Simulations and Numerical Methods

Project

Part I

In this section, you'll explore a system of three first-order ordinary differential equations (ODEs). The equations involve both linear and nonlinear terms with various coefficients. The system is defined as follows:

$$\begin{cases} x'(t) = 3x - 2y + 2z + a_1 x^2 z \\ y'(t) = 2y + z + a_2 xy \\ z'(t) = 4y + 2z + a_3 y^2 \end{cases}$$

along with its initial conditions:

$$x(0) = 1, \quad y(0) = 0, \quad z(0) = -1$$

Case I – Linear System

In Case I, the nonlinear terms are ignored (set to zero). You are asked to carry out the following tasks:

1. Solve the system analytically using linear algebra techniques covered in the Mathematics for Engineering 7 course.

2. Apply both Euler's Method with a fixed step size $h = 0.1$ and the Runge-Kutta Method (RK45) to solve the problem over the time interval $t \in [0, 5]$

3. Plot the numerical and exact solutions for $x(t), y(t)$, and $z(t)$.

4. Plot the absolute error between the numerical and exact solutions.

5. Discuss: Accuracy of each method, stability over time and computational cost (number of steps, if relevant)

Case II: Nonlinear System

In Case II, we introduce the nonlinear terms back into the system, where the coefficients are:

$$a_1 = 0.5, \quad a_2 = -1, \quad a_3 = 0.25$$

1. Implement Euler's method and RK45 for solving the nonlinear system over the interval $t \in [0, 5]$.

2. Plot the numerical solutions for $x(t), y(t)$, and $z(t)$.

3. Compare the stability, and efficiency of Euler's method and RK45 in solving the nonlinear system. How do the nonlinear terms influence the error growth and performance of each method?

Part II

This part focuses on the numerical simulation of a simplified nonlinear Boundary Value Problem (BVP) modeling the behavior of an elastic beam under an external distributed load. The governing equation for the transverse deflection $u(x)$ of the beam is given by:

$$EI\,u''(x) + \alpha\,(u'(x))^3 + q(x) = 0, \quad x \in [0, L]$$

where:

- E: Young's modulus of the material,
- I: moment of inertia of the beam's cross-section,
- $u(x)$: transverse deflection of the beam,
- α: a parameter representing geometric nonlinearity,
- $q(x) = q_0 \sin\left(\frac{\pi x}{L}\right)$: external distributed load per unit length.

The beam is clamped at both ends, so the boundary conditions are:

$$u(0) = 0, \quad u(L) = 0$$

The following parameters are used for solving the beam deflection problem:

- Beam length: $L = 1.0\,\mathrm{m}$
- Young's modulus: $E = 2 \times 10^{11}\,\mathrm{Pa}$
- Moment of inertia: $I = 1 \times 10^{-6}\,\mathrm{m}^4$
- Distributed load magnitude: $q_0 = 1000\,\mathrm{N/m}$

Case I – Linear BVP

You are requested to:

1. Find the analytical solution for $u(x)$ by integrating the equation, applying boundary conditions, and plotting the result.
2. Solve the problem numerically using the shooting method, applying a Runge-Kutta solver with iterative adjustment and plot the solution.
3. Apply the finite difference method by discretizing the domain, approximating $u''(x)$ with central differences, solving the resulting linear system, and plot the solution.
4. Compare all three solutions (analytical, shooting, and finite difference) by plotting them together, calculating the maximum absolute error and discuss the method that performed best.

Case II: Nonlinear BVP

You are requested to:

1. Solve the problem numerically using the shooting method, applying a Runge-Kutta solver with iterative adjustment and plot the solution.
2. Apply the finite difference method by discretizing the domain, approximating $u''(x)$ with central differences, solving the resulting linear system, and plot the solution.
3. Compare both solutions (shooting and finite difference) by plotting them together and calculating the maximum absolute error.

Deliverables

1. **Report:** In the report you must include followings
   (a) General introduction.
   (b) Mathematical description of each problem.
   (c) Details about all applied analytical and numerical methods (Euler, Runge-Kutta, Shooting, linear and nonlinear Finite difference methods) according to the specified problems.
   (d) Comparison graphs between numerical and exact solutions and between numerical methods.
   (e) Graphs of absolute errors.
   (f) Interpretations.
   (g) Conclusion.
2. **Code:** A folder that contains the following $.m$ files
   (a) A script called **Linear_diff_system** in which you solve the linear differential system using Euler and Runge-kutta methods. The script should show a graph that compares both numerical methods with the exact solution and another graph for the absolute errors.
   (b) A script called **Nonlinear_diff_system** in which you solve the nonlinear differential system using Euler and Runge-kutta methods. The script should show a graph that compares both numerical methods and another graph for the errors between methods.
   (c) A script called **Linear_BVP** in which you solve the linear BVP using the shooting and the finite difference methods. The script should show a graph that compares both numerical methods with the exact solution and another graph for the absolute errors.
   (d) A script called **Nonlinear_BVP** in which you solve the nonlinear BVP using the shooting and the finite difference methods. The script should show a graph that compares both numerical methods and another graph for the errors between methods.

Remarks

1. In the last page of your report, you have to mention the details of involvement of each group member.

Deadline

The deadline to upload your projects on Moodle in one compressed file is **June the 2^{nd}**.