Project Two Template

MAT-350: Applied Linear Algebra

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Problem 1

Use the svd() function in MATLAB to compute A_1, the **rank-1 approximation of** A**.** Clearly state what A_1 is, rounded to 4 decimal places. Also, **compute** the root-mean square error (RMSE) between A and A_1.

Solution:

```
%We will be using singular value decomposition (SVD) to compress an image to
%smaller size for more efficient storage. In this example we are using a
%smaller matrix to demonstrate the compression.
%SVD is used to generate a low-rank approximation of the original matrix.

A = [1 2 3; 3 3 4; 5 6 7]
```

```
A = 3×3
     1     2     3
     3     3     4
     5     6     7
```

```
%SVD command is used to factor the original matrix into 3 parts.
%The orthogonal matrices U and V along with S, a diagonal matrix.
[U, S, V] =  svd(A)
```

```
U = 3×3
   -0.2904    0.9504   -0.1114
   -0.4644   -0.2418   -0.8520
   -0.8367   -0.1957    0.5115
S = 3×3
   12.5318         0         0
         0    0.9122         0
         0         0    0.3499
V = 3×3
   -0.4682   -0.8261   -0.3136
   -0.5581    0.0012    0.8298
   -0.6851    0.5635   -0.4616
```

```
%Per the decomposition, these can be multiplied together to rebuild the
%initial Matrix (at rank k). k variable designates how many rows from each
%factored U S V matrice are desired. This determines the similarity of our
%product matrix to the origin matrix A.
k = 1
```

```
k = 1
```

```
AK = U(:,1:k)*S(1:k,1:k)*V(:,1:k).'
```

```
AK = 3×3
    1.7039    2.0313    2.4935
    2.7243    3.2477    3.9867
    4.9087    5.8517    7.1832
```

```
[m,n] = size(A)
```

```
m = 3
n = 3
```

```
RMSE_rank_1 = norm(double(A)-double(AK),'fro')/sqrt(m*n)
```

```
RMSE_rank_1 = 0.3257
```

Problem 2

Use the svd() function in MATLAB to compute A_2, the **rank-2 approximation of** A. Clearly state what A_2 is, rounded to 4 decimal places. Also, **compute** the root-mean square error (RMSE) between A and A_2. Which approximation is better, A_1 or A_2? Explain.

Solution:

```
k = 2
```

```
k = 2
```

```
AK = U(:,1:k)*S(1:k,1:k)*V(:,1:k).'
```

```
AK = 3×3
    0.9878    2.0324    2.9820
    2.9065    3.2474    3.8624
    5.0561    5.8515    7.0826
RMSE_rank_2 = 0.1166
```

```
RMSE_rank_2 = norm(double(A)-double(AK),'fro')/sqrt(m*n)
```

```
RMSE_rank_2 = 0.1166
```

Explain: We will be using singular value decomposition (SVD) to compress an image to smaller size for more efficient storage. In this example we are using the matrix A to demonstrate the compression. SVD is used to generate a low-rank approximation of the original matrix.

The rank selected determines what rank of the data is being preserved. It equates to the number of linearly independent columns. The higher the rank, the better the quality at the cost of increased storage requirements.

i.e if I take rank 1 using SVD I'm compressing the matrix to 1 linearly independent column. Rank 2 is two independent columns.

To compare the compressed vs uncompressed matrix we use a Root Mean Squared Error (RMSE). This is the square root of the average squared error, which is a value indicating the average distance between

the compressed values and the actual values of the original matrix.

When comparing a Rank 1 RMSE 0.3257 to the Rank 2 RMSE 0.1166 we can see a marked decrease in the RMSE as we use more columns. This was expected considering the increased detail of the data as we use more columns.

Problem 3

For the 3×3 matrix A, the singular value decomposition is $A = USV'$ where $U = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$. Use MATLAB to **compute** the dot product $d_1 = dot(\mathbf{u}_1, \mathbf{u}_2)$.

Also, use MATLAB to **compute** the cross product $\mathbf{c} = cross(\mathbf{u}_1, \mathbf{u}_2)$ and dot product $d_2 = dot(\mathbf{c}, \mathbf{u}_3)$. Clearly state the values for each of these computations. Do these values make sense? **Explain**.

Solution:

```
%define u1 and u2
u1 = U(:,1)
```

```
u1 = 3×1
   -0.2904
   -0.4644
   -0.8367
```

```
u2 = U(:,2)
```

```
u2 = 3×1
    0.9504
   -0.2418
   -0.1957
```

```
%calculate dot product of u1 & u2
d1 = dot(u1, u2)
```

```
d1 = 5.5511e-17
```

```
%calculate the cross product of u1 and u2
c = cross(u1, u2)
```

```
c = 3×1
   -0.1114
   -0.8520
    0.5115
```

```
%dot product d2 of and u3
d2 = dot(c, U(:,3))
```

```
d2 = 1
```

Explain: The dot product measures how much two vectors orientate towards the same direction, while the cross product measures how much two vectors orient in different directions. Using these two measures can help to determine the orientation of u1 and u2. A dot product of 1 indicates they are orthonormal, pointing in the same direction and their length are reciprocals. This is verified by the cross product equating to value u3 from the rank 1 matrix.

Problem 4

Using the matrix $U = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$, **determine whether or not the columns of U span $\mathbb{R}^3$. Explain your approach.**

Solution:

```
%Use the reduced row echelon form
span_r3 = rref(U)
```

```
span_r3 = 3×3
     1     0     0
     0     1     0
     0     0     1
```

Explain: The reduced row echelon form of a matrix helps to determine the span of a matrix. Every linearly independent vector is a leading value in the identity matrix. Since there are 3 leading values here, it is a rank 3 matrix with 3 linearly independent columns.

Problem 5

Use the MATLAB imshow() function to load and display the image A stored in the image.mat file, available in the Project Two Supported Materials area in Brightspace. For the loaded image, **derive the value of k** that will result in a compression ratio of $CR \approx 2$. For this value of k, **construct the rank-k approximation of the image**.

Solution:

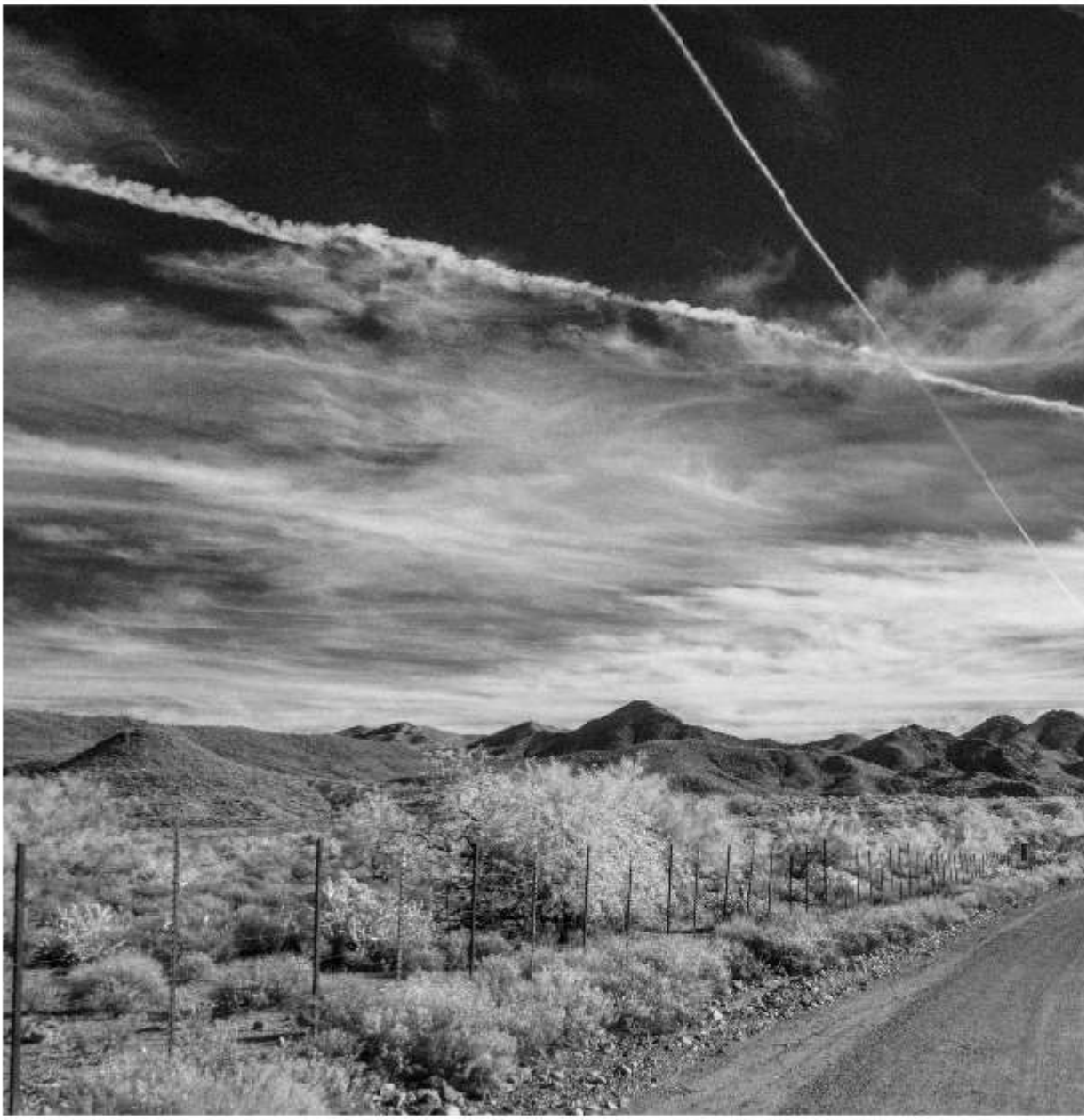
```
%image upload
figure
image = load("image.mat")
```

```
image = struct with fields:
    A: [2583×4220 uint8]
```

```
%image to matrix
A = image.A
```

```
A = 2583×4220 uint8 matrix
   23   23   31   34   22   22   35   31   30   29   32   34   31   28   30   31   29   31   31
   30   30   31   36   30   25   31   31   29   25   24   30   38   41   38   36   33   28   23
   38   33   23   21   19   16   25   35   31   26   18   23   37   41   35   32   25   27   30
   29   30   29   29   24   20   26   28   31   30   21   21   31   32   27   29   25   30   33
   20   24   32   37   32   32   33   17   26   30   26   26   34   33   28   31   26   29   30
   26   23   24   26   24   34   38   17   22   27   28   30   37   36   30   28   23   30   33
   30   29   25   25   24   31   36   23   25   28   29   30   32   33   30   27   28   33   32
   32   35   29   23   21   22   30   31   29   30   32   29   25   29   34   34   29   29   24
   27   25   22   27   28   24   28   30   23   35   34   26   21   25   33   32   33   26   22
   23   20   16   23   29   29   32   31   29   32   25   20   16   18   25   25   26   28   23
```

```
imshow(A)
```



```
%svd
%Convert matrix of image to double-data type for compatibility with svd
%function
[U, S, V] = svd(double(A))
```

U = 2583×2583

-0.0106	-0.0360	-0.0006	0.0032	-0.0032	0.0041	-0.0066	0.0022	-0.0199	0.00
-0.0105	-0.0361	-0.0006	0.0030	-0.0035	0.0049	-0.0062	0.0020	-0.0204	0.00
-0.0105	-0.0362	-0.0006	0.0034	-0.0037	0.0042	-0.0064	0.0025	-0.0203	0.00
-0.0105	-0.0362	-0.0009	0.0029	-0.0035	0.0052	-0.0056	0.0028	-0.0194	0.00
-0.0106	-0.0361	-0.0011	0.0034	-0.0035	0.0046	-0.0061	0.0022	-0.0208	0.00
-0.0106	-0.0363	-0.0011	0.0031	-0.0030	0.0049	-0.0061	0.0031	-0.0205	0.00
-0.0106	-0.0364	-0.0008	0.0032	-0.0032	0.0043	-0.0057	0.0033	-0.0201	0.00
-0.0106	-0.0365	-0.0006	0.0029	-0.0033	0.0050	-0.0052	0.0031	-0.0197	0.00
-0.0106	-0.0366	-0.0007	0.0033	-0.0031	0.0040	-0.0053	0.0033	-0.0199	0.00
-0.0106	-0.0368	-0.0009	0.0030	-0.0034	0.0044	-0.0052	0.0032	-0.0196	0.00

S = 2583×4220

10^5 ×

4.0600	0	0	0	0	0	0	0	0	
0	0.8702	0	0	0	0	0	0	0	
0	0	0.4169	0	0	0	0	0	0	
0	0	0	0.4104	0	0	0	0	0	
0	0	0	0	0.3405	0	0	0	0	
0	0	0	0	0	0.2992	0	0	0	
0	0	0	0	0	0	0.2550	0	0	
0	0	0	0	0	0	0	0.2268	0	
0	0	0	0	0	0	0	0	0.2092	
0	0	0	0	0	0	0	0	0	0.20

V = 4220×4220

-0.0130	0.0044	-0.0358	0.0028	-0.0085	0.0177	0.0128	-0.0163	0.0298	0.00
-0.0130	0.0045	-0.0357	0.0024	-0.0079	0.0184	0.0134	-0.0162	0.0309	0.00
-0.0130	0.0045	-0.0359	0.0025	-0.0078	0.0181	0.0124	-0.0168	0.0317	0.00
-0.0130	0.0046	-0.0361	0.0030	-0.0087	0.0185	0.0125	-0.0158	0.0314	0.00
-0.0130	0.0045	-0.0366	0.0032	-0.0095	0.0182	0.0116	-0.0149	0.0310	0.00
-0.0129	0.0046	-0.0369	0.0034	-0.0095	0.0185	0.0112	-0.0151	0.0302	0.00
-0.0130	0.0047	-0.0372	0.0046	-0.0104	0.0178	0.0103	-0.0141	0.0302	0.00
-0.0130	0.0048	-0.0377	0.0045	-0.0097	0.0179	0.0108	-0.0145	0.0298	0.00
-0.0130	0.0046	-0.0375	0.0049	-0.0094	0.0170	0.0102	-0.0147	0.0306	0.00
-0.0130	0.0045	-0.0379	0.0043	-0.0090	0.0166	0.0095	-0.0142	0.0304	0.00

```
%per documentation, compression ratio (CR) is computed
% CR = mn / k(m+n+1)
%Since we desire a compresion ratio of 2, CR = 2
%m and n are derived from the row and column counts of A using size()

[m, n] = size(A)
```

m = 2583
n = 4220

```
%set compression ratio to 2
CR = 2
```

CR = 2

```
%solve for k
k = (m*n)/(CR*(m+n+1))
```

k = 801.0185

```
%Round value of K so it can be used in below equation
k = round(k)
```

k = 801

```
A2 = U(:,1:k) * S(1:k,1:k) * V(:,1:k)'
```

A2 = 2583×4220

```
26.4896   27.2541   30.5810   28.9530   23.3828   25.7705   35.1037   29.3968   32.9167   35.72
32.6831   34.0733   28.4258   30.5682   27.6882   28.4547   35.6629   31.2002   29.6536   27.39
35.5230   30.8250   18.7994   19.9743   19.8952   17.0400   24.6764   26.0911   29.5376   23.26
33.6440   29.7702   26.1173   29.8858   26.5095   15.9739   24.7017   25.8886   27.9959   25.37
27.7165   26.0012   30.5018   36.7547   35.3434   29.8915   34.5078   25.1416   24.9395   25.38
27.3996   25.5183   26.6092   29.4463   25.5795   28.8615   33.9147   23.0624   21.5388   26.76
32.0484   32.4889   27.5089   22.7250   20.3684   25.0803   33.3388   26.7700   26.6038   29.15
26.0954   32.2044   27.4183   18.1894   21.2836   28.1417   31.7244   26.2813   29.0047   32.42
23.2187   25.5412   22.1689   25.1362   29.2165   30.3222   34.5845   30.5812   29.6937   35.34
21.3048   20.9797   19.1568   25.5860   26.9030   24.8220   30.0068   30.1700   28.2604   34.43
```

```
%convert back to 8-bit binary
A2 = uint8(round(A2))
```

```
A2 = 2583×4220 uint8 matrix
   26   27   31   29   23   26   35   29   33   36   30   26   32   27   28   34   31   31   28
   33   34   28   31   28   28   36   31   30   27   22   28   40   34   34   34   33   33   28
   36   31   19   20   20   17   25   26   30   23   22   27   35   35   37   34   28   29   29
   34   30   26   30   27   16   25   26   28   25   22   24   33   35   32   30   25   29   31
   28   26   31   37   35   30   35   25   25   25   23   27   33   33   27   27   25   31   28
   27   26   27   29   26   29   34   23   22   27   29   34   33   33   28   26   28   33   31
   32   32   28   23   20   25   33   27   27   29   35   33   27   28   31   29   28   34   31
   26   32   27   18   21   28   32   26   29   32   36   31   28   31   34   34   23   27   30
   23   26   22   25   29   30   35   31   30   35   34   27   20   24   34   34   24   20   21
   21   21   19   26   27   25   30   30   28   34   29   20   16   22   29   32   23   23   23
```

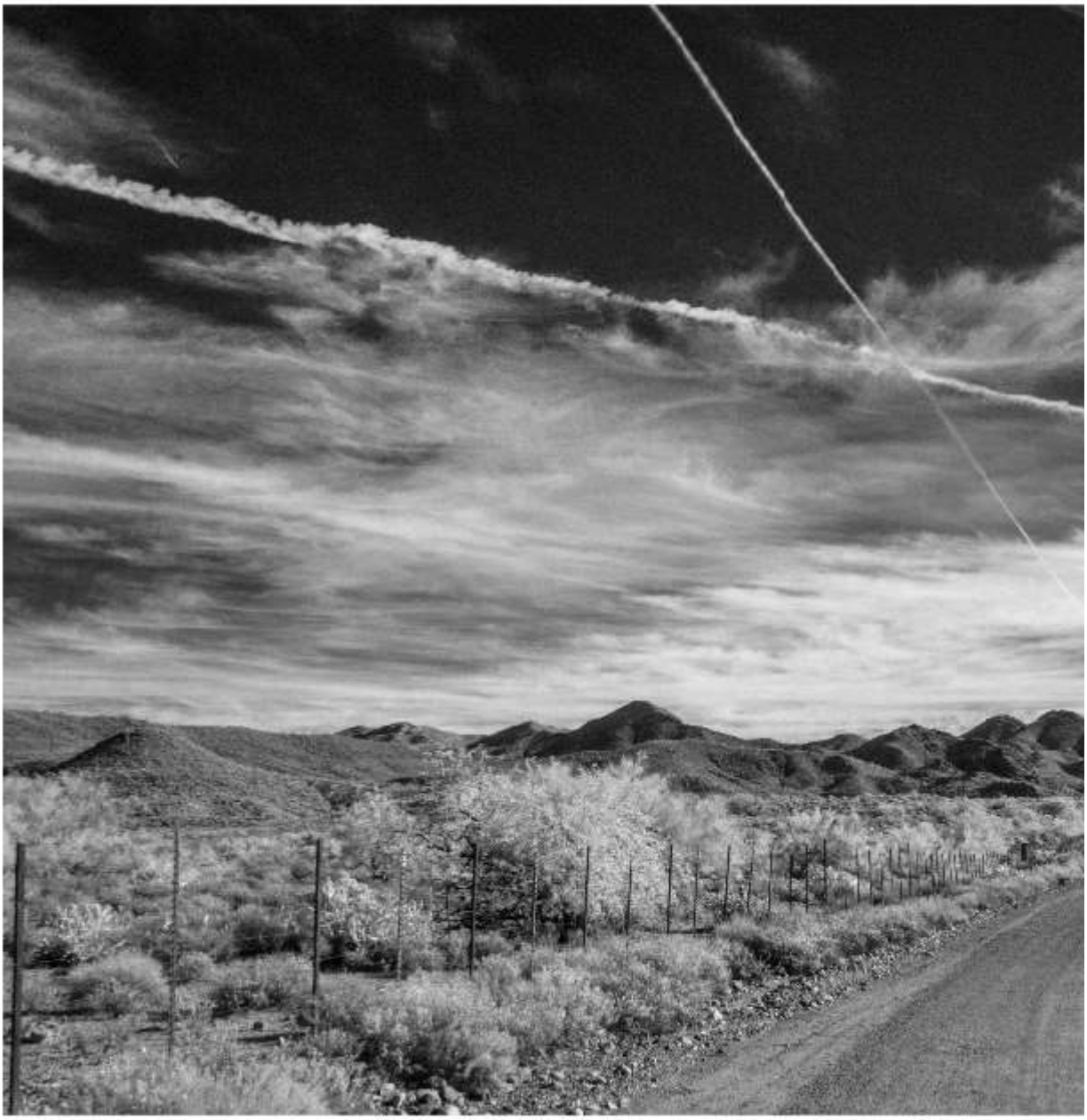
Explain: Using a combination of size function, re-arranging the compression ratio equation, and svd function we have created an algorithm to create a compressed image matrice based upon our desired compression ratio.

Problem 6

Display the image and compute the root mean square error (RMSE) between the approximation and the original image. Make sure to include a copy of the approximate image in your report.

Solution:

```
%image
imshow(A2)
```



```
%rmse
RSME2 = norm(double(A)-double(A2),'fro')/sqrt(m*n)
```

```
RSME2 = 3.1664
```

Problem 7

Repeat Problems 5 and 6 for $CR \approx 10$, $CR \approx 25$, and $CR \approx 75$. **Explain** what trends you observe in the image approximation as CR increases and provide your recommendation for the best CR based on your observations. Make sure to include a copy of the approximate images in your report.

Solution:

Compression Rate 10

```
%set compression ratio to 10
```

```
CR = 10
```

```
CR = 10
```

```
%solve for k
k = (m*n)/(CR*(m+n+1))
```

```
k = 160.2037
```

```
%Round value of K so it can be used in below equation
k = round(k)
```

```
k = 160
```

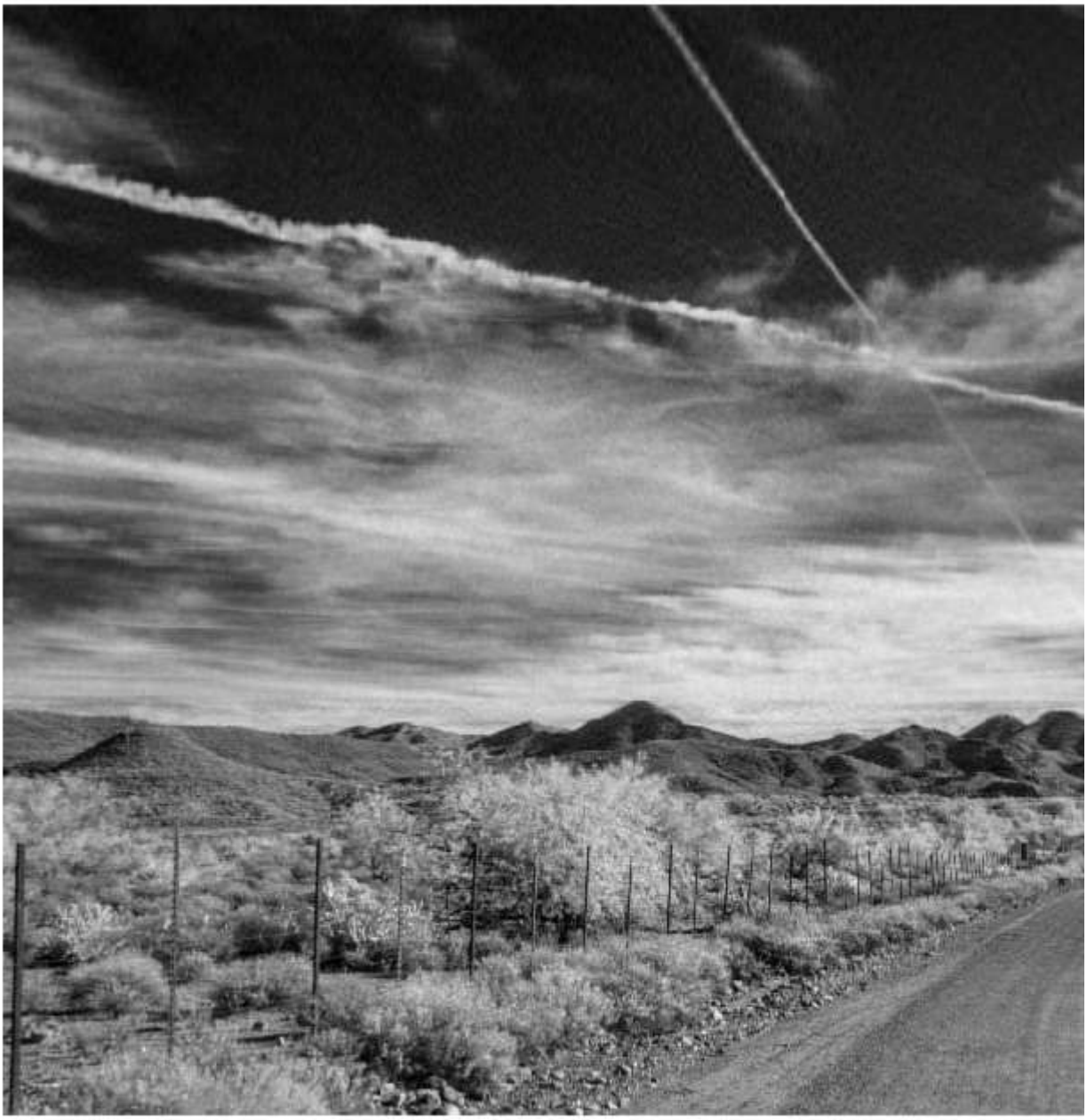
```
A10 = U(:,1:k) * S(1:k,1:k) * V(:,1:k)'
```

```
A10 = 2583×4220
   29.0415   29.0508   30.3481   28.6840   28.4125   25.8146   26.5310   25.2709   27.6396   26.63
   29.4990   29.1068   29.9563   28.2658   28.2759   26.2124   27.5634   26.0996   27.8406   27.01
   26.0110   25.7091   26.5596   24.7097   24.6572   23.0067   24.2242   22.2803   24.1519   23.60
   28.3636   28.5688   29.5533   27.3636   27.6176   25.7857   27.0061   24.7262   26.4571   25.90
   28.5023   28.4627   29.3196   27.9087   28.6184   26.5532   27.9189   25.7023   27.1868   26.36
   26.6380   26.7335   27.2446   25.9346   26.6994   25.3844   27.4323   25.3185   26.7060   25.83
   27.7105   26.9342   27.3695   25.8611   26.2616   24.8003   26.8496   24.7796   26.1347   25.19
   26.6047   26.2936   26.6370   25.8935   26.9915   25.8782   27.7421   25.4926   26.7357   25.37
   25.4761   25.3280   25.3720   24.6828   25.3964   24.6848   26.1862   23.9901   24.9284   23.76
   24.0066   23.4106   23.1904   22.8845   23.3535   22.7318   23.9445   22.0238   22.6494   21.06
```

```
%convert back to 8-bit binary
A10 = uint8(round(A10))
```

```
A10 = 2583×4220 uint8 matrix
   29   29   30   29   28   26   27   25   28   27   26   25   28   27   27   28   26   32   34
   29   29   30   28   28   26   28   26   28   27   26   25   28   27   28   29   26   31   33
   26   26   27   25   25   23   24   22   24   24   24   23   26   25   27   28   26   30   32
   28   29   30   27   28   26   27   25   26   26   26   24   28   27   27   28   26   31   33
   29   28   29   28   29   27   28   26   27   26   27   25   28   27   27   28   26   30   32
   27   27   27   26   27   25   27   25   27   26   26   24   26   26   27   27   25   30   32
   28   27   27   26   26   25   27   25   26   25   26   24   26   26   27   27   25   30   31
   27   26   27   26   27   26   28   25   27   25   25   23   25   24   25   25   23   28   30
   25   25   25   25   25   25   26   24   25   24   24   21   23   23   23   23   21   25   26
   24   23   23   23   23   23   24   22   23   21   21   19   21   21   22   23   22   25   27
```

```
imshow(A10)
```



```
%RSME test
RSME10 = norm(double(A)-double(A10),'fro')/sqrt(m*n)
```

```
RSME10 = 8.2127
```

Compression Rate 25

```
%set compression ratio to 25
CR = 25
```

```
CR = 25
```

```
%solve for k
k = (m*n)/(CR*(m+n+1))
```

```
k = 64.0815
```

```
%Round value of K so it can be used in below equation
k = round(k)
```

```
k = 64
```

```
A25 = U(:,1:k) * S(1:k,1:k) * V(:,1:k)'
```

```
A25 = 2583×4220
   24.9993   24.3447   22.0100   22.2062   21.9511   21.9226   23.0593   23.7963   24.8462   25.98
   25.2032   24.4691   22.2505   22.4141   22.3977   22.5232   23.5883   24.2551   25.3147   26.41
   23.7727   23.0131   20.8102   21.1738   20.9707   21.3994   22.4306   23.1625   24.0254   25.13
   24.3150   23.8209   21.6649   21.8529   21.8114   22.1064   23.1593   23.8830   24.8460   25.97
   24.8134   24.0232   21.8824   22.0873   22.2178   22.3894   23.4309   24.2761   25.0236   26.13
   24.1187   23.4903   21.2471   21.5617   21.6885   21.8974   22.7524   23.5456   24.2134   25.11
   24.0810   23.2159   21.0505   21.3372   21.4513   21.7045   22.5713   23.3211   23.9904   24.95
   25.0993   24.3390   22.2436   22.6802   23.0230   23.0038   23.7316   24.4781   25.2091   26.12
   23.7390   22.9025   20.8467   21.1917   21.5129   21.5312   22.1834   23.0650   23.5944   24.62
   22.2805   21.5541   19.4265   20.0297   20.3143   20.3821   20.8926   21.9840   22.5538   23.62
```

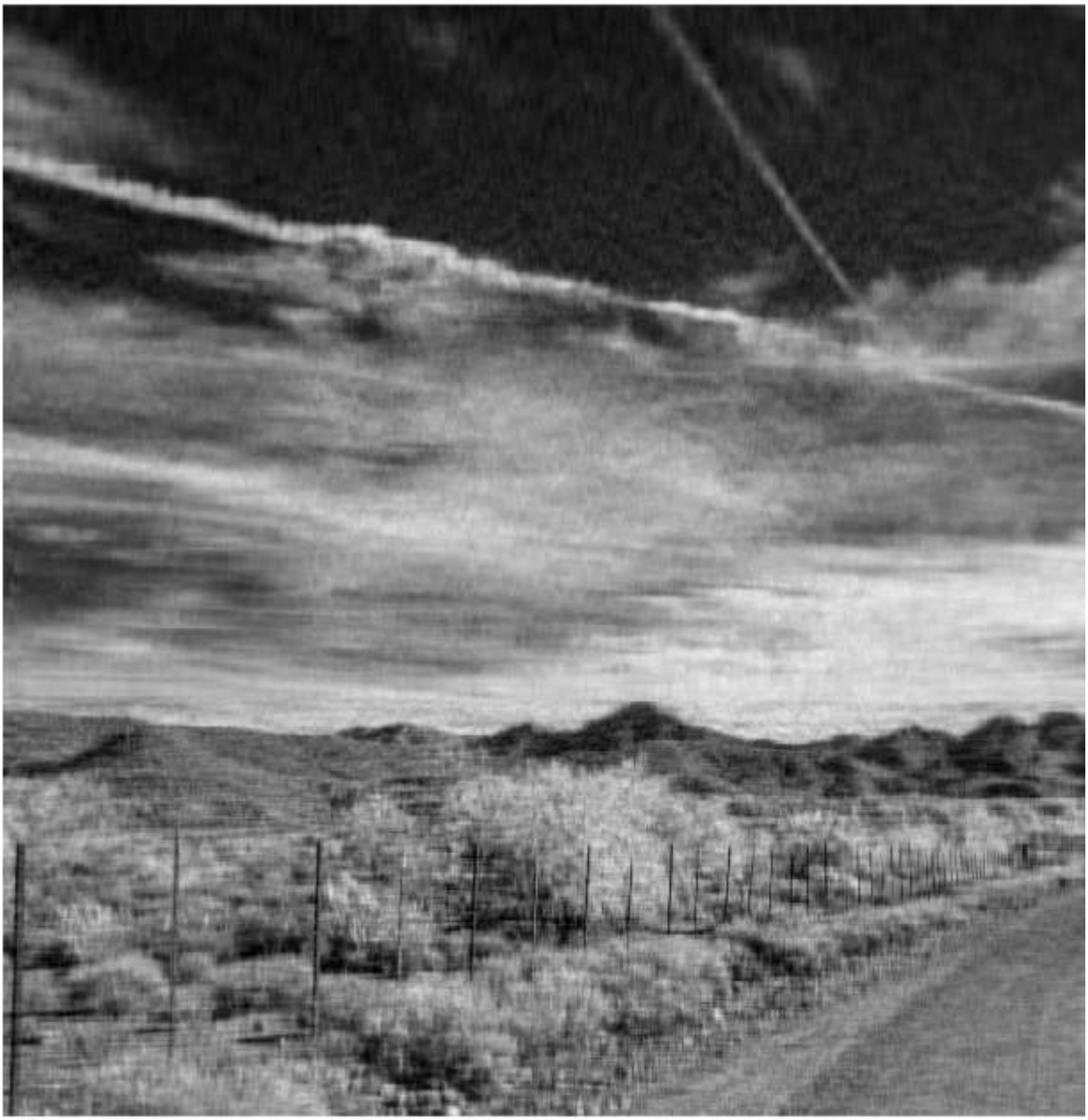
```
%convert back to 8-bit binary
A25 = uint8(round(A25))
```

```
A25 = 2583×4220 uint8 matrix
   25   24   22   22   22   22   23   24   25   26   27   27   29   29   28   30   28   30   32
   25   24   22   22   22   23   24   24   25   26   27   27   30   29   29   31   30   31   33
   24   23   21   21   21   21   22   23   24   25   26   26   29   28   28   29   28   30   31
   24   24   22   22   22   22   23   24   25   26   27   27   29   28   28   29   27   29   31
   25   24   22   22   22   22   23   24   25   26   27   27   29   29   28   29   28   30   31
   24   23   21   22   22   22   23   24   24   25   26   26   28   28   27   28   27   29   30
   24   23   21   21   21   22   23   23   24   25   26   26   28   27   26   27   26   27   29
   25   24   22   23   23   23   24   24   25   26   27   27   29   28   27   28   26   28   30
   24   23   21   21   22   22   22   23   24   25   26   25   27   27   26   27   26   27   29
   22   22   19   20   20   20   21   22   23   24   25   25   27   26   26   26   25   26   28
```

```
A25 = uint8(round(A25))
```

```
A25 = 2583×4220 uint8 matrix
   25   24   22   22   22   22   23   24   25   26   27   27   29   29   28   30   28   30   32
   25   24   22   22   22   23   24   24   25   26   27   27   30   29   29   31   30   31   33
   24   23   21   21   21   21   22   23   24   25   26   26   29   28   28   29   28   30   31
   24   24   22   22   22   22   23   24   25   26   27   27   29   28   28   29   27   29   31
   25   24   22   22   22   22   23   24   25   26   27   27   29   29   28   29   28   30   31
   24   23   21   22   22   22   23   24   24   25   26   26   28   28   27   28   27   29   30
   24   23   21   21   21   22   23   23   24   25   26   26   28   27   26   27   26   27   29
   25   24   22   23   23   23   24   24   25   26   27   27   29   28   27   28   26   28   30
   24   23   21   21   22   22   22   23   24   25   26   25   27   27   26   27   26   27   29
   22   22   19   20   20   20   21   22   23   24   25   25   27   26   26   26   25   26   28
```

```
imshow(A25)
```



```
%RSME test
RSME25 = norm(double(A)-double(A25),'fro')/sqrt(m*n)
```

```
RSME25 = 12.3020
```

Compression Rate 75

```
%set compression ratio to 75
CR = 75
```

```
CR = 75
```

```
%solve for k
k = (m*n)/(CR*(m+n+1))
```

```
k = 21.3605
```

```
%Round value of K so it can be used in below equation
k = round(k)
```

```
k = 21
```

```
A75 = U(:,1:k) * S(1:k,1:k) * V(:,1:k)'
```

```
A75 = 2583×4220
```

26.2163	25.6337	24.9907	25.1005	25.5472	26.3732	27.4089	27.1685	28.3767	28.46
26.5129	25.9007	25.2598	25.3699	25.8707	26.7271	27.7391	27.5112	28.6565	28.66
26.3007	25.6709	25.0385	25.1923	25.6988	26.5671	27.6553	27.4353	28.5952	28.61
27.0439	26.5387	25.9096	26.0592	26.5085	27.3315	28.3511	28.1032	29.2320	29.27
26.9099	26.2511	25.6362	25.6966	26.2742	27.0451	28.0548	27.8644	28.9925	29.10
26.6065	25.9442	25.3392	25.4571	26.0535	26.8352	27.8248	27.6089	28.7083	28.75
26.8196	26.1732	25.5741	25.6539	26.1657	26.8895	27.8945	27.6443	28.7723	28.79
27.2339	26.6413	26.0681	26.0830	26.6782	27.3193	28.2429	27.9771	29.1202	29.18
26.0167	25.3796	24.8159	24.7942	25.3968	25.9820	26.8889	26.6155	27.7070	27.83
27.3611	26.7141	26.1752	26.1811	26.7707	27.3992	28.3091	28.0254	29.0480	29.06

```
%convert back to 8-bit binary
A75 = uint8(round(A75))
```

```
A75 = 2583×4220 uint8 matrix
```

26	26	25	25	26	26	27	27	28	28	29	29	30	31	30	32	32	34	36
27	26	25	25	26	27	28	28	29	29	30	29	30	31	31	32	32	34	36
26	26	25	25	26	27	28	27	29	29	29	29	30	31	31	32	32	34	36
27	27	26	26	27	27	28	28	29	29	30	30	31	32	31	33	33	35	37
27	26	26	26	26	27	28	28	29	29	30	30	31	32	31	33	33	35	37
27	26	25	25	26	27	28	28	29	29	30	29	30	31	31	32	32	34	36
27	26	26	26	26	27	28	28	29	29	30	29	30	31	31	32	32	34	36
27	27	26	26	27	27	28	28	29	29	30	30	31	32	32	33	33	35	37
26	25	25	25	25	26	27	27	28	28	29	28	30	30	30	32	32	34	36
27	27	26	26	27	27	28	28	29	29	30	30	31	32	31	33	33	35	37

```
imshow(A75)
```

```
%RSME test
RSME75 = norm(double(A)-double(A75),'fro')/sqrt(m*n)
```

```
RSME75 = 18.2650
```

Explain: Per our earlier discussions of rank and the effect of adding or removing linearly independent columns, we gain an excellent visual perspective on the inverse relationship between compression ratio and granularity of image matrice as it relates to the viewer experience. For viewer experience any compression ratio higher than 10 would be detrimental. However, this is all negotiable based upon what the desired viewer experience is, and what our storage needs are as an organization.