Digital Electronics and Microprocessor Systems (ELEC211)

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Digital 12: The Quine-McCluskey method

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Outline

- Define
  - Implicants
  - Prime Implicants (PIs)
  - Essential Prime Implicants (EPIs)
- Quine-McCluskey v. Karnaugh maps
- Find PIs using Q-M…
- … and using a PI chart, find EPIs and a minimum SOP expression for a function

Previous material

Karnaugh maps✓

Shannon's expansion✓

Minterms ✓

Minimum sum of products ✓

Use VITAL!:

- Stream lectures
- Handouts
- Notes and Q&A each week
- Discussion Board
- Exam resources

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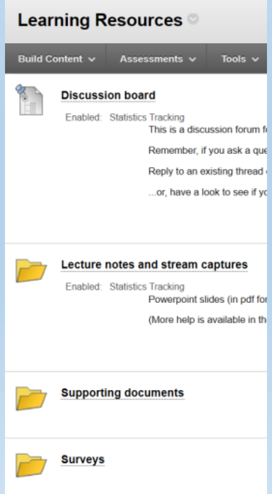
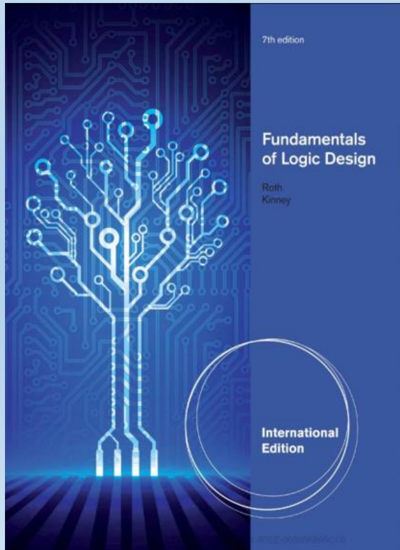


Course textbook – 7th ed. available as e-book!



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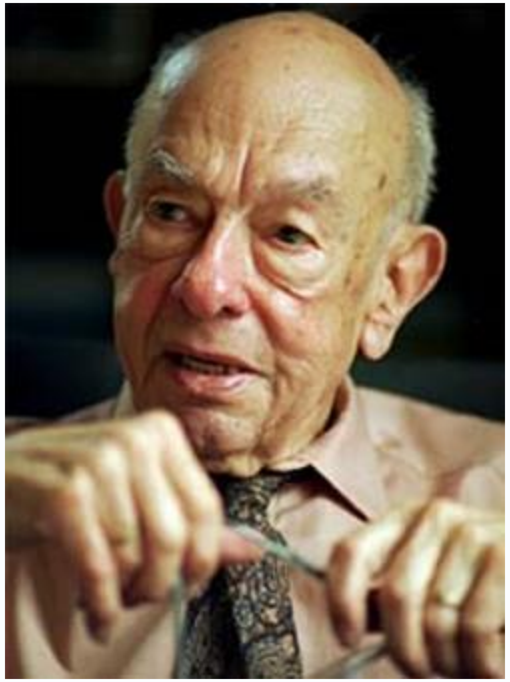
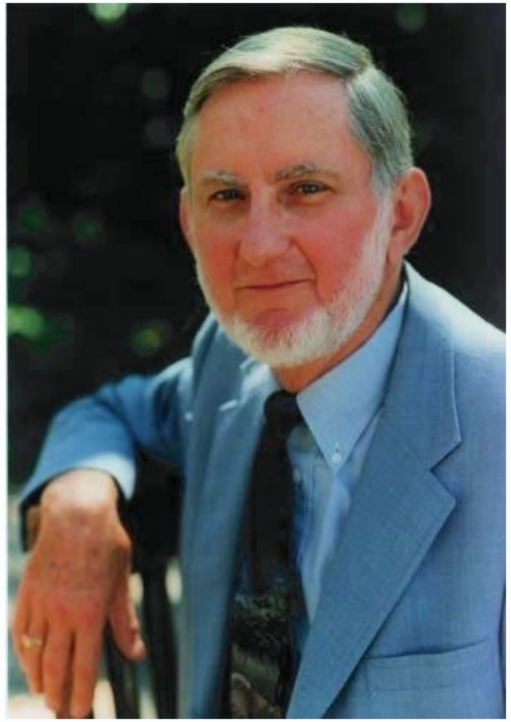
Quine McCluskey



Willard Van Orman Quine (1908 - 2000)
Professor of Mathematics at Harvard



Edward J. McCluskey (1929 – 2016)
Professor of Electrical Eng. at Stanford

Literal

- Each appearance of a variable or its complement in an expression

a and b’ are *literals* in the expression a.b’

- The literal **b’** is any appearance of the variable b in complemented form

Minterm (reminder)

- A minterm of *n* variables is a 'product' (logical AND) of *n* literals in which each variable appears exactly once in either true or complemented form.

a.b'.c.d is a minterm of 4 variables

the minterm

- It is a unique combination of literals which can only take a TRUE (logic 1) value for one set of variable values

the set of values for which the minterm takes a TRUE value

a.b'.c.d = 1 *only when* a = 1, b = 0, c = 1, d = 1 (when abcd = 1011)

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Definitions another way…

$f = a'.b'.c'.d' + a'.b.c'.d' + a'.b.c'.d + a'.b.c.d + a.b.c'.d + a.b.c.d$

These are all **minterms**

the set of values for which the minterm takes a TRUE value

f

cd \ ab	00	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	1	1	0
10	0	0	0	0

Implicant

- A product term **P** is an implicant of function *f* if for every combination of the values of the n variables for which **P** = 1, then *f* = 1

- For example **a.d** and **a.b’** are implicants of the function

$$\boldsymbol{f = a.d + a.b'}$$

Prime implicant

- A product term implicant which is no longer an implicant if any literal is removed from the product term

CONFUSED
???

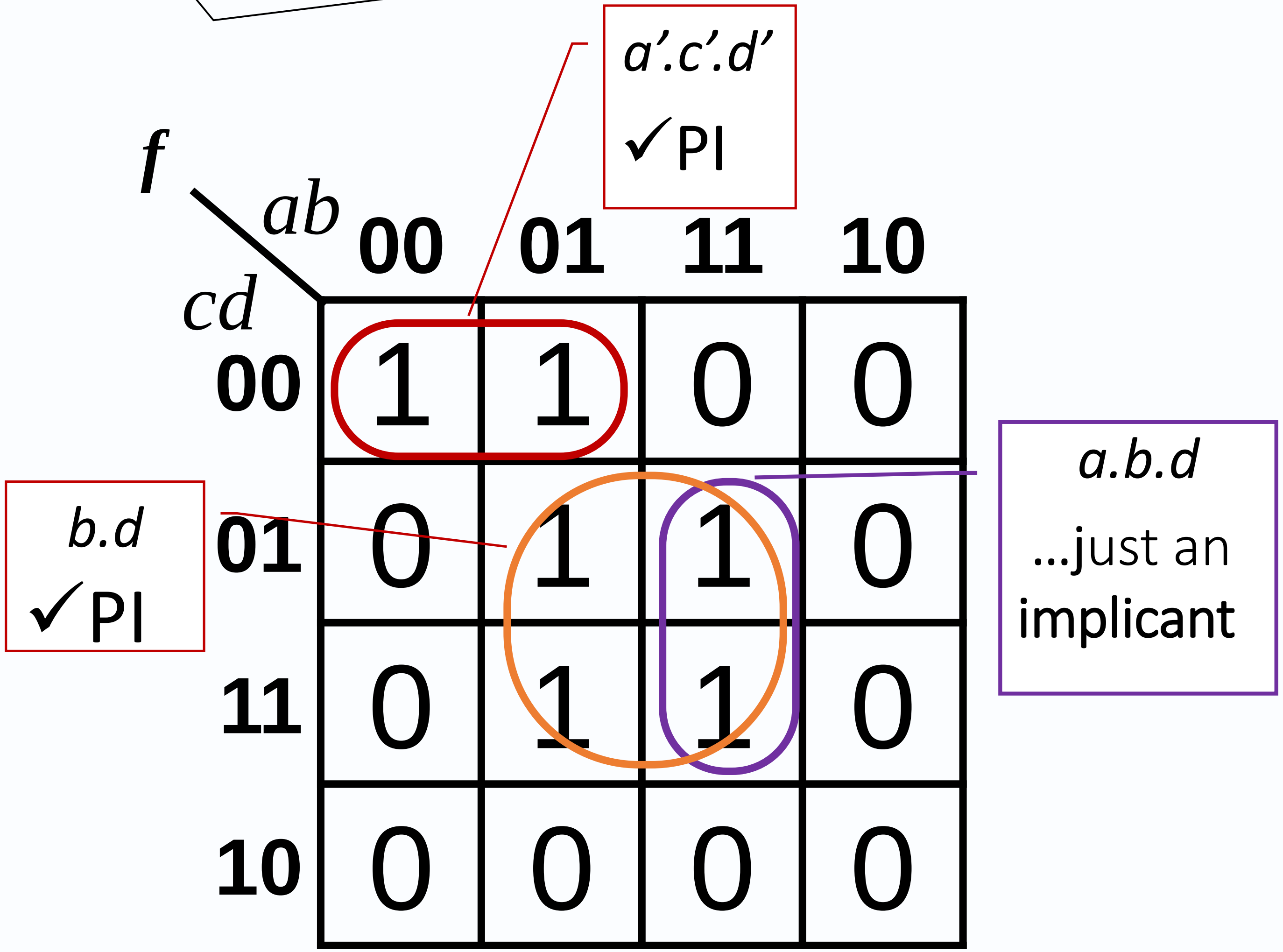
Definitions another way…

f = a’.b’.c’.d’ + a’.b.c’.d’ + a’.b.c’.d + a’.b.c.d + a.b.c’.d + a.b.c.d

These are all **minterms**

- All permissible K-map groupings are **IMPLICANTS** (even non-maximal ones)
- ONLY <u>maximal</u> groupings are called **PRIME IMPLICANTS**
- We look for PIs when simplifying with K-maps…

  ***f* = a’.c’.d’ + b.d**
- This is the **minimum SOP**



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But hang on…

An **ESSENTIAL PRIME IMPLICANT** is a Prime Implicant which is essential to cover a particular minterm - *without it, that minterm would not be covered by any other Prime Implicant*

E.g. m_{15} is **only** covered by *b.d*

b.d is an ESSENTIAL PI

E.g. m_0 is **only** covered by *a′.c′.d′*

a′.c′.d′ is an ESSENTIAL PI

Isn’t a’.b.c’ a maximal grouping too?

f

cd \ *ab*	**00**	**01**	**11**	**10**
00	1	1	0	0
01	0	1	1	0
11	0	1	1	0
10	0	0	0	0

a’.b.c’ is a Prime Implicant, but not an Essential Prime Implicant

Careful…

You **don't have** to be an Essential Prime Implicant to be part of the minimum SOP!

Here, the minimum SOP could be

a'.b.d + **b.c.d** + a.c.d'

OR

a'.b.d + **a.b.c** + a.c.d'

… even though neither **b.c.d** nor **a.b.c** is an EPI, we need <u>one of them!</u>

cd \ ab	**00**	**01**	**11**	**10**
00	0	0	0	0
01	0	1	0	0
11	0	1	1	0
10	0	0	1	1

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Quine-McCluskey method

- Karnaugh maps are an effective way to simplify functions which have up to 6 variables
  - *(5 or 6 variables can be mapped using Map-Entered Variables… upcoming topic)*
- A computerised method would be preferable:
  - for a large number of variables, or
  - for simplifying several functions
- **The Quine-McCluskey algorithm provides a systematic simplification procedure which can be readily programmed for a computer**
- It reduces the minterm expansion* of a function to obtain a minimum sum of products

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(*i.e. expression of the function purely as minterms)

Quine-McCluskey Method

- Just like K-maps, this algorithm eliminates as many literals as possible from each term by applying the rule:

$$X.Y + X.Y' = X$$

- The resulting terms are **Prime Implicants**
- After this, a Prime Implicant chart is used to select a minimum set* of prime implicants , which, when ORed together, are equal to the original function
- The resulting expression is the **minimum sum of products**

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*occasionally there is more than one minimum set

Determination of Prime Implicants

First the function must be defined as a sum of minterms, e.g. $f = m_4 + m_5$

(If the function is not in its minterm form, the minterm expansion must be found first.)

In the Q-M method, minterms are systematically combined using the XY + XY' = X rule.

A	B	C	Minterm	Value
0	0	0	A'.B'.C'	0
0	0	1	A'.B'.C	1
0	1	0	A'.B.C'	2
0	1	1	A'.B.C	3
1	**0**	**0**	**A.B'.C'**	**4**
1	**0**	**1**	**A.B'.C**	**5**
1	1	0	A.B.C'	6
1	1	1	A.B.C	7

SOP (minterms) e.g. A.B'.C' + A.B'.C = A.B'

Set of values for which SOP is true 1.0.0 + 1.0.1 = 1.0.–

Example

$$f(a,b,c,d)=\sum m\,(0,1,4,5,10,13,15)$$

f **defined as a sum of minterms**

In order to find all of the prime implicants, all possible pairs of minterms should be compared and combined whenever possible.

To organise the comparisons, the minterms are arranged into groups according to the number of 1's in each term …

(… i.e. in each associated set of values for which that minterm is true).

Column 1						
	Value	a	b	c	d	
Group 0	0	0	0	0	0	
Group 1	1	0	0	0	1	
	4	0	1	0	0	
Group 2	5	0	1	0	1	
	10	1	0	1	0	
Group 3	13	1	1	0	1	
Group 4	15	1	1	1	1	

Quine McCluskey v. Karnaugh map

Group 0: m_0

Group 1: $m_1\ m_2\ m_4\ m_8$

Group 2: $m_3\ m_5\ m_6\ m_9\ m_{10}\ m_{12}$

Group 3: $m_7\ m_{11}\ m_{13}\ m_{14}$

Group 4: m_{15}

	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

Number represents the value of each minterm.

Colours represents the group to which that minterm belongs.

Example - continued

$$f(a,b,c,d)=\sum m(0,1,4,5,10,13,15)$$

$cd \backslash ab$	00	01	11	10
00	1	1	0	0
01	1	1	1	0
11	0	0	1	0
10	0	0	0	1

Column 1						
	Value	a	b	c	d	
Group 0	0	0	0	0	0	
Group 1	1	0	0	0	1	
	4	0	1	0	0	
Group 2	5	0	1	0	1	
	10	1	0	1	0	
Group 3	13	1	1	0	1	
Group 4	15	1	1	1	1	

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$$f(a,b,c,d)=\sum m(0,1,4,5,10,13,15)$$

Compare entry in Group 0 with all entries in Group 1.

Terms 0000 and 0001 can be combined to eliminate the variable d, which gives 000–

Terms 0000 and 0100 can be combined to eliminate the variable b, which gives 0-00

Column 1						
	Value	a	b	c	d	
Group 0	0	0	0	0	0	✔
Group 1	1	0	0	0	1	✔
	4	0	1	0	0	✔

Column 2…….						
$a'.b'.c'$	0,1	0	0	0	-	
$a'.c'.d'$	0,4	0	-	0	0	

$cd \backslash ab$	00	01	11	10
00	1	1	0	0
01	1	1	1	0
11	0	0	1	0
10	0	0	0	1

Column I → Column 2

$$f(a,b,c,d)=\sum m(0,1,4,5,10,13,15)$$

Column 1						
	Value	a	b	c	d	
Group 0	0	0	0	0	0	✓
Group 1	1	0	0	0	1	✓
	4	0	1	0	0	✓
Group 2	5	0	1	0	1	✓
	10	1	0	1	0	PI
Group 3	13	1	1	0	1	✓
Group 4	15	1	1	1	1	✓

Column 2						
	Value	a	b	c	d	
Group 0	0,1	0	0	0	-	
	0, 4	0	-	0	0	
Group 1	1,5	0	-	0	1	
	4, 5	0	1	0	-	
Group 2	5, 13	-	1	0	1	
Group 3	13,15	1	1	-	1	

One prime implicant, m_{10}

Column I → Column 2

$$f(a,b,c,d)=\sum m(0,1,4,5,10,13,15)$$

cd \ ab	00	01	11	10
00	1	1	0	0
01	1	1	1	0
11	0	0	1	0
10	0	0	0	1

Column 2						
	Value	a	b	c	d	
Group 0	0,1	0	0	0	-	
	0, 4	0	-	0	0	
Group 1	1,5	0	-	0	1	
	4, 5	0	1	0	-	
Group 2	5, 13	-	1	0	1	
Group 3	13,15	1	1	-	1	

Prime implicant in prev. slide = m_{10} (associated with $abcd$ = 1010)

Column 2 → Column 3

$$f(a,b,c,d)=\sum m\left(0,1,4,5,10,13,15\right)$$

Column 2						
	Value	a	b	c	d	
Group 0	0,1	0	0	0	-	✓
	0, 4	0	-	0	0	✓
Group 1	1,5	0	-	0	1	✓
	4, 5	0	1	0	-	✓
Group 2	5, 13	-	1	0	1	PI
Group 3	13, 15	1	1	-	1	PI

Two more prime implicants.

Column 3						
	Value	a	b	c	d	
	0,1,4,5	0	-	0	-	PI
	~~0,4,1,5~~	~~0~~	~~-~~	~~0~~	~~-~~	

Note (above) that there are duplicate terms; one can be deleted.

..and this is also a prime implicant.

Summarising the Prime Implicants

$$f(a,b,c,d)=\sum m(0,1,4,5,10,13,15)$$

$cd \backslash ab$	00	01	11	10
00	1	1	0	0
01	1	1	1	0
11	0	0	1	0
10	0	0	0	1

Prime implicants						
	Value	a	b	c	d	
$a.b'.c.d'$	10	1	0	1	0	
$b.c'.d$	5,13	-	1	0	1	
$a.b.d$	13,15	1	1	-	1	
$a'.c'$	0,1,4,5	0	-	0	-	

The terms which were not ticked off (could not be combined) are the **prime implicants**.

Question

- Use the Quine McCluskey method to find the prime implicants of the following expression:

$$f(a,b,c,d) = \sum m(7,10,11,13,14,15)$$

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Solution

$$f(a,b,c,d) = \sum m(7,10,11,13,14,15)$$

minterm	a b c d	Group
7	0 1 1 1	3
10	1 0 1 0	2
11	1 0 1 1	3
13	1 1 0 1	3
14	1 1 1 0	3
15	1 1 1 1	4

Column 1						
	Value	a	b	c	d	
Group 2	10	1	0	1	0	✓
Group 3	7	0	1	1	1	✓
	11	1	0	1	1	✓
	13	1	1	0	1	✓
	14	1	1	1	0	✓
Group 4	15	1	1	1	1	✓

Column 2						
	Value	a	b	c	d	
Group 2	10, 11	1	0	1	-	
	10, 14	1	-	1	0	
Group 3	7, 15	-	1	1	1	
	11, 15	1	-	1	1	
	13, 15	1	1	-	1	
	14, 15	1	1	1	-	

Solution

$$f(a,b,c,d)=\sum m(7,10,11,13,14,15)$$

Column 1						
	Value	a	b	c	d	
up 2	10	1	0	1	0	✓
up 3	7	0	1	1	1	✓
	11	1	0	1	1	✓
	13	1	1	0	1	✓
	14	1	1	1	0	✓
up 4	15	1	1	1	1	✓

Column 2						
	Value	a	b	c	d	
Group 2	10, 11	1	0	1	-	✓
	10, 14	1	-	1	0	✓
Group 3	7, 15	-	1	1	1	**PI**
	11, 15	1	-	1	1	✓
	13, 15	1	1	-	1	**PI**
	14, 15	1	1	1	-	✓

Column 3						
	Value	a	b	c	d	
Group 2	10,11,14,15	1	-	1	-	**PI**
	~~10,14,11,15~~	~~1~~	~~-~~	~~1~~	~~-~~	

Prime implicants are
$b.c.d$, $a.b.d$ and $a.c$

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The Prime Implicant Chart

For finding the essential prime implicants – which helps with the minimum SOP

The minterms are listed across the top of the chart

The prime implicants are listed down the side, both as a sum of minterms and as a product term

If a prime implicant covers a given minterm, an X is placed at the intersection of the corresponding row and column.

			7	10	11	13	14	15
(7,15)	$b.c.d$		X					X
(13,15)	$a.b.d$					X		X
(10,11,14,15)	$a.c$			X	X		X	X

The Prime Implicant Chart

Remember, if a minterm is covered by only one prime implicant, then that prime implicant is an ***essential prime implicant***.

So $\boldsymbol{b.c.d}$, $\boldsymbol{a.b.d}$ and $\boldsymbol{a.c}$ are all **essential prime implicants**.

			7	10	11	13	14	15
(7,15)	$b.c.d$		X					X
(13,15)	$a.b.d$					X		X
(10,11,14,15)	$a.c$			X	X		X	X

Question

- Using a prime implicant chart, find the essential prime implicants of the following expression:

$$f(a,b,c,d) = \sum m(0,1,4,5,10,13,15)$$

Given: the prime implicants found using QM are:

$a.b'.c.d'$	10
$b.c'.d$	5,13
$a.b.d$	13,15
$a'.c'$	0,1,4,5

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Solution

		0	1	4	5	10	13	15
10	$a.b'.c.d'$					X		
(5,13)	$b.c'.d$				X		X	
(13,15)	$a.b.d$						X	X
(0,1,4,5)	$a'.c'$	X	X	X	X			

$a.b'.c.d'$, $a.b.d$ and $a'.c'$ are **essential** prime implicants.

$b.c'.d$ is not an essential prime implicant because minterms m_5 and m_{13} are covered by other prime implicants.

Question

- Use the Quine McCluskey method to simplify the following expression: $f(a,b,c,d)=\sum m(0,1,2,5,6,7,8,9,10,14)$

We assume this means "find the minimum SOP" i.e. **fully** simplify

f (cd \ ab)	00	01	11	10
00	1	0	0	1
01	1	1	0	1
11	0	1	0	0
10	1	1	1	1

Column 1							
	Value		*a*	*b*	*c*	*d*	
Group 0	0		0	0	0	0	
Group 1	1		0	0	0	1	
	2		0	0	1	0	
	8		1	0	0	0	
Group 2	5		0	1	0	1	
	6		0	1	1	0	
	9		1	0	0	1	
	10		1	0	1	0	
Group 3	7		0	1	1	1	
	14		1	1	1	0	
Group 4	No members in this example						

Question

- Use the Quine McCluskey method to simplify the following expression: $f(a,b,c,d) = \sum m(0, 1, 2, 5, 6, 7, 8, 9, 10, 14)$

We assume this means "find the minimum SOP" i.e. **fully** simplify

minterm	a b c d	Group
0	0 0 0 0	0
1	0 0 0 1	1
2	0 0 1 0	1
5	0 1 0 1	2
6	0 1 1 0	2
7	0 1 1 1	3
8	1 0 0 0	1
9	1 0 0 1	2
10	1 0 1 0	2
14	1 1 1 0	3

Column 1					
	Value	*a*	*b*	*c*	*d*
Group 0	0	0	0	0	0
Group 1	1	0	0	0	1
	2	0	0	1	0
	8	1	0	0	0
Group 2	5	0	1	0	1
	6	0	1	1	0
	9	1	0	0	1
	10	1	0	1	0
Group 3	7	0	1	1	1
	14	1	1	1	0
Group 4	No members in this example				

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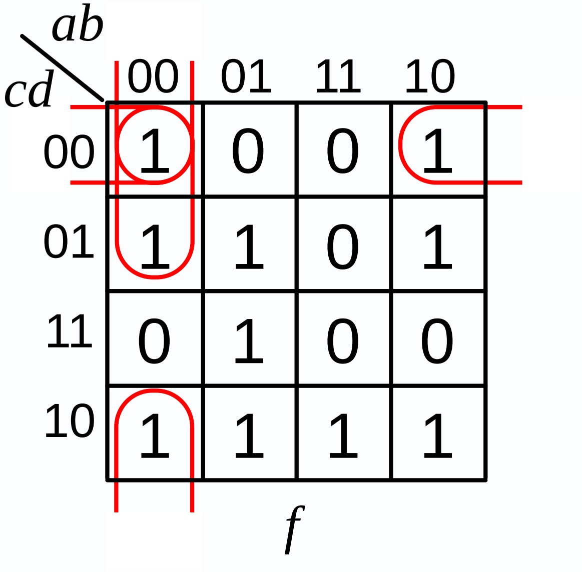
Step 1: determining Prime Implicants



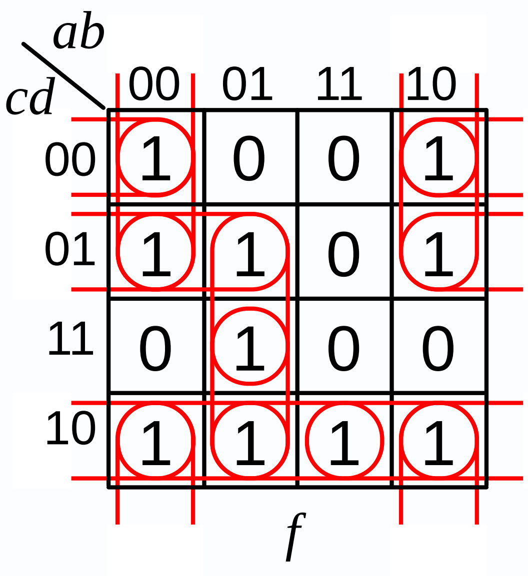
Pair minterms from group 0 with minterms from group 1.

(In general, pair group n with group (n+1).)

Column 1							
	Value		*a*	*b*	*c*	*d*	
Group 0	0		0	0	0	0	✔
Group 1	1		0	0	0	1	✔
	2		0	0	1	0	✔
	8		1	0	0	0	✔

Column 2							
	Value		*a*	*b*	*c*	*d*	
Group 0	0, 1		0	0	0	-	
	0, 2		0	0	-	0	
	0, 8		-	0	0	0	

Step 1: determining Prime Implicants



Column 2							
	Value		a	b	c	d	
Group 0	0,1		0	0	0	-	✓
	0,2		0	0	-	0	✓
	0,8		-	0	0	0	✓
Group 1	1,5		0	-	0	1	PI
	1,9		-	0	0	1	✓
	2,6		0	-	1	0	✓
	2,10		-	0	1	0	✓
	8,9		1	0	0	-	✓
	8,10		1	0	-	0	✓
Group 2	5,7		0	1	-	1	PI
	6,7		0	1	1	-	PI
	6,14		-	1	1	0	✓
	10,14		1	-	1	0	✓

Step 1: determining Prime Implicants

cd \ ab	00	01	11	10
00	1	0	0	1
01	1	1	0	1
11	0	1	0	0
10	1	1	1	1

f

Column 3							
	Values		a	b	c	d	
Group 0	0,1,8,9		-	0	0	-	PI
	0,2,8,10		-	0	-	0	PI
	~~0,8,1,9~~			~~0~~	~~0~~		~~PI~~
	~~0,8,2,10~~		~~-~~	~~0~~	~~-~~	~~0~~	~~PI~~
Group 1	2,6,10,14		-	-	1	0	PI
	~~2,10,6,14~~		~~-~~	~~-~~	~~1~~	~~0~~	~~PI~~

The prime implicants are given below.

$$f = a'c'd + a'bd + a'bc + b'c' + b'd' + cd'$$

(1,5) (5,7) (6,7) (0,1,8,9) (0,2,8,10) (2,6,10,14)

In this expression, each term has a minimum number of literals, but the number of terms is not minimum.

Step 2: The Prime Implicant Chart / EPIs

			0	1	2	5	6	7	8	9	10	14
(0,1,8,9)	b’c’		X	X					X	X		
(0,2,8,10)	b’d’		X		X				X		X	
(2,6,10,14)	cd’				X		X				X	X
(1,5)	a’c’d			X		X						
(5,7)	a’bd					X		X				
(6,7)	a’bc						X	X				

Note that minterms, m_9 and m_{14}, are covered by only one prime implicant, b’c’ and cd’. Therefore these must be essential prime implicants.

Step 2: The Prime Implicant Chart / EPIs

			0	1	2	5	6	7	8	9	10	14
(0,1,8,9)	b'c'		X	X					X	X		
(0,2,8,10)	b'd'		X		X				X		X	
(2,6,10,14)	cd'				X		X				X	X
(1,5)	a'c'd			X		X						
(5,7)	a'bd					X		X				
(6,7)	a'bc						X	X				

When a prime implicant is determined to be an essential prime implicant, all minterms in that row are covered.

That leaves m_5 and m_7 uncovered.

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Step 3: The minimum SOP

			5	7
(1,5)	a’c’d		X	
(5,7)	a’bd		X	X
(6,7)	a’bc			X

Prime implicant a’.b.d covers both minterms, m_5 and m_7, so it is chosen as part of the minimum sum of products expression (below) **even though it is not an essential prime implicant.**

cd \ ab	00	01	11	10
00	1	0	0	1
01	1	1	0	1
11	0	1	0	0
10	1	1	1	1

f

$$f = b'c' + cd' + a'bd$$

Question

- Use the Quine McCluskey method to simplify the following expression:

$$f(D,C,B,A) = \sum m(0,3,5,7,11,12,13,15)$$

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Method

$$f(D,C,B,A)=\sum m(0,3,5,7,11,12,13,15)$$

Column I

	Value		D	C	B	A	
Group 0	0		0	0	0	0	PI
Group 1							
Group 2	3		0	0	1	1	✓
	5		0	1	0	1	✓
	12		1	1	0	0	✓
Group 3	7		0	1	1	1	✓
	11		1	0	1	1	✓
	13		1	1	0	1	✓
Group 4	15		1	1	1	1	✓

Column II

Value		D	C	B	A	
3,7		0	-	1	1	✓
3,11		-	0	1	1	✓
5,7		0	1	-	1	✓
5,13		-	1	0	1	✓
12,13		1	1	0	-	PI
7,15		-	1	1	1	✓
11,15		1	-	1	1	✓
13,15		1	1	-	1	✓

Column III

Value		D	C	B	A	
3,7,11,15		-	-	1	1	PI
~~3,11,7,15~~		~~-~~	~~-~~	~~1~~	~~1~~	
5,7,13,15		-	1	-	1	PI
~~5,13,7,15~~		~~-~~	~~1~~	~~-~~	~~1~~	

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Answer

$$f(D,C,B,A)=\sum m(0,3,5,7,11,12,13,15)$$

			0	3	5	7	11	12	13	15
(0)	*D'C'B'A'*		X							
(12,13)	*DCB'*							X	X	
(3,7,11,15)	*BA*			X		X	X			X
(5,7,13,15)	*CA*				X	X			X	X

All prime implicants are essential prime implicants, THIS TIME...

$$f=CA+BA+DCB'+D'C'B'A'$$

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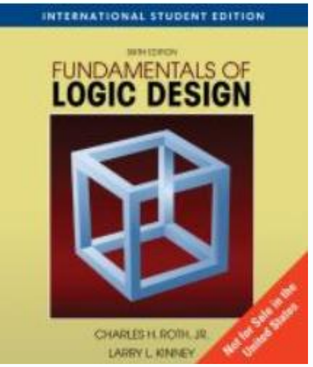
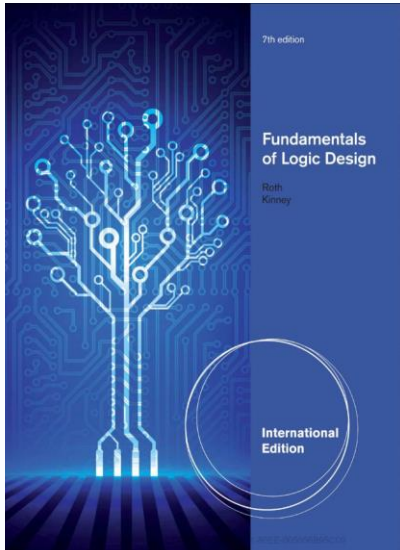
Summary and suggested reading



- Section 4.2-3 Minterm & maxterm expansions
- Section 6.1 Determination of Prime Implicants
- Section 6.2 Prime Implicant charts



Roth and Kinney *Fundamentals of Logic Design*

….. 7th ed. is available as an e-book!

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