Synchronization & Cycle Consistency
in Computer Vision

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in collaboration with

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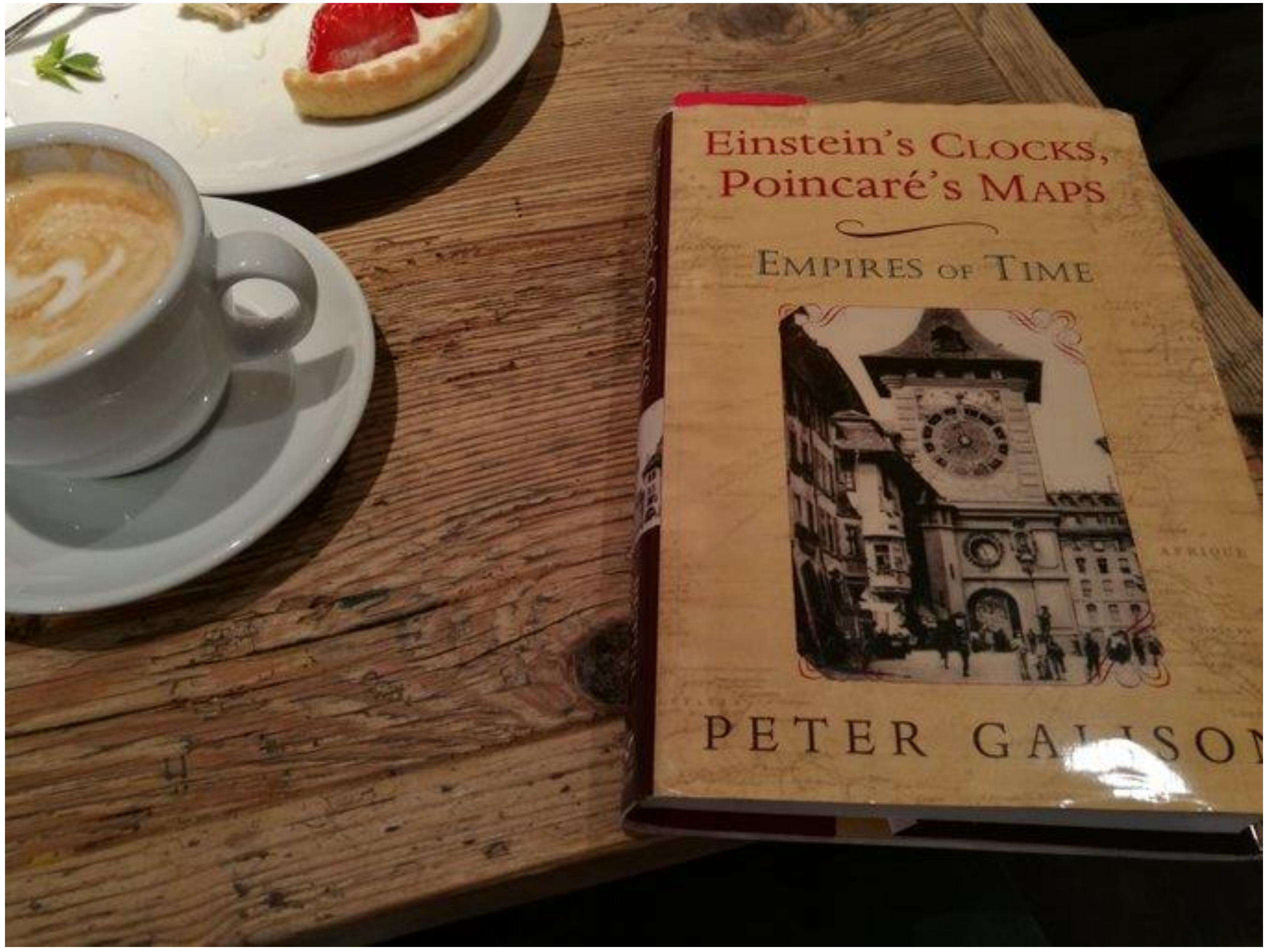
Synchronization

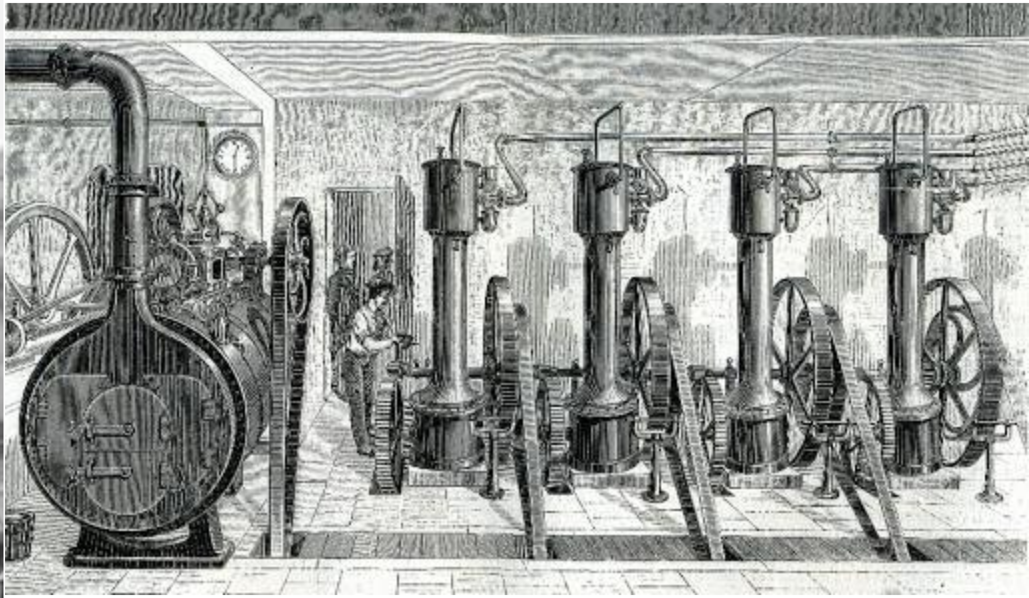
- The coordination of events to operate a system in unison
- Finding simultaneity

It all began with…

Clocks.



Parisian Pneumatic Clocks

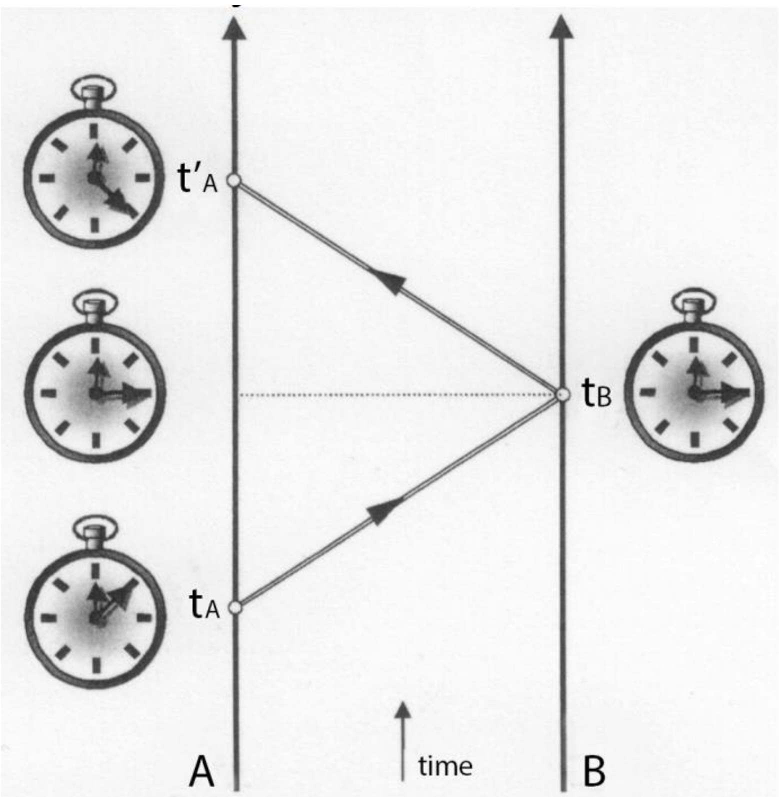


Paris Pneumatic Clock Network
http://www.douglas-self.com/MUSEUM/COMMS/airclock/airclock.htm

The town of Einstein, Bern is no short of clocks.

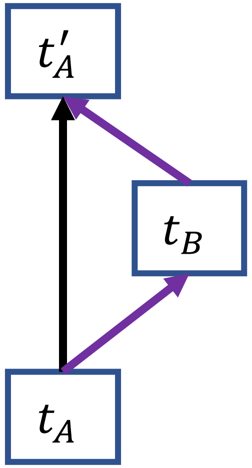
wdl.org

Einstein - Poincaré Synchronization

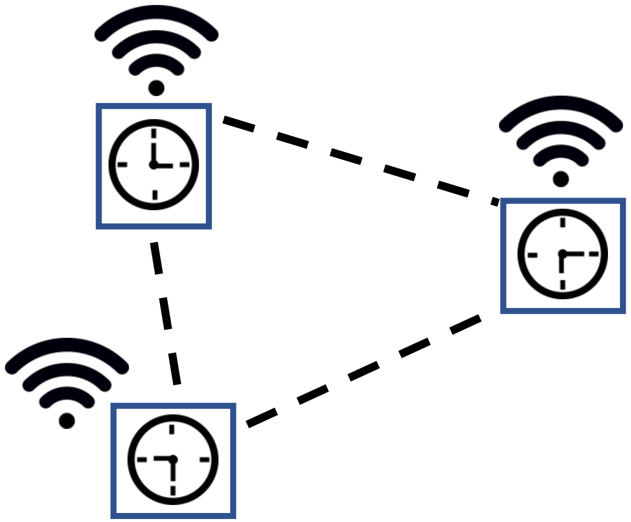


$$t_B = t_A + 1/2\left(t'_A - t_A\right)$$

Synchronization & Cycle Consistency pre-Computer Vision



Clock Synchronization



Wireless Communication

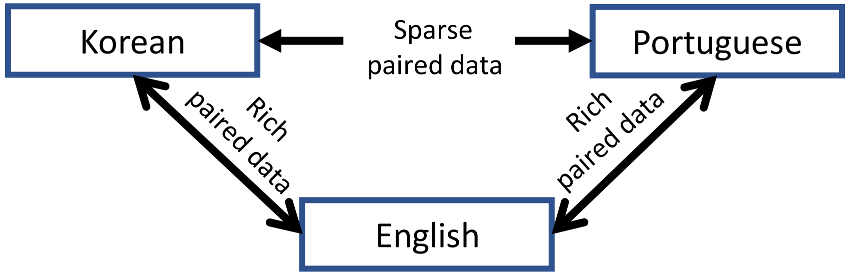


Image-to-Image Translation

Applications of Cycle Consistency in Computer Vision

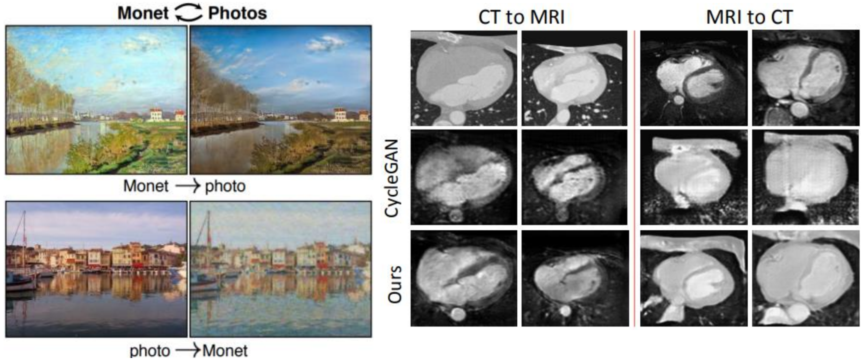
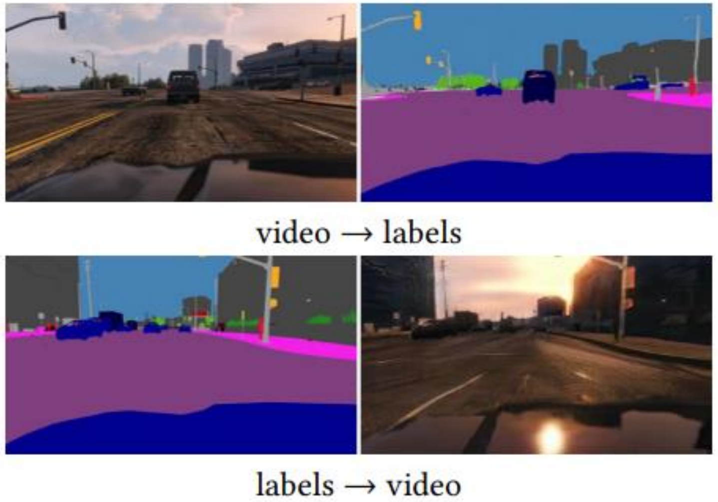


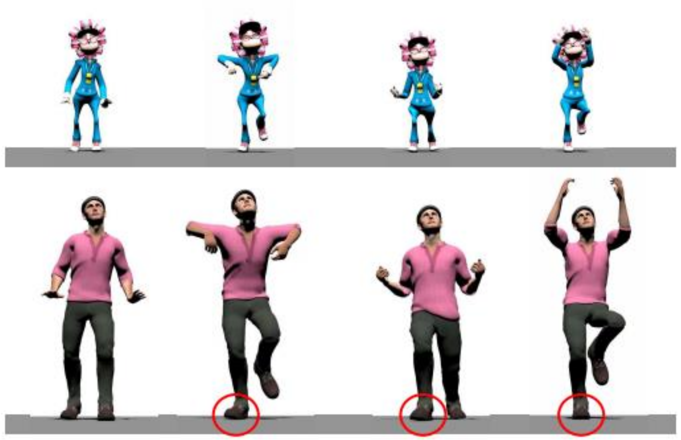
Image-to-Image Translation
Zu et al. ICCV'17

Video Understanding
Lai & Xie BMVC'19

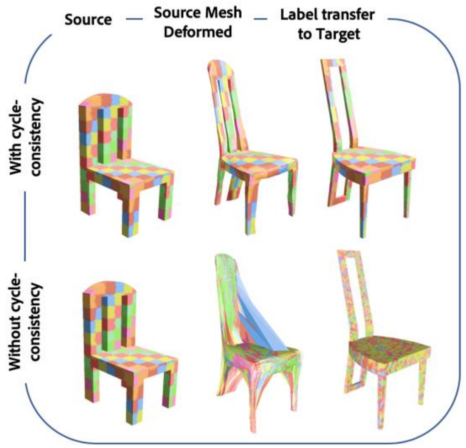
3D Geometric Registration
Kulkarni ICCV'19



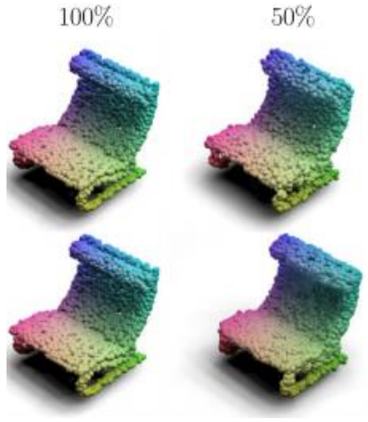
Video-to-Video Translation
Chen et al. ACMMM'19



Motion Retargeting
Villegas et al. '19

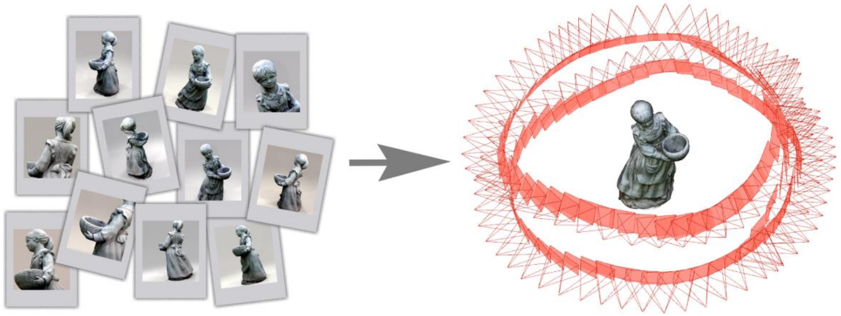


Self Supervised Deep Deformation
Groueix Eurographics'19

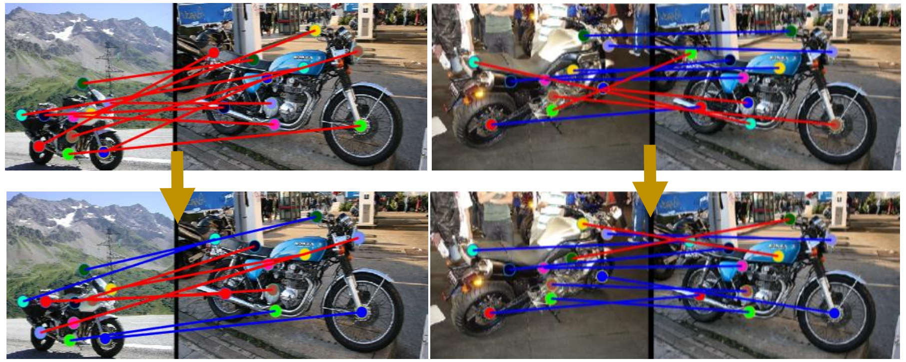


3D Generative Modeling
Pumarola CVPR'20

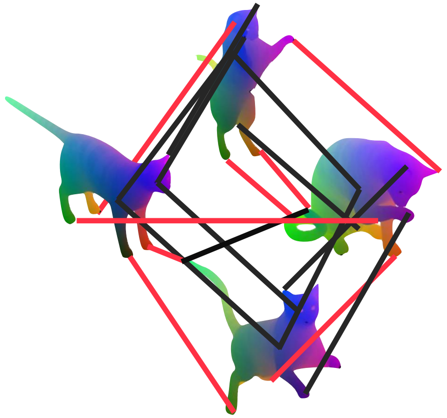
Applications of Synchronization in Computer Vision



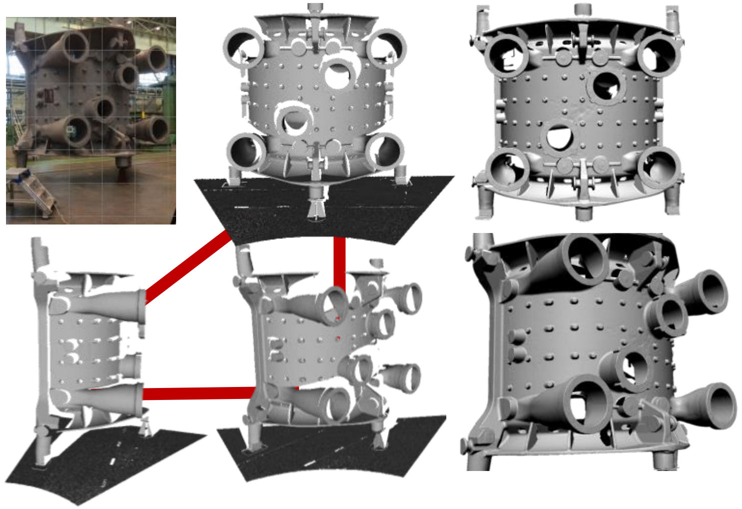
Structure from Motion
Bianco '18



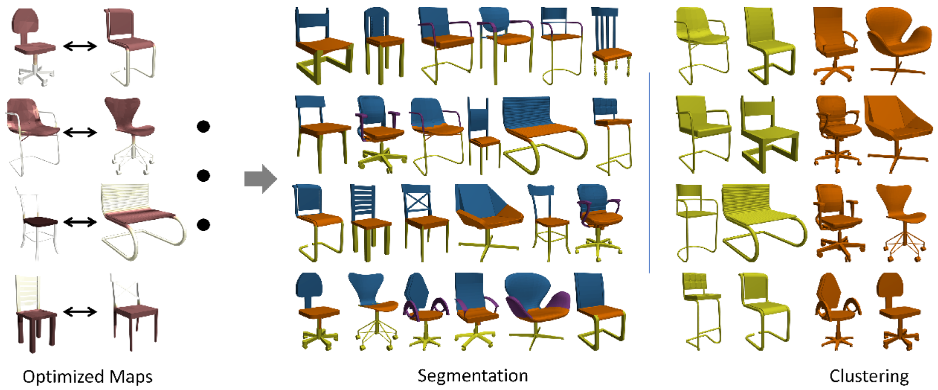
Feature Matching
Birdal et al. CVPR'19



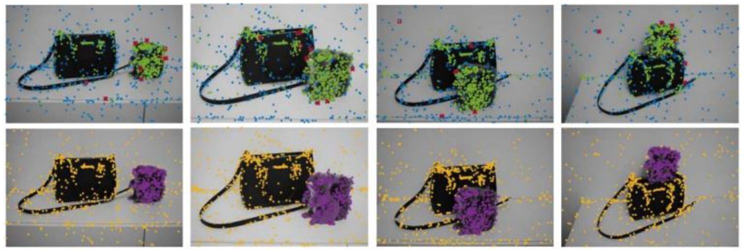
3D Shape Matching
Birdal et al. CVPR'19



3D Reconstruction from Scans
Birdal et al. ICCV'17



Co-Segmentation / Clustering
Huang ACM'19



Simultaneous Segmentation & Registration
Arrigoni et al. ICCV'19

Applications of Synchronization in Computer Vision

Villegas, Ruben, et al. "Neural kinematic networks for unsupervised motion retargetting." *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2018.

A connected graph

Domain, map, labeling

Notation
(To be filled)

graph *vertices* *edges*

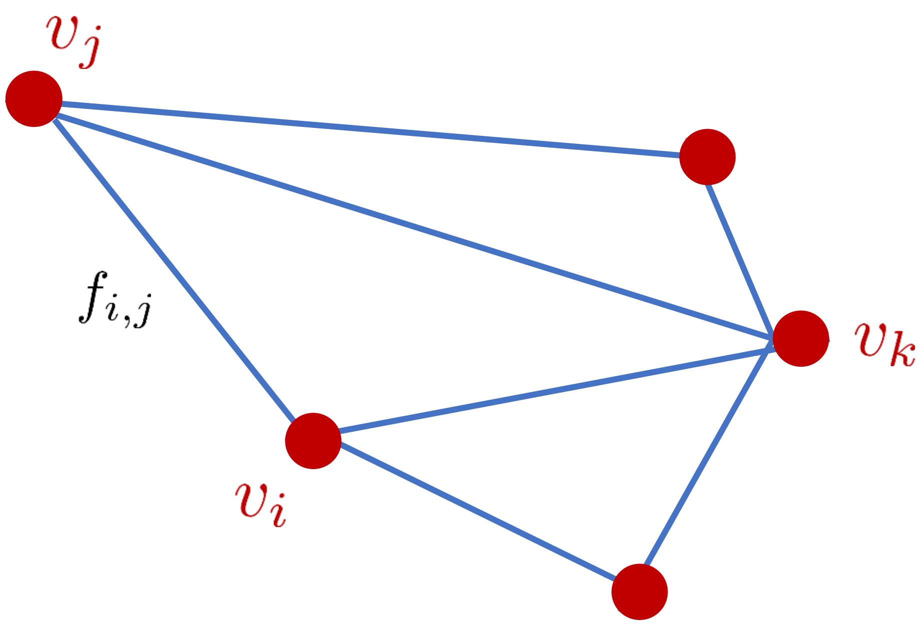
$$\mathcal{G} \triangleq (\mathcal{V} = \{v_1, v_2, \ldots, v_n\}, \mathcal{E} \subset [n] \times [n])$$

domain $\mathcal{D}_2 \xrightarrow[\text{map}]{f_{1,n}} \mathcal{D}_n$

(relative)

a vertex labeling

$$f_i : \mathcal{V} \to \mathcal{D}_i$$



Path, Cycle, Null Cycle

Path: $p = \{(i_1, i_2), (i_2, i_3), \cdots, (i_{n-1}, i_n)\} \in \mathcal{P}$

Cycle: $c = \{(i_1, i_2), (i_2, i_3), \cdots, (i_{n-1}, i_n), (i_n, i_1)\} \in \mathcal{C}$

Null-cycle: A cycle where the composition of functions lead to identity:

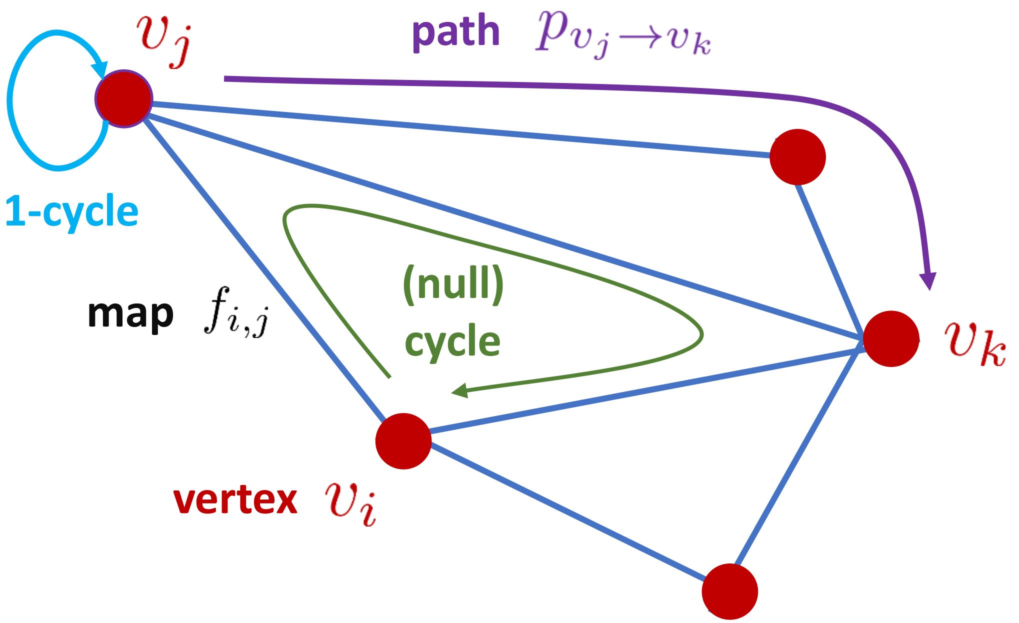
$$f_c = f_{1,2} \circ f_{2,3} \cdots \circ f_{(n-1),n} \circ f_{n,1} = f_{\varnothing}$$

k-cycle:

1-cycle: $f_{i,i} = f_{\varnothing}$

2-cycle: $f_c = f_{i,j} \circ f_{j,i}$

$$f_{\varnothing}(\mathbf{x}) = \mathbf{x}, \; \forall \mathbf{x}$$



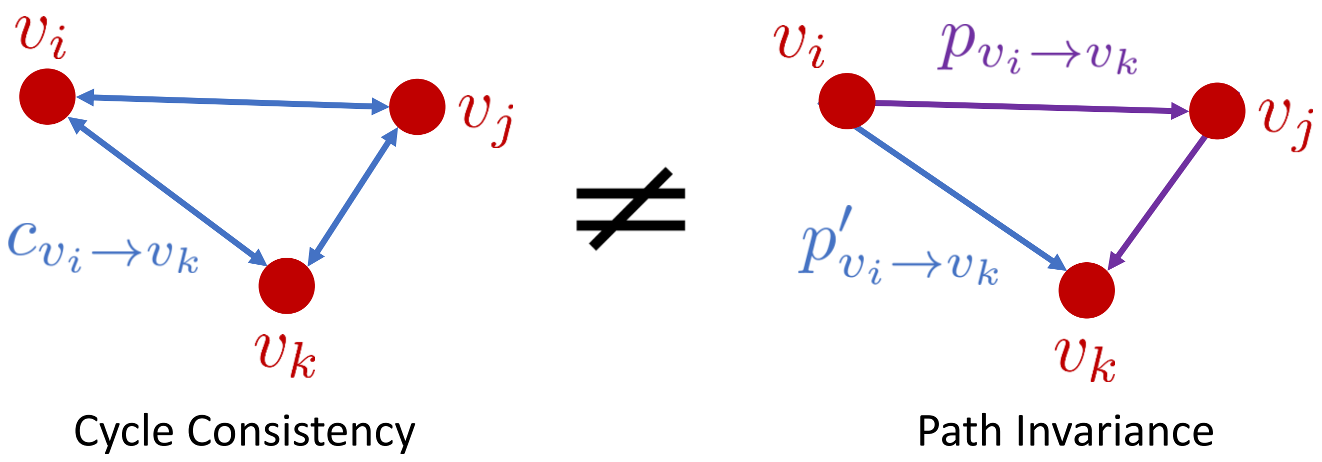
All possible paths between u and v: $\mathcal{G}_{\text{path}}(u,v)$

All possible path pairs: $\mathcal{G}_{\text{pairs}} = \bigcup_{(u,v)\in\mathcal{V}^2} \{(p,q) \, : \, p,q \in \mathcal{G}_{\text{path}}(u,v)\}$

Graph is Path Invariant if: $f_p = f_q \quad \forall (p,q) \in \mathcal{G}_{\text{pairs}}$

Graph is Cycle Consistent if: $f_c = f_{\varnothing} \quad \forall c \in \mathcal{C}$

Path Invariance (PI)
&
Cycle Consistency (CC)



Guibas, Leonidas J., Qixing Huang, and Zhenxiao Liang. "A Condition Number for Joint Optimization of Cycle-Consistent Networks." *Advances in Neural Information Processing Systems*. 2019.

Common Assumptions on f

1. $f_{i,j}$ is assumed to be an *isomorphism* *i.e.* $f_{j,i} = f_{i,j}^{-1}$.
2. $f_{i,j}$ are linear.
3. $f_{i,j}$ is *group valued* *i.e.* relative maps are *closed*, *invertible* and *associative* under the group operation $\circ$ with the existence of an identity element. Note that this induces (1).

Cycle Consistency by Construction

Consistent vertex labeling: $f_{i,j} = f_i \circ f_j^{-1} \quad \forall (i,j) \in \mathcal{E}$

Simple proof:

$$\begin{aligned} f_c &= f_{1,2} \circ f_{2,3} \circ \cdots \circ f_{(n-1),n} \circ f_{n,1} \\ &= \left(f_1 \circ f_2^{-1}\right) \circ \left(f_2 \circ f_3^{-1}\right) \circ \cdots \circ \left(f_n \circ f_1^{-1}\right) \\ &= f_1 \circ \left(f_2^{-1} \circ f_2\right) \circ \left(f_3^{-1} \circ f_3\right) \circ \cdots \circ \left(f_n^{-1} \circ f_n\right) \circ f_1^{-1} \\ &= f_1 \circ f_1^{-1} = f_{null} \quad \forall c \in \mathcal{C}. \end{aligned}$$

Gauge Freedom

"Conventionality of Simultaneity"

Without loss of generality the following holds for group-valued f:

$$(f_i \circ f_g) \circ (f_j \circ f_g)^{-1} = (f_i \circ f_g) \circ (f_g^{-1} \circ f_j) = f_i \circ f_j^{-1}.$$

"That light requires the same time to traverse the path A → M as for the path B → M is in **reality neither a supposition nor a hypothesis about the physical nature of light, but a stipulation which I can make of my own free will in order to arrive at a definition of simultaneity** " – Albert Einstein

("Ueber die spezielle und die allgemeine Relativitaetstheorie", Vieweg, Braunschweig (1988))

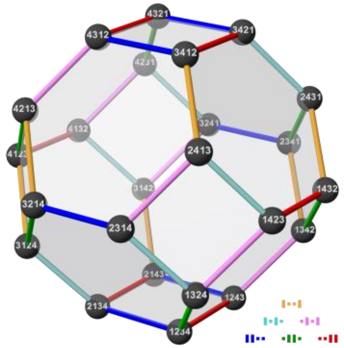
Can fix one of the functions arbitrarily to reduce the DoF.

Anderson, R., I. Vetharaniam, and G. Stedman, 1998. "Conventionality of Synchronisation, Gauge Dependence and Test Theories of Relativity," *Physics Reports*, 295: 93–180

What are those maps f ?

Rotation Matrices

Zhao et al.'20

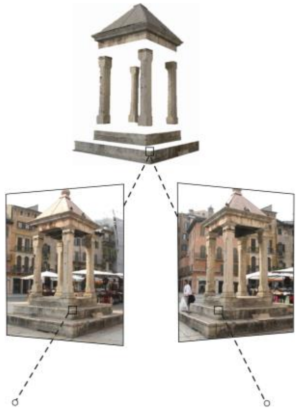


Permutations

wikipedia.org/wiki/Permutohedron

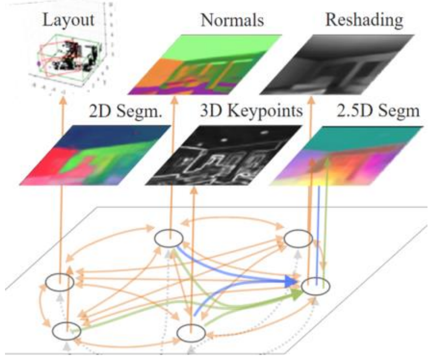
Functional Maps

Guibas '20



Essential Matrices

3dflow.net



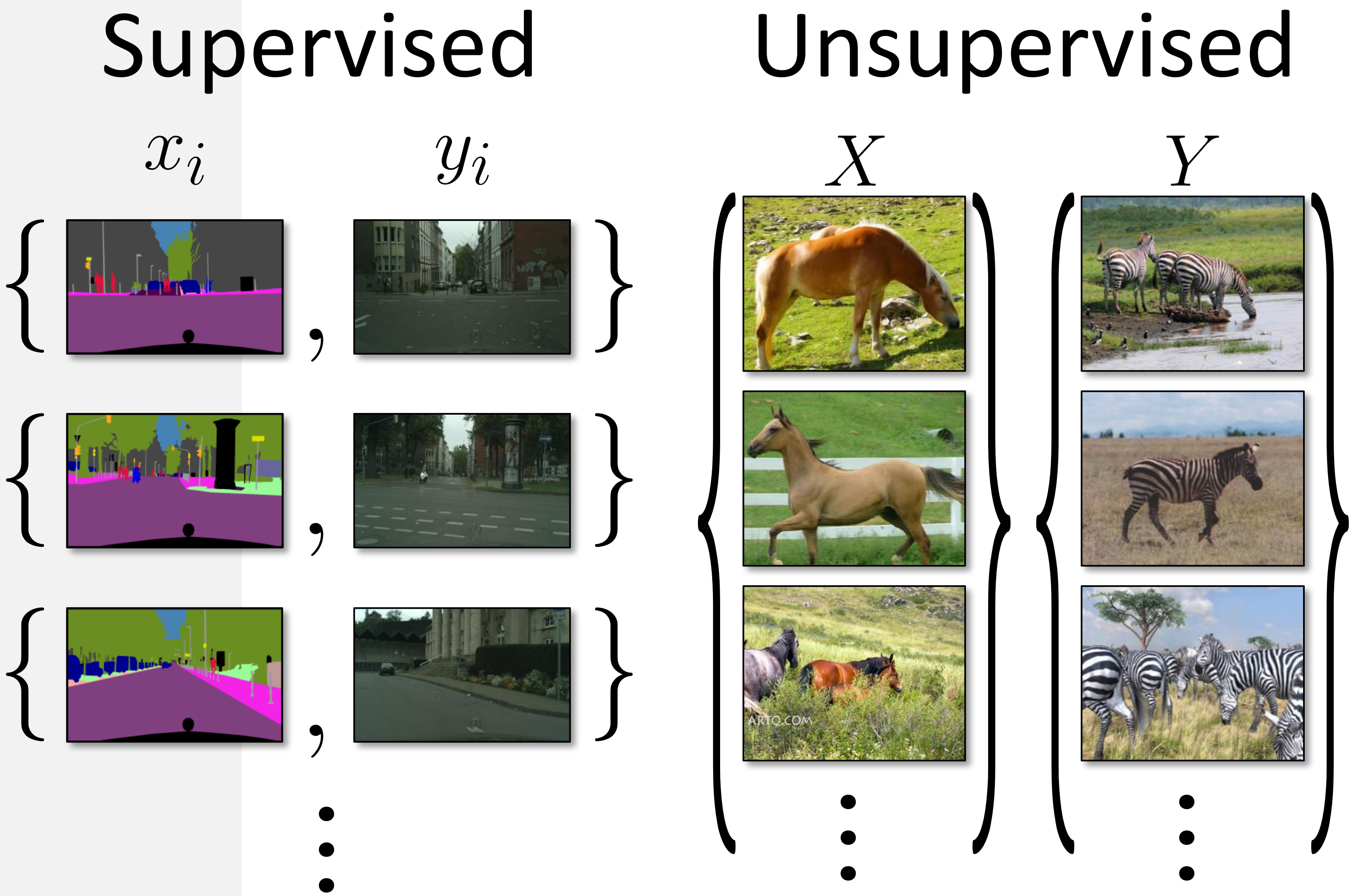
Neural Networks

Zamir et al.'18

Tensor Maps

webstockreview.net

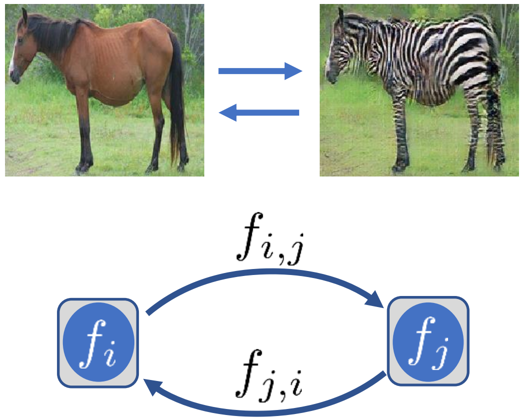
When to Use Path Invariance (PI) and Cycle Consistency (CC)



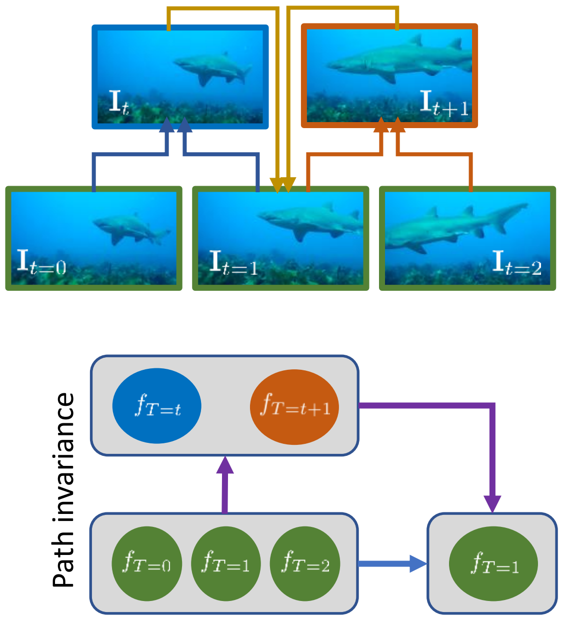
Zhu, Jun-Yan, et al. "Unpaired image-to-image translation using cycle-consistent adversarial networks." *Proceedings of the IEEE international conference on computer vision*. 2017.

PI & CC as Unsupervised Losses

Cycle-GAN, ICCV’17

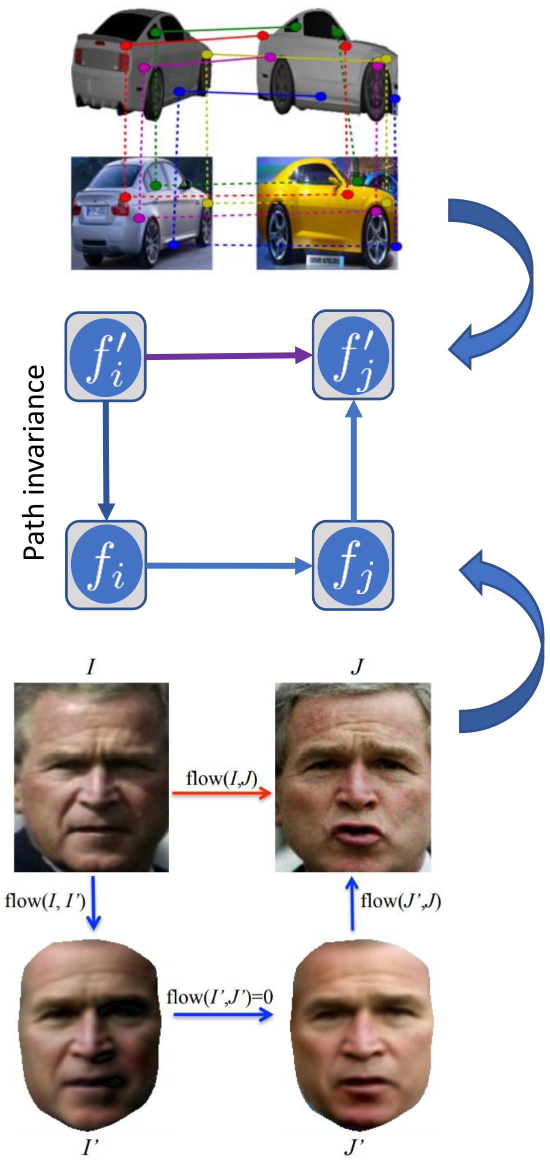


Vertex label is an image, relative map is a neural network.



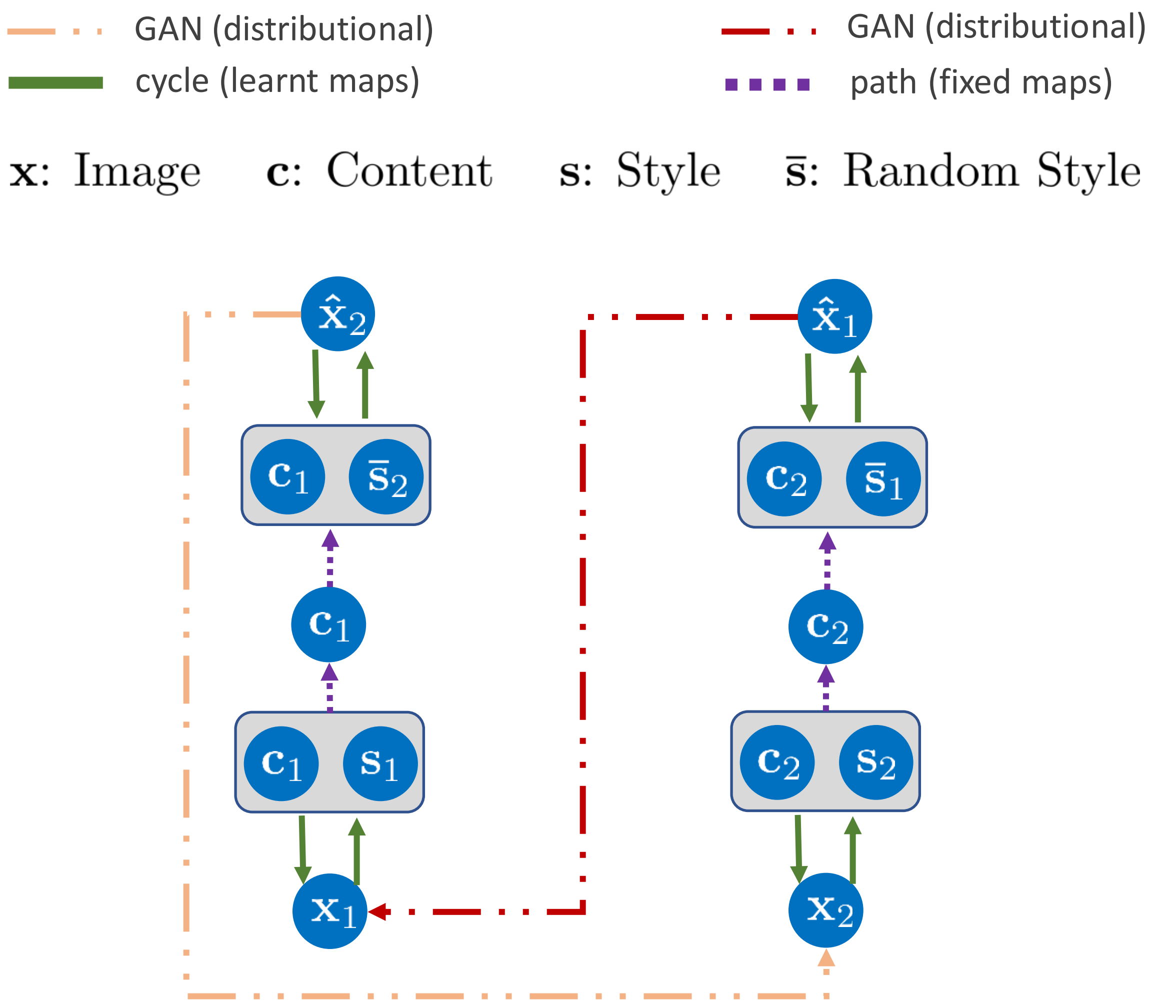
Unsupervised Video Interpolation Using **Cycle Consistency**, Reda et al. ICCV 2019

Learning Dense Correspondence via 3D-guided **Cycle Consistency**, Zhou et al.



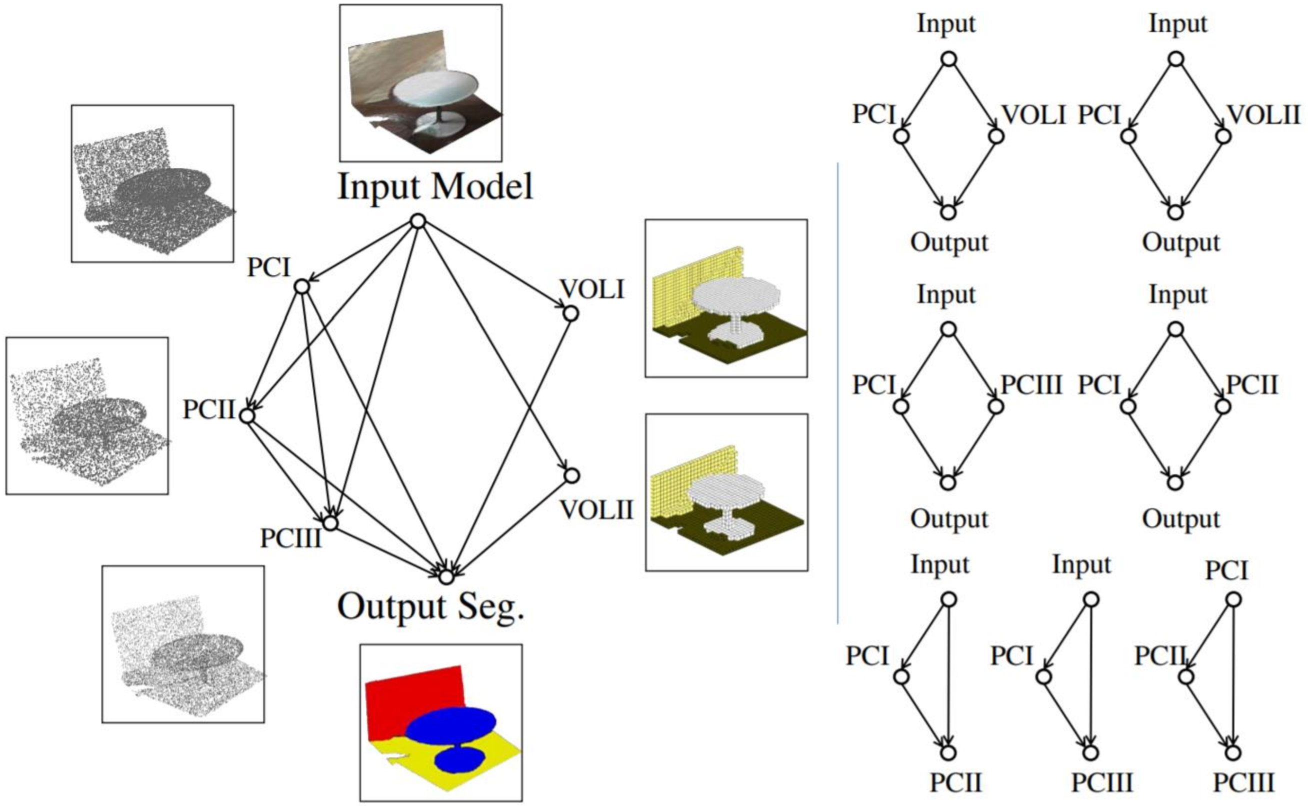
Collection Flow, Shlizerman et al.

PI & CC as Unsupervised Losses



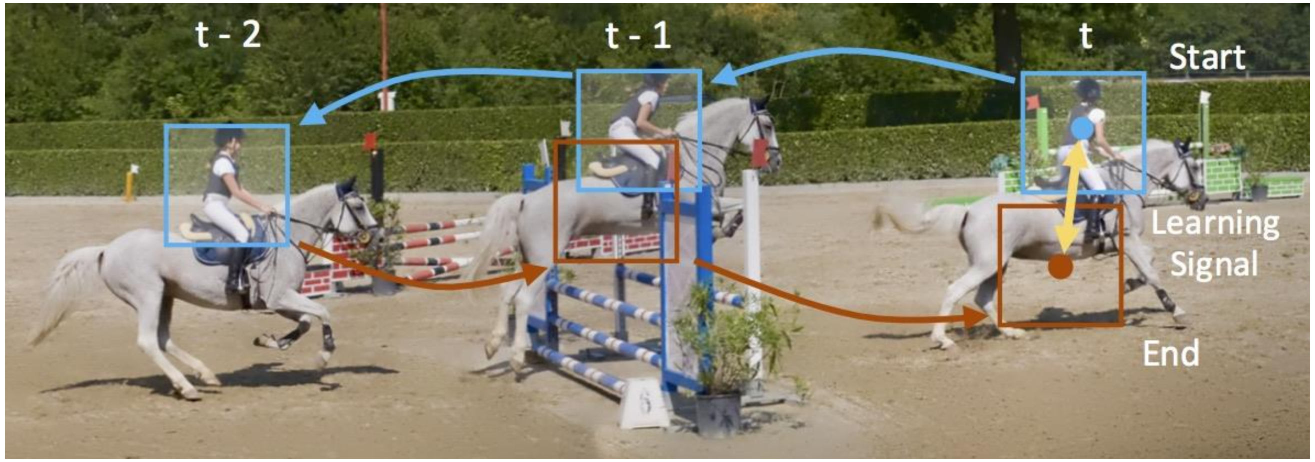
Huang, Xun, et al. "Multimodal unsupervised image-to-image translation." *Proceedings of the European Conference on Computer Vision (ECCV)*. 2018.

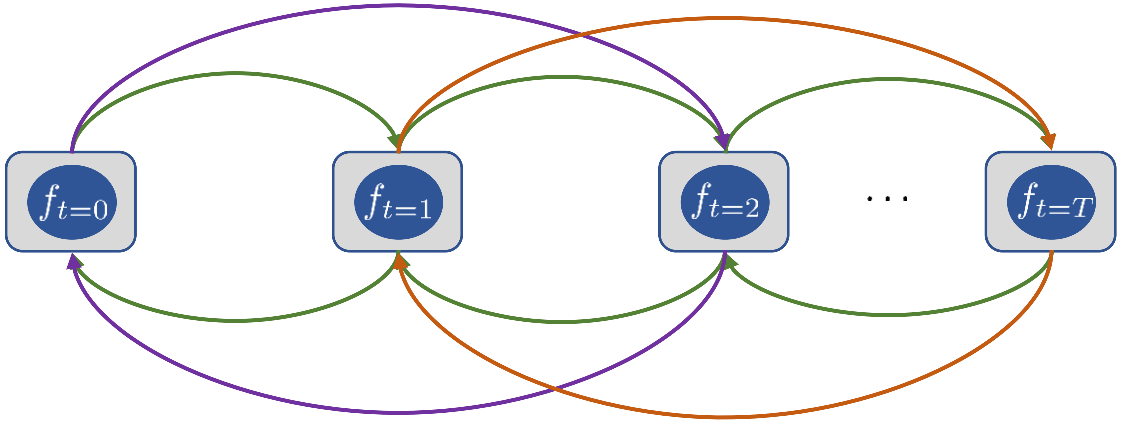
PI & CC as Unsupervised Losses



Zhang, Zaiwei, et al. "Path-invariant map networks."
Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition. 2019.

PI & CC as Unsupervised Losses





Wang, Xiaolong et al. "Learning correspondence from the cycle-consistency of time." *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2019.

Synchronization

the art of handshaking

Given noisy relative maps $\{f_{i,j}\}$ as observations, recover the absolute maps $\{f_i\}$.

Given relative maps, find a consistent vertex labeling.

Synchronization

the art of handshaking

Naïve, over all the paths:

$$\arg\min_{\{f_k\}_k} \underbrace{\sum_{c \in \bar{\mathcal{C}}} \epsilon(f_c, f_\varnothing)}_{\text{All cycles are null.}} + \sum_{(i,j)\in\mathcal{E}} \underbrace{d_f(f_i \circ f_{i,j}, f_j)}_{\text{Pairwise consistency}}$$

Group valued version:

$$\arg\min_{\{f_k\}_k} \sum_{(i,j)\in\mathcal{E}} \rho\big(d(f_i \circ f_j^{-1}, f_{i,j})\big)$$

Sync via path invariance:

$$\arg\min_{\{f_k\}_k} \sum_{(i,j)\in\mathcal{E}} d_f(f_i \circ f_{i,j}, f_j) + \underbrace{\sum_{(p,q)\in\mathcal{B}} \mathrm{E}\epsilon_B(f_p, f_q)}_{\text{Path Invariance}}.$$

Synchronization

the art of handshaking

Naïve, over all the paths:

$$\arg\min_{\{f_k\}_k} \underbrace{\sum_{c \in \bar{\mathcal{C}}} \epsilon(f_c, f_\varnothing)}_{\text{All cycles are null.}} + \sum_{(i,j)\in\mathcal{E}} \underbrace{d_f(f_i \circ f_{i,j}, f_j)}_{\text{Pairwise consistency}}$$

Group valued version:

$$\arg\min_{\{f_k\}_k} \sum_{(i,j)\in\mathcal{E}} \rho\big(d(f_i \circ f_j^{-1}, f_{i,j})\big)$$

Synchronization: Algorithms

The case of linear maps.

$$\arg\min_{\{f_k\}_k} \sum_{(i,j)\in\mathcal{E}} \rho\left(d(f_i \circ f_j^{-1}, f_{i,j})\right) \longrightarrow$$

Similar to a Dirichlet energy when $\rho(\mathbf{x}) = \mathbf{x}$

*i.e. we seek to find a **harmonic**.*

Matrix point of view for linear maps:

- $\mathbf{F}_{i,j} \in G$ represents $f_{i,j}$
- $\mathbf{I}$ is the identity

$$\longrightarrow \quad \mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{F}_{1,2} & \mathbf{F}_{1,3} & \cdots & \mathbf{F}_{1,n} \\ \mathbf{F}_{2,1} & \mathbf{0} & \mathbf{F}_{2,3} & \cdots & \mathbf{F}_{2,n} \\ \cdots & \cdots & \vdots & \cdots & \\ \mathbf{F}_{n,1} & \mathbf{F}_{n,2} & \mathbf{F}_{n,3} & \cdots & \mathbf{F}_{n,n} \end{bmatrix} \succcurlyeq 0 \qquad \mathbf{Z} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \vdots \\ \mathbf{F}_n \end{bmatrix} \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \cdots & \mathbf{F}_n \end{bmatrix}$$

Synchronization: Algorithms

The case of invertible linear maps.

$$\arg\min_{\{f_k\}_k} \sum_{(i,j)\in\mathcal{E}} \rho\big(d(f_i \circ f_j^{-1}, f_{i,j})\big) \qquad \equiv \qquad \arg\min_{\{\mathbf{F}_k\in\mathcal{M}\}_{k=1}^{K}} \sum_{(i,j)\in\mathcal{E}} w_{ij} \|\mathbf{F}_i\mathbf{F}_j^{-1} - \mathbf{F}_{ij}\|_F^2$$

Semidefinite Relaxation (SDP)

*The case of **orthogonal** linear maps.*

$$\underset{\{\mathbf{F}_k \in \mathcal{M}\}_{k=1}^K}{\arg\min} \sum_{(i,j)\in\mathcal{E}} w_{ij} \|\mathbf{F}_i \mathbf{F}_j^{-1} - \mathbf{F}_{ij}\|_F^2 \quad \equiv \quad \underset{\{\mathbf{F}_k \in \mathcal{M}\}_{k=1}^K}{\arg\max} \sum_{(i,j)\in\mathcal{E}} w_{ij} \operatorname{Tr}(\mathbf{F}_i^\top \mathbf{F}_{i,j} \mathbf{F}_j)$$

which can be re-arranged into:

$$\max_{\mathbf{F}} \operatorname{Tr}(\mathbf{F}^\top \mathbf{G} \mathbf{F})$$

semi-definite relaxation:

$$\max_{\hat{\mathbf{G}} \succcurlyeq 0, \mathbf{G}_{ii} = \mathbf{I}} \operatorname{Tr}(\mathbf{G}\hat{\mathbf{G}})$$

where $\hat{\mathbf{G}} = \mathbf{F}^\top \mathbf{F}$

Spectral Solutions

$$\max_{\mathbf{G}} \mathrm{Tr}(\mathbf{F}^\top \mathbf{G}\mathbf{F}) \ \text{ s.t. } \ \mathbf{F}^\top \mathbf{F} = \mathbf{I}$$

Rayleigh Problem

Solution is given by the eigenvectors of **G***.*

Optimization on Riemannian Manifolds

Riemannian Metric :

$\mathbf{G} = \langle \cdot, \cdot \rangle_{\mathcal{T}_X M} : \mathcal{T}_X M \times \mathcal{T}_X M \rightarrow \mathbb{R}$. e. g. In $\mathbb{R}^2, G = dx^2 + dy^2$.

Exponential Map :

$\exp_X : \mathcal{T}_X M \rightarrow X$ maps a tangent vector to the endpoint of a geodesic path.

Logarithmic Map :

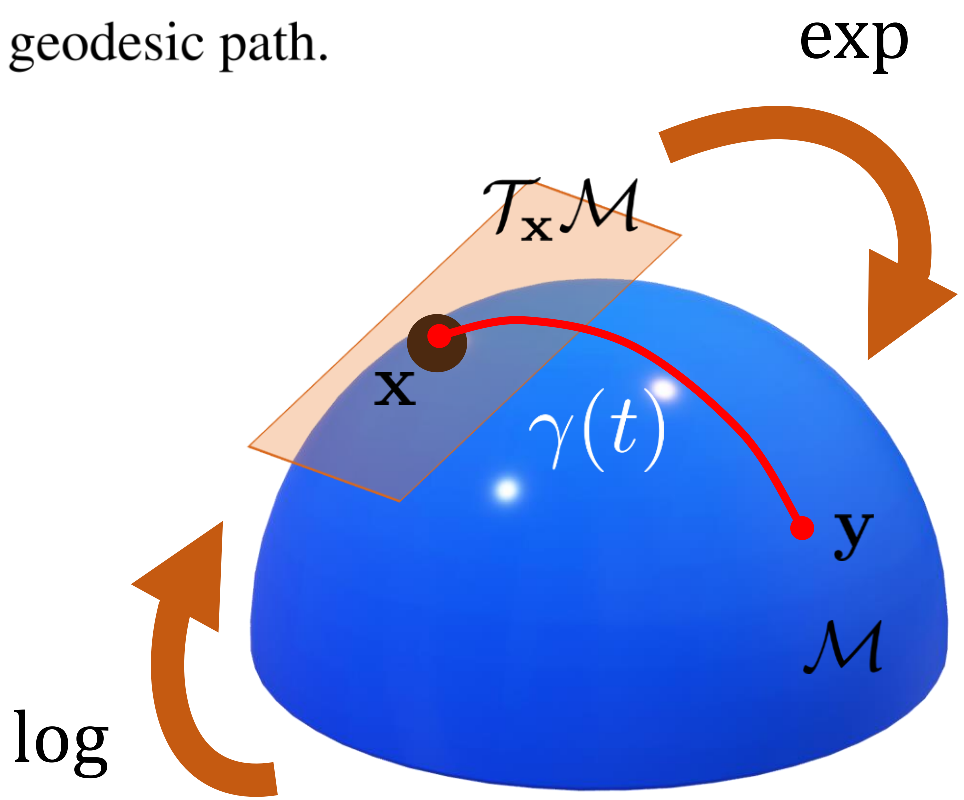
Inverse of the exponential map $\log_X : X \rightarrow \mathcal{T}_X M$

Isometry is a metric-preserving deformation.

A geodesic path γ is length minimizing and constant speed

$d(x, y) = \min l_x^y(\gamma)$

On $\mathcal{M} : l_x^y = \int_x^y \sqrt{G(\dot{\gamma}(t), \dot{\gamma}(t))} dt$



Riemannian Descent

Riemannian Manifolds + Group Structure = Lie Group

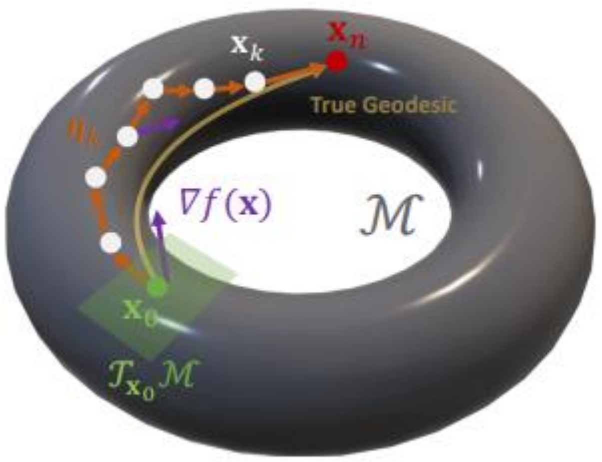
$$\underset{\boxed{\{\mathbf{F}_k \in \mathcal{M}\}_{k=1}^{K}}}{\arg\min} \sum_{(i,j)\in\mathcal{E}} w_{ij} \|\mathbf{F}_i\mathbf{F}_j^{-1} - \mathbf{F}_{ij}\|_F^2$$

Algorithm 1: General Riemannian Line Search Minimizer

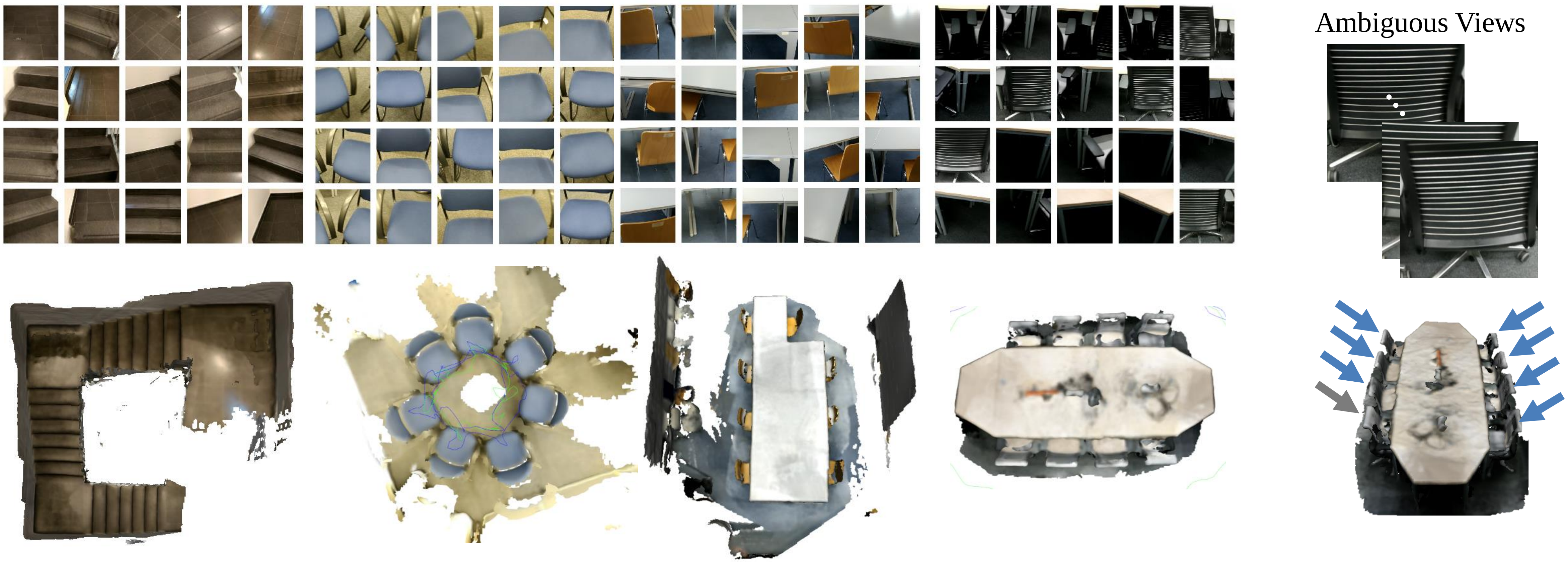
input: A Riemannian manifold $\mathcal{M}$, a retraction operator R and initial iterate $\mathbf{x}_k \in \mathcal{M}$ where $k = 0$.

while $\mathbf{x}_k$ *does not sufficiently minimize* f **do**

- Pick a gradient related descent direction $\eta_k \in \mathcal{T}_{\mathbf{x}_k}\mathcal{M}$.
- Choose a retraction $R_{\mathbf{x}_k} : \mathcal{T}_{\mathbf{x}_k}\mathcal{M} \to \mathcal{M}$.
- Choose a step length $\tau_k \in \mathbb{R}$.
- Set $\mathbf{x}_{k+1} \leftarrow R_{\mathbf{x}_k}(\tau_k \boldsymbol{\eta}_k)$.
- $k \leftarrow k + 1$.



Real life is challenging.



Uncertainty

Ambiguities

Bui, Mai, Tolga Birdal, Haowen Deng, Shadi Albarqouni, Leonidas Guibas, Slobodan Ilic, and Nassir Navab. "6D Camera Relocalization in Ambiguous Scenes via Continuous Multimodal Inference." *arXiv preprint arXiv:2004.04807* (2020).

Multiple Rotation Averaging

Camera poses form non-Euclidean parameter spaces

$$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \in SE(3)$$

$$\mathbf{R} \in SO(3) \qquad \mathbf{t} \in \mathbb{R}^3$$

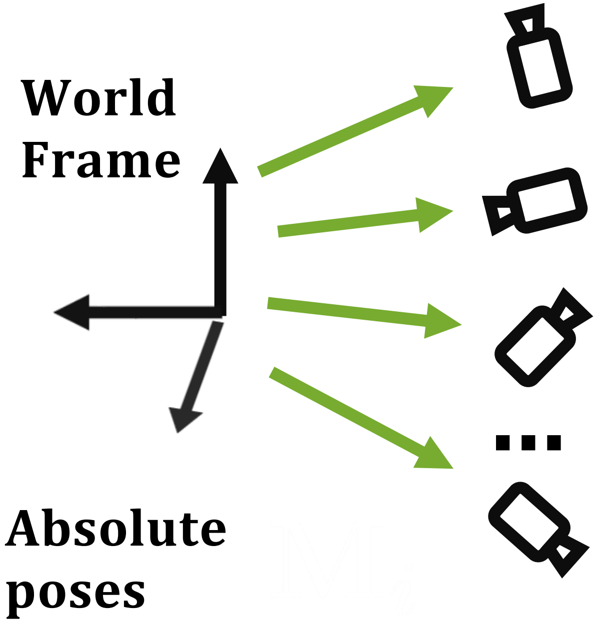
$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

The manifold of rotations can be parameterized in many ways.
We will use **quaternions**.

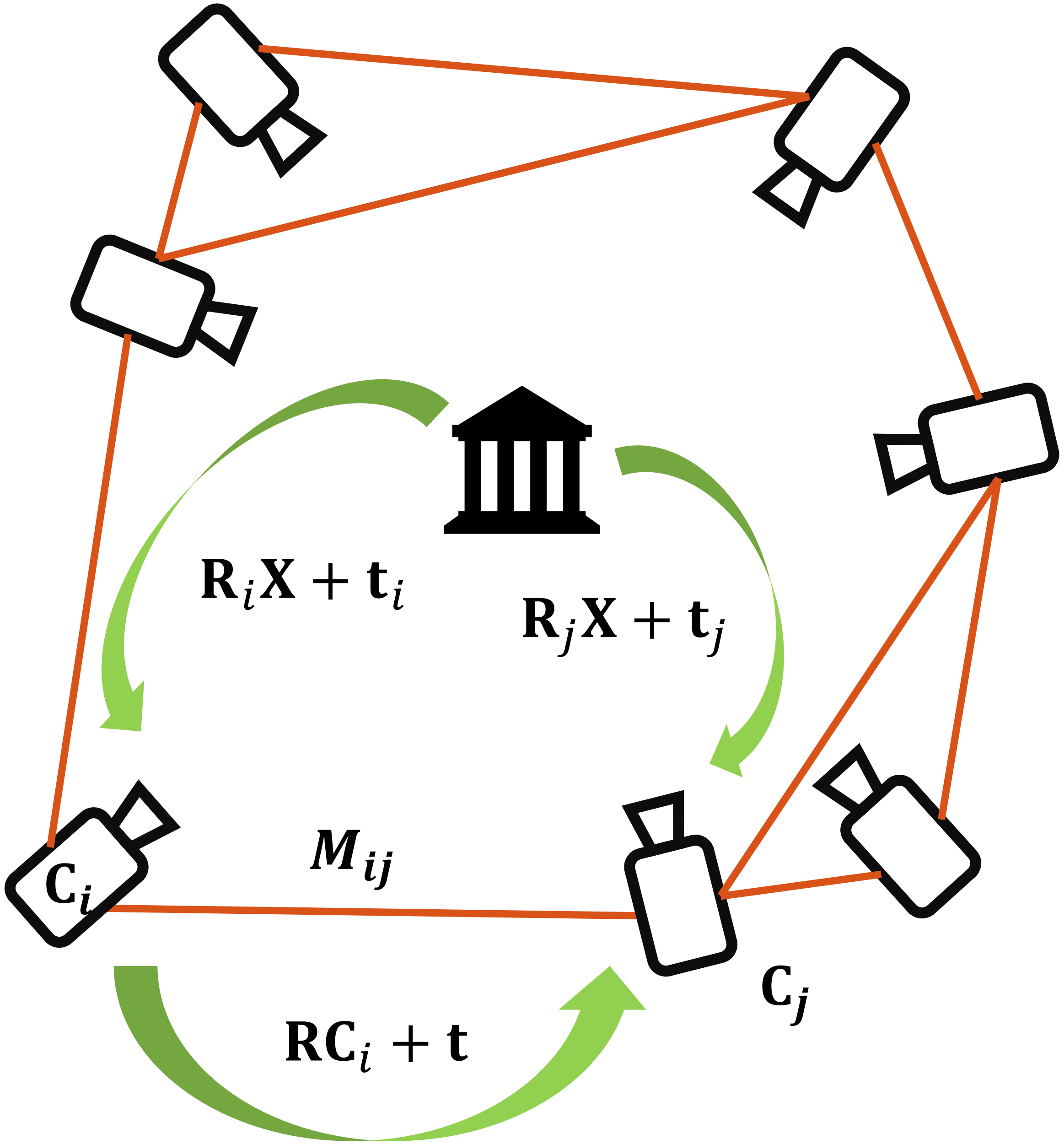
Multiple Motion Averaging

$$\mathbf{M}_{ij} \approx \mathbf{M}_j \mathbf{M}_i^{-1}, \forall i \neq j$$

(cycle consistency constraint)



+ estimate **uncertainties**



Birdal, Tolga, et al. "Synchronizing Probability Measures on Rotations via Optimal Transport" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2020.

Quaternions

- Introduced by Hamilton
- Extends complex numbers:
- Quaternions lie on a 4D unit sphere:
- Quaternions are antipodally symmetric
- Natural and good way to parameterize
- Rotate a point by quaternion:
- Quaternion distance:
- Easily relate to axis-angle:

$$\mathbf{q} = \cos(\tfrac{\theta}{2}) + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin(\tfrac{\theta}{2})$$

$$\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

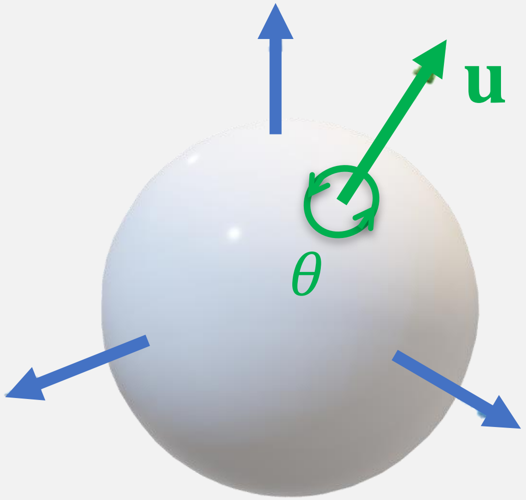
$$\|\mathbf{q}\| = 1$$

$$\mathbf{q} \equiv -\mathbf{q}$$

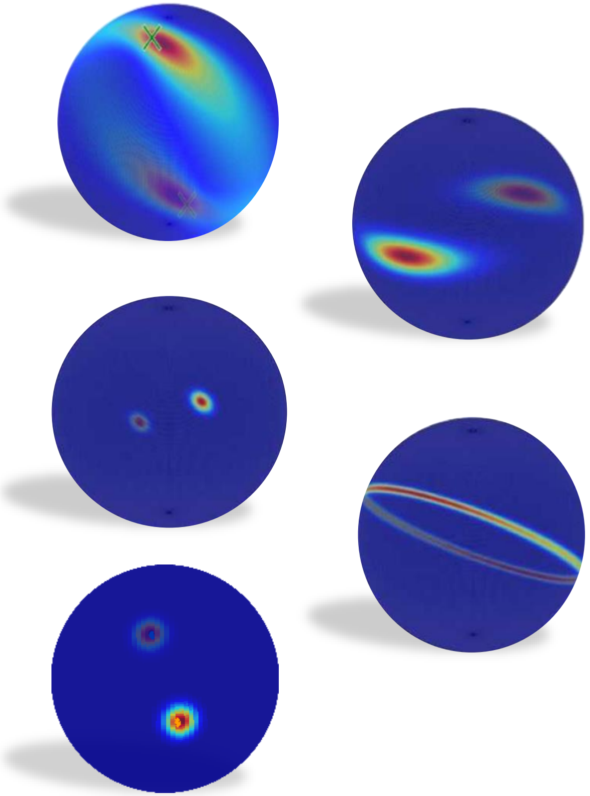
$$\mathbf{R} \leftrightarrow \mathbf{q}$$

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$$

$$d(\mathbf{q}_1, \mathbf{q}_2) = 2\arccos(|\mathbf{q}_1\bar{\mathbf{q}}_2|)$$



Bingham Distribution



$$\mathcal{B}(\mathbf{x};\boldsymbol{\Lambda},\mathbf{V}) = (1/F)\exp(\mathbf{x}^T\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T\mathbf{x})$$
$$= (1/F)\exp\left(\sum_{i=1}^{d}\lambda_i(\mathbf{v}_i^T\mathbf{x})^2\right)$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} 0, & & & & \\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_{d-1} \end{bmatrix}_{d\times d}$$

$(0 \geq \lambda_1 \geq \ldots \geq \lambda_{d-1})$

$$F \triangleq |S_{d-1}| \cdot {}_1F_1\left(\frac{1}{2}, \frac{d}{2}, \boldsymbol{\Lambda}\right)$$

- $\mathbf{V} \in \mathbb{R}^{d\times d}$: an orthogonal matrix
- $\boldsymbol{\Lambda} \in \mathbb{R}^{d\times d}$: *concentration matrix*, a diagonal matrix with a set of non-positive values
- $F(\boldsymbol{\Lambda})$: the normalization constant
- A variant of zero-mean Gaussian distribution conditioned on $\mathbb{S}^{d-1}$

Bingham, Christopher. "An antipodally symmetric distribution on the sphere." The Annals of Statistics (1974): 1201-1225.

Tempered Geodesic MCMC Framework for Sampling and Optimization

A Bayesian Formulation for Synchronization on *SE(3)*

1. The maximum a-posteriori (MAP) estimate:

$$(\mathbf{Q}^\star, \mathbf{T}^\star) = \arg\max_{\mathbf{Q},\mathbf{T}} p(\mathbf{Q}, \mathbf{T} | \mathcal{D}) =$$

$$\arg\max_{\mathbf{Q},\mathbf{T}} \Bigg(\underbrace{\sum_{(i,j)\in E} \Big\{ \log p(\mathbf{q}_{ij} | \mathbf{Q}, \mathbf{T}) + \log p(\mathbf{t}_{ij} | \mathbf{Q}, \mathbf{T}) \Big\}}_{\text{Likelihood}} + \underbrace{\sum_i \log p(\mathbf{q}_i) + \sum_i \log p(\mathbf{t}_i)}_{\text{Prior}} \Bigg),$$

2. The full posterior distribution:

$$p(\mathbf{Q}, \mathbf{T} | \mathcal{D}) \propto p(\mathcal{D} | \mathbf{Q}, \mathbf{T}) \times p(\mathbf{Q}) \times p(\mathbf{T})$$

Birdal, Tolga, et al. “Bayesian Pose Graph Pptimization via Bingham Distributions and Tempered Geodesic MCMC" *Proceedings of the Neural Information Processing Systems*. 2018.

Inference: Tempered Geodesic MCMC

Markov process on the Riemannian manifold can be simulated in the embedded Euclidean space:

Inverse temperature

$$d\tilde{\mathbf{x}}_t = \mathbf{G}(\tilde{\mathbf{x}}_t)^{-1}\mathbf{p}_t dt$$

$$d\mathbf{p}_t = -\Big(\nabla_{\tilde{\mathbf{x}}} U_\lambda(\tilde{\mathbf{x}}_t) + \frac{1}{2}\nabla_{\tilde{\mathbf{x}}} \log|\mathbf{G}| + c\mathbf{p}_t + \frac{1}{2}\nabla_{\tilde{\mathbf{x}}}(\mathbf{p}_t^\top \mathbf{G}^{-1}\mathbf{p}_t)dt + \sqrt{(2c/\beta)\mathbf{M}^\top\mathbf{M}}\, dW_t$$

Hamiltonian Dynamics

momentum

Riemannian metric

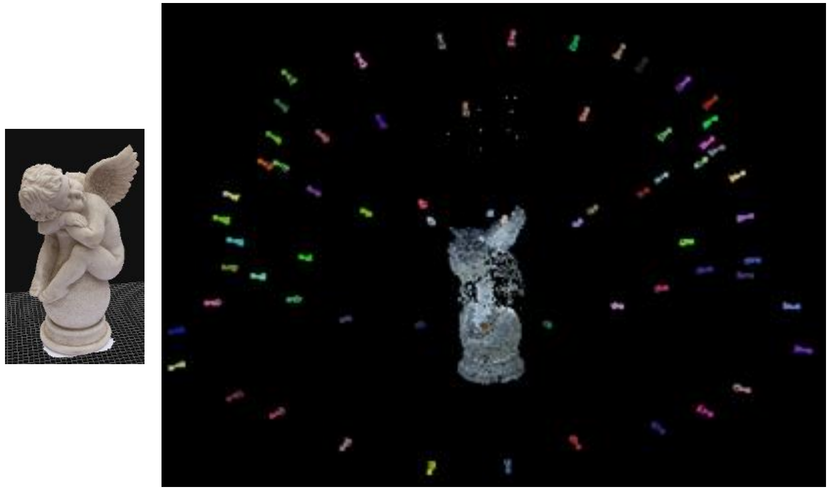
friction

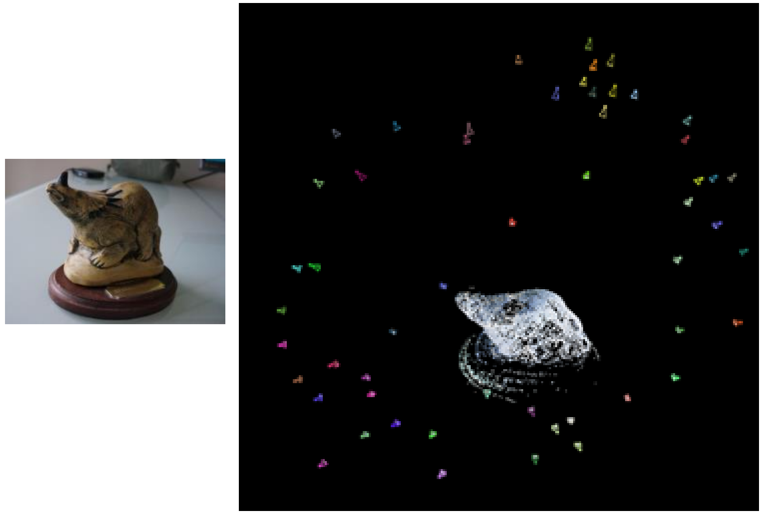
Brownian motion

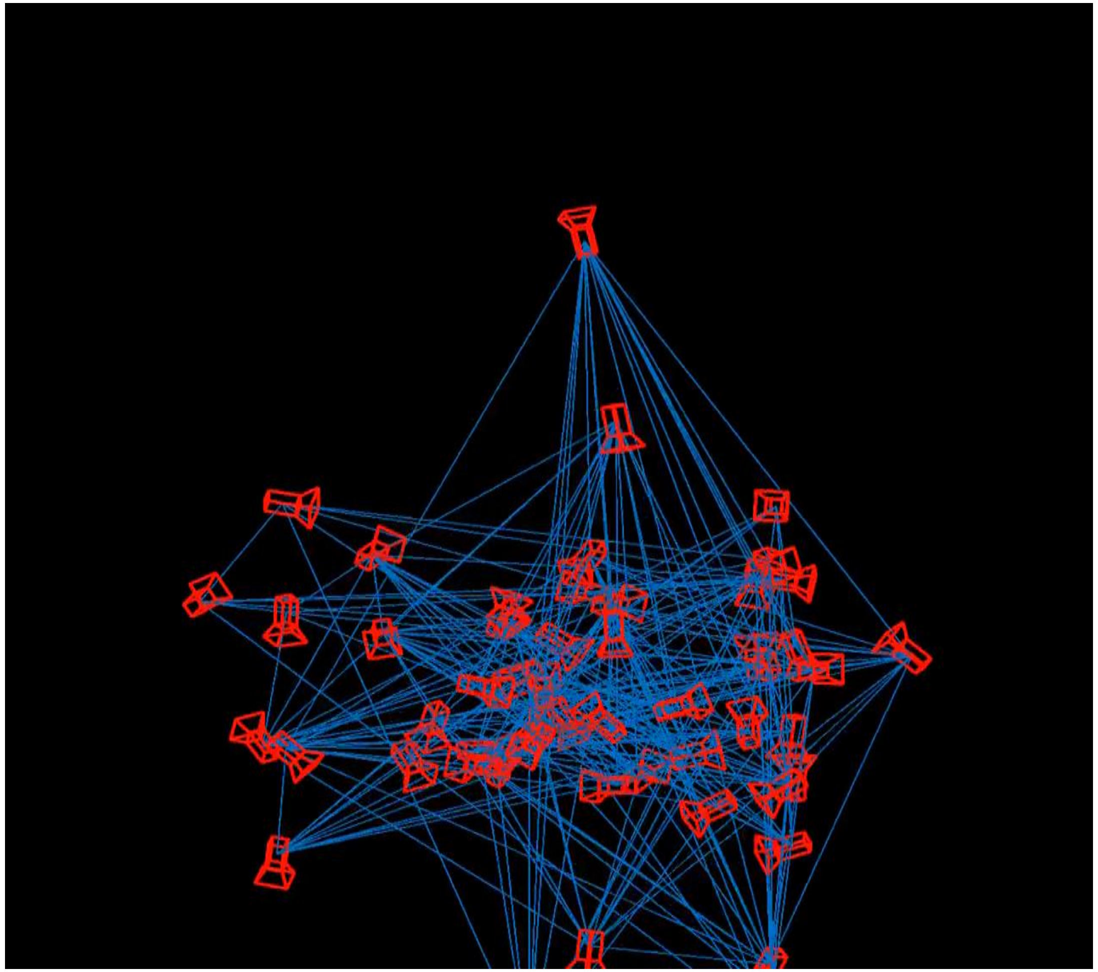
1. Eventually, we will provide samples close to the global minimum (even when non-convex).
2. Suffers from *meta-stability phenomenon* (exponential time required).

Birdal, Tolga, et al. "Bayesian Pose Graph Pptimization via Bingham Distributions and Tempered Geodesic MCMC" *Proceedings of the Neural Information Processing Systems*. 2018.

Solving the Classical Problem

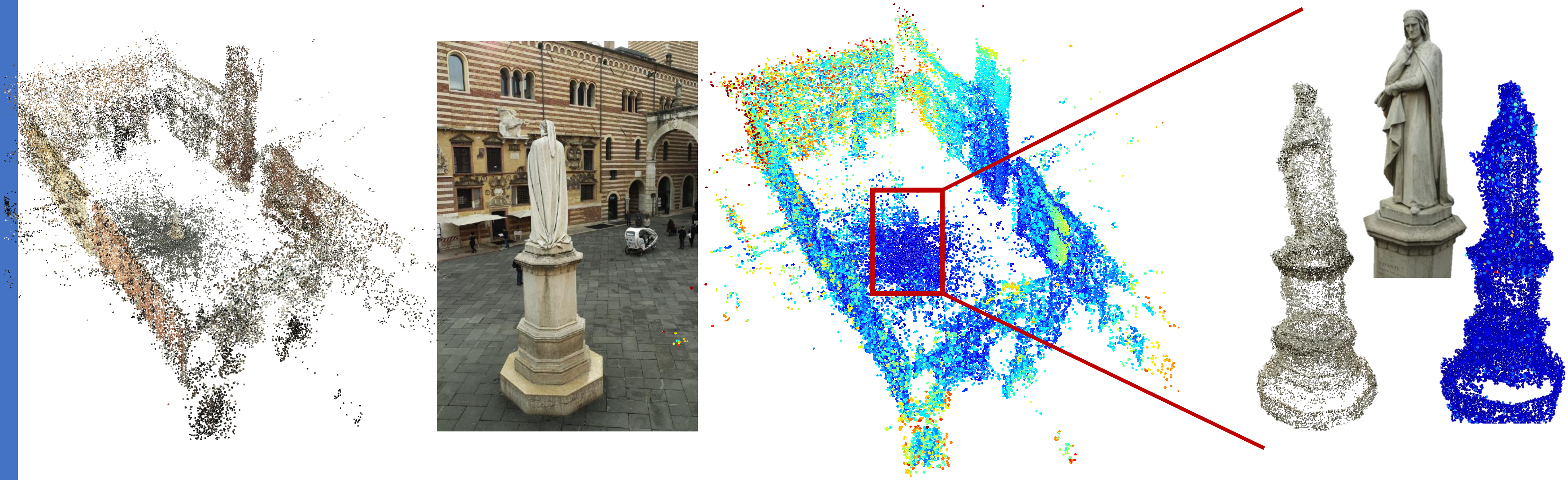






Birdal, Tolga, et al. “Bayesian Pose Graph Pptimization via Bingham Distributions and Tempered Geodesic MCMC" *Proceedings of the Neural Information Processing Systems*. 2018.

Estimating Uncertainty



Birdal, Tolga, et al. “Bayesian Pose Graph Pptimization via Bingham Distributions and Tempered Geodesic MCMC" *Proceedings of the Neural Information Processing Systems*. 2018.

Estimating Uncertainty

(a) Madrid Metropolis **(b)** 3D Reconstruction **(c)** Uncertainty Map

Birdal, Tolga, et al. “Bayesian Pose Graph Pptimization via Bingham Distributions and Tempered Geodesic MCMC" *Proceedings of the Neural Information Processing Systems*. 2018.

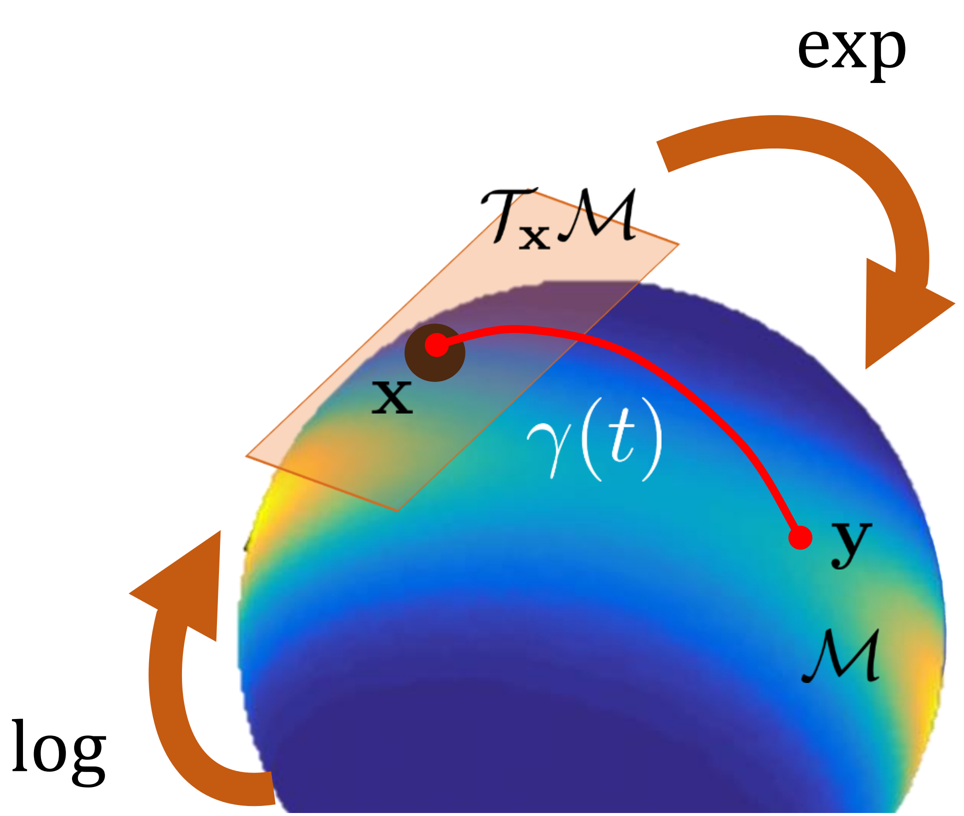
A New Probabilistic Look to Synchronization

Given a Riemannian Manifold $\mathcal{M}$

Analytical Geodesic Flow

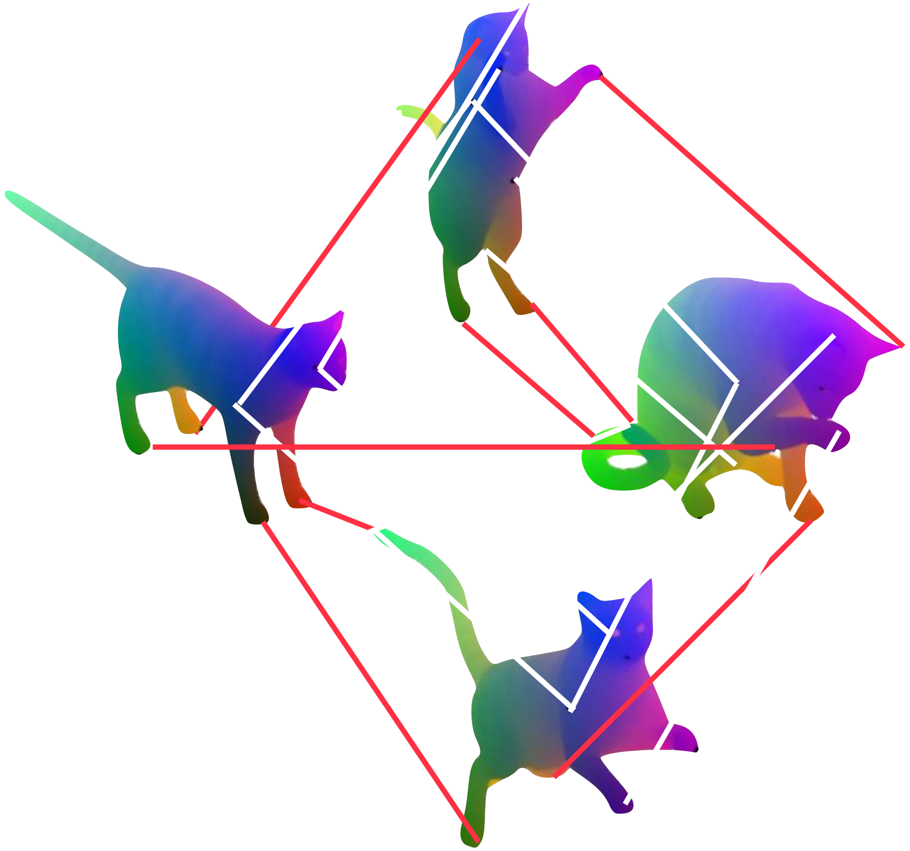
A Suitable Probability Distribution

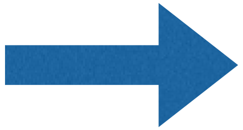
Configurable
Sampling & Optimization

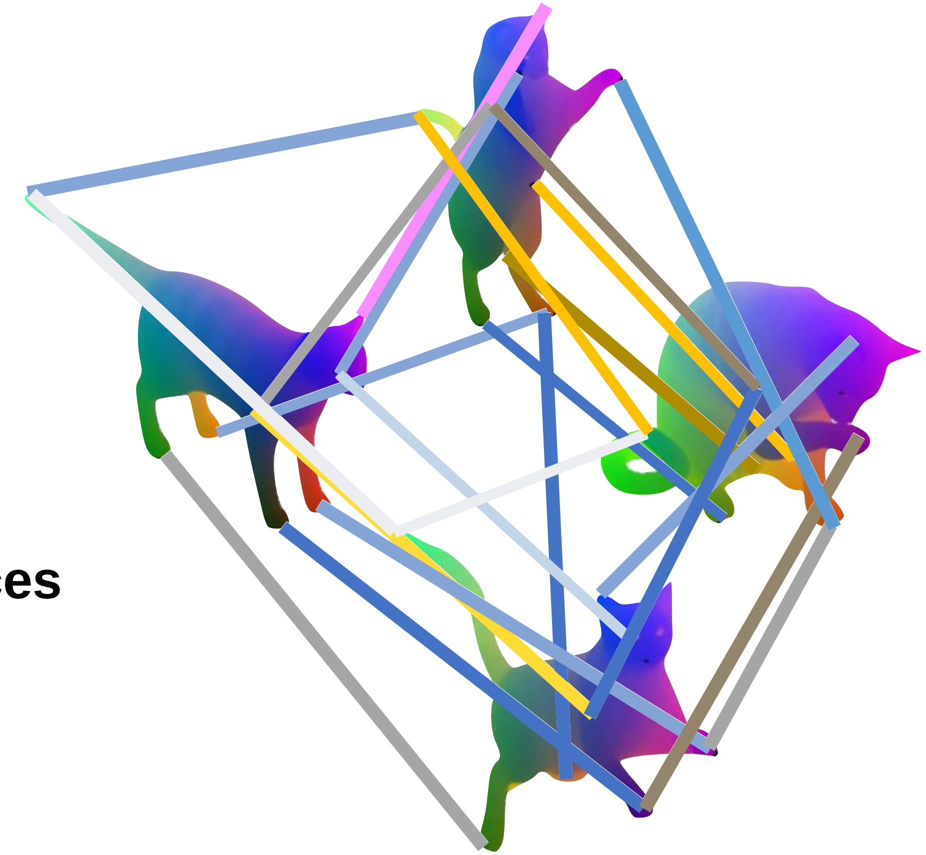


Birdal, Tolga, et al. "Bayesian Pose Graph Pptimization via Bingham Distributions and Tempered Geodesic MCMC" *Proceedings of the Neural Information Processing Systems*. 2018.

Extension to Correspondence Synchronization (Multi-Graph Matching)







Birdal, Tolga, et al. "Probabilistic Permutation Synchronization using the Riemannian Structure of the Birkhoff Polytope" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2019.

Extension to Correspondence Synchronization (Multi-Graph Matching)

$$\mathbf{X}^\star = \mathop{\arg\min}_{\boxed{\mathbf{X}\in\mathcal{P}_n^N}} \left\{ U(\mathbf{X}) := \sum_{(i,j)\in E} \|\mathbf{P}_{ij} - \mathbf{X}_i \mathbf{X}_j^\top\|_\mathrm{F}^2 \right\}$$

Direct optimization on permutations is difficult.

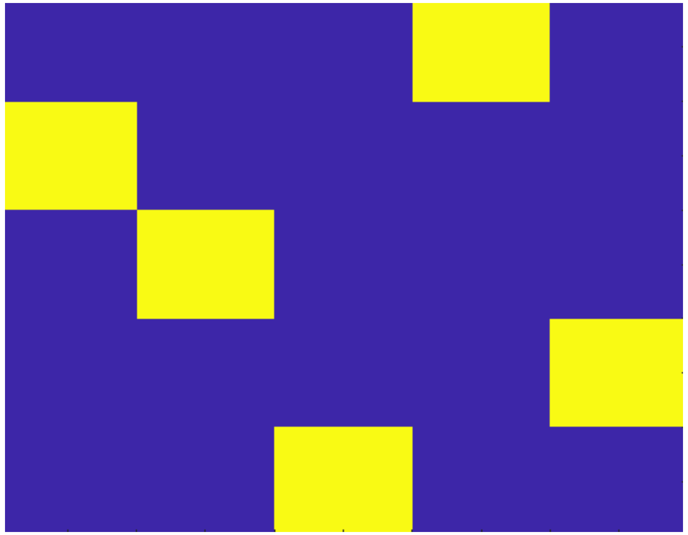
Birdal, Tolga, et al. "Probabilistic Permutation Synchronization using the Riemannian Structure of the Birkhoff Polytope" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2019.

Extension to Correspondence Synchronization (Multi-Graph Matching)

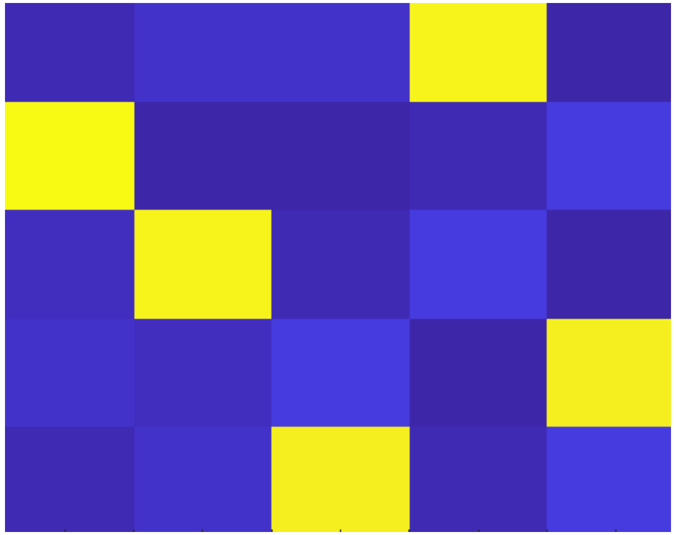
Relaxation onto Doubly Stochastic Matrices

$$\mathcal{DP}_n = \{\ \mathbf{X} \in \mathbb{R}_{+}^{n\times n} : \sum_{i=1}^{n} x_{ij} = 1 \wedge \sum_{j=1}^{n} x_{ij} = 1\ \}.$$

P



X



Birdal, Tolga, et al. "Probabilistic Permutation Synchronization using the Riemannian Structure of the Birkhoff Polytope" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2019.

Extension to Correspondence Synchronization (Multi-Graph Matching)

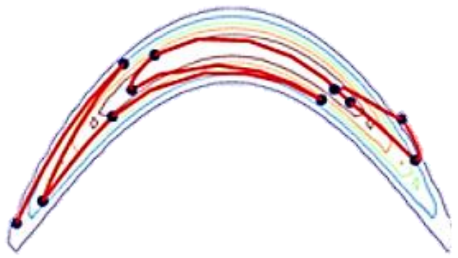
$$\mathbf{X}^\star = \underset{\boxed{\mathbf{X}\in\mathcal{DP}_n^N}}{\arg\min}\left\{U(\mathbf{X}) := \sum_{(i,j)\in E} \|\mathbf{P}_{ij} - \mathbf{X}_i\mathbf{X}_j^\top\|_{\mathrm{F}}^2\right\}$$

Optimize using Riemannian LBFGS

Sample using Birkhoff RLMC

Birdal, Tolga, et al. "Probabilistic Permutation Synchronization using the Riemannian Structure of the Birkhoff Polytope" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2019.

Riemannian Langevin Monte Carlo



Riemannian Metric

$$\mathrm{d}\tilde{\mathbf{X}}_t = (-\mathbf{G}^{-1}\nabla_{\tilde{\mathbf{X}}} U_\lambda(\tilde{\mathbf{X}}_t) + \mathbf{\Gamma}_t)\mathrm{d}t + \sqrt{2/\beta\mathbf{G}^{-1}}\mathrm{dB}_t$$

Euclidean Embedding

Riemannian Correction

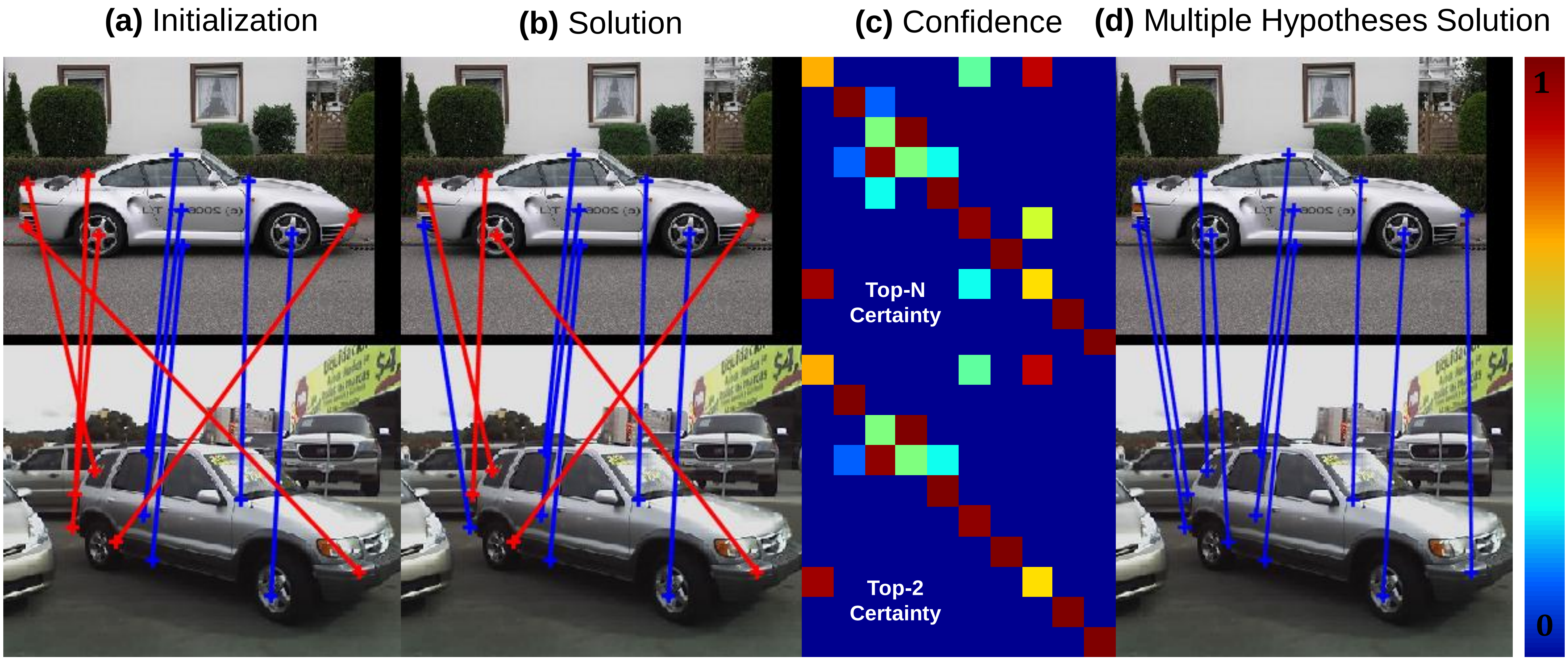
Brownian Motion

Embedded Posterior Density

$$\pi_{\mathcal{H}}(\mathbf{x}) = \pi_\lambda(\tilde{\mathbf{X}})/\sqrt{|\mathbf{G}(\tilde{\mathbf{X}})|}$$

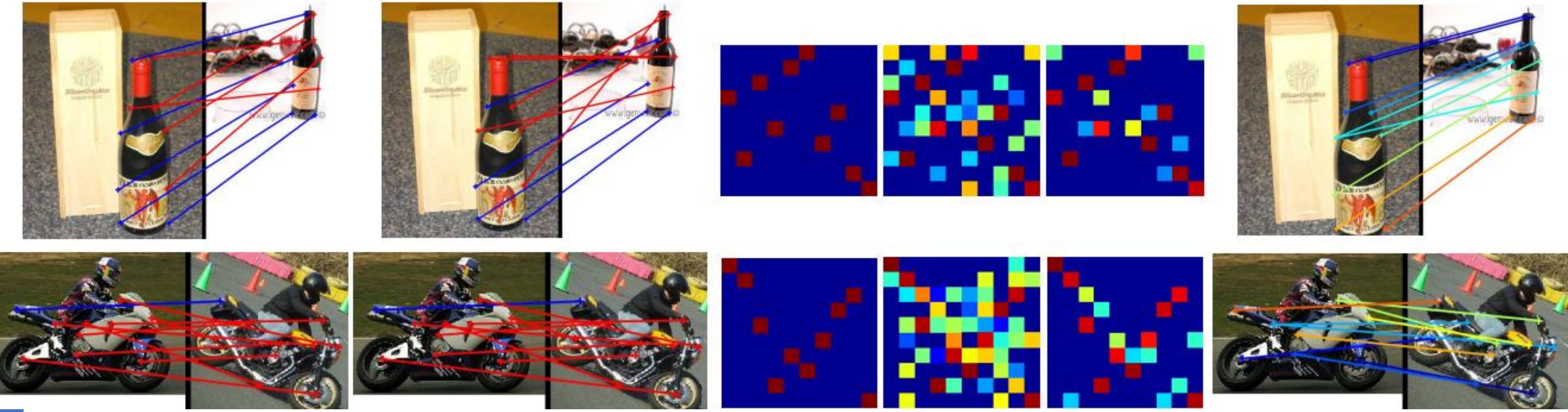
Birdal, Tolga, et al. "Probabilistic Permutation Synchronization using the Riemannian Structure of the Birkhoff Polytope" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2019.

Minimizing Objective & Estimating Uncertainty



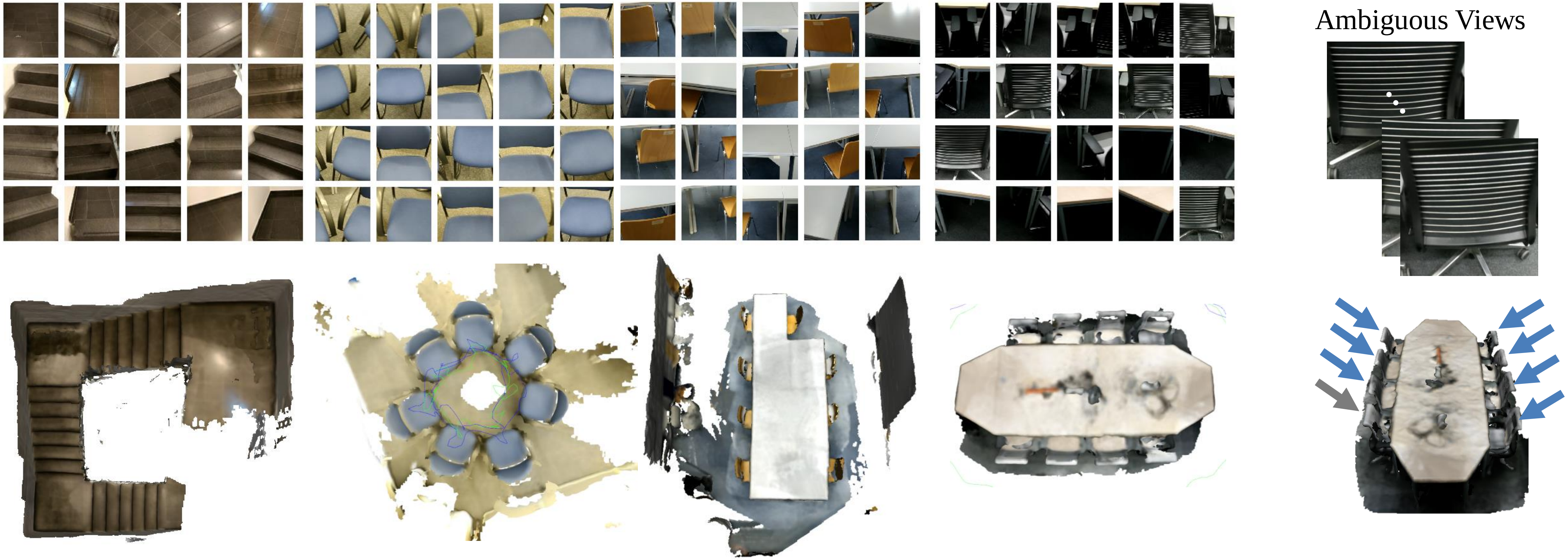
Birdal, Tolga, et al. "Probabilistic Permutation Synchronization using the Riemannian Structure of the Birkhoff Polytope" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2019.

Minimizing Objective & Estimating Uncertainty



Birdal, Tolga, et al. "Probabilistic Permutation Synchronization using the Riemannian Structure of the Birkhoff Polytope" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2019.

Real life is challenging.



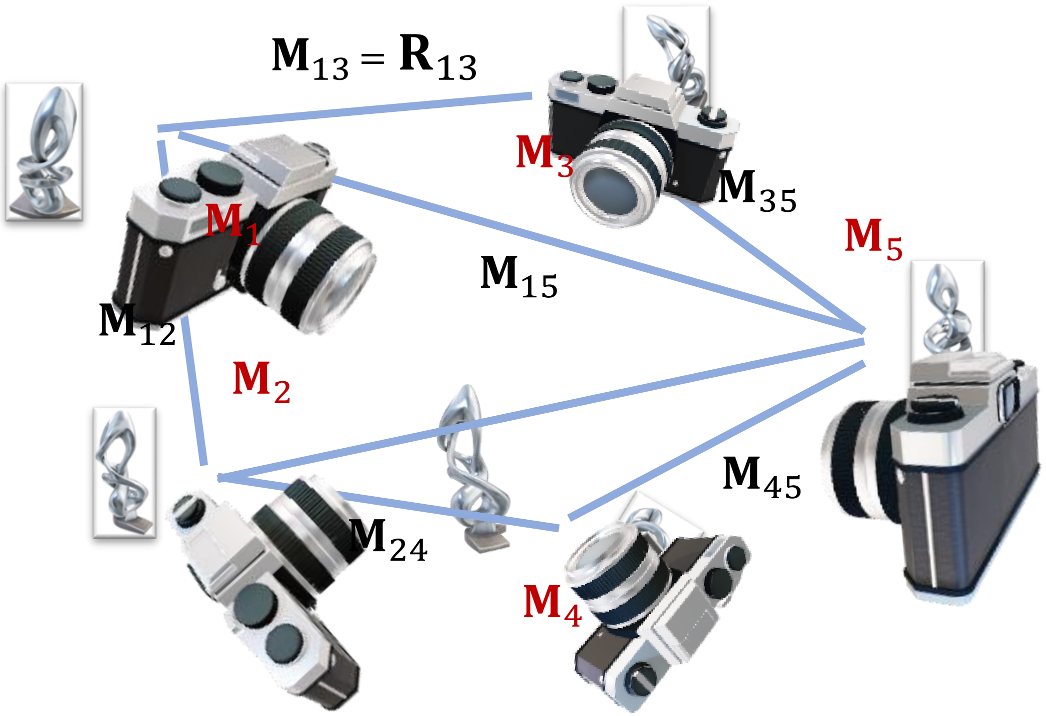
Uncertainty

Ambiguities

Bui, Mai, Tolga Birdal, Haowen Deng, Shadi Albarqouni, Leonidas Guibas, Slobodan Ilic, and Nassir Navab. "6D Camera Relocalization in Ambiguous Scenes via Continuous Multimodal Inference." *arXiv preprint arXiv:2004.04807* (2020).

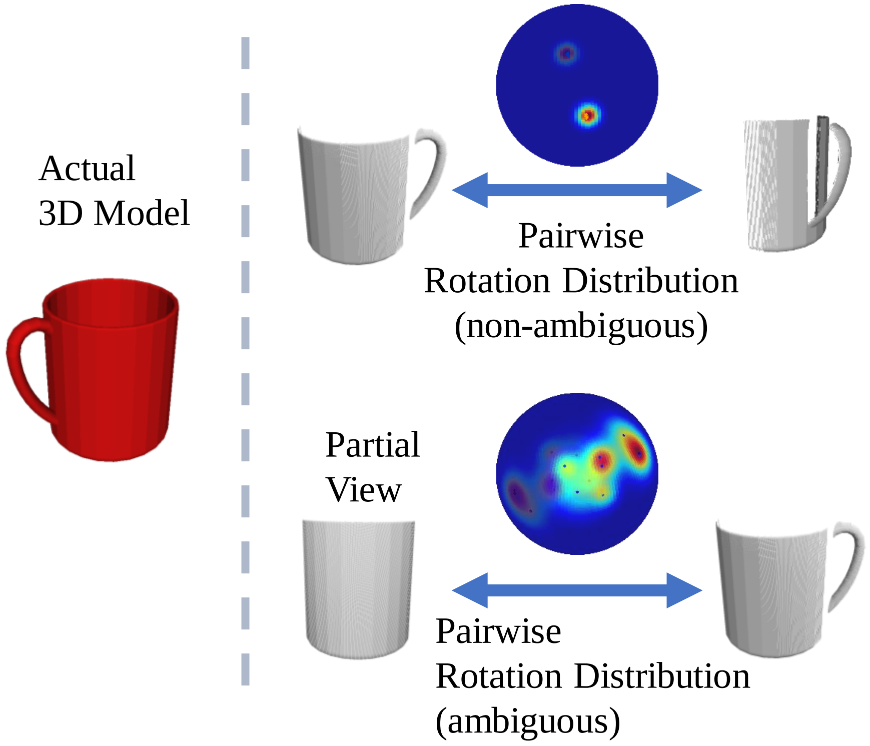
Synchronizing Distributions

Classical Synchronization



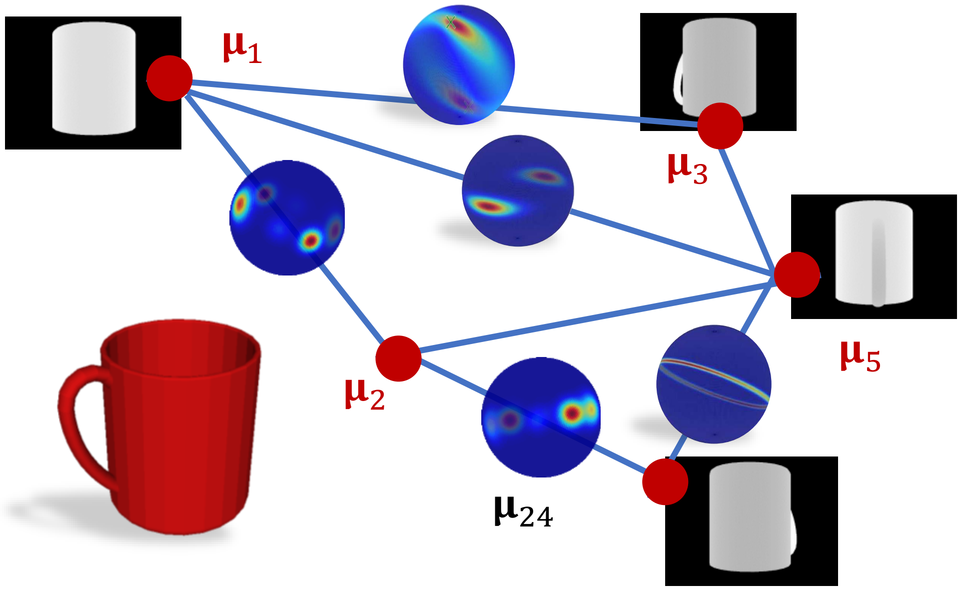
$\mathbf{M}_i$: **Absolute** rotation $\mathbf{M}_{ij}$: **Relative** rotation

The Ambiguity Problem



Gojcic, Zan, Caifa Zhou, Jan D. Wegner, Leonidas J. Guibas, and Tolga Birdal. "Learning multiview 3D point cloud registration." *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2020.

Synchronizing Distributions = Measure Synchronization



relative empirical probability measure $\quad \boldsymbol{\mu}_{ij} \triangleq \sum_{k=1}^{K_{ij}} w_{ij}^{(k)} \delta_{\mathbf{q}_{ij}^{(k)}}$

absolute (sought) probability measure $\quad \boldsymbol{\mu}_{i} \triangleq \sum_{k=1}^{K_{i}} w_{i}^{(k)} \delta_{\mathbf{q}_{i}^{(k)}}$

Classical Rotation Synchronization

$$\mathbf{R}_{ij} \approx \mathbf{R}_j \mathbf{R}_i^{-1}, \forall i \neq j$$

Measure Synchronization

$$\mathcal{D}(g_{ij}(\boldsymbol{\mu}), \boldsymbol{\mu}_{ij}) \to 0$$

Divergence (e.g. Wasserstein distance)

composition functions (relative rotation)

coupling (of absolute measures)

observed relative rotation

Birdal, Tolga, et al. "Synchronizing Probability Measures on Rotations via Optimal Transport" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2020.

How to Solve?

Classical Rotation Synchronization

$$\arg\min_{\{\mathbf{R}_i\}_i} \sum_{(i,j)\in\mathcal{E}} d_{\text{geo}}\left(\mathbf{R}_{ij}\ ,\ \mathbf{R}_j\mathbf{R}_i^T\right)$$

$$\text{s.t.} \qquad \mathbf{R}_i \in SO(3),\ \forall i \in \{1,\dots,n\}$$

Solved by many existing methods:

Spectral, Low-rank, Riemannian descent, Semi-definite programming, Lie algebraic and etc.

Riemannian Measure Synchronization

$$\min_{\boldsymbol{\mu}} \sum_{(i,j)\in\mathcal{E}} \mathcal{D}(g_{ij}(\boldsymbol{\mu}), \boldsymbol{\mu}_{ij}) + \mathcal{R}(\boldsymbol{\mu})$$

$$\begin{aligned}\text{s.t.}\quad & [w_i^1,\dots,w_i^{K_i}]^\top \in \mathcal{C}_i, && \forall i \in \{1,\dots,n\}\\ & \mu_i \in \mathbb{H}_1, && \forall i \in \{1,\dots,n\}\end{aligned}$$

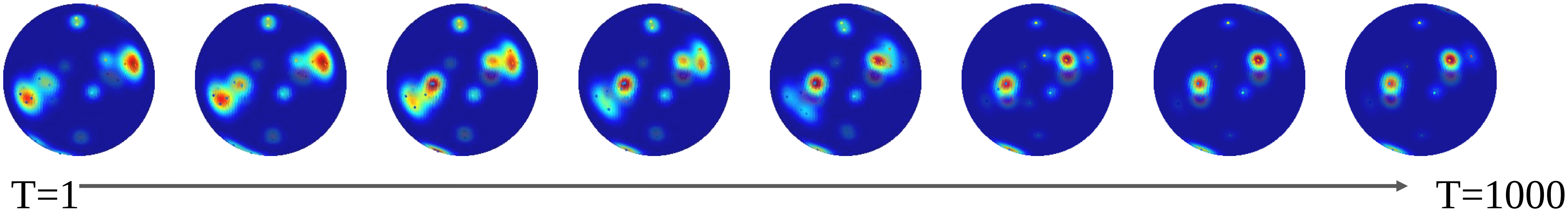
Multiple Optimal Transport Problems on the Lie Group of Quaternions

Proposed

Riemannian Particle Gradient Descent via **Geodesic Sinkhorn Divergences** with convergence guarantees

Birdal, Tolga, et al. "Synchronizing Probability Measures on Rotations via Optimal Transport" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2020.

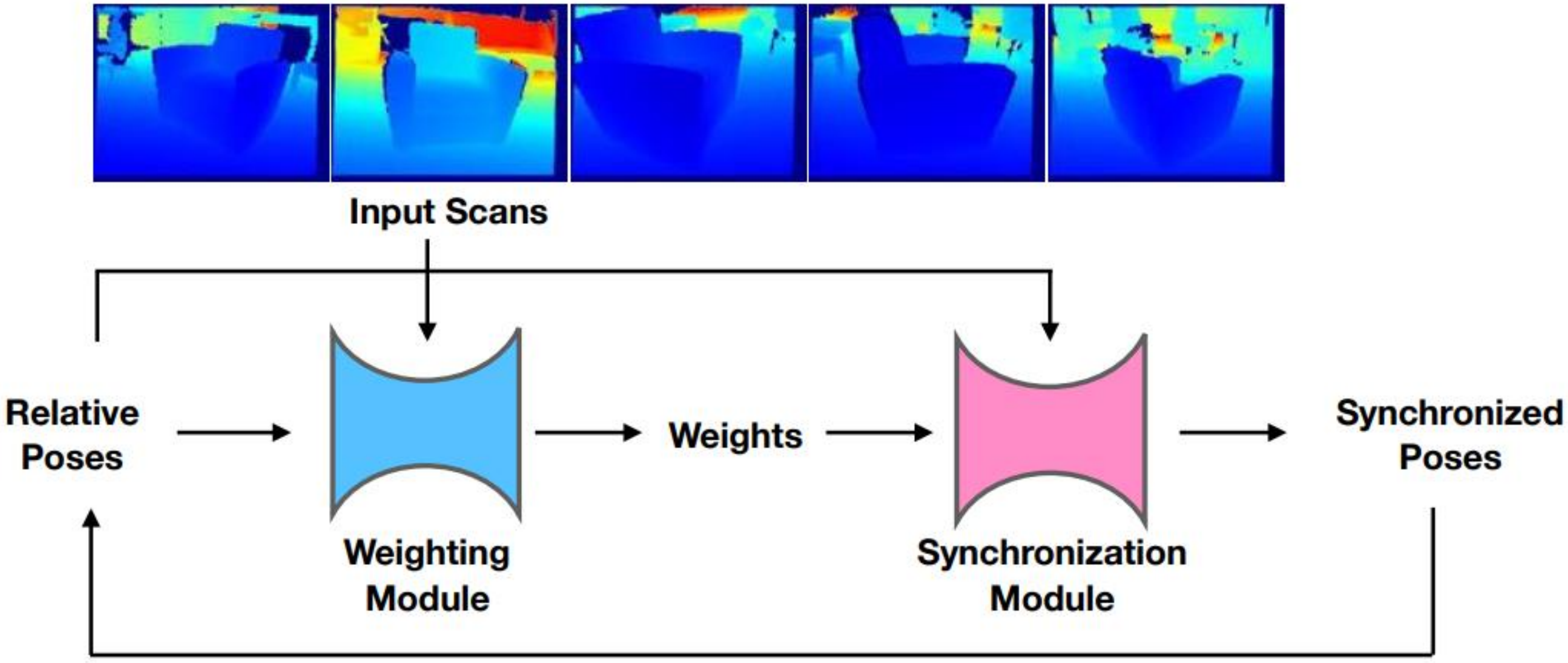
Theoretical Guarantees

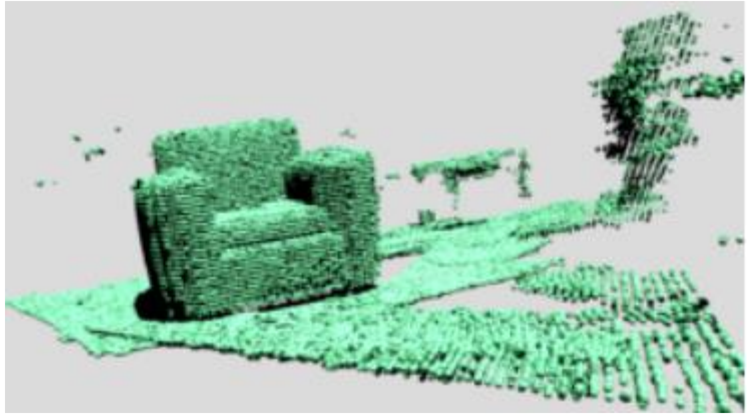


Our algorithm is shown to converge to the global solution under mild assumptions.

Birdal, Tolga, et al. "Synchronizing Probability Measures on Rotations via Optimal Transport" *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2020.

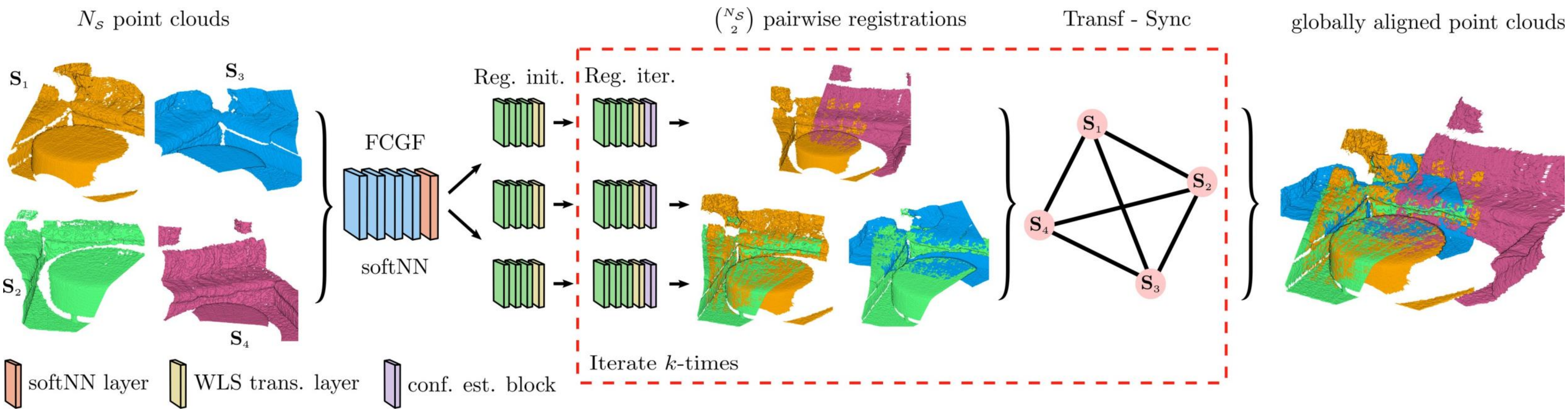
Use of Differentiable Synchronization in Neural Networks





Huang, Xiangru, et al. "Learning transformation synchronization." *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2019.

Use of Differentiable Synchronization in Neural Networks



Feature Extraction → Iterative Pairwise Registration → Synchronization

Gojcic, Zan, Caifa Zhou, Jan D. Wegner, Leonidas J. Guibas, and Tolga Birdal. "Learning multiview 3D point cloud registration." *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2020.

Learning to Synchronize Robustly

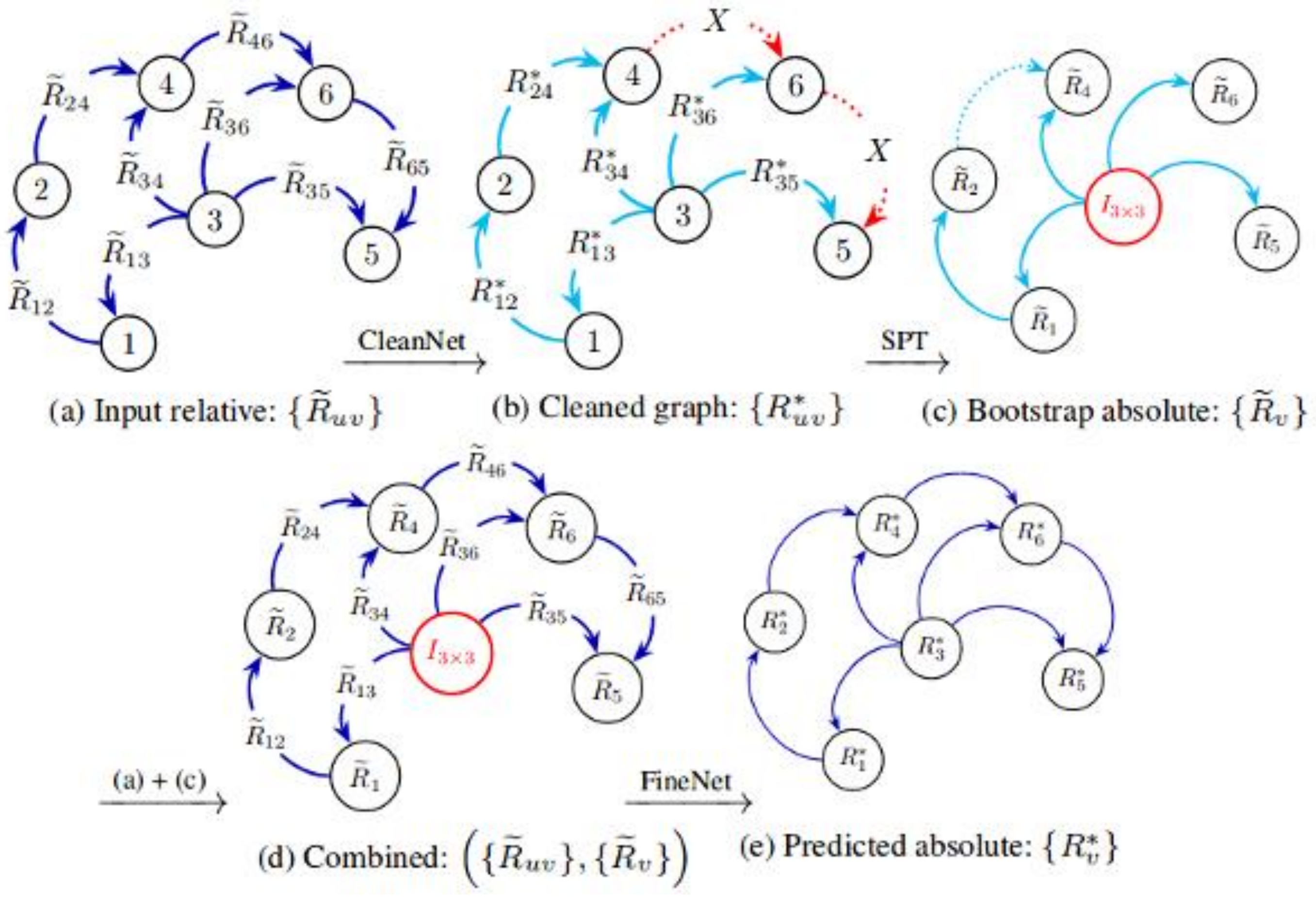


Table 2. Results of Rotation averaging on a test synthetic dataset. The average angular error on all the view-graphs in our dataset is displayed. The proposed method NeuRoRA is remarkably faster than the baselines while producing better results. NeuRoRA-$v2$ is a variation of NeuRoRA where the initialization from CleanNet is fine-tuned in a two-step of FineNet. There is no GPU implementations of [5] and [16] available, thus these methods are excluded from the runtime comparisons on cuda.

	Angular Error			Runtime (seconds)		
Baseline Methods	mn	md	rms	cpu		cuda
Chatterjee [5]	2.17°	1.25°	4.55	5.38s	(1×)	—
Weiszfeld [16]	3.35°	1.02°	9.74	50.92s	(0.11×)	—
Proposed Methods	mn	md	rms	cpu		cuda
CleanNet-SPT + [5]	2.11°	1.26°	4.04	5.41s	(0.99×)	—
CleanNet-SPT + [16]	1.74°	1.01°	**3.53**	50.36s	(0.11×)	—
NeuRoRA	**1.45°**	**0.74°**	**3.53**	**0.21s**	(24×)	0.0016s
NeuRoRA-$v2$	**1.30°**	**0.68°**	**3.28**	**0.30s**	(18×)	0.0021s
Other Methods	mn	md	rms	cpu		cuda
CleanNet-SPT	2.93°	1.47°	5.34	**0.11s**	(47×)	0.0007s
SPT-FineNet	3.00°	1.57°	6.12	**0.11s**	(47×)	0.0007s
SPT-FineNet + [5]	2.12°	1.26°	4.11	5.41s	(0.99×)	—
SPT-FineNet + [16]	1.78°	1.01°	3.95	50.36s	(0.11×)	—
NeuRoRA + [5]	2.11°	1.26°	4.04	5.51s	(0.97×)	—
NeuRoRA + [16]	1.73°	1.01°	**3.51**	50.46s	(0.10×)	—

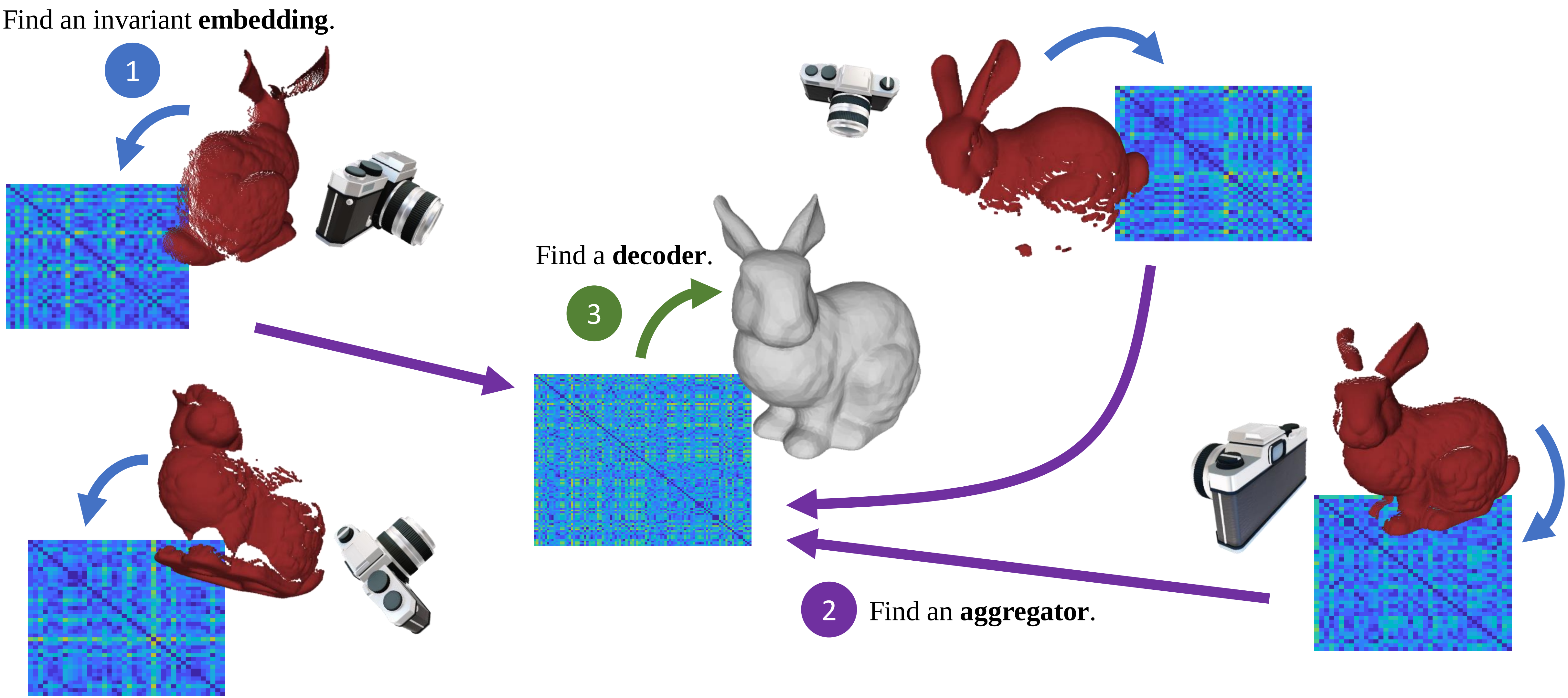
mn: mean of the angular error, md: median of the angular error,
rms: root mean square angular error, and cpu: the runtime of the method on a cpu.
MethodA + MethodB: MethodB is initialized by the solution of MethodA

Gojcic, Zan, Caifa Zhou, Jan D. Wegner, Leonidas J. Guibas, and Tolga Birdal. "Learning multiview 3D point cloud registration." *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2020.

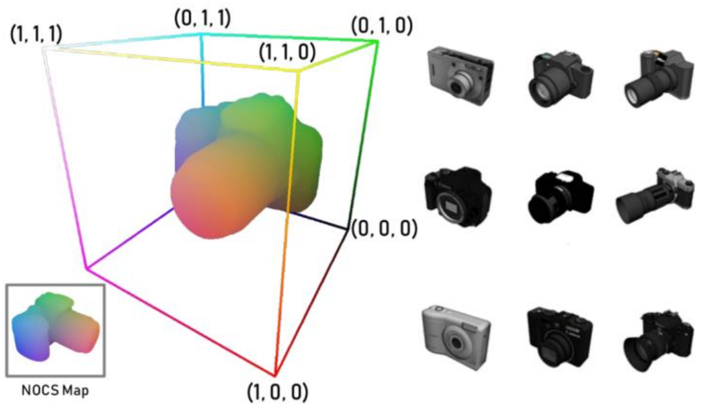
Where to go next?

- Training Neural Networks via Synchronization
- A Complete Treatment of Uncertainty

Synchronization without Synchronization



Synchronization without Synchronization: Canonicalizers



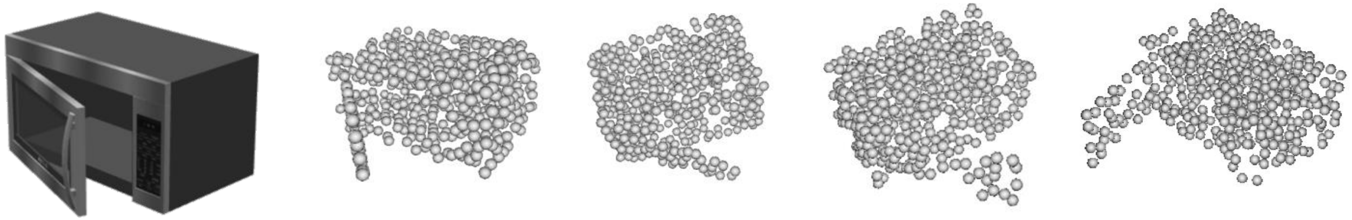
Normalized Object Coordinates

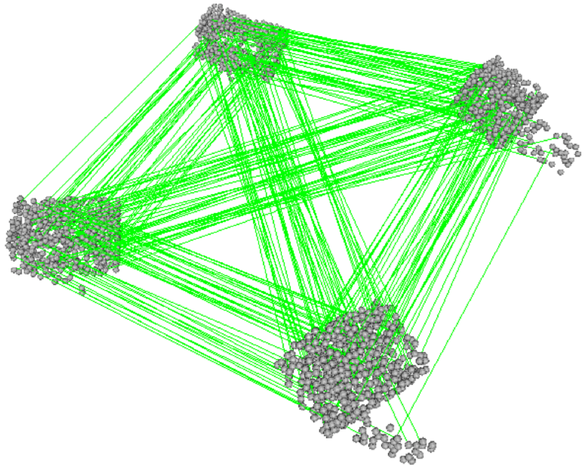
Canonicalization → **Invariance**

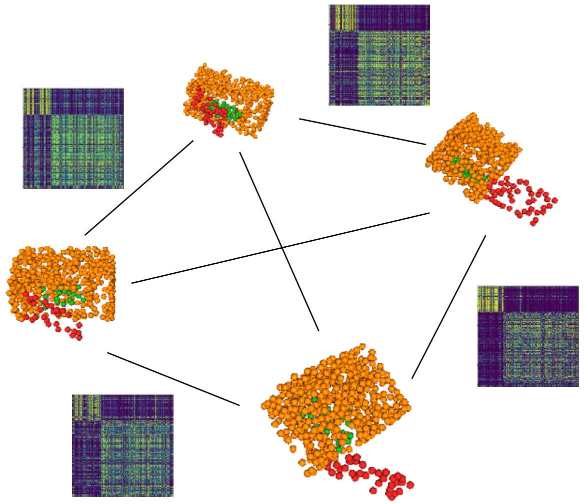
Synchronization → **Equivariance?**

Hybrid methods?

Dealing with Articulation:
Simultaneous Synchronization of Motion and Segmentation

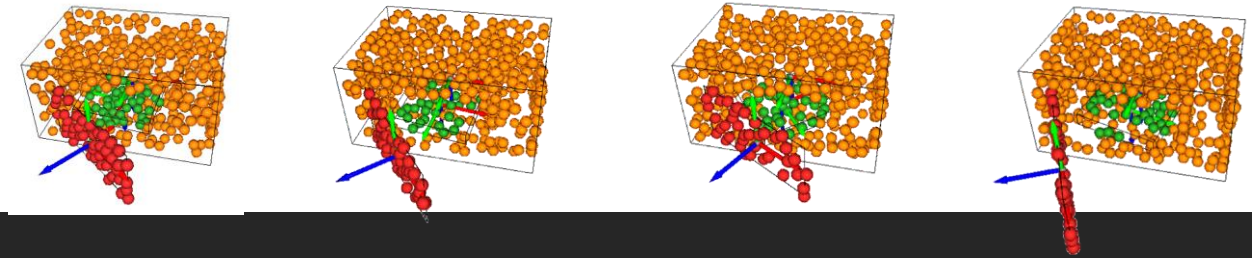






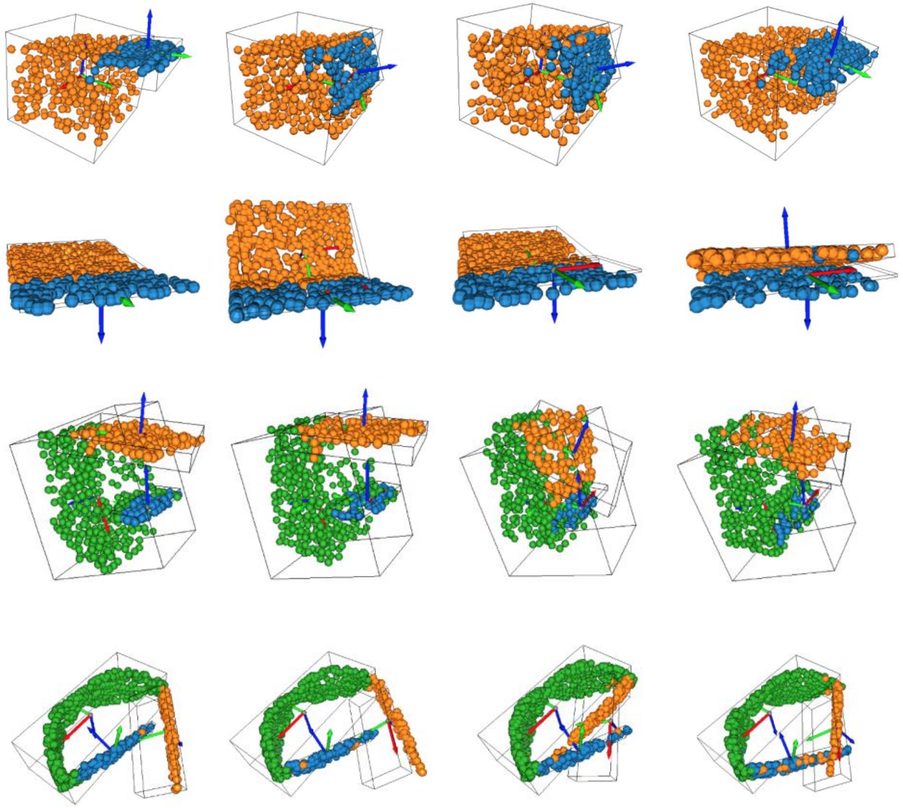
Synchronization over
Correspondence

Synchronization over
Rigid Motion Segmentation



Motion based analysis can naturally generalize across shapes from different categories!

Results



That’s all folks.