ELEC 207 Part B

Control Theory Lecture 9: Control System Design (1)

Prof Simon Maskell
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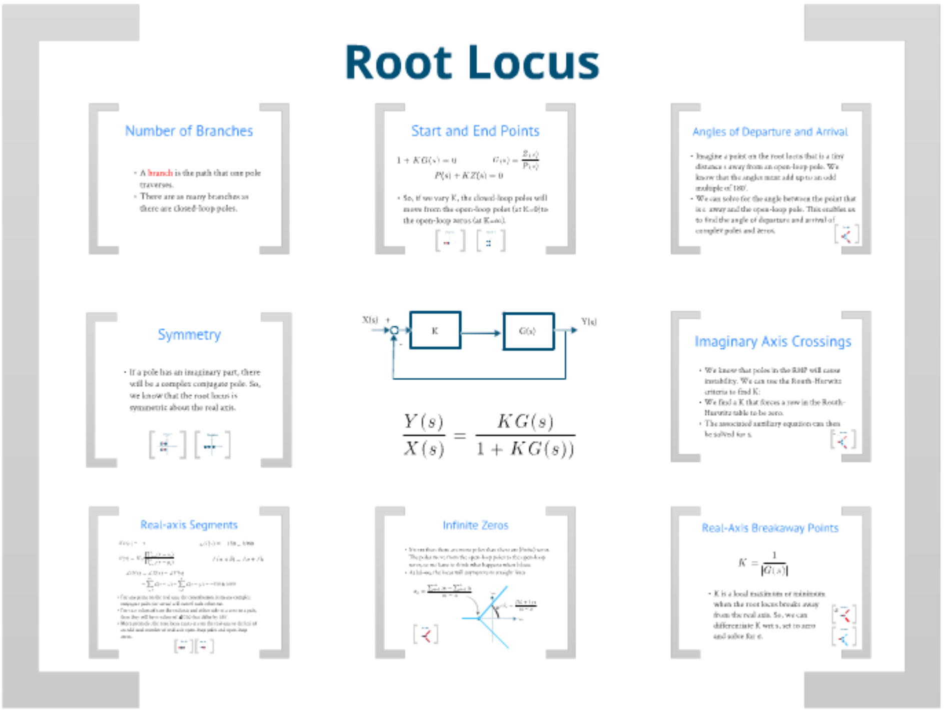
UNIVERSITY OF LIVERPOOL



This lecture covers:

- *Root locus diagram representation for a closed loop system with variable gain.*

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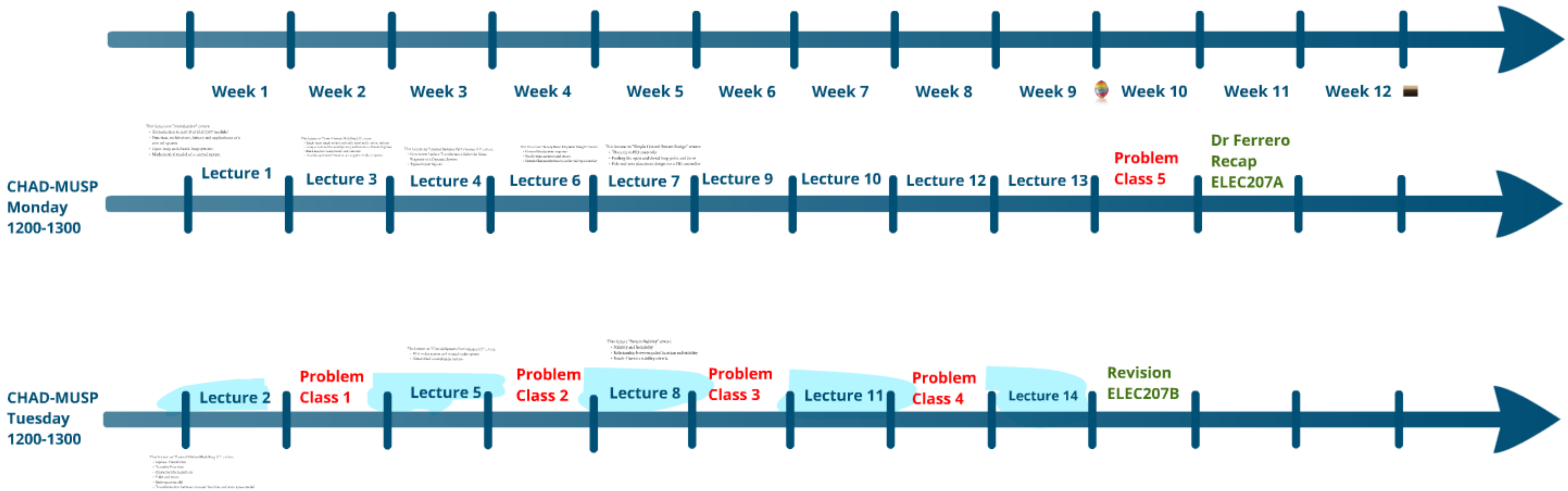
ELEC 207B: Timeline



This lecture covers:

- Root locus diagram representation for a closed loop system with variable gain.

Root Locus

Number of Branches

- A branch is the path that one pole traverses.
- There are as many branches as there are closed-loop poles.

Start and End Points

$$1 + KG(s) = 0 \qquad G(s) = \frac{Z(s)}{P(s)}$$

$$P(s) + KZ(s) = 0$$

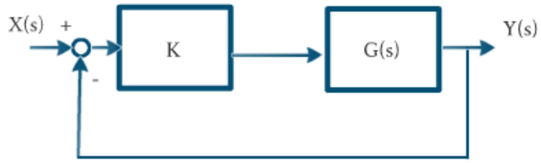
- So, if we vary K, the closed-loop poles will move from the open-loop poles (at K=0)to the open-loop zeros (at K=∞).

Angles of Departure and Arrival

- Imagine a point on the root locus that is a tiny distance ε away from an open-loop pole. We know that the angles must add up to an odd multiple of 180°.
- We can solve for the angle between the point that is ε away and the open-loop pole. This enables us to find the angle of departure and arrival of complex poles and zeros.

Symmetry

- If a pole has an imaginary part, there will be a complex conjugate pole. So, we know that the root locus is symmetric about the real axis.



$$\frac{Y(s)}{X(s)} = \frac{KG(s)}{1 + KG(s))}$$

Imaginary Axis Crossings

- We know that poles in the RHP will cause instability. We can use the Routh-Hurwitz criteria to find K:
- We find a K that forces a row in the Routh-Hurwitz table to be zero.
- The associated auxiliary equation can then be solved for s.

Real-axis Segments

$$KG(s) = -1 \qquad \angle G(s) = -180 \pm k360$$

$$G(s) = K_G \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)} \qquad \angle(a \times b) = \angle a + \angle b$$

$$\angle G(s) = \angle Z(s) - \angle P(s)$$
$$= \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = -180 \pm k360$$

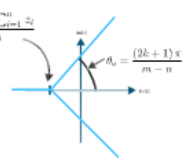
- For any point on the real axis, the contribution from any complex conjugate poles (or zeros) will cancel each other out.
- For two values of s on the real axis and either side of a zero or a pole, then they will have values of $\angle G(s)$ that differ by 180°.
- More precisely, the root locus exists at s on the real-axis to the left of an odd total number of real-axis open-loop poles and open-loop zeros.

Infinite Zeros

- If $n<m$ then there are more poles than there are (finite) zeros. The poles move from the open-loop poles to the open-loop zeros, so we have to think what happens when |s|=∞.
- As |s|→∞, the locus will asymptote to straight lines

$$\sigma_a = \frac{\sum_{i=1}^{m} p_i - \sum_{i=1}^{n} z_i}{m - n}$$

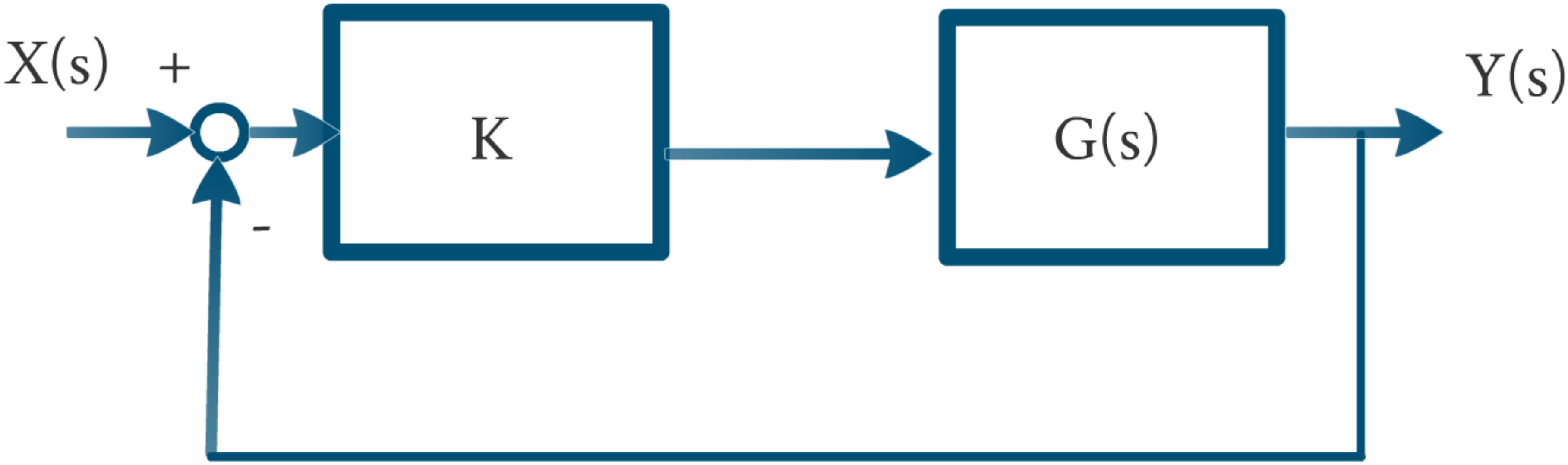
$$\theta_a = \frac{(2k+1)\pi}{m - n}$$



Real-Axis Breakaway Points

$$K = \frac{1}{|G(s)|}$$

- K is a local maximum or minimum when the root locus breaks away from the real axis. So, we can differentiate K wrt s, set to zero and solve for σ.



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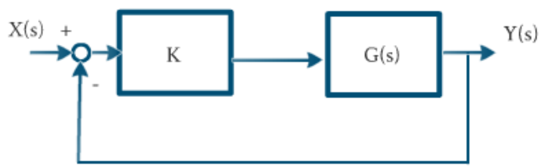

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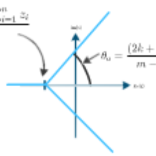

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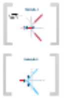

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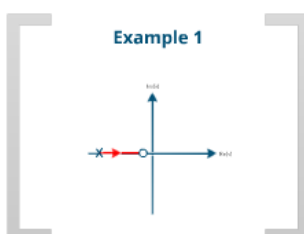
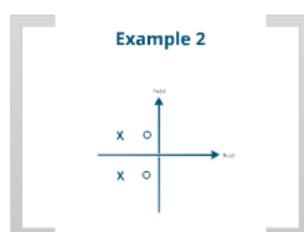
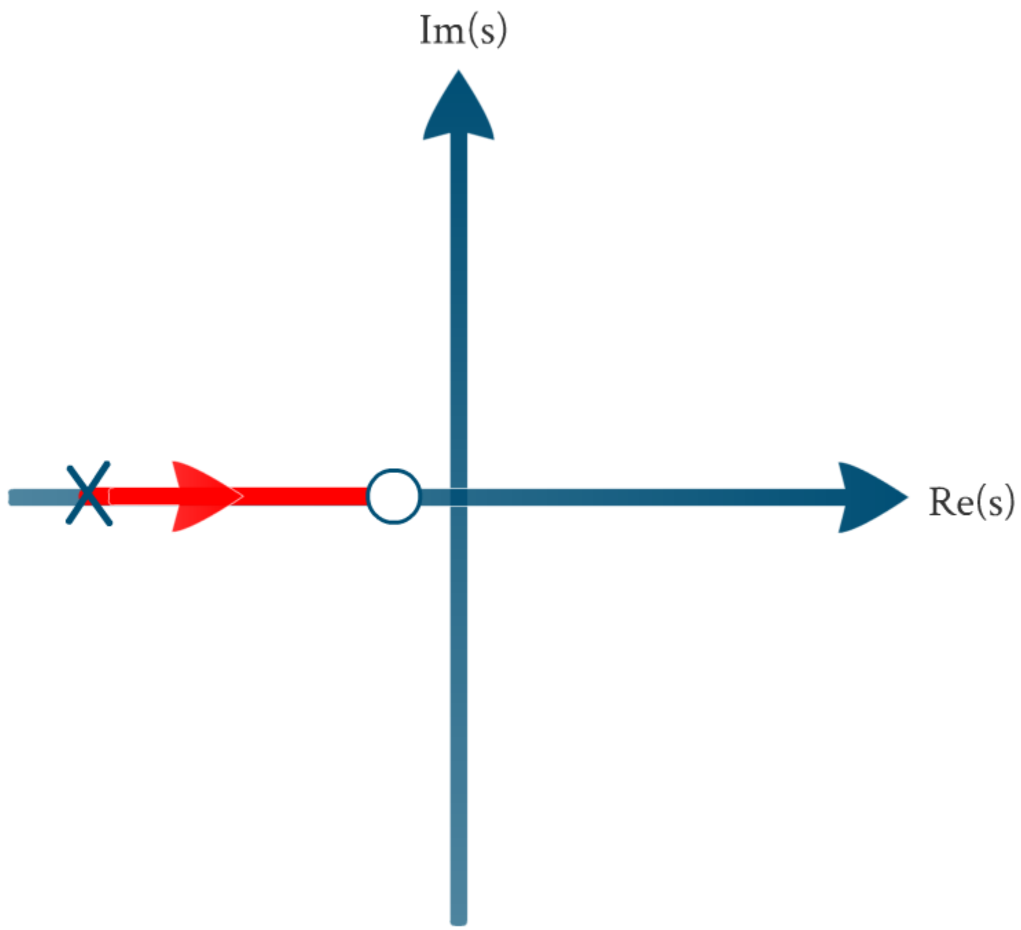
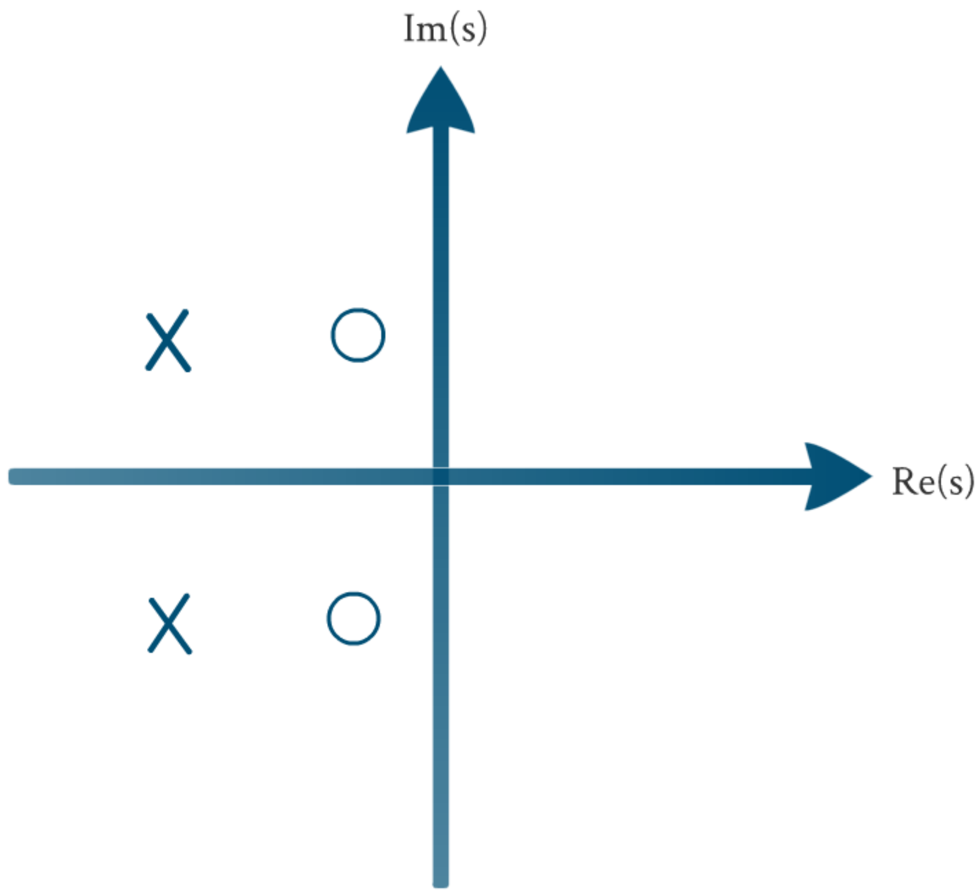




Example 1



Example 2



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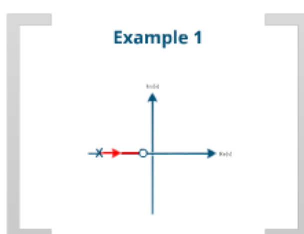

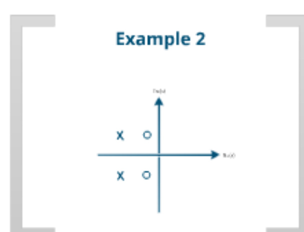

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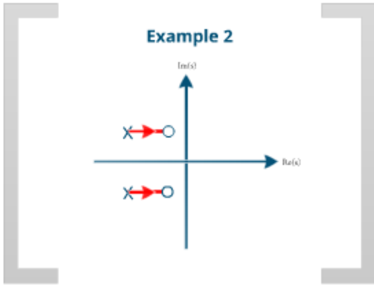

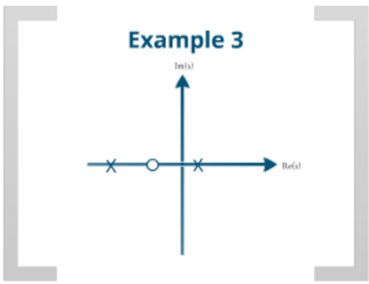

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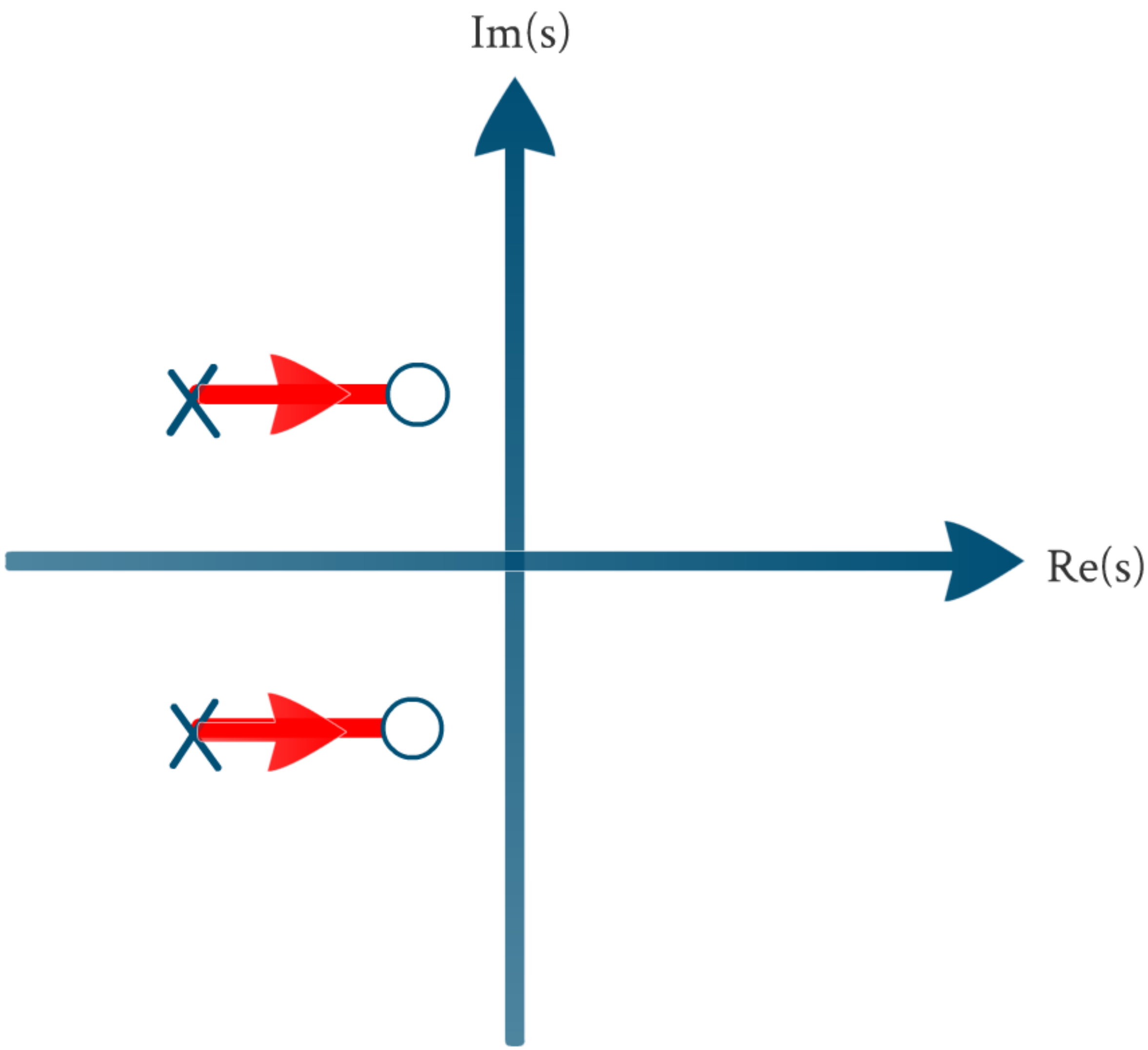

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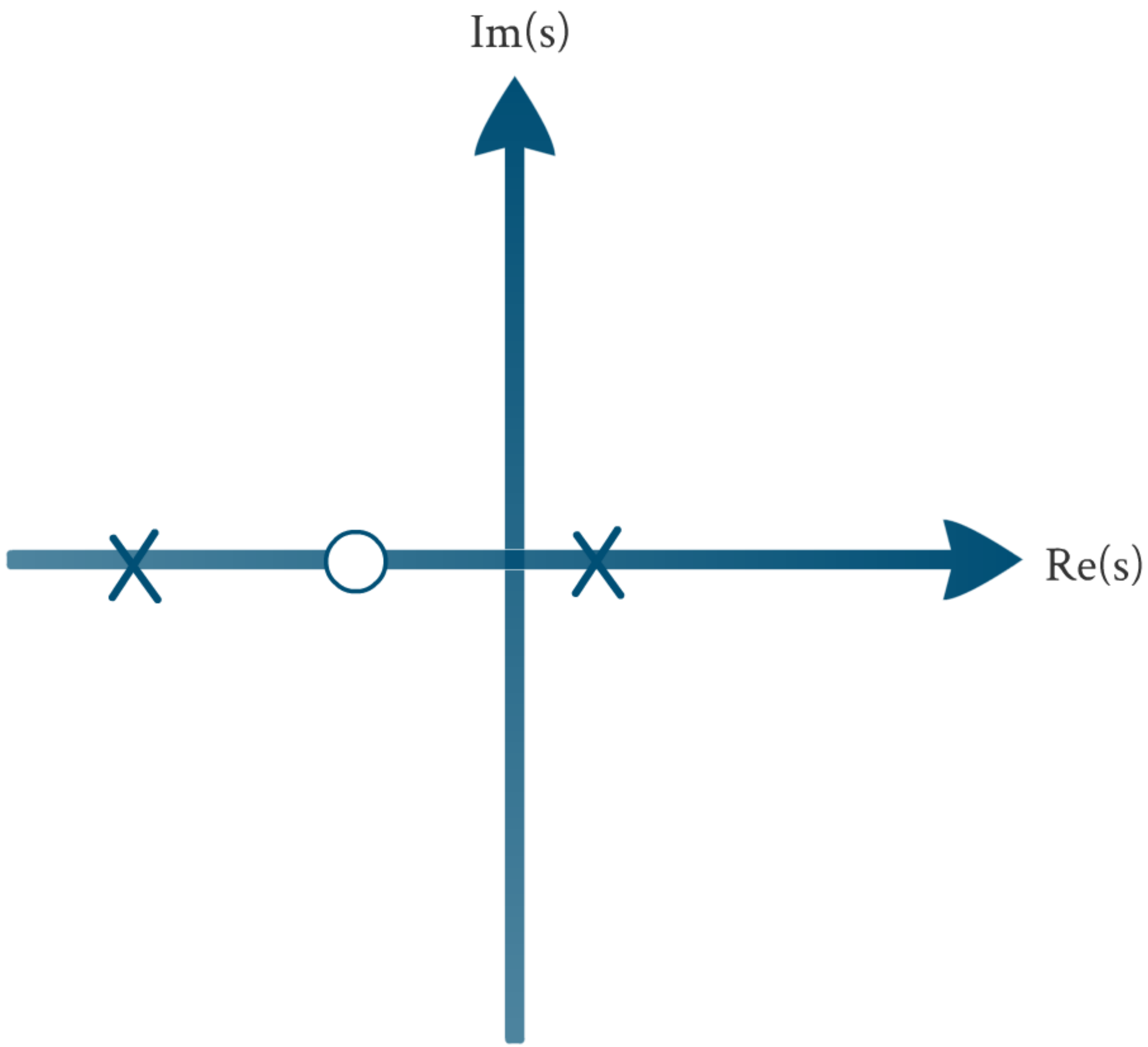

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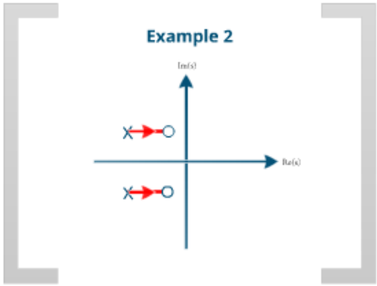

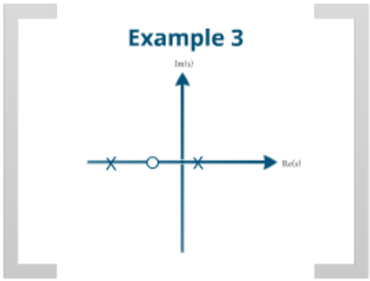

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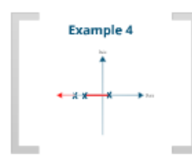

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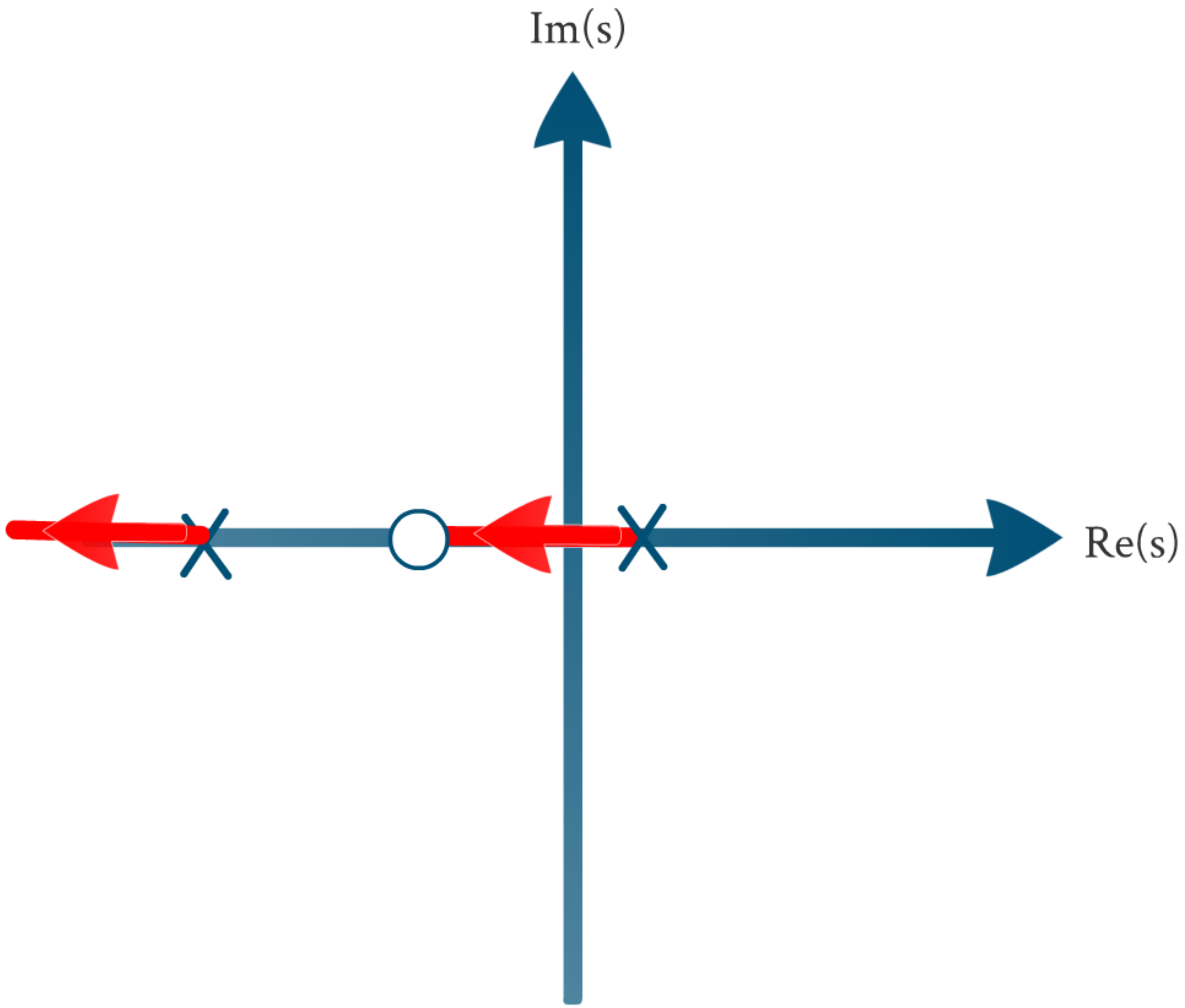

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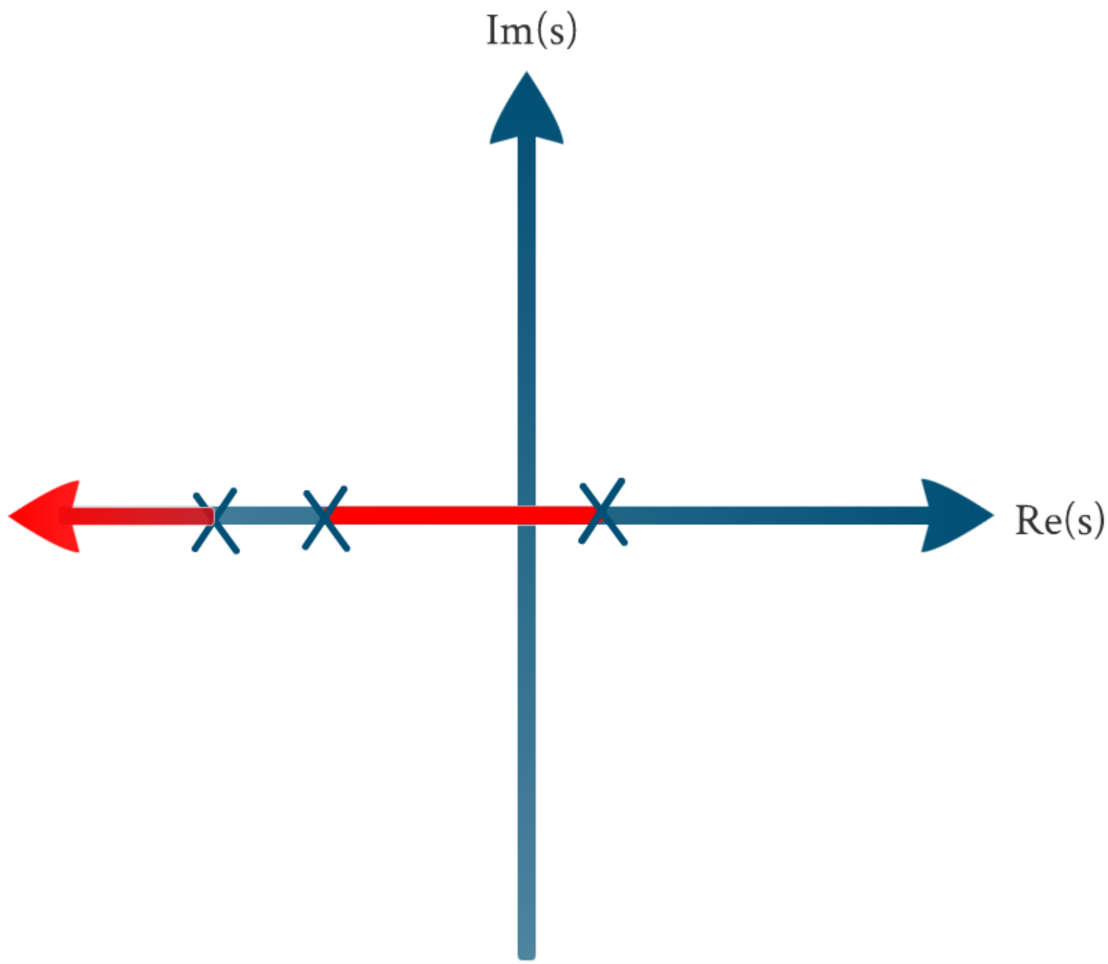

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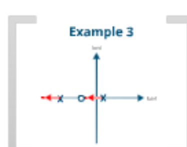

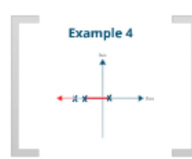

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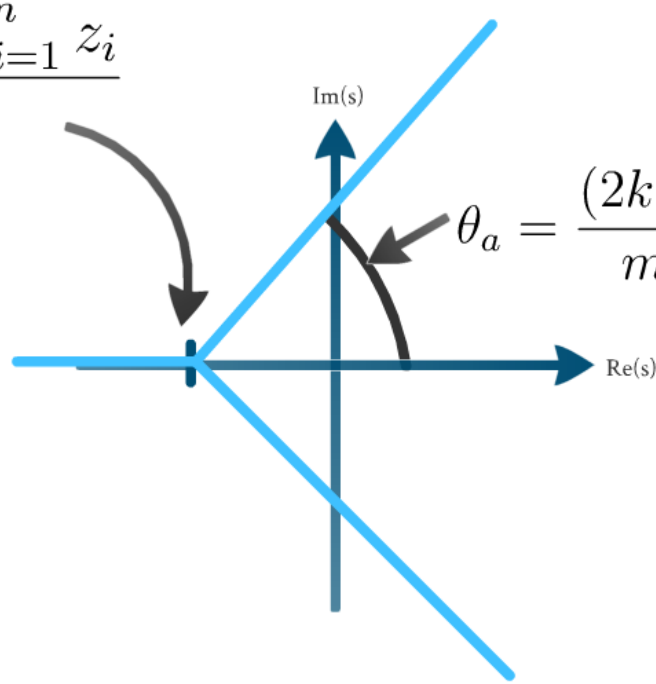

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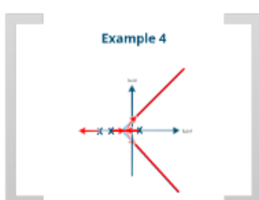

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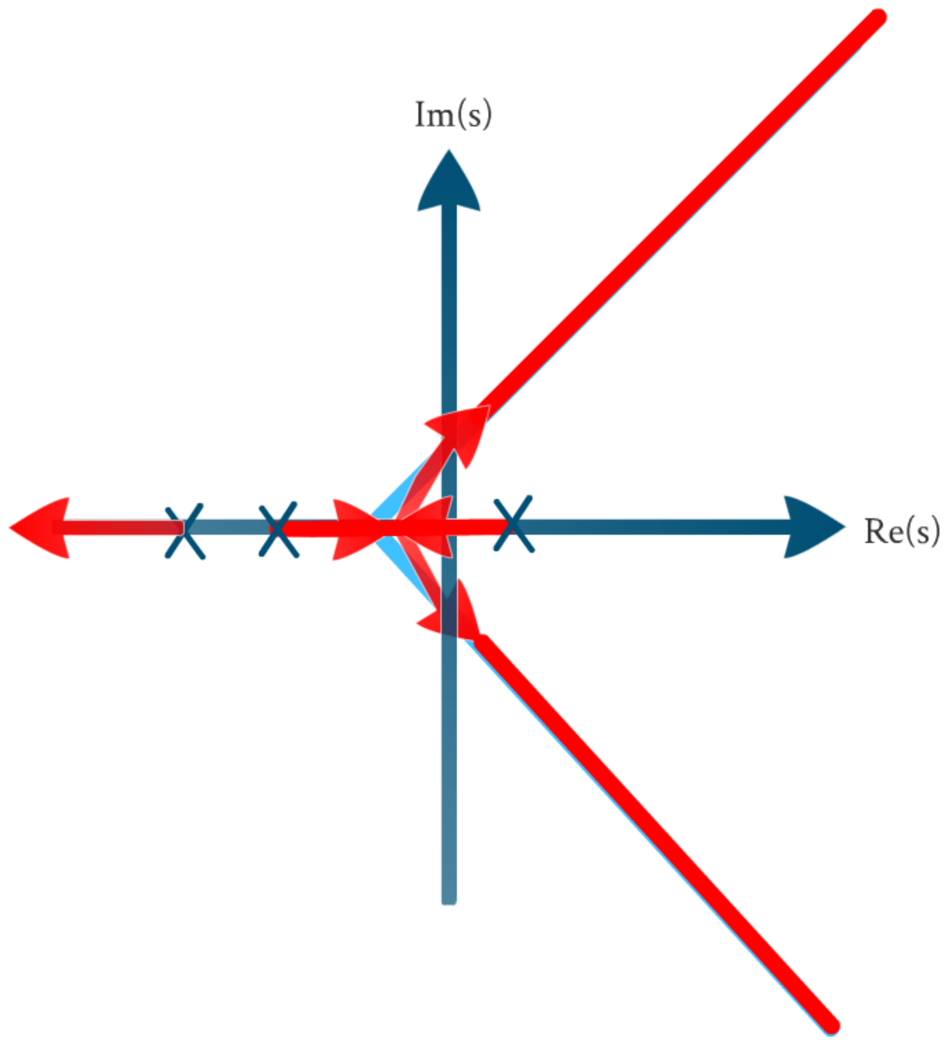

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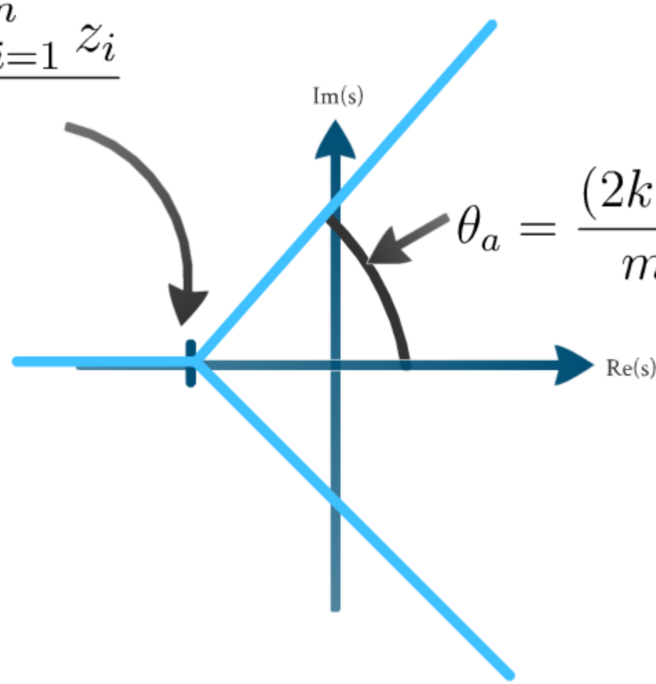

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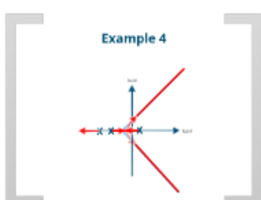

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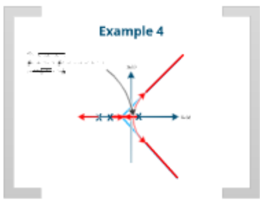

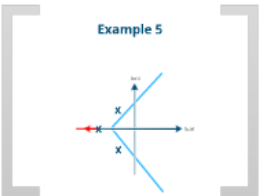

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$$G(s) = \frac{1}{(s+3)(s+1)(s-1)}$$

$$\frac{d}{ds}(s+3)(s+1)(s-1) = \frac{d}{ds}(s^3+3s^2-s-3) = 3s^2+6s-1 = 0$$

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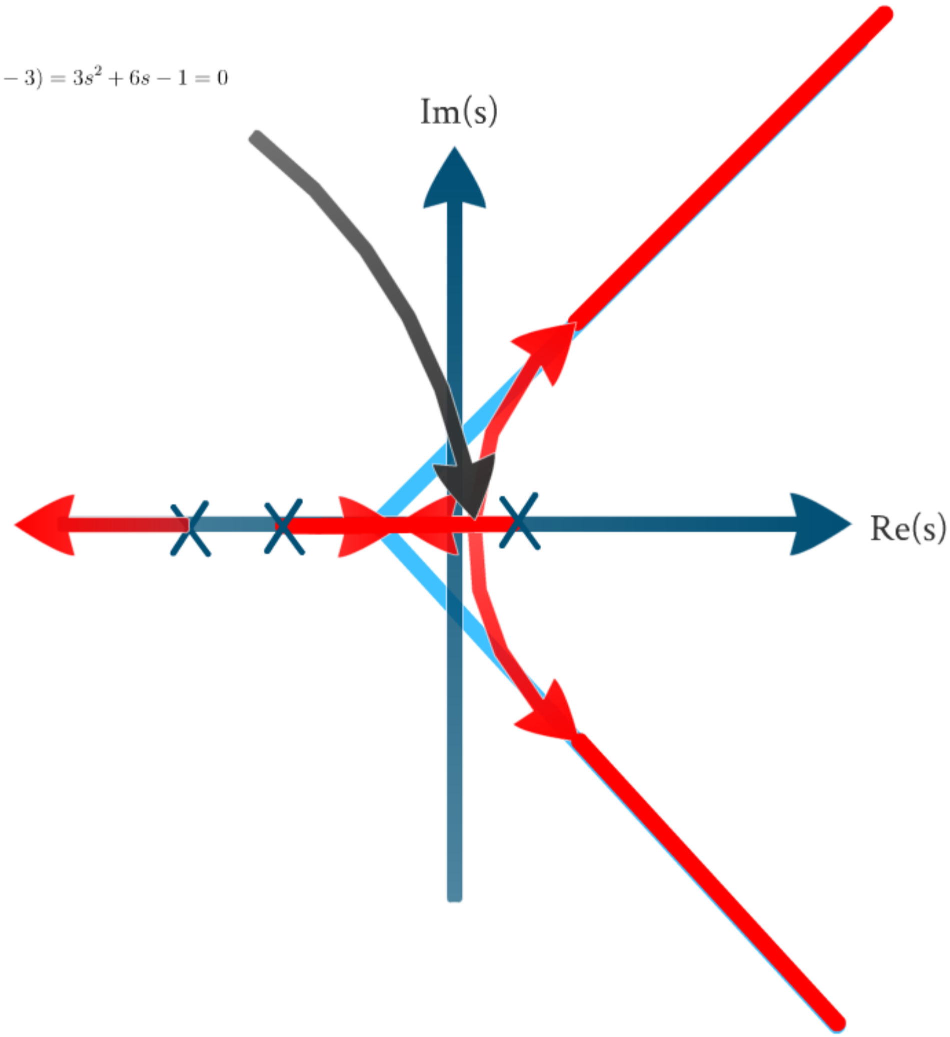

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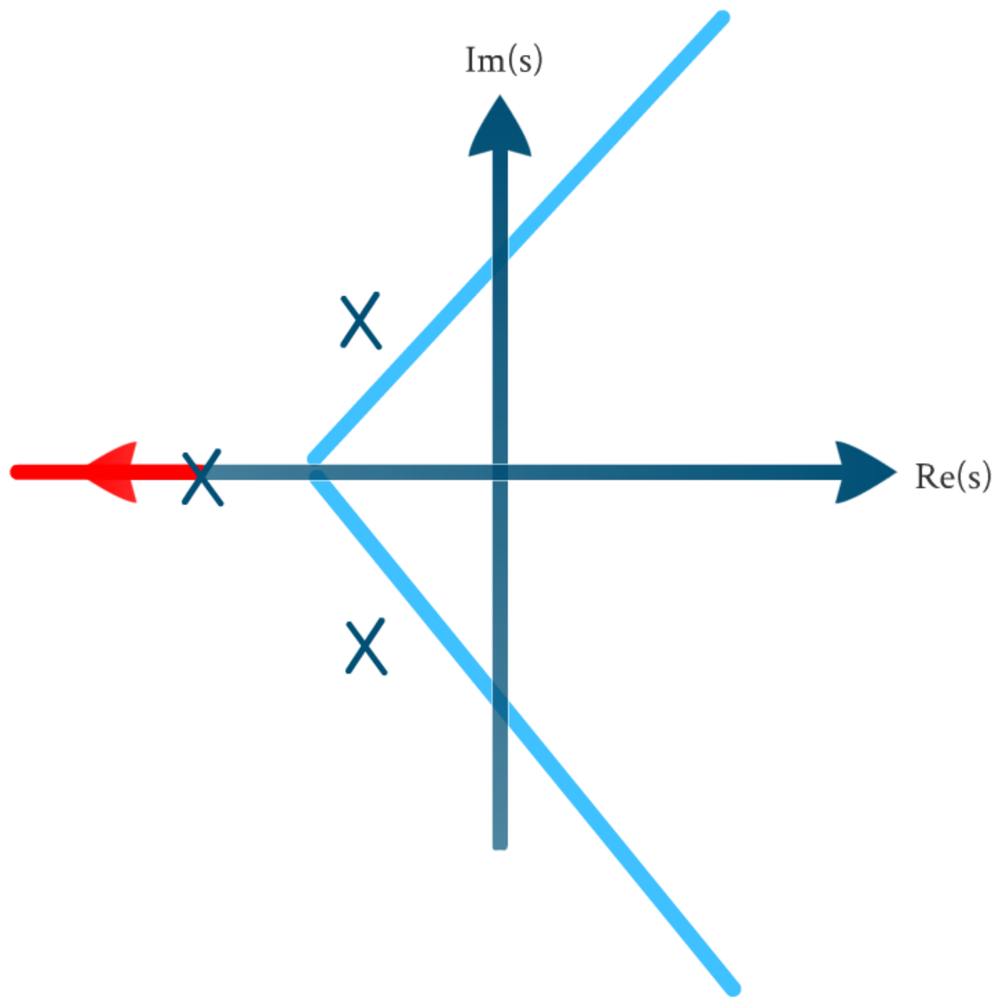

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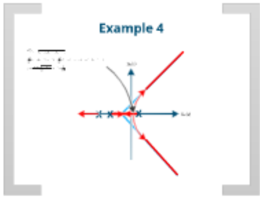

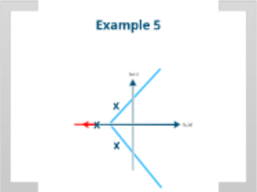

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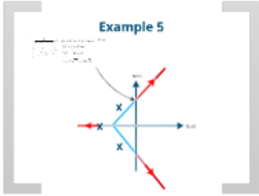


Example 5

$$1 + \frac{K}{s^3 + 5s^2 + 8s + 6} = 0$$

s^3	1	8
s^2	5	$6+K$
s^1	$-\frac{1}{5}\begin{vmatrix} 1 & 8 \\ 5 & 6+K \end{vmatrix}$	0
s^0		

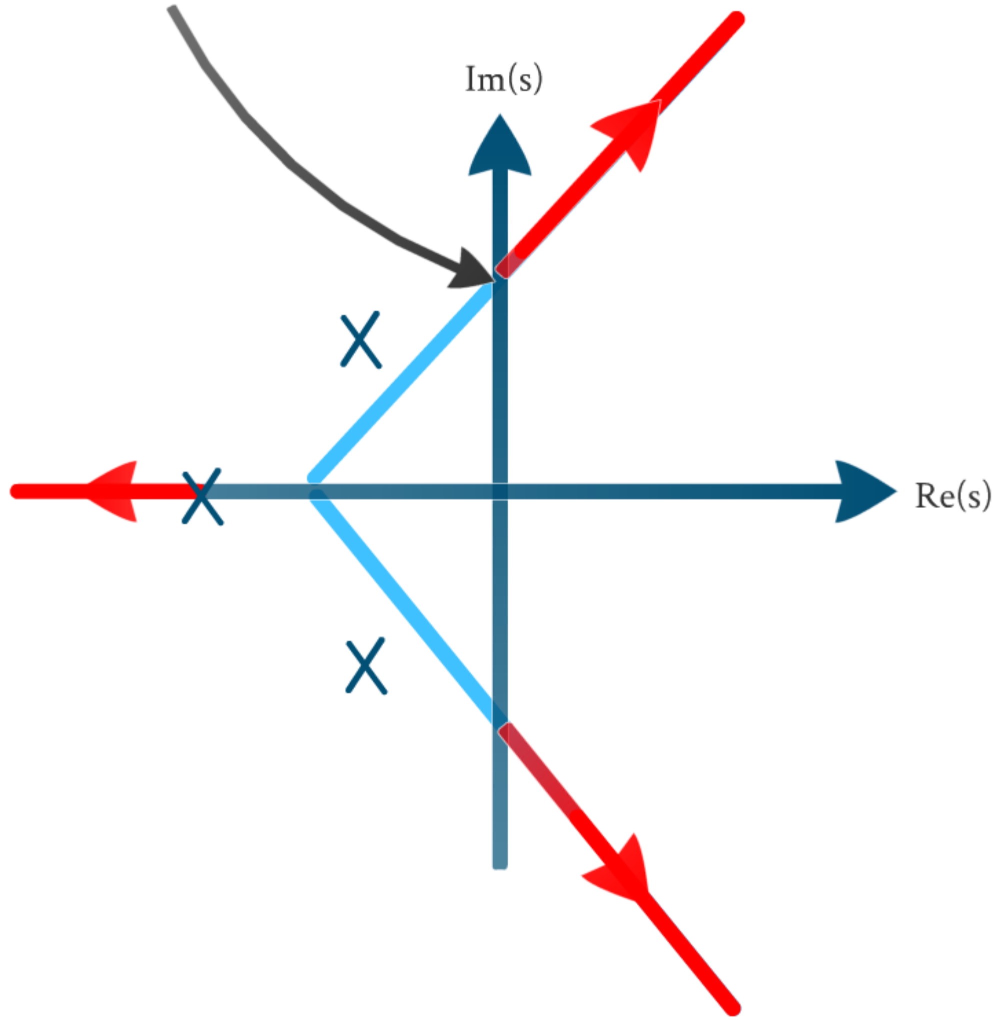
$$(6 + K) - 8 \times 5 = 0$$

K=34

$$5s^2 + 6 + K = 0$$

$$5s^2 + 40 = 0$$

$$s = \pm\sqrt{-8} = \pm 2\sqrt{2}j$$



Exa

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K=34

$$5s^2+6+K=0$$

$$5s^2+40=0$$

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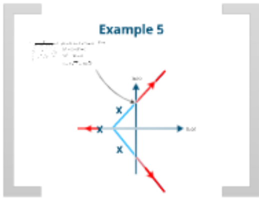

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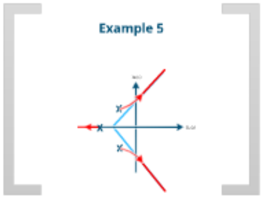
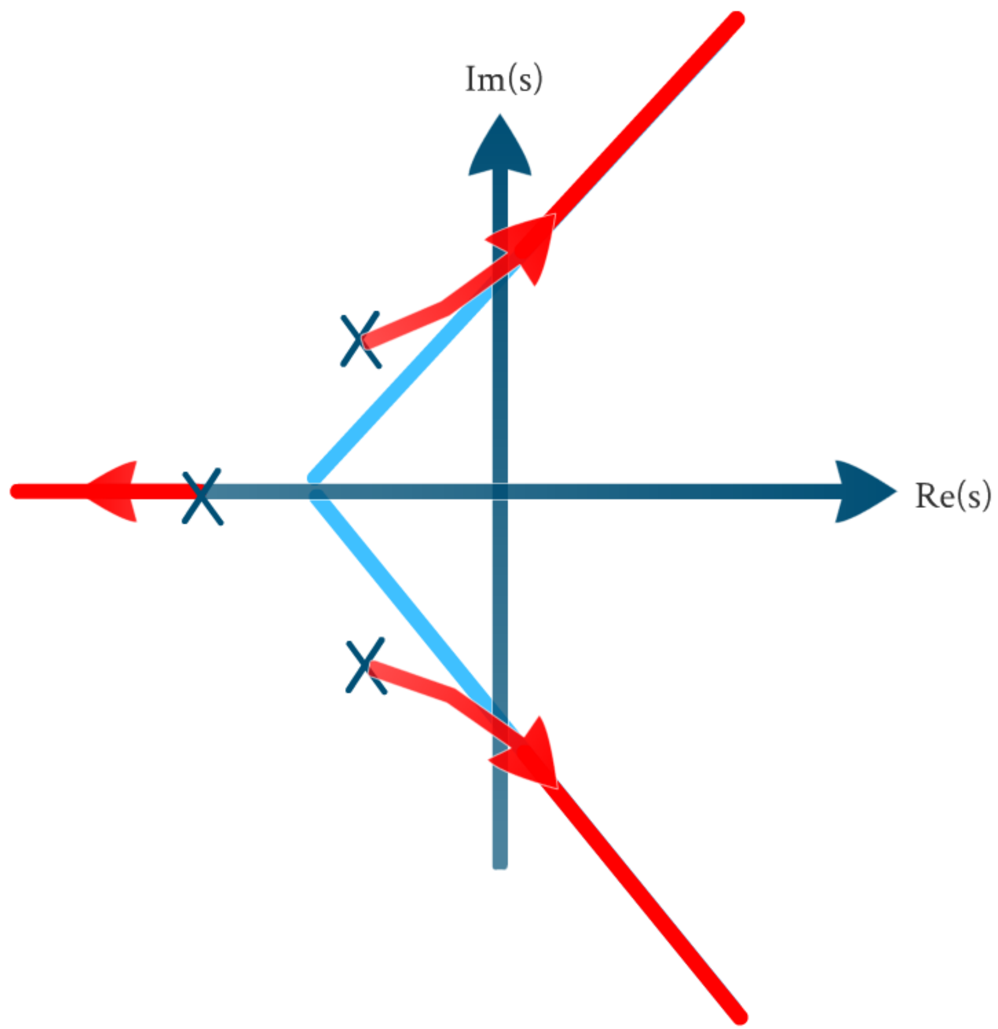


Example 5



Angles of Departure and Arrival

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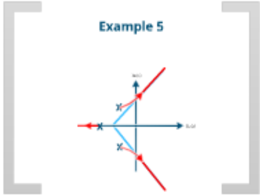


Root Locus

Number of Branches

- A branch is the path that one pole traverses.
- There are as many branches as there are closed-loop poles.

Start and End Points

$$1 + KG(s) = 0 \qquad G(s) = \frac{Z(s)}{P(s)}$$

$$P(s) + KZ(s) = 0$$

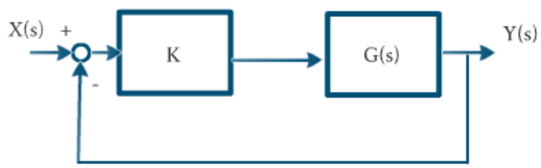
- So, if we vary K, the closed-loop poles will move from the open-loop poles (at K=0)to the open-loop zeros (at K=∞).

Angles of Departure and Arrival

- Imagine a point on the root locus that is a tiny distance ε away from an open-loop pole. We know that the angles must add up to an odd multiple of 180°.
- We can solve for the angle between the point that is ε away and the open-loop pole. This enables us to find the angle of departure and arrival of complex poles and zeros.

Symmetry

- If a pole has an imaginary part, there will be a complex conjugate pole. So, we know that the root locus is symmetric about the real axis.



$$\frac{Y(s)}{X(s)} = \frac{KG(s)}{1 + KG(s))}$$

Imaginary Axis Crossings

- We know that poles in the RHP will cause instability. We can use the Routh-Hurwitz criteria to find K:
- We find a K that forces a row in the Routh-Hurwitz table to be zero.
- The associated auxiliary equation can then be solved for s.

Real-axis Segments

$$KG(s) = -1 \qquad \angle G(s) = -180 \pm k360$$

$$G(s) = K_G \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)} \qquad \angle(a \times b) = \angle a + \angle b$$

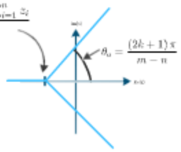
$$\angle G(s) = \angle Z(s) - \angle P(s)$$
$$= \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = -180 \pm k360$$

- For any point on the real axis, the contribution from any complex conjugate poles (or zeros) will cancel each other out.
- For two values of s on the real axis and either side of a zero or a pole, then they will have values of ∠G(s) that differ by 180°.
- More precisely, the root locus exists at s on the real-axis to the left of an odd total number of real-axis open-loop poles and open-loop zeros.

Infinite Zeros

- If n<m then there are more poles than there are (finite) zeros. The poles move from the open-loop poles to the open-loop zeros, so we have to think what happens when |s|=∞.
- As |s|→∞, the locus will asymptote to straight lines

$$\sigma_a = \frac{\sum_{i=1}^{m} p_i - \sum_{i=1}^{n} z_i}{m - n}$$



$$\theta_a = \frac{(2k+1)\pi}{m - n}$$

Real-Axis Breakaway Points

$$K = \frac{1}{|G(s)|}$$

- K is a local maximum or minimum when the root locus breaks away from the real axis. So, we can differentiate K wrt s, set to zero and solve for σ.

This lecture covers:

- Root locus diagram representation for a closed loop system with variable gain.