ELEC 207 Part B

Control Theory Lecture 4: Control System Performance (1)

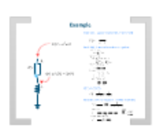
Prof Simon Maskell
CHAD-G68
s.maskell@liverpool.ac.uk
0151 794 4573

UNIVERSITY OF LIVERPOOL

Solving for the Time Response

Step-by-step process:

- Find X(s), Laplace Transform of input x(t)
- Find H(s), Transfer Function of System
- Y(s) = H(s)X(s)
- Find y(t), inverse Laplace Transform of Y(s)
  - This will often involve partial fractions



This lecture covers:

- How to use Laplace Transforms to Solve the Time Response of a Dynamic System
- Typical Input Signals

ELEC 207 Part B

Control Theory Lecture 4: Control System Performance (1)

Prof Simon Maskell
CHAD-G68
s.maskell@liverpool.ac.uk
0151 794 4573

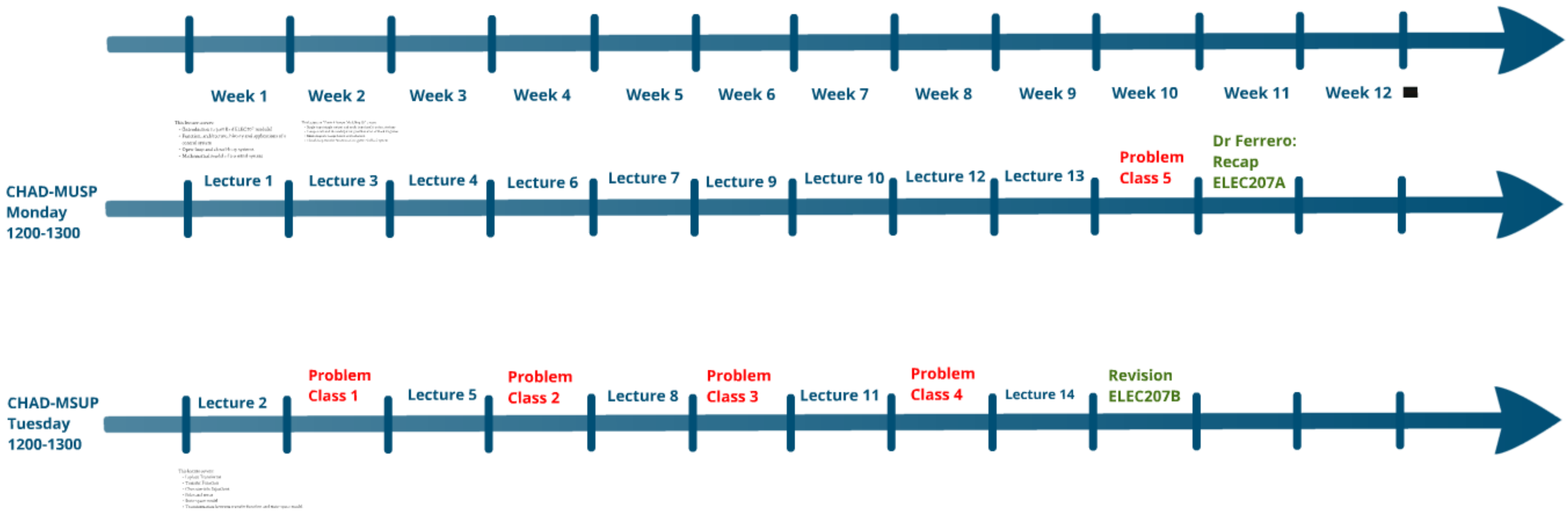
UNIVERSITY OF LIVERPOOL

This lecture covers:

- How to use Laplace Transforms to Solve the Time Response of a Dynamic System
- Typical Input Signals

rt B

ELEC 207B: Timeline



Week 1

This lecture covers:

- (Introduction to part B of ELEC207 module)
- Function, architecture, history and applications of a control system
- Open-loop and closed-loop systems
- Mathematical model of a control system

Lecture 1

This lecture on "Co

- Single-input sin
- Components an
- Block diagram r
- Closed-loop tra

Le

Problem Class 1

Lecture 2

This lecture covers:

- Laplace Transforms
- Transfer Function
- Characteristic Equations
- Poles and zeros
- State-space model
- Transformation between transfer function and state-space model

d

Week 2

Week

ule)

cations of a

This lecture on "Control System Modelling (2)" covers:

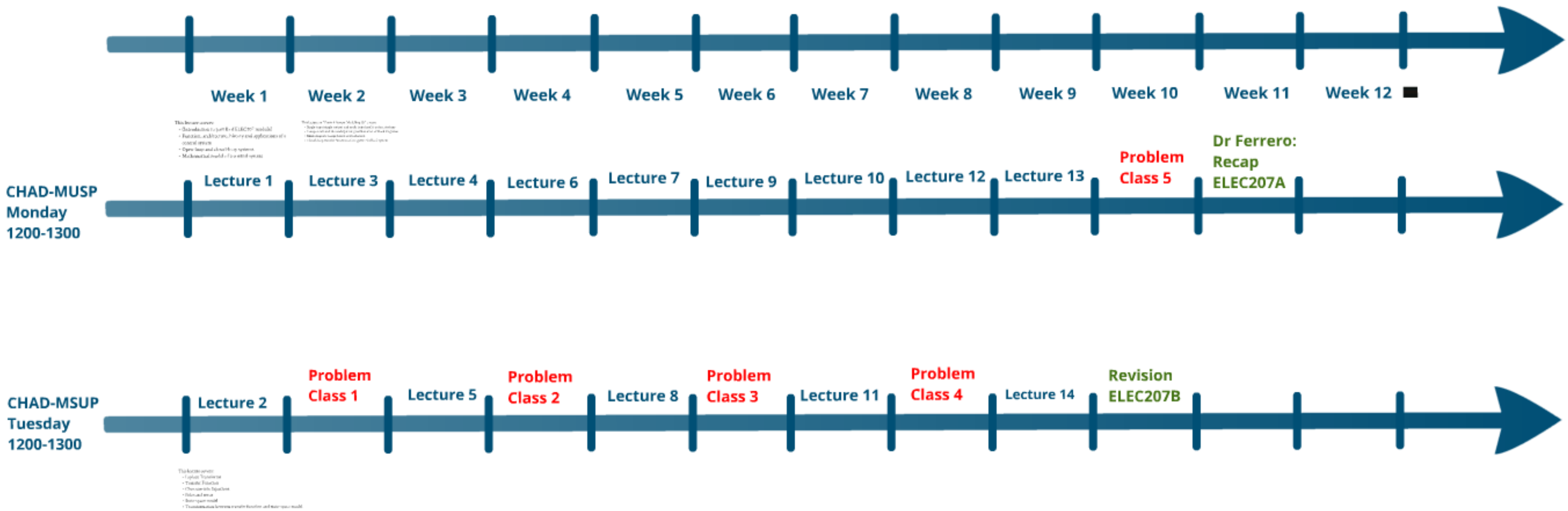
- Single-input single-output and multi-input multi-output systems
- Components and the underpinning mathematics of block diagrams
- Block diagram manipulation and reduction
- Closed-loop transfer function of a negative feedback system

1

Lecture 3

Lectu

ELEC 207B: Timeline



This lecture covers:

- How to use Laplace Transforms to Solve the Time Response of a Dynamic System
- Typical Input Signals

rt B

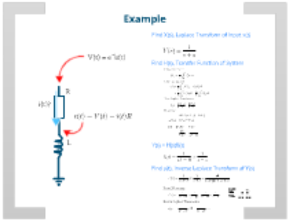
Solving for the Time Response

Step-by-step process:

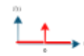
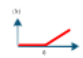
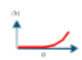
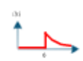
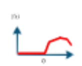
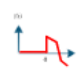
- Find X(s), Laplace Transform of input x(t)
- Find H(s), Transfer Function of System
- Y(s) = H(s)X(s)
- Find y(t), inverse Laplace Transform of Y(s)
  - This will often involve partial fractions

f(t)	F(s)
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin(\omega t)\,u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)\,u(t)$	$\frac{s}{s^2+\omega^2}$

This table (without sketches) will be provided in the exam

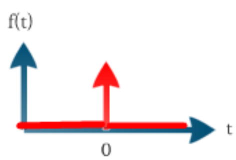
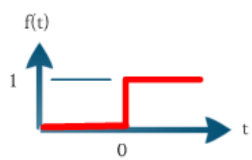


t)

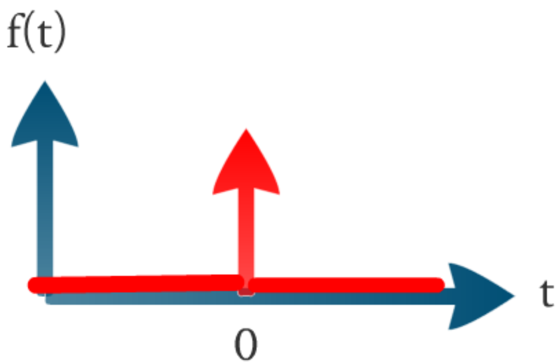
f(t)	F(s)
 $\delta(t)$	1
 $u(t)$	$\frac{1}{s}$
 $tu(t)$	$\frac{1}{s^2}$
 $t^n u(t)$	$\frac{n!}{s^{n+1}}$
 $e^{-at}u(t)$	$\frac{1}{s+a}$
 $\sin(\omega t)\,u(t)$	$\frac{\omega}{s^2+\omega^2}$
 $\cos(\omega t)\,u(t)$	$\frac{s}{s^2+\omega^2}$

This table (without sketches)
will be provided
in the exam

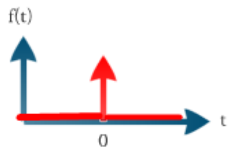
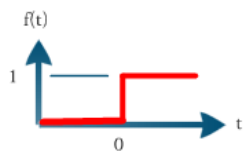
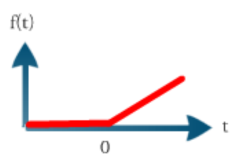
u(t) often gets dropped
from the discussion
since we assume t>0

f(t)	F(s)
 $\delta(t)$	1
 $u(t)$	$\frac{1}{s}$

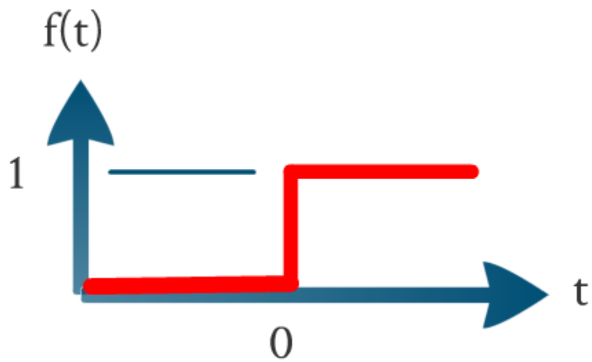
$$F(s)$$



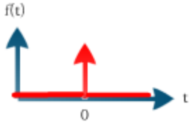
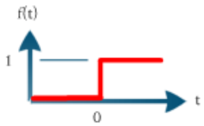
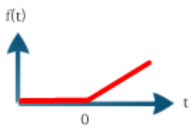
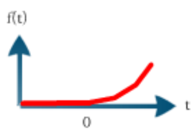
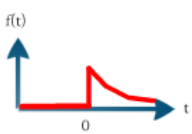
$$\delta(t)$$

$f(t)$		$F(s)$
	$\delta(t)$	1
	$u(t)$	$\frac{1}{s}$
	$tu(t)$	$\frac{1}{s^2}$

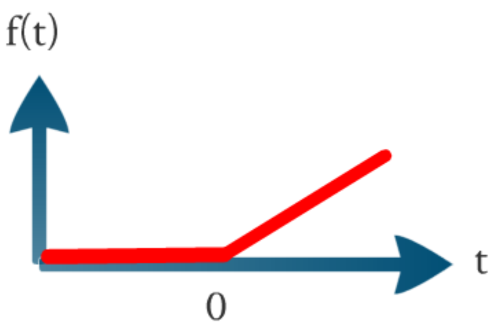
f(t)



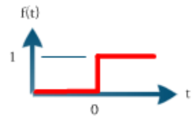
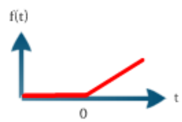
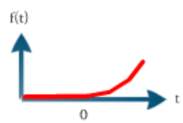
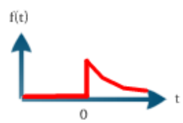
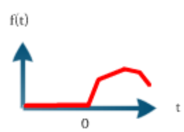
$$u(t)$$

 $\delta(t)$	1
 $u(t)$	$\frac{1}{s}$
 $tu(t)$	$\frac{1}{s^2}$
 $t^n u(t)$	$\frac{n!}{s^{n+1}}$
 $e^{-at}u(t)$	1

$$u(t)$$

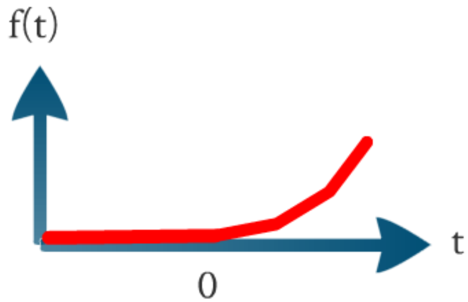


$$tu(t)$$

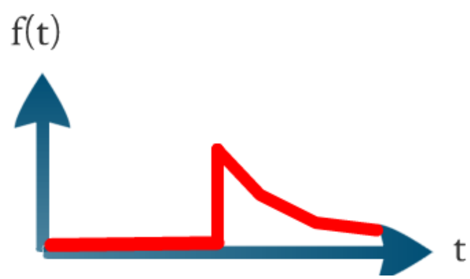
 $u(t)$	$\frac{1}{s}$
 $tu(t)$	$\frac{1}{s^2}$
 $t^n u(t)$	$\frac{n!}{s^{n+1}}$
 $e^{-at}u(t)$	$\frac{1}{s+a}$
 $\sin(\omega t)\, u(t)$	$\frac{\omega}{s^2+\omega^2}$

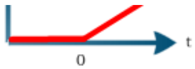
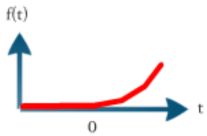
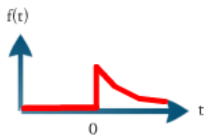
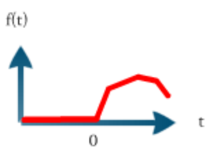
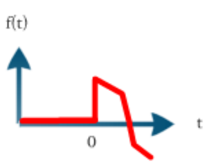
0

$$tu(t)$$

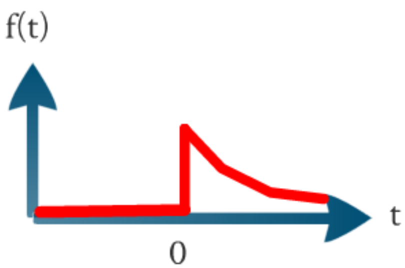


$$t^n u(t)$$

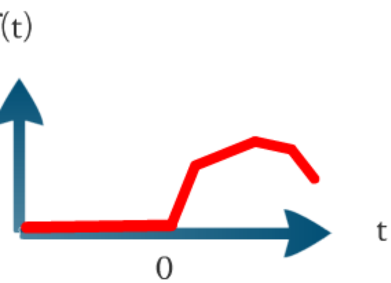


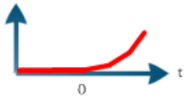
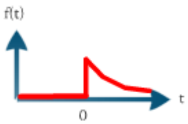
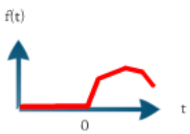
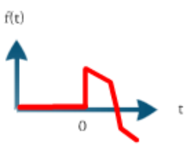
 $tu(t)$	$\frac{1}{s^2}$
 $t^n u(t)$	$\frac{n!}{s^{n+1}}$
 $e^{-at}u(t)$	$\frac{1}{s+a}$
 $\sin(\omega t)\,u(t)$	$\frac{\omega}{s^2+\omega^2}$
	s

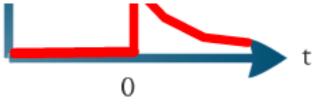
$$t^n u(t)$$



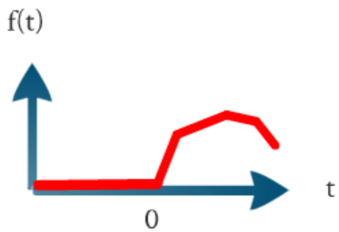
$$e^{-at}u(t)$$



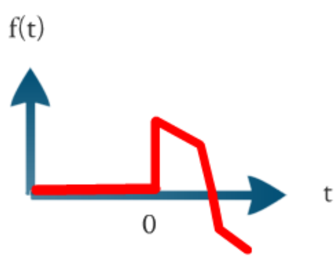
 $t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
 $e^{-at} u(t)$	$\dfrac{1}{s+a}$
 $\sin(\omega t)\, u(t)$	$\dfrac{\omega}{s^2+\omega^2}$
 $\cos(\omega t)\, u(t)$	$\dfrac{s}{s^2+\omega^2}$



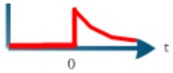
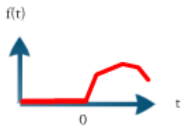
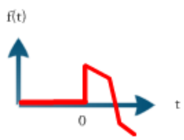
$$e^{-at}u(t)$$



$$\sin(\omega t)\,u(t)$$



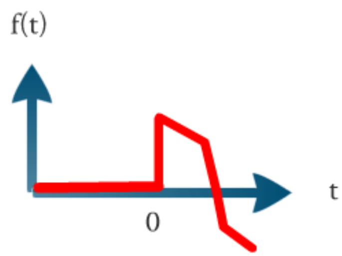
$$\cos(\omega t)\,u(t)$$

 $e^{-at}u(t)$	$\frac{1}{s+a}$
 $\sin(\omega t)\,u(t)$	$\frac{\omega}{s^2+\omega^2}$
 $\cos(\omega t)\,u(t)$	$\frac{s}{s^2+\omega^2}$

This table (without sketches) will be provided

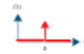
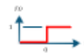
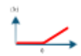
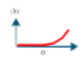
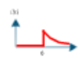
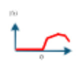
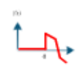
u(t) often ge
from the disc
since we assu

$$\sin(\omega t)\, u(t)$$



$$\cos(\omega t)\, u(t)$$

t)

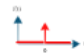
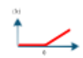
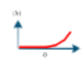
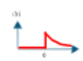
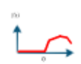
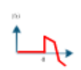
f(t)	F(s)
 $\delta(t)$	1
 $u(t)$	$\frac{1}{s}$
 $tu(t)$	$\frac{1}{s^2}$
 $t^n u(t)$	$\frac{n!}{s^{n+1}}$
 $e^{-at}u(t)$	$\frac{1}{s+a}$
 $\sin(\omega t)\, u(t)$	$\frac{\omega}{s^2+\omega^2}$
 $\cos(\omega t)\, u(t)$	$\frac{s}{s^2+\omega^2}$

This table (without sketches)
will be provided
in the exam

u(t) often gets dropped
from the discussion
since we assume t>0

$u(t)$ often gets dropped from the discussion since we assume $t>0$

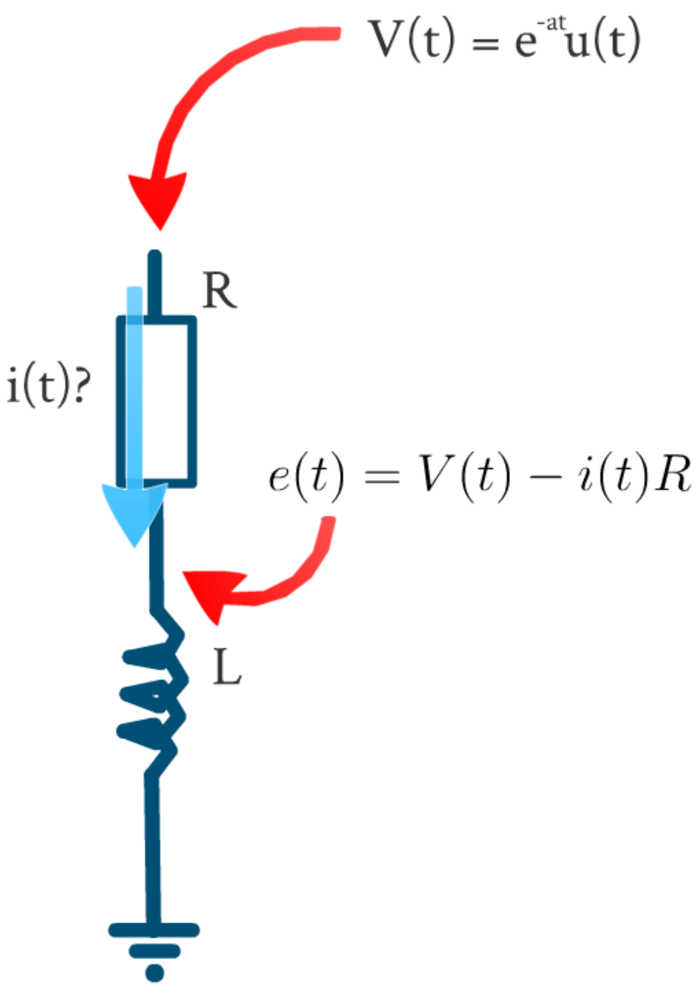
t)

f(t)	F(s)
 $\delta(t)$	1
 $u(t)$	$\frac{1}{s}$
 $tu(t)$	$\frac{1}{s^2}$
 $t^n u(t)$	$\frac{n!}{s^{n+1}}$
 $e^{-at}u(t)$	$\frac{1}{s+a}$
 $\sin(\omega t)\, u(t)$	$\frac{\omega}{s^2+\omega^2}$
 $\cos(\omega t)\, u(t)$	$\frac{s}{s^2+\omega^2}$

This table (without sketches)
will be provided
in the exam

u(t) often gets dropped
from the discussion
since we assume t>0

Example



Find X(s), Laplace Transform of input x(t)

$$V(s) = \frac{1}{s+a}$$

Find H(s), Transfer Function of System

Consider current:

$$i(t) = \frac{1}{L}\int_0^t e(t)dt$$

Substitute for e(t):

$$\begin{aligned} i(t) &= \frac{1}{L}\int_0^t (V(t) - i(t)R)\,dt \\ &= \frac{1}{L}\int_0^t V(t)\,dt - \frac{R}{L}\int_0^t i(t)\,dt \end{aligned}$$

Take Laplace Transforms:

$$I(s) = \frac{1}{L}\frac{V(s)}{s} - \frac{R}{L}\frac{I(s)}{s}$$

Rearrange:

$$sLI(s) = V(s) - RI(s)$$

$$sLI(s) + RI(s) = V(s)$$

$$(sL + R)\,I(s) = V(s)$$

$$\frac{I(s)}{V(s)} = \frac{1}{sL+R}$$

Y(s) = H(s)X(s)

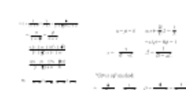
$$I(s) = \frac{1}{sL+R} \times \frac{1}{s+a}$$

Find y(t), inverse Laplace Transform of Y(s)

$$I(s) = \frac{1}{sL+R} \times \frac{1}{s+a} = \frac{\frac{1}{L}}{(s+\frac{R}{L}) \times (s+a)}$$

Partial Fractions:

$$I(s) = -\frac{1}{R-aL} \times \frac{1}{s+\frac{R}{L}} + \frac{1}{R-aL} \times \frac{1}{s+a}$$



Inverse Laplace Transforms:

$$i(t) = -\frac{1}{R-aL}e^{-\frac{R}{L}t} + \frac{1}{R-aL}e^{-at}$$

Example

- $V(t) = e^{-at}u(t)$

Find X(s), Laplace Transform of input x(t)

$$V(s) = \frac{1}{s+a}$$

Find H(s), Transfer Function of System

Consider current:

$$i(t) = \frac{1}{L}\int_0^t e(t)dt$$

Substitute for e(t):

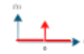
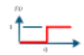
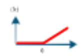
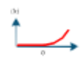
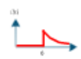
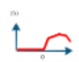
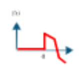
$$\begin{aligned} i(t) &= \frac{1}{L}\int_0^t (V(t) - i(t)R)\,dt \\ &= \frac{1}{L}\int_0^t V(t)\,dt - \frac{R}{L}\int_0^t i(t)\,dt \end{aligned}$$

Take Laplace Transforms:

$$I(s) = \frac{1}{L}\frac{V(s)}{s} - \frac{R}{L}\frac{I(s)}{s}$$

Rearrange:

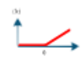
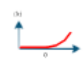
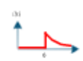
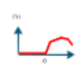
t)

f(t)	F(s)
 $\delta(t)$	1
 $u(t)$	$\frac{1}{s}$
 $tu(t)$	$\frac{1}{s^2}$
 $t^n u(t)$	$\frac{n!}{s^{n+1}}$
 $e^{-at}u(t)$	$\frac{1}{s+a}$
 $\sin(\omega t)\, u(t)$	$\frac{\omega}{s^2+\omega^2}$
 $\cos(\omega t)\, u(t)$	$\frac{s}{s^2+\omega^2}$

This table (without sketches)
will be provided
in the exam

u(t) often gets dropped
from the discussion
since we assume t>0

t)

f(t)	F(s)
 $\delta(t)$	1
 $u(t)$	$\frac{1}{s}$
 $tu(t)$	$\frac{1}{s^2}$
 $t^n u(t)$	$\frac{n!}{s^{n+1}}$
 $e^{-at}u(t)$	$\frac{1}{s+a}$
 $\sin(\omega t)\, u(t)$	$\frac{\omega}{s^2+\omega^2}$
 $\cos(\omega t)\, u(t)$	$\frac{s}{s^2+\omega^2}$

This table (without sketches)
will be provided
in the exam

u(t) often gets dropped
from the discussion
since we assume t>0

Example

$V(t) = e^{-at}u(t)$

Find X(s), Laplace Transform of input x(t)

$$V(s) = \frac{1}{s+a}$$

Find H(s), Transfer Function of System

Consider current:

$$i(t) = \frac{1}{L}\int_0^t e(t)dt$$

Substitute for e(t):

$$\begin{aligned} i(t) &= \frac{1}{L}\int_0^t (V(t) - i(t)R)\,dt \\ &= \frac{1}{L}\int_0^t V(t)\,dt - \frac{R}{L}\int_0^t i(t)\,dt \end{aligned}$$

Take Laplace Transforms:

$$I(s) = \frac{1}{L}\frac{V(s)}{s} - \frac{R}{L}\frac{I(s)}{s}$$

Rearrange:

Find H(s), Transfer Function of System

Consider current:

$$i(t) = \frac{1}{L}\int_0^t e(t)dt$$

Substitute for e(t):

$$\begin{aligned} i(t) &= \frac{1}{L}\int_0^t \left(V\left(t\right) - i\left(t\right)R\right)dt \\ &= \frac{1}{L}\int_0^t V\left(t\right)dt - \frac{R}{L}\int_0^t i\left(t\right)dt \end{aligned}$$

Take Laplace Transforms:

$$I(s) = \frac{1}{L}\frac{V\left(s\right)}{s} - \frac{R}{L}\frac{I\left(s\right)}{s}$$

Rearrange:

$$sLI(s) = V\left(s\right) - RI\left(s\right)$$

$$sLI(s) + RI(s) = V\left(s\right)$$

$$\left(sL + R\right)I(s) = V\left(s\right)$$

$$\frac{I(s)}{V(s)} = \frac{1}{sL + R}$$

$$\frac{I(s)}{V(s)} = \frac{1}{sL + R}$$

Y(s) = H(s)X(s)

$$I(s) = \frac{1}{sL + R} \times \frac{1}{s + a}$$

Find y(t), inverse Laplace

$$I(s) = \frac{1}{sL + R} \times \frac{1}{s + a} = \frac{\quad}{(s + \cdots}$$

$$I(s) = \frac{}{sL+R} \times \frac{}{s+a}$$

Find y(t), inverse Laplace Transform of Y(s)

$$I(s) = \frac{1}{sL+R} \times \frac{1}{s+a} = \frac{\frac{1}{L}}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

Partial Fractions:

$$I(s) = -\frac{1}{R-aL} \times \frac{1}{s+\frac{R}{L}} + \frac{1}{R-aL} \times \frac{1}{s+a}$$

Inverse Laplace Transforms:

$$i(t) = -\frac{1}{R-aL} e^{-\frac{R}{L}t} + \frac{1}{R-aL} e^{-at}$$

$$I(s) = \frac{1}{sL+R} \times \frac{1}{s+a} = \frac{\frac{1}{L}}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

$$= \frac{\alpha}{s+\frac{R}{L}} + \frac{\beta}{s+a}$$

$$= \frac{\alpha(s+a) + \beta\left(s+\frac{R}{L}\right)}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

$$= \frac{s(\alpha+\beta) + \left(a\alpha + \frac{R}{L}\beta\right)}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

$$I(s) = -\frac{1}{R-aL} \times \frac{1}{s+\frac{R}{L}} + \frac{1}{R-aL} \times \frac{1}{s+a}$$

$$\alpha + \beta = 0 \qquad a\alpha + \frac{R}{L}\beta = \frac{1}{L}$$

$$-aL\beta + R\beta = 1$$

$$\alpha = -\frac{1}{R-aL} \qquad \beta = \frac{1}{R-aL}$$

"Cover up" method:

$$\alpha = \frac{\frac{1}{L}}{-\frac{R}{L}+a} = -\frac{1}{R-aL} \qquad \beta = \frac{\frac{1}{L}}{-a+\frac{R}{L}} = \frac{1}{R-aL}$$

$$I(s) = \frac{1}{sL+R} \times \frac{1}{s+a} = \frac{\frac{1}{L}}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

$$= \frac{\alpha}{s+\frac{R}{L}} + \frac{\beta}{s+a}$$

$$= \frac{\alpha\,(s+a) + \beta\left(s+\frac{R}{L}\right)}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

$$= \frac{s\,(\alpha+\beta) + \left(a\alpha + \frac{R}{L}\beta\right)}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

$$\alpha + \beta = 0 \qquad a\alpha + \frac{R}{L}\beta = \frac{1}{L}$$

$$-aL\beta + R\beta = 1$$

$$\alpha = -\frac{1}{R-aL} \qquad \beta = \frac{1}{R-aL}$$

$$I(s) = -\frac{1}{R-aL} \times \frac{1}{s+\frac{R}{L}} + \frac{1}{R-aL} \times \frac{1}{s+a}$$

"Cover up" method:

$$\alpha = \frac{\frac{1}{L}}{-\frac{R}{L}+a} = -\frac{1}{R-aL} \qquad \beta = \frac{\frac{1}{L}}{-a+\frac{R}{L}} = \frac{1}{R-aL}$$

$$I(s) = \frac{}{sL+R} \times \frac{}{s+a}$$

Find y(t), inverse Laplace Transform of Y(s)

$$I(s) = \frac{1}{sL+R} \times \frac{1}{s+a} = \frac{\frac{1}{L}}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

Partial Fractions:

$$I(s) = -\frac{1}{R-aL} \times \frac{1}{s+\frac{R}{L}} + \frac{1}{R-aL} \times \frac{1}{s+a}$$

Inverse Laplace Transforms:

$$i(t) = -\frac{1}{R-aL} e^{-\frac{R}{L}t} + \frac{1}{R-aL} e^{-at}$$

$$I(s) = \frac{1}{sL+R} \times \frac{1}{s+a} = \frac{\frac{1}{L}}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

$$= \frac{\alpha}{s+\frac{R}{L}} + \frac{\beta}{s+a}$$

$$= \frac{\alpha(s+a) + \beta\left(s+\frac{R}{L}\right)}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

$$= \frac{s(\alpha+\beta) + \left(a\alpha+\frac{R}{L}\beta\right)}{\left(s+\frac{R}{L}\right) \times (s+a)}$$

$$I(s) = -\frac{1}{R-aL} \times \frac{1}{s+\frac{R}{L}} + \frac{1}{R-aL} \times \frac{1}{s+a}$$

$$\alpha + \beta = 0 \qquad a\alpha + \frac{R}{L}\beta = \frac{1}{L}$$

$$-aL\beta + R\beta = 1$$

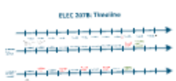
$$\alpha = -\frac{1}{R-aL} \qquad \beta = \frac{1}{R-aL}$$

"Cover up" method:

$$\alpha = \frac{\frac{1}{L}}{-\frac{R}{L}+a} = -\frac{1}{R-aL} \qquad \beta = \frac{\frac{1}{L}}{-a+\frac{R}{L}} = \frac{1}{R-aL}$$

This lecture covers:

- How to use Laplace Transforms to Solve the Time Response of a Dynamic System
- Typical Input Signals



rt B