Graph Algorithms II

Summer 2018 • Lecture 07/24

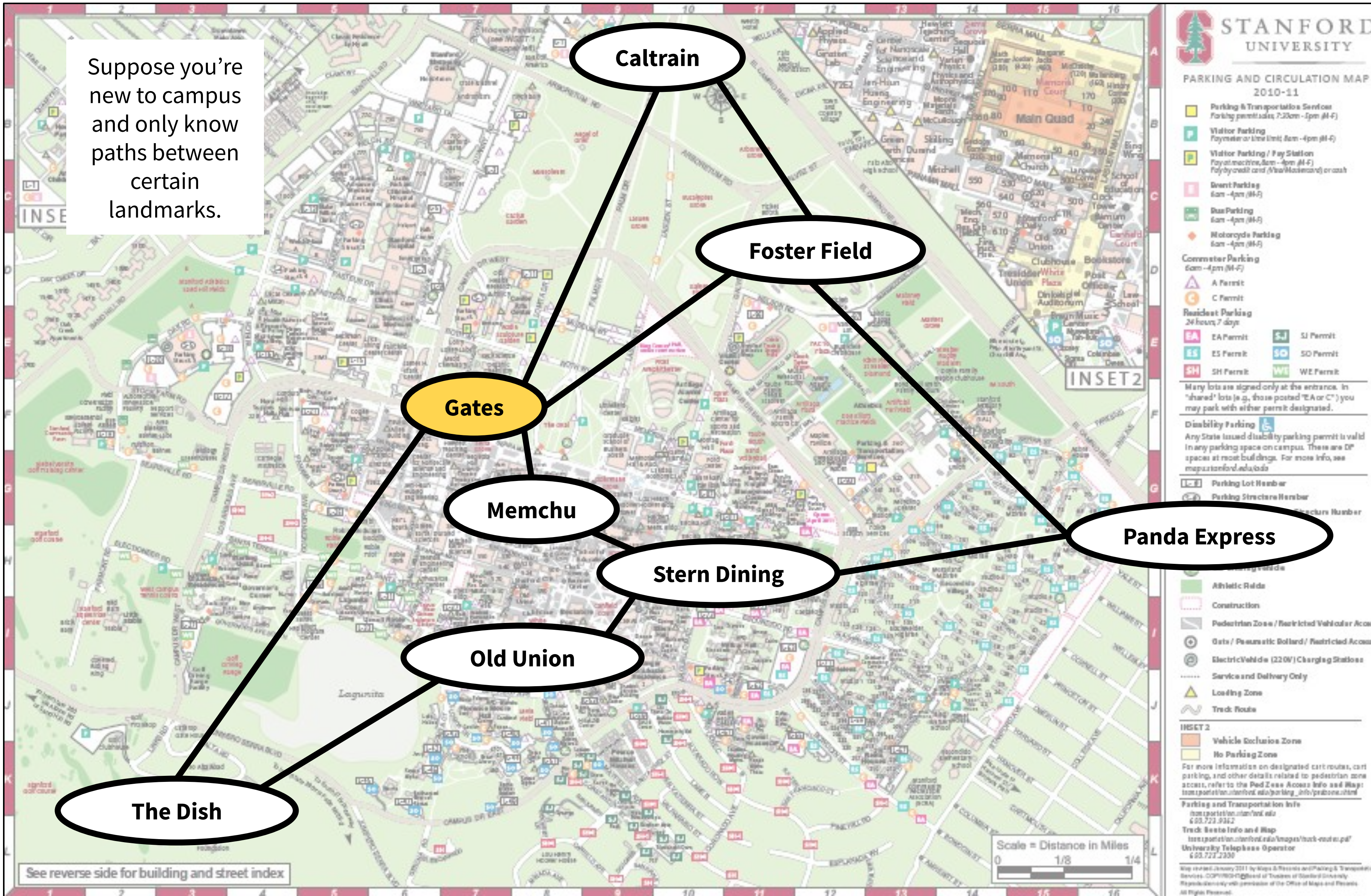
A Few Notes

- Midterm Review Session office hours
  - Today 12-1:20 p.m. in STLC 115
- Homework 3
  - Due today **(you can't use late days!)**

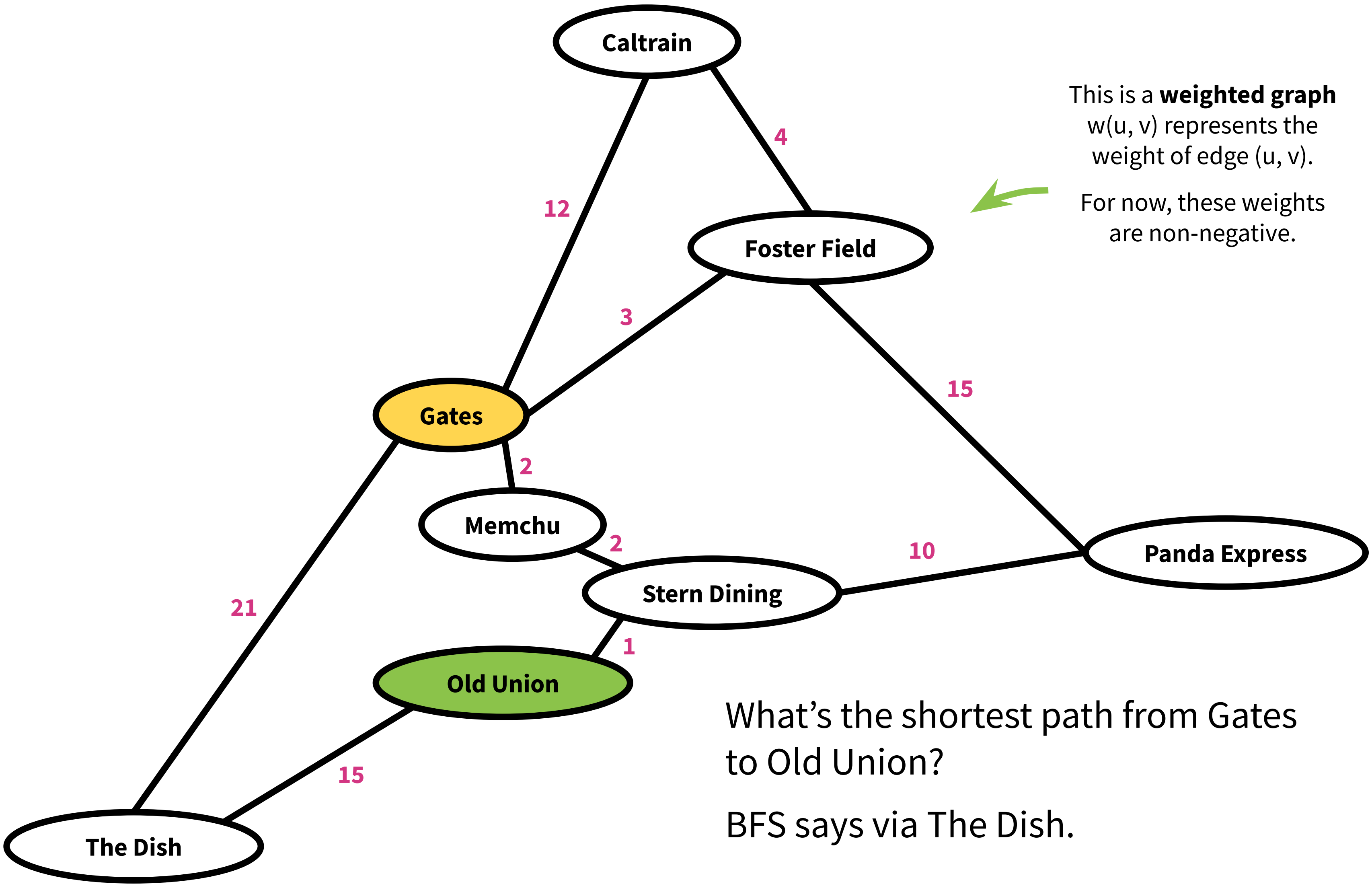
Dijkstra’s Algorithm

Shortest Path

Suppose you're new to campus and only know paths between certain landmarks.



Shortest Path



This is a **weighted graph**
w(u, v) represents the weight of edge (u, v).

For now, these weights are non-negative.

What's the shortest path from Gates to Old Union?

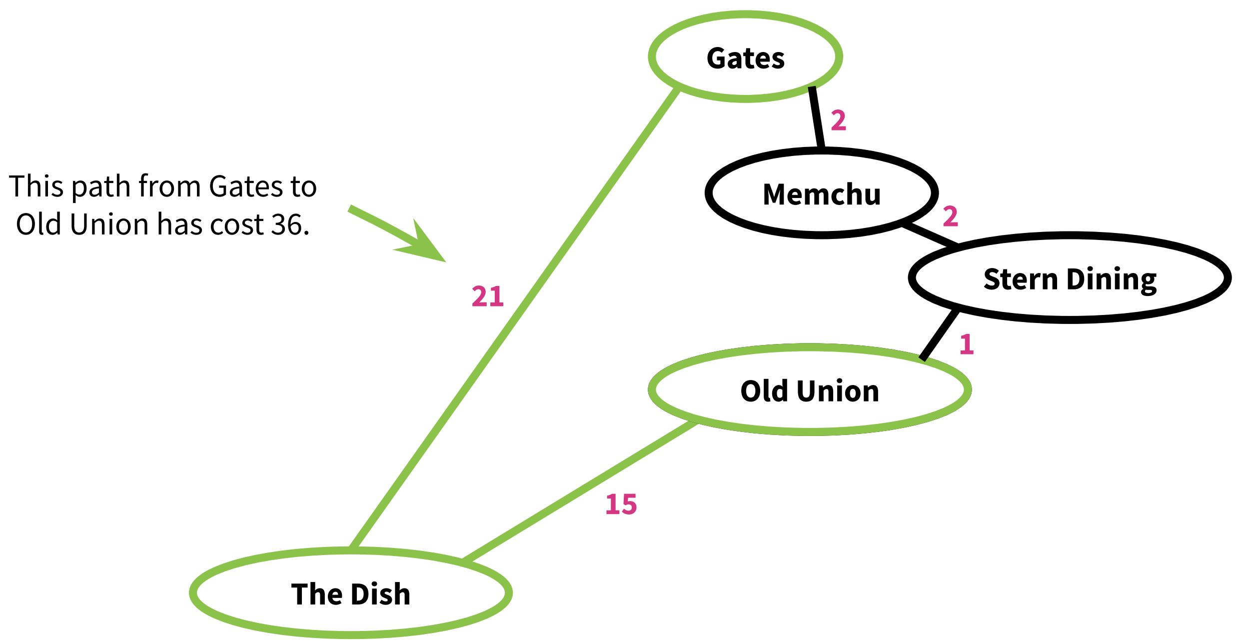
BFS says via The Dish.

Shortest Path

What is the **shortest path** between u and v in a weighted graph?

The cost of a path is the sum of the weights along that path.

The shortest path is the one with the minimum cost.



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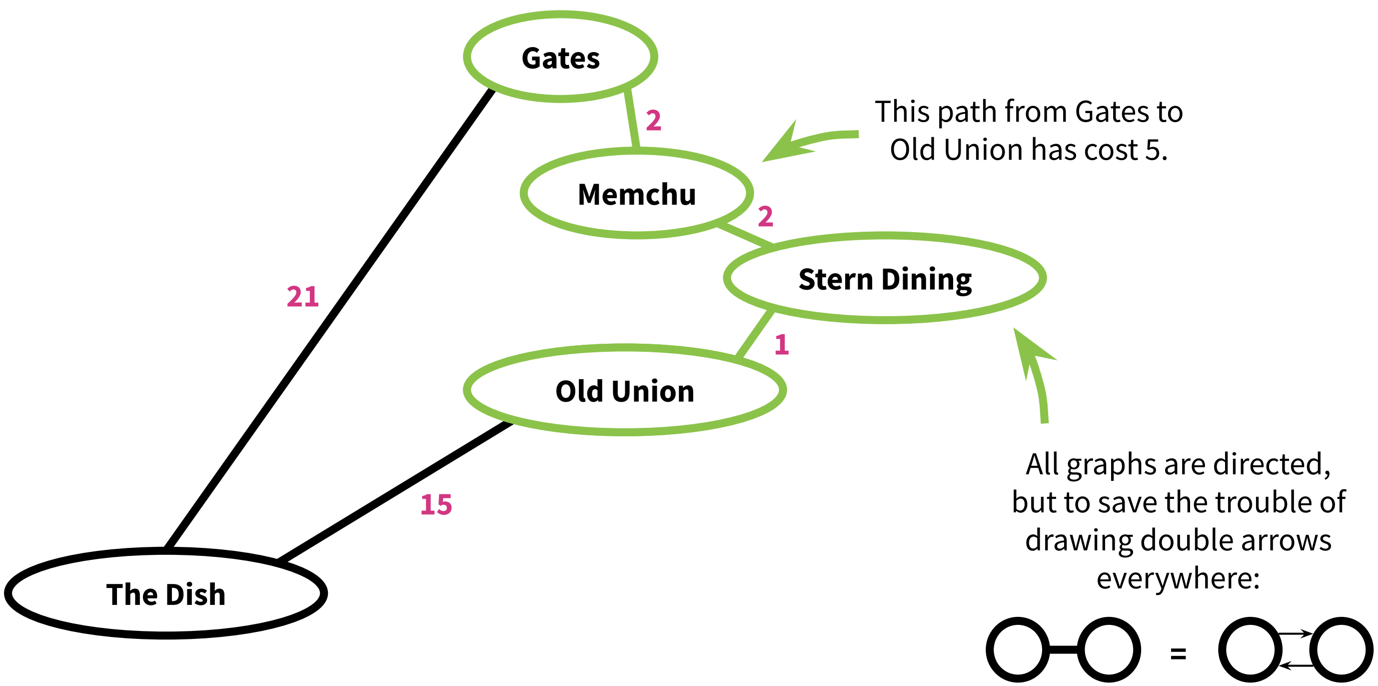
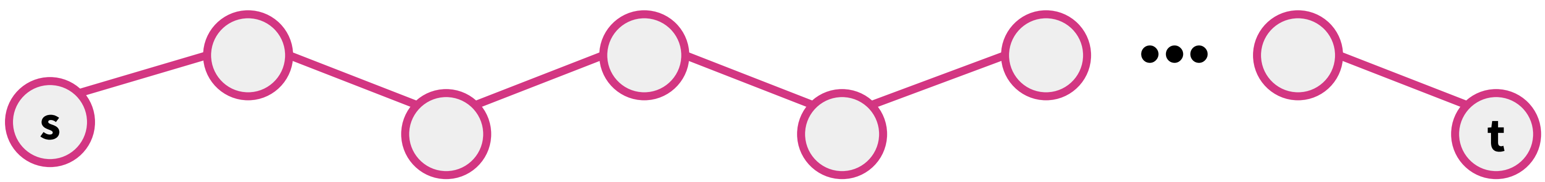


Shortest Path

Claim: A subpath of a shortest path is also a shortest path.

Intuition:



Suppose **this** is a shortest path from **s** to **t**.

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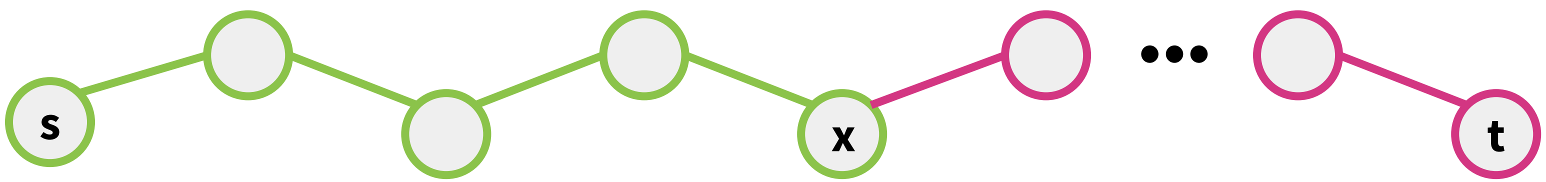

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Then **this** is a shortest path from **s** to **x**.

Why? 🤔

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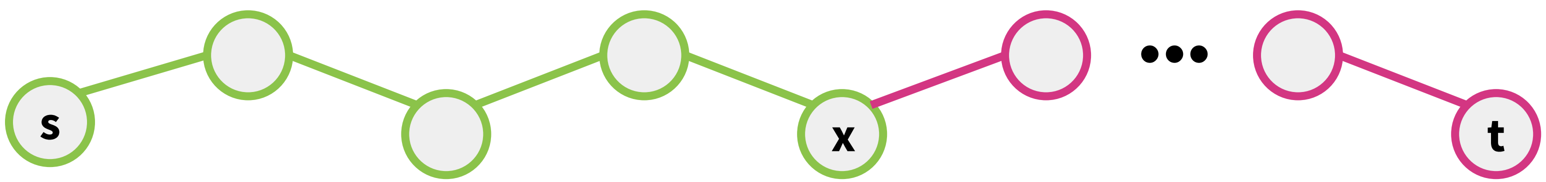

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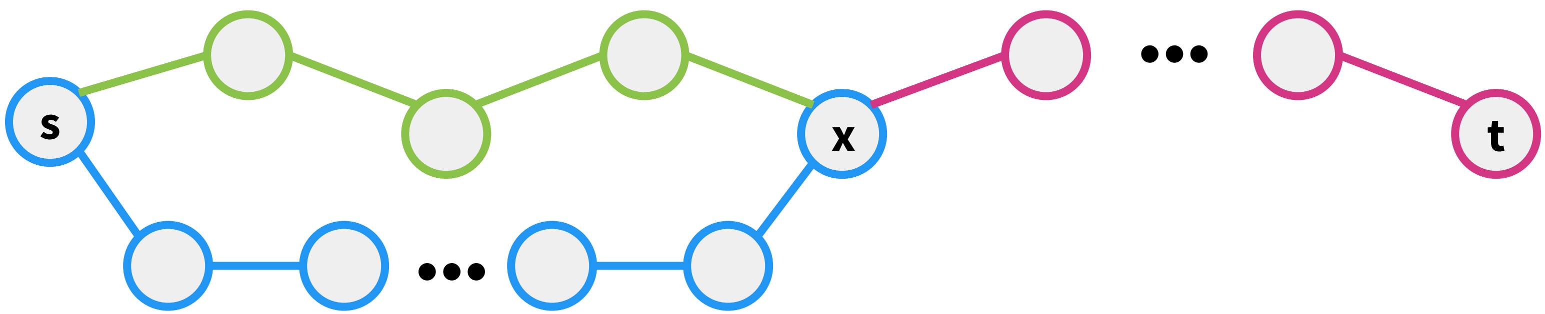

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Then **this** is a shortest path from **s** to **x**.

Why? 🤔 By contradiction, suppose there exists a shorter path from **s** to **x**, namely **this** one.

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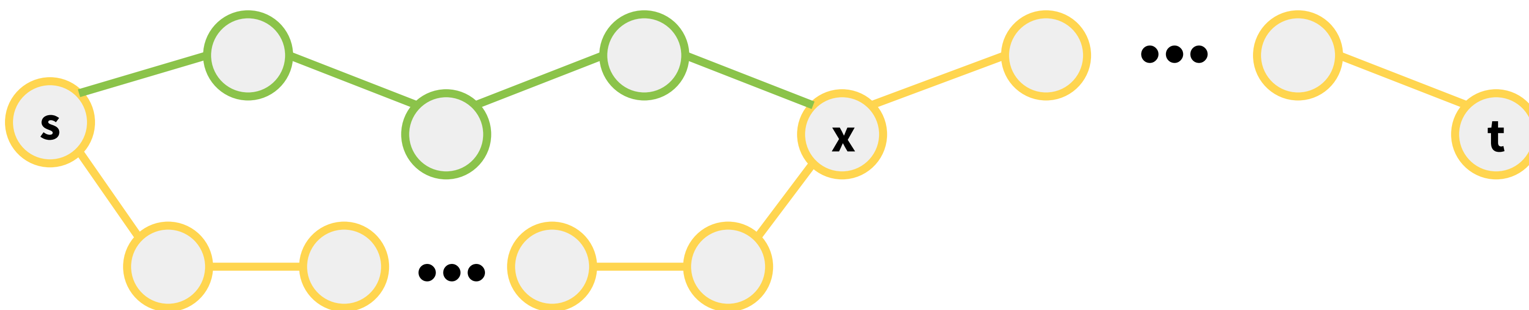

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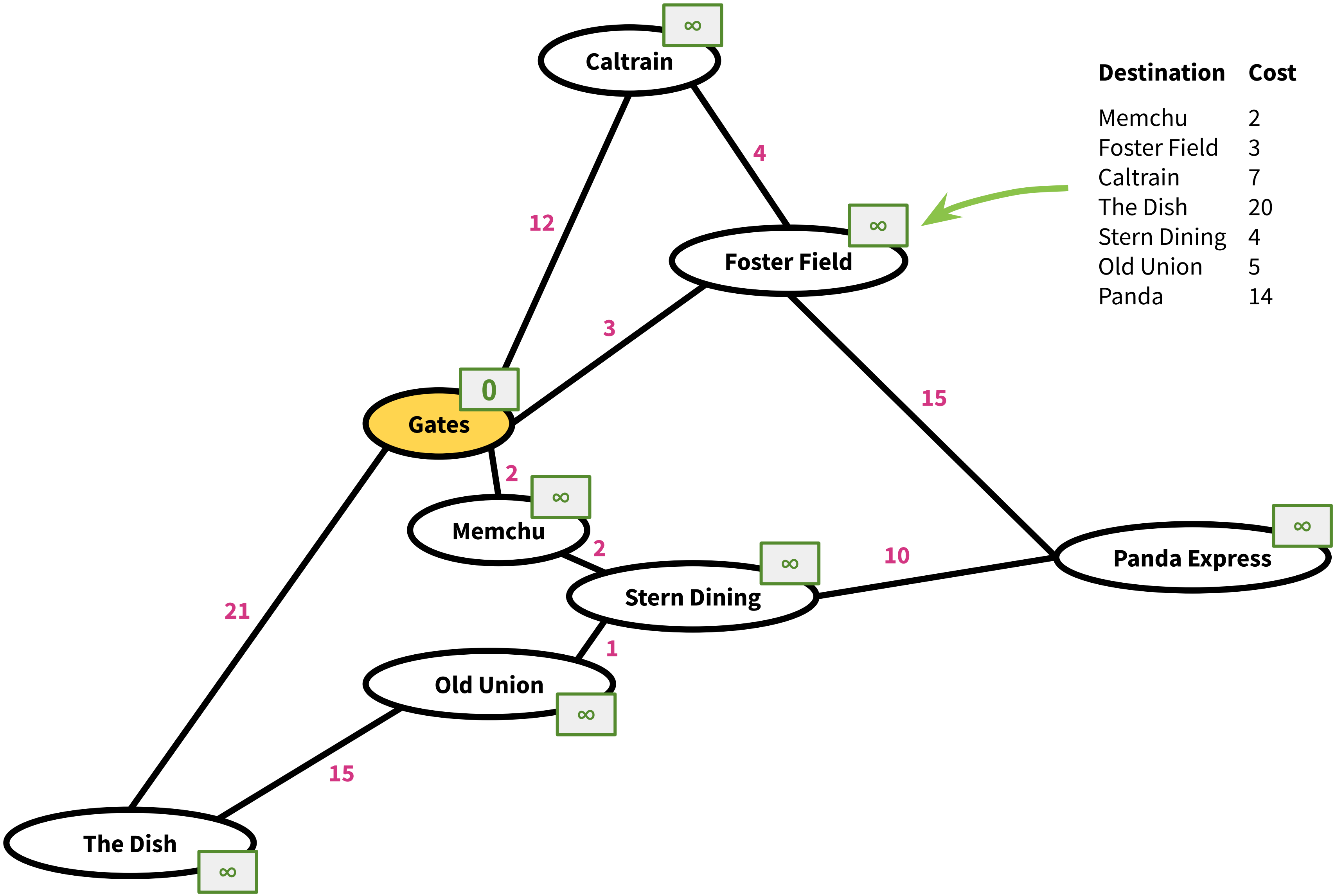
Then **this** is a shortest path from **s** to **x**.

Why? 🤔 By contradiction, suppose there exists a shorter path from **s** to **x**, namely **this** one.

But then **this** is shorter than **this** shortest path from **s** to **t**.

Single-Source Shortest Path



Destination	Cost
Memchu	2
Foster Field	3
Caltrain	7
The Dish	20
Stern Dining	4
Old Union	5
Panda	14

Single-Source Shortest Path

Application: Finding the shortest path from Palo Alto to [somewhere else] for a commuter using BART, Caltrain, bike, walking, Uber, Lyft, etc.

- Edge weights are a function of time, money, hassle that change depending on the commuter's mood on that day.

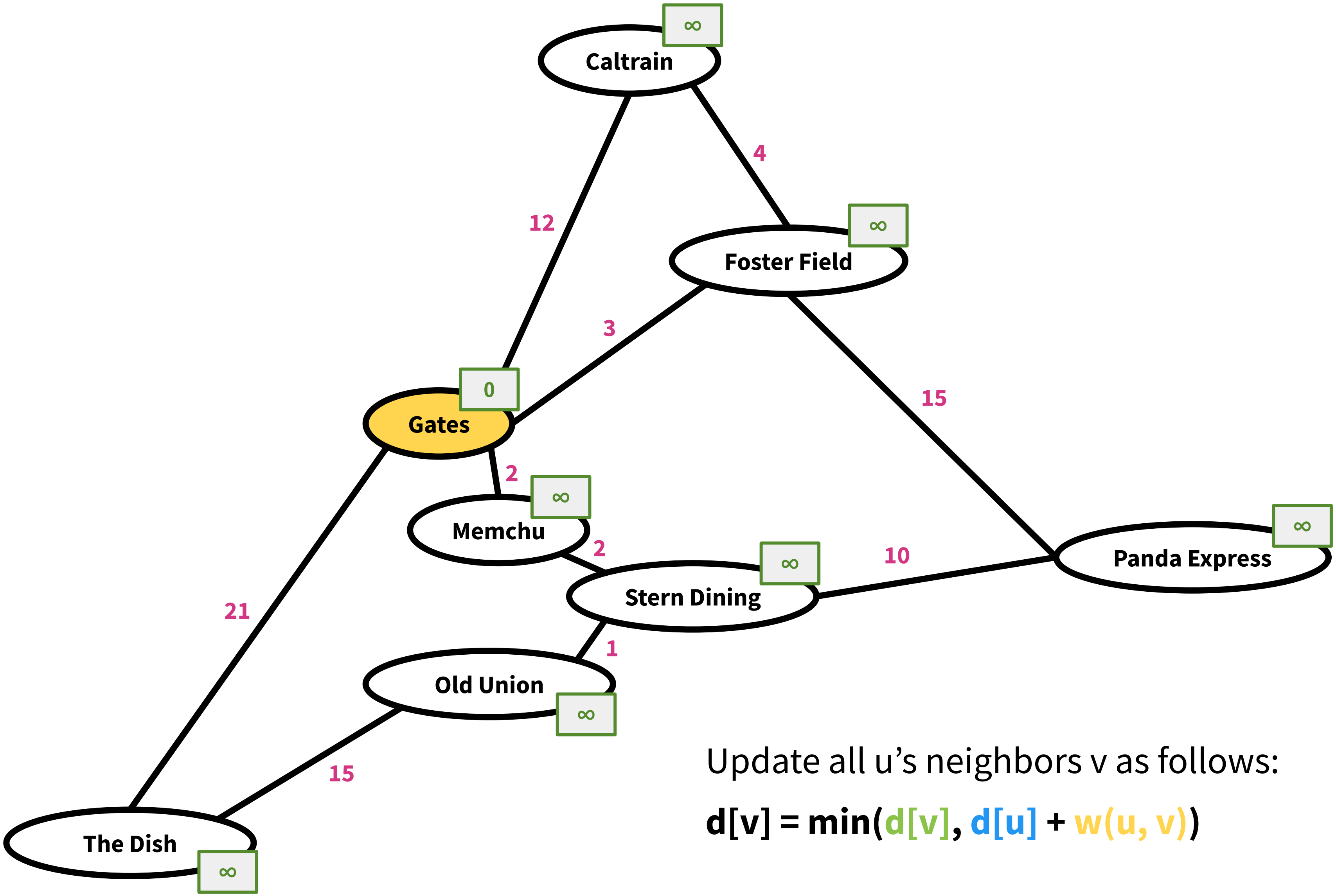
Application: Finding the shortest path from my computer to the desired server for packets using the Internet.

- Edge weights are a function of link length, traffic, other costs, etc.

Dijkstra’s Algorithm

Dijkstra’s Algorithm solves the single-source shortest path problem.

Dijkstra's Algorithm



Update all u's neighbors v as follows:

d[v] = min(d[v], d[u] + w(u, v))

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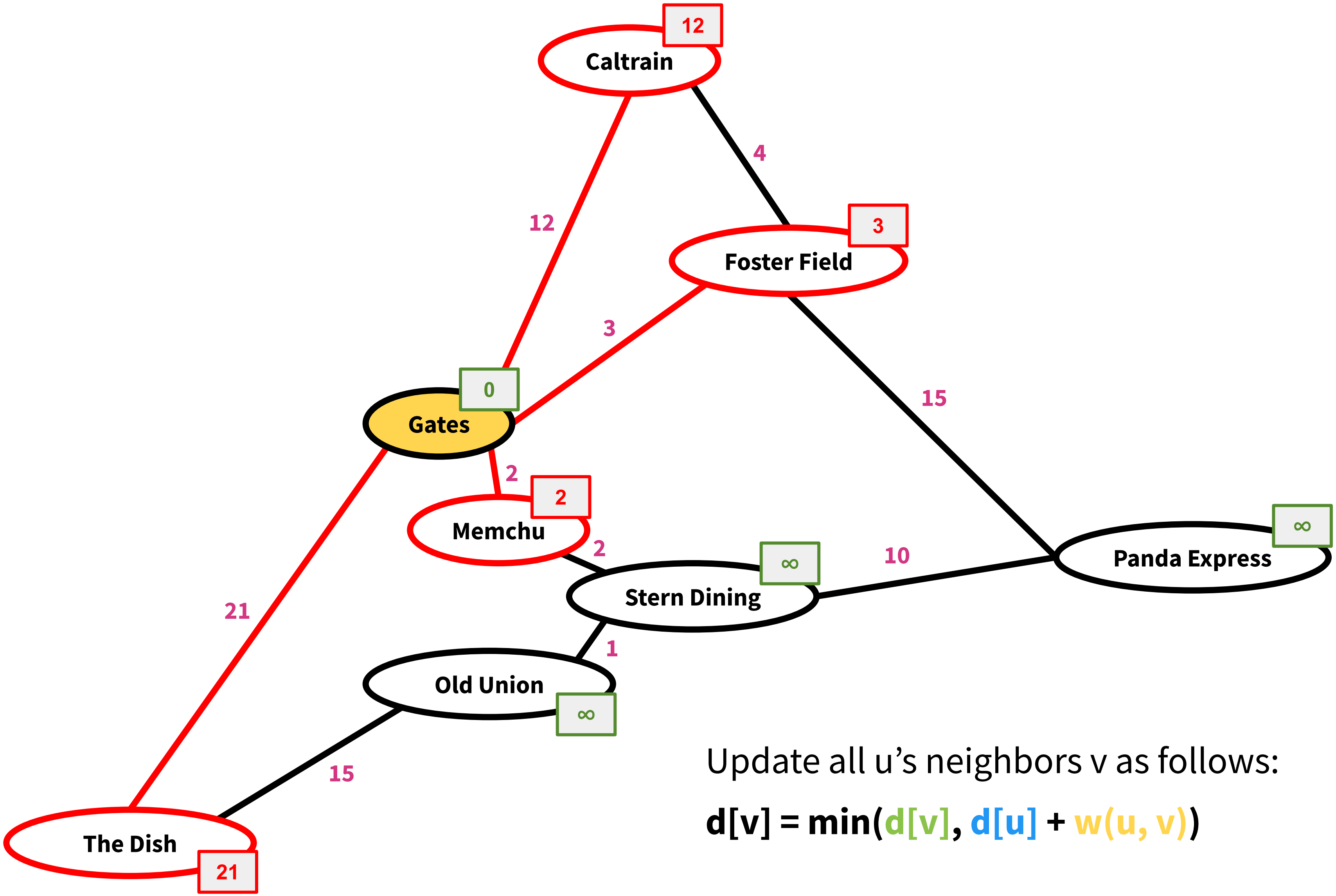

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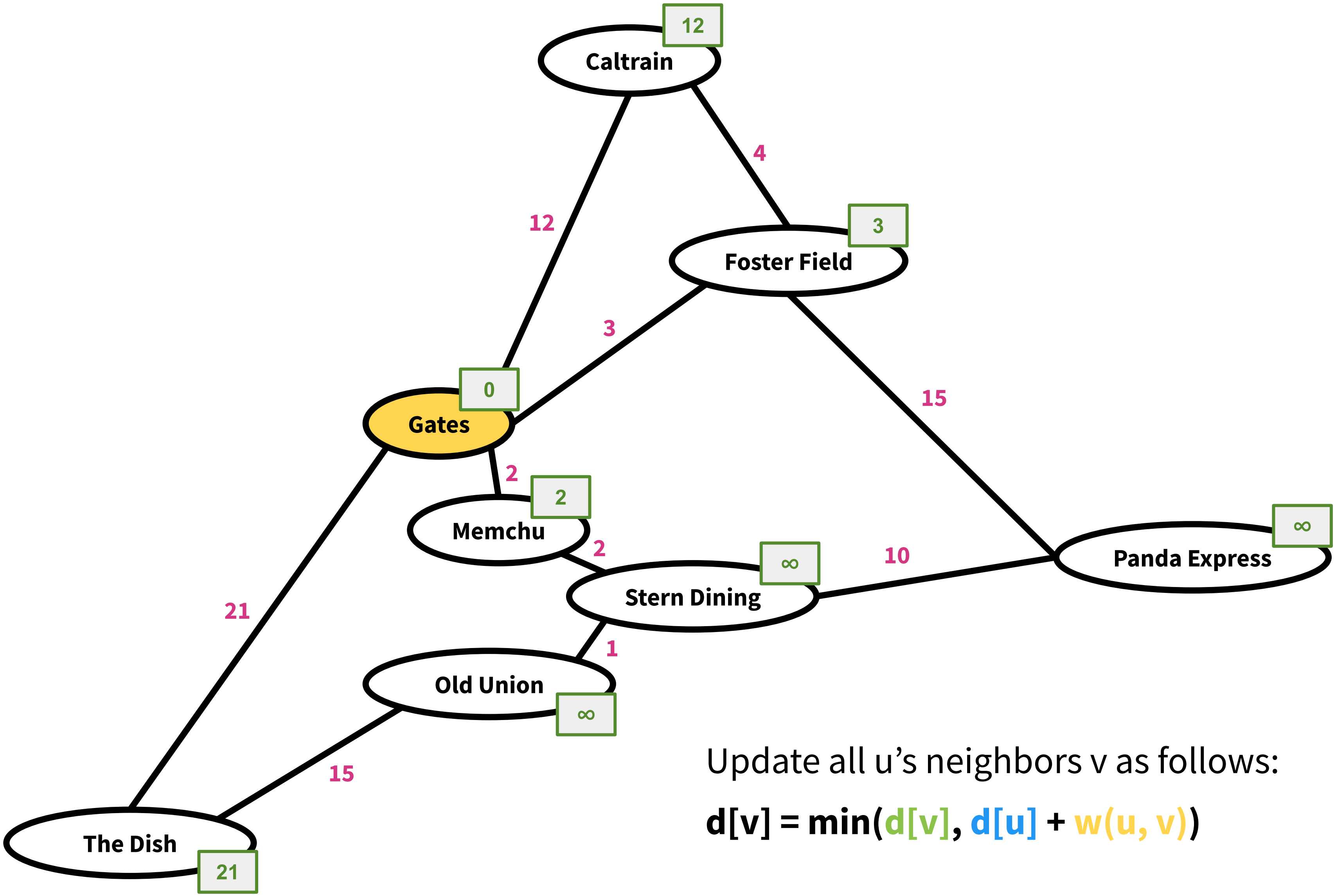

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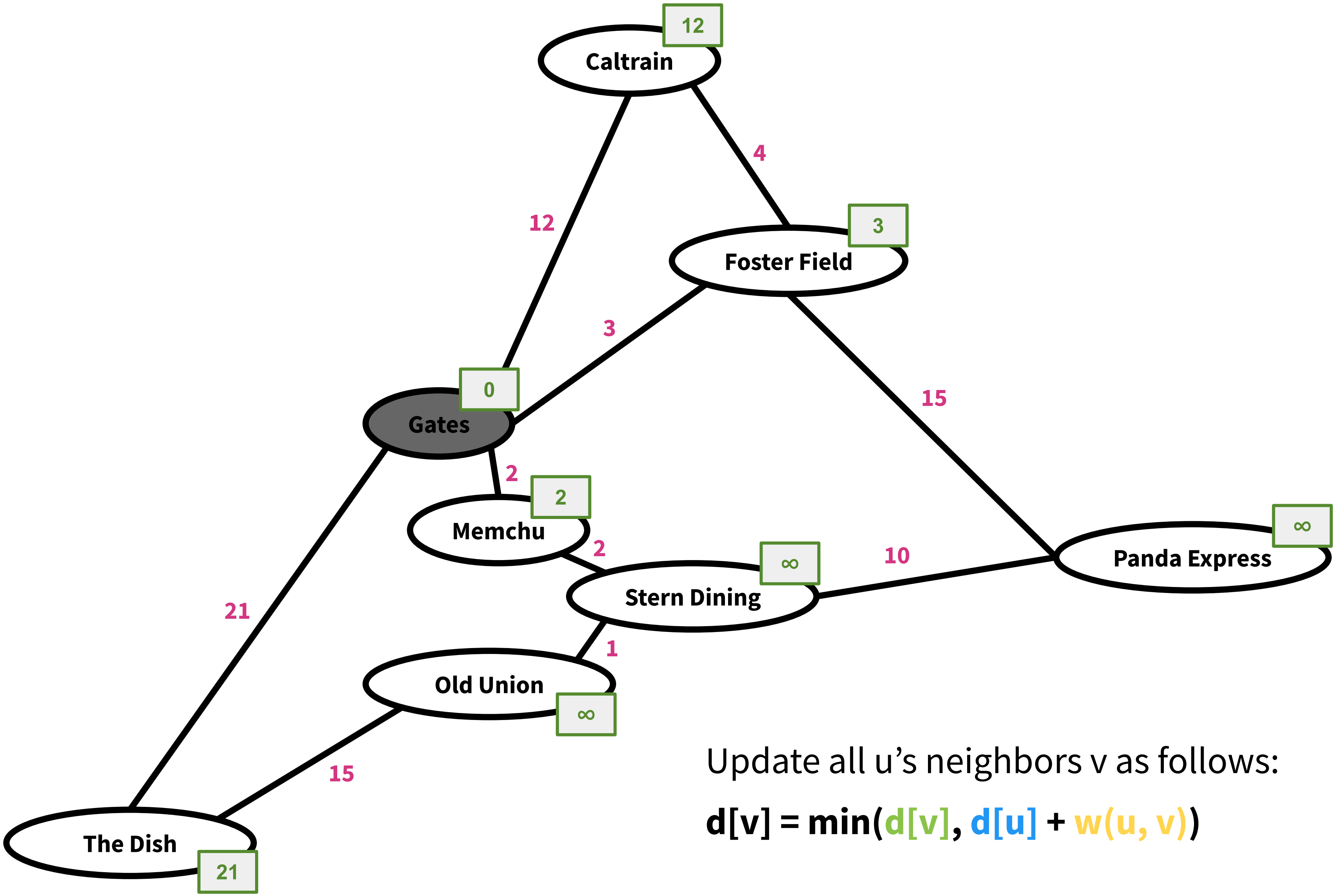

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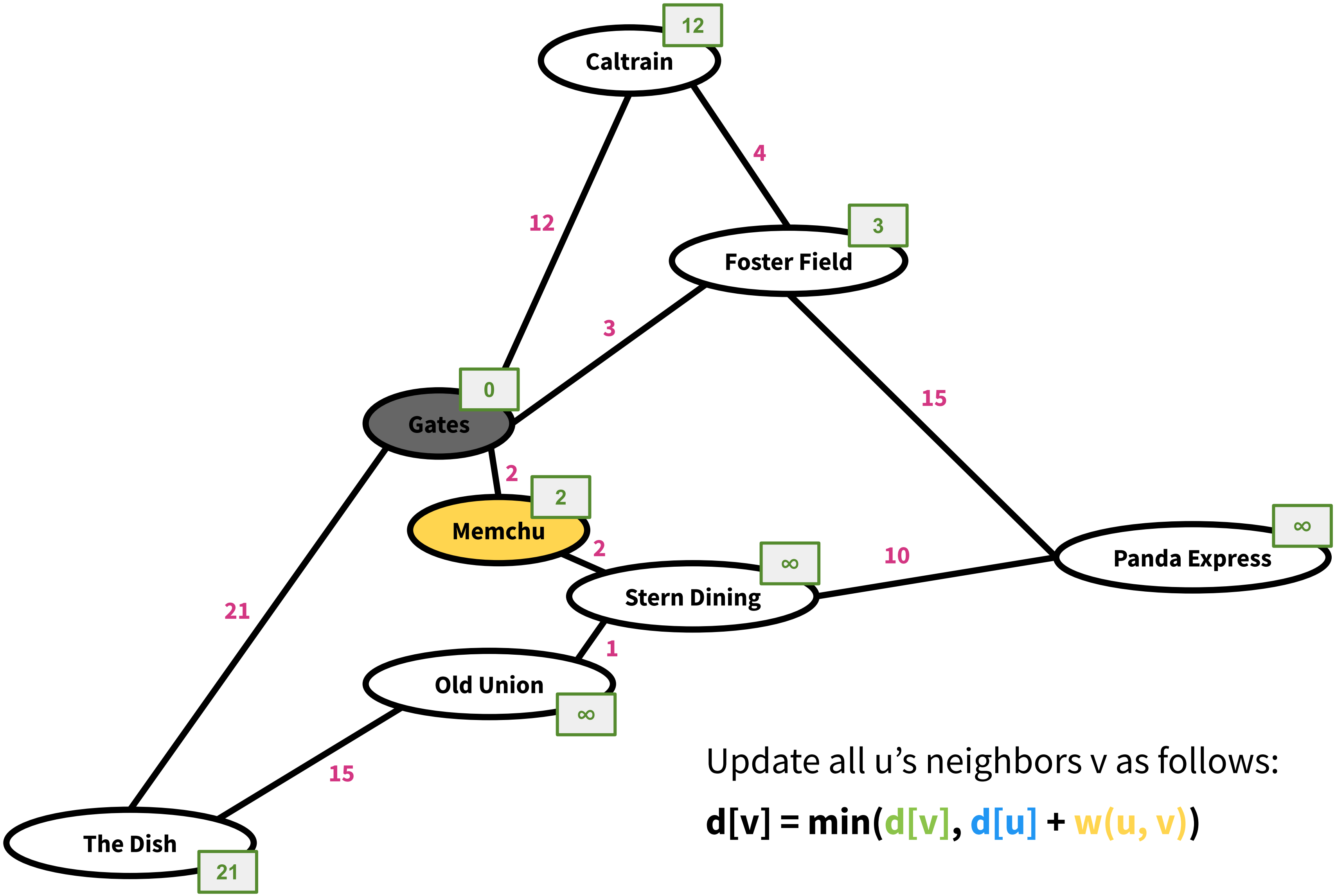

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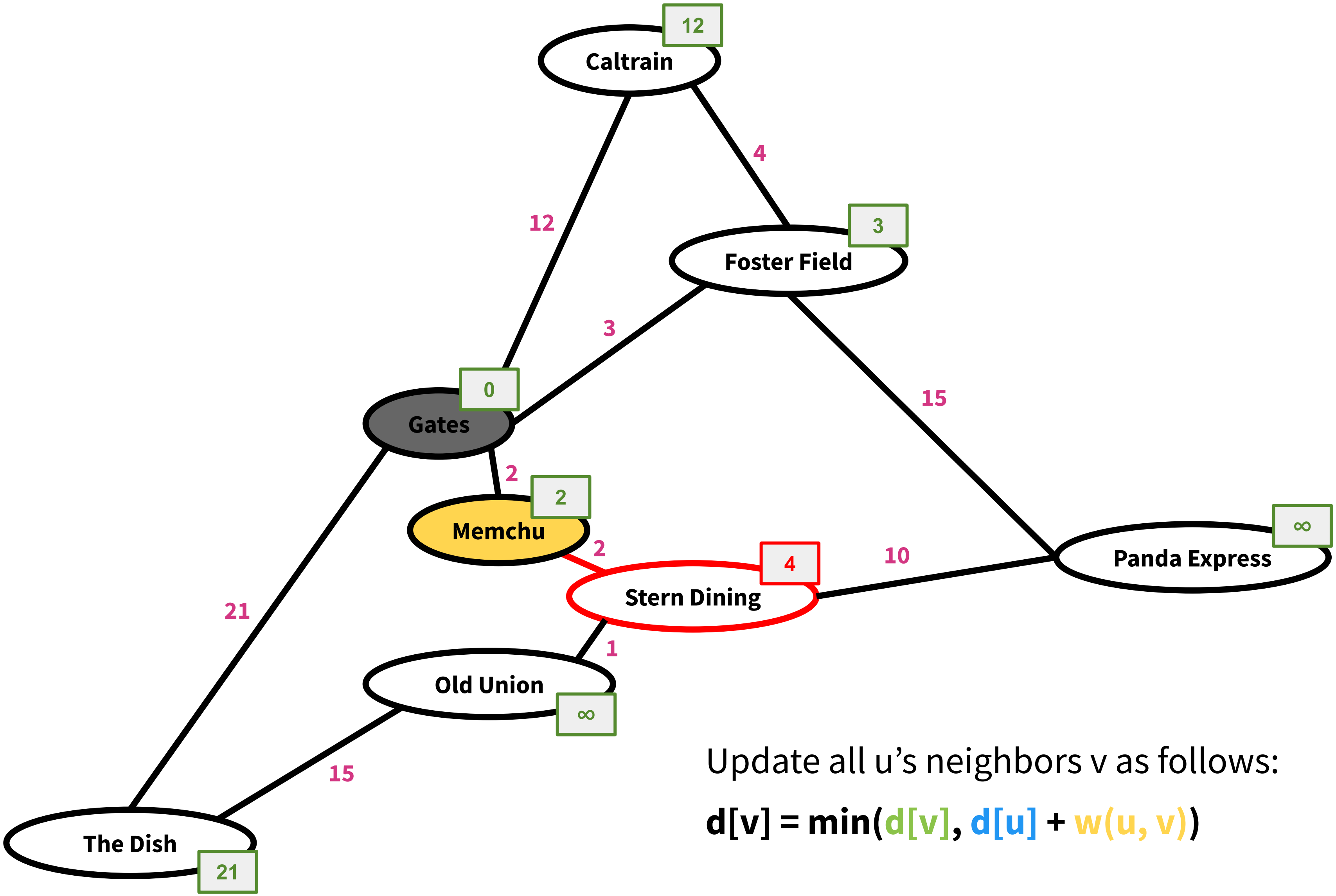

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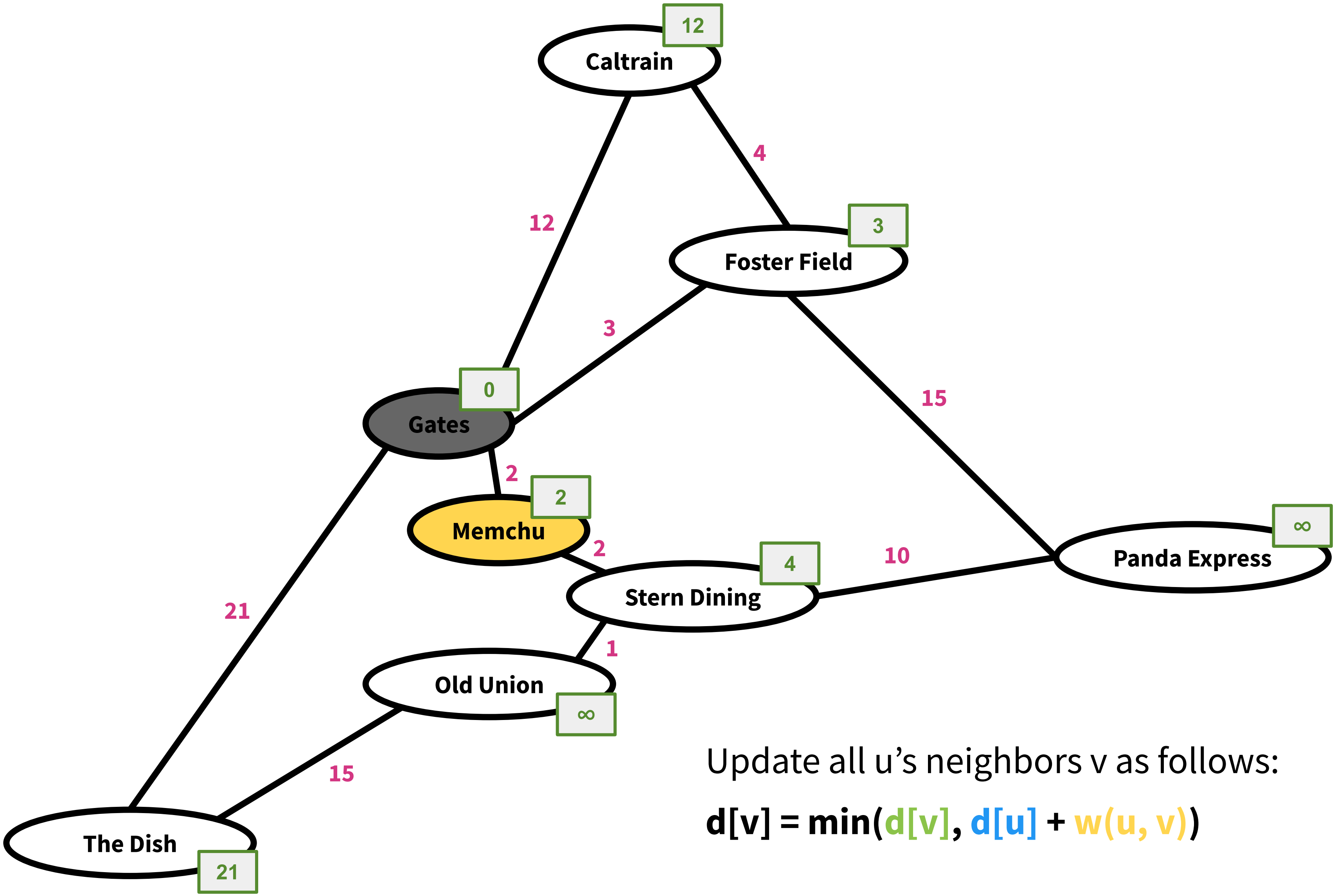

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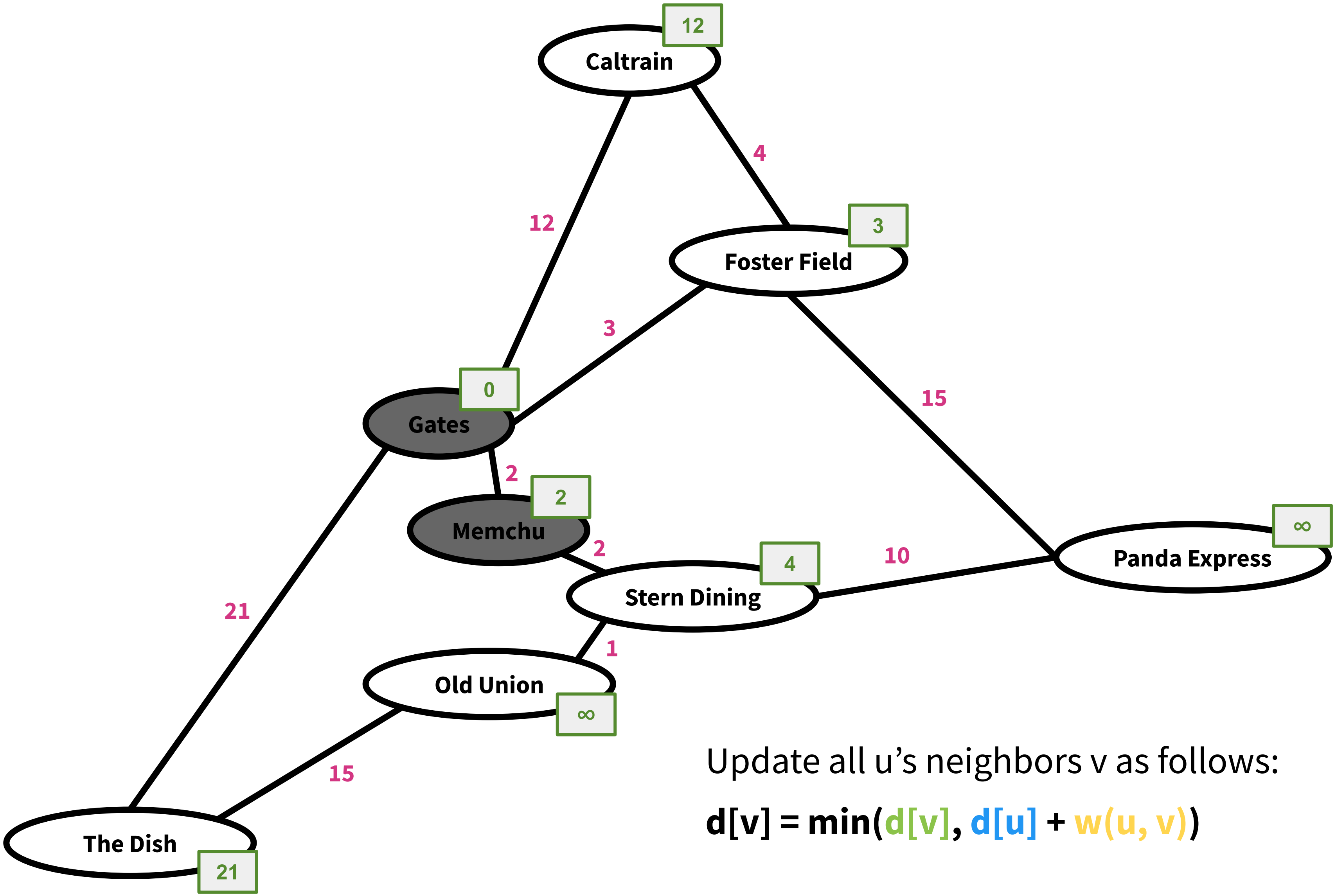

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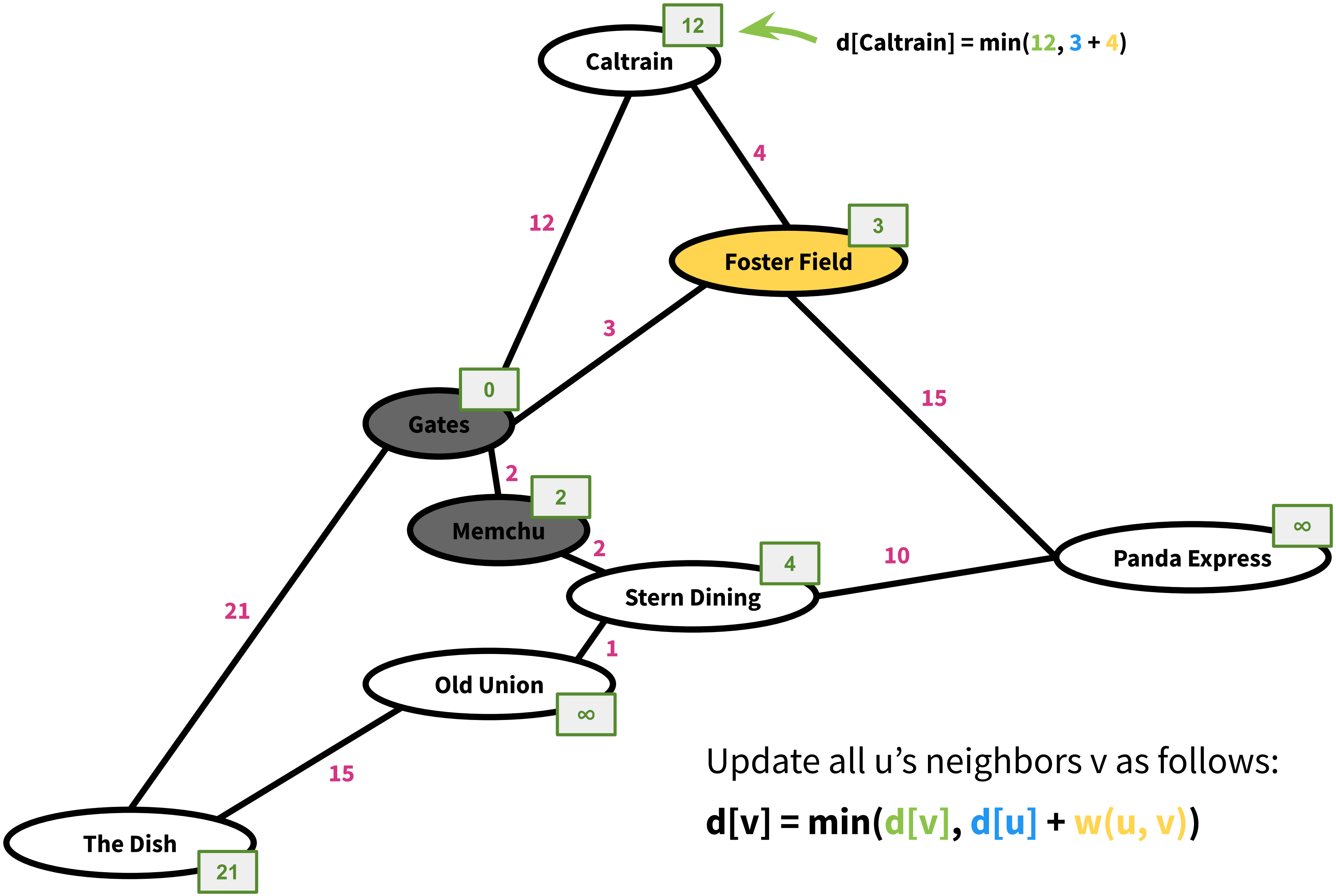

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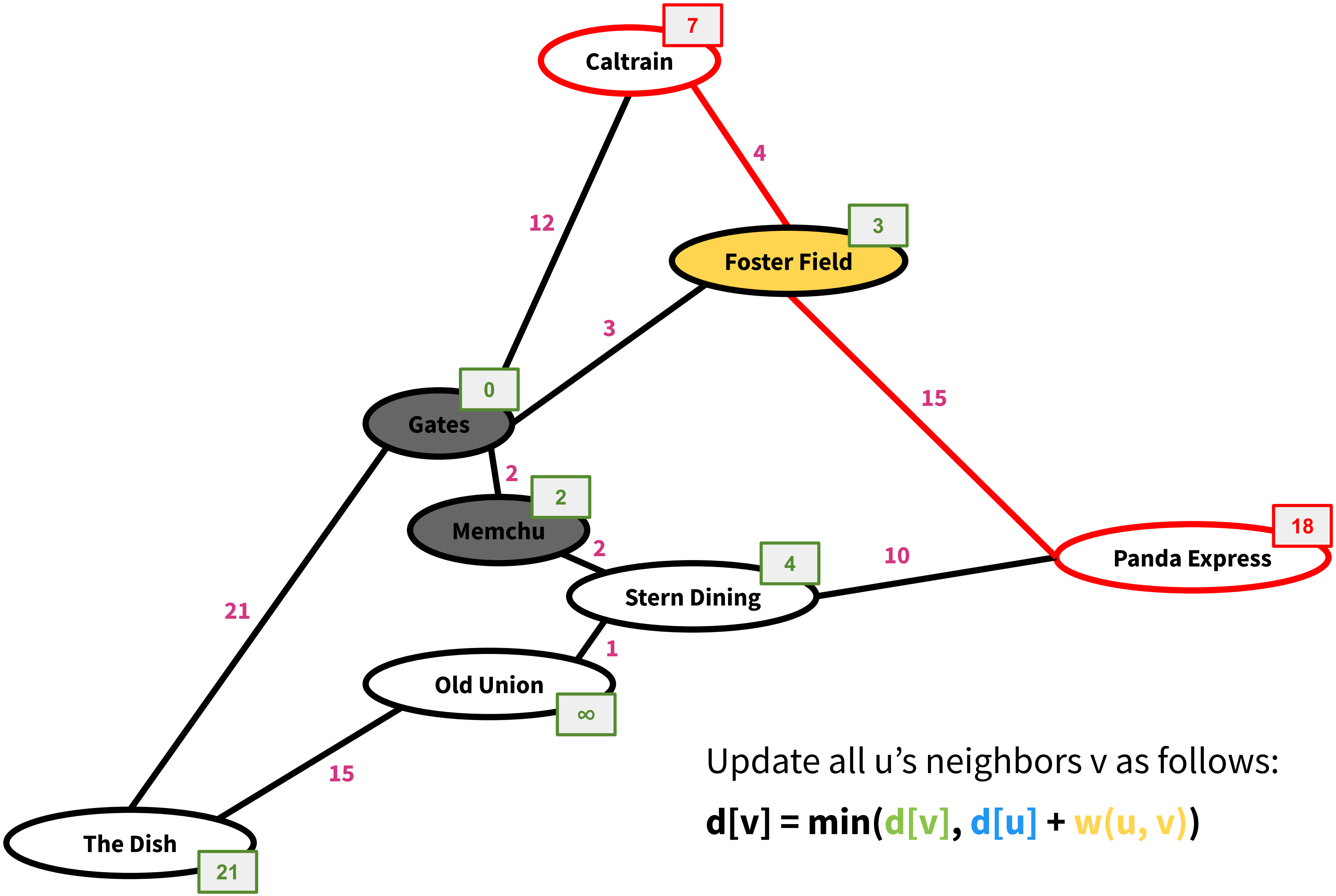

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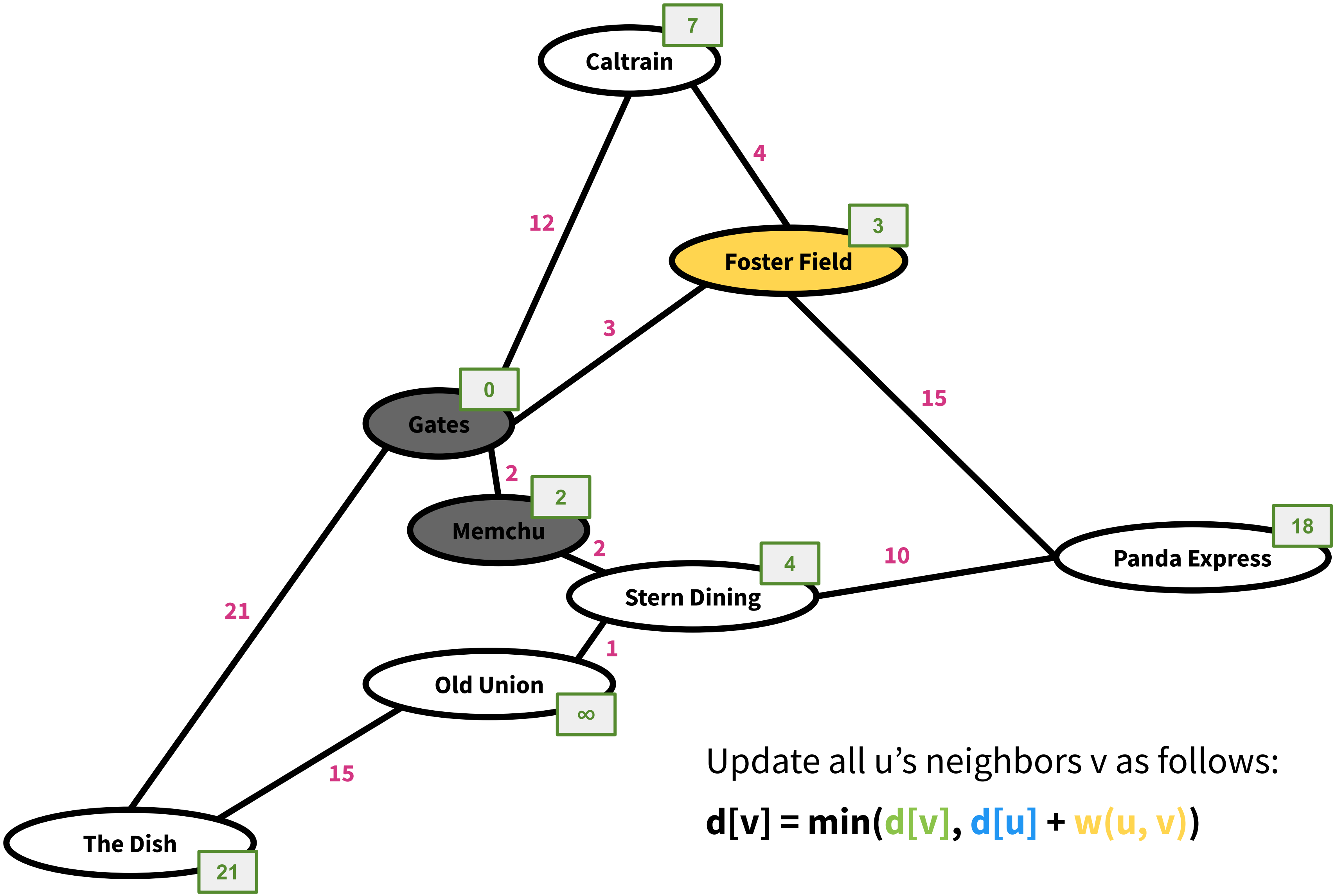

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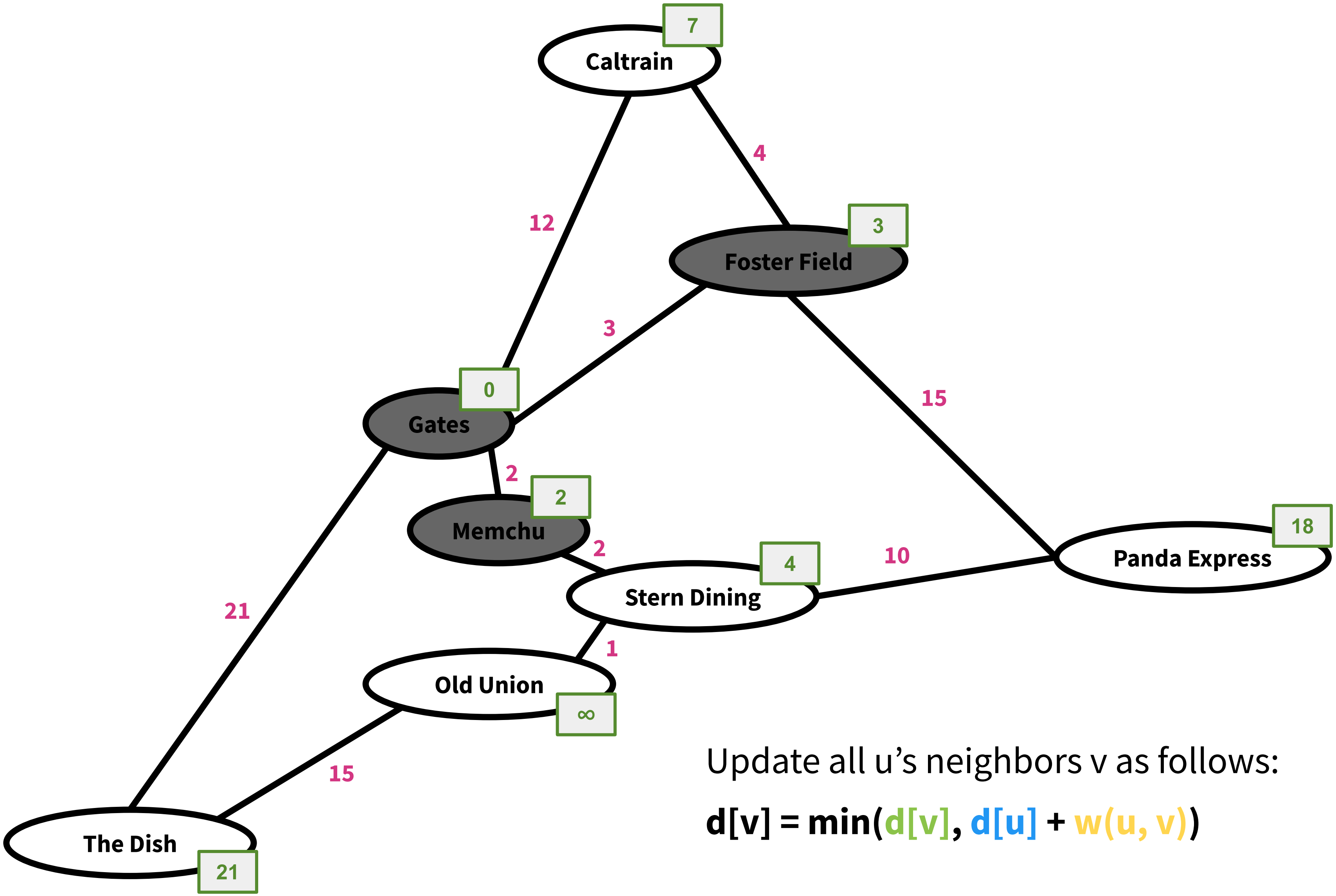

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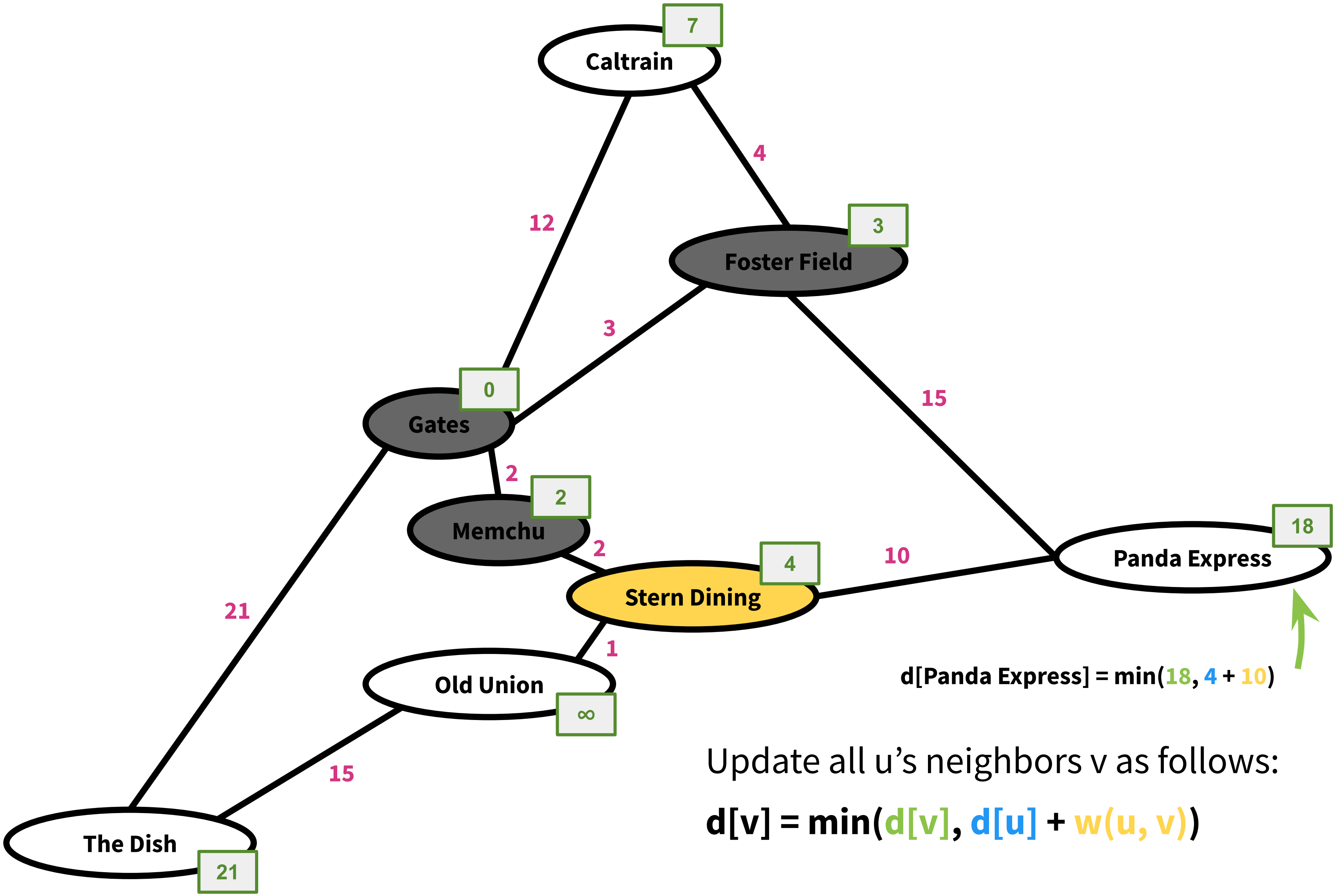

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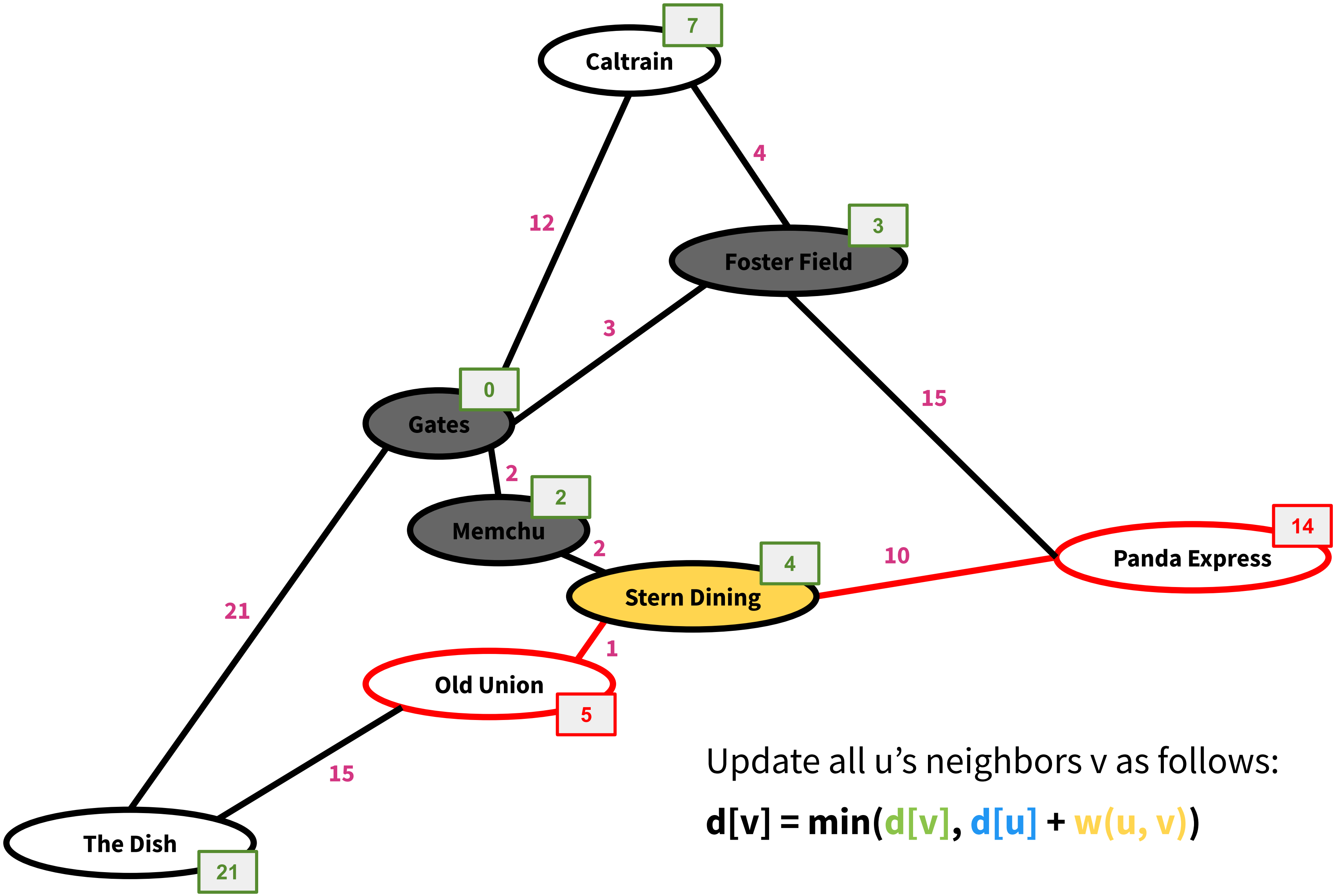

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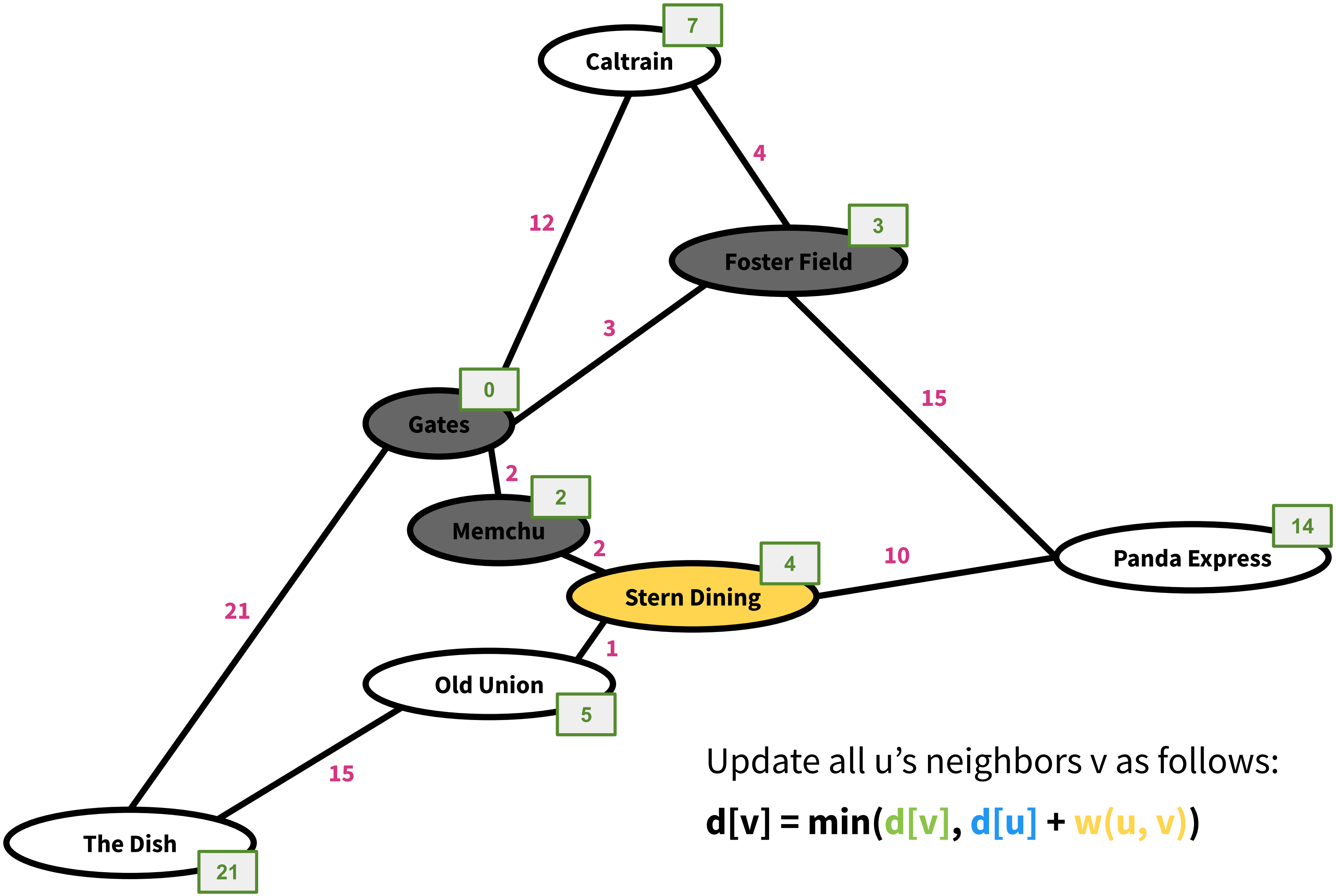

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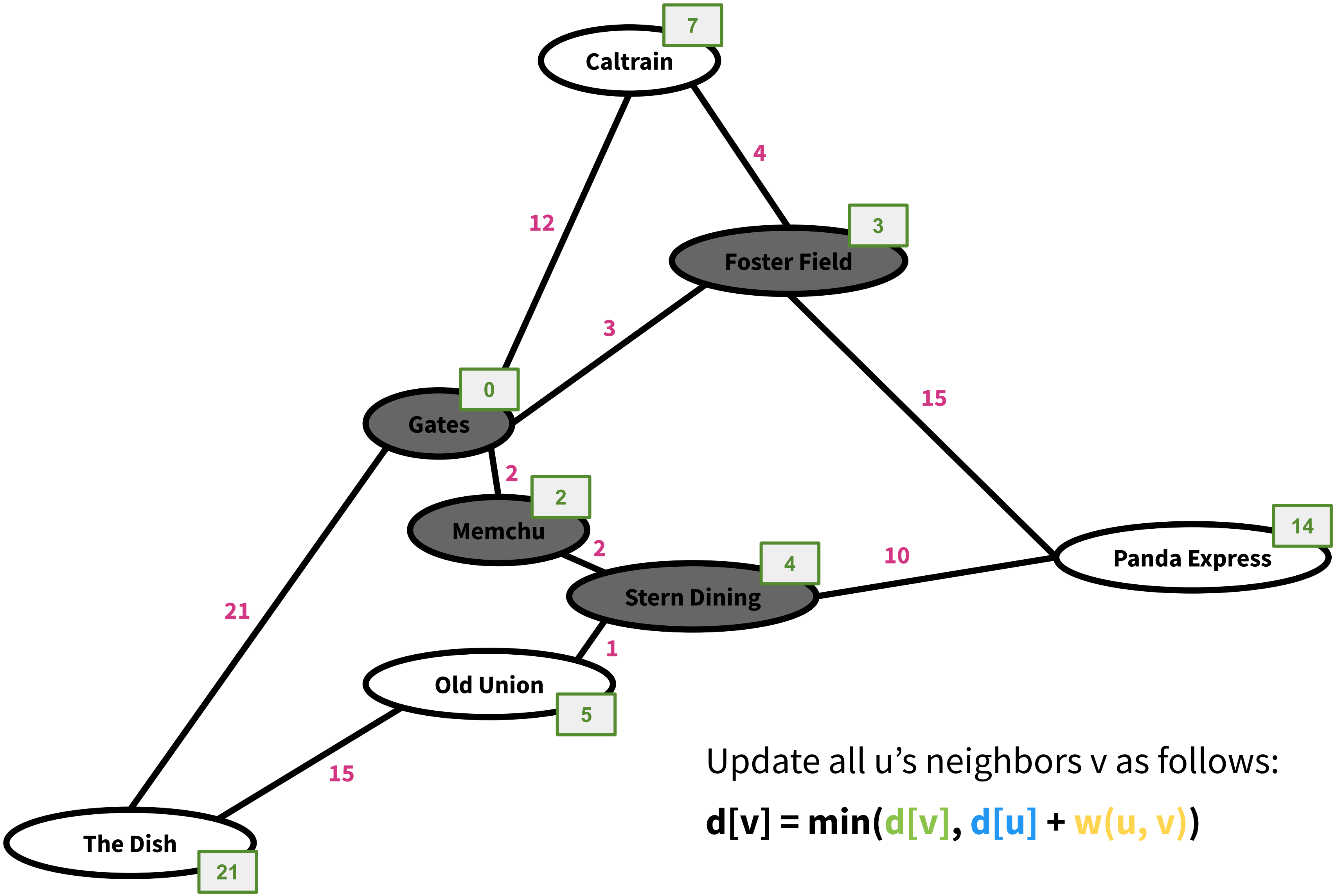

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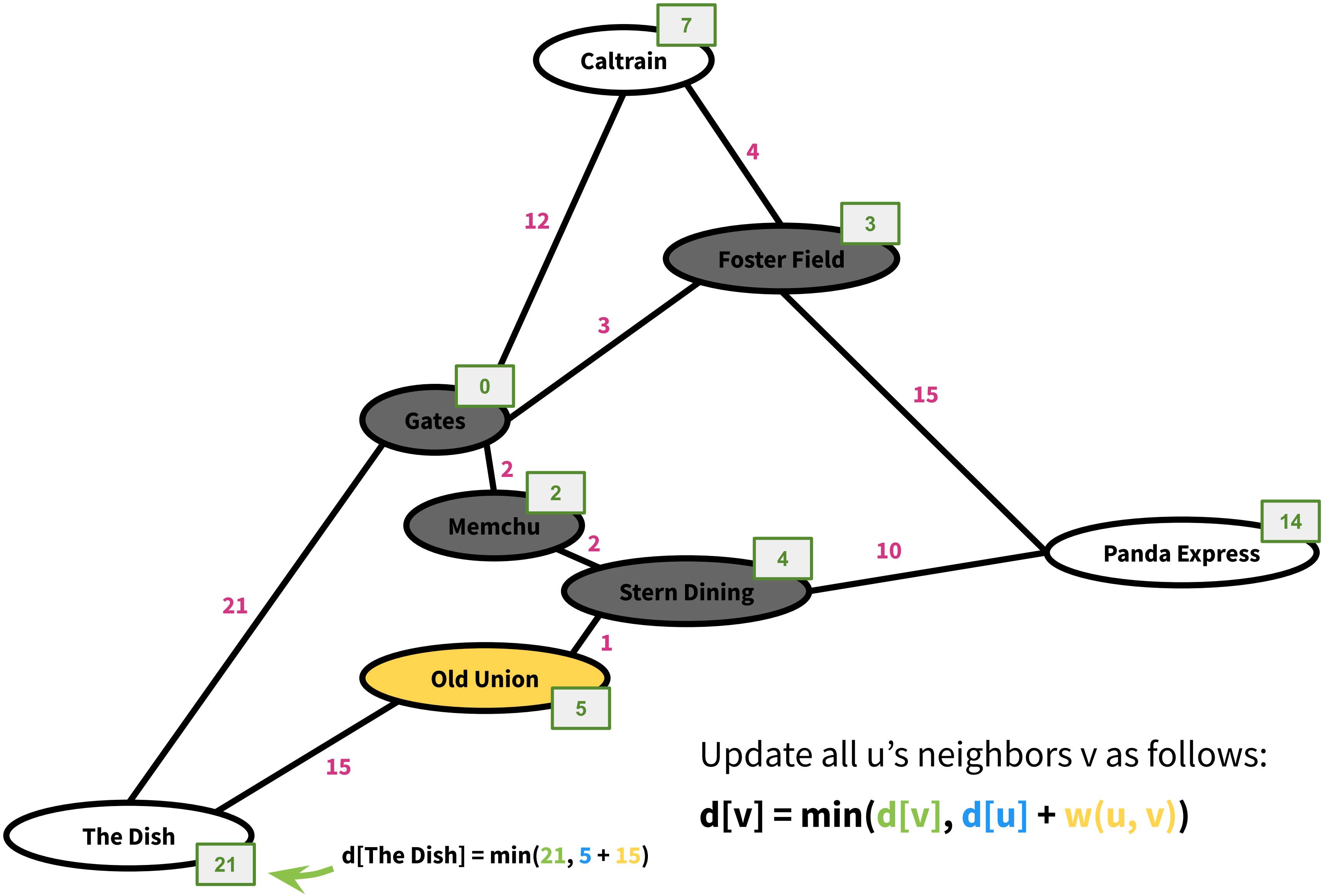

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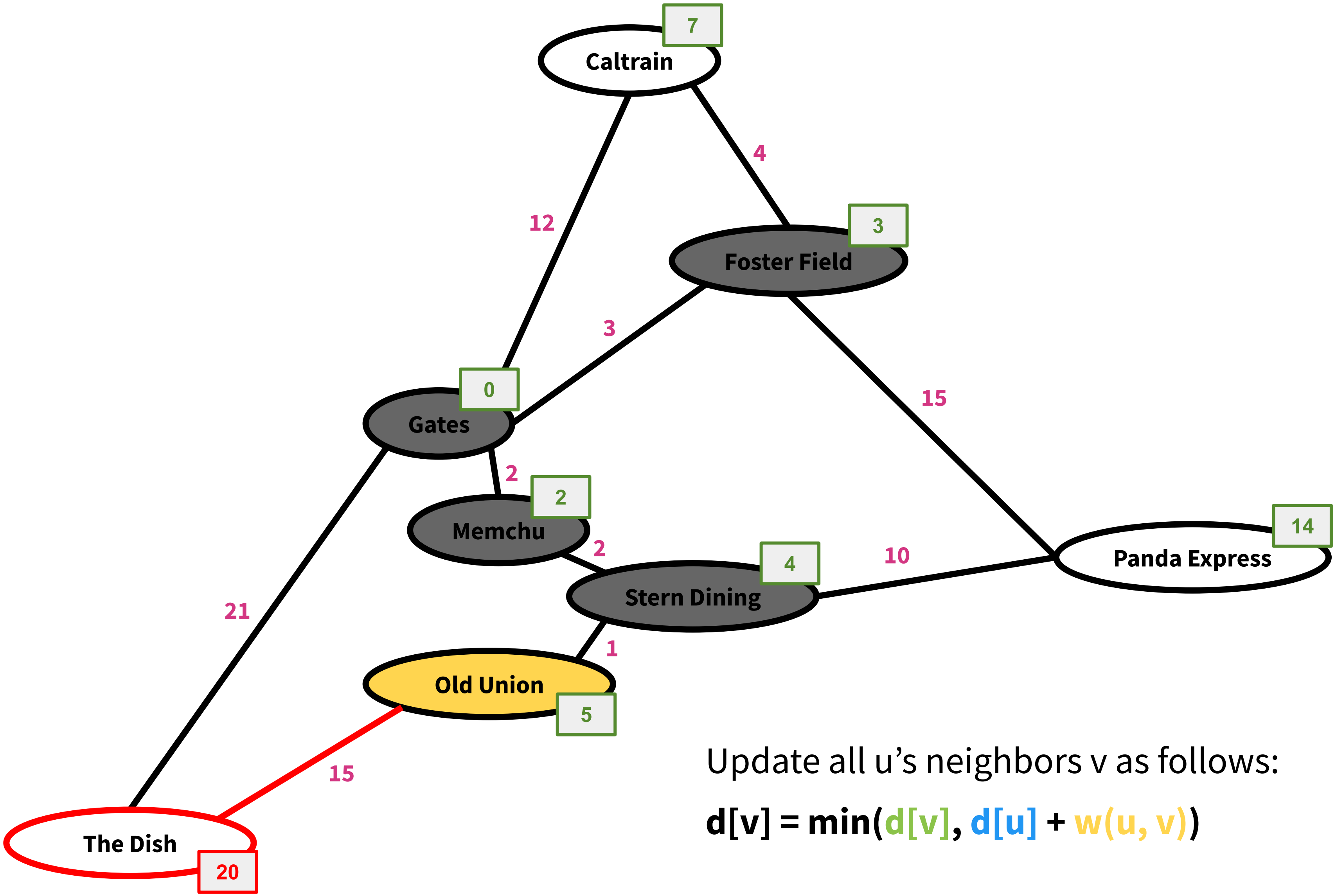

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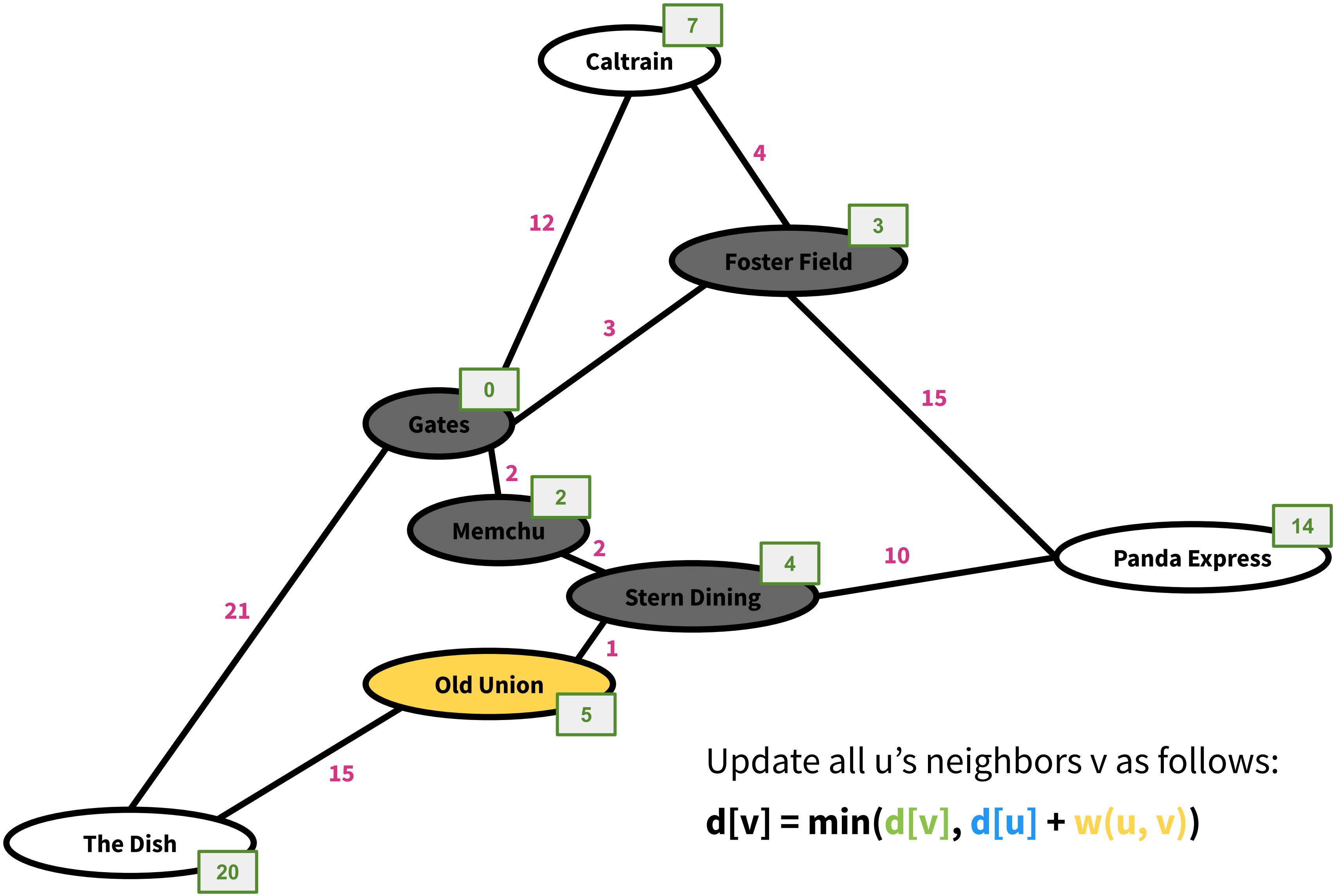

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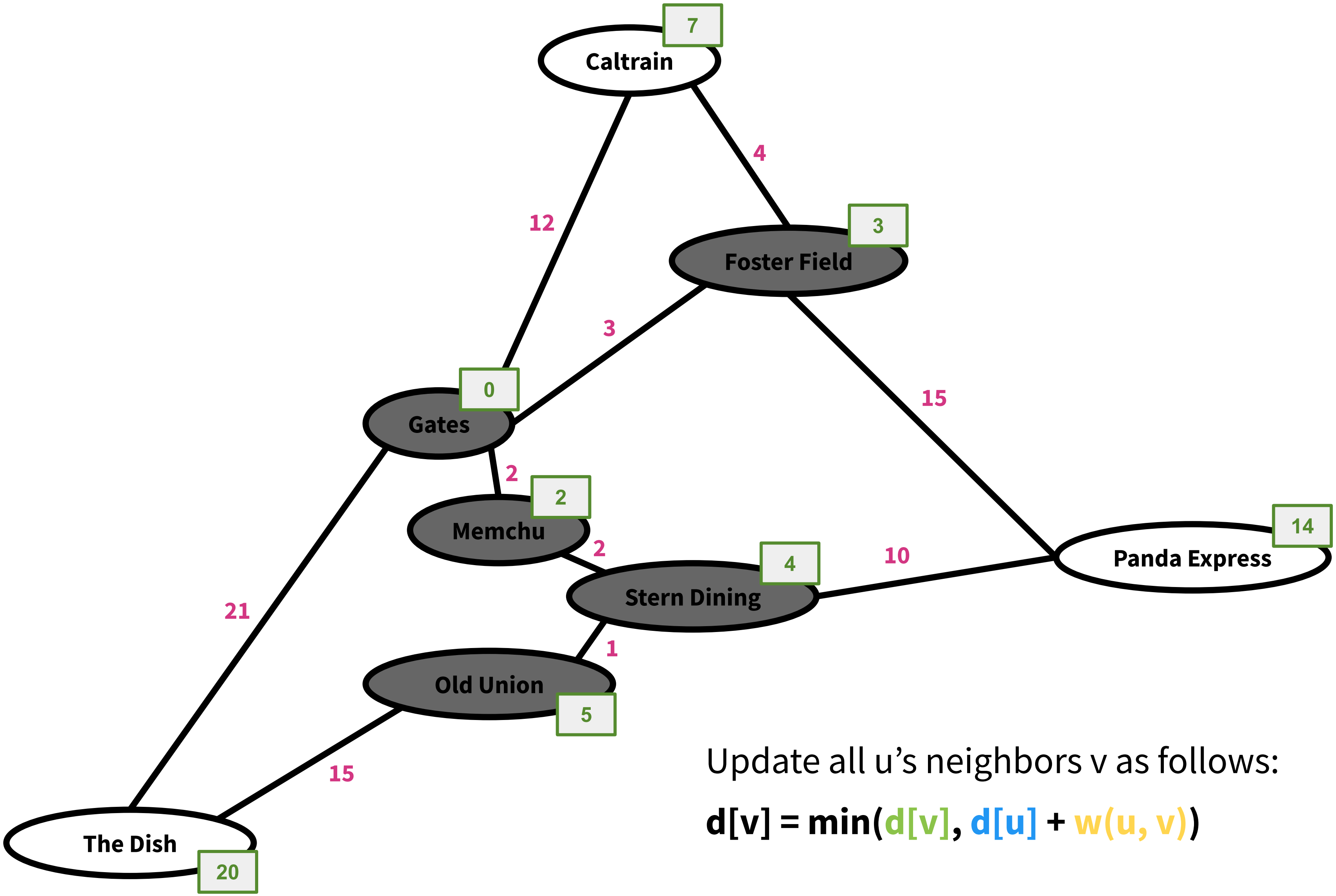

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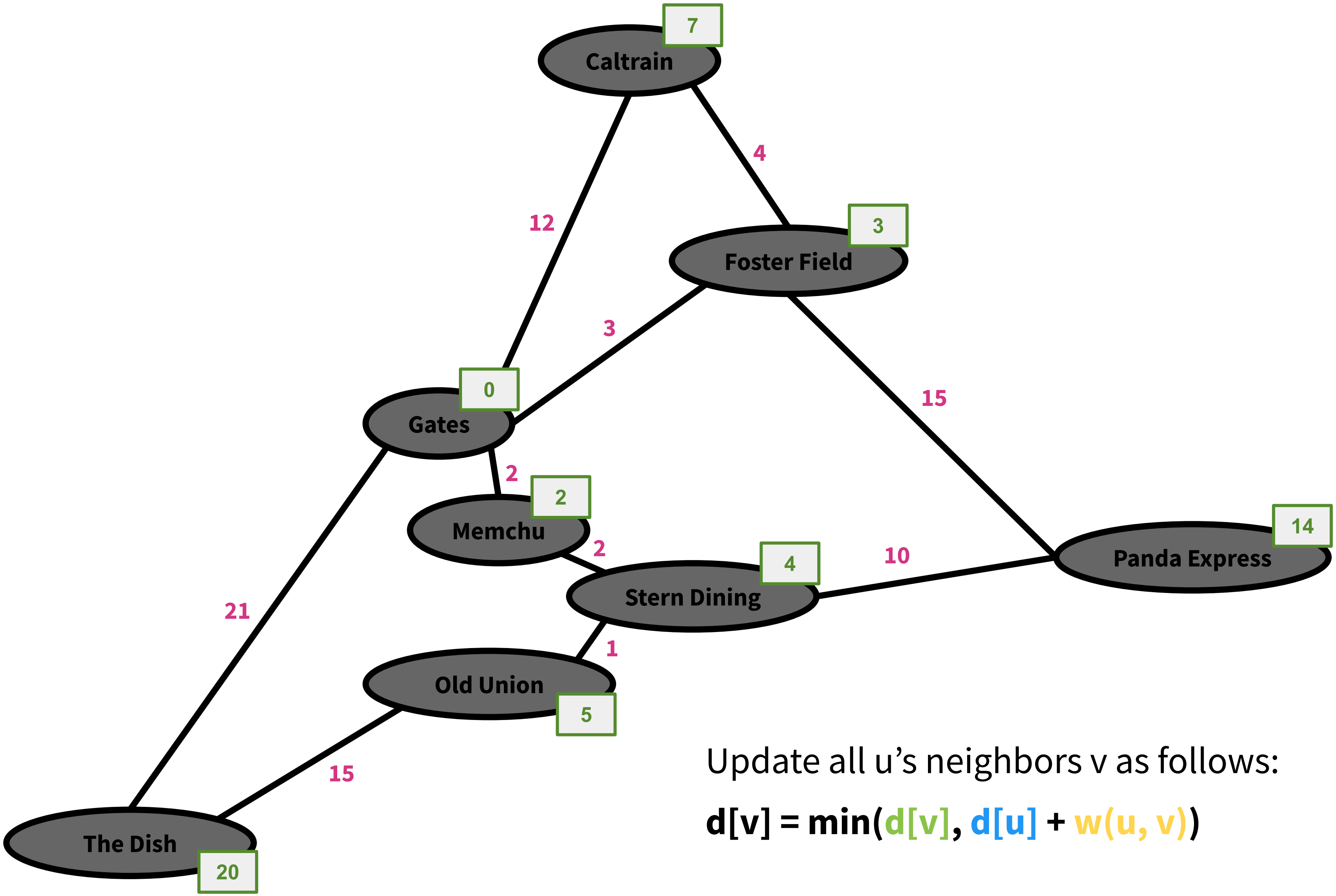

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Why does this work?

Let s be the single source.

Theorem: After running Dijkstra's Algorithm, the estimate d[v] is the actual distance d(s, v).

Proof Outline:

Claim 1: For all v, d[v] ≥ d(s, v).

Claim 2: When a vertex v gets marked "done", d[v] = d(s, v).

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All vertices are eventually "done" (stopping condition in algorithm).

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Therefore, all vertices end up with d[v] = d(s, v).

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After t = 0 iterations, d(s, s) = 0 and d(s, v) ≤ ∞ which satisfy d[v] ≥ d(s, v).

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For the inductive step, suppose the inductive hypothesis holds for iteration t. Then at iteration $t + 1$, the algorithm picks a vertex u and for each of its neighbors v sets: $\mathbf{d[v] = \min(d[v], d[u] + w(u, v))} \geq d(s, v)$.

By induction, $d[v] \geq d(s, v)$

$d(s, v) \leq d(s, u) + d(u, v)$
$\leq d[u] + w(u, v)$

Thus, the induction holds for $t + 1$.

Dijkstra's Algorithm

Why does this work?

Claim 2: When a vertex v gets marked “done”, d[v] = d(s, v).

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We proceed by induction on t, the number of vertices marked as “done.”

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We proceed by induction on t, the number of vertices marked as "done."

For the base case, note that after s is marked as "done", d[s] = d(s, s) = 0, which satisfies d[v] = d(s, v).

For the inductive step, assume that for all vertices v already marked as "done", d[v] = d(s, v). Let x be the vertex with minimum distance estimate. We must prove d[x] = d(s, x).

Dijkstra's Algorithm

Why does this work?

Claim 2: When a vertex v gets marked "done", d[v] = d(s, v).

Proof, cont.:

We proceed by contradiction. Suppose $d[x] \neq d(s, x)$.

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Claim 2: When a vertex v gets marked "done", d[v] = d(s, v).

Proof, cont.:

We proceed by contradiction. Suppose d[x] ≠ d(s, x).

Let p be the shortest path from s to x. There must exist some z on p such that d[z] = d(s, z). Let z be the closest such vertex to x. We know $d[z] = d(s, z) \leq d(s, x) < d[x]$.

z must exist since, at the very least, s is part of the shortest path, and d[s] = d(s, s).

Weights are non-negative.

Claim 1 implies d(s, x) ≤ d[x] and we assumed that d[x] ≠ d(s, x).

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Weights are non-negative.

Claim 1 implies d(s, x) ≤ d[x] and we assumed that d[x] ≠ d(s, x).

Otherwise, z would be the vertex with minimum distance estimate.

Therefore, d[z] < d[x]. But this can't be the case. Why not? Since d[z] < d[x] and x is the vertex with minimum distance estimate, z must be already marked "done."

Dijkstra's Algorithm

Why does this work?

Claim 2: When a vertex v gets marked "done", d[v] = d(s, v).

Proof, cont.:

Since z is already marked "done," the edges out of z, including the edge (z, z') (where z' is also on p) have been relaxed by the algorithm

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$d[z'] \leq d(s, z) + w(z, z') = d(s, z')$ since z is on the shortest path from s to z' and the distance estimate of z' must be correct.

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Since z is already marked "done," the edges out of z, including the edge (z, z') (where z' is also on p) have been relaxed by the algorithm i.e. d[z'] ≤ d(s, z) + w(z, z') = d(s, z') since z is on the shortest path from s to z' and the distance estimate of z' must be correct.

However, this contradicts z being the closest vertex on p to x satisfying d[z] = d(s, z). Thus, our assumption that d[z] < d[x] must be false, and it follows that d[x] = d(s, x). ■

Dijkstra’s Algorithm

Another wording of Claim 2

- When a vertex v gets marked “done”, d[v] must be d(s, v).

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- When a vertex v gets marked "done", d[v] must be d(s, v).
- By contradiction, assume there exists an x such that when it gets marked as "done," d[x] ≠ d(s, x).
- Consider the shortest path p from s to x.

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- By contradiction, assume there exists an x such that when it gets marked as "done," d[x] $\neq$ d(s, x).
- Consider the shortest path p from s to x.
- There must exist a vertex z closest to x on p for which d[z] = d(s, z). Notice, by our assumption that z $\neq$ x.

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- By contradiction, assume there exists an x such that when it gets marked as “done,” d[x] ≠ d(s, x).
- Consider the shortest path p from s to x.
- There must exist a vertex z closest to x on p for which d[z] = d(s, z). Notice, by our assumption that z ≠ x.
- But z cannot be the closest vertex to x on p; simply consider the next vertex z’ along the path.

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- By contradiction, assume there exists an x such that when it gets marked as “done,” d[x] ≠ d(s, x).
- Consider the shortest path p from s to x.
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- But z cannot be the closest vertex to x on p; simply consider the next vertex z’ along the path.
    - Since the subpath of a shortest path is also a shortest path, and p is a shortest path from s to z’ to x, then the subpath s to z’ must also be a shortest path.

Dijkstra’s Algorithm

Another wording of Claim 2

- When a vertex v gets marked “done”, d[v] must be d(s, v).
- By contradiction, assume there exists an x such that when it gets marked as “done,” d[x] ≠ d(s, x).
- Consider the shortest path p from s to x.
- There must exist a vertex z closest to x on p for which d[z] = d(s, z). Notice, by our assumption that z ≠ x.
- But z cannot be the closest vertex to x on p; simply consider the next vertex z’ along the path.
    - Since the subpath of a shortest path is also a shortest path, and p is a shortest path from s to z’ to x, then the subpath s to z’ must also be a shortest path.
    - z must have been marked as “done” since it’s on the subpath to x and weights are non-negative.

Dijkstra's Algorithm

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- But z cannot be the closest vertex to x on p; simply consider the next vertex z' along the path.
    - Since the subpath of a shortest path is also a shortest path, and p is a shortest path from s to z' to x, then the subpath s to z' must also be a shortest path.
    - z must have been marked as "done" since it's on the subpath to x and weights are non-negative.
    - Before z was marked as "done", d[z'] was updated to d[z] + w(z, z'), which must equal d(s, z') since s to z to z' is a shortest path.

Dijkstra's Algorithm

Another wording of Claim 2

- When a vertex v gets marked "done", d[v] must be d(s, v).
- By contradiction, assume there exists an x such that when it gets marked as "done," d[x] $\neq$ d(s, x).
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- There must exist a vertex z closest to x on p for which d[z] = d(s, z). Notice, by our assumption that z $\neq$ x.
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  - Before z was marked as "done", d[z'] was updated to d[z] + w(z, z'), which must equal d(s, z') since s to z to z' is a shortest path.
- **Thus, contradiction!**

Bellman-Ford

Bellman-Ford Algorithm

- Dijkstra's algorithm solves the single-source shortest path problem in weighted graphs.
  - Sometimes it works on graphs with negative edge weights, but sometimes it doesn't work.
- Bellman-Ford also solves the SSSP problem in weighted graphs.
  - Always works on graphs with negative edge weights (when a solution exists).

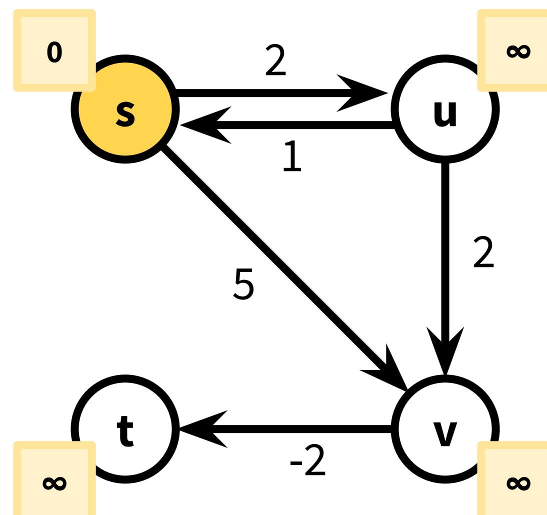
Bellman-Ford Algorithm

We maintain a list $d^{(k)}$ of length n for each $k = 0, 1, \ldots, |V|-1$.

$d^{(k)}[b]$ is the cost of the shortest path from s to b with at most k edges.

We know k = 0
i.e. shortest
paths to each
vertex with at
most 0 edges
in it.

	s	**u**	**v**	**t**
$\mathbf{d^{(0)}}$	**0**	**∞**	**∞**	**∞**
$\mathbf{d^{(1)}}$				
$\mathbf{d^{(2)}}$				
$\mathbf{d^{(3)}}$				



Bellman-Ford Algorithm

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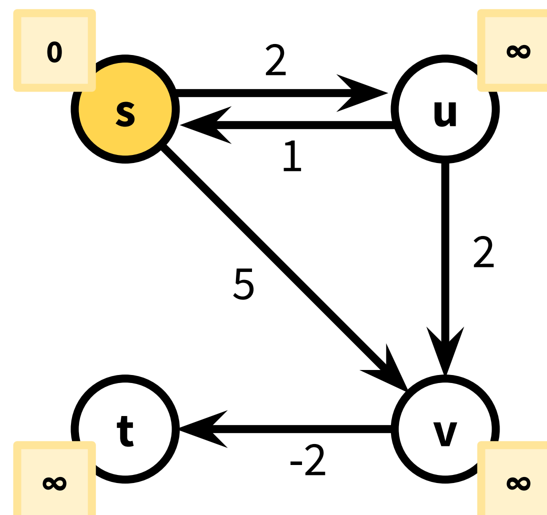
$d^{(k)}[b]$ is the cost of the shortest path from s to b with at most k edges.

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$				
$d^{(2)}$				
$d^{(3)}$				

We will use table $d^{(0)}$ to fill in $d^{(1)}$. More generally, we will use table $d^{(k-1)}$ to fill in $d^{(k)}$.

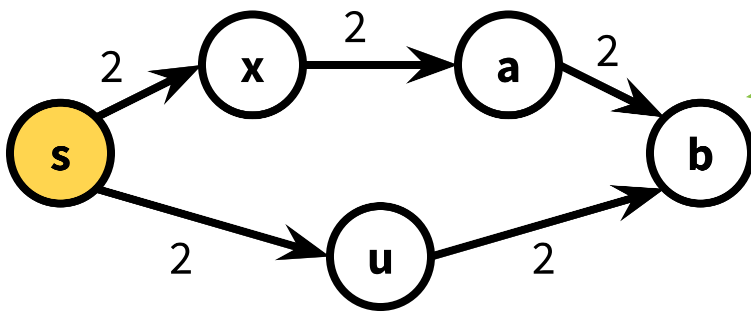


Bellman-Ford Algorithm

How do we use $d^{(k-1)}$ to fill in $d^{(k)}[b]$?

Recall $d^{(k)}[b]$ is the cost of the shortest path from s to b with at most k edges.

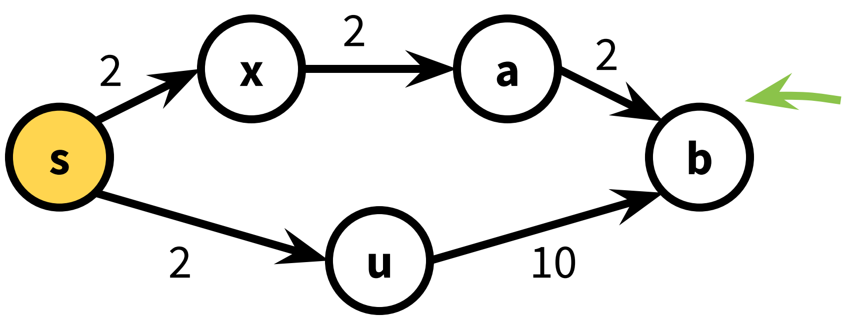
Case 1: the shortest path from s to b with at most k edges actually has at most k - 1 edges.



Suppose k = 3.

$d^{(k)}[b] = d^{(k-1)}[b]$ i.e. the shortest path of at most k - 1 edges is at least as short as any path of at most k edges.

Case 2: the shortest path from s to b with at most k edges really has k edges.



Suppose k = 3.

$d^{(k)}[b] = \min_a\{d^{(k-1)}[a] + w(a, b)\}$ i.e. the shortest path of at most k edges is shorter than any path of at most k - 1 edges.

Bellman-Ford Algorithm

```
def bellman_ford(G):
  d^(k) = [] for k = 0 to |V|-1
  d^(0)[v] = ∞ for all v ≠ s
  d^(0)[s] = 0
  for k = 1 to |V|-1:
    for b in V:
      d^(k)[b] = min{d^(k-1)[b], min_a{d^(k-1)[a] + w(a,b)} }
  return d^(|V|-1)
```

Runtime: $O(|V||E|)$

Bellman-Ford Algorithm

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This is a simplification to make the pseudocode nice. In reality, we'd only keep two of them at a time.

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Minimum over all a such that (a, b) ∈ E.

Runtime: $O(|V||E|)$

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  return d^(|V|-1)
```

$$d^{(k)}[b] = \min\{\underbrace{d^{(k-1)}[b]}_{\text{Case 1}},\ \min_a\{d^{(k-1)}[a] + w(a,b)\}\ \}$$

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Minimum over all a such that (a, b) ∈ E.

Runtime: O(|V||E|)

Bellman-Ford Algorithm

```
def bellman_ford(G):
  d(k) = [] for k = 0 to |V|-1
  d(0)[v] = ∞ for all v ≠ s
  d(0)[s] = 0
  for k = 1 to |V|-1:
    for b in V:
      d(k)[b] = min{d(k-1)[b], min_a{d(k-1)[a] + w(a,b)} }
  return d(|V|-1)
```

This is a simplification to make the pseudocode nice. In reality, we'd only keep two of them at a time.

Minimum over all a such that (a, b) ∈ E.

Case 1: $d^{(k-1)}[b]$

Case 2: $\min_a\{d^{(k-1)}[a] + w(a,b)\}$

Runtime: $O(|V||E|)$

Bellman-Ford Algorithm

```
def bellman_ford(G):
  d^(k) = [] for k = 0 to |V|-1
  d^(0)[v] = ∞ for all v ≠ s
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```

$$d^{(k)}[b] = \min\{\underbrace{d^{(k-1)}[b]}_{\text{Case 1}},\ \underbrace{\min_a\{d^{(k-1)}[a] + w(a,b)\}}_{\text{Case 2}}\}$$

This is a simplification to make the pseudocode nice. In reality, we'd only keep two of them at a time.

Minimum over all a such that (a, b) ∈ E.

Case 1

Case 2

Runtime: $O(|V||E|)$

Slower than Dijkstra's $O(|E| + |V|\log(|V|))$

Bellman-Ford Algorithm

We maintain a list $d^{(k)}$ of length n for each k = 0, 1, …, |V|-1.

```
for k = 1 to |V|-1:
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```

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$				
$d^{(2)}$				
$d^{(3)}$				

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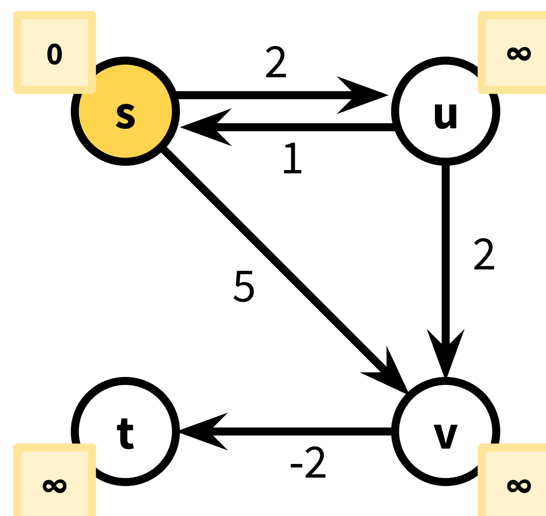

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```

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$				
$d^{(3)}$				

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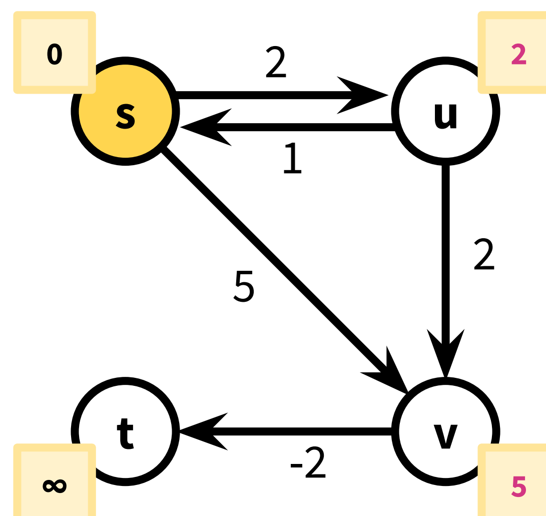

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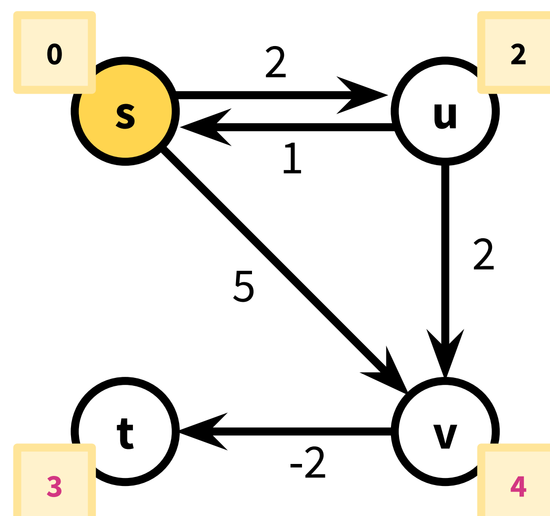
```

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
$d^{(3)}$				



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$d^{(0)}$	0	∞	∞	∞
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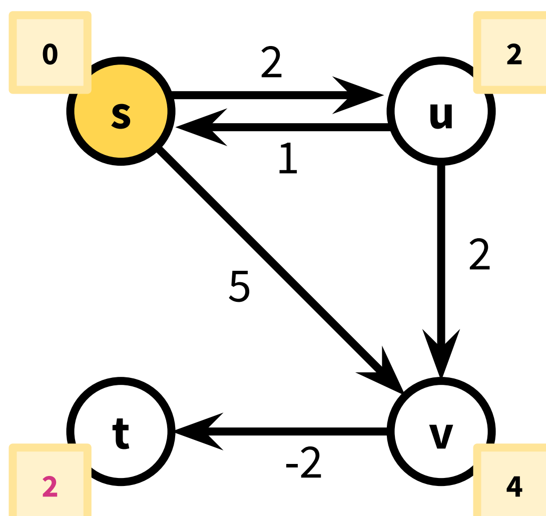

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Recall $d^{(k)}[b]$ is the cost of the shortest path from s to b with at most k edges.

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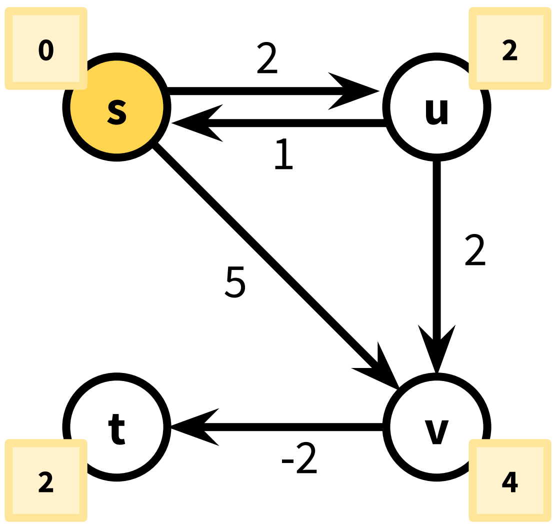

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The shortest path from s to t with 1 edge has cost ∞ (no path exists).

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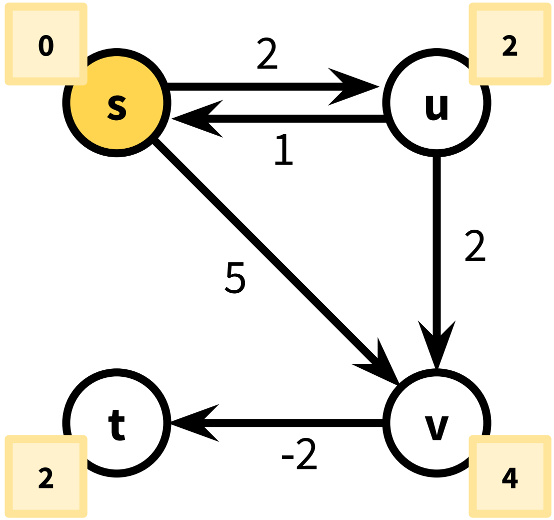

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The shortest path from s to t with 2 edges has cost **3** (s-v-t).

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
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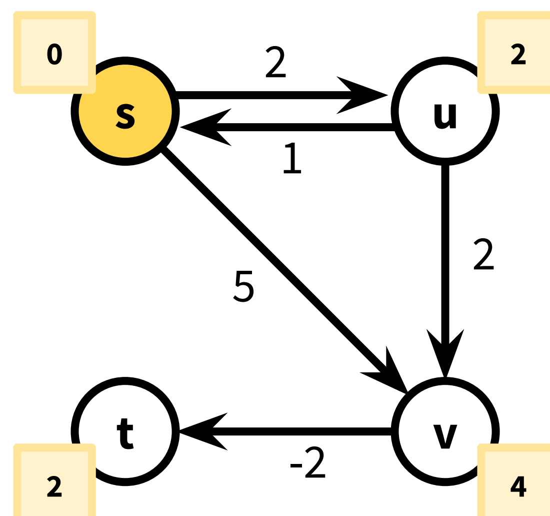

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The shortest path from s to t with 2 edges has cost **3** (s-v-t).

The shortest path from s to t with 3 edges has cost **2** (s-u-v-t).

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
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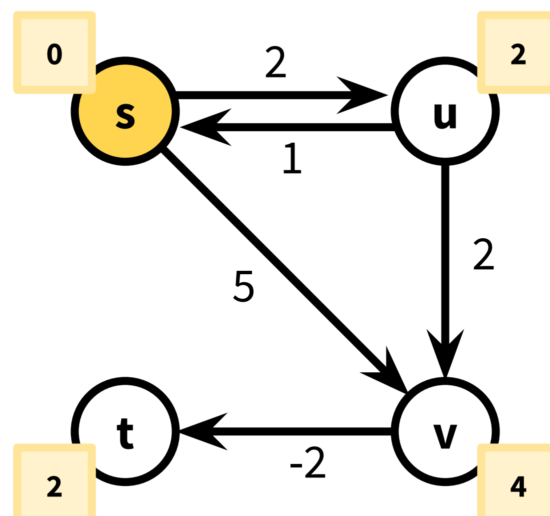

BF Proof of Correctness

We need to prove our main argument.

- $d^{(|V|-1)}[b]$ is the cost of the shortest path from s to b with at most |V|-1 edges.

BF Proof of Correctness

Lemma: $d^{(|V|-1)}[b]$ is the cost of the shortest path from s to b with at most |V|-1 edges.

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Proof: We proceed by induction on k, the number of iterations completed by the algorithm.

For our base case, at the start of iteration k = 1, the shortest path from s to s with 0 edges has cost 0. The path from s to all vertices $v \neq s$ contains at least 1 edge; there doesn't exist a path from s to v with 0 edges, and this path costs ∞. Therefore, $d^{(0)}$ is correct.

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For our inductive step, assume that at the start of iteration k, $d^{(k-1)}[b]$ is the cost of the shortest path from s to b with at most k - 1 edges. We consider two cases:

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Case 1: $d^{(k-1)}[b] < \min_a\{d^{(k-1)}[a] + w(a, b)\}$. This corresponds to the case in which the shortest path contains fewer than k edges. Then our algorithm correctly sets $d^{(k)}[b] = d^{(k-1)}[b]$.

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Case 2: $d^{(k-1)}[b] \geq \min_a\{d^{(k-1)}[a] + w(a, b)\}$. This corresponds to the case in which the shortest path contains exactly k edges. Then our algorithm correctly sets $d^{(k)}[b] = \min_a\{d^{(k-1)}[a] + w(a, b)\}$, which minimizes the sum of the shortest path with at most k-1 edges to an in-neighbor of b and the weight from a to b.

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At the start of iteration k = |V|, the algorithm terminates and $d^{(|V|-1)}$ is correct.

BF Proof of Correctness

We need to prove our main argument.

$d^{(|V|-1)}[b]$ is the cost of the shortest path from s to b with at most |V|-1 edges. 👌

What else to do? 🤔

BF Proof of Correctness

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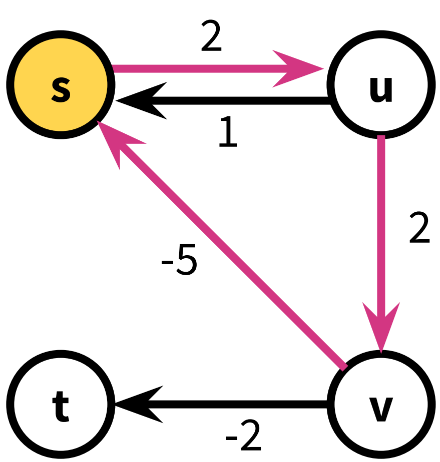
What else to do? 🤔

We still need to prove that this argument implies `bellman_ford` is correct i.e. $d^{(|V|-1)}[a]$ = distance(s, a).

To show this, we'll prove that the shortest path with at most |V|-1 edges is the shortest path with any number of edges (if a shortest path exists).

If the graph has a negative cycle, a shortest path might not exist!

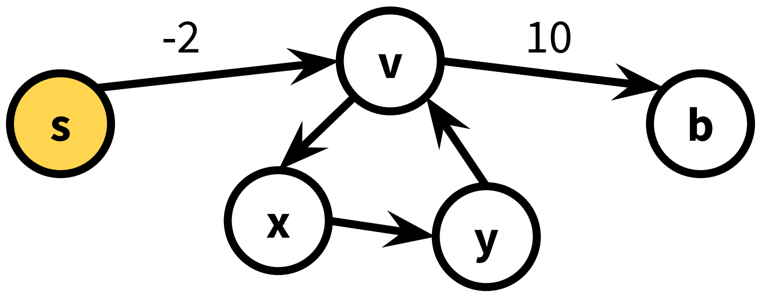


BF Proof of Correctness

But if there's no negative cycle.

There's always a simple shortest path.

A simple path has
no cycles.



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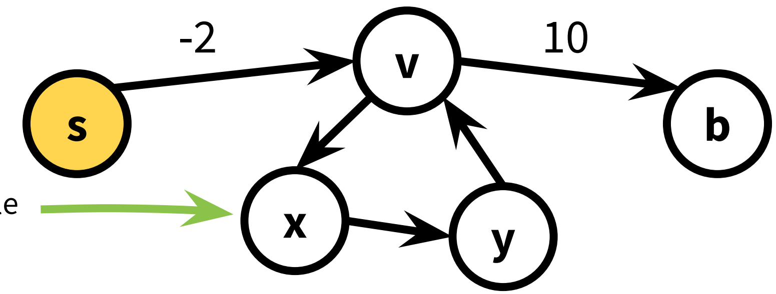

How do we know this cycle doesn't help? 🤔

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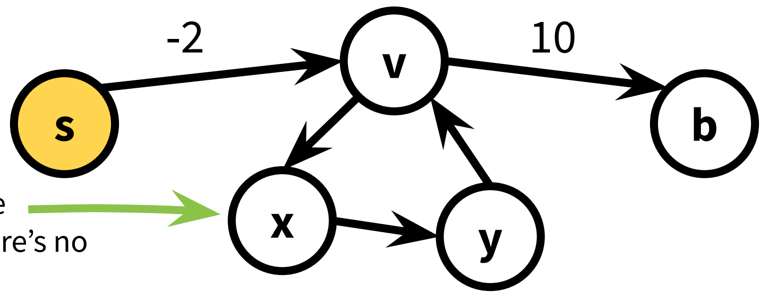

How do we know this cycle
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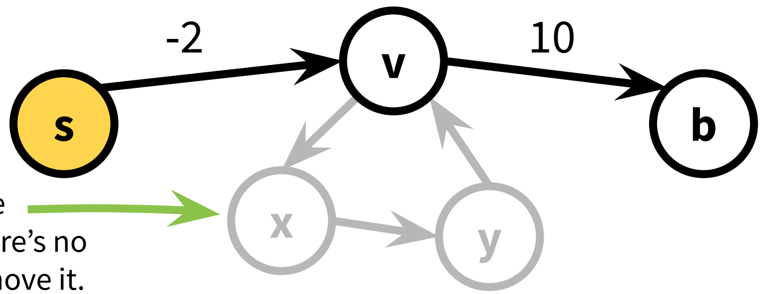


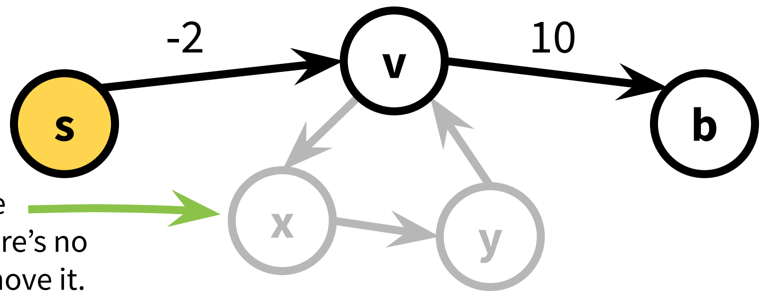
How do we know this cycle doesn't help? 🤔 Since there's no negative cycles! So we remove it.

BF Proof of Correctness

But if there’s no negative cycle.

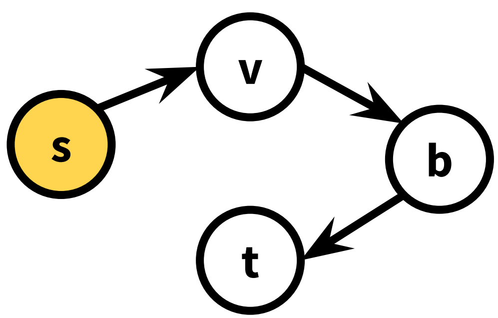
There’s always a simple shortest path.

A simple path has no cycles.



How do we know this cycle doesn’t help? 🤔 Since there’s no negative cycles! So we remove it.

A simple path in a graph with |V| vertices has at most |V|-1 edges in it.

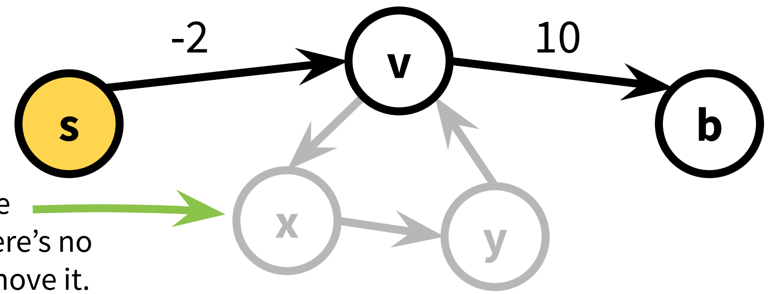


BF Proof of Correctness

But if there’s no negative cycle.

There’s always a simple shortest path.

A simple path has no cycles.



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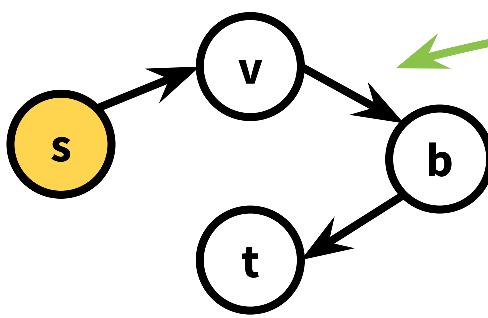

We can’t add another edge to this s-t path without making a cycle (an edge from s to b wouldn’t be along the path).

BF Proof of Correctness

Theorem: `bellman_ford` is correct as long as the graph has no negative cycles.

Proof:

By our lemma, $d^{(|V|-1)}[b]$ contains the cost of the shortest path from s to b with at most |V|-1 edges. If there are no negative cycles, then the shortest path must be simple, and all simple paths have at most |V|-1 edges. Therefore, the value the algorithm returns, $d^{(|V|-1)}[b]$, is also the cost of the shortest path from s to b with any number of edges.

Bellman-Ford Algorithm

Bellman-Ford gets used in practice.

- e.g. Routing Information Protocol (RIP) uses it. Each router keeps a table of distances to every other router. Periodically, we do a Bellman-Ford update.

Dynamic Programming

Bellman-Ford is an example of **dynamic programming**!

Dynamic programming is an algorithm design paradigm.

- Often it's used to solve optimization problems e.g. **shortest** path.