Tutorial 3

Exercise

Is the following an exact potential for Prisoner's Dilemma?

① \ ②	Q	F
Q	2,2	0,3
F	3,0	1,1

Prisoner's Dilemma

	Q	F
Q	0	1
F	1	2

Function P.

By definition (see lecture notes), if and only if it is an exact potential, then: the following 4 equations are true.

- $U_1(Q,Q) - U_1(F,Q) = P(Q,Q) - P(F,Q)$
  $\Leftrightarrow \quad 2 - 3 = 0 - 1 \longrightarrow$ True

- $U_1(Q,F) - U_1(F,F) = P(Q,F) - P(F,F)$
  $\Leftrightarrow \quad 0 - 1 = 1 - 2 \longrightarrow$ True

- $U_2(Q,Q) - U_2(Q,F) = P(Q,Q) - P(Q,F)$
  $\Leftrightarrow \quad 2 - 3 = 0 - 1 \longrightarrow$ True

- $U_2(F,Q) - U_2(F,F) = P(F,Q) - P(F,F)$
  $\Leftrightarrow \quad 0 - 1 = 1 - 2 \longrightarrow$ True

So, P is an exact potential of the given game.

Remark:
In fact any function P' in the next matrix is an exact potential for every $a \in \mathbb{R}$.

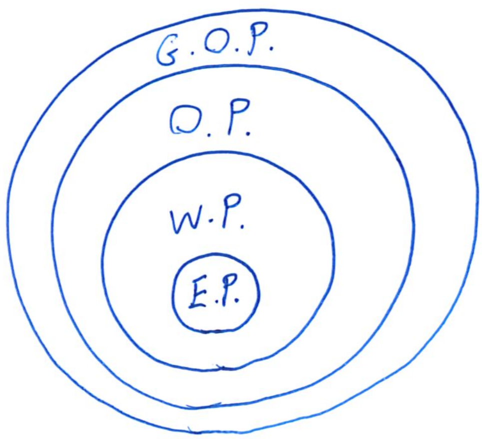
	Q	F
Q	a	$a+1$
F	$a+1$	$a+2$

Useful Lemma (see lecture notes)

Let Γ be a finite game. Then, Γ has the Finite Improvement Property (F.I.P.) if and only if Γ has a generalized ordinal potential.

Useful Theorem (see lecture notes)

Every unweighted congestion game admits an exact potential.



Exact Potential $\Longrightarrow$ Weighted Potential $\Longrightarrow$ Ordinal Potential $\Longrightarrow$ Generalized. Ordinal Pot.

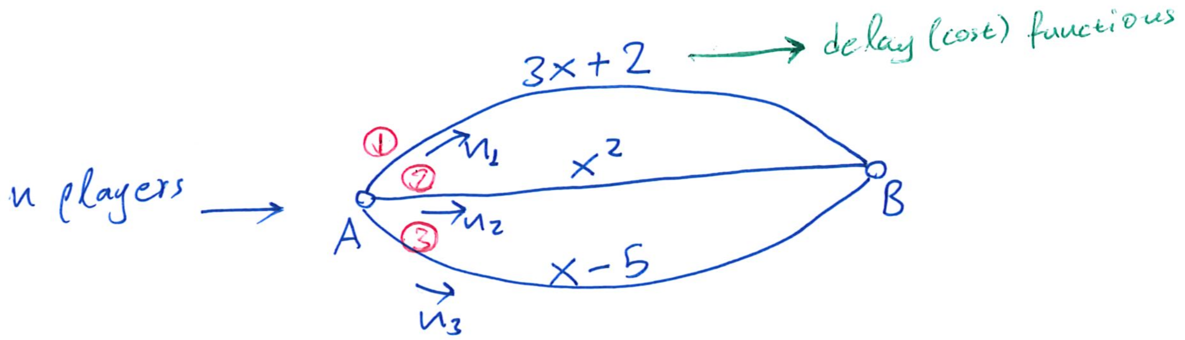
Exercise

In the following congestion game, where n players ($n \geq 4$) can use the 3 edges to go from A to B (1 edge each):

a) What would be the maximum value of the Rosenthal Potential (R.P.)?

b) What is the value of the R.P. when the players split equally to the 3 edges?

c) What is the running time of the algorithm for finding a PNE in the worst case?



Formula for the R.P.:

$$P(A) = \sum_{j \in \bigcup_{i=1}^{n} A_i} \left(\sum_{k=1}^{\sigma_j(A)} d_j(k) \right)$$

delay function of j for k players

$A = (A_i)_{i \in N}$ strategy profile

$\sigma_j(A)$ = # players who use the resource j in A

all the resources used in A.

a) In our problem, the R.P. can be simplified:

$$P(A) = \sum_{k=1}^{n_1}(3k+2) + \sum_{k=1}^{n_2}(k^2) + \sum_{k=1}^{n_3}(k-5)$$

where n_1, n_2, n_3 are the # players using edges 1,2,3 respectively in profile A.

It is clear that the maximum potential value is achieved when all n players choose the second edge (we will name this profile A_0)
In this case, the R.P. function has value:

$$P(A_0) = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \in O(n^3).$$

b) $n_1 = n_2 = n_3 = \frac{n}{3}$. (we will name this profile A_1)
In this case, the R.P. has value:

$$P(A_1) = \sum_{k=1}^{\frac{n}{3}}(3k+2) + \sum_{k=1}^{\frac{n}{3}}(k^2) + \sum_{k=1}^{\frac{n}{3}}(k-5)$$

$$= \left[3\cdot\sum_{k=1}^{\frac{n}{3}} k + 2\cdot\frac{n}{3}\right] + \left[\sum_{k=1}^{\frac{n}{3}} k^2\right] + \left[\sum_{k=1}^{\frac{n}{3}} k - 5\cdot\frac{n}{3}\right]$$

$$= 3\cdot\frac{\frac{n}{3}\left(\frac{n}{3}+1\right)}{2} + \frac{\frac{n}{3}\left(\frac{n}{3}+1\right)\left(2\frac{n}{3}+1\right)}{6} + \frac{\frac{n}{3}\left(\frac{n}{3}+1\right)}{2} - 3\cdot\frac{n}{3}$$

$$= 2\cdot\frac{n}{3}\left(\frac{n}{3}+1\right) + \frac{\frac{n}{3}\left(\frac{n}{3}+1\right)\left(2\frac{n}{3}+1\right)}{6} - n$$

$$= \frac{12\cdot\frac{n}{3}\left(\frac{n}{3}+1\right) + \frac{n}{3}\left(\frac{n}{3}+1\right)\left(2\frac{n}{3}+1\right)}{6} - n$$

$$= \frac{\frac{n}{3}\left(\frac{n}{3}+1\right)\left(2\frac{n}{3}+13\right)}{6} - n$$

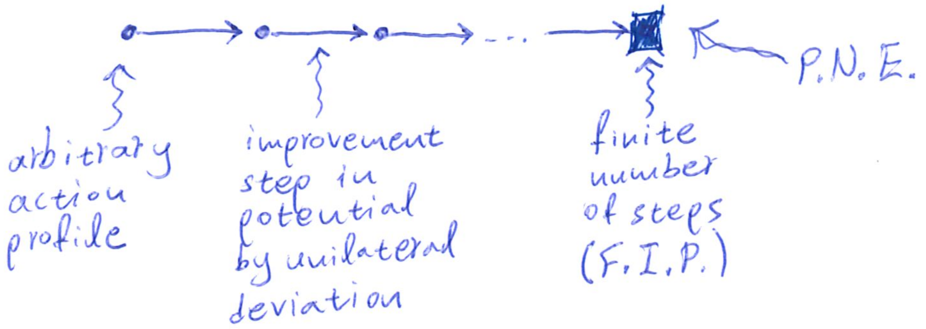
$$= \frac{\left(\frac{n^2}{9}+\frac{n}{3}\right)\left(2\frac{n}{3}+13\right)}{6} - n$$

$$= \frac{\frac{2}{27}n^3 + \frac{13}{9}n^2 + \frac{2}{9}n^2 + \frac{13}{3}n}{6} - n$$

$$= \frac{2}{162}n^3 + \frac{15}{54}n^2 + \frac{13}{18}n - n$$

$$= \frac{1}{81}n^3 + \frac{5}{18}n^2 - \frac{5}{18}n$$

c) Our algorithm for finding a PNE works as follows:



- We know that any value of the R.P. is ≥ 0 by definition, so let's suppose the final action profile has potential 0.
- The maximum possible value of the R.P. is $O(n^3)$ due to (a).
- The smallest possible improvement (difference) in potential is 1 by definition of the R.P.

Therefore, the worst case number of steps (running-time) of our algorithm is $O(n^3)$.