ELEC 207 Part B

Control Theory Lecture 11: Frequency Response (1)

Prof Simon Maskell
CHAD-G68
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UNIVERSITY OF LIVERPOOL

Determination of Frequency Response

For a linear system, the output will be at the same frequency as the input.

Experimental

Use a "wave analyser":
- Frequency of a sinusoidal input is varied
- Amplitude and relative phase of the output measured

Theoretical

Laplace transform of a cos(wt) input

$$\frac{s}{s^2+\omega^2} = \frac{s}{(s+j\omega)(s-j\omega)}$$

Output from a system with transfer function G(s)

Taking inverse Laplace Transforms

So (relative to the input) the output has:
- Amplitude of $|G(j\omega)|$
- Phase of $\angle G(j\omega)$

This lecture covers:

- Experimental and theoretical determination of frequency response
- Diagrammatic representations using Bode plot

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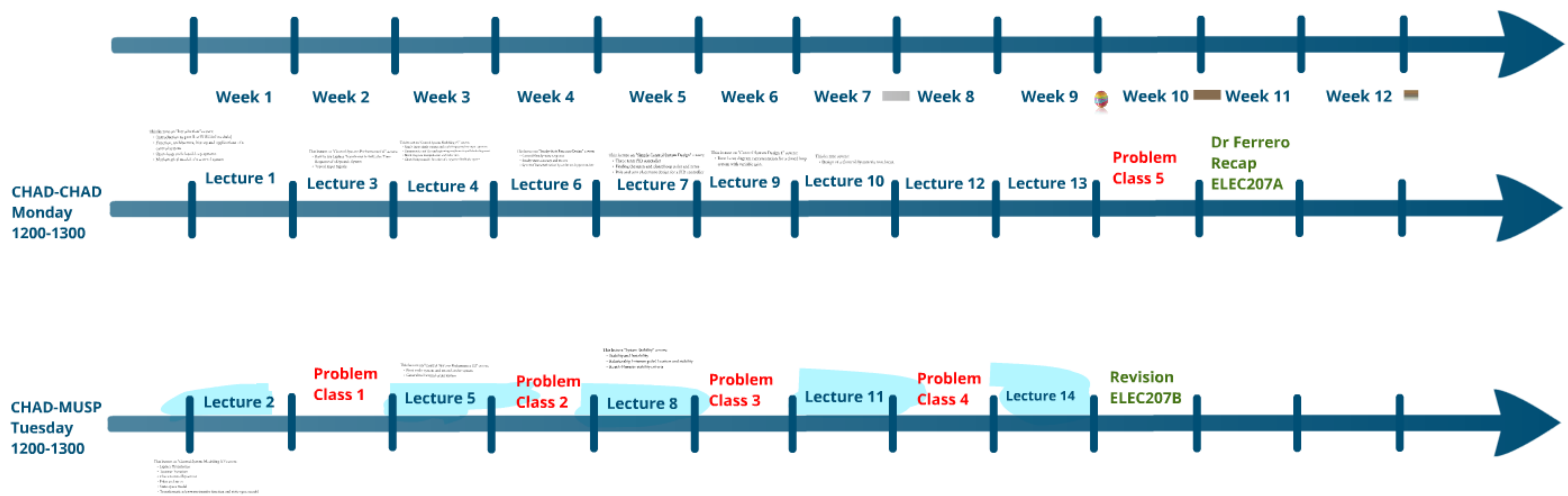
ELEC 207B: Timeline



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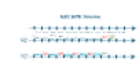

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Taking inverse Laplace Transforms

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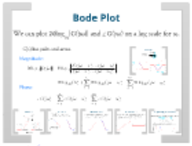

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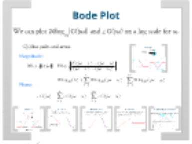

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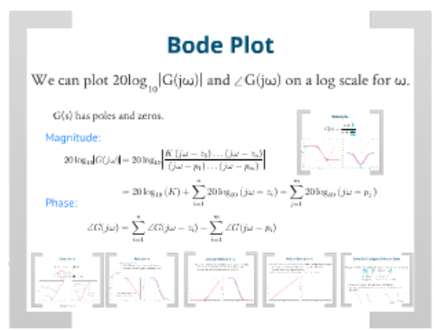

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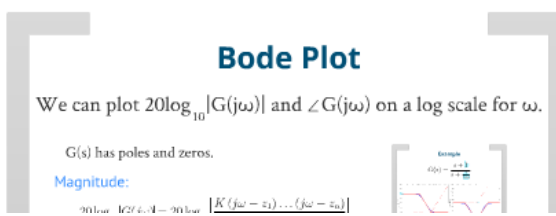

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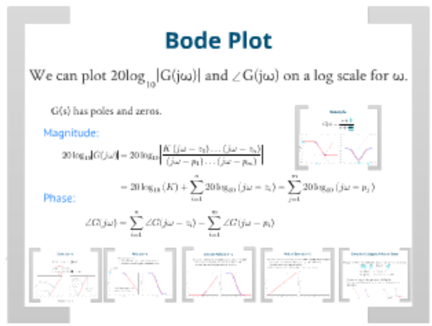

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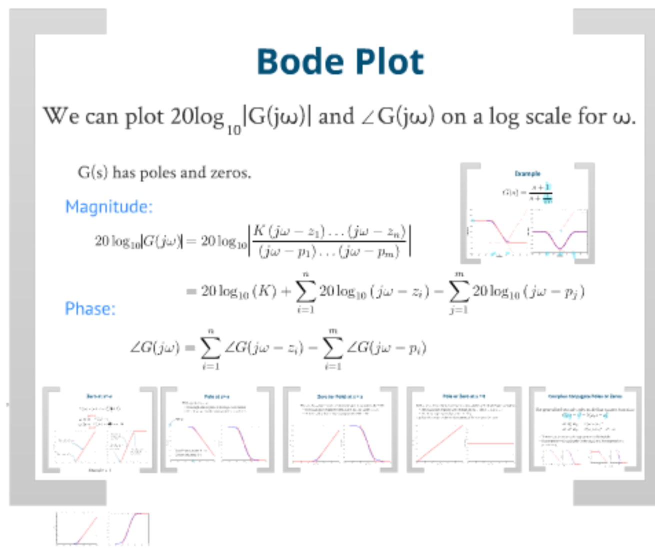

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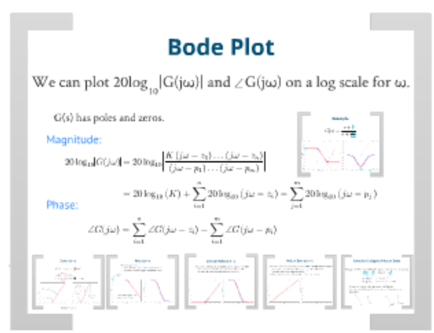
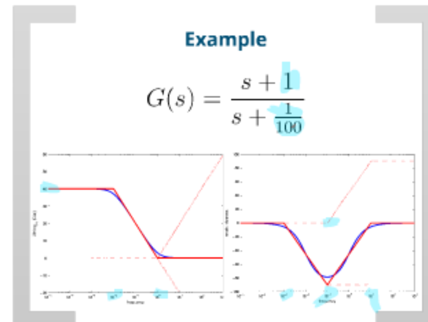


Bode Plot

We can plot $20\log_{10}|G(j\omega)|$ and $\angle G(j\omega)$ on a log scale for ω.

G(s) has poles and zeros.



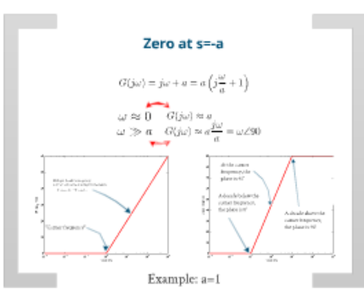
Magnitude:

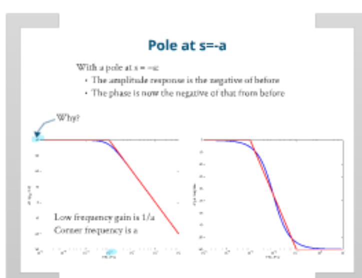
$$20\log_{10}|G(j\omega)| = 20\log_{10}\left|\frac{K(j\omega - z_1)\ldots(j\omega - z_n)}{(j\omega - p_1)\ldots(j\omega - p_m)}\right|$$

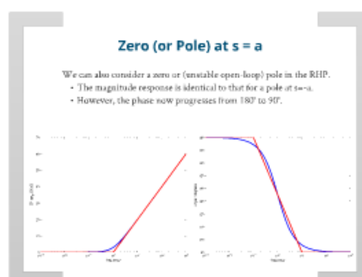
$$= 20\log_{10}(K) + \sum_{i=1}^{n} 20\log_{10}(j\omega - z_i) - \sum_{j=1}^{m} 20\log_{10}(j\omega - p_j)$$

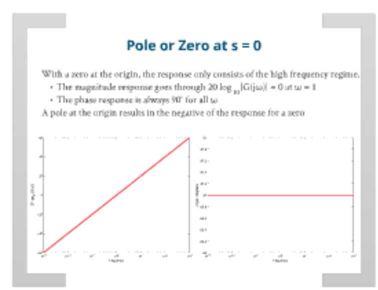
Phase:

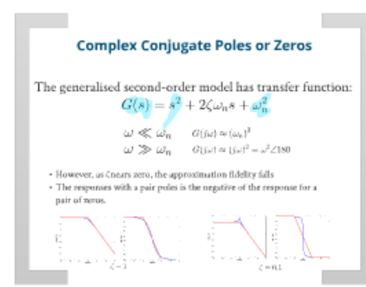
$$\angle G(j\omega) = \sum_{i=1}^{n} \angle G(j\omega - z_i) - \sum_{i=1}^{m} \angle G(j\omega - p_i)$$









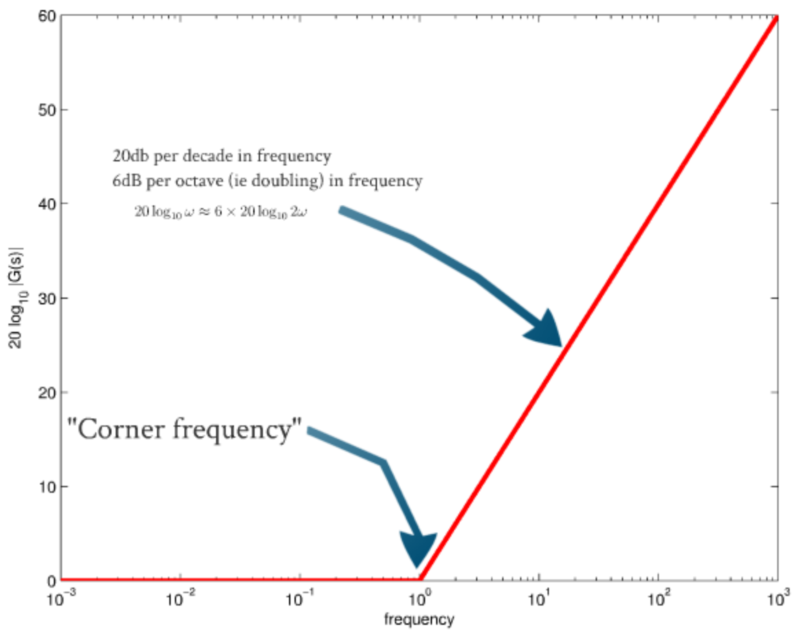


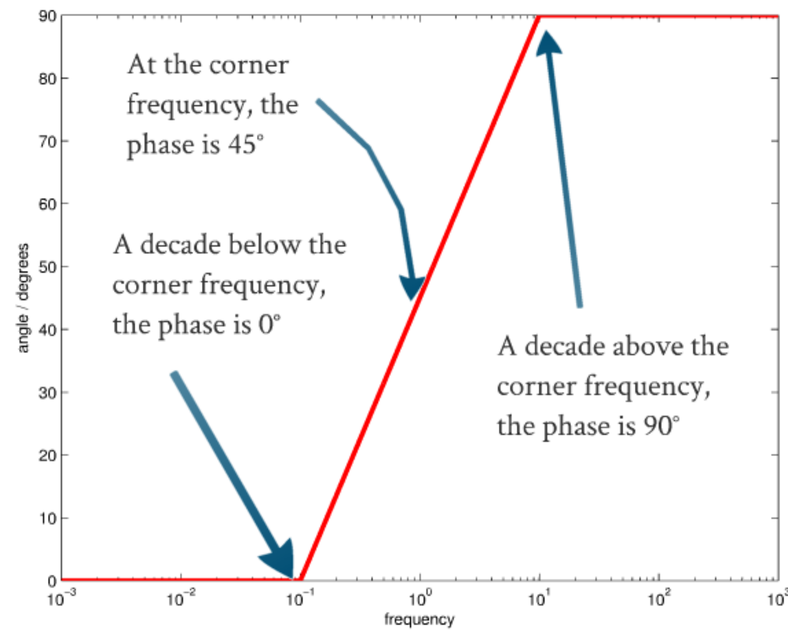
Zero at s=-a

$$G(j\omega) = j\omega + a = a\left(j\frac{\omega}{a} + 1\right)$$

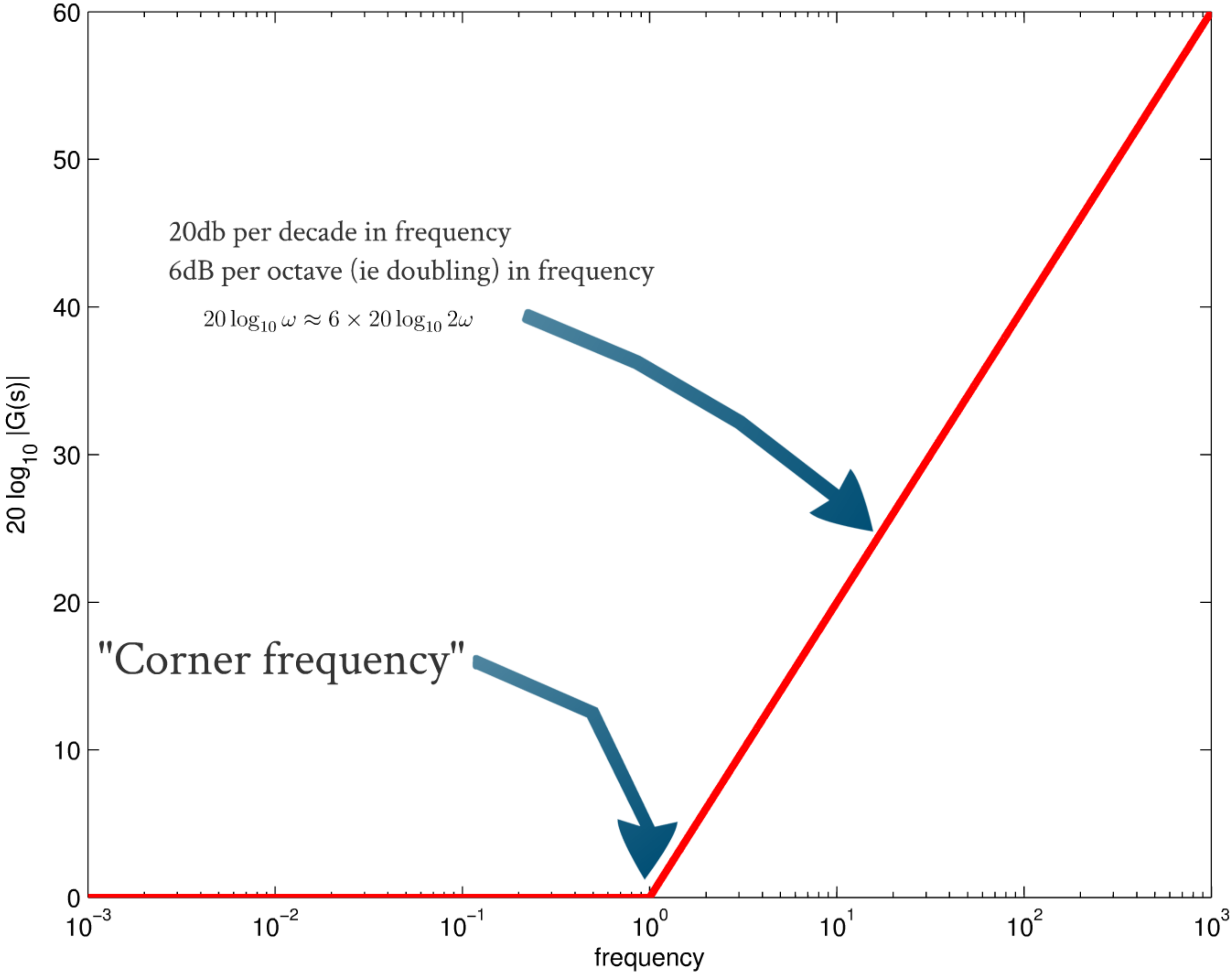
$$\omega \approx 0 \qquad G(j\omega) \approx a$$

$$\omega \gg a \qquad G(j\omega) \approx a\frac{j\omega}{a} = \omega\angle 90$$

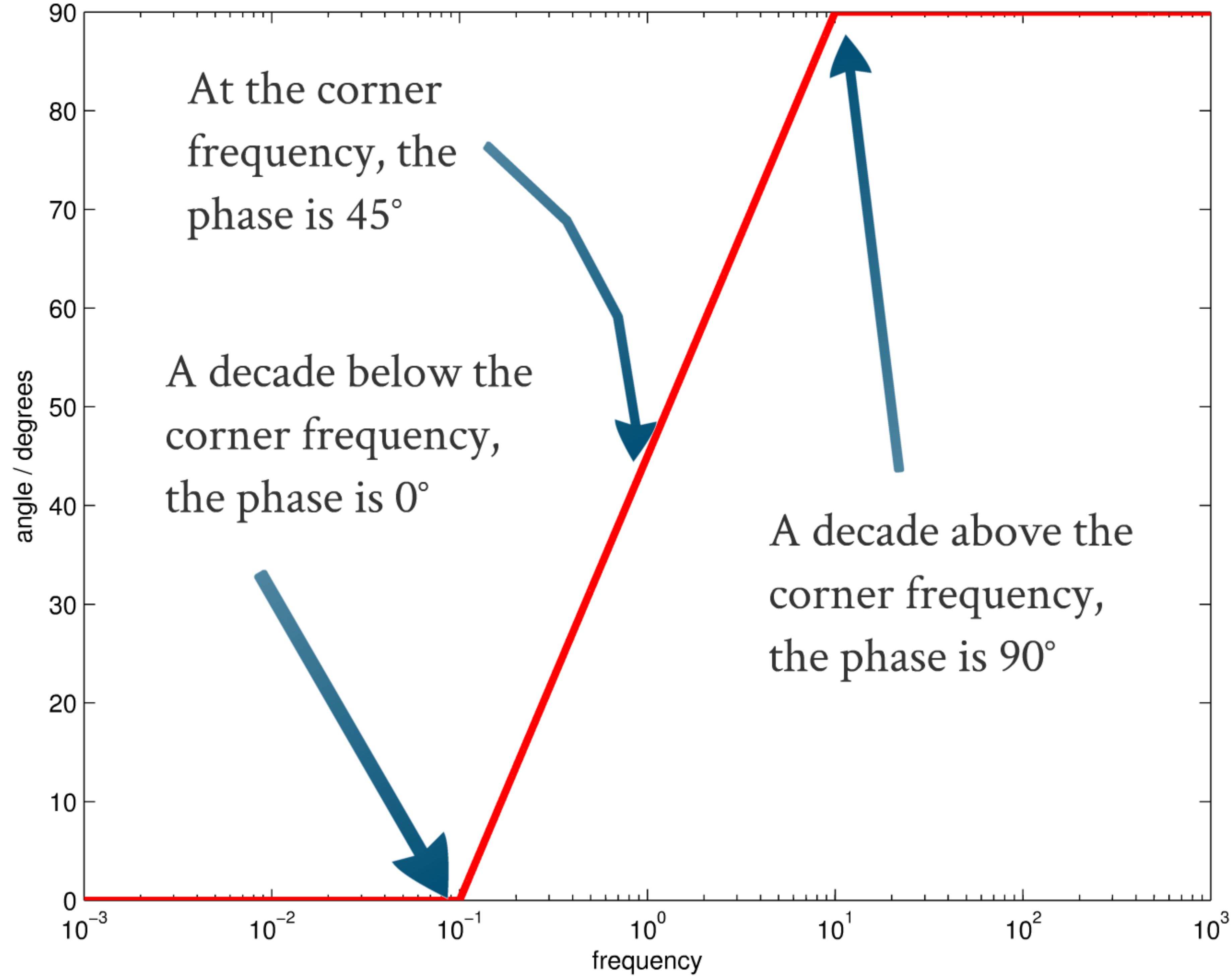




Example: a=1



60
50
40
30
20
10
0
20 log10 |G(s)|
20db per decade in frequency
6dB per octave (ie doubling) in frequency
$20 \log_{10} \omega \approx 6 \times 20 \log_{10} 2\omega$
"Corner frequency"
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
frequency


At the corner frequency, the phase is 45°
A decade below the corner frequency, the phase is 0°
A decade above the corner frequency, the phase is 90°
angle / degrees
0
10
20
30
40
50
60
70
80
90
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
frequency

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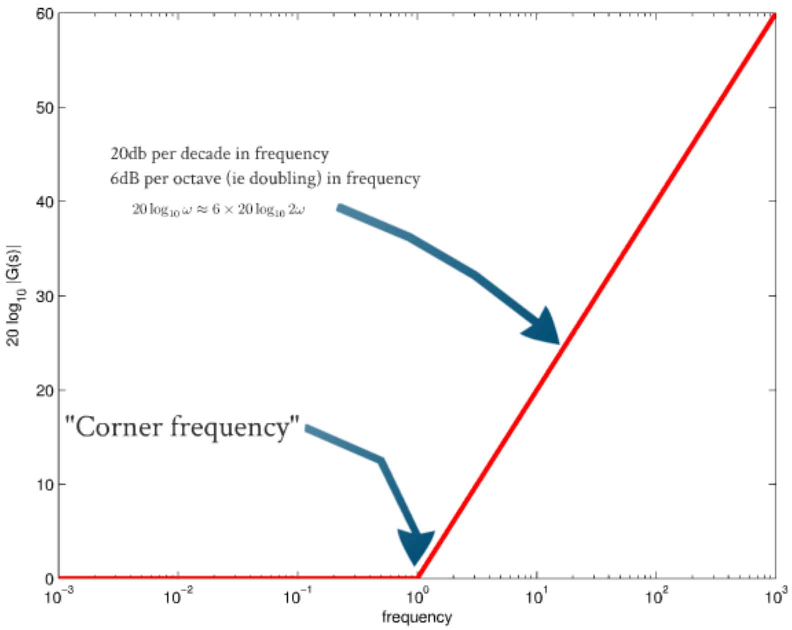

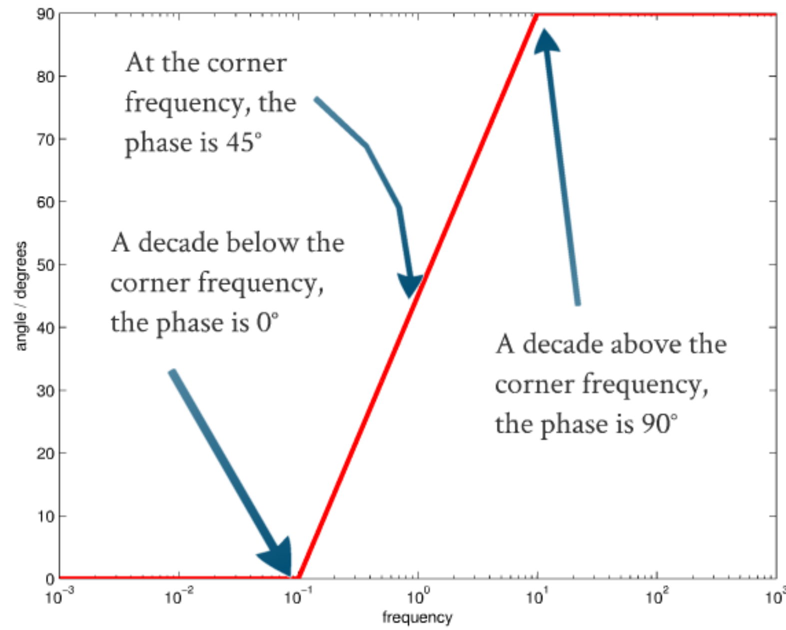

Example: a=1

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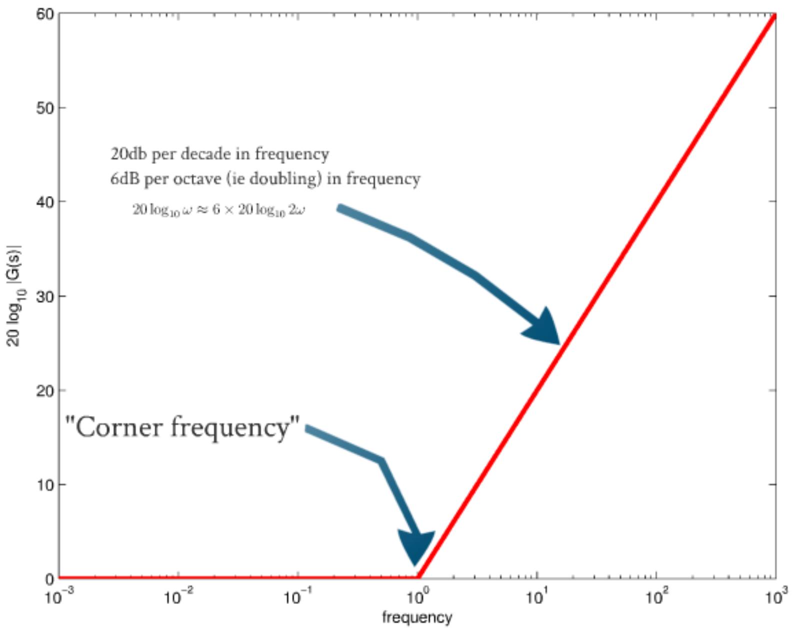

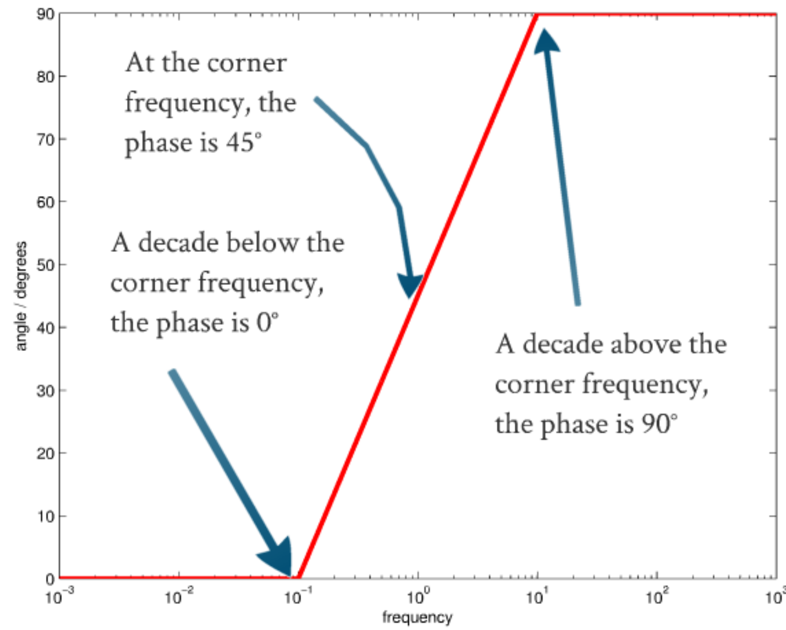

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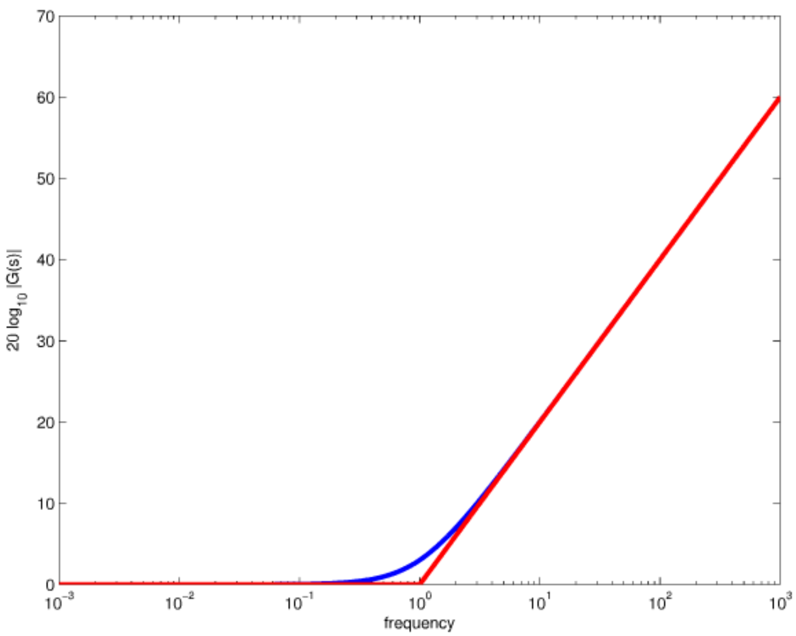
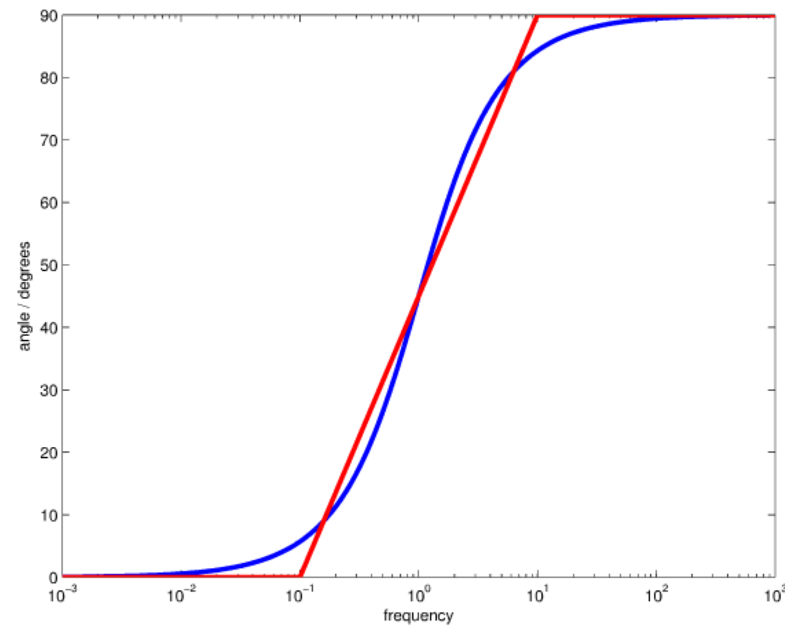


70
60
50
40
30
20
10
0
20 log10 |G(s)|
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3
frequency



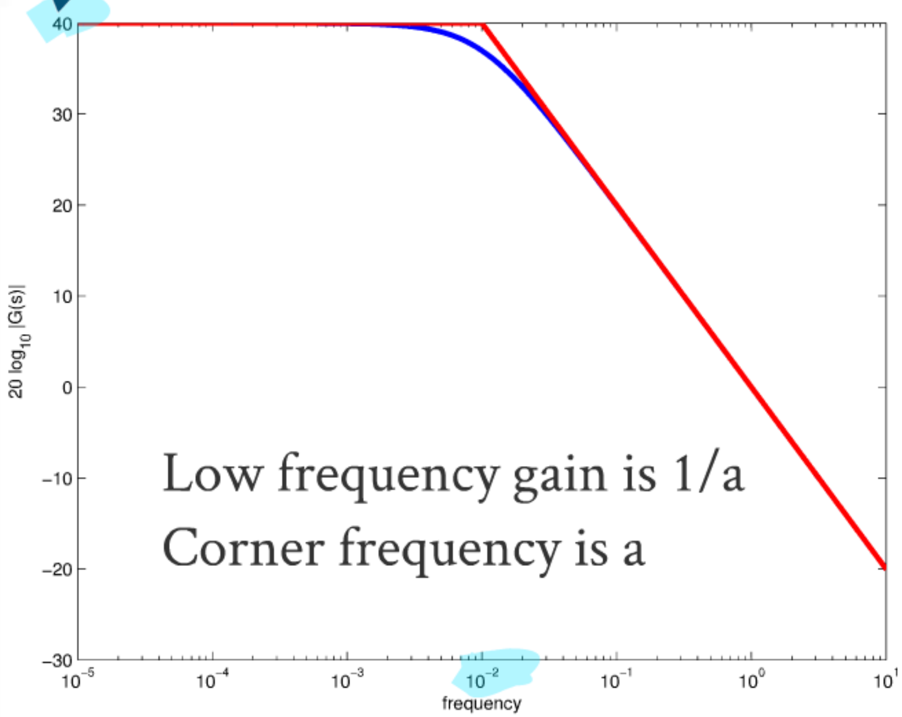
90
80
70
60
50
40
30
20
10
0
angle / degrees
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3
frequency

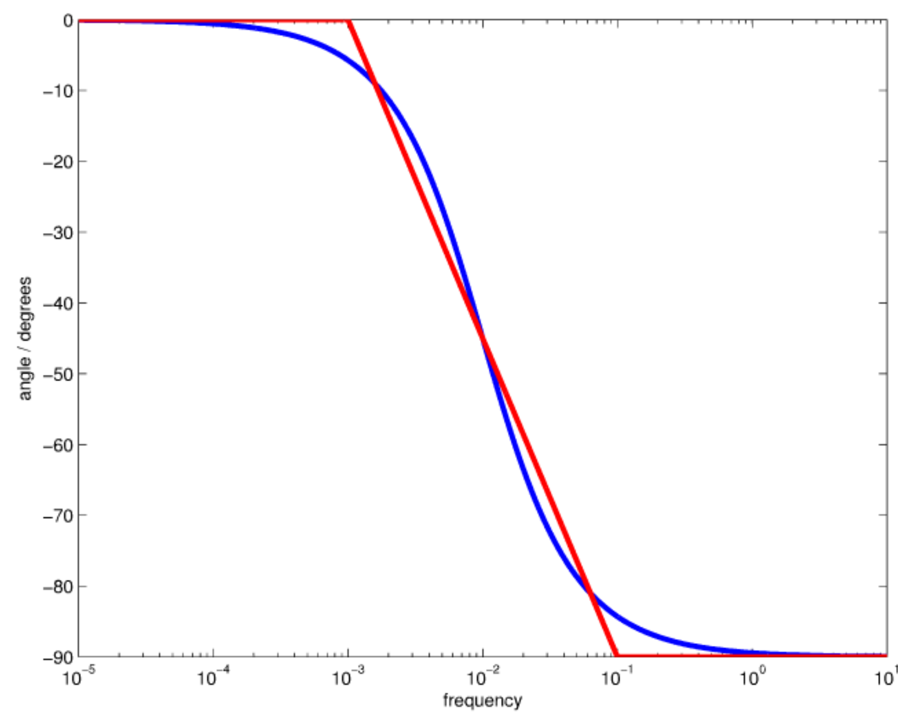
Pole at s=-a

With a pole at s = −a:

- The amplitude response is the negative of before
- The phase is now the negative of that from before

Why?

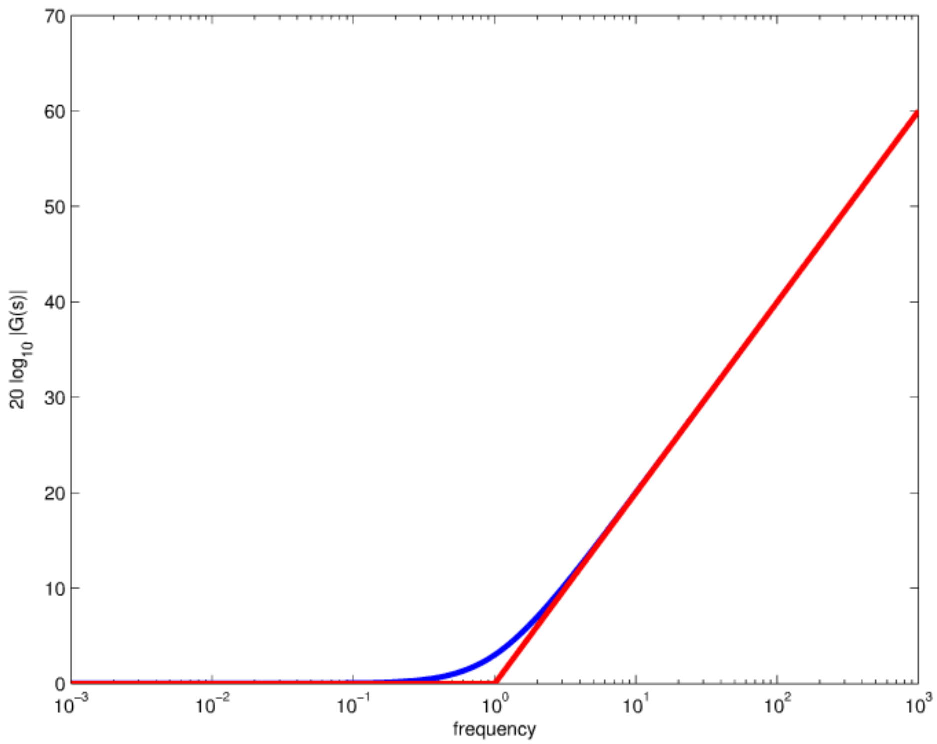


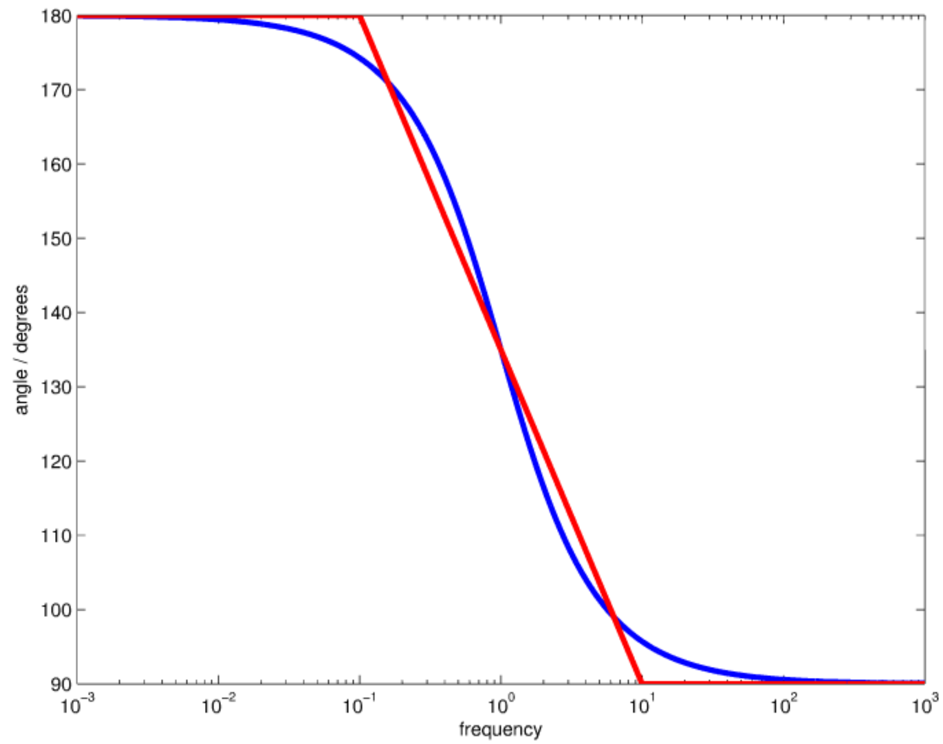


Zero (or Pole) at s = a

We can also consider a zero or (unstable open-loop) pole in the RHP.

- The magnitude response is identical to that for a pole at s=-a.
- However, the phase now progresses from 180° to 90°.



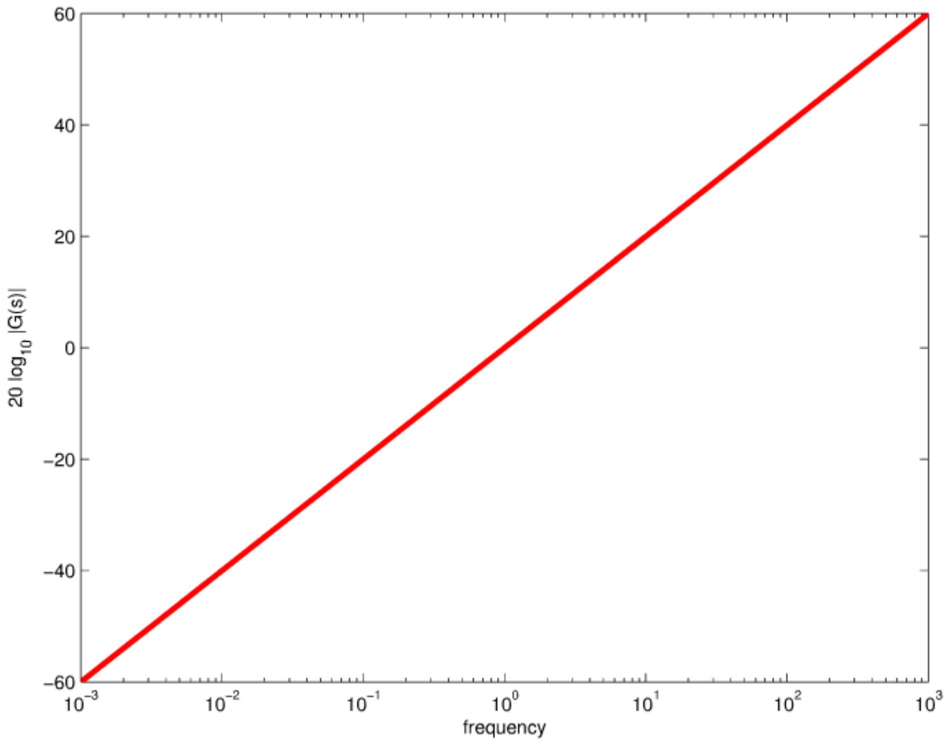


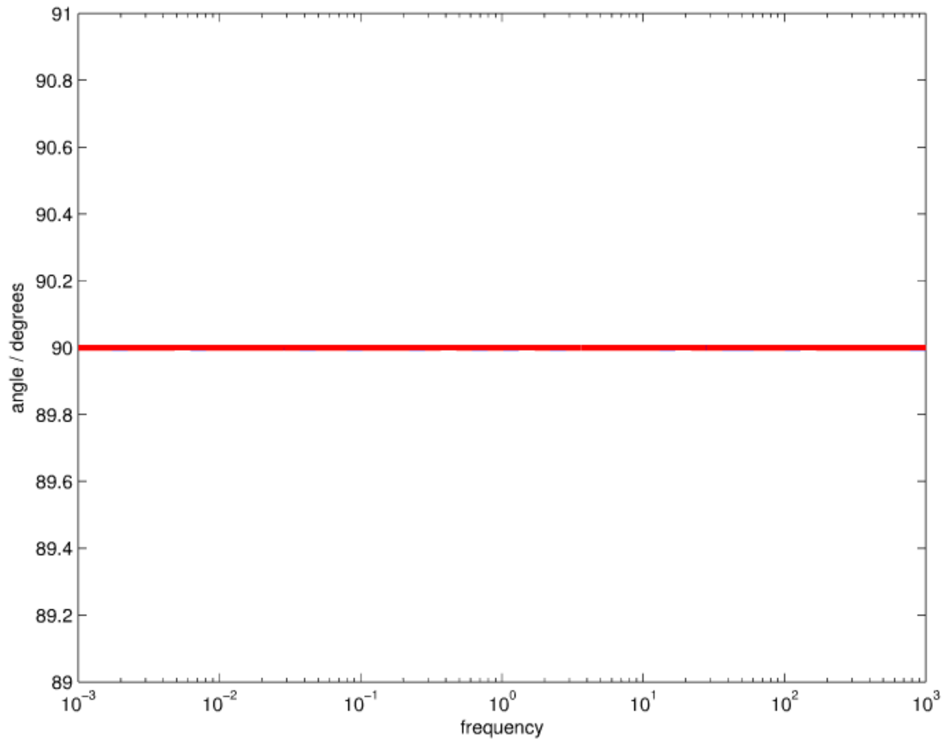
Pole or Zero at s = 0

With a zero at the origin, the response only consists of the high frequency regime.

- The magnitude response goes through $20 \log_{10} |G(j\omega)| = 0$ at $\omega = 1$
- The phase response is always 90° for all ω

A pole at the origin results in the negative of the response for a zero





Complex Conjugate Poles or Zeros

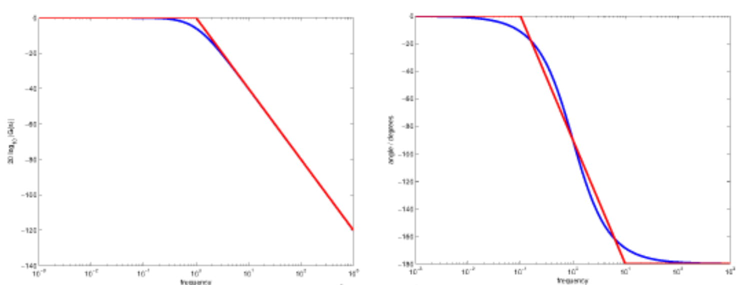
The generalised second-order model has transfer function:

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

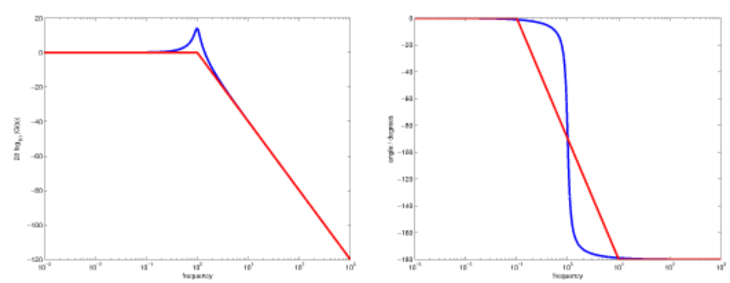
$$\omega \ll \omega_n \qquad G(j\omega) \approx (\omega_n)^2$$

$$\omega \gg \omega_n \qquad G(j\omega) \approx (j\omega)^2 = \omega^2 \angle 180$$

- However, as ζ nears zero, the approximation fidelity falls
- The responses with a pair poles is the negative of the response for a pair of zeros.

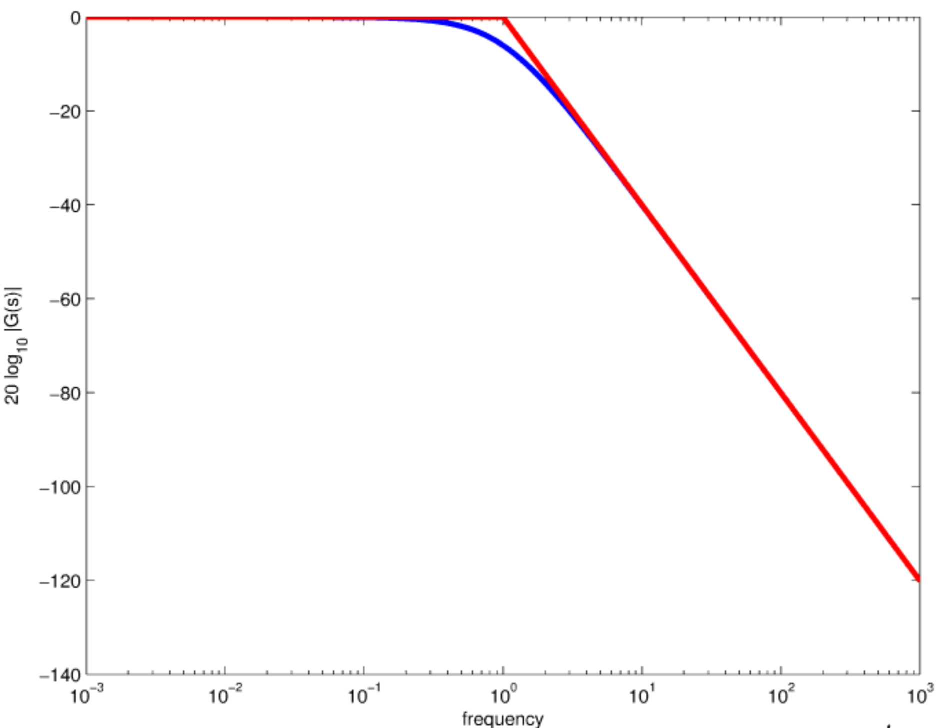


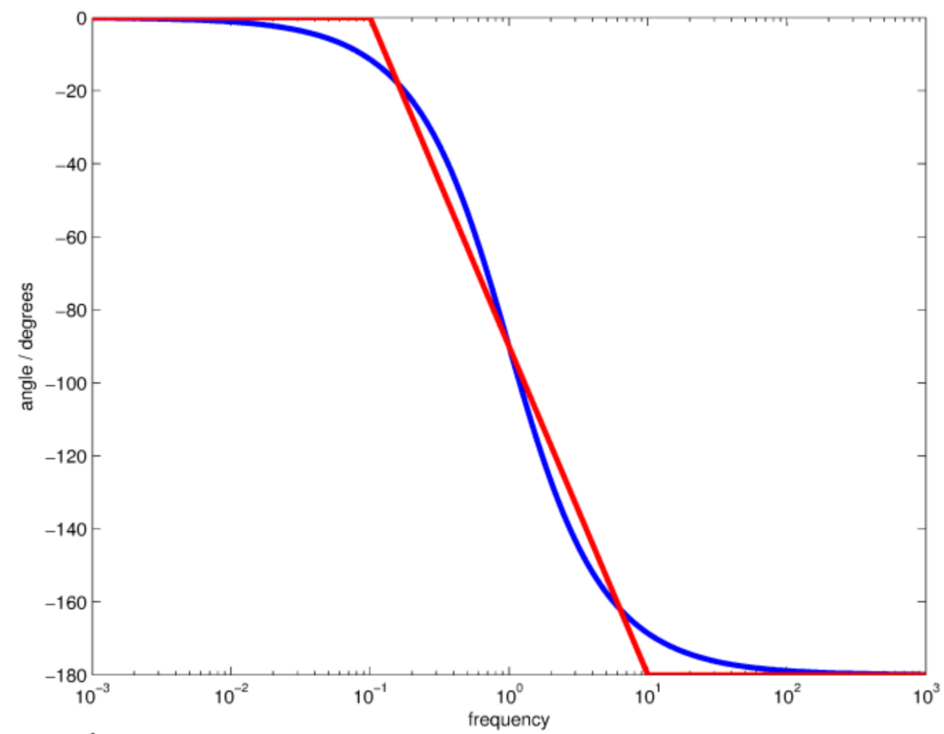
$\zeta = 1$



$\zeta = 0.1$

pair of zeros.





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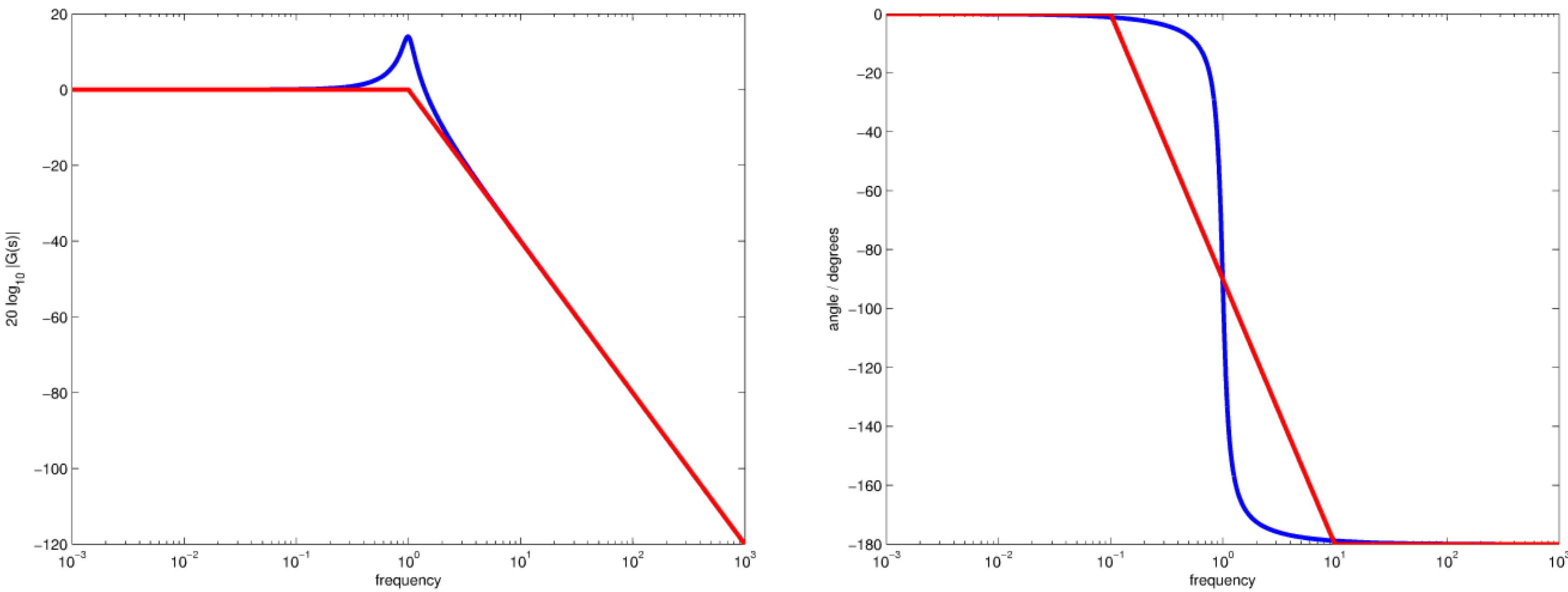

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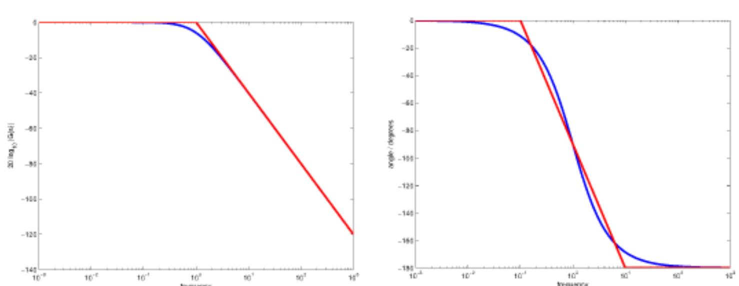
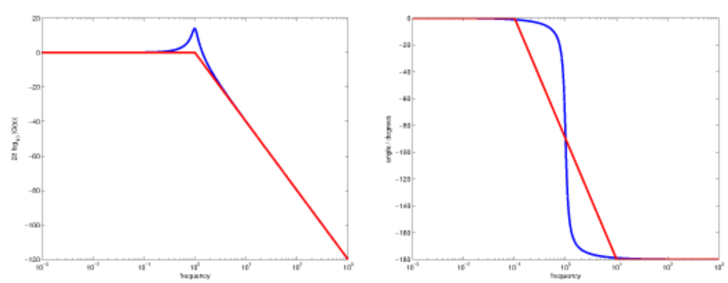


$\zeta = 1$

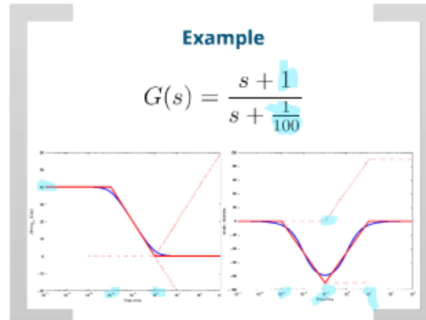


$\zeta = 0.1$

Bode Plot

We can plot $20\log_{10}|G(j\omega)|$ and $\angle G(j\omega)$ on a log scale for ω.

G(s) has poles and zeros.



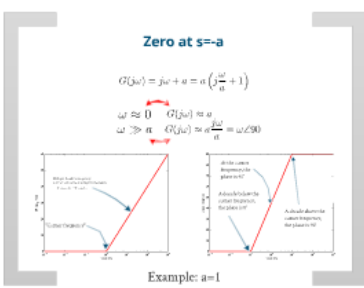
Magnitude:

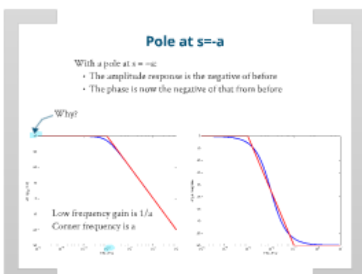
$$20\log_{10}|G(j\omega)| = 20\log_{10}\left|\frac{K\,(j\omega - z_1)\ldots(j\omega - z_n)}{(j\omega - p_1)\ldots(j\omega - p_m)}\right|$$

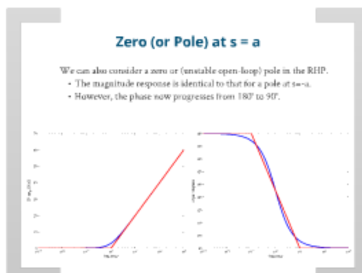
$$= 20\log_{10}(K) + \sum_{i=1}^{n} 20\log_{10}(j\omega - z_i) - \sum_{j=1}^{m} 20\log_{10}(j\omega - p_j)$$

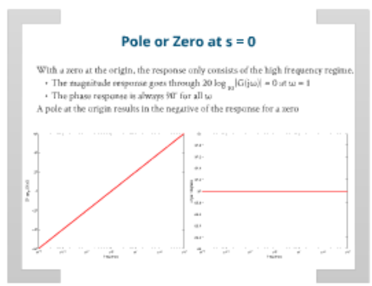
Phase:

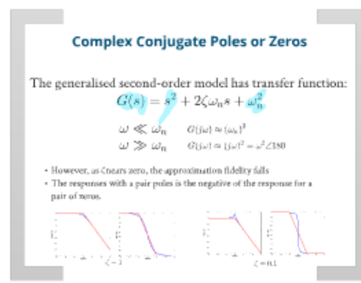
$$\angle G(j\omega) = \sum_{i=1}^{n} \angle G(j\omega - z_i) - \sum_{i=1}^{m} \angle G(j\omega - p_i)$$





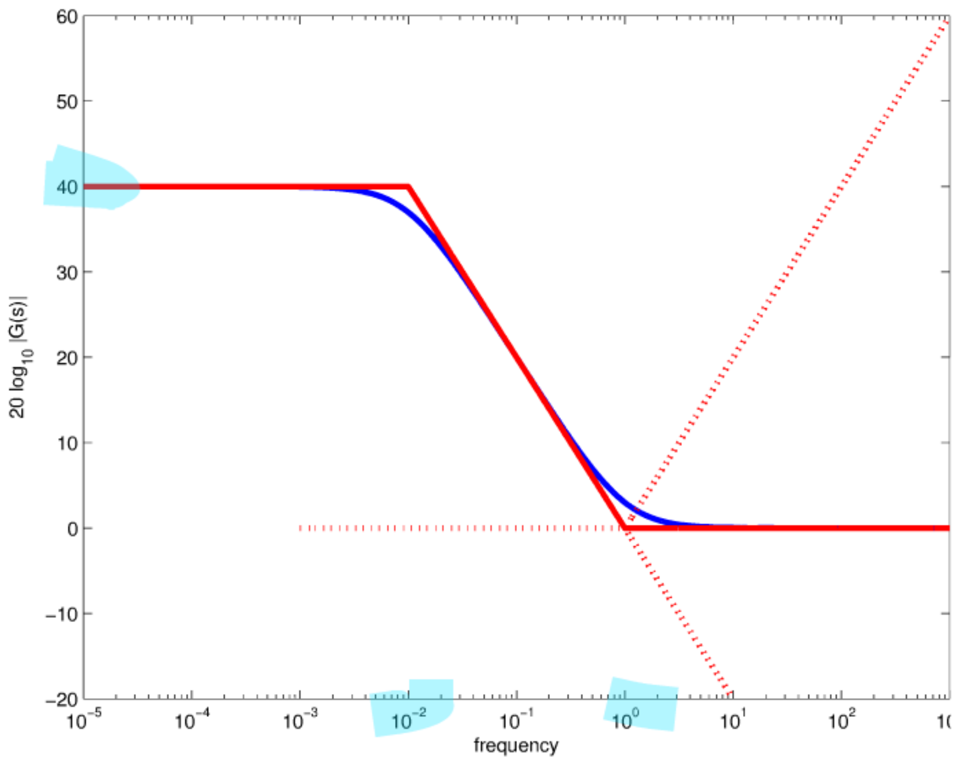


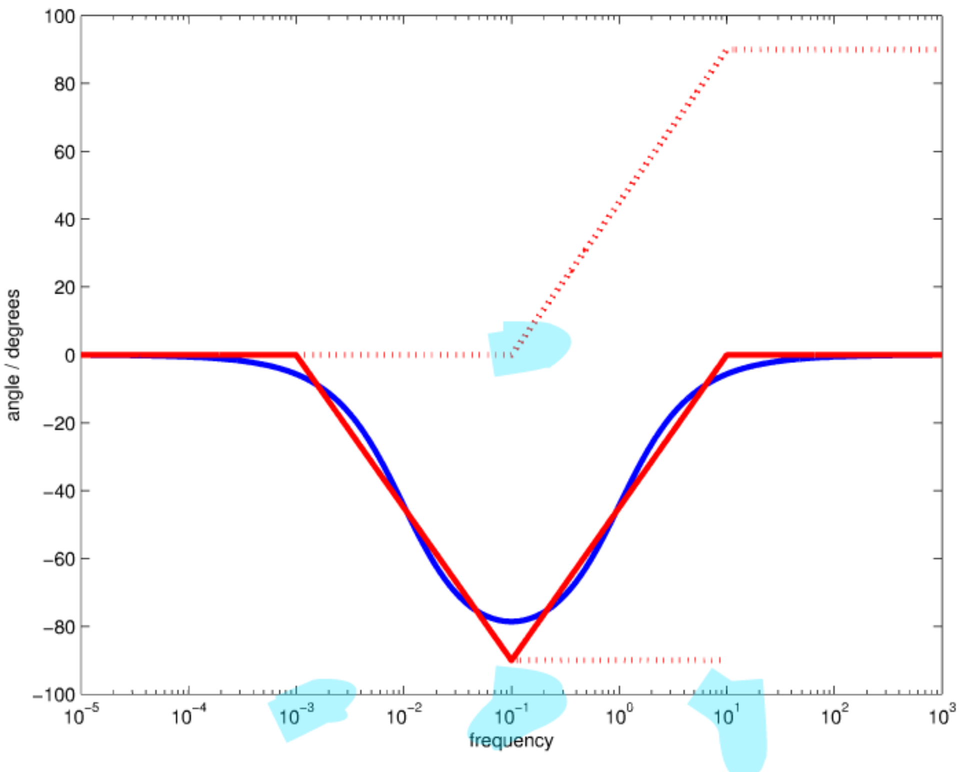




Example

$$G(s) = \frac{s + 1}{s + \frac{1}{100}}$$





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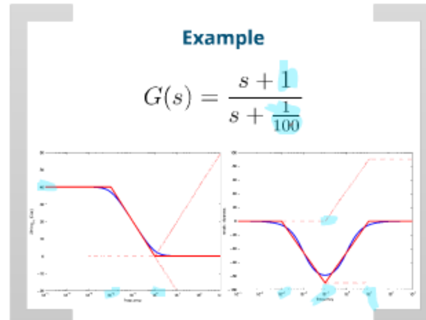

Magnitude:

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Phase:

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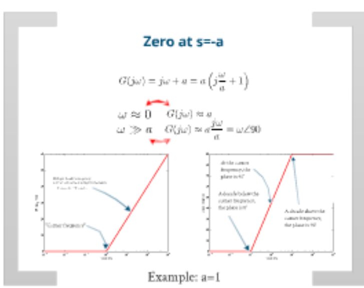

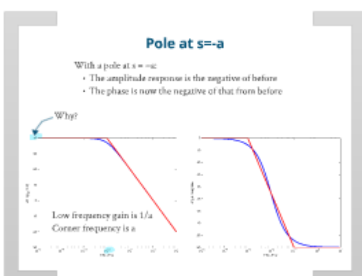

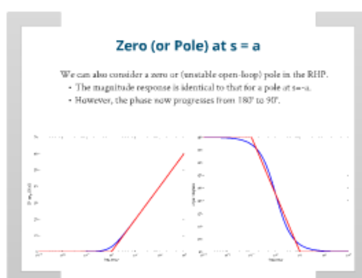

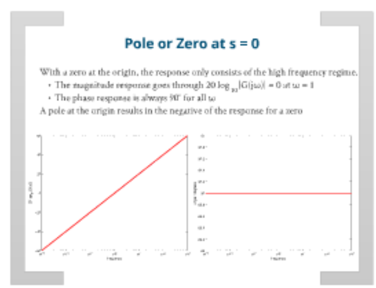

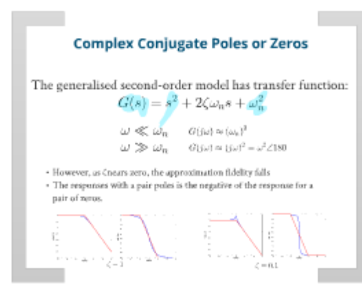

This lecture covers:

- Experimental and theoretical determination of frequency response
- Diagrammatic representations using Bode plot