Tutorial 6

1st price sealed-bid auction

- n players submit bids simultaneously in envelopes.
- The single item is won by the player with the highest bid.
- If multiple players have the same bid, the player with the highest valuation wins (there are no equal valuations, by assumption)
- The winner pays her bid.

Note: To make analysis easier, we name the players according to their valuations for the item. I.e. $v_1 > v_2 > \ldots > v_n$

We can formulate it as a strategic form game where each player i's utility is:

$$u_i(b_1, b_2, \ldots, b_n) = \begin{cases} v_i - b_i & \text{, if } i \text{ wins} \\ 0 & \text{, if } i \text{ looses} \end{cases}$$

Exercise 1

Consider a 1st price sealed-bid auction with 4 players where $v_1 = 8$, $v_2 = 6$, $v_3 = 2$, $v_4 = 1$. Consider also the following action profiles:

(a) $(b_1, b_2, b_3, b_4) = (8, 6, 2, 1)$

(b) $(b_1, b_2, b_3, b_4) = (8, 0, 0, 0)$

(c) $(b_1, b_2, b_3, b_4) = (5, 4, 5, 1)$

(d) $(b_1, b_2, b_3, b_4) = (6, 4, 1, 6)$

Which of the above is a NE? Justify your answer

Solution

(a) $u_1 = u_2 = u_3 = u_4 = 0$

The action profile is not a NE, because pl.① could choose $b_1' = 7$ and get $u_1' = 1 > 0 = u_1$.

(b) $u_1 = u_2 = u_3 = u_4 = 0$

The action profile is not a NE, because pl.① could choose $b_1' = 5$ and get $u_1' = 3 > 0 = u_1$.

(c) $u_1 = 3$, $u_2 = u_3 = u_4 = 0$

The action profile is not a NE, because pl.② could choose $b_2' = 5.5$ and get $u_2' = 0.5 > 0 = u_2$.

(d) $u_1 = 2$, $u_2 = u_3 = u_4 = 0$

The action profile is a NE, because:

- If $b_1' > 6$ then pl.① still wins, but gets $u_1' = 8 - b_1' < 2 = u_1$
  If $b_1' < 6$ then pl.① looses and gets $u_1' = 0 < 2 = u_1$
- If pl.② could get some utility better than $u_2 = 0$, she would have to win. If she won she would have chosen some $b_2' > 6$ and she would get $u_2' = 6 - b_2' < 0 = u_2$.
- For pl.③ and pl.④ a similar argument as for pl.② holds (Homework)

Useful Theorem (see lecture notes)

An action profile $(b_1, b_2, \dots, b_n)$ is a NE of a 1st price sealed-bid auction if and only if all of the below hold:

- the two highest bids are the same.
- one of these bids is submitted by pl.①.
- the highest bid is at least v_2 and at most v_1.

2nd price sealed-bid auction

- n players submit bids simultaneously in envelopes.
- The single item is won by the player with the highest bid
- If multiple players have the same bid, the player with the highest valuation wins (there are no equal valuations, by assumption)
- The winner pays the second highest bid.

Note: To make analysis easier, we name the players according to their valuations for the item. I.e. $v_1 > v_2 > \ldots > v_n$

We can formulate it as a strategic form game where each player i's utility is:

$$u_i(b_1, b_2, \ldots, b_n) = \begin{cases} v_i - b_k, & \text{if } i \text{ wins, where } b_k \text{ is the second highest bid} \\ 0, & \text{if } i \text{ looses} \end{cases}$$

Exercise 2

Consider a 2nd price sealed-bid auction with 4 players with valuations $v_1 = 8$, $v_2 = 6$, $v_3 = 2$, $v_4 = 1$. Consider also the action profiles of Exercise 1.

Which of them is a NE? Justify your answer.

Solution

(a) $u_1 = 2, u_2 = u_3 = u_4 = 0$

The action profile is a NE, because:

- If $b_1' \geq 6$ then pl.① still wins and gets $u_1' = 2 = u_1$.
- If $b_1' < 6$ then pl.① looses and gets $u_1' = 0 < 2 = u_1$.
- If pl.② could get some utility better than $u_2 = 0$, she would have to win. If she won she would have chosen some $b_2' > 8$ which would make $b_1 = 8$ the second highest bid, and she would get $u_2' = 6 - 8 = -2 < 0 = u_2$.
- For pl.③ and pl.④ a similar argument as for pl.② holds. (Homework)

(b) $u_1 = 8, u_2 = u_3 = u_4 = 0$

The action profile is a NE, because:

- If $b_1' \geq 0$ then pl.① still wins and gets $u_1' = 8 = u_1$. (She cannot bid a negative number)
- If pl.② could get some utility better than $u_2 = 0$, she would have to win. If she won she would have chosen some $b_2' > 8$ which would make $b_1 = 8$ the second highest bid, and she would get $u_2' = 6 - 8 = -2 < 0 = u_2$
- For pl.③ and pl.④ a similar argument as for pl.② holds. (Homework)

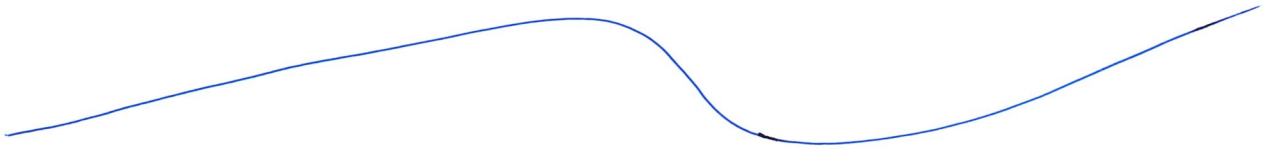
(c) $u_1 = 3, u_2 = u_3 = u_4 = 0$

The action profile is not a NE, because pl.② could choose $b_2' = 6$ and get $u_2' = 1 > 0 = u_2$.

(d) $u_1 = 2$, $u_2 = u_3 = u_4 = 0$

The action profile is a NE, because:

- If $b_1' > 6$ then pl.① still wins and gets $u_1' = 2 = u_1$.
  If $b_1' < 6$ then pl.① looses and gets $u_1' = 0 < 2 = u_1$.
- If pl.② could get some utility better than $u_2 = 0$, she would have to win. If she won she would have chosen some $b_2' > 6$ which would make $b_1 = 6$ the second highest bid, and she would get $u_2' = 6 - 6 = 0 = u_2$.
- If pl.③ could get some utility better than $u_3 = 0$, she would have to win. If she won she would have chosen some $b_3' > 6$ which would make $b_1 = 6$ the second highest bid, and she would get $u_3' = 2 - 6 = -4 < 0 = u_3$.
- For pl.④ a similar argument as for pl.③ holds. (Homework)



Remark (see lecture notes)

In a 2nd price sealed-bid auction, the following action profiles are NE:

- $(b_1, b_2, \dots, b_n) = (v_1, v_2, \dots, v_n)$ ⇝ Also, truthful bidding weakly dominates all other bids for a player.
- $(b_1, b_2, \dots, b_n) = (v_1, 0, \dots, 0)$
- $(b_1, b_2, \dots, b_n) = (v_2, v_1, 0, \dots, 0)$