=(ab)^n$$
$$a^n\div b^n=\left(\frac{a}{b}\right)^n$$
$$(a^m)^n=a^{m\times n}$$
$$a^{\frac{m}{n}}=\left(\sqrt[n]{a}\right)^m=\sqrt[n]{a^m}$$

- Exponential equations can be used to model growth and decay of some simple real-life systems, taking as a general form the function:

$$N=Ba^{\left(\frac{t}{k}\right)}+c$$

- $\log_a b$ asks the question: 'What exponent do I have to raise a to in order to get b?'

$$b=a^x\Leftrightarrow x=\log_a b$$

- e is the mathematical constant, (Euler's number): e = 2.71828182849…
- $\log_{10} x$ (common logarithm) is often written as $\log x$.

- $\log_e x$ (natural logarithm) is often written as $\ln x$.
- **Logarithms** undo the effect of exponentiating and vice versa:

$$\log_a(a^x) = x \Leftrightarrow x = a^{\log_a x}$$

- Logarithms obey these rules, which are *not* covered in the Formula booklet.

$$\log_a xy = \log_a x + \log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a\left(\frac{1}{x}\right) = -\log_a x$$
$$\log_a x^p = p\log_a x$$
$$\log_a 1 = 0$$

- Taking the logarithm of a negative number or zero does not give a real value.
- There is also the change of base formula (not in the Formula booklet): $\log_b a = \dfrac{\log_c a}{\log_c b}$ and a related rule for exponents (*is* in the Formula booklet): $a^x = e^{x\ln a}$
- The following properties of the logarithm function can be deduced from its graph, if $y = \log_a x$ then:
  - The graph of y against x crosses the x-axis at (1,0)
  - Log x is negative for $0 < x < 1$ and positive for $x > 1$
  - The graph lies to the right of the y-axis; the y-axis is an asymptote to the curve
  - The logarithm graph increases throughout; as x tends to infinity so does y.
- Logarithms are used to solve many exponential equations.

Introductory problem revisited

A radioactive substance has a half-life of 72 years. A 1 kg block of the substance is found to have a radioactivity of 25 million Becquerel (Bq). How long, to the nearest 10 years, would it take for the radioactivity to have fallen below 10 000 Bq?

Establish the exponential equation

Let R be the radioactivity after t years

$R = Ba^{\frac{t}{k}}$

Initial condition gives B

When $t = 0$, $R = 25 \times 10^6 = B$

Every time t increases by 72, R falls by 50%

$a = 0.5, k = 72$

$C = 25 \times 10^6 \times 0.5^{\frac{t}{72}}$

continued...

Require $R \le 10^4$

$25 \times 10^6 \times 0.5^{\frac{t}{72}} \le 10^4$

$\Leftrightarrow 0.5^{\frac{t}{72}} \le 0.0004$

The unknown is in the exponent so take logarithms

$\Leftrightarrow \log\left(0.5^{\frac{t}{72}}\right) \le \log(0.0004)$

$\frac{t}{72}\log(0.5) \le \log(0.0004)$

Now rearrange to find t.
Note that $\log(0.5) < 0$, so the inequality is reversed when dividing through

$\Leftrightarrow t \ge \frac{72\log(0.0004)}{\log(0.5)} = 812.7$

It takes around 810 years for the radioactivity to fall below 10 000 Bq.

Mixed examination practice 2

Short questions

1. Solve $\log_5\left(\sqrt{x^2+49}\right)=2$. *[4 marks]*

2. If $a=\log x$, $b=\log y$ and $c=\log z$ (all logs base 10) find in terms of a, b, c and integers:

(a) $\log\frac{x^2\sqrt{y}}{z}$ (b) $\log\sqrt{0.1x}$ (c) $\log_{100}\left(\frac{y}{z}\right)$ *[6 marks]*

3. Solve the simultaneous equations:

$$\ln x+\ln y^2=8$$
$$\ln x^2+\ln y=6$$ *[6 marks]*

4. If $y=\ln x-\ln(x+2)+\ln(4-x^2)$, express x in terms of y. *[6 marks]*

5. Find the exact value of x satisfying the equation

$$2^{3x-2}\times 3^{2x-3}=36^{x-1}$$

giving your answer in simplified form $\frac{\ln p}{\ln q}$, where $p,q\in\mathbb{Z}$. *[5 marks]*

6. Given $\log_a b^2=c$ and $\log_b a=c-1$ for some value c, where $0<a<b$, express a in terms of b. *[6 marks]*

7. Solve the equation $9\log_5 x=25\log_x 5$, expressing your answers in the form $5^{\frac{p}{q}}$, where $p,q\in\mathbb{Z}$. *[6 marks]*

(*© IB Organization 2006*)

8. Find the exact solution to the equation $\ln x=4\log_x \mathrm{e}$. *[5 marks]*

Long questions

1. The speed of a parachutist (V) in metres per second, t seconds after jumping is modelled by the expression:

$$V=42(1-\mathrm{e}^{-0.2t})$$

(a) Sketch a graph of V against t.

(b) What is the initial speed?

(c) What is the maximum speed that the parachutist could reach?

When the parachutist reaches 22 ms^{-1} he opens the parachute.

(d) How long is he falling before he opens his parachute? *[9 marks]*

2. Scientists think that the global population of tigers is falling exponentially. Estimates suggest that in 1970 there were 37 000 tigers but by 1980 the number had dropped to 22 000.

(a) Form a model of the form $T = ka^n$ connecting the number of tigers (T) with the number of years after 1970 (n).

(b) What does the model predict that the population will be in 2020?

(c) When the population reaches 1000 the tiger population will be described as 'near extinction'. In which year will this happen?

In the year 2000 a worldwide ban on the sale of tiger products was implemented, and it is believed that by 2010 the population of tigers had recovered to 10 000.

(d) If the recovery has been exponential find a model of the form $T = ka^m$ connecting the number of tigers (T) with the number of years after 2000 (m).

(e) If in each year since 2000 the rate of growth has been the same, find the percentage increase each year. *[12 marks]*

3. (a) If $\ln y = 2\ln x + \ln 3$ find y in terms of x.

(b) If the graph of $\ln y$ against $\ln x$ is a straight line with gradient 4 and y-intercept 6, find the relationship between x and y.

(c) If the graph of $\ln y$ against x is a straight line with gradient 3 and it passes through the point (1, 2), express y in terms of x.

(d) If the graph of e^y against x^2 is a straight line through the origin with gradient 4, find the gradient of the graph of y against $\ln x$. *[10 marks]*