Tutorial 5

Exercise

Suppose we have 3 machines M_1, M_2, M_3 with speeds $s_1 = 3$, $s_2 = 10$, $s_3 = 13$, respectively. We also have 6 tasks (players), with weights $w_1 = 30$, $w_2 = 27$, $w_3 = 21$, $w_4 = 18$, $w_5 = 14$, $w_6 = 12$. Find a pure Nash equilibrium using the LPT rule (see lecture slides). Is this also an optimum allocation?

Solution

According to the LPT rule, first we sort the weights in a non-increasing order: $w_1 \geq w_2 \geq w_3 \geq w_4 \geq w_5 \geq w_6$. In that order, starting with the greatest weight, we place them one by one to the machine where, if put, it pays the least delay among all delays of all machines at that point.

weight	latency M_1	M_2	M_3
W_1	$\frac{30}{3}$	$\frac{30}{10}$	$\frac{30}{13}$
W_2	$\frac{27}{3}$	$\frac{27}{10}$	$\frac{30+27}{13}=\frac{57}{13}$
W_3	$\frac{21}{3}$	$\frac{27+21}{10}=\frac{48}{10}$	$\frac{30+21}{13}=\frac{51}{13}$
W_4	$\frac{18}{3}$	$\frac{27+18}{10}=\frac{45}{10}$	$\frac{30+21+18}{13}=\frac{69}{13}$
W_5	$\frac{14}{3}$	$\frac{27+18+14}{10}=\frac{59}{10}$	$\frac{30+21+14}{13}=\frac{65}{13}$
W_6	$\frac{14+12}{3}=\frac{26}{3}$	$\frac{27+18+12}{10}=\frac{57}{10}$	$\frac{30+21+12}{13}=\frac{63}{13}$

For each weight the minimum delay (at that point) is circled.
Therefore, the allocation given by the LPT rule is
$M_1: \{W_5\}$, $M_2: \{W_2, W_4\}$, $M_3: \{W_1, W_3, W_6\}$.
The respective latency (delay) for each player on each of the machines is:

$$l_1 = \frac{14}{3}, \quad l_2 = \frac{45}{10}, \quad l_3 = \frac{63}{13}.$$

Let us call the allocation given by the LPT rule "A".
The cost (see lecture slides) of the allocation is

$$cost(A) = \max\{l_1, l_2, l_3\} = l_3 = \frac{63}{13}.$$

However, this is not the optimal allocation, since the following allocation "A′" is better: $M_1: \{W_5\}$, $M_2: \{W_1, W_4\}$, $M_3: \{W_2, W_3, W_6\}$
with $cost(A') = \max\{\frac{14}{3}, \frac{30+18}{10}, \frac{27+21+12}{13}\} = \frac{48}{10}$

Exercise

Consider the following instance of the load balancing game where the number of tasks is equal to the number of machines, and in particular we have:

- m identical machines $M_1, M_2, \dots, M_m$ (all of speed 1),
- m identical tasks with weights $w_1 = w_2 = \dots = w_m = 1$.

Consider also the mixed strategy profile A where each of the tasks is assigned to all machines equiprobably (i.e. with probability $\frac{1}{m}$). As shown in the set of problems of the previous week, A is a Nash equilibrium. Calculate the ratio $\frac{cost(A)}{cost(OPT)}$ in the following special cases:

(a) $m = 2$

(b) $m = 3$

Discuss what this ratio is for arbitrary m. What does this imply about the Price of Anarchy on identical machines for mixed Nash equilibria?

Solution

Recall that, since we are in the setting with mixed strategy profiles, by definition, the cost of a strategy profile P is

$$\mathrm{cost}(P) = \mathop{\mathbb{E}}_{\mathrm{cost}(H)\sim P}[\mathrm{cost}(H)] = \mathbb{E}\left[\max_{j\in[m]}(\ell_j)\right],$$

where H is an allocation that is realised with probability specified by P, and ℓ_j is the latency (delay) of machine $j\in[m]$.

Observe that an optimum allocation OPT is one with $\mathrm{cost}(OPT)=1$ and is a pure allocation where only one task is placed on each machine.

We are given the strategy profile A. Its cost is

$$\mathrm{cost}(A) = \mathop{\mathbb{E}}_{\mathrm{cost}(H)\sim A}[\mathrm{cost}(H)] = \sum_{\substack{\text{all}\\ \text{allocations } H}} \mathrm{cost}(H)\cdot \Pr\{H \text{ is realised under } A\}$$

(a) $cost(A) = 2 \cdot Pr\{2 \text{ tasks allocated on the same machine}\}$

$+ 1 \cdot Pr\{1 \text{ task allocated on each machine}\}$

$$= 2 \cdot \frac{2}{4} + 1 \cdot \frac{2}{4}$$

$$= \frac{3}{2}$$

Therefore it is $\boxed{\frac{cost(A)}{cost(OPT)} = \frac{3}{2}}$.

(b) $cost(A) = 3 \cdot Pr\{3 \text{ tasks allocated on the same machine}\}$

$+ 2 \cdot Pr\{2 \text{ tasks allocated on the same machine, } 1 \text{ on some other}\}$

$+ 1 \cdot Pr\{1 \text{ task allocated on each machine}\}$

$$= 3 \cdot \frac{3}{27} + 2 \cdot \frac{18}{27} + 1 \cdot \frac{6}{27}$$

$$= \frac{51}{27}$$

Therefore it is $\boxed{\frac{cost(A)}{cost(OPT)} = \frac{17}{9}}$.

By Proposition 20.11 in "Algorithmic Game Theory – Nisan et al.", for arbitrary m, it is $cost(A) = \Theta\left(\frac{\ln m}{\ln(\ln m)}\right)$. Since, as mentioned earlier A is a NE, by definition of the Price of Anarchy in the special class of identical machines and mixed NE, it is

$$PoA = \Omega\left(\frac{\log m}{\log(\log m)}\right)$$, where $m = \#$ of machines.

That is because, since some NE (A) yields cost $\Theta\left(\frac{\log m}{\log(\log m)}\right)$, the maximum cost over all NE cannot be less than that. Also, as we mentioned earlier $cost(OPT) = 1$.