Randomized Algorithms I

Summer 2018 • Lecture 07/10

Announcements

- Homework 1
  - **`hw1.zip`** is due today!
  - We'll grade them by Sunday night.
- Homework 2
  - **`hw2.zip`** is live!
  - It's due next Tuesday 7/17, but start early!
- Tutorial 3
  - Friday, 7/13 3:30-4:50 p.m. in STLC 115.
  - RSVP, so I can print enough copies for everyone: https://goo.gl/forms/NRPZi87GS9v7meJa2 (requires Stanford email).

Course Overview

- Algorithmic Analysis
- Divide and Conquer
- **Randomized Algorithms**
- Tree Algorithms
- Graph Algorithms
- Dynamic Programming
- Greedy Algorithms
- Advanced Algorithms

Today's Outline

- Randomized Algorithms I
  - Comparison-based sorting lower bounds
  - *Algorithms: Randomized select and randomized quicksort*
  - Reading: CLRS: 5, 7

Randomized Algorithms

- A randomized algorithm is an algorithm that incorporates randomness as part of its operation.
- Often aim for properties like:
  - Good average-case behavior
  - Getting exact answers with high probability
  - Getting answers that are close to the right answer
- Monte Carlo vs. Las Vegas
  - Las Vegas algorithms guarantee correctness, but not runtime. **We'll focus on these algorithms today.**
  - Monte Carlo algorithms guarantee runtime, but not correctness. **We'll revisit this next week when we see Karger's algorithm.**

Bogosort

```
def bogosort(A):
  # Randomly permutes A until it's sorted
  while True:
    random.shuffle(A)
    sorted = True
    for i in range(len(A)-1):
      if A[i] > A[i+1]:
        sorted = False
    if sorted:
      return A
```

Worst-case runtime $O(\infty)$

Bogosort

- Unlike the deterministic algorithms that we’ve studied so far, when analyzing Las Vegas randomized algorithms, we’re interested in:
  - What’s the average-case runtime of the algorithm?
  - How does this compare to the worst-case runtime of the algorithm?

Bogosort

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      if A[i] > A[i+1]:
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    if sorted:
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```

Worst-case $O(\infty)$

Think of this as the adversary chooses the randomness.

Expected $O(n \cdot n!)$

Pr[randomly list sorted] = 1/n!
By the expectation of geometric distribution, we expect to permute **A** n! times before it's sorted. Each permutation requires $O(n)$-time.

Quicksort

Summer 2018 • Lecture 07/10

Quicksort

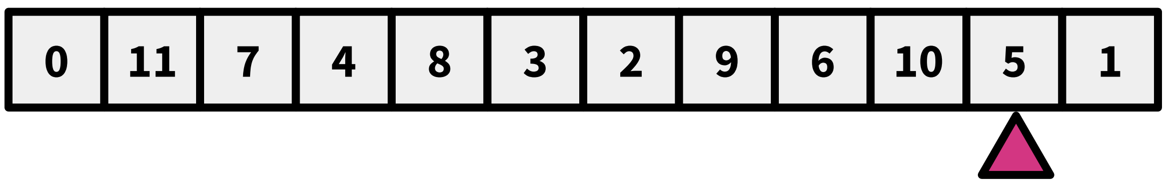
Our next example of a randomized algorithm is `quicksort`.

It's pretty smart.

It behaves as follows:

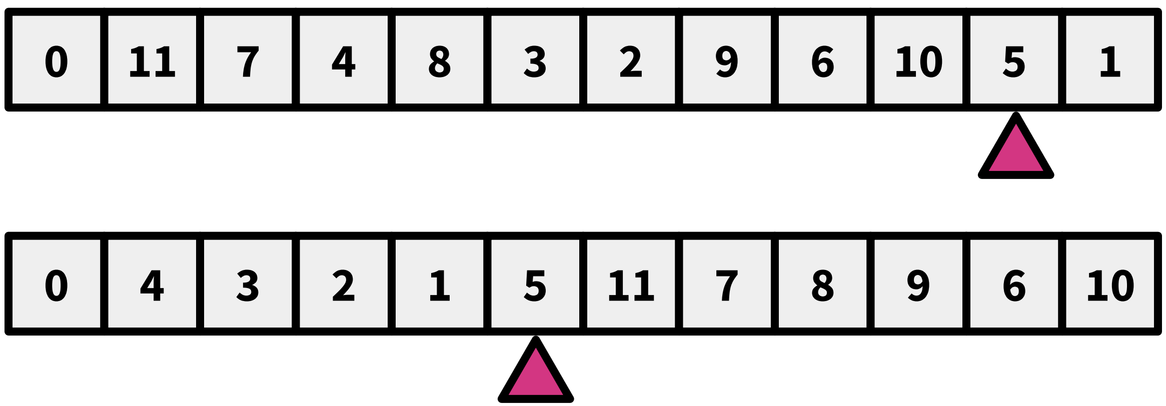
- If the list has 0 or 1 elements it's sorted.
- Otherwise, choose a pivot and partition around it.
- Recursively apply `quicksort` to the sublists to the left and right of the pivot.

Quicksort



Choose a pivot.

Quicksort



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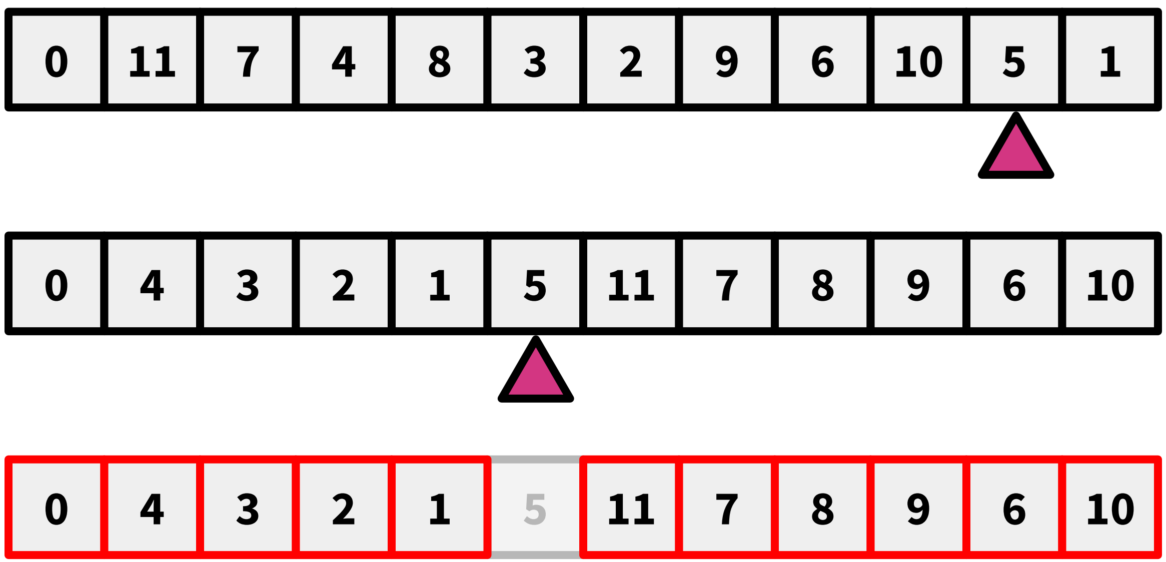
Partition around it.

Quicksort

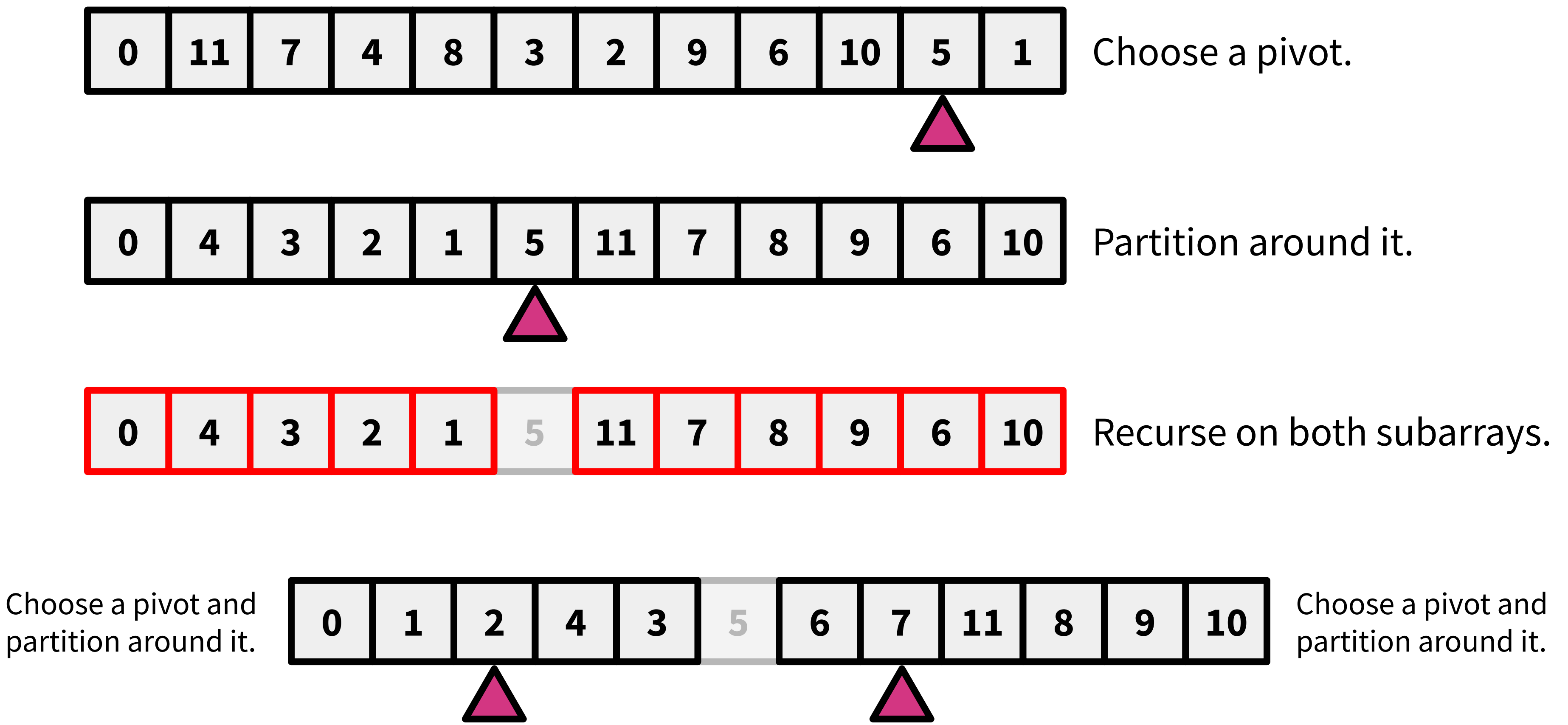


Choose a pivot.

Partition around it.

Recurse on both subarrays.

Quicksort


0 11 7 4 8 3 2 9 6 10 5 1
Choose a pivot.
0 4 3 2 1 5 11 7 8 9 6 10
Partition around it.
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Recurse on both subarrays.
Choose a pivot and partition around it.
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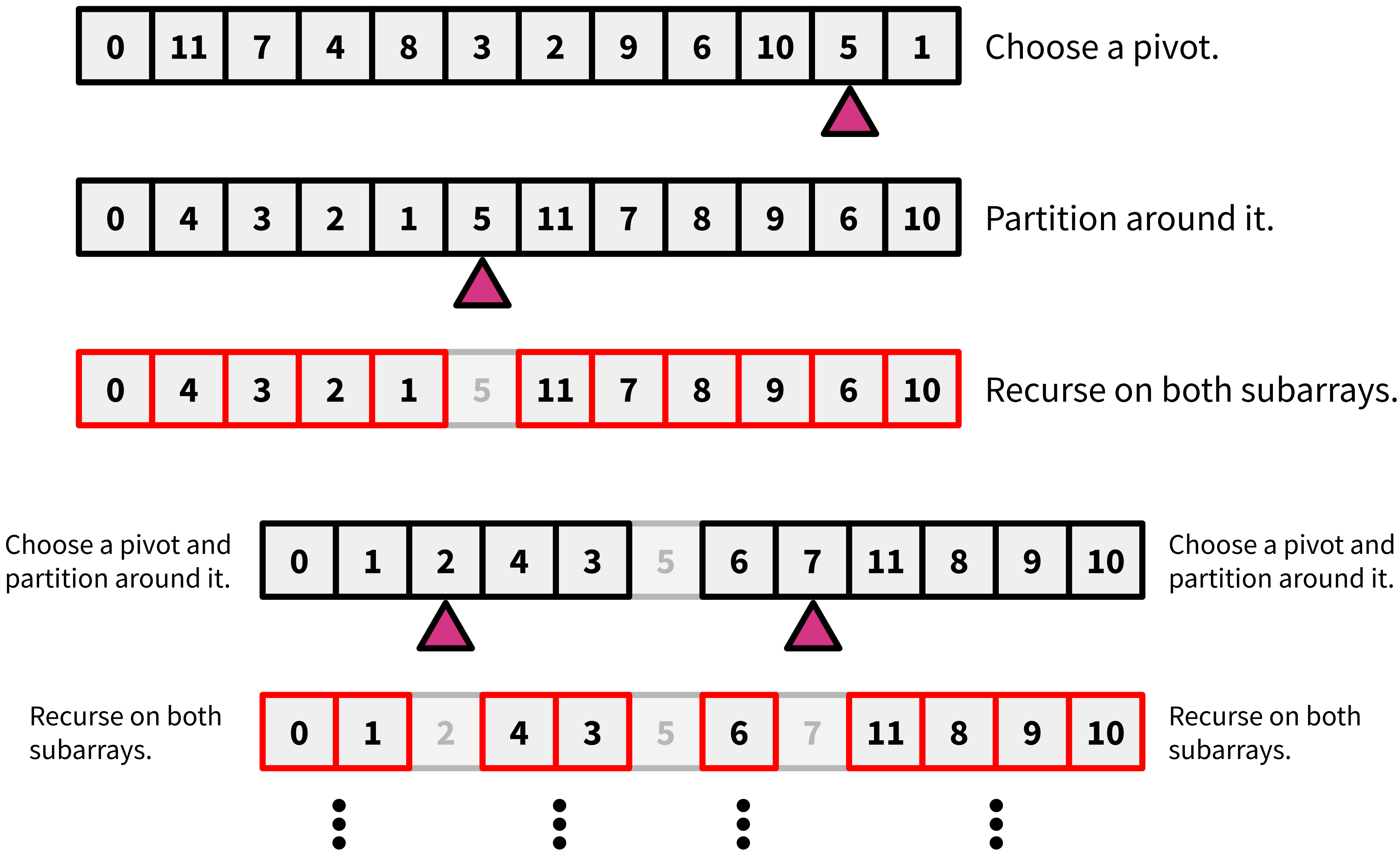
Quicksort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Recurse on both subarrays.

Quicksort

```
def quicksort(A):
  if len(A) <= 1:
    return
  pivot = A[0]
  left, right = partition_about_pivot(A, pivot)
  quicksort(left)
  quicksort(right)
```

Worst-case runtime $\Theta(n^2)$

Randomized Quicksort

```
def randomized_quicksort(A):
  if len(A) <= 1:
    return
  pivot = random.choice(A)
  left, right = partition_about_pivot(A, pivot)
  quicksort(left)
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```

<u>Worst-case</u> $\Theta(n^2)$

<u>Expected</u> $\Theta(n \log(n))$

Think of this as the adversary chooses the randomness.

Initial Observations

There’s a really good case, in which partition always picks the median element as the pivot.

- What’s the recurrence relation? 🤔

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What's the recurrence relation? 🤔

$\mathbf{T(0) = T(1) = \Theta(1)}$

$\mathbf{T(n) = 2T(\lfloor n/2 \rfloor) + \Theta(n)}$ Runtime of `partition`.

$\mathbf{= O(n \log n)}$ Master method a = 1, b = 2, d = 1.

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T(0) = T(1) = Θ(1)

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= **`O(nlogn)`** Master method a = 1, b = 2, d = 1.

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What's the recurrence relation? 🤔

$\mathbf{T(0) = T(1) = \Theta(1)}$

$\mathbf{T(n) = T(n\text{-}1) + \Theta(n)}$

$= O(n^2)$ Draw the recursion tree.

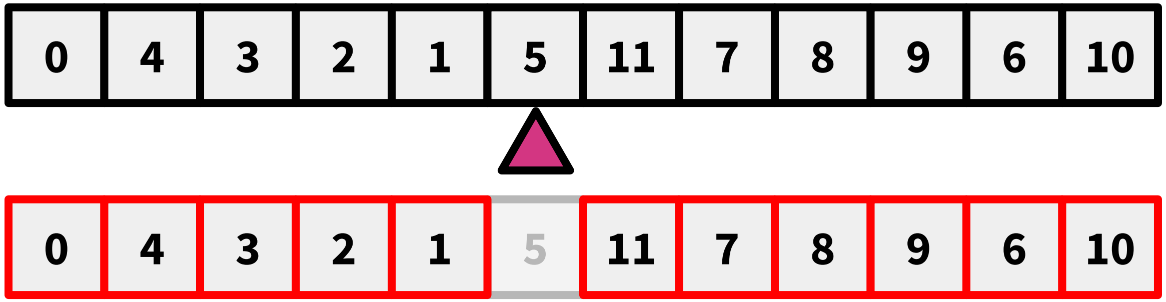
Expected Runtime of Randomized Quicksort

How do we know the expected runtime of `quicksort` is **`O(nlogn)`**?

To answer this question, let's count the number of times two elements get compared!

- This might not seem intuitive at first, but it's an approach you can use to analyze runtime of randomized algorithms.

Expected Runtime of Randomized Quicksort



Partition around it.

Recurse on both subarrays.

All elements were compared to **5** in the top recursive call, and then never again.

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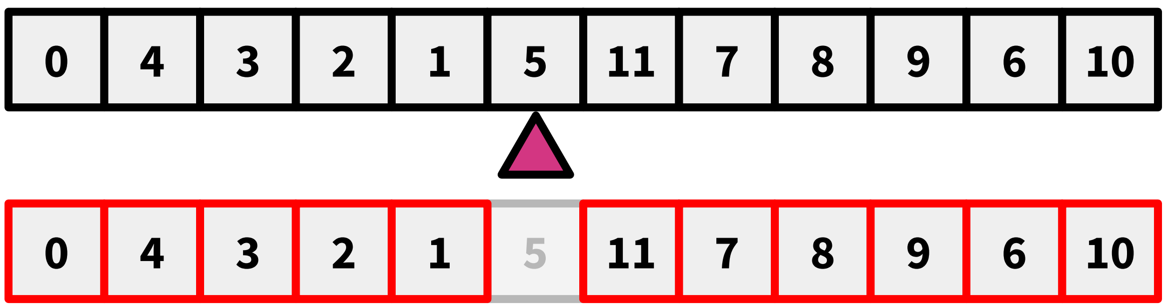

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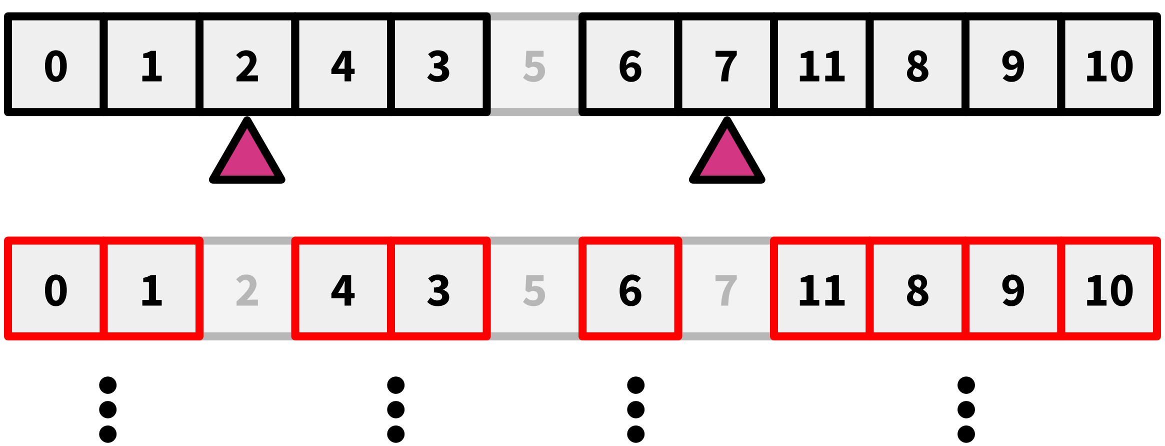
Choose a pivot and partition around it.



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Only the elements to the left of **5**, the original pivot, were compared to **2** in the left recursive call; only the elements to the right of the original pivot were compared to **7** in the right recursive call.

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Each pair of elements **a** and **b** is compared 0 or 1 times.
Which is it?

Expected Runtime of Randomized Quicksort

Each pair of elements **a** and **b** is compared 0 or 1 times. Which is it?

Let $X_{a,b}$ be random variable that depends on choice of pivots, such that:

$$X_{a,b} = \begin{cases} 1 & \text{if } \mathbf{a} \text{ and } \mathbf{b} \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$$

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In the previous example, $X_{3,5} = 1$ since **3** and **5** are compared but $X_{4,6} = 0$ since **4** and **6** are not compared.

Notice that these assignments of $X_{3,5}$ and $X_{4,6}$ both depended on our random choice of pivot **5**.

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The total number of comparisons?

We need to figure out this value!

$$E\left[\sum_{a=0}^{n-2}\sum_{b=a+1}^{n-1} X_{a,b}\right] = \sum_{a=0}^{n-2}\sum_{b=a+1}^{n-1} E[X_{a,b}]$$

By linearity of expectation

Expected Runtime of Randomized Quicksort

So what's $E[X_{a,b}]$?

By definition of expectation

$$E[X_{a,b}] = P(X_{a,b} = 1) \cdot 1 + P(X_{a,b} = 0) \cdot 0 = P(X_{a,b} = 1)$$

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To determine $P(X_{a,b} = 1)$, consider an example …

0	**6**	**8**	**11**	**1**	**10**	**2**	**7**	**3**	**9**	**4**	**5**

$P(X_{a,b} = 1)$ is the probability that **a** and **b** are compared.

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Expected Runtime of Randomized Quicksort

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Why doesn't this depend on the length of the overall list, 12? Consider an analogy: let's say you're playing the game: roll a die; if it's 1 you win, if it's 2 you lose, else roll again. You will win with probability 1/2, regardless of how many sides of the die!

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So, we can see that $\mathbf{P(X_{a,b} = 1) = 2 / (b - a + 1)}$

Expected Runtime of Randomized Quicksort

This gives that $E[X_{a,b}] = P(X_{a,b} = 1) = 2 / (b - a + 1)$. Thus,

$$\sum_{a=0}^{n-2} \sum_{b=a+1}^{n-1} E[X_{a,b}] = \sum_{a=0}^{n-2} \sum_{b=a+1}^{n-1} 2 / (b - a + 1)$$

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$$= \sum_{a=0}^{n-2} \sum_{c=1}^{n-a-1} 2 / (c + 1)$$

$$\leq \sum_{a=0}^{n-1} \sum_{c=1}^{n-1} 2 / (c + 1)$$

This is the hard part, and it's a useful skill.

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$$= 2n \sum_{c=1}^{n-1} 1 / (c+1)$$

Expected Runtime of Randomized Quicksort

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Harmonic series

Expected Runtime of Randomized Quicksort

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Harmonic series

$$= O(n \log n)$$

Randomized Quicksort

```
def randomized_quicksort(A):
  if len(A) <= 1:
    return
  pivot = random.choice(A)
  left, right = partition_about_pivot(A, pivot)
  quicksort(left)
  quicksort(right)
```

Worst-case $\Theta(n^2)$

Think of this as the adversary chooses the randomness.

Expected $\Theta(n \log(n))$

We can lower-bound it with the sorting lower bound!

Quicksort vs. Randomized Quicksort

Quicksort has worst-case **inputs**, producing the algorithm's worst-case runtimes.

Randomized quicksort does not have worst-case **inputs**. It's worst-case runtimes result from being unlucky.

Better Quicksort?

Any ideas to make `randomized_quicksort` better? It still has worst-case $O(n^2)$-time.

- Recall that worst-case for randomized algorithms allows the adversary to control the randomness.

Better Quicksort?

Any ideas to make `randomized_quicksort` better? It still has worst-case $O(n^2)$-time.

- Recall that worst-case for randomized algorithms allows the adversary to control the randomness.
- We can borrow ideas from `select` and instead partition around the median of medians. It might also be a good idea to partition about the actual median or the median of three.

Randomized Selection

Randomized Selection

Our next example of a randomized algorithm is `randomized_select`.

You’ve actually seen it before.

Randomized Selection

```
def select randomized_select(A, k, c=100):
  if len(A) <= c:
    return naive_select(A, k)
  pivot = random.choice(A)
  left, right = partition_about_pivot(A, pivot)
  if len(left) == k:
    # The pivot is the kth smallest element!
    return pivot
  elif len(left) > k:
    # The kth smallest element is left of the pivot
    return select(left, k, c)
  else:
    # The kth smallest element is right of the pivot
    return select(right, k-len(left)-1, c)
```

<u>"Worst-case"</u> runtime $\Theta(n^2)$

I didn't give you the whole story...

Randomized Selection

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Worst-case $\Theta(n^2)$

Expected $\Theta(n)$

Expected Runtime of Randomized Selection

How do we know the expected runtime of `quickselect` is **`O(n)`**?

Let's refer to how we bounded the worst-case runtime for `select` with `median_of_medians`!

Expected Runtime of Randomized Selection

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- `select` with `median_of_medians` upper-bounds the length of the list on which it recurses with 7n/10+c.

Expected Runtime of Randomized Selection

How do we know the expected runtime of `quickselect` is `O(n)`?

Let's refer to how we bounded the worst-case runtime for `select` with `median_of_medians`!

- `select` with `median_of_medians` upper-bounds the length of the list on which it recurses with 7n/10+c.
- Here, let's estimate the expected runtime of shrinking the length of the list to, say, 75% of the original length.

Expected Runtime of Randomized Selection

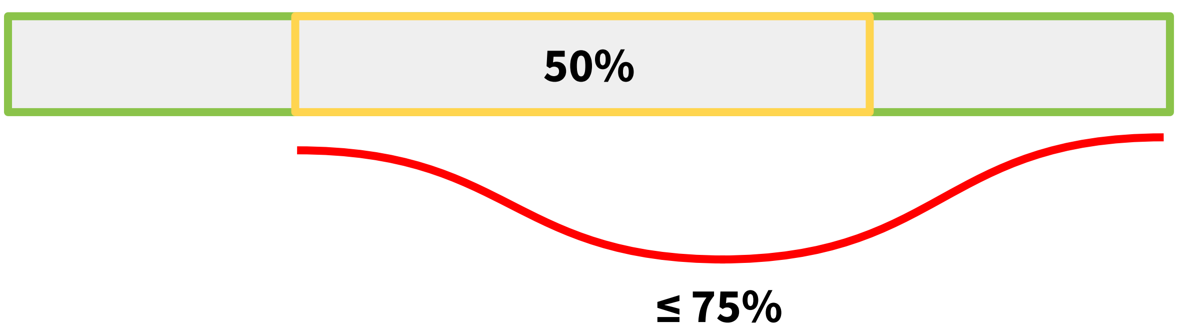
Let's define one "phase" of `randomized_select` to be when it decreases the length of the input list to 75% of the original length or less.

Expected Runtime of Randomized Selection

Let's define one "phase" of `randomized_select` to be when it decreases the length of the input list to 75% of the original length or less.

Why 75%?

Selecting a pivot in the middle 50% of all list values guarantees that the length of the input list decreases to below 75%.



A phase ends as soon as `randomized_select` picks a pivot in the middle 50% of values.

Expected Runtime of Randomized Selection

Why at most?

If we number the phases 0, 1, 2, …

in phase k, the length of the list is at most $n(3/4)^k$ and the last phase is numbered $\lceil \log_{4/3} n \rceil$.

Expected Runtime of Randomized Selection

Why at most?

If we number the phases 0, 1, 2, …

in phase k, the length of the list is at most $n(3/4)^k$ and the last phase is numbered $\lceil \log_{4/3} n \rceil$.

Let X_k be a random variable equal to the number of recursive calls in phase k, and W be a random variable equal to the runtime.

Expected Runtime of Randomized Selection

Why at most?

If we number the phases 0, 1, 2, …

in phase k, the length of the list is at most $n(3/4)^k$ and the last phase is numbered $\lceil \log_{4/3} n \rceil$.

Let X_k be a random variable equal to the number of recursive calls in phase k, and W be a random variable equal to the runtime.

The runtime of phase k is at most $X_k \cdot cn(3/4)^k$, so:

$$W \leq \sum_{k=0}^{\lceil \log_{4/3} n \rceil} X_k \cdot cn(3/4)^k = cn \sum_{k=0}^{\lceil \log_{4/3} n \rceil} X_k \cdot (3/4)^k$$

Expected Runtime of Randomized Selection

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And the expected runtime must be:

$$E[W] \leq E[cn \sum_{k=0}^{\lceil \log_{4/3} n \rceil} X_k \cdot (3/4)^k]$$

Expected Runtime of Randomized Selection

Simplifying the expression gives …

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$$= cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} E[X_k](3/4)^k$$

The important part: How might we solve for $E[X_k]$?

Expected Runtime of Randomized Selection

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- Recall X_k represents a random variable equal to the number of recursive calls in phase k.

Expected Runtime of Randomized Selection

How might we solve for $E[X_k]$?

Recall X_k represents a random variable equal to the number of recursive calls in phase k.

Since all pivot choices are independent, we have a geometric random variable with probability of success of $\geq 1/2$ (since a phase ends as soon as `randomized_select` picks a pivot in the middle 50% of values).

The first trial, probability of success is 1/2. If it fails, then the probability of success will be > 1/2 thereafter.

Expected Runtime of Randomized Selection

How might we solve for $E[X_k]$?

Recall X_k represents a random variable equal to the number of recursive calls in phase k.

Since all pivot choices are independent, we have a geometric random variable with probability of success of ≥1/2 (since a phase ends as soon as `randomized_select` picks a pivot in the middle 50% of values).

$E[X_k] \le 1/(1/2) = 2$.

Expected Runtime of Randomized Selection

Simplifying the expression gives …

$$E[W] \le cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} E[X_k](3/4)^k$$

Expected Runtime of Randomized Selection

Simplifying the expression gives …

$$E[W] \leq cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} E[X_k](3/4)^k$$

$$\leq cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} 2(3/4)^k$$

Expected Runtime of Randomized Selection

Simplifying the expression gives …

$$E[W] \le cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} E[X_k](3/4)^k$$

$$\le cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} 2(3/4)^k$$

$$\le cn \cdot \sum_{k=0}^{\infty} 2(3/4)^k$$

Expected Runtime of Randomized Selection

Simplifying the expression gives …

$$
\begin{aligned}
E[W] &\leq cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} E[X_k](3/4)^k \\
&\leq cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} 2(3/4)^k \\
&\leq cn \cdot \sum_{k=0}^{\infty} 2(3/4)^k \\
&= 8cn \\
&= O(n)
\end{aligned}
$$

This is the hard part, and it's a useful skill.

By the sum of infinite geometric series.

Randomized Selection

```
def randomized_select(A, k, c=100):
  if len(A) <= c:
    return naive_select(A, k)
  pivot = random.choice(A)
  left, right = partition_about_pivot(A, pivot)
  if len(left) == k:
    # The pivot is the kth smallest element!
    return pivot
  elif len(left) > k:
    # The kth smallest element is left of the pivot
    return select(left, k, c)
  else:
    # The kth smallest element is right of the pivot
    return select(right, k-len(left)-1, c)
```

Worst-case $\Theta(n^2)$ **Expected** $\Theta(n)$

Select vs. Randomized Select

Select has worst-case **inputs**, producing the algorithm's worst-case runtimes.

Randomized select does not have worst-case **inputs**. It's worst-case runtimes result from being unlucky.

3 min break

Majority Element

Majority Element

The **majority element problem** is the following: Given an input list A, find the element that occurs at least $\lfloor n/2 \rfloor + 1$ times, provided one exists.

Input accepts a list **A** and its length n.

Try to solve the same problem, but return NIL when one doesn't exist.

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Additionally, suppose we can only perform the `equals` operation on the list, which accepts two values **a** and **b** and returns True if **a** equals **b**; otherwise returns False.

1	0	3	1	1	5	2	1	1	1	4	1

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`equals(A[0], A[3])` returns True

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`equals(A[0], 1)` returns True

Majority Element

We will visit two solutions to this problem.

The first will be a divide-and-conquer algorithm; the second will be a randomized algorithm.

Majority Element

The divide-and-conquer approach …

- Recursive calls should return the majority element of a list's sublists.
- How might we merge two majority elements into a single majority element for this list? 🤔

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0	**1**	**2**	**3**	**4**	**5**	**2**	**2**	**2**	**2**	**2**	**2**

`m_left = 5`   `m_right = 2`

Majority Element

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0	1	2	3	4	5	2	2	2	2	2	2

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Key insight: The majority element of entire list (if it exists) must be the same as the majority element as one of the sublists (otherwise it would occur at most $\lfloor n/2 \rfloor$ times).

Majority Element

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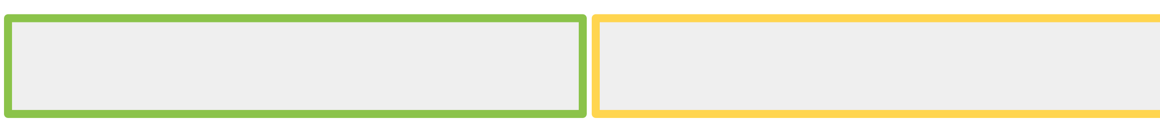
Key insight: The majority element of entire list (if it exists) must be the same as the majority element as one of the sublists (otherwise it would occur at most $\lfloor n/2 \rfloor$ times). To convince yourself of this case, consider if it's possible for recursive calls to return these sublists if the majority element of the entire list isn't 5 or 2.



`m_left = 5`   `m_right = 2`

Majority Element

```
def majority_element(A):
  # divide and conquer
  n = len(A), mid = (n-1)/2
  if n <= 1:
    return A[0]
  m1 = majority_element(A[:mid])
  m2 = majority_element(A[mid+1:])
  count = 0
  for a in A:
    if equals(m1, a): count += 1
  if count > n/2+1: return m1
  else: return m2
```

Runtime: `O(nlogn)`

Count the number of calls to equals.

Recurrence: **$T(n) = 2T(n/2) + O(n)$**

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Inductive step Suppose the algorithm works on input lists of length $n/2 = 2^{i-1}$. Now consider an input of length $n = 2^i$.

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Inductive step Suppose the algorithm works on input lists of length $n/2 = 2^{i-1}$. Now consider an input of length $n = 2^i$. The majority element of the entire array, if it exists, must be the majority element of at least one of **`A[:mid]`** or **`A[mid+1:]`**; otherwise it would occur at most $\lfloor n/2 \rfloor$ times.

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Conclusion Since the **`majority_element`** is called on the entire array, it can correctly find it, given that one exists.

Majority Element

The randomized approach …

- Think about low-hanging fruit: will an algorithm similar to `bogosort` work?

Majority Element

The randomized approach …

- Think about low-hanging fruit: will an algorithm similar to `bogosort` work?
  - Choose a random index from 1 to n.
  - Is the element at that index the majority element?

Majority Element

```
def majority_element(A):
  # randomized
  while True:
    guess = random.choice(A)
    count = 0
    for a in A:
      if equals(guess, a): count += 1
    if count > n/2+1: return guess
```

Runtime

Expected: 🤔 Worst-case: 🤔

Majority Element

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```

Runtime

Expected: $O(n)$ Worst-case: $O(\infty)$

Not all randomized algorithms have expected runtime O(n log n)!!! I don't want to see this everrr.

Expected Runtime of Majority Element

Provided there exists a majority element, this element must occur at least $\lfloor n/2 \rfloor + 1$ times.

- Let X be a geometric random variable for which success corresponds to finding the majority element; otherwise, failure.
- Since the algorithm finds the majority element with $p > 1/2$,
- E[# iterations through the while loop] = $1/p < 2$.
- Each iteration requires n `equals` queries, so the expected runtime is **`O(n)`**.

Majority Element

Divide and Conquer Runtime

Expected & Worst-case: `O(nlogn)`

Randomized Runtime

Expected: `O(n)` Worst-case: `O(∞)`

Can you think of a deterministic algorithm that finds the majority element and only uses at most `n` - 1 calls to `equals`?

Get Hyped!

The randomized algorithmic paradigm appears everywhere in computer science.

As such, it will reappear throughout the quarter, starting next week with graph algorithms!