Greedy Algorithms I

Summer 2018 • Lecture 08/02

A Few Notes

Homework 4

Homework 5

- Original soft deadline was 8/9. I’m extending it to 8/10 at 5 p.m. due to the lecture schedule.

Course Overview

- Algorithmic Analysis
- Divide and Conquer
- Randomized Algorithms
- Tree Algorithms
- Graph Algorithms
- Dynamic Programming
- **Greedy Algorithms**
- Advanced Algorithms

Outline for Today

Greedy algorithms

- Frog Hopping
- Minimum Spanning Trees

Frog Hopping

Greedy Algorithms

Greedy algorithms construct solutions one step at a time, at each step choosing the locally best option.

Advantages: simple to design, often efficient

Disadvantages: difficult to verify correctness or optimality

Freddie the Frog

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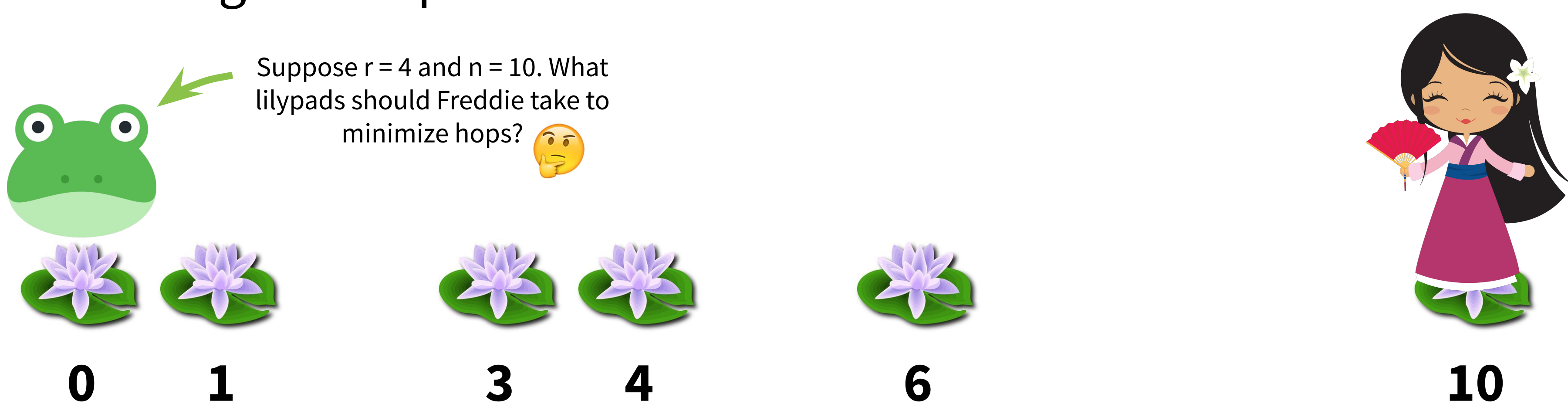

Frog Hopping

```python
def frog_hopping(lilys, r, n):
  # lilys = [0, 1, 3, 4, 6, 10] in the previous example
  H = [0]  # contains hops
  cur_lily = {"index": 0, "position": 0}
  while cur_lily["position"] < n:
    next_lily = furthest_reachable_lily(
      cur_lily, lilys, r
    )  # finds the furthest lilypad still reachable
       # from cur_lily
    H.append(next_lily["position"])
    cur_lily = next_lily
  return H
```

You should be able to implement this function yourself.

Runtime: `O(n)`

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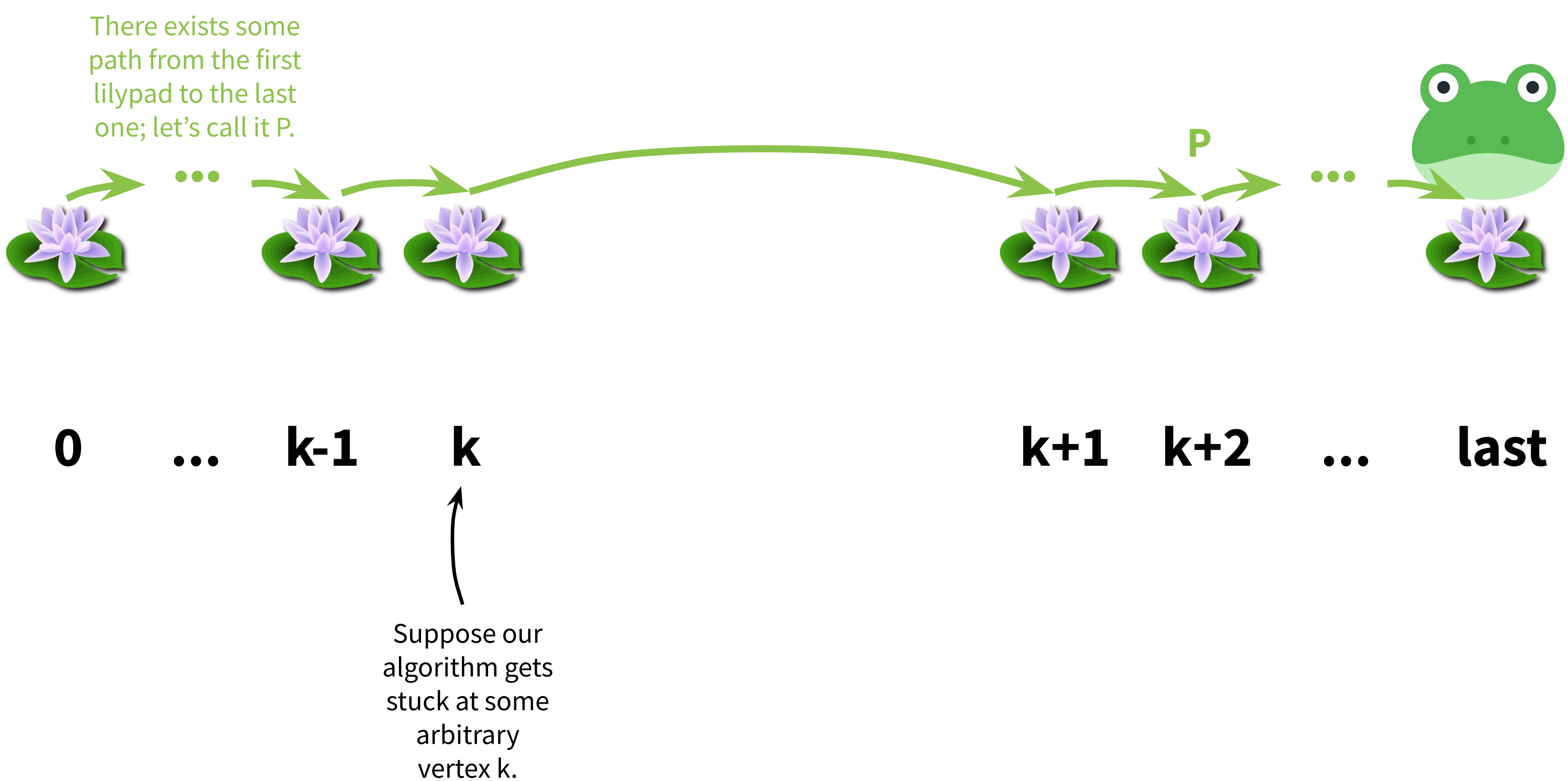

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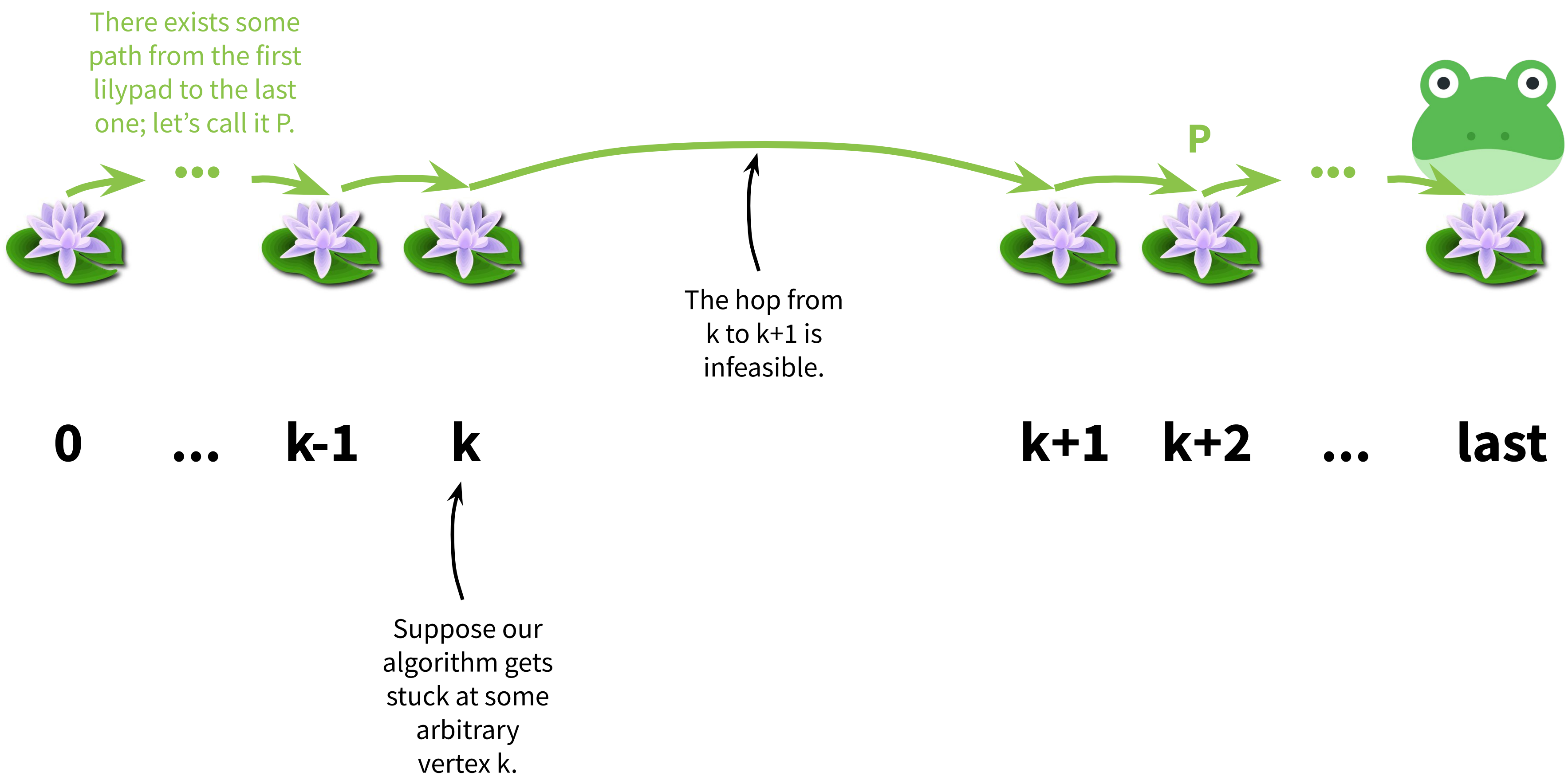

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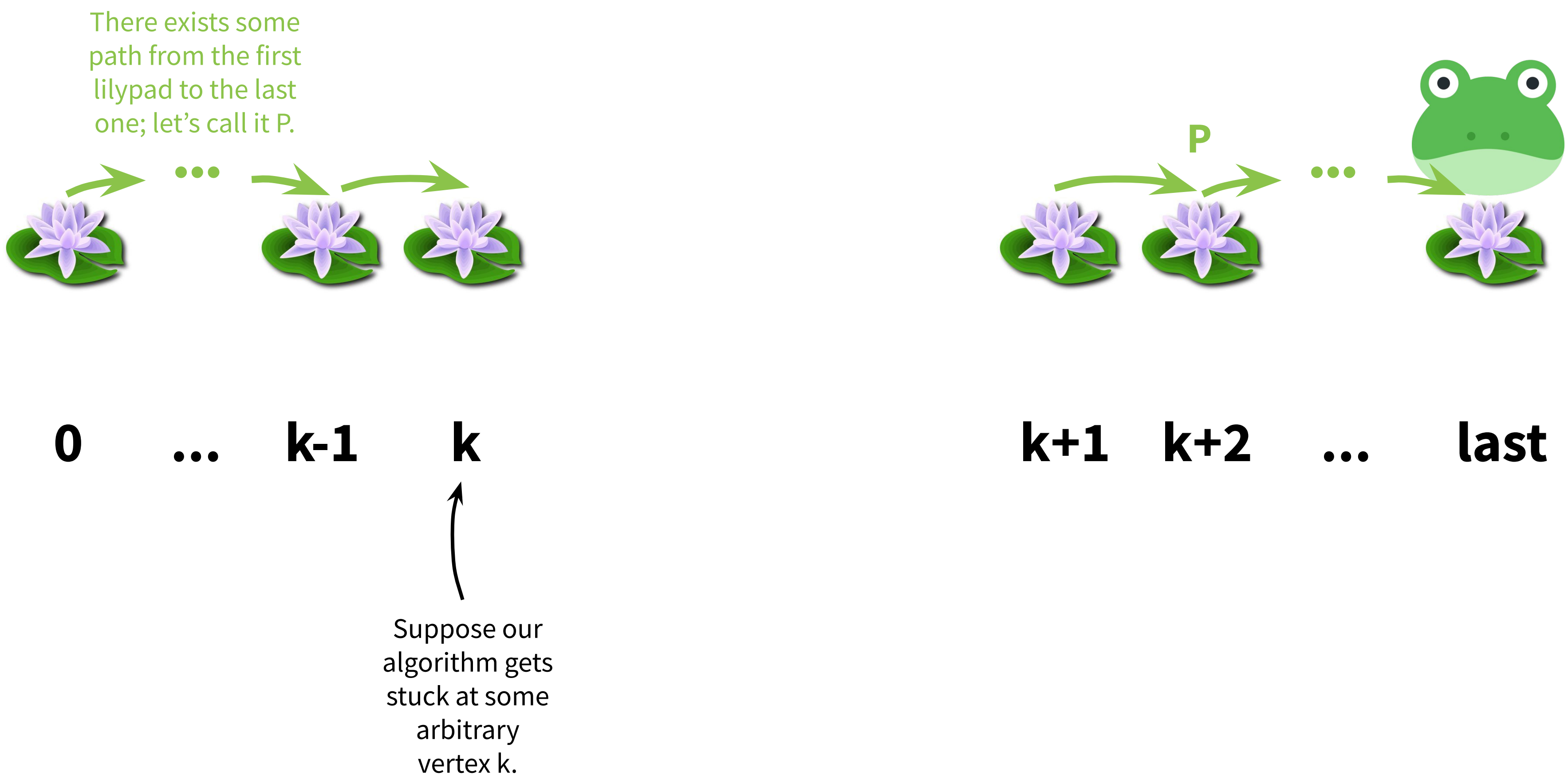

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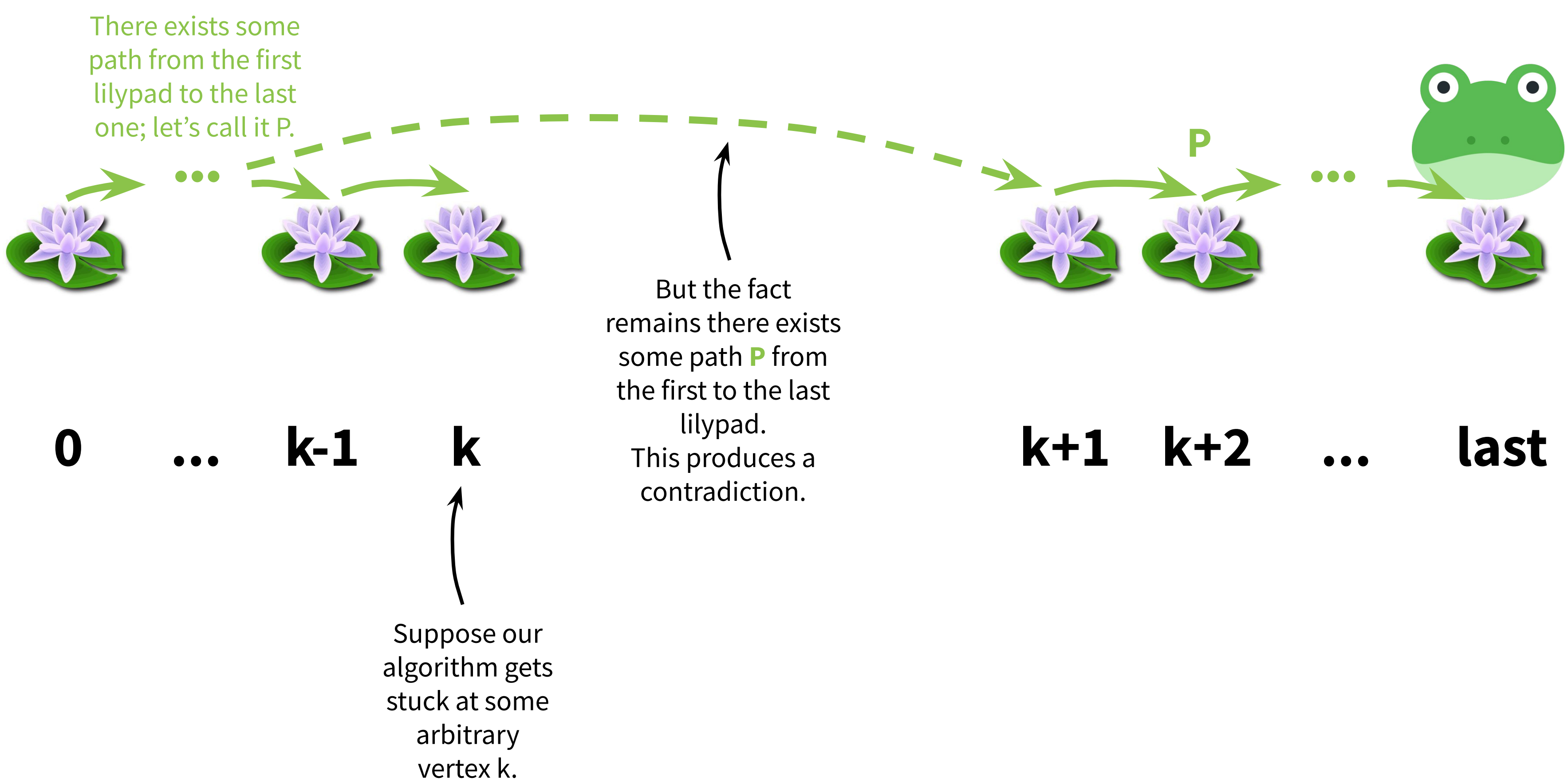

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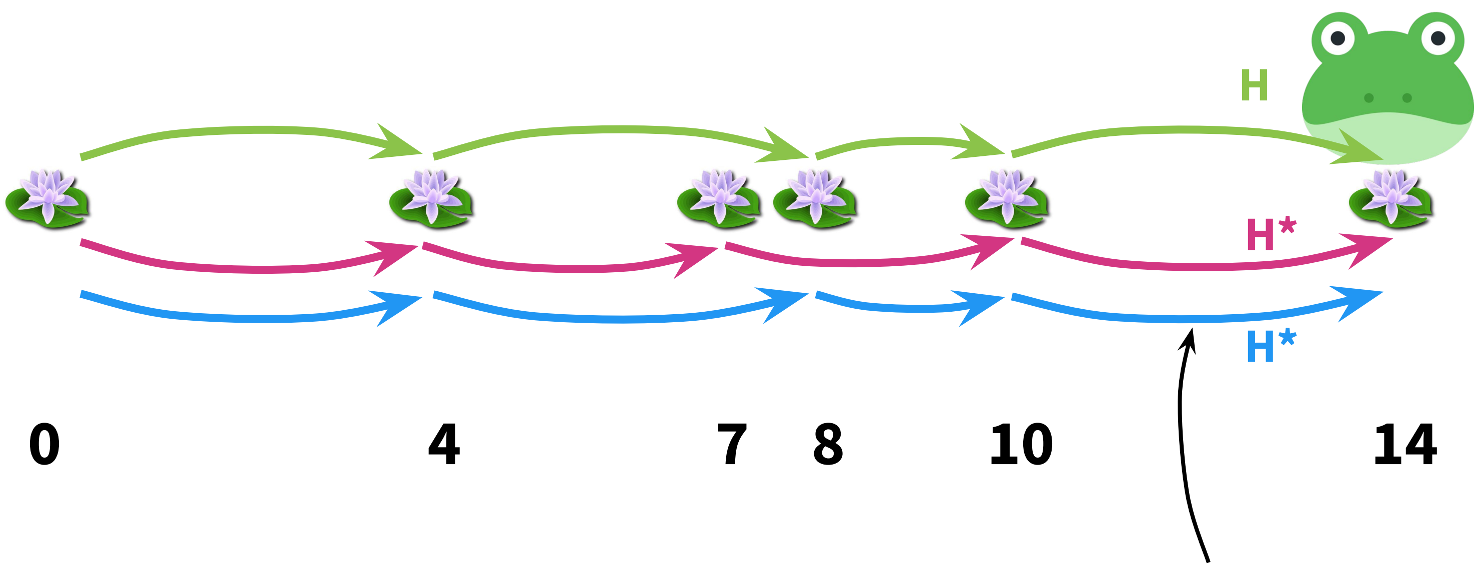
Intuition: Consider an arbitrary optimal series of hops H*, then show that our greedy algorithm produces a series of hops H no worse than H*.

What Does Arbitrary H* Mean?



There could be many optimal H* (this series of lilypads has 2); this proof relies on an arbitrary choice from among this H*.

Suppose we choose **H***.

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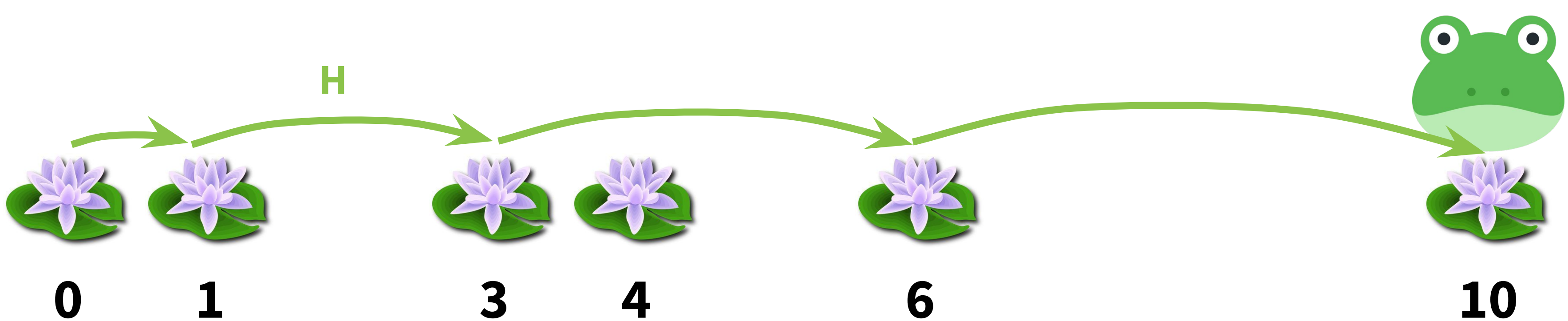

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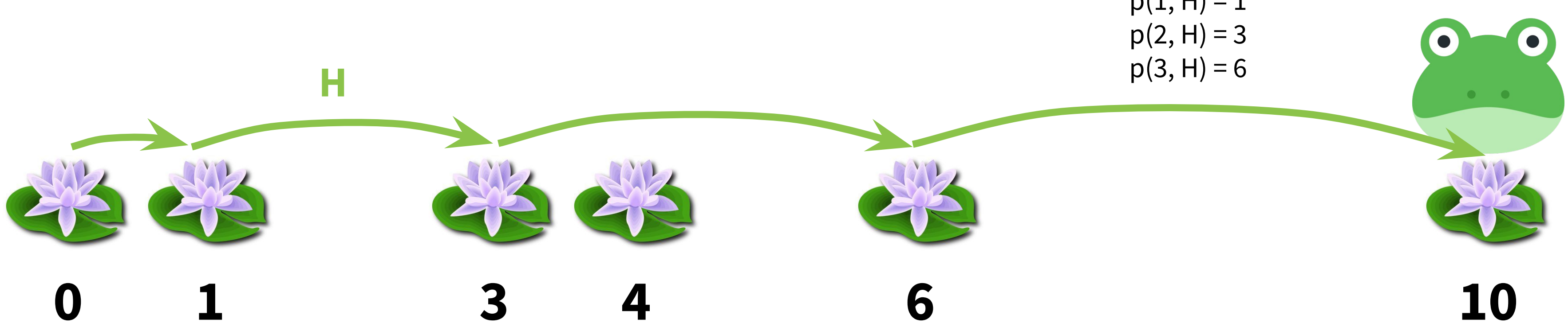

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Let's formalize this using induction.

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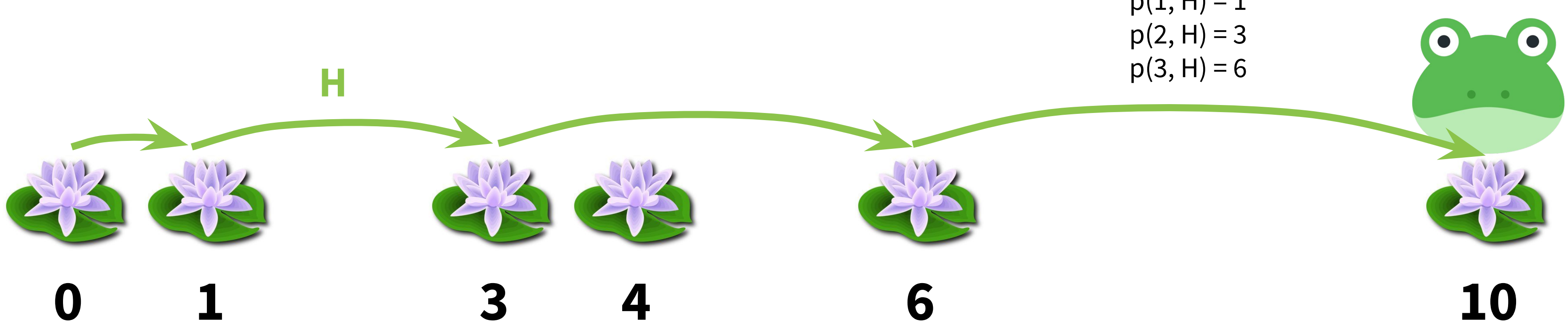

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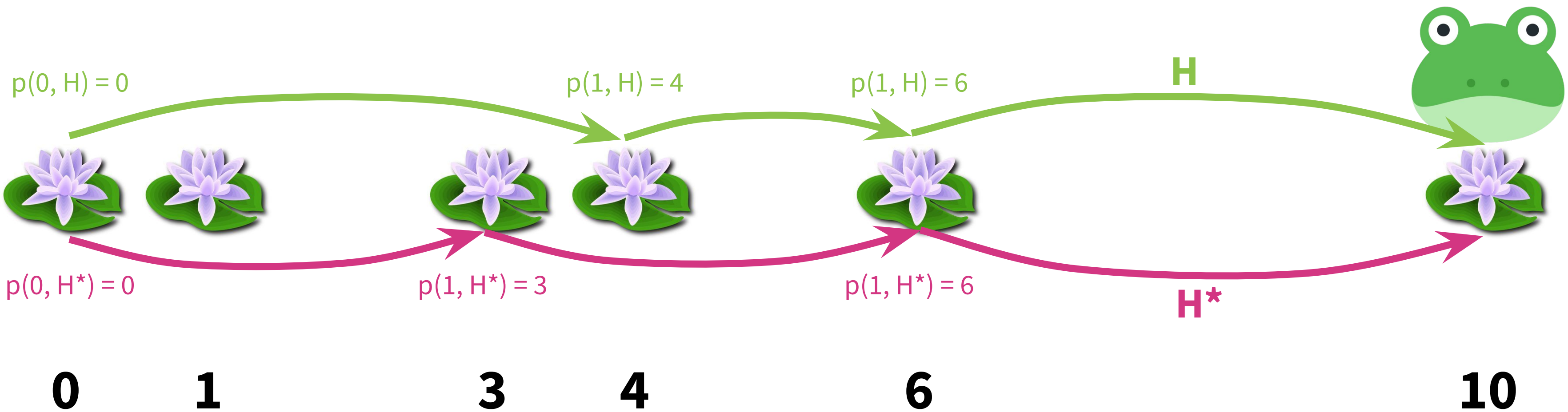

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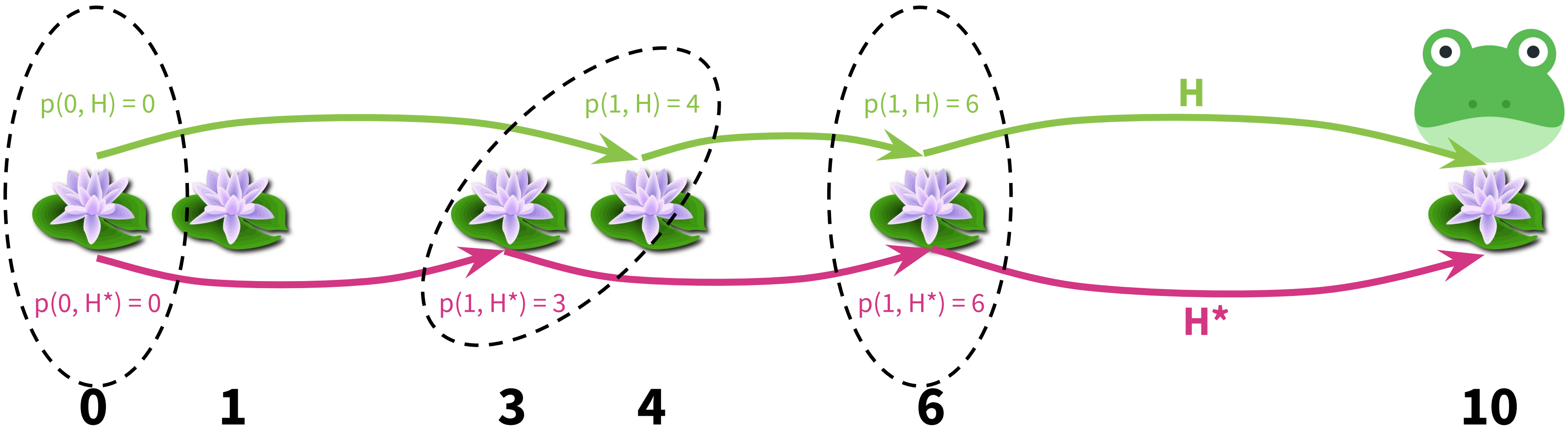

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Lemma 2: For all $0 \leq i \leq |H^*|$, we have $p(i, H) \geq p(i, H^*)$, constructing H from `frog_hopping`.

Proof: We proceed by induction.

As a base case, if $i = 0$, then $p(0, H) = 0 \geq 0 = p(0, H^*)$ since the frog hasn't moved.

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Frog Hopping

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Proof: We proceed by induction.

As a base case, if $i = 0$, then $p(0, H) = 0 \geq 0 = p(0, H^*)$ since the frog hasn't moved.

For the inductive step, assume that the claim holds for some $0 \leq i < |H^*|$. We'll prove the claim holds for $i + 1$ by considering two cases:

Case 1: $p(i, H) \geq p(i+1, H^*)$. Since each hop moves forward, we have $p(i+1, H) \geq p(i, H)$, so we have $p(i+1, H) \geq p(i+1, H^*)$.

Case 2: $p(i, H) < p(i+1, H^*)$. Each hop is of size at most r, so $p(i+1, H^*) \leq p(i, H^*) + r$. By our inductive hypothesis, we know $p(i, H) \geq p(i, H^*)$, so $p(i+1, H^*) \leq p(i, H) + r$; i.e. position $p(i+1, H^*)$ is reachable from position $p(i, H)$. Since the greedy algorithm hops to the furthest lilypad still reachable from position $p(i, H)$, it hops to at least position $p(i+1, H^*)$. Therefore, $p(i+1, H) \geq p(i+1, H^*)$.

So $p(i+1, H) \geq p(i+1, H^*)$, completing the induction. ■

Frog Hopping

Now for the theorem: `frog_hopping` produces an optimal solution for Freddie.

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Since H^* is an optimal solution, we know that $|H^*| \leq |H|$. We will prove $|H^*| = |H|$.

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Here, we proved this step using a direct proof. You should be able to structure the proof by contradiction here too.

Frog Hopping

We need to prove two properties about the algorithm to guarantee correctness.

(1) **Feasibility.** The algorithm finds a feasible (aka legal) series of hops (i.e. it doesn't "get stuck" or break any rules). 👌

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Greedy Stays Ahead

The style of proof we just wrote is an example of a **greedy stays ahead** proof.

(1) Find intermediate values that evaluate the solution produced by any algorithm, including the greedy one.

What's our values in `frog_hopping`? 🤔 The position after i hops.

(2) Show the greedy algorithm produces values at least as good as any solution's (using induction).

(3) Prove that since the greedy algorithm produces values at least as good as any solution's, it must be optimal (using direct proof or proof by contradiction).

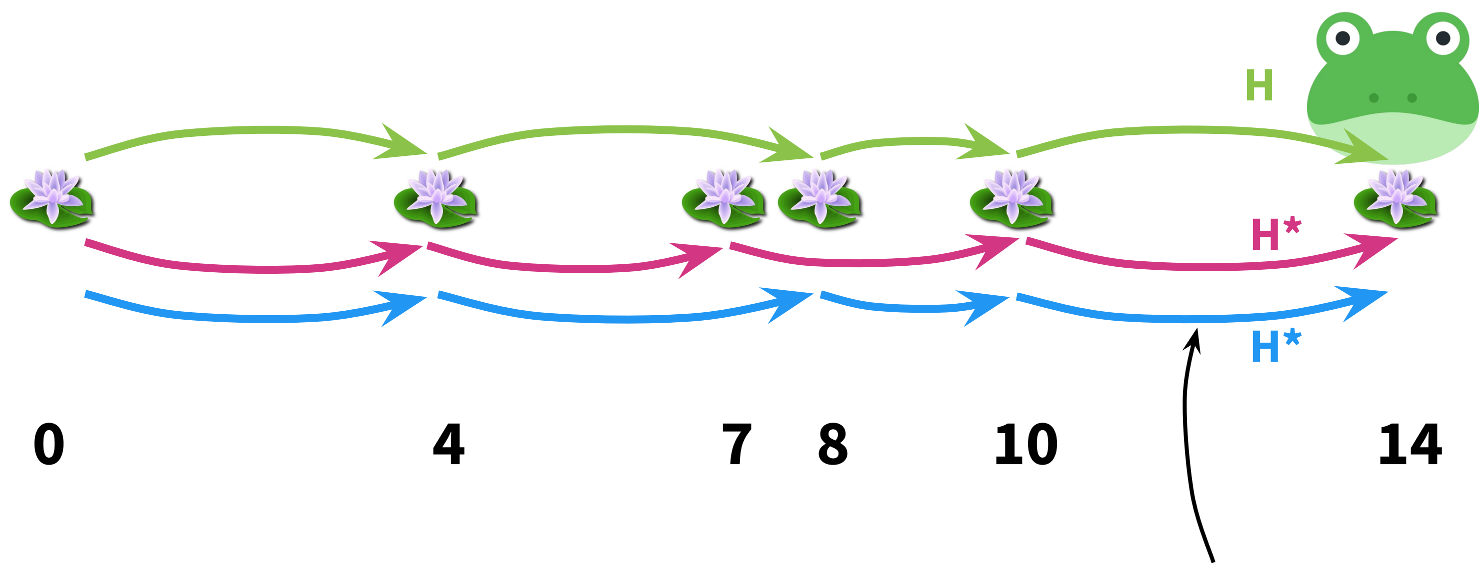
Greedy Exchange Argument

There's another style of proof that uses **greedy exchange argument**.

- If we swap an optimal solution out for the greedy solution, argue that we're still optimal.

Greedy Exchange Argument

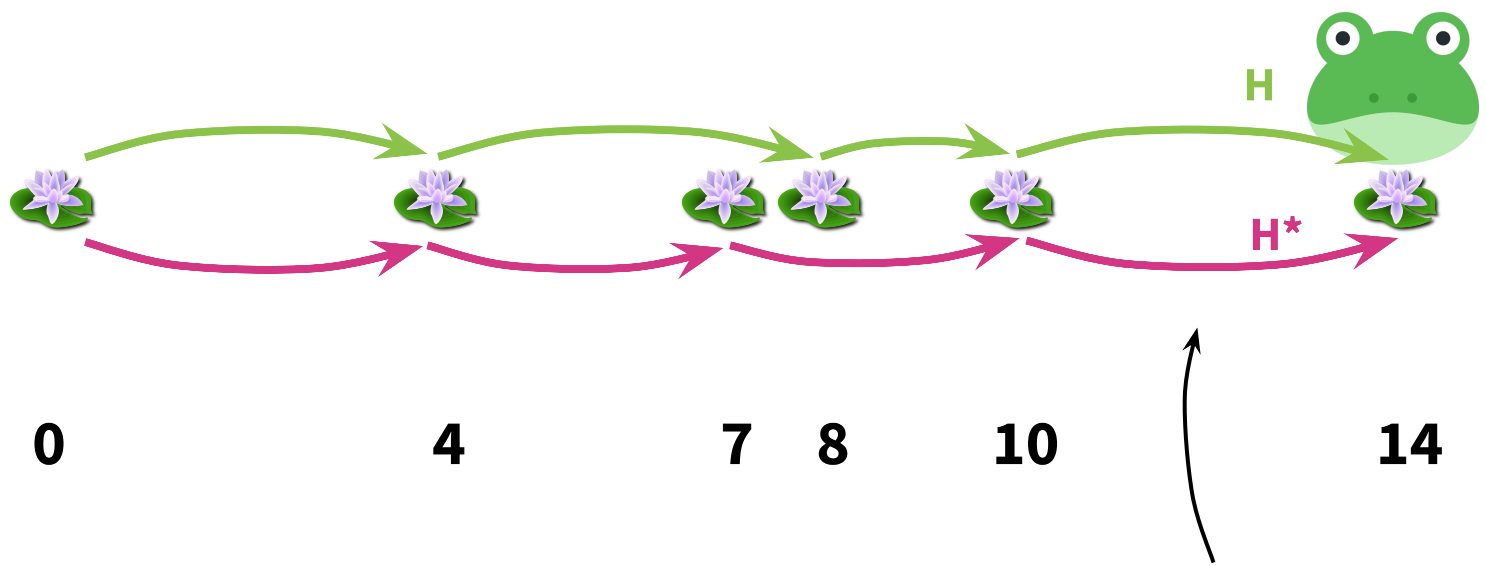
Again, this proof will rely on an arbitrary choice of H*.



There could be many optimal H* (this series of lilypads has 2); this proof relies on an arbitrary choice from among this H*.

Suppose we choose **H***.

Greedy Exchange Argument



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Theorem: `frog_hopping` produces an optimal solution.

Proof: We proceed by induction.

As a base case, we initialize H to `[0]` and all feasible hops H* must have H*[0] = 0.

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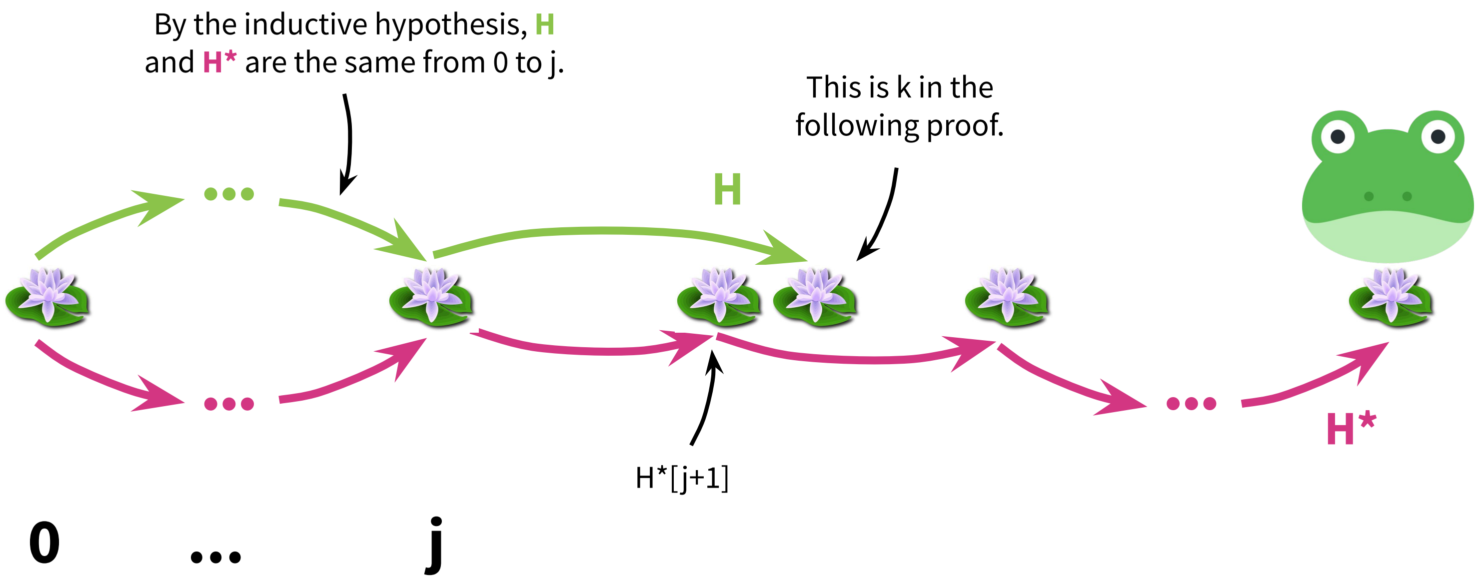

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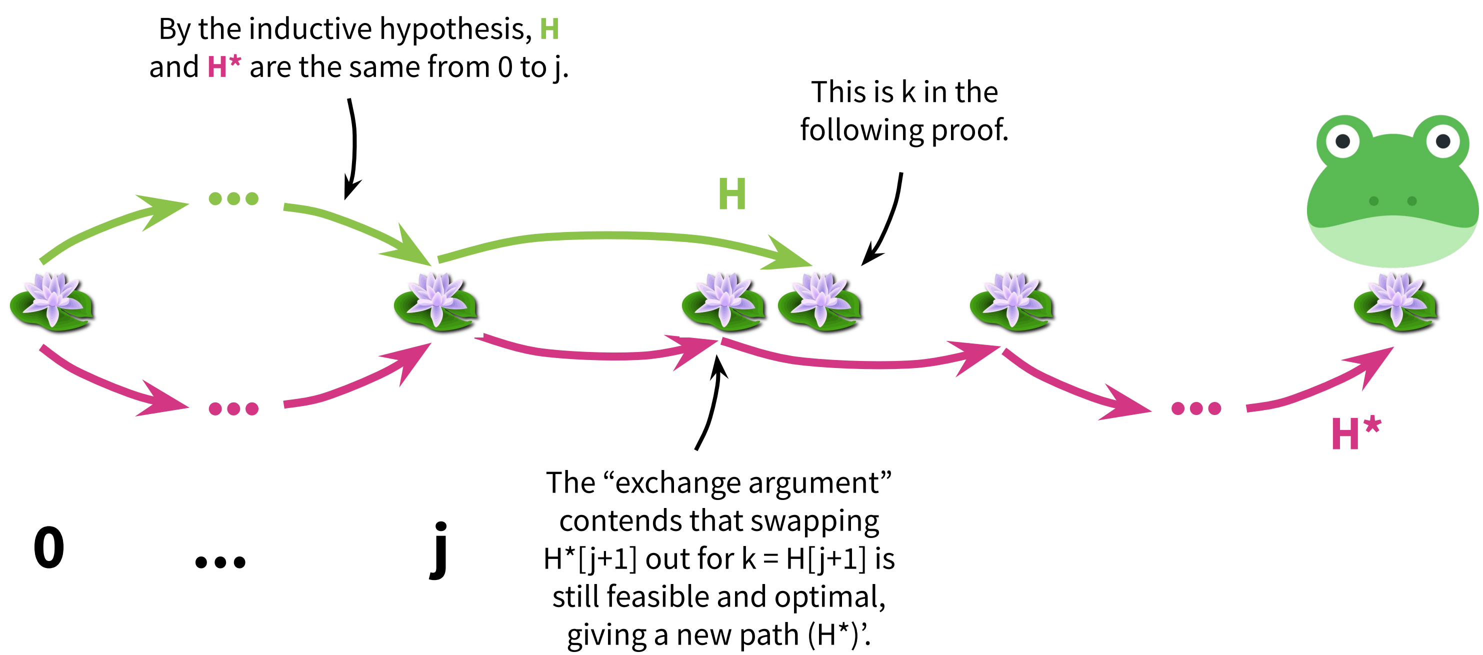

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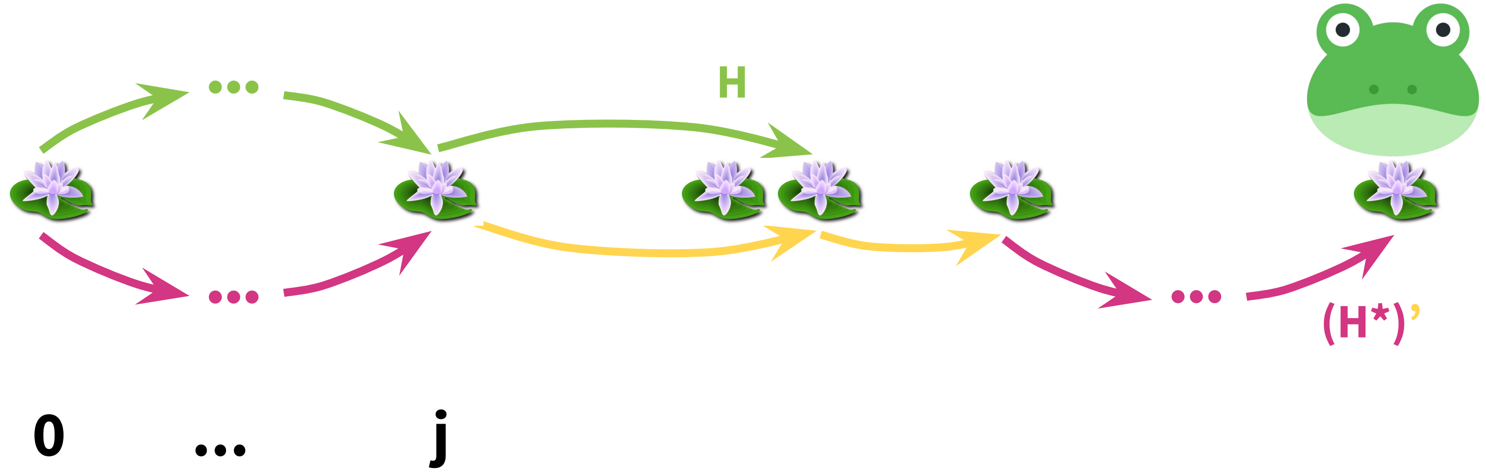

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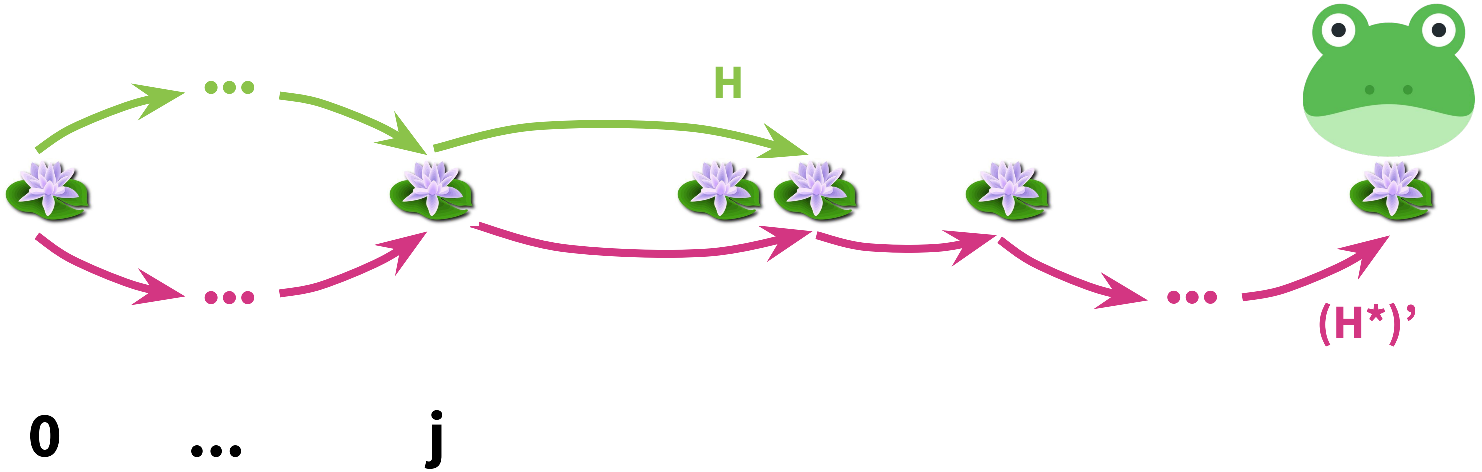

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This is still feasible since (H*)'[j+1] = k ≤ r + (H*)'[j] and (H*)'[j+2] = H*[j+2] ≤ r + H*[j+1] ≤ r + k = r + (H*)'[j].

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Since H* and (H*)' are the same except at position j+1.

This is still optimal since (H*)' has the same number of hops as H*.

3 min break

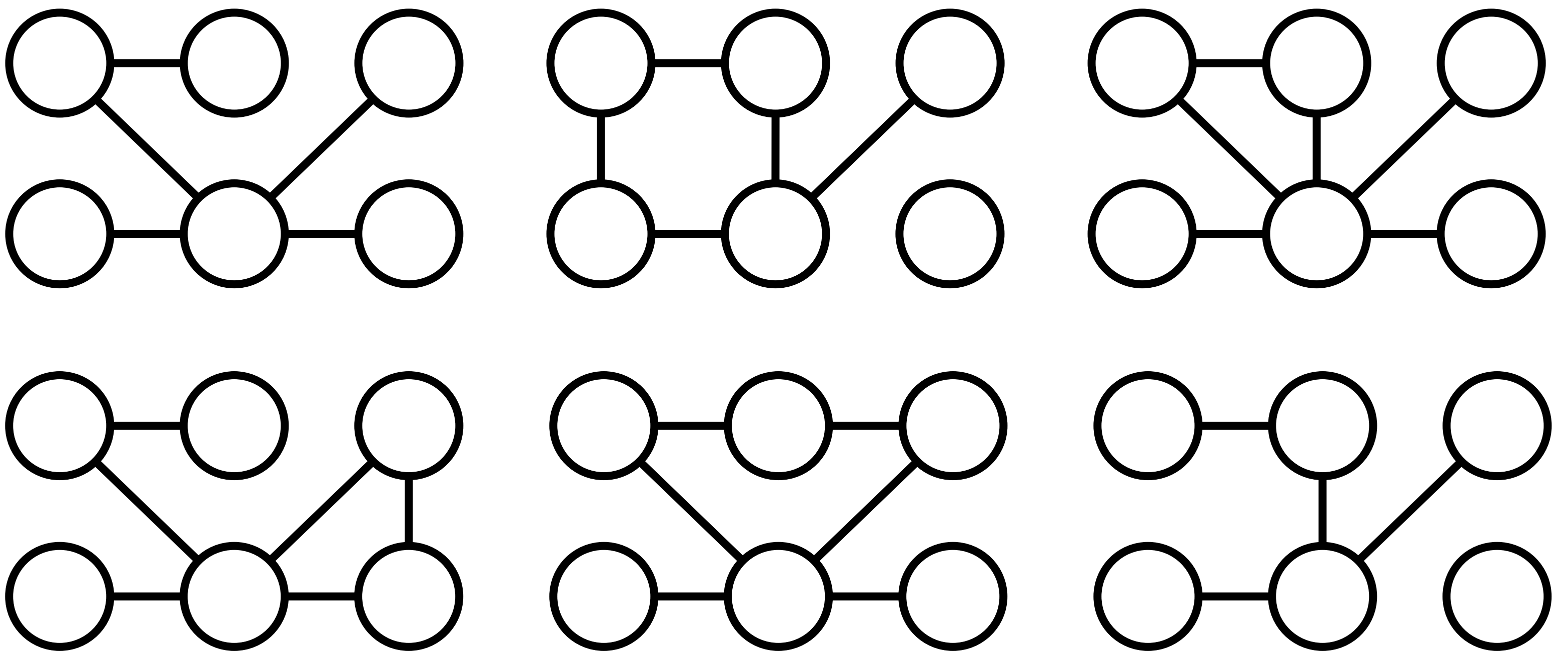
Minimum Spanning Trees

MSTs

A tree is an undirected, acyclic, connected graph.

In Lecture 3, we studied trees with directed edges from parent to children vertices. In this lecture, edges will be undirected.

Which of these graphs contain connected components which are trees? 🤔



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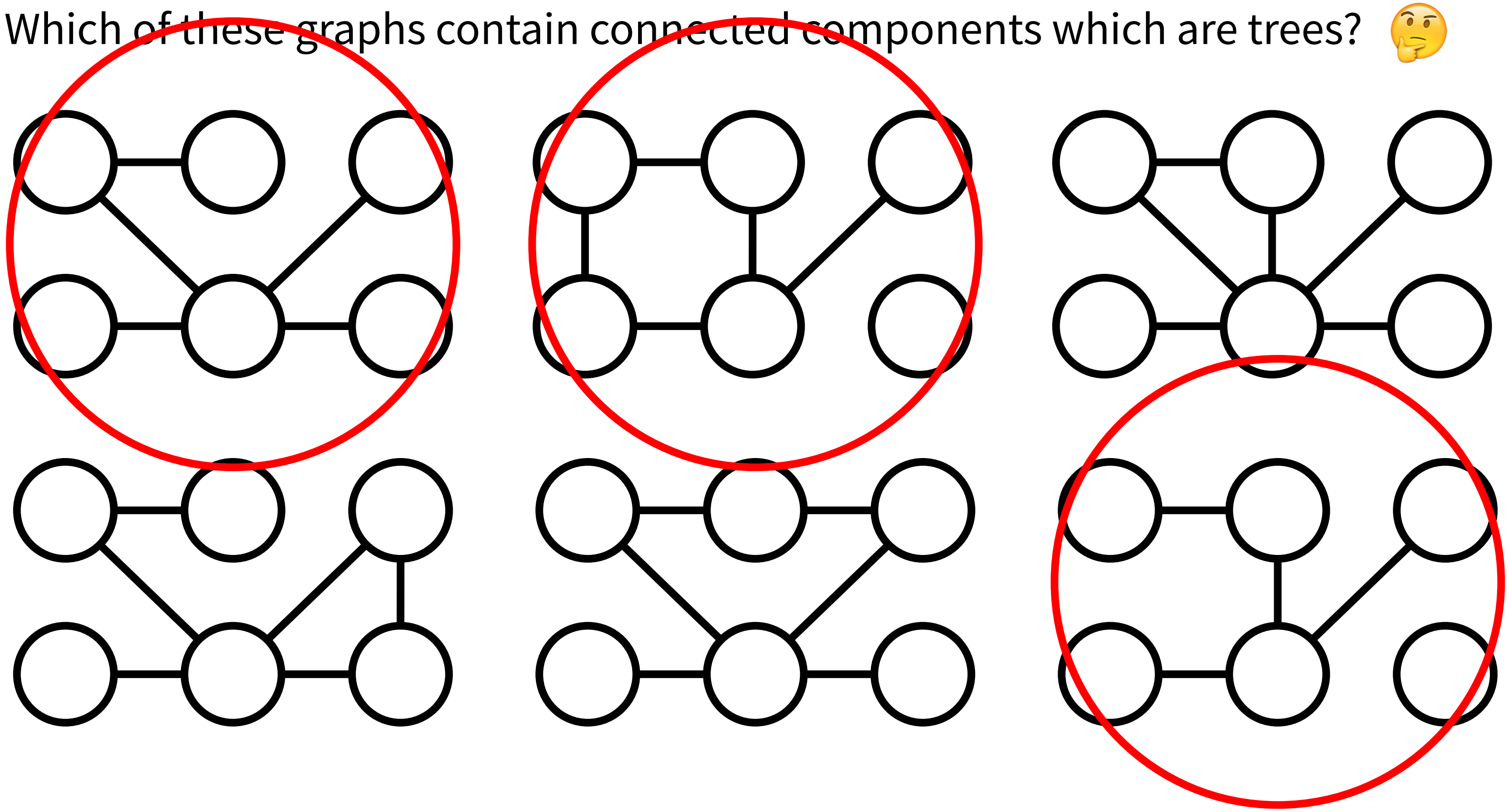
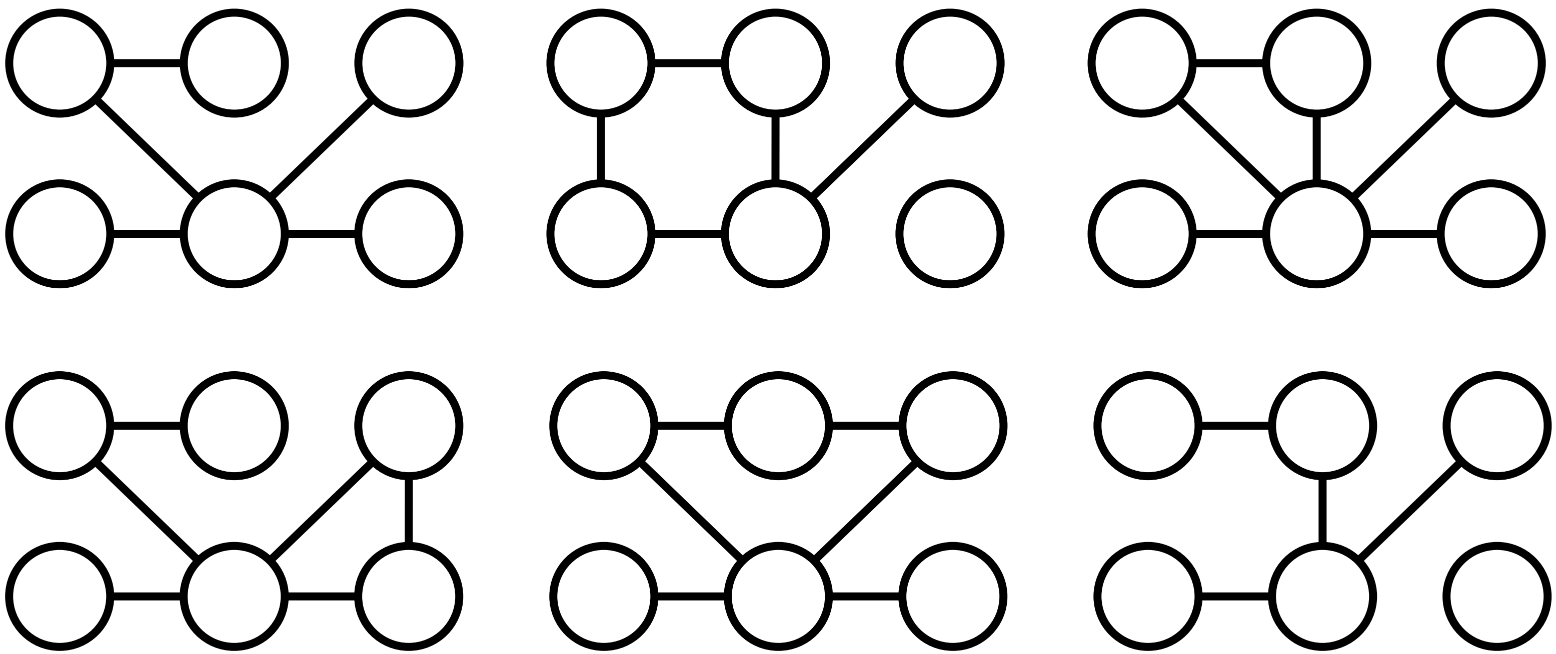


MSTs

A spanning tree is a tree that connects all of the vertices.

Which of these graphs are spanning trees? 🤔



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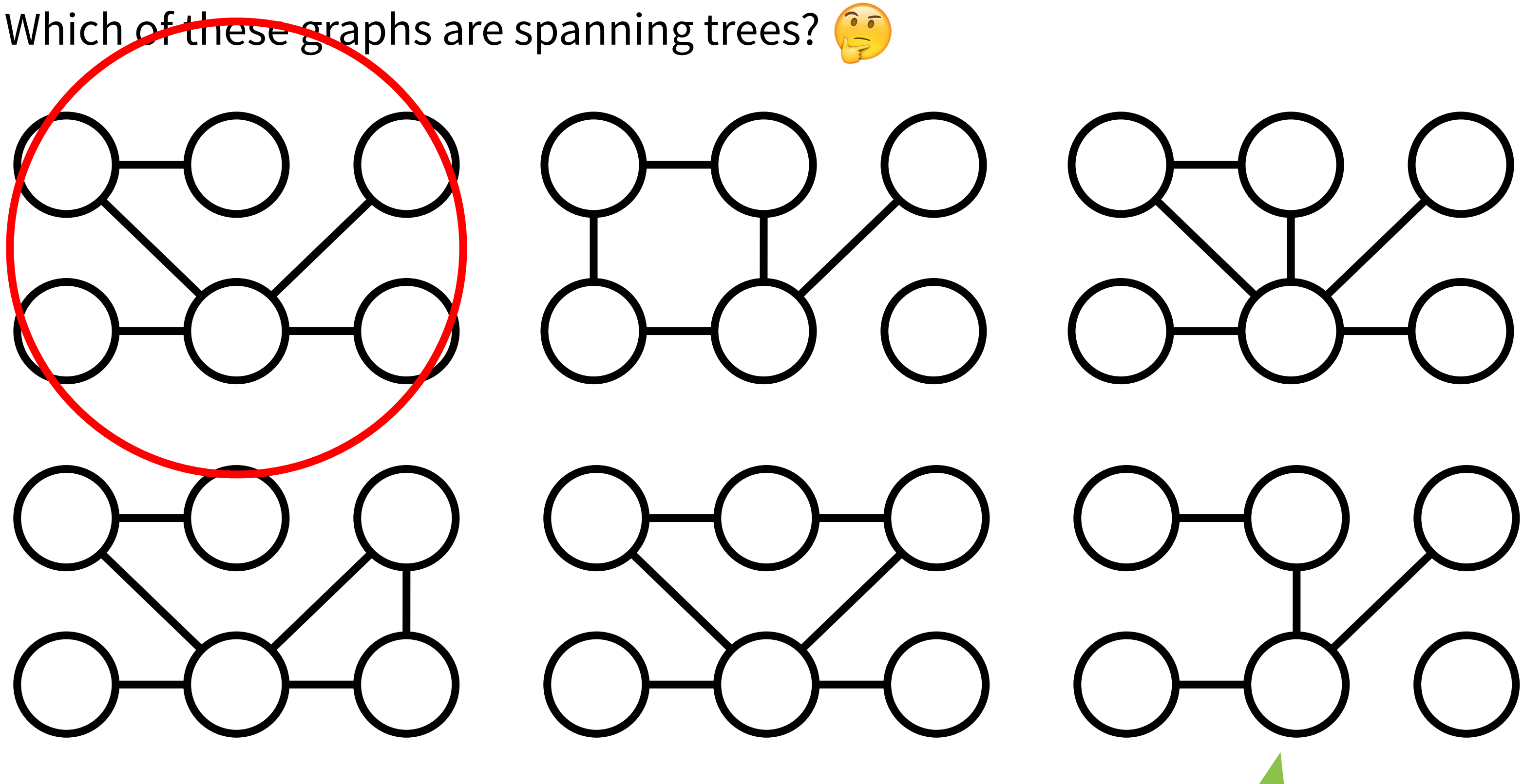
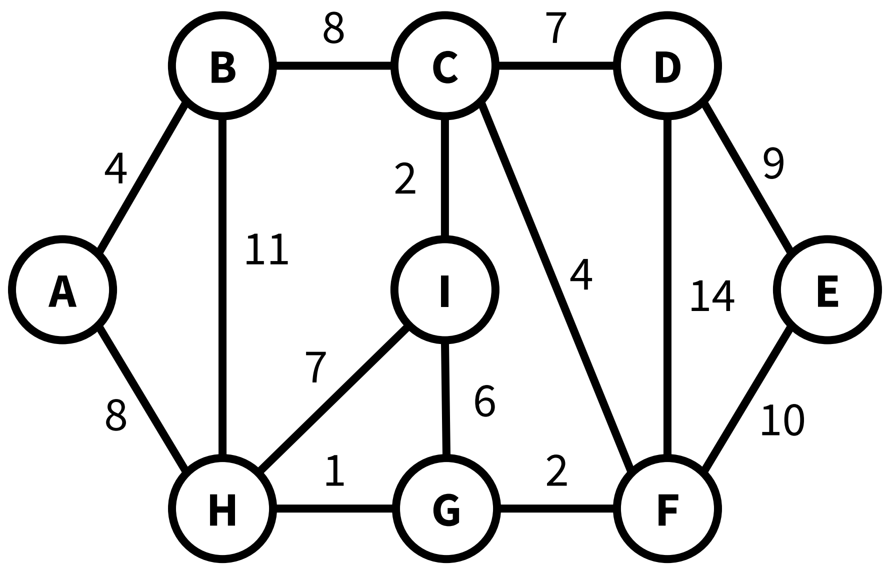


This connected component of the graph is a tree, but it doesn't include all of the vertices.

MSTs

A spanning tree is a tree that connects all of the vertices.

The cost of a spanning tree is the sum of the weights on the edges.



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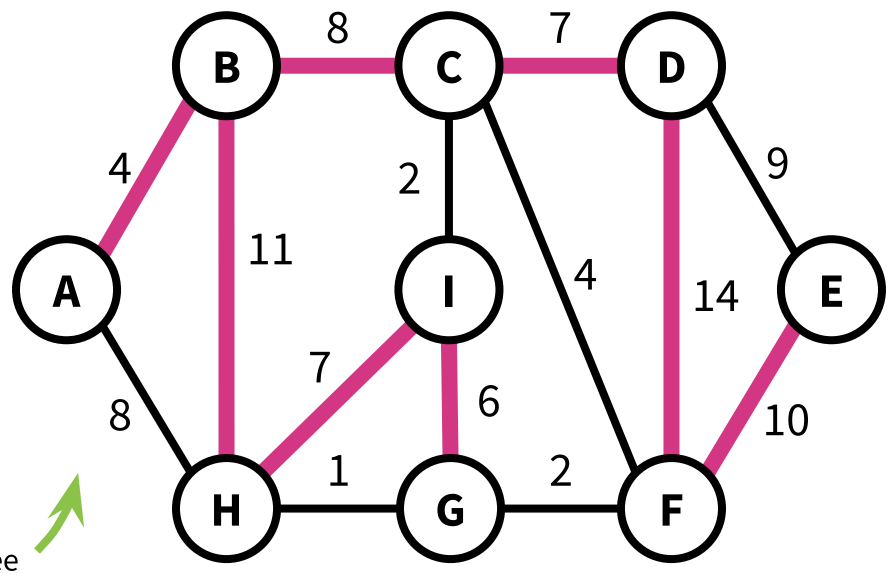

This spanning tree has a cost of 67.

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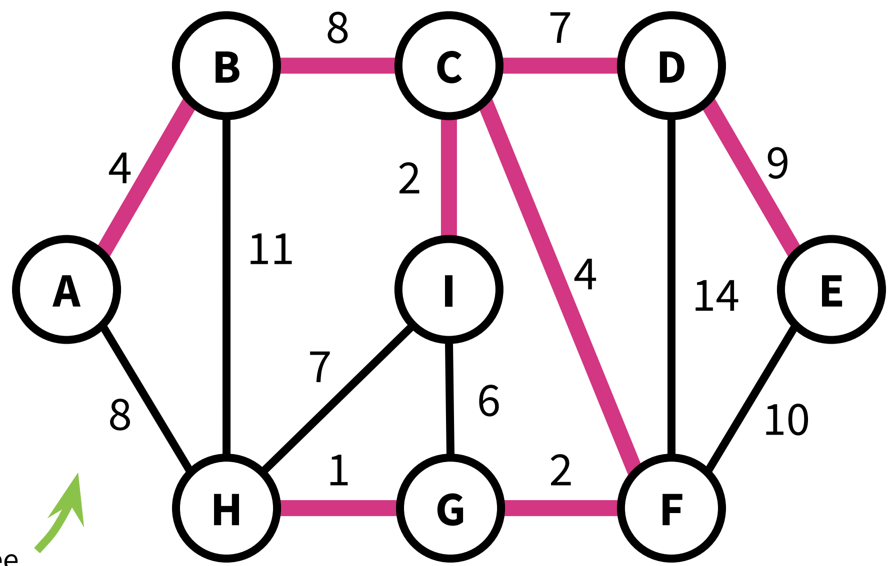

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MSTs

mininmum

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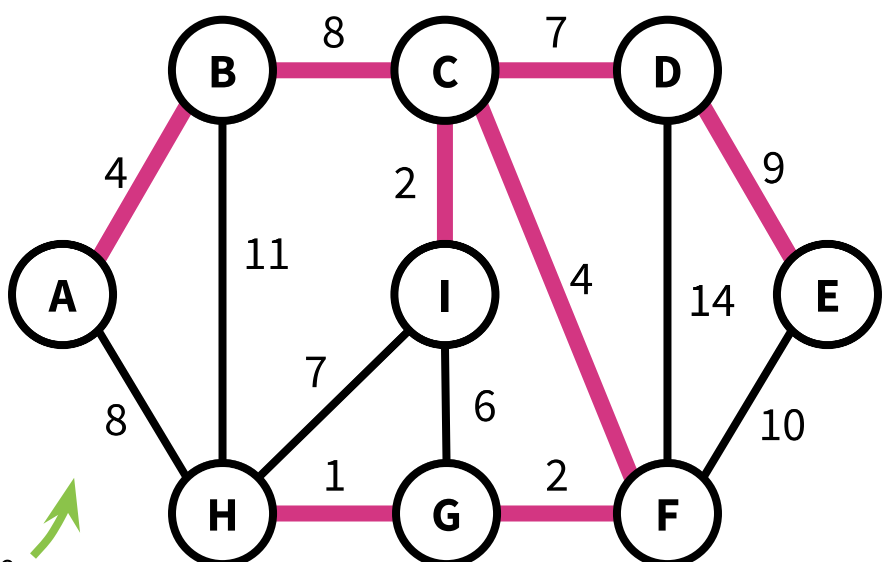


This spanning tree has a cost of 37. This is a minimum spanning tree.

MSTs

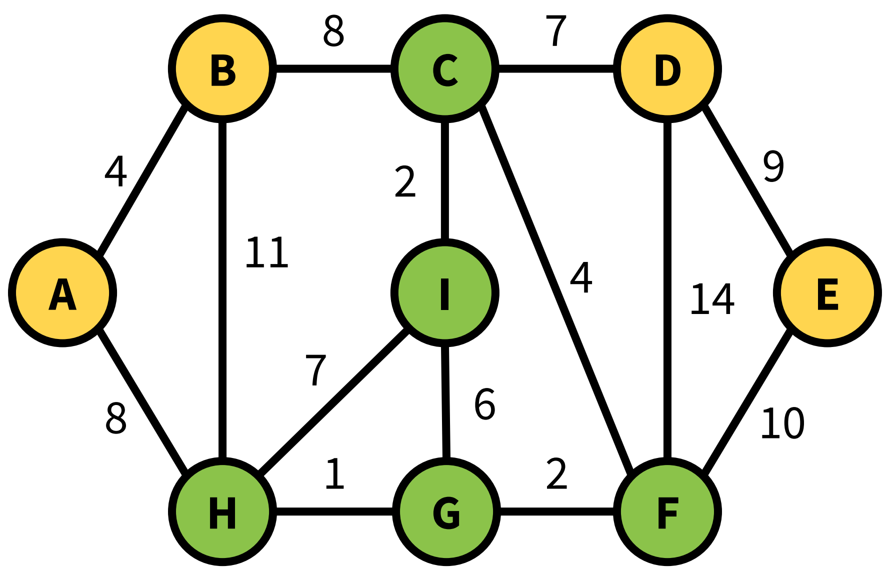
How might we find an MST?

- Today, we'll see two greedy algorithms that find an MST.

MSTs

A **cut** is a partition of the vertices into two nonempty parts.

e.g. This is the cut "{A, B, D, E} and {C, I, F, G, H}".



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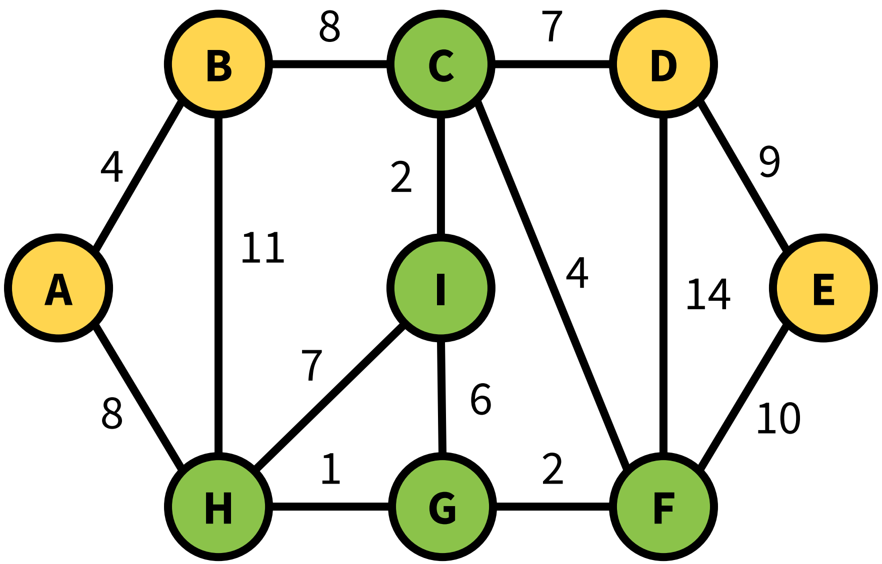

A cut respects a set of edges if no edges in the set cross the cut.

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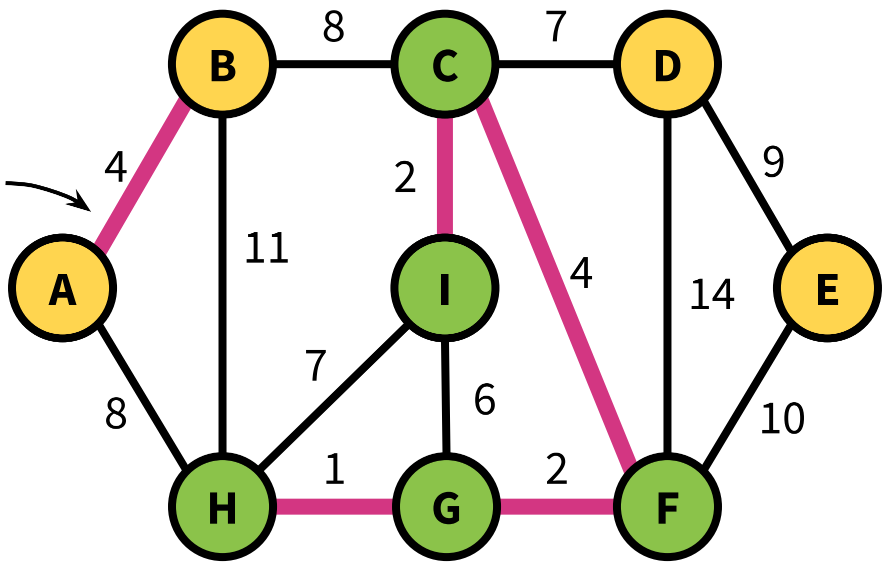

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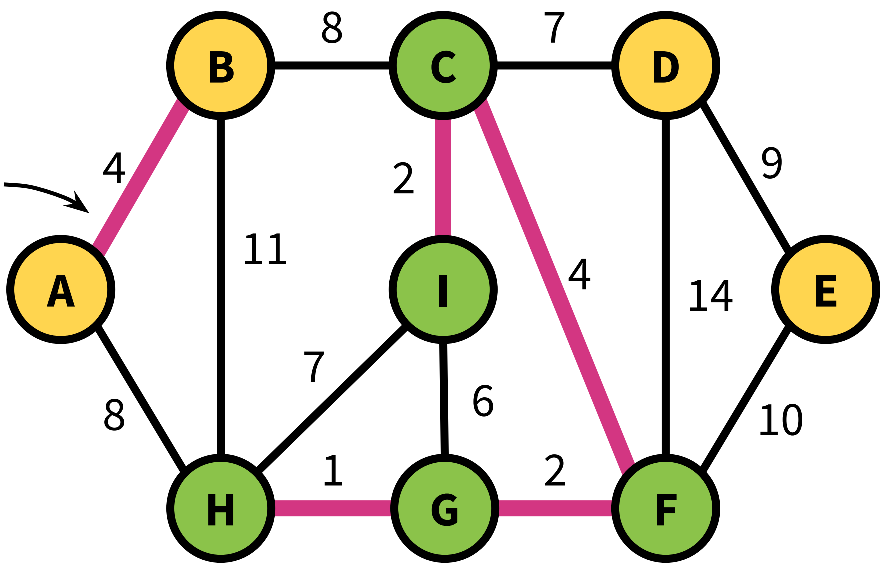

A cut respects a set of edges if no edges in the set cross the cut.

An edge is light if it has the smallest weight of any edge crossing the cut.

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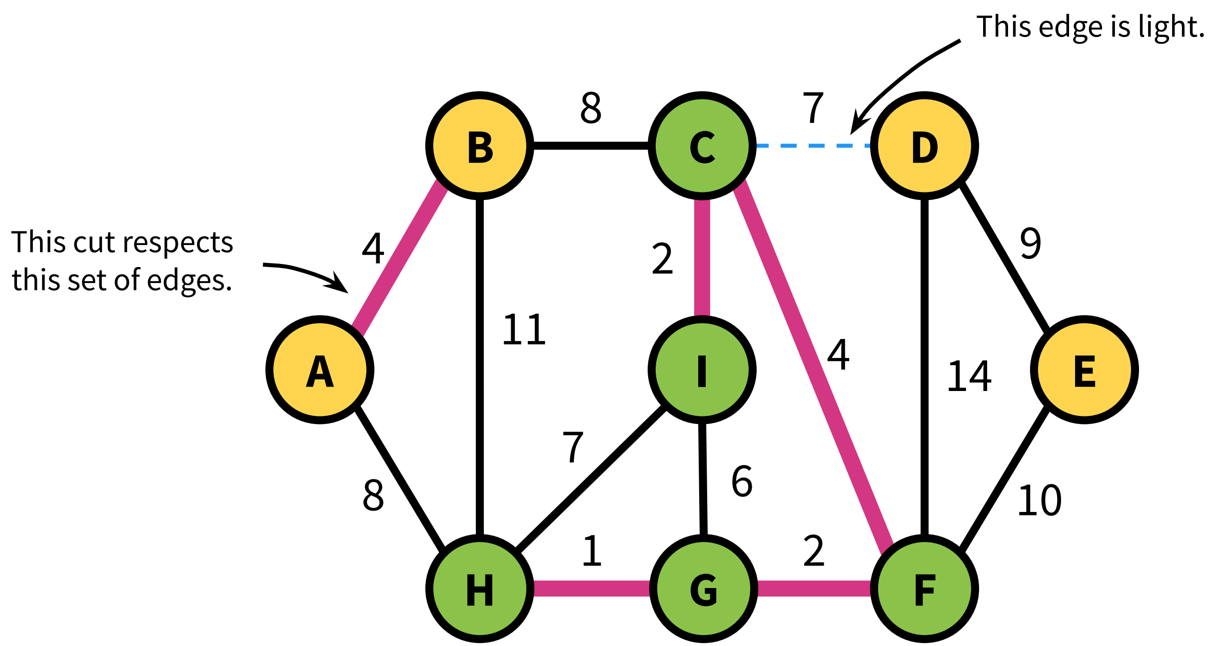

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An edge is light if it has the smallest weight of any edge crossing the cut.

Lemma

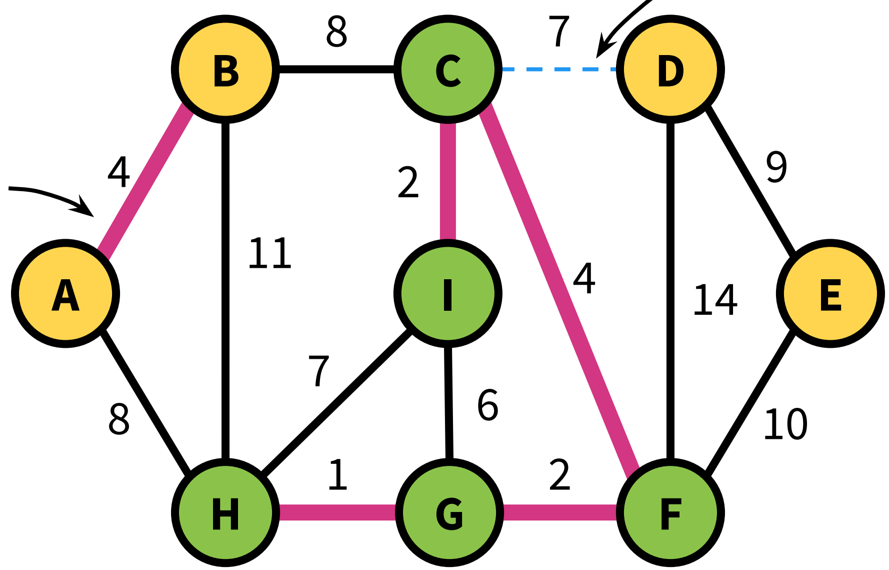
Consider a cut that respects a set of edges **A**.

Suppose there exists an MST containing **A**.

Let (u, v) be a light edge.

This edge is light.

This cut respects this set of edges.



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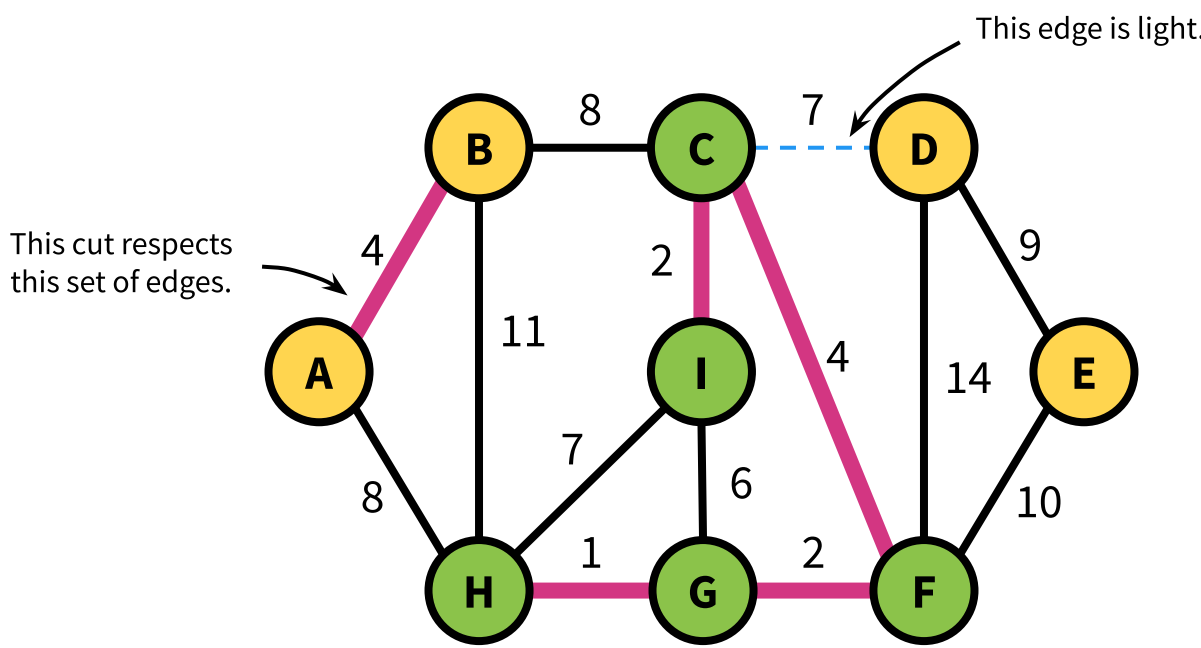
Then there exists an MST containing **A** ∪ {(u, v)}.



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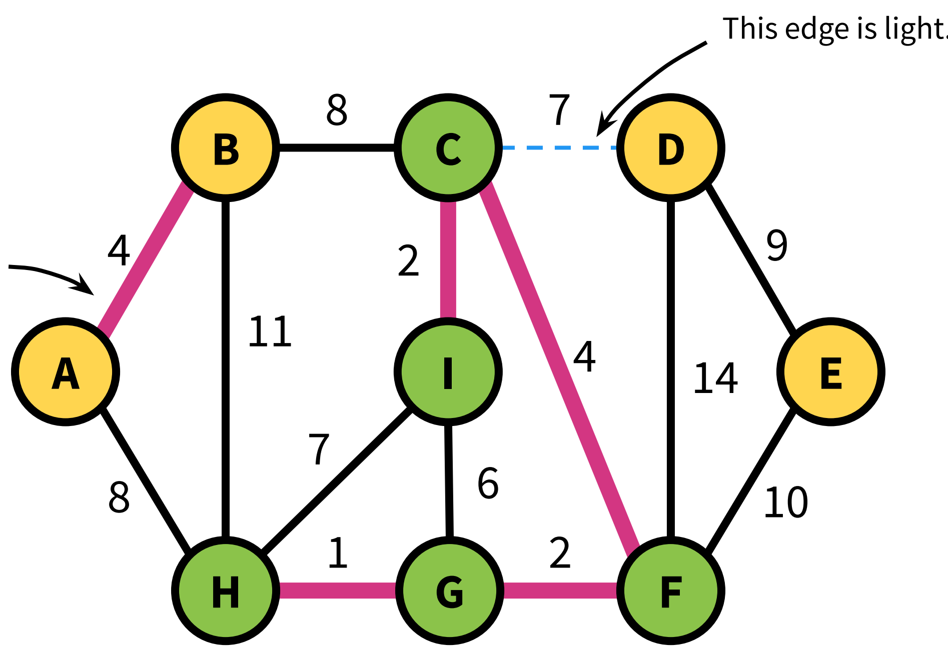
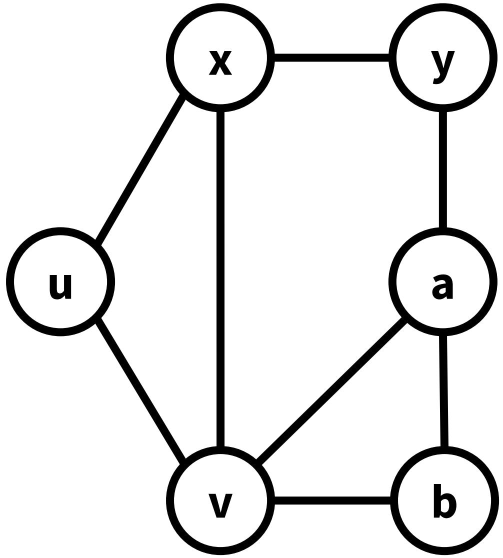
This is precisely the sort of statement we need for a greedy algorithm: If we haven't ruled out the possibility of success so far, then adding a light edge won't rule it out.

This cut respects this set of edges.



Proof of Lemma

Consider a graph with …



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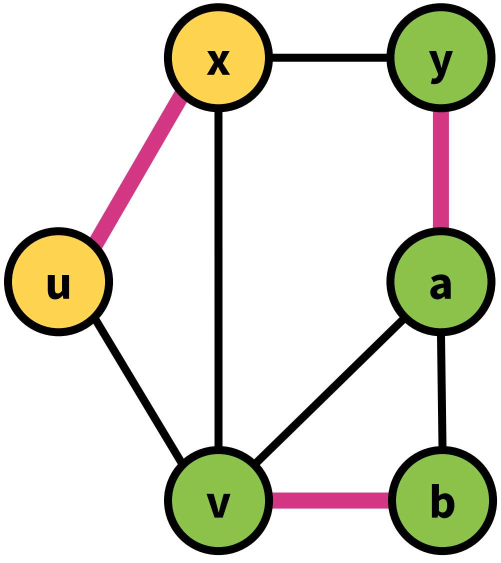

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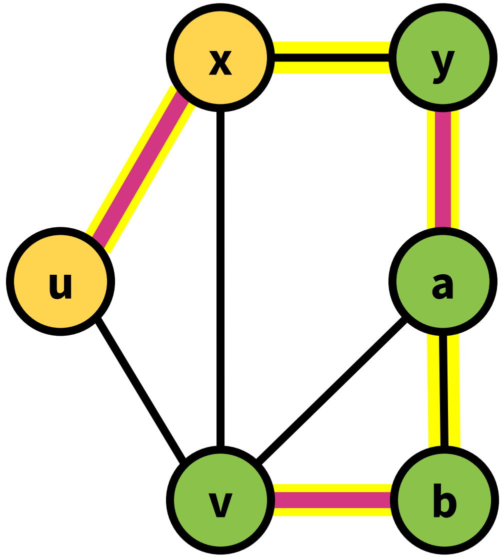

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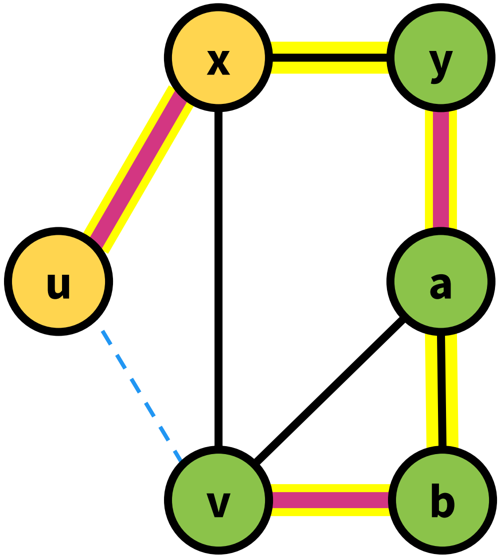

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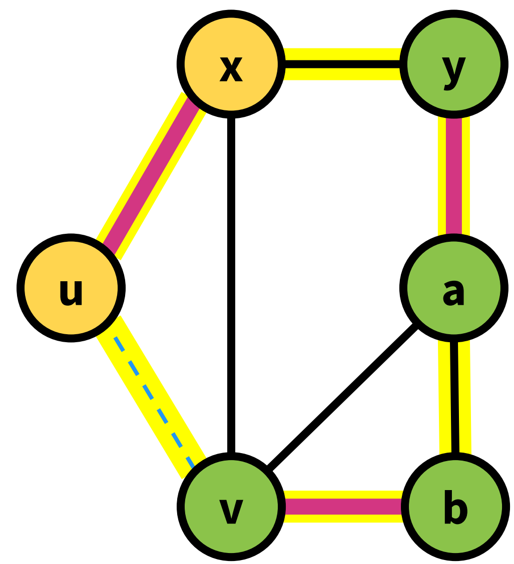

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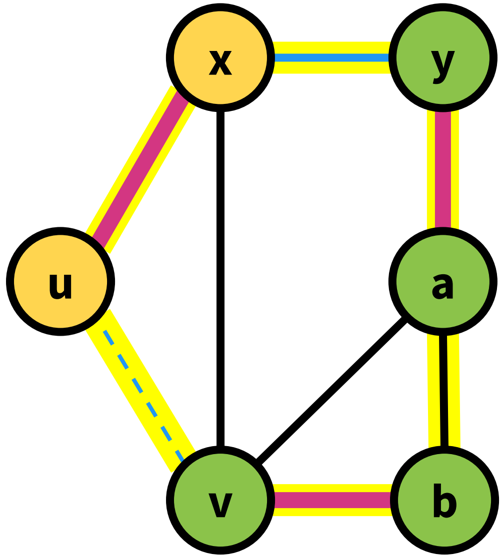

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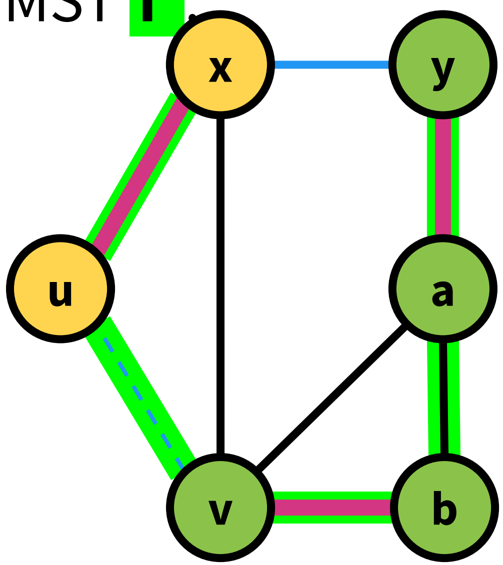

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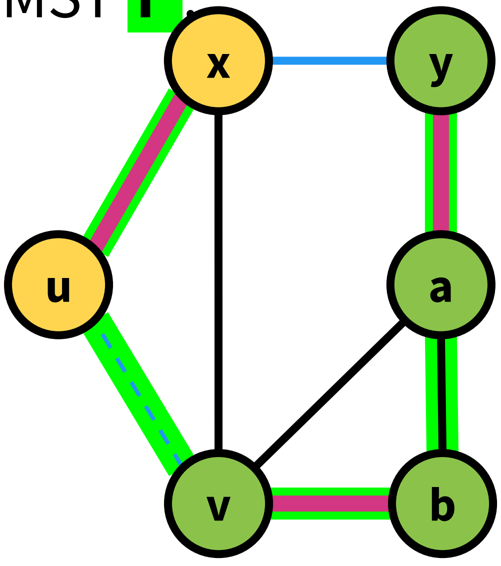
Since (u, v) is light, **T'** has cost at most that of **T**.



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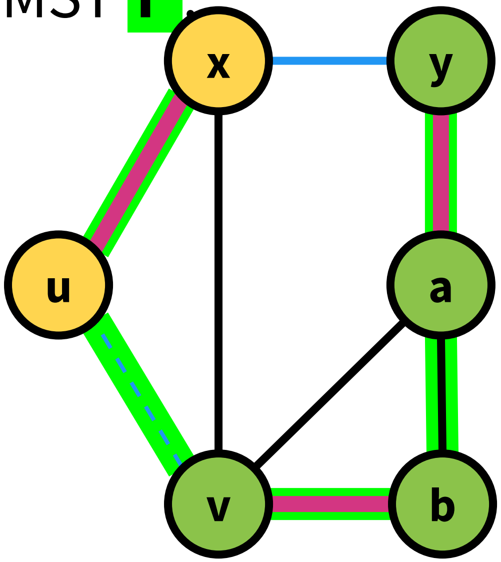
Thus, there exists an MST containing **A** ∪ {(u, v)}.



Prim’s Algorithm

Any Ideas?

Recall our lemma:

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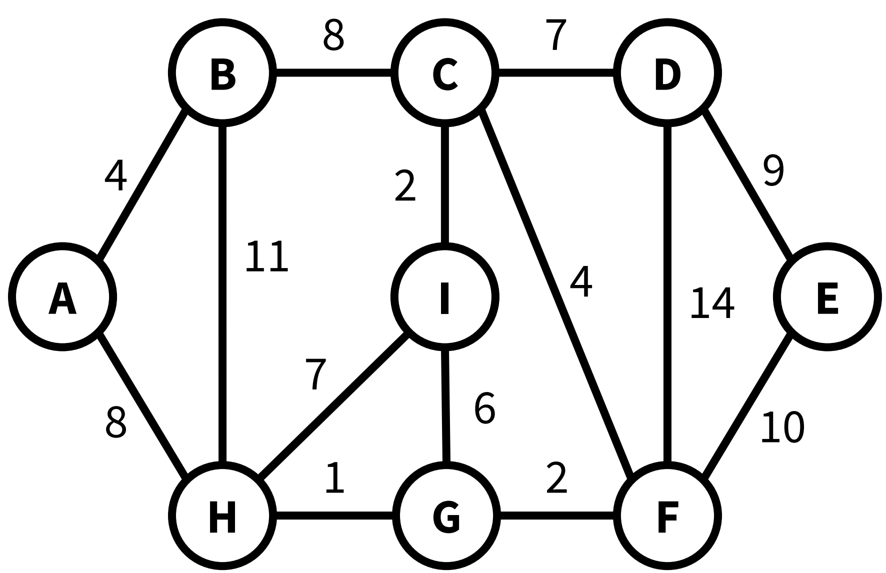
Any ideas about what to greedily choose?



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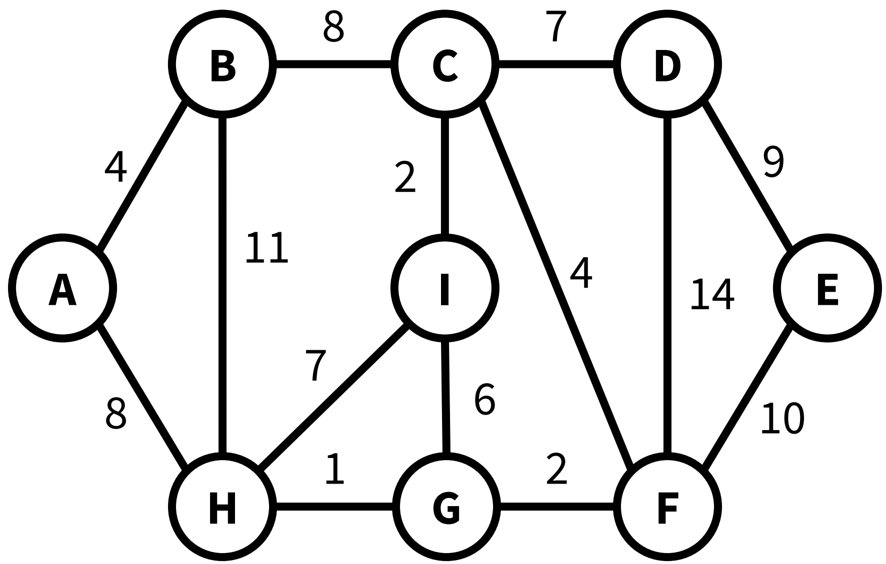
Main idea: Extend a single tree of visited vertices by greedily adding the closest vertex.



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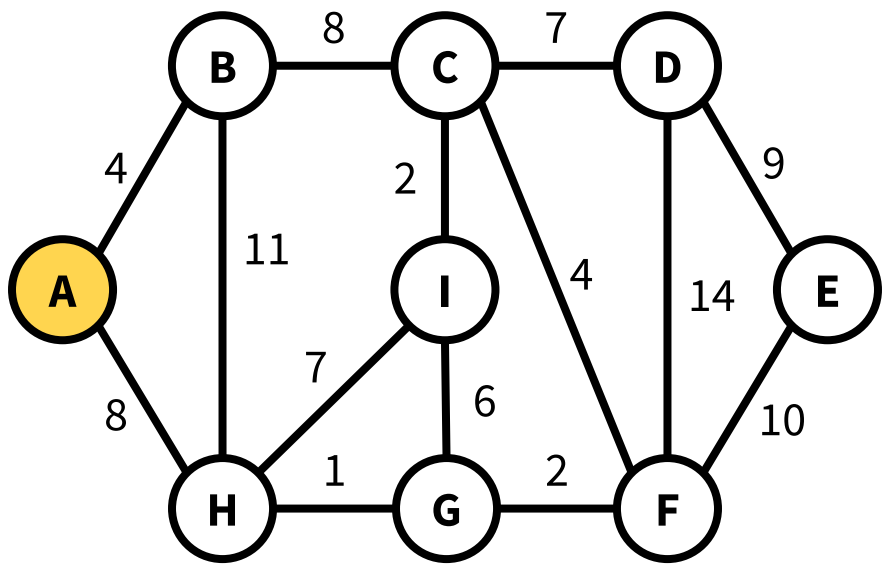

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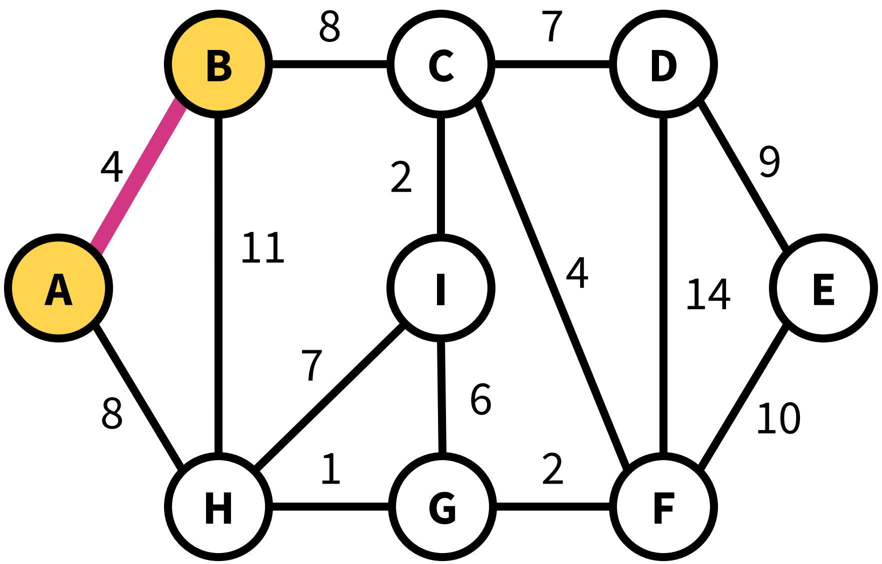

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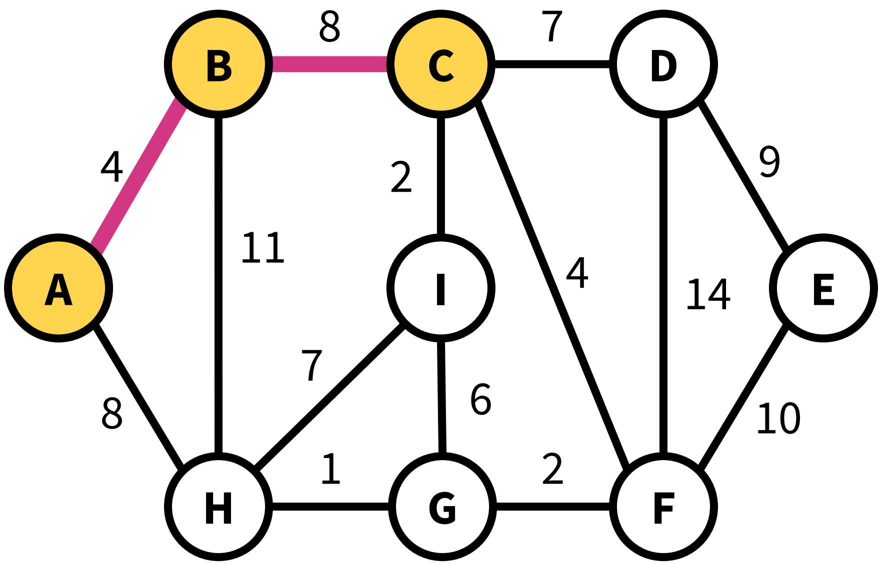

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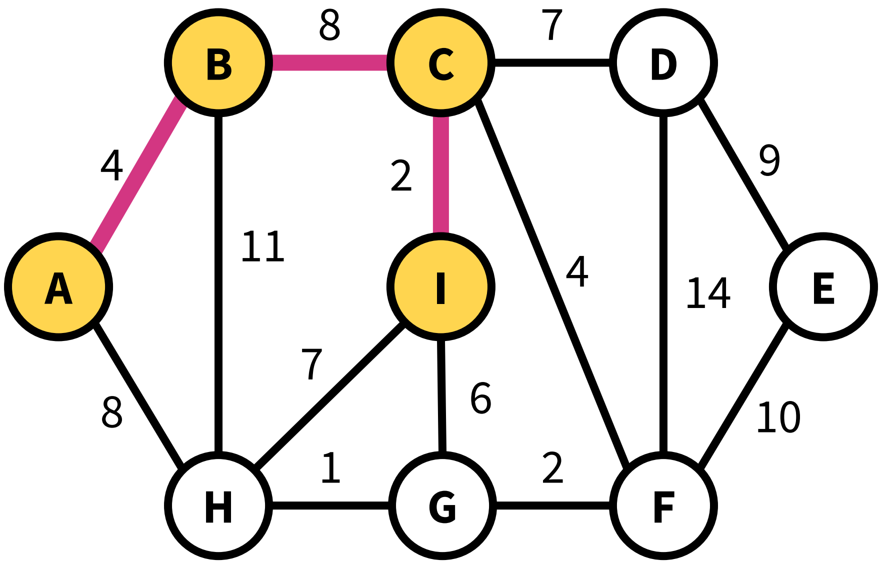

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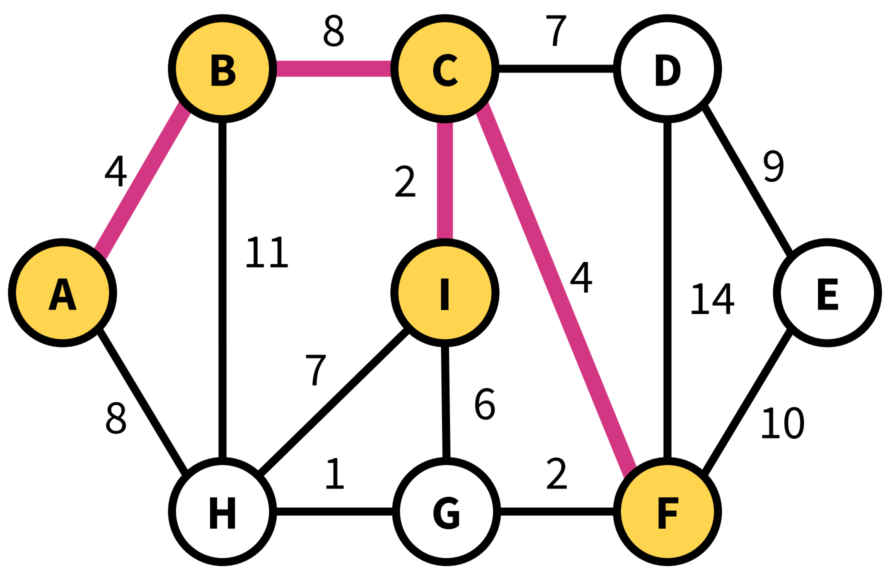

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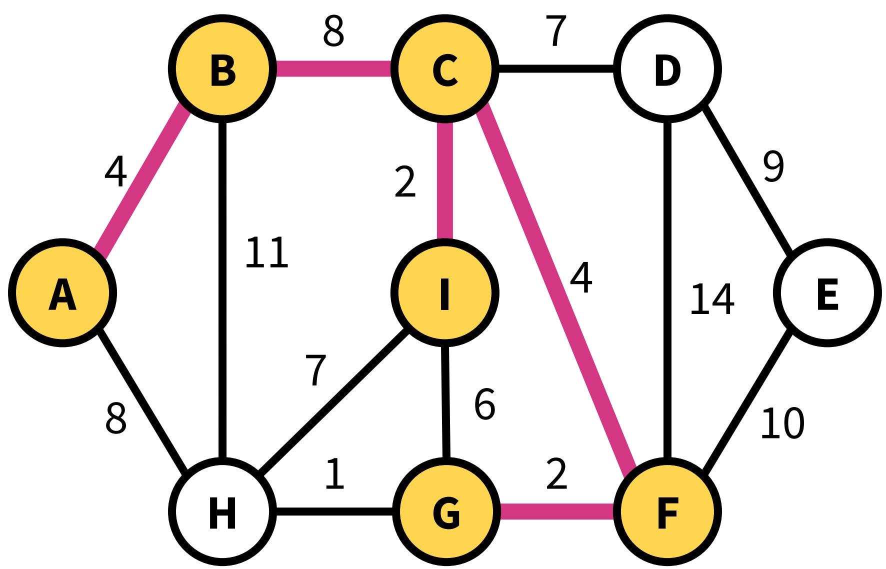

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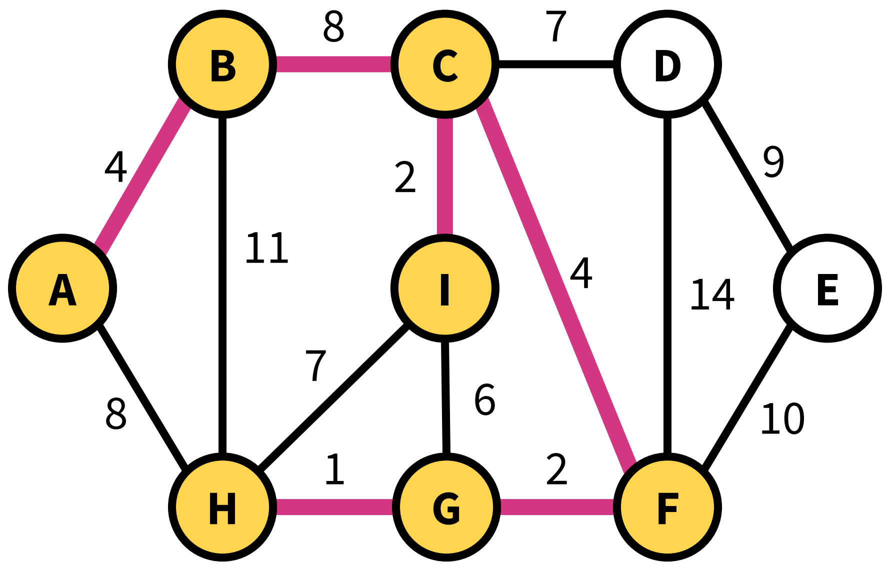

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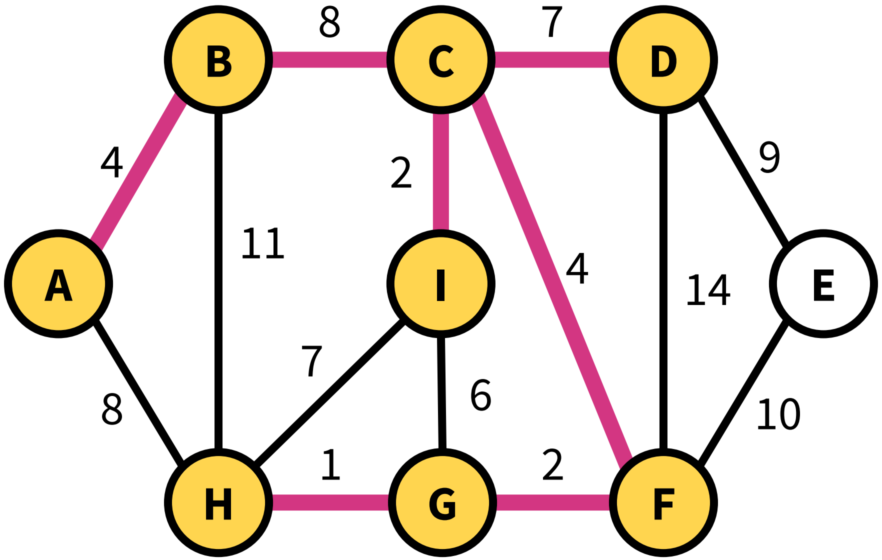

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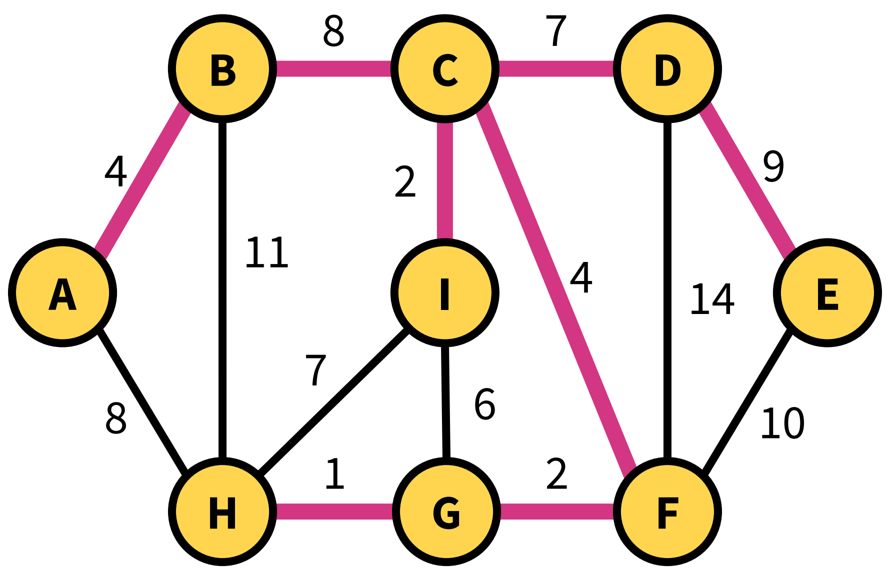

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def slow_prim(G):
  s = random vertex in G
  MST = {}
  visited_vertices = {s}
  while |visited_vertices| < |V|:
    (x, v) = lightest_edge(G, visited_vertices)
    MST.add((x, v))
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Runtime: $O(|V| \cdot |E|)$

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For each of the |V| iterations of the while loop, might need to iterate through all edges.

Proving Feasibility

Theorem: `prim` finds a feasible spanning tree.

Proof:

To prove this statement, we prove the loop invariant: `MST` contains edges of a spanning tree of the vertices in `visited_vertices`.

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Now, we prove the inductive step. Suppose that the invariant holds at the start of iteration i, so the edges in `MST` are (1) acyclic and (2) connect all vertices in `visited_vertices`. Then prim adds an edge (x, v) to `MST` and vertex v to `visited_vertices`.

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At the termination of the loop, `visited_vertices` contains all of the vertices, so `MST` contains a spanning tree over the entire graph. ⬛

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Recall our lemma:

Consider a cut that respects a set of edges **A**, such that there's an MST **T** containing **A**, and a light edge (u, v) not in **T**.

Lemma: There exists an MST containing **A** ∪ {(u, v)}.

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After adding the the (n-1)st edge, we have a spanning tree; therefore, `MST` contains a minimum spanning tree. ■

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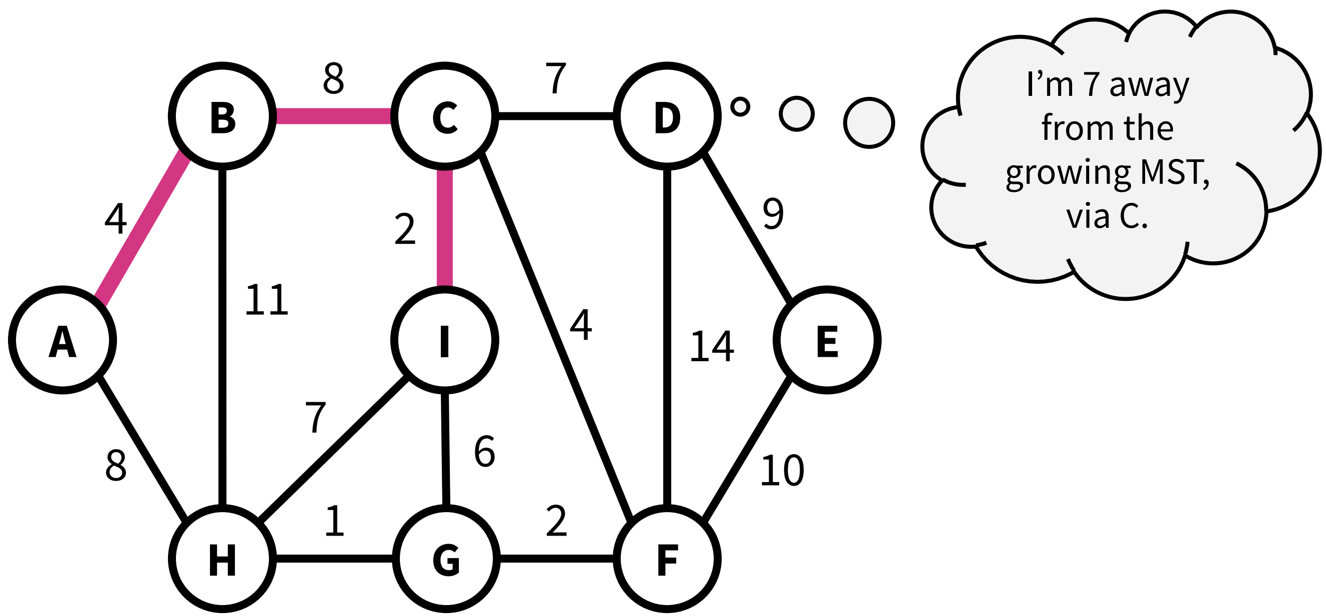
We called the algorithm `slow_prim`. There's a more efficient implementation.

Main idea: vertices maintain information about the distance from itself to the growing spanning tree (if one edge away) and how to get there.



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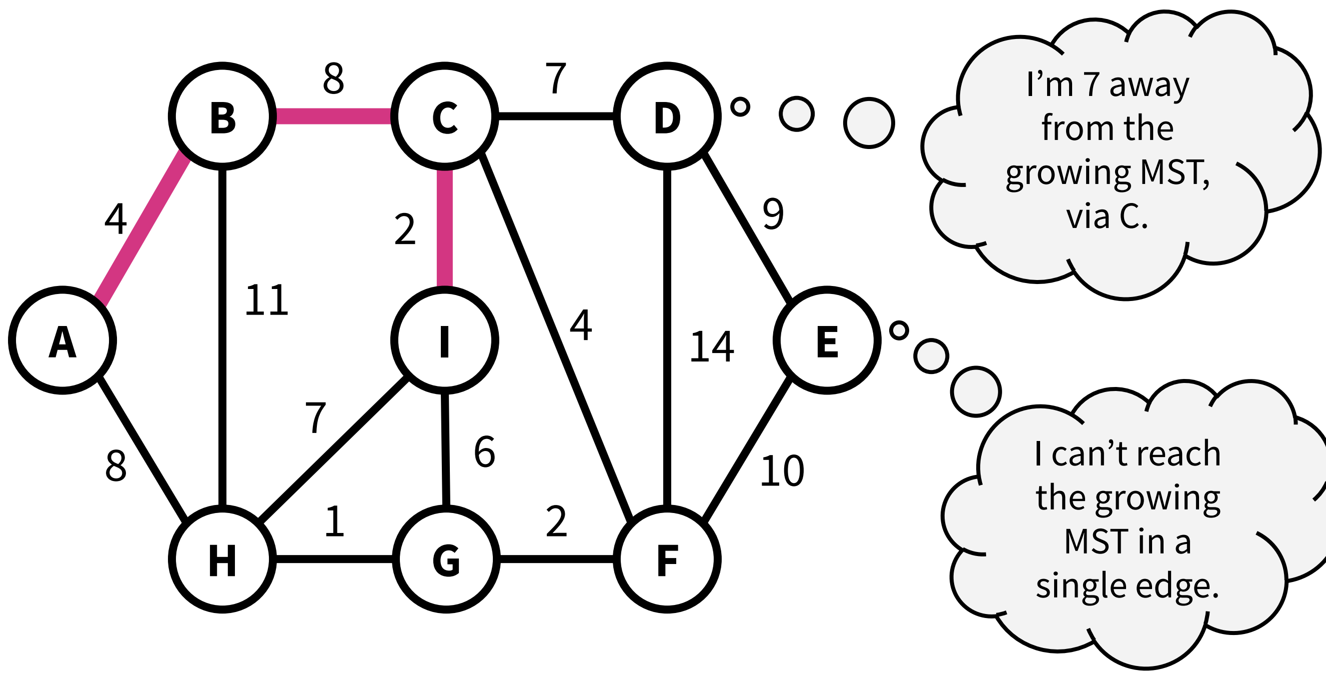

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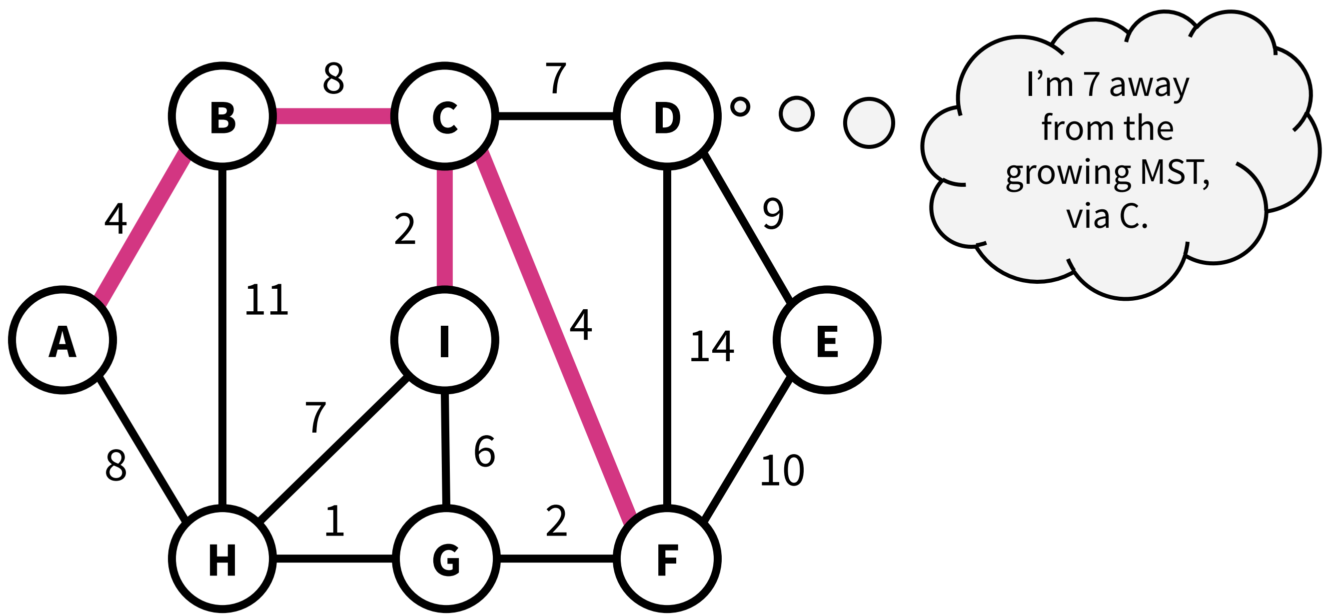

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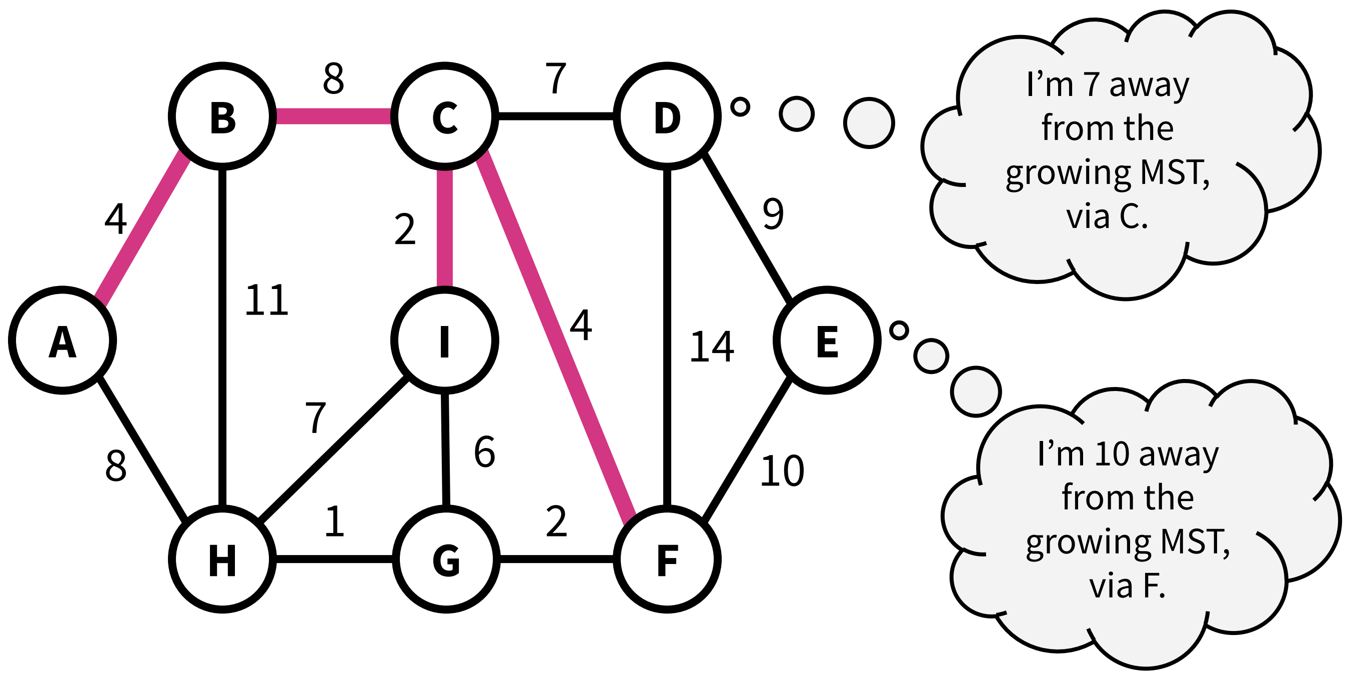

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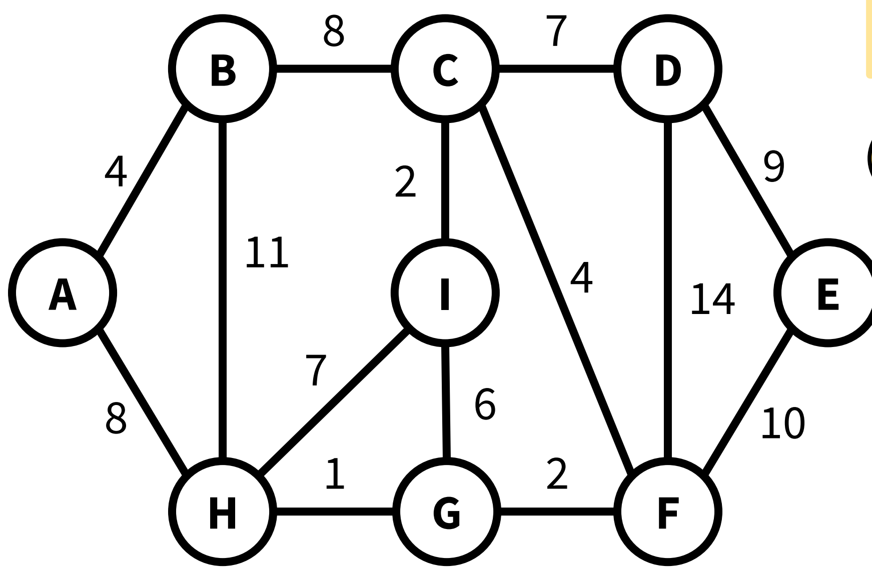

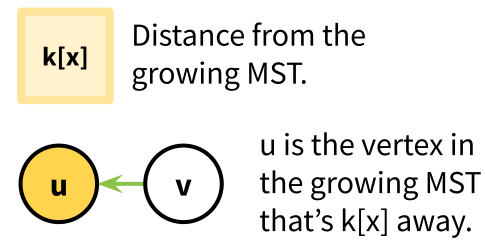

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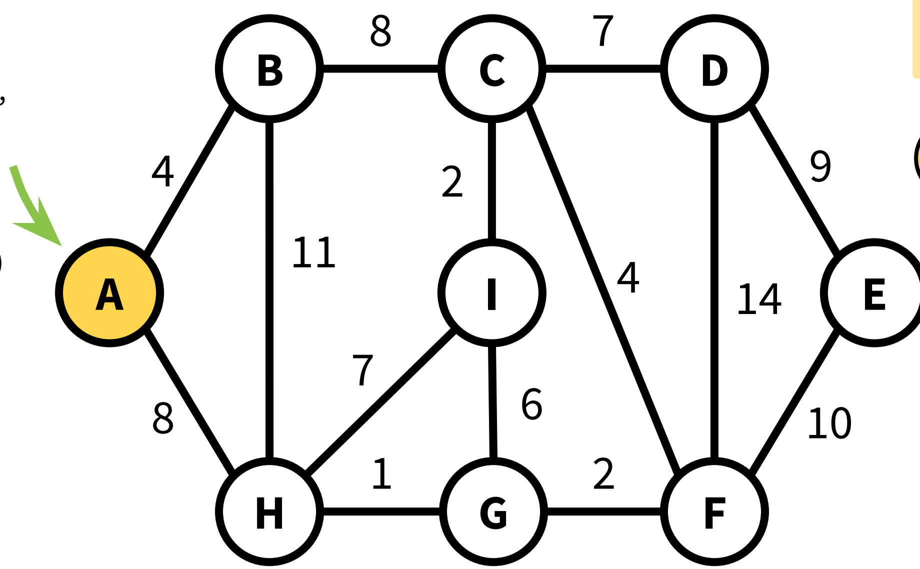
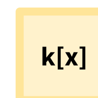
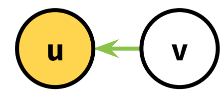
In addition to visiting A, update information of its neighbors with:

for v in u.neighbors:
  k[v] = min(k[v], w(u, v))





Distance from the growing MST.



u is the vertex in the growing MST that's k[x] away.

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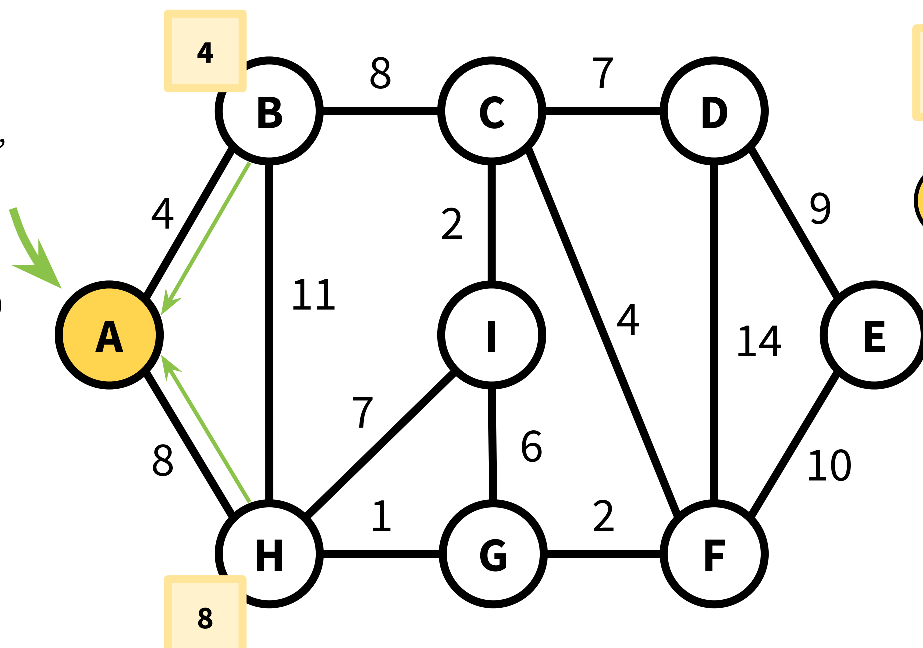

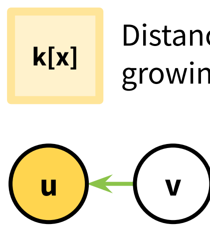

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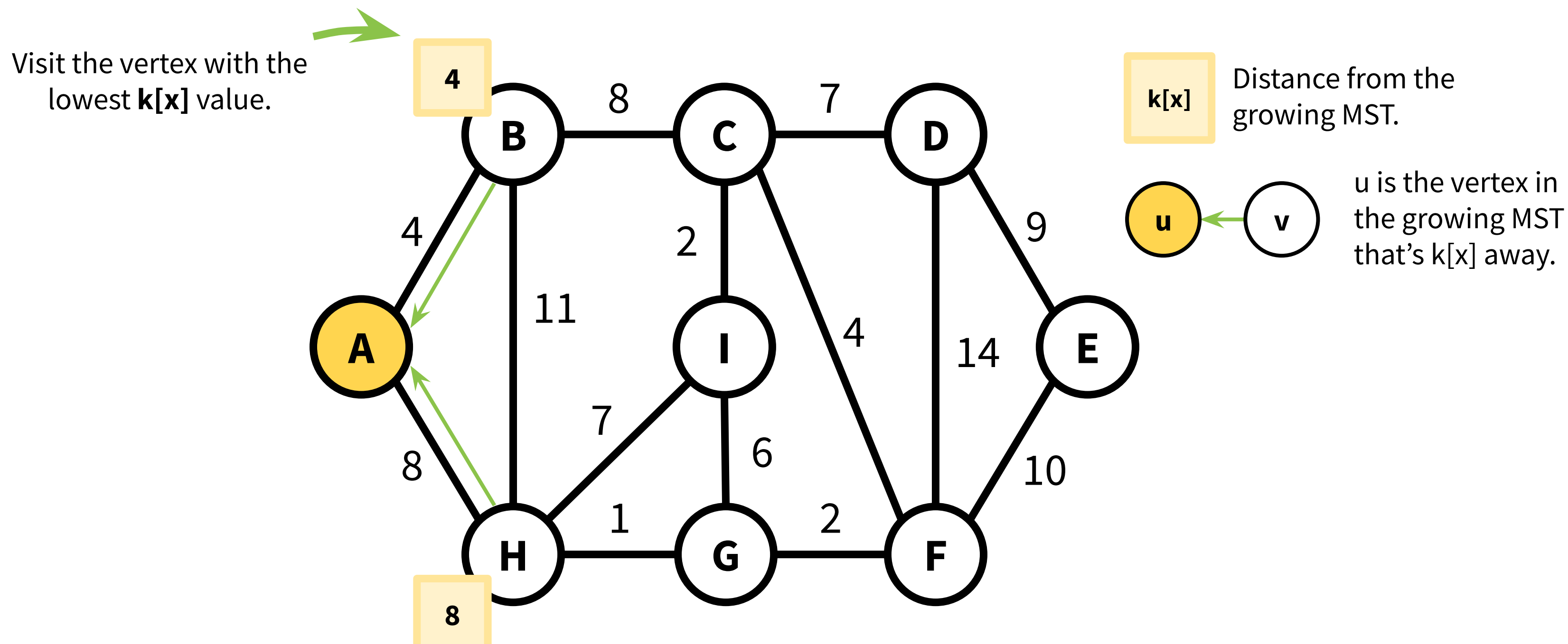

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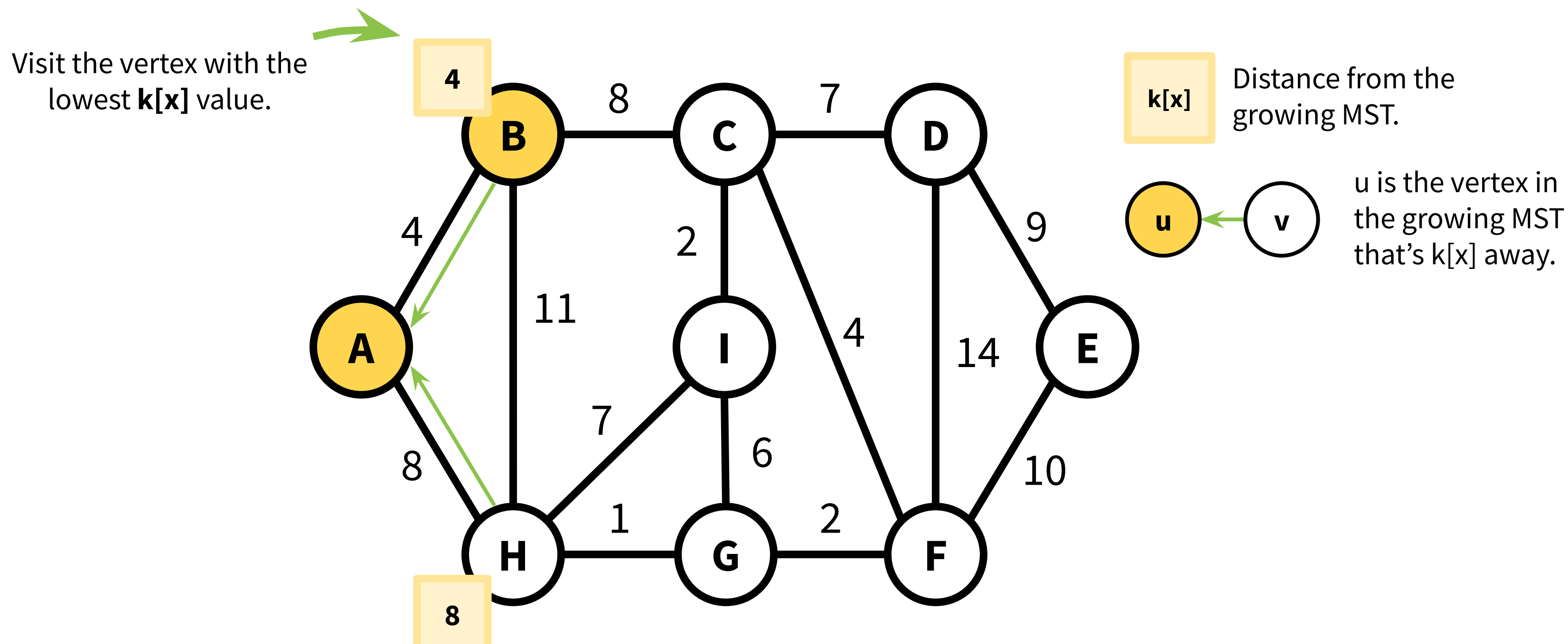

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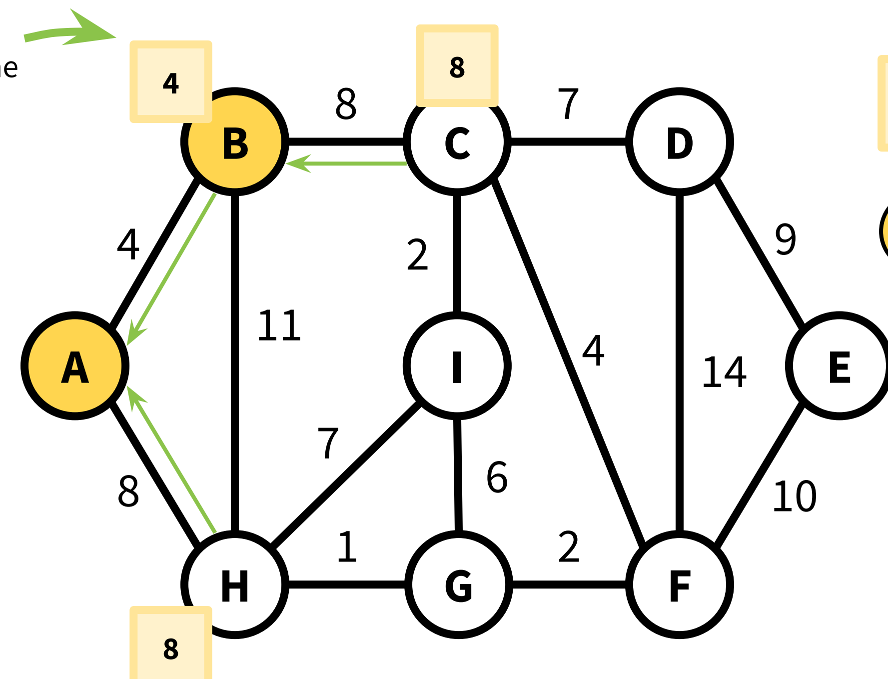
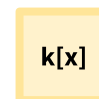
Visit the vertex with the lowest **k[x]** value.





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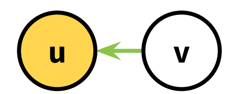

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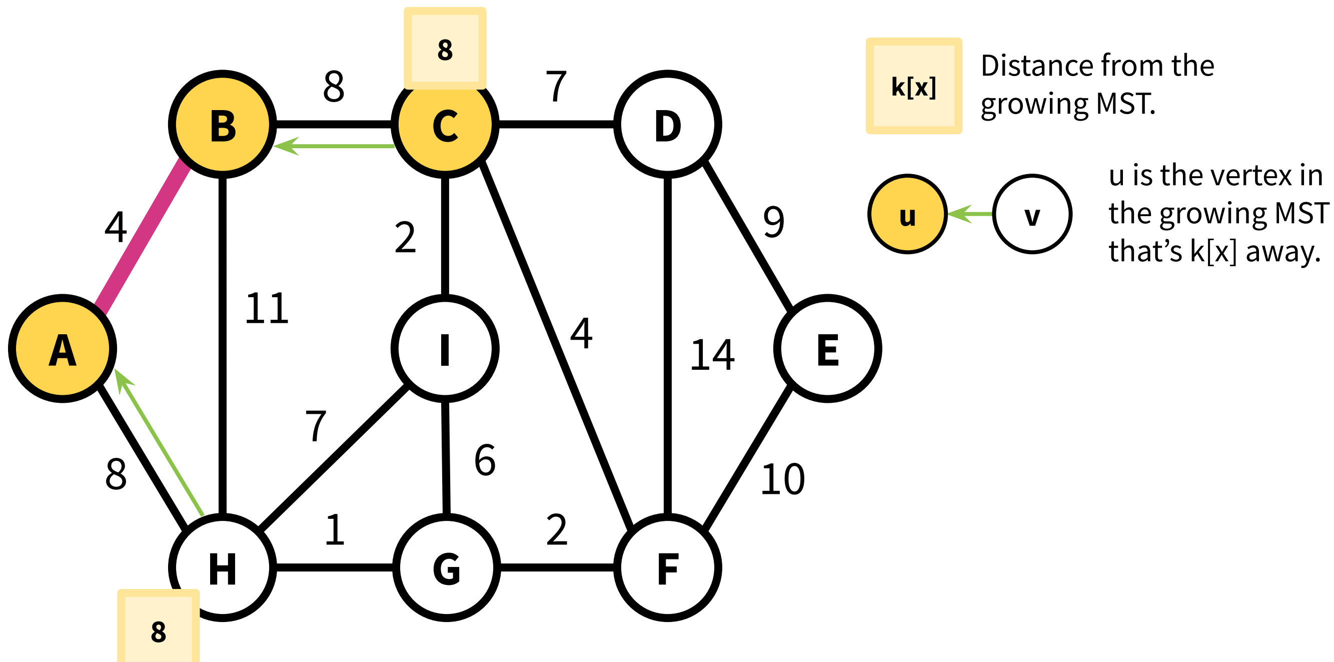

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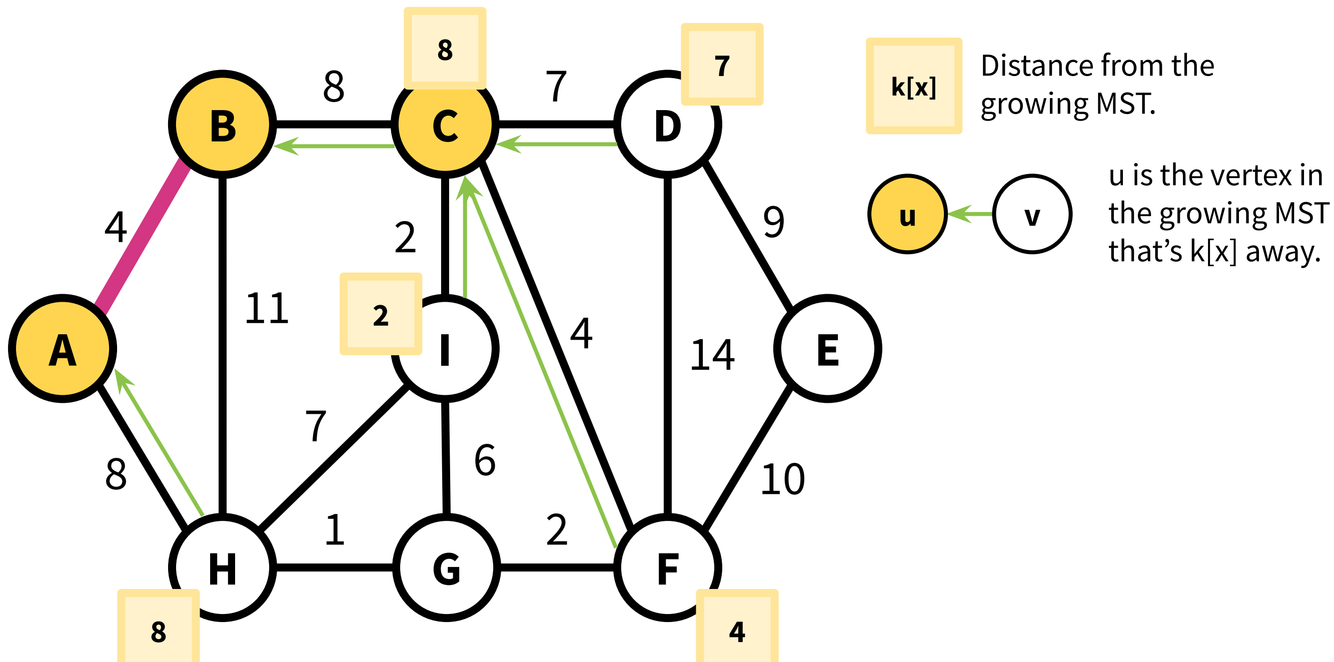

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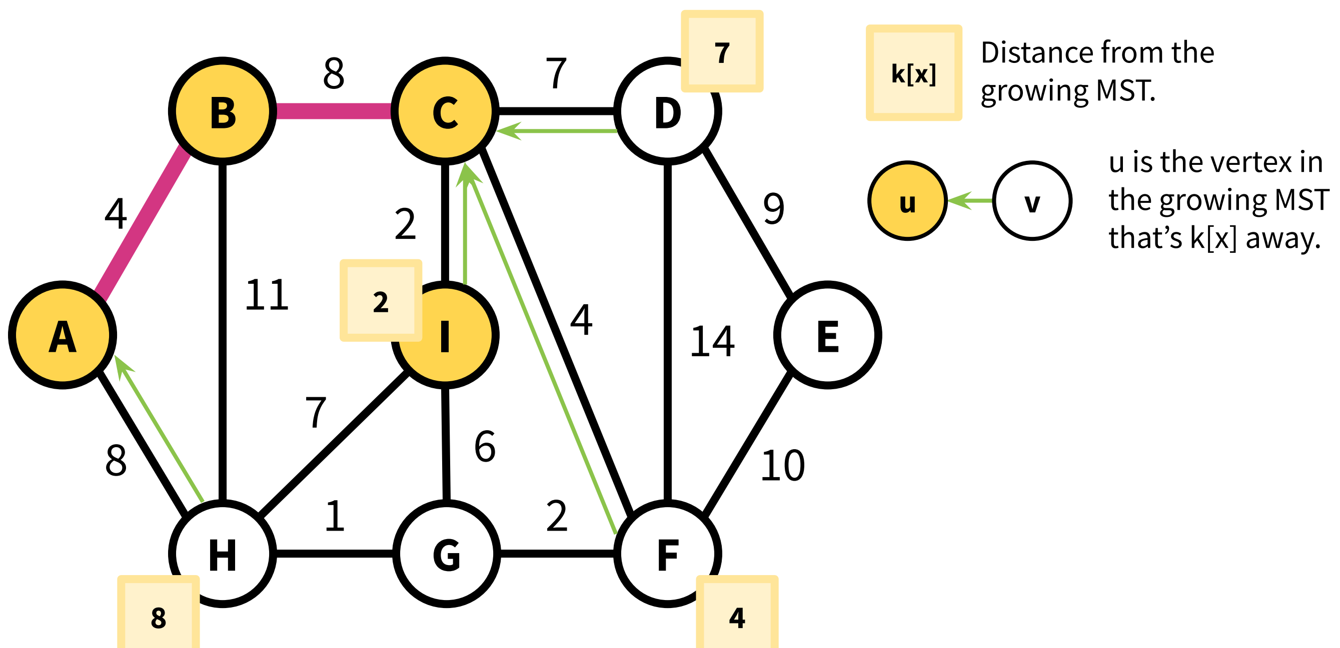

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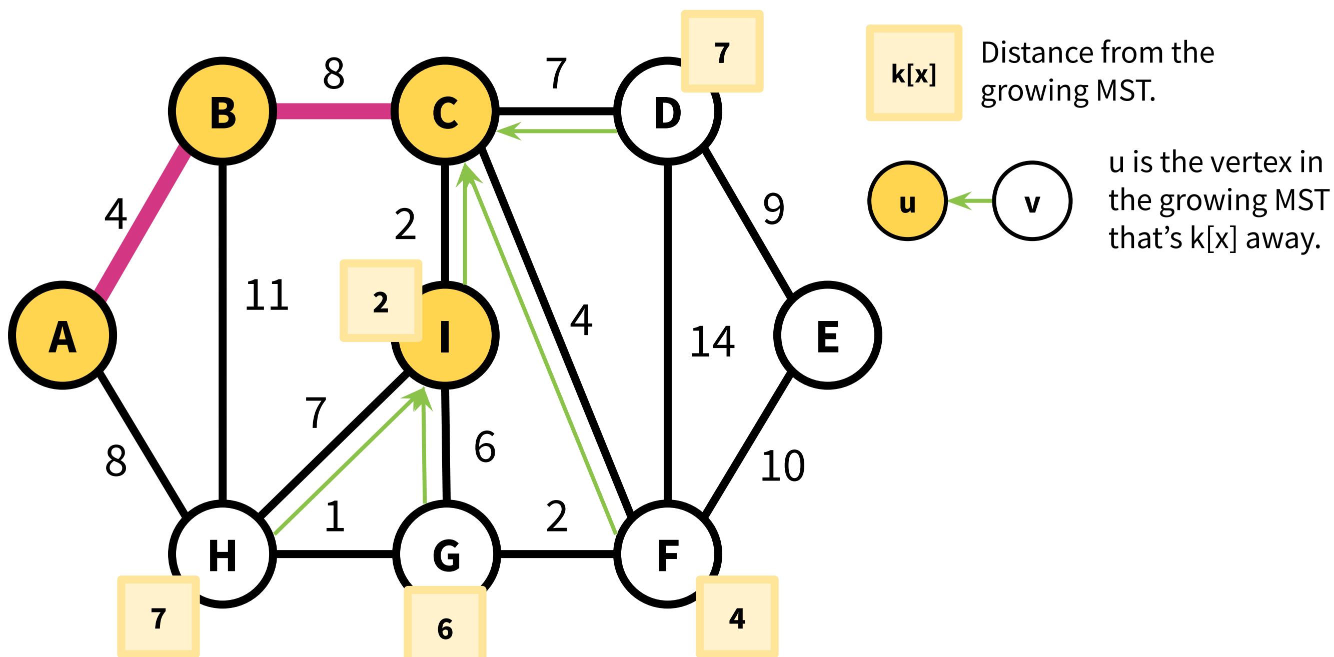

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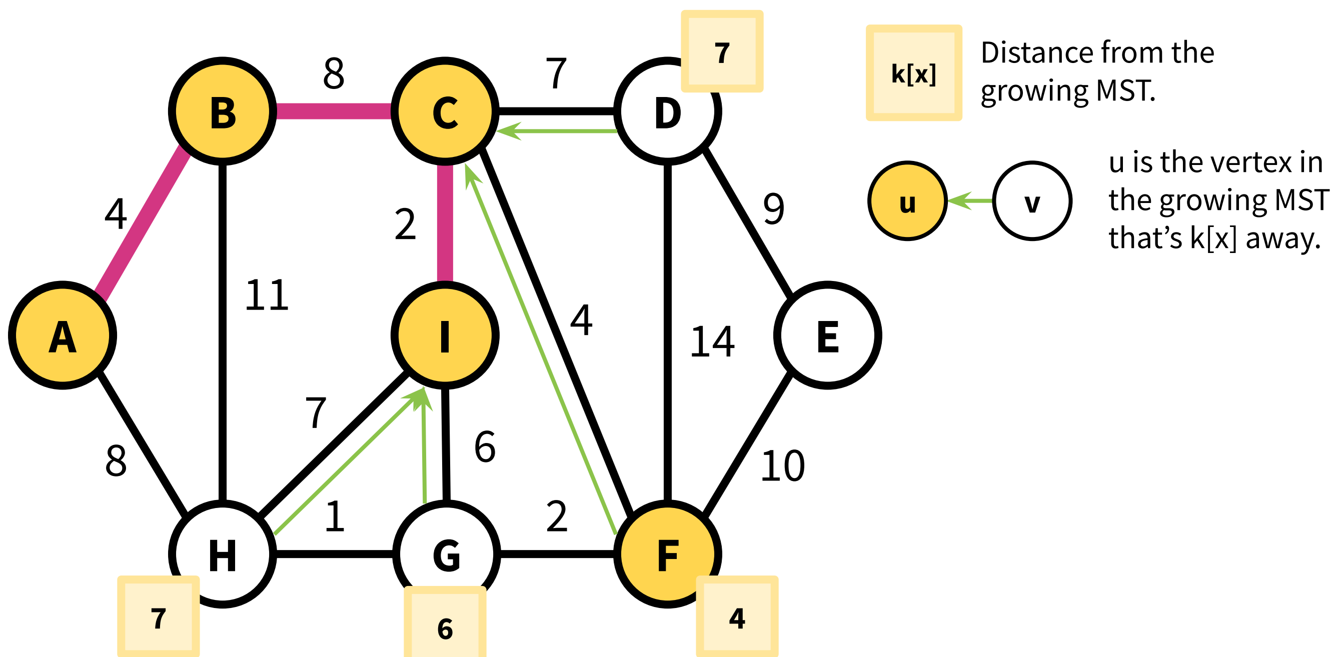

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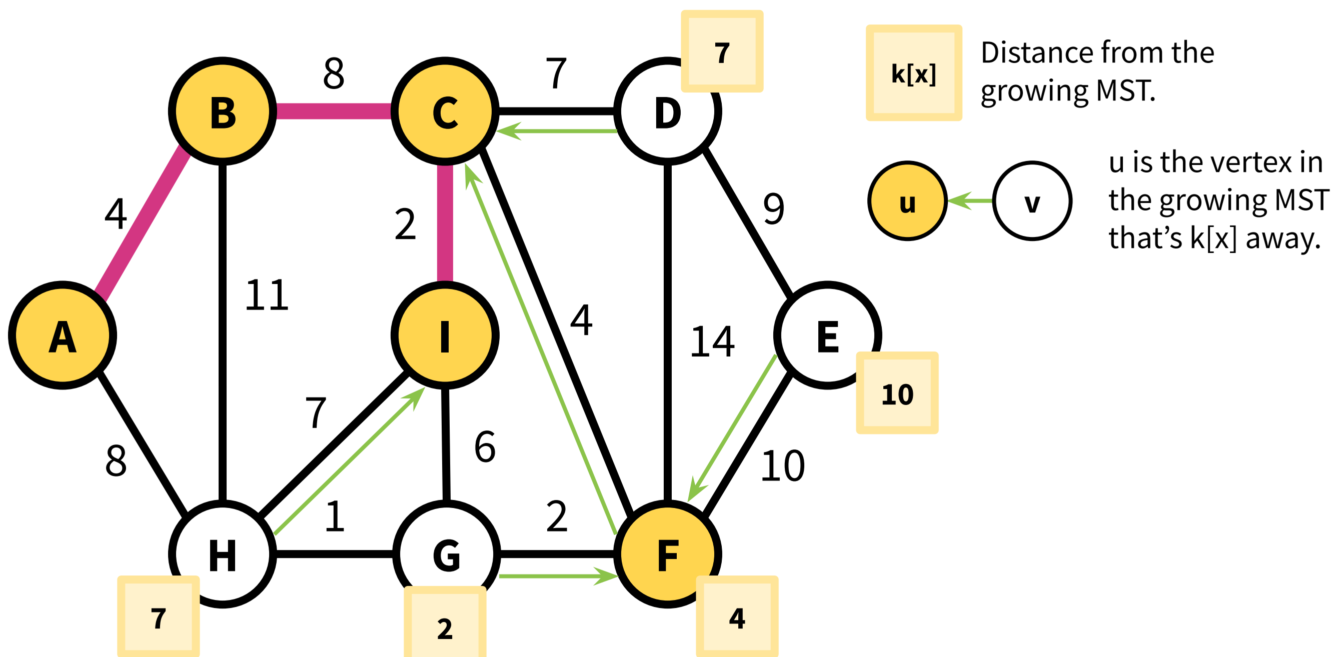

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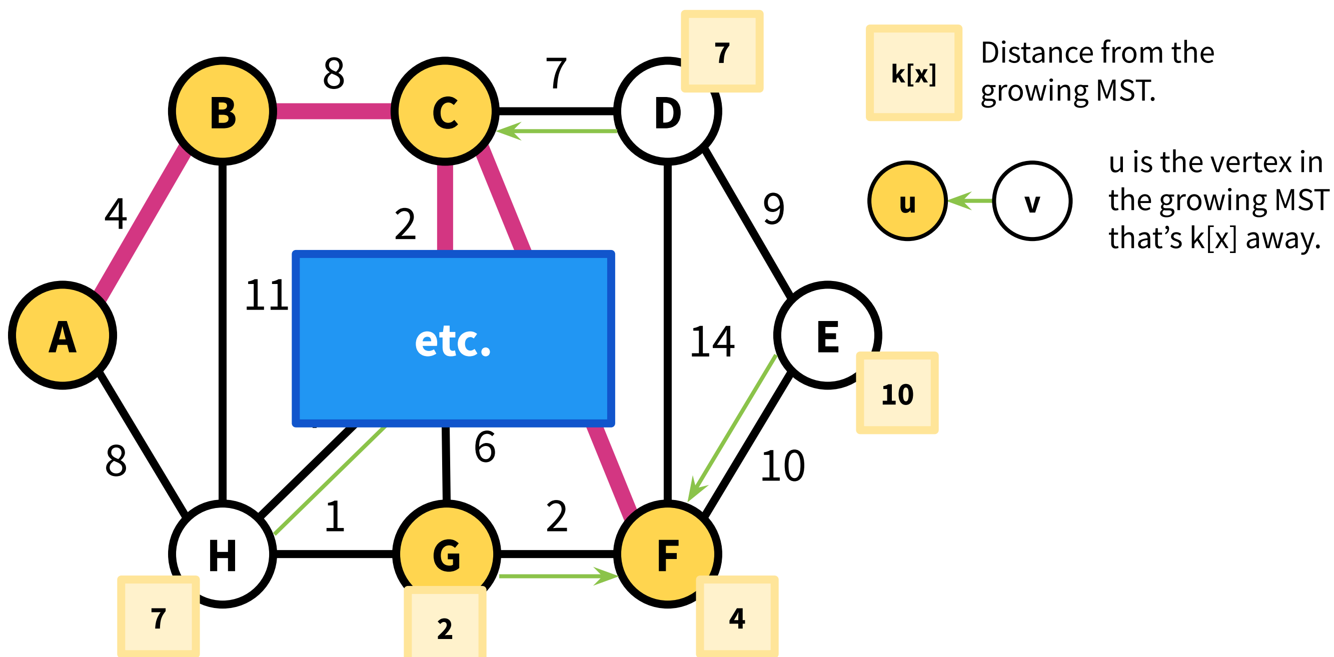

Prim's Algorithm

```
def prim(G):
  s = random vertex in G
  MST = {}
  visited_vertices = {s}
  update_info(G, s)
  while |visited_vertices| < |V|:
    (x, v) = lightest_edge(G, visited_vertices)
    MST.add((x, v))
    visited_vertices.add(v)
    update_info(G, v)
  return MST
```

Updates information about distance from the growing MST.

Runtime:

$O(|E|\log(|V|))$ Using a red-black tree as a priority queue

$O(|E|+|V|\log(|V|))$ Using a fibonacci heap