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DATE: Monday, 14 December 2020

MODULE: COMP323
ASSIGNMENT: Class Test 2
EXAMINER: Prof. Paul G. Spirakis

The class test is held online and will account for 15% of your final mark.

**EXERCISE 1.** (30%) We consider the case of selfish load balancing where the available machines have speeds. The LPT rule for placing the tasks is as follows:
"Insert the tasks in a nonincreasing order of weights; each task is assigned to a machine that minimizes the cost of the task at its insertion time."
It is known that this rule produces a pure Nash equilibrium assignment of tasks.

As an application, we are given two machines, $M_1, M_2$, with speeds $s_1 = 2$ and $s_2 = 3$. We are also given tasks 1, 2, 3, 4, 5 with weights $w_1 = 6, w_2 = 12, w_3 = 18, w_4 = 24, w_5 = 30$, respectively.

(a) (25%) For each task, find the machine in which the task goes (by applying the LPT) and justify why below.

Task ..5.. goes to machine ..2...
Explanation:
Task 5 is the task with the heaviest weight, it should go to the machine with the highest efficiency, which is machine 2, the time taken is 10 (10<15)

Task .4.. goes to machine ..1...
Explanation:
Since if task 4 goes to machine 1, the time taken by machine 1 is 12, while since machine 2 has taken tasks 5, thus, the result time taken up to now is $\frac{24+30}{3} = 18$, thus, task 4 goes to machine 1.
(12<18)

Task ..3... goes to machine .2...
Explanation:
If task 3 goes to machine 1, since machine 1 takes task 4, the time is (24+18)/2 = 21, if task 3 goes to machine 2, the time is (30+18)/2 = 16. Thus, the task 3 should go to task 2 (16<21)

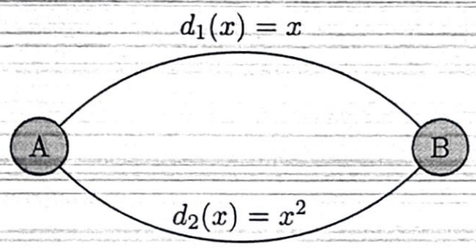
Task .2... goes to machine ..1...
Explanation:
If task 2 goes to machine 1, since machine 1 takes 4, the total time is (24+12)/2 = 18, If the task 2 goes to machine 2, since W5, W3 have already been in machine 2, the total time is (30+18+12)/3 = 20.
Thus, Task 2 goes to machine 1 (18<20)

Task ..1... goes to machine ..2..
Explanation:
If task 1 goes to machine 1, since machine 1 takes task 4, 2, the total time is $18 + \frac{6}{2} = 21$, If task 1 goes to machine 2, the time is $16 + \frac{6}{3} = 18$, since at the time task 3 is finish, the time stamp is 16. thus, task 1 goes to machine 2 (16<18)

(b) (5%) Find the makespan (cost) of the pure Nash equilibrium produced by the LPT rule and fill in the answer below.

The makespan (cost) of the pure Nash equilibrium produced by the LPT rule is the cost here referes to the time, which ~~is~~ the longest time of the machine dominates the total cost, thus, the cost of the pure Nash Equilibrium is max (18, 18) = 18

**EXERCISE 2.**(30%) Consider the following network congestion game. There are 7 cars that wish to go from $A$ to $B$ as shown in the figure below. There are two possible roads for each car, road 1 (the upper), and road 2 (the bottom), as shown in the Figure. Each road has a delay function (determines the delay per car) which is a function of the number of cars taking that road. For example, if 5 cars use road 2, then the delay per car in that road is $5^2 = 25$. An allocation $\pi$ of cars is a split of the 7 cars to two disjoint populations $\sigma_1(\pi), \sigma_2(\pi)$ so that population $\sigma_i(\pi)$ uses the $i^{th}$ road.




For this congestion game instance, the Rosenthal Potential under an allocation $\pi$ is:

$$\Phi(\pi) = \sum_{i=1}^{2} \sum_{k=1}^{\sigma_i(\pi)} d_i(k),$$ where $d_i(k)$ is the delay of the $i^{th}$ road when $k$ players use it, and $\sigma_i(\pi)$ is the total number of players that use road $i$ under allocation $\pi$.

Given that $\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$ and $\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$:

(a) (10%) If all cars use road 1, find the Rosenthal Potential value of the game in that situation.

(b) (10%) Find a pure Nash equilibrium of the game (hint: you can use the Rosenthal Potential).

(c) (10%) Justify your answer to (b).

(a) Since all Cars use road 1

$$\Phi(\pi) = \sum_{i=1}^{2} \sum_{k=1}^{\sigma_i(\pi)} d_i(k)$$
$$= \sum_{k=1}^{7} k$$
$$= \frac{7 \times 8}{2}$$
$$= 28$$

(b) One Nash Equilibrium is 2 of Cars go to road 2 and 5 of cars go to road 1.

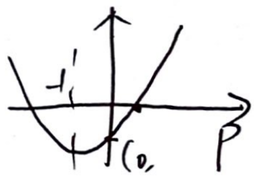
(c) $$\Phi(\pi) = \sum_{i=1}^{2} \sum_{k=1}^{\sigma_i(\pi)} d_i(k)$$
$$= \sum_{k=1}^{\sigma_1(\pi)} d_1(k) + \sum_{k=1}^{\sigma_2(\pi)} d_2(k)$$
$$= \sum_{k=1}^{\sigma_1(\pi)} k + \sum_{k=1}^{\sigma_2(\pi)} k^2.$$

Since $\sigma_1(\pi) + \sigma_2(\pi) = 7$

Let $\sigma_1(\pi) = P$, $\sigma_2(\pi) = 7 - P$.

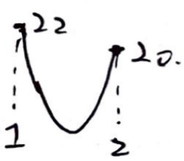
$$= \frac{(7-P)(8-P)}{2} + \frac{P(P+1)(2P+1)}{6}$$
$$= \frac{1}{3}P^3 + P^2 - \frac{22}{3}P + 28$$

$$\frac{d\Phi(\pi)}{dP} = P^2 + 2P - \frac{22}{3}.$$




$$P = \frac{-2}{2 \cdot 1} = -1$$

$P^2 + 2P - \frac{22}{3} = 0$, $P > 0$




$P \approx 1.8$

Since $P$ is only integer.

if $P = 1$, $\phi(\pi) = 22$

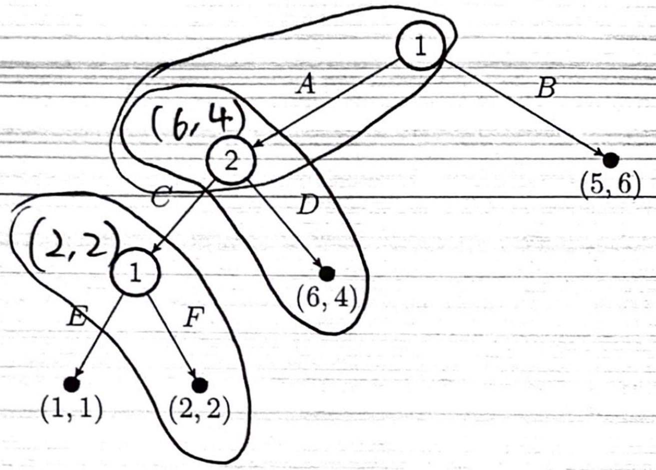
$P = 2$, $\phi(\pi) = 20$

Thus, 2 cars go to road 2, 5 cars go to road 1

**EXERCISE 3.**(20%) Consider the following extensive form game of two players with perfect information. When the game starts, player 1 has two available actions, namely $A$ and $B$. If she chooses $B$ then the game ends, while if she chooses $A$ then player 2 plays. If player 2 chooses $D$ then the game ends, while if she chooses $C$ then player 1 chooses either $E$ or $F$. After this move the game ends. In the following tree the payoffs for the two players are shown for each terminal history; the leftmost is player 1's payoff, and the rightmost is player 2's payoff.

(a) (10%) Find all subgame perfect equilibria of the game.

(b) (10%) Justify your answer to (a).




(a). The SPE is (AF, D)

(b) Since the SPE should be found backwards.
See the tree. E, F, with the actor of 1.
actor 1 will choose F. since actor 1 receives 2, greater than 1
see the subtree, C,D, with the actor 2.
actor 2 will choose D since actor 2 receives 4 greater than 2
See the subtree, A, B, with actor 1,
actor 1 will choose A, since actor 1 receives 6 greater than 5.
Thus, this backward approach gives SPE of (AF, D)

**EXERCISE 4.**(20%) Suppose that we have a second-price sealed bid auction of 3 players 1, 2, 3 and a single item. Let $v_i$ be the valuation of player $i$ for the item, and suppose that $v_1 = 30$, $v_2 = 20$, and $v_3 = 5$. Let $(b_1, b_2, b_3)$ be an action profile where player $i$ bids $b_i$. In case more than one bids are maximum, the item goes to the player with the smallest index.

(a) (10%) Give a profile that is a Nash equilibrium, where player 3 wins the item.

(b) (10%) Justify your answer to (a).

(a) The Profile I choose, which is a Nash Equilibrium for this game is $(4, 5, 30) = (b_1, b_2, b_3)$.

(b). In order to prove this profile is a Nash Equilibrium I need to show for every player, there is no need (higher Payoff) for changing their action profile, which means itself is the optimal at the game.

$V_1 = 30$ $V_2 = 20$ $V_3 = 5$.

$b_1 = 4$ $b_2 = 5$ $b_3 = 30$.

For bider 1., if he wants to win, he must give a higher price say, 31. what is the expected pay off? The pay off is $V_1 - b_3 = 30 - 30 = 0$, which is No better than its original action.
On the other hand, if $b_1$ is decreasing, there is no point in increasing the possibility in losing. Thus, For player one, it is a Nash Equilibrium.
For bider 2, ~~if~~ as indicated before, he wants to win, he gives a higher price say, 31, the pay off is $V_2 - b_3 = -10$, it is even worse.
Also, bider 2 does not want to increase the chance to lose.
So, bider 2 is in Nash Equilibrium
For bider 3, if he increases his value, say, 31. ~~it is~~ bider 3 itself is already a winer, there is no point in doing so,

If bider goes lower, provider it is higher than the ~~secon~~ other biders, the corresponding Payoff does not change there is No point in doing so. thus, player 3 reach a Nash Equalibrium. There whole ~~system~~ action profile meets the Nash Equalibrium Since all three players reach their own Nash Equalibrium