COMP323 – Introduction to Computational Game Theory

Auctions

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Outline

Part I of lecture notes 6 : **Introduction**

Auctions

- In an auction, a good is sold to the party who submits the highest bid.
- It is a kind of economic activity used to allocate significant resources, such as
    - works of art;
    - radio spectrum used for wireless communication;
    - Treasury bills and timber and oil leases.
- Auctions take many forms:
    - bids may be called out sequentially or submitted in sealed envelopes;
    - the price paid may be the highest bid or some other price;
    - if more than one unit of good is being sold, bids may be taken on all units simultaneously or the units may be sold sequentially.
- A game-theoretic analysis of auctions helps us to understand the consequences of various auction designs:
    - e.g., it suggests the design likely to be the most effective on allocating resources, and
    - the design likely to raise the most revenue.

Types of auctions

- We will consider the case of a seller auctioning one item to a set of buyers.
- The underlying assumption we make when modeling auctions is that each bidder has an intrinsic value for the item being auctioned: she is willing to purchase the item for a price up to this value, but not for any higher price. We will also refer to this intrinsic value as the bidder's true value or simply valuation for the item.
- There are four main types of auctions when a single item is being sold (and many variants of these types).
  1. Ascending-bid auctions, also called English auctions;
  2. Descending-bid auctions, also called Dutch auctions;
  3. First-price sealed-bid auctions;
  4. Second-price sealed-bid auctions auctions, also called Vickrey auctions.

Ascending-bid or English auctions

- Ascending-bid or English auctions are carried out interactively in real time.
- Bidders are present either physically or electronically.
- The seller gradually raises the price.
- Bidders drop out until finally only one bidder remains, and that bidder wins the object at this final price.
- Oral auctions in which bidders shout out prices, or submit them electronically, are forms of ascending-bid auctions.

Descending-bid or Dutch auctions

- Descending-bid auctions are also carried out interactively in real time.
- The seller gradually lowers the price from some high initial value until the first moment when some bidder accepts and pays the current price.
- These auctions are called Dutch auctions because flowers have long been sold in the Netherlands using this procedure.

First-price sealed-bid auctions

- In a first-price sealed-bid auction, bidders submit simultaneous sealed bids to the seller.
- The terminology comes from the original format for such auctions, in which bids were written down and provided in sealed envelopes to the seller, who would then open them all together.
- The highest bidder wins the object and pays the value of her bid.

Second-price sealed-bid or Vickrey auctions

- In a second-price sealed-bid auction, bidders submit simultaneous sealed bids to the sellers.
- The highest bidder wins the object and pays the value of the second-highest bid.
- These auctions are called Vickrey auctions in honor of William Vickrey, who wrote the first game-theoretic analysis of auctions (including the second-price auction). Vickrey won the Nobel Memorial Prize in Economics in 1996 for this body of work.

Comparing auction formats

- A purely superficial comparison of the first-price and second-price sealed-bid auctions might suggest that the seller would get more money for the item if she ran a first-price auction: after all, she'll get paid the highest bid rather than the second-highest bid.
- It may seem strange that in a second-price auction, the seller is intentionally undercharging the bidders.
- But such reasoning ignores one of the main messages from our study of game theory: that when you make up rules to govern people's behavior, you have to assume that they will adapt their behavior in light of the rules.
- Here, the point is that bidders in a first-price auction will tend to bid lower than they do in a second-price auction, and in fact this lowering of bids will tend to offset what would otherwise look like a difference in the size of the winning bid.

Part II of lecture notes 6 : **Second-price sealed-bid auctions**

Introduction

- The sealed-bid second-price auction is particularly interesting, and there are a number of examples of it in widespread use:
  - the auction form used on eBay is essentially a second-price auction;
  - the pricing mechanism that search engines use to sell keyword-based advertising is a generalization of the second-price auction.
- One of the most important results in auction theory is that with independent, private values, bidding your true value is a dominant strategy in a second price sealed-bid auction: the best choice of bid is exactly what the object is worth to you.

Motivation

- Every person is certain for her valuation of the object before the bidding begins.
- Therefore we can assume that each person decides, before bidding begins, the most she is willing to bid (her maximal bid).
- When the players carry out their plans, the winner is the person whose maximal bid is highest.
- How much does she need to bid? To win, she needs to bid slightly more than the second highest maximal bid.
- If the bidding increment is small, we can take the price the winner pays to be equal to the second highest maximal bid.

Formulation as a strategic game

We can model such a second-price sealed-bid auction as a strategic game in which

- each player chooses an amount of money, interpreted as the maximal amount she is willing to bid, and
- the player who chooses the highest amount obtains the object and pays a price equal to the second highest amount.

To define the second-price sealed-bid auction precisely, denote

- v_i the value player i attaches to the object;
- if i obtains the object at the price p her payoff is $v_i - p$.

Formulation as a strategic game

- The players' valuations of the object are assumed to be all different and all positive.
- The set of players is denoted $N = \{1, 2, \ldots, n\}$ so that

$$v_1 > v_2 > \cdots > v_n > 0 \ .$$

- Each player i submits a (sealed) bid b_i.
- If player i's bid is higher than every other bid, she obtains the object at a price equal to the second-highest bid, say b_j, and receives payoff $v_i - b_j$.
- If some other bid is higher than player i's bid, player i does not obtain the object and receives the payoff of zero.
- If player i is in a tie for the highest bid, her payoff depends on the way ties are broken: a simple assumption is that the winner is the bidder of the smallest number (i.e., highest valuation) among those submitting the highest bid.

Formulation as a strategic game

In summary, a second-price sealed-bid auction is the following strategic game:

Players: The set $N = \{1, 2, \dots, n\}$ of the n bidders.

Actions: The set of actions of each player is the set of possible bids (nonnegative numbers).

Payoffs: Denote by b_i the bid of player i and by $\overline{b}$ the highest submitted by a player other than i.

- If (a) $b_i > \overline{b}$ or (b) $b_i = \overline{b}$ and the number of every other player who bids $\overline{b}$ is greater than i, then player's i payoff is $v_i - \overline{b}$.
- Otherwise, player i's payoff is 0.

Nash equilibria of second-price sealed-bid auction

The game has many Nash equilibria. One equilibrium is

$$(b_1, \ldots, b_n) = (v_1, \ldots, v_n) \ ,$$

i.e., each player's bid is equal to her valuation of the object:

- Since $v_1 > \cdots > v_n$, the outcome is that player 1 obtains the object at the price b_2, her payoff is $v_1 - b_2 = v_1 - v_2 > 0$ and every other player's payoff is 0.
- This profile is a Nash equilibrium because:
  - If player 1 changes her bid to some $b_1' \geq b_2$, then the outcome does not change. If she changes her bid to some $b_1' < b_2$ then she loses and receives the payoff of zero.
  - If some other player i lowers her bid or raises it so some price at most equal to b_1, then she remains a loser. If she raises her bid to some $b_i' > b_1$ then she wins the object but receives the payoff of $v_i - b_1 = v_i - v_1 < 0$.

Nash equilibria of second-price sealed-bid auction

Another equilibrium is

$$(b_1, \ldots, b_n) = (v_1, 0, \ldots, 0) \ .$$

- In this profile, player 1 gets the object and her payoff is v_1.
- The profile is an equilibrium because:
  - If player 1 changes her bid, then the outcome remains the same.
  - If any other player i raises her bid to b_i', then either the outcome remains the same (if $b_i' \leq v_1$) or causes player i to obtain the object at a price that exceeds her valuation (if $b_i' > v_1$).

Nash equilibria of second-price sealed-bid auction

In both equilibria we just described, player 1 obtains the object. But there are also equilibria in which player 1 does not obtain the object, e.g.,

$$(b_1, \ldots, b_n) = (v_2, v_1, 0, \ldots, 0) \ ,$$

where player 2 obtains the object at the price v_2 and every player (including player 2) receives the payoff of zero. This profile is an equilibrium because

- If player 1 raises her bid to v_1 or more, she wins the object but her payoff remains 0, because she pays v_1 (the bid of player 2). Any other change in her bid has no effect on the outcome.
- If player 2 changes her bid to some other price greater than v_2, the outcome does not change. If she changes her bid to v_2 or less she loses and her payoff remains 0.
- If any other player raises her bid to at most v_1, the outcome does not change. If she raises her bid above v_1, then she wins, but in paying price v_1 (bid by player 2) she obtains negative payoff.

Analysis of equilibria

We saw that

$$(v_1, v_2, \ldots, v_n), (v_1, 0, \ldots, 0), (v_2, v_1, 0, \ldots, 0)$$

are equilibria of the second-price sealed-bid auction.

- Player 2's bid in the last equilibrium exceeds her valuation ($b_2 = v_1 > v_2$).
- If player 1 were to increase her bid to any value less than v_1, player's 2 payoff would be negative (she would obtain the object at a price greater than her valuation).
- This property does not affect the fact that the profile is a Nash equilibrium.
- But the property does suggest that this equilibrium is less plausible as the outcome of the auction than the first equilibrium, in which every player bids her valuation.

Truth-telling is a dominant strategy

The weakness of the last equilibrium is reflected in the fact that player 2's bid v_1 is weakly dominated by the bid v_2.

Theorem

In a second-price sealed-bid auction, a player's bid equal to her valuation weakly dominates all her other bids.

- In other words, truth-telling is a weakly dominant strategy.
- That is, for any bid $b_i \neq v_i$, player i's bid v_i is at least as good as b_i, no matter what the other players bid, and is better than b_i for some actions of the other players.
- A player who bids less than her valuation stands not to win in some cases in which she could profit by winning (when the highest of the bids is between her bid and her valuation).

Truth-telling is a dominant strategy

Proof

Theorem

In a second-price sealed-bid auction, a player's bid equal to her valuation weakly dominates all her other bids.

Proof.

- We need to show that if player i bids $b_i = v_i$, then no deviation from this bid would improve her payoff, regardless of what strategy everyone else is using.
- There are two cases to consider: deviations in which i raises her bid, and deviations in which i lowers her bid.

Truth-telling is a dominant strategy

Proof

Theorem

In a second-price sealed-bid auction, a player's bid equal to her valuation weakly dominates all her other bids.

Proof (continued).

- The key point in both cases is that the value of i's bid only affects whether i wins or loses, but never affects how much i pays in the event that she wins (the amount paid is determined entirely by the other bids, and in particular by the largest among the other bids).
- Since all other bids remain the same when i changes her bid, a change to i's bid only affects her payoff if it changes her win/loss outcome.

Truth-telling is a dominant strategy

Proof

Theorem

In a second-price sealed-bid auction, a player's bid equal to her valuation weakly dominates all her other bids.

Proof (continued). First, suppose that instead of bidding v_i, player i chooses a bid $b_i' > v_i$.

- This only affects player i's payoff if i would lose with bid v_i but would win with bid b_i'.
- In order for this to happen, the highest other bid b_j must be between b_i and b_i'.
- In this case, the payoff to i from deviating would be at most $v_i - b_j \leq 0$, and so this deviation to bid b_i' does not improve i's payoff.

Truth-telling is a dominant strategy

Proof

Theorem

In a second-price sealed-bid auction, a player's bid equal to her valuation weakly dominates all her other bids.

Proof (continued). Next, suppose that instead of bidding v_i, player i chooses a bid $b'_i < v_i$.

- This only affects player is payoff if i would win with bid v_i but would lose with bid b'_i.
- So before deviating, v_i was the winning bid, and the second-place bid b_j was between v_i and b'_i.
- In this case, i's payoff before deviating was $v_i - b_j \geq 0$, and after deviating it is 0 (since i loses), so again this deviation does not improve i's payoff.

Truth-telling is a dominant strategy

In summary:

A second-price sealed-bid auction has many Nash equilibria, but the equilibrium

$$(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$$

in which each player's bid is equal to her valuation of the object, is distinguished by the fact that every player's action weakly dominates all her other strategies.

Truth-telling is a dominant strategy

Discussion

- The fact that truthfulness is a dominant strategy makes second-price auctions conceptually very clean.
- Truthful bidding is the best thing to do regardless of what the other bidders are doing.
- So in a second-price auction, it makes sense to bid your true value even if other bidders are overbidding, underbidding, colluding, or behaving in other unpredictable ways.
- In other words, truthful bidding is a good idea even if the competing bidders in the auction don't know that they ought to be bidding truthfully as well.

Second-price sealed-bid auction with two bidders

Let us compute *all* Nash equilibria of a second-price sealed-bid auction with two bidders.

- If player 2's bid b_2 is less than v_1 then any bid of b_2 or more is a best response of player 1 (she wins and pays the price b_2).
- If player 2's bid is equal to v_1 then every bid of player 1 yields her the payoff zero (either she wins and pays v_1 , or she loses), so every bid is a best response.
- If player 2's bid b_2 exceeds v_1 then any bid of less than b_2 is a best response of player 1. (If she bids b_2 or more she wins, but pays the price $b_2 > v_1$, and hence obtains a negative payoff.)

Second-price sealed-bid auction with two bidders

In summary, player 1's best response function is

$$B_1(b_2) = \begin{cases} \{b_1 : b_1 \geq b_2\} & \text{if } b_2 < v_1 \\ \{b_1 : b_1 \geq 0\} & \text{if } b_2 = v_1 \\ \{b_1 : 0 \leq b_1 < b_2\} & \text{if } b_2 > v_1 \end{cases}$$

and, by similar arguments, player 2's best response function is

$$B_2(b_1) = \begin{cases} \{b_2 : b_2 \geq b_1\} & \text{if } b_1 < v_2 \\ \{b_2 : b_2 \geq 0\} & \text{if } b_1 = v_2 \\ \{b_2 : 0 \leq b_2 < b_1\} & \text{if } b_1 > v_2 \end{cases} .$$

Therefore

The set of Nash equilibria is the set of pairs (b_1, b_2) such that

$$\text{either} \quad b_1 \leq v_2 \text{ and } b_2 \geq v_1$$

$$\text{or} \quad b_1 \geq v_2\,,\; b_1 \geq b_2\,,\text{ and } b_2 \leq v_1 \;.$$

Part III of lecture notes 6 : **First-price sealed-bid auctions**

Introduction

- A first-price sealed-bid auction differs from a second-price sealed bid auction only in that the winner pays the price she bids, not the second highest bid.
- This means that the value of your bid not only affects whether you win but also how much you pay.
- As a result, most of the reasoning from the second-price sealed-bid auction has to be redone, and the conclusions are now different.

Motivation

- A first-price sealed-bid auction models an auction in which people submit sealed bids and the highest bid wins.
- It is also a model for a dynamic auction in which the auctioneer begins by announcing the highest price, which she gradually lowers until someone indicates her willingness to buy the object.
- A bid is interpreted as the price at which the bidder will indicate her willingness to buy the object in the dynamic auction.

Formulation as a strategic game

A first-price sealed-bid auction is the following strategic game:

Players: The set $N = \{1, 2, \ldots, n\}$ of the n bidders.

Actions: The set of actions of each player is the set of possible bids (nonnegative numbers).

Payoffs: Denote by b_i the bid of player i and by $\overline{b}$ the highest submitted by a player other than i.

- If (a) $b_i > \overline{b}$ or (b) $b_i = \overline{b}$ and the number of every other player who bids $\overline{b}$ is greater than i, then player's i payoff is $v_i - b_i$.
- Otherwise, player i's payoff is 0.

Analysis of equilibria

A first-price sealed-bid auction has many Nash equilibria, but in all equilibria the winner is the player who values the object most highly (player 1):

- In any action profile $(b_1, \ldots, b_n)$ in which some player $i \neq 1$ wins, we have $b_i > b_1$.
- If $b_i > v_2$, then i's payoff is negative, so that she can do better by reducing her bid to 0.
- If $b_i \leq v_2$, then player 1 can increase her payoff from 0 to $v_1 - b_i$ by bidding b_i, in which case she wins.
- Thus no such action profile is a Nash equilibrium.

Characterization of equilibria

The following theorem characterizes the set of Nash equilibria of a first-price sealed-bid auction:

Theorem

An action profile $(b_1, \ldots, b_n)$ is a Nash equilibrium of a first-price sealed-bid auction if and only if

- *the two highest bids are the same;*
- *one of these bids is submitted by player 1; and*
- *the highest bid is at least v_2 and at most v_1.*

Characterization of equilibria

Proof

Proof ($\Longrightarrow$).

- A profile of bids in which the two highest bids are not the same is not a Nash equilibrium because the player naming the highest bid can reduce her bid slightly, continue to win, and pay a lower price.
- Recall that, in any equilibrium, player 1 wins the object. Thus she submits one of the highest bids.
- If the highest bid is less than v_2, then player 2 can increase her bid to a value between the highest bid and v_2, win, and obtain a positive payoff. Thus in an equilibrium the highest bid is at least v_2.
- If the highest bid exceeds v_1, player 1's payoff is negative, and she can increase this payoff by reducing her bid. Thus in an equilibrium the highest bid is at most v_1.

Characterization of equilibria

Proof

Proof ($\Longleftarrow$). Any profile $(b_1, \ldots, b_n)$ of bids that satisfies the conditions is a Nash equilibrium because:

- If player 1 increases her bid she continues to win, and reduces her payoff.
- If player 1 decreases her bid she loses and obtains the payoff 0, which is at most her payoff at $(b_1, \ldots, b_n)$.
- If any other player increases her bid she either does not affect the outcome, or wins and obtains a negative payoff.
- If any other player decreases her bid she does not affect the outcome.

□

Dominated bids in first-price sealed-bid auctions

- In any equilibrium in which the winning bid exceeds v_2, at least one player's bid exceeds her valuation.
- Such a bid seems "risky" because it would yield the bidder a negative payoff if it were to win.
- In the equilibrium there is no risk, because the bid does not win, but (as in a second-price sealed-bid auction) the fact that the bid has this property reduces the plausibility of the equilibrium.
- This potential "riskiness" to player i of a bid $b_i > v_i$ is reflected in the fact that it is weakly dominated by the bid v_i.

Dominated bids in first-price sealed-bid auctions

In a first-price sealed-bid auction, a player i's bid $b_i > v_i$ is weakly dominated by the bid v_i.

Proof.

- If the other players' bids are such that player i loses when she bids b_i, then the outcome is the same whether she bids b_i or v_i.
- If the other players' bids are such that player i wins when she bids b_i, then her payoff is negative when she bids b_i and zero when she bids v_i (regardless of whether this bid wins.

$\square$

Dominated bids in first-price sealed-bid auctions

However, in a first-price auction, unlike a second-price auction, a bid $b_i < v_i$ of player i is not weakly dominated by the bid v_i. In fact,

In a first-price sealed-bid auction, a player i's bid $b_i < v_i$ is not weakly dominated by *any* bid.

Proof.

- A bid $b_i < v_i$ is not weakly dominated by a bid $b_i' < b_i$ because if the other players' highest bid is between b_i' and b_i, then b_i' loses, whereas b_i wins and yields player i a positive payoff.
- A bid $b_i < v_i$ is not weakly dominated by a bid $b_i' > b_i$ because if the other players' highest bid is less than b_i, then both b_i and b_i' win and b_i yields a lower price.

□

Dominated bids in first-price sealed-bid auctions

Further, even though the bid v_i weakly dominates higher bids, this bid is itself weakly dominated, by a lower bid!

In a first-price sealed-bid auction, a player i's bid v_i is weakly dominated by any bid $b_i < v_i$.

Proof.

- If player i bids v_i her payoff is 0 regardless of the other players' bids.
- If player i bids $b_i < v_i$ her payoff is either 0 (if she loses) or positive (if she wins).

□

Dominated bids in first-price sealed-bid auctions

In summary,

Theorem

In a first-price sealed-bid auction, a player's bid of at least her valuation is weakly dominated, and a bid of less than her valuation is not weakly dominated.

An implication of this result is that

In every Nash equilibrium of a first-price sealed-bid auction at least one player's action is weakly dominated.

Part IV of lecture notes 6 : **Variants**

All-pay auctions

- Some situations may be modeled as all-pay auctions in which every bidder, not only the winner, pays.
- An example is competition between lobby groups for government attention: each group spends resources in attempt to win favor; the one that spends the most is successful.
- We will study both first- and second-price versions of an all-pay auction with two bidders, in which both bidders pay the winning price.

Second-price all-pay auction

Two bidders, both pay the winning price

We start with the second-price version of an all-pay auction with two bidders, in which both bidders pay the winning price, and we will try to compute the set of Nash equilibria of the game.
The payoff function of bidder 1 is

$$u_1(b_1, b_2) = \begin{cases} -b_1 & \text{if } b_1 < b_2 \\ v_1 - b_2 & \text{if } b_1 \geq b_2 \end{cases}$$

and that of bidder 2 is

$$u_2(b_1, b_2) = \begin{cases} -b_2 & \text{if } b_2 \leq b_1 \\ v_2 - b_1 & \text{if } b_2 > b_1 \end{cases}$$

Second-price all-pay auction

Two bidders, both pay the winning price

- A profile (b, b) is not a Nash equilibrium for any value of b because player 2 can increase her payoff by either increasing her bid slightly or by reducing it to 0.
- A profile (b_1, b_2) with $0 < b_1 < b_2$ is not a Nash equilibrium, because player 1 can lower her bid to 0 and receive a payoff of $0 > -b_1$.
- A profile (b_1, b_2) with $b_1 > b_2 > 0$ is not a Nash equilibrium, because player 2 can lower her bid to 0 and receive a payoff of $0 > -b_2$.
- It remains to consider the profiles $(0, b_2)$ and $(b_1, 0)$ where $b_1, b_2 > 0$.

Second-price all-pay auction

Two bidders, both pay the winning price

A profile $(0, b_2)$ where $b_2 > 0$ is a Nash equilibrium if and only if $b_2 \geq v_1$:

($\Rightarrow$) Assume $(0, b_2)$ is a Nash equilibrium. Player 1 receives a payoff of 0. If player 1 raised her bid to b_2 she would receive a payoff of $v_1 - b_2$. If $v_1 - b_2 > 0$ then $(0, b_2)$ could not be an equilibrium. Therefore $b_2 \geq v_1$.

($\Leftarrow$) Assume $b_2 \geq v_1$. Then $(0, b_2)$ is a Nash equilibrium:

- If player 1 raises her bid to some b'_1 so that $0 < b'_1 < b_2$ she would receive a payoff of $-b'_1 < 0$. If she raises her bid to some $b'_1 \geq b_2$ she would receive a payoff of $v_1 - b_2 \leq 0$.
- If player 2 raises her bid, or lowers it to some $b'_2 > 0$, her payoff remains the same. If she lowers her bid to 0 she would receive a payoff of 0, instead of $v_2 - b_1 = v_2 \geq 0$.

Second-price all-pay auction

Two bidders, both pay the winning price

Similarly, we can show that a profile $(b_1, 0)$ where $b_1 > 0$ is a Nash equilibrium if and only if $b_1 \geq v_2$.
Therefore, to summarize:

The set of all Nash equilibria of a second-price all-pay auction with two bidders, where both bidders pay the winning price, is the set of all pairs $(0, b_2)$ and $(b_1, 0)$ where $b_1 > 0$, $b_2 > 0$, $b_2 \geq v_1$, and $b_1 \geq v_2$.

First-price all-pay auction

Two bidders, both pay the winning price

We now consider the first-price version of an all-pay auction with two bidders, in which both bidders pay the winning price, and we will try to compute the set of Nash equilibria of the game.

- In any Nash equilibrium the two highest bids are equal, otherwise the player with the higher bid can increase her payoff by reducing her bid a little (keeping it larger than the other player's bid).
- But no profile of bids in which the two highest bids are equal is a Nash equilibrium, because the player with the higher index who submits this bid can increase her payoff by slightly increasing her bid, so that she wins rather than loses.

Therefore

A first-price all-pay auction with two bidders in which both bidders pay the winning price has no Nash equilibrium.

Multiunit auctions

- In multiunit auctions, many units of an object are available, and each bidder may value positively more than one unit.
- Each bidder submits a bid for each unit of the good.
- An action is a list of bids $(b_i^1, \ldots, b_i^k)$, where b_i^1 is player i's bid for the first unit of the good, b_i^2 is her bid for the second unit, and so on.
- The player who submits the highest bid for any given unit obtains that unit.

Multiunit auctions

Types of multiunit auctions

We will study three types of multiunit auctions, which differ in the prices paid by the winners.

Discriminatory auction: The price paid for each unit is the winning bid for that unit.

Uniform-price auction: The price paid for each unit is the same, equal to the highest rejected bid among all the bids for all units.

Vickrey auction: A bidder who wins k objects pays the sum of the k highest rejected bids submitted by the *other* bidders.

- Note that the first type generalizes a first-price auction, whereas the last two generalize a second-price auction.
- We will study these auctions when two units of an object are available.

Two-unit auctions

- Assume that two units of an object are available.
- There are n bidders.
- Bidder i values the first unit that she obtains at v_i and the second unit at w_i, where $v_i > w_i > 0$.
- Each bidder submits two bids; the two highest bids win.

We will show that, in this setting,

In discriminatory and uniform-price auctions, player i's action of bidding v_i and w_i does not dominate all her other actions, whereas in a Vickrey auction it does.

Two-unit auctions

Discriminatory auction

In a discriminatory two-unit auction, player i's action of bidding v_i and w_i does not dominate all her other actions.

Proof. Recall that the price paid for each unit is the winning bid for that unit.

- To show that the action of bidding v_i and w_i is not dominant for player i, we need only find actions for the other players and alternative bids for player i such that player i's payoff is higher under the alternative bids than it is under the v_i and w_i, given the other players' actions.
- Suppose that each of the other players submits two bids of 0.
- Then if player i submits one bid between 0 and v_i and one bid between 0 and w_i she still wins two units, and pays less than when she bids v_i and w_i.

Two-unit auctions

Uniform-price auction

In a uniform-price two-unit auction, player i's action of bidding v_i and w_i does not dominate all her other actions.

Proof. Recall that the price paid for each unit is the same, equal to the highest rejected bid among all the bids for all units.

- Suppose that some bidder other than i submits one bid between w_i and v_i and one bid of 0, and all the remaining bidders submit two bids of 0.
- Then bidder i wins one unit, and pays the price w_i.
- If she replaces her bid of w_i with a bid between 0 and w_i then she pays a lower price, and hence is better off.

□

Two-unit auctions

Vickrey auction

In a Vickrey two-unit auction, player i's action of bidding v_i and w_i dominates all her other actions.

Proof. Recall that a bidder who wins k objects pays the sum of the k highest rejected bids submitted by the *other* bidders.
Suppose that player i bids v_i and w_i. We will consider separately the cases in which the bids of the players other than i are such that player i wins 0, 1, and 2 units.

Player i wins 0 units:

- In this case the second highest of the other players' bids is at least v_i.
- If player i changes her bids so that she wins one or more units, for any unit she wins she pays at least v_i.
- Thus no change in her bids increases her payoff from its current value of 0 (and some changes lower her payoff).

Two-unit auctions

Vickrey auction

In a Vickrey two-unit auction, player i's action of bidding v_i and w_i dominates all her other actions.

Proof (continued).

Player i wins 1 unit:

- If player i raises her bid of v_i then she still wins one unit and the price remains the same.
- If she lowers this bid then either she still wins and pays the same price, or she does not win any units.
- If she raises her bid of w_i then either the outcome does not change, or she wins a second unit. In the latter case the price she pays is the previously-winning bid she beat, which is at least w_i, so that her payoff either remains zero or becomes negative.

Two-unit auctions

Vickrey auction

In a Vickrey two-unit auction, player i's action of bidding v_i and w_i dominates all her other actions.

Proof (continued).

Player i wins 2 units:

- Player i's raising either of her bids has no effect on the outcome.
- Her lowering a bid either has no effect on the outcome or leads her to lose rather than to win, leading her to obtain the payoff of zero.

□

An application of multiunit auctions

Internet pricing

A proposal to deal with congestion on electronic message pathways is that each message should include a field stating an amount of money the sender is willing to pay for the message to be sent.

Suppose that:

- During some time interval, each of n people wants to send one message.
- The capacity of the pathway is $k < n$ messages.
- The k messages whose bids are highest are the ones sent.
- Each of the persons sending these messages pays a price equal to the $(k+1)$st highest bid.

Note that the auction differs from the two-unit auctions we studied before because each person submits only one bid.

An application of multiunit auctions

Internet pricing

However, the situation may be modeled as a multiunit auction in which

- k units are available;
- each player attaches a positive value to only one unit and submits a bid for only one unit;
- the k highest bids win; and
- each winner pays the $(k+1)$st highest bid.

By a variant of the argument for a second-price auction, in which highest of the other players' bids is replaced by highest rejected bid, we can show that

A player's action of bidding her value weakly dominates all her other actions.

Part V of lecture notes 6 : **Auctions with imperfect information**

Imperfect information

Auctions with imperfect information:

- So far, we have assumed every bidder knows every other bidder's valuation of the object for sale.
- We will analyze auctions in which bidders are not perfectly informed about each others' valuations.

General setting:

- We assume a single object is for sale, and each bidder independently receives some information (a signal) about the value of the object to her.
  - If each bidder's signal is simply her valuation of the object, we say that the bidders' valuations are private.
  - If each bidder's valuation depends on other bidders' signal as well as her own, we say that the valuations are common.

Imperfect information

Examples:

- The assumption of private values are appropriate for a work of art whose beauty than resale value interests the buyers. Each bidder knows her valuation of the object, but not that of any other bidder, and the other bidders' valuations have no bearing on her valuation.
- The assumption of common values is appropriate for an oil tract containing unknown reserves on which each bidder has conducted a test. Each bidder i's test result gives her some information about the size of the reserves, and hence her valuation, but the other bidders' test results, if known to bidder i, would typically improve this information.

First-price auction with private values

- We want to capture a setting in which bidders know how many competitors they have, and they have partial information about their competitors' values for the item. However, they do not know their competitors' values exactly.
- We will focus on auctions with private values in which:
  - bids for a single object are submitted simultaneously (sealed-bid);
  - the participant who submits the highest bid obtains the object; and
  - the winner pays the price she bid (first-price auctions).

First-price auction with two bidders

The model

In the simple case, suppose that:

- There are two bidders.
- Each bidder has a private value that is independently and uniformly distributed between 0 and 1.
    - Note that the fact that the 0 and 1 are the lowest and highest possible values is not crucial; by shifting and re-scaling these quantities, we could equally well consider values that are uniformly distributed between any other pair of endpoints.
- This information is common knowledge among the two bidders.

First-price auction with two bidders

Strategies

A strategy for a bidder is a function $s(v) = b$ that maps her true value v to a non-negative bid b. We will make the following simple assumptions about the strategies the bidders are using:

1. $s(\cdot)$ is a strictly increasing, differentiable function; so in particular, if two bidders have different values, then they will produce different bids.
2. $s(v) \leq v$ for all v: bidders can shade their bids down, but they will never bid above their true values. Notice that since bids are always non-negative, this also means that $s(0) = 0$.

First-price auction with two bidders

Strategies

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These two assumptions permit a wide range of strategies:

- The strategy of bidding your true value is represented by $s(v) = v$.
- The strategy of shading your bid downward to by a factor of $c < 1$ times your true value is represented by $s(v) = c \cdot v$.
- More complex strategies such as $s(v) = v^2$ are also allowed (although we will see that in first-price auctions they are not optimal).

First-price auction with two bidders

Strategies

The two assumptions help us narrow the search for equilibrium strategies:

- The assumption of strictly increasing strategies restricts the scope of possible equilibrium strategies, but it makes the analysis easier while still allowing us to study the important issues.
- Since the two bidders are identical in all ways except the actual value they draw from the distribution, we will narrow the search for equilibria in one further way: we will consider the case in which the two bidders follow the same strategy $s(\cdot)$.

First-price auction with two bidders

Payoffs

- The assumption of strictly increasing strategies says that the bidder with the higher value will also produce the higher bid.
- If bidder i has a value of v_i, the probability that this is higher than the value of i's competitor in the interval $[0, 1]$ is exactly v_i.
- Therefore, i will win the auction with probability v_i.
- If i does win, i receives a payoff of $v_i - s(v_i)$.

Putting all this together, we see that i's expected payoff is

$$u(v_i) = v_i(v_i - s(v_i)) \ .$$

First-price auction with two bidders

Equilibrium strategies

What does it mean for $s(\cdot)$ to be a (symmetric) equilibrium strategy?

s$(\cdot)$ is an equilibrium strategy if, for each bidder i, there is no incentive for i to deviate from strategy $s(\cdot)$ if i's competitor is also using strategy $s(\cdot)$.

- It is not immediately clear how to analyze deviations to an arbitrary strategy.
- Fortunately, there is an elegant device that lets us reason about deviations as follows: rather then actually switching to a different strategy, bidder i can implement her deviation by keeping the strategy $s()$ but supplying a different "true value" to it.

First-price auction with two bidders

The revelation principle

Here is how this works.

- If i's competitor is also using strategy $s(\cdot)$, then i should never announce a bid above $s(1)$, since i can win with bid $s(1)$ and get a higher payoff with bid $s(1)$ than with any bid $b > s(1)$.
- So in any possible deviation by i, the bid she will actually report will lie between $s(0) = 0$ and $s(1)$.
- Therefore, for the purposes of the auction, she can simulate her deviation to an alternate strategy by first pretending that her true value is v_i' rather than v_i, and then applying the existing function $s(\cdot)$ to v_i' instead of v_i.
- This is a special case of a much broader idea known as the Revelation Principle; for our purposes, we can think of it as saying that deviations in the bidding strategy function can instead be viewed as deviations in the "true value" that bidder i supplies to her current strategy $s(\cdot)$.

First-price auction with two bidders

Equilibrium

We can now write the condition that i does not want to deviate from strategy $s(\cdot)$ as follows:

$$v_i(v_i - s(v_i)) \geq v(v_i - s(v)) \quad \forall v \in [0,1] \ .$$

- In order for $s(\cdot)$ to satisfy the above inequality, it must have the property that for any true value v_i, the expected payoff function $u(v) = v(v_i - s(v))$ is maximized by setting $v = v_i$.
- The first derivative of $u(\cdot)$ is $u'(v) = v_i - s(v) - vs'(v)$.
- Therefore, v_i should satisfy $u'(v_i) = 0$ or equivalently

$$s'(v_i) = 1 - \frac{s(v_i)}{v_i} \ .$$

- This differential equation is solved by the function $s(v) = v/2$.

First-price auction with two bidders

Equilibrium

- Thus, if two bidders know they are competing against each other, and know that each has a private value drawn uniformly at random from the interval $[0, 1]$, then it is an equilibrium for each to shade their bid down by a factor of 2. Bidding half your true value is optimal behavior if the other bidder is doing this as well.
- Unlike the case of the second-price auction with complete information, we have not identified a dominant strategy, only an equilibrium. In solving for a bidder's optimal strategy we used each bidder's expectation about her competitor's bidding strategy. In an equilibrium, these expectations are correct. But if other bidders for some reason use non-equilibrium strategies, then any bidder should optimally respond and potentially also play some other bidding strategy.

Part VI of lecture notes 6 : **Extensions**

First-price auction with many bidders

Payoffs

Now suppose that there are n bidders, where $n \geq 2$.

- Each bidder i draws her true value v_i independently and uniformly at random from the interval $[0, 1]$.
- The assumptions on the strategies still imply that the bidder with the highest true value will produce the highest bid and hence win the auction.
- For a given bidder i with true value v_i, what is the probability that her bid is the highest?
- This requires each other bidder to have a value below v_i; since the values are chosen independently, this event has a probability of v_i^{n-1}.
- Therefore, bidder i's expected payoff is

$$u(v_i) = v_i^{n-1}(v_i - s(v_i)) \ .$$

First-price auction with many bidders

Equilibrium strategies

- The condition for $s(\cdot)$ to be an equilibrium strategy remains the same as it was in the case of two bidders.
- Using the Revelation Principle, we view a deviation from the bidding strategy as supplying a "fake" value v to the function $s(\cdot)$.
- Given this, we require that the true value v_i produces an expected payoff at least as high as the payoff from any deviation:

$$v_i^{n-1}(v_i - s(v_i)) \geq v^{n-1}(v_i - s(v)) \quad \forall v \in [0,1] \ .$$

First-price auction with many bidders

Equilibrium strategies

- We can derive the form of the bidding function $s(\cdot)$ using the differential-equation approach that worked for two bidders.
- The expected payoff function $u(v) = v^{n-1}(v_i - s(v))$ must be maximized by setting $v = v_i$.
- Setting the derivative $u'(v_i) = 0$ we get

$$(n-1)v^{n-2}v_i - (n-1)v^{n-2}s(v_i) - v_i^{n-1}s'(v_i) = 0$$

or equivalently

$$s'(v_i) = (n-1)\left(1 - \frac{s(v_i)}{v_i}\right) \quad \forall v_i \in [0,1] \ .$$

- This differential equation is solved by the function

$$s(v) = \left(\frac{n-1}{n}\right) v \ .$$

First-price auction with many bidders

Equilibrium strategies

- So if each bidder shades her bid down by a factor of $(n-1)/n$, then this is optimal behavior given what everyone else is doing.
- Notice that when $n = 2$ this is our two-bidder strategy.
- The form of this strategy highlights an important principle in first-price auctions: as the number of bidders increases, you generally have to bid more "aggressively", shading your bid down less, in order to win.
- For the simple case of values drawn independently from the uniform distribution, our analysis here quantifies exactly how this increased aggressiveness should depend on the number of bidders n.

General distributions

- In addition to considering larger numbers of bidders, we can also relax the assumption that bidders' values are drawn from the uniform distribution on an interval.
- Suppose that each bidder has her value drawn from a probability distribution over the non-negative real numbers.
- We can represent the probability distribution by its cumulative distribution function $F(\cdot)$: for any x, the value $F(x)$ is the probability that a number drawn from the distribution is at most x.
- We will assume that F is a differentiable function.

General distributions

Payoffs and equilibrium condition

Most of the earlier analysis continues to hold at a general level:

- The probability that a bidder i with true value v_i wins the auction is the probability that no other bidder has a larger value, so it is equal to $F(v_i)^{n-1}$.
- Therefore, the expected payoff to v_i is

$$F(v_i)^{n-1}(v_i - s(v_i)) \ .$$

- Then, the requirement that bidder i does not want to deviate from this strategy becomes

$$F(v_i)^{n-1}(v_i - s(v_i)) \geq F(v)^{n-1}(v_i - s(v)) \quad \forall v \in [0, 1] \ .$$

General distributions

Equilibrium strategies

- The equilibrium condition

$$F(v_i)^{n-1}(v_i - s(v_i)) \geq F(v)^{n-1}(v_i - s(v)) \quad \forall v \in [0,1]$$

  can be used to write a differential equation just as before, using the fact that the function of v should be maximized when $v = v_i$.
- The derivative of the cumulative distribution function $F(\cdot)$ is the probability density function $f(\cdot)$; proceeding by analogy with the analysis for the uniform distribution, we get the differential equation

$$s'(v_i) = (n-1)\left(\frac{f(v_i)v_i - f(v_i)s(v_i)}{F(v_i)}\right) .$$

- Finding an explicit solution isn't possible unless we have an explicit form for the distribution of values, but it provides a framework for taking arbitrary distributions and solving for equilibrium strategies.

Seller revenue

Let us now try to compare the revenue a seller should expect to make in first-price and second-price auctions.

There are two competing forces at work here:

- In a second-price auction, the seller explicitly commits to collecting less money, since she only charges the second-highest bid.
- In a first-price auction, the bidders reduce their bids, which also reduces what the seller can collect.

Seller revenue

- To understand how these opposing factors trade off against each other, suppose we have n bidders with values drawn independently from the uniform distribution on the interval $[0, 1]$.
- Since the seller's revenue will be based on the values of the highest and second-highest bids, which in turn depend on the highest and second-highest values, we need to know the expectations of these quantities.
- Computing these expectations is complicated, but the form of the answer is very simple. The basic statement is:

Suppose n numbers are drawn independently from the uniform distribution on the interval $[0, 1]$ and then sorted from smallest to largest. The expected value of the number in the kth position on this list is $\frac{k}{n+1}$.

Seller revenue

Comparison between first- and second-price auctions

- If the seller runs a second-price auction, and the bidders follow their dominant strategies and bid truthfully, the seller's expected revenue will be the expectation of the second-highest value.
- Since this will be the value in position $n-1$ in the sorted order of the n random values from smallest to largest, the expected value of the seller's revenue is
$$\frac{n-1}{n+1} \quad .$$

Seller revenue

Comparison between first- and second-price auctions

- If the seller runs a first-price auction, then in equilibrium we expect the winning bidder to submit a bid that is $(n-1)/n$ times her true value.
- Her true value has an expectation of $n/(n+1)$ (since it is the largest of n numbers drawn independently from the unit interval), and so the seller's expected revenue is

$$\frac{n-1}{n} \cdot \frac{n}{n+1} = \frac{n-1}{n+1} \quad .$$

The two auctions provide exactly the same expected revenue to the seller!

Revenue equivalence

The fact that the two auctions provide the same expected value to the seller is a reflection of a much broader and deeper principle known in the auction literature as revenue equivalence, which, roughly speaking, asserts that

> A seller's revenue will be the same across a broad class of auctions and arbitrary independent distributions of bidder values, when bidders follow equilibrium strategies.

Reserve prices

So far, we have implicitly assumed that the seller must sell the object. But how does the seller's expected revenue change if she has the option of holding onto the item and choosing not to sell it?

- We assume that the seller values the item at $u \geq 0$, which is thus the payoff she gets from keeping the item rather than selling it.
- Clearly, if $u > 0$, then the seller should not use a simple first-price or second-price auction: in either case, the winning bid might be less than u, and the seller would not want to sell the object.
- Instead, the seller announces a reserve price of r before running the auction.
- The item is sold to the highest bidder if the highest bid is above r; otherwise, the item is not sold.

Reserve prices

Models

Models of auctions with a reserve price:

- In a first-price auction with a reserve price, the winning bidder (if there is one) still pays her bid.
- In a second-price auction with a reserve price, the winning bidder (if there is one) pays the maximum of the second-place bid and the reserve price r.

As we will see, it is in fact useful for the seller to declare a reserve price even if her value for the item is $u = 0$.

Reserve prices

Second-price auctions

We focus on the the case of a second-price auction with a reserve price.

- It is not hard to go back over the argument that truthful bidding is a dominant strategy in second-price auctions and check that it still holds in the presence of a reserve price.
- Essentially, it is as if the seller were another "simulated" bidder who always bids r; and since truthful bidding is optimal regardless of how other bidders behave, the presence of this additional simulated bidder has no effect on how any of the real bidders should behave.

Reserve prices

Second-price auctions

What value should the seller choose for the reserve price?

- If the item is worth u to the seller, then clearly she should set $r \geq u$.
- But in fact the reserve price that maximizes the seller's expected revenue is strictly greater than u.

To see why this is true, we will first consider a very simple case:

- a second-price auction with a single bidder, whose value is uniformly distributed on $[0, 1]$, and
- a seller whose value for the item is $u = 0$.

Reserve prices

Second-price auctions

- With only one bidder, the second-price auction with no reserve price will sell the item to the bidder at a price of 0.
- On the other hand, suppose the seller sets a reserve price of $r > 0$. In this case:
  - with probability $1 - r$, the bidder's value is above r, and the object will be sold to the bidder at a price of r;
  - with probability r, the bidder's value is below r, and so the seller keeps the item, receiving a payoff of $u = 0$.

  Therefore, the seller's expected revenue is $r(1 - r)$, and this is maximized at $r = 1/2$.

Reserve prices

Second-price auctions

- If the seller's value u is greater than zero, then her expected payoff is $r(1 - r) + ru$ (since she receives a payoff of u when the item is not sold), and this is maximized by setting $r = (1 + u)/2$.
- So with a single bidder, the optimal reserve price is halfway between the value of the object to the seller and the maximum possible bidder value.
- With more intricate analyses, one can similarly determine the optimal reserve price for a second-price auction with multiple bidders, as well as for a first-price auction with equilibrium bidding strategies of the form we derived earlier.

Further reading

- Martin J. Osborne: An Introduction to Game Theory. Oxford University Press, 2004.
- David Easley and Jon Kleinberg: Networks, Crowds, and Markets: Reasoning About a Highly Connected World. Cambridge University Press, 2010.