MATH230: Tutorial Five

Natural Deductions in First Order Logic

Key ideas

- Interpret formulae in particular models.
- Natural deductions with $\forall$ $\exists$ rules.

Relevant lectures: Lectures 10,11,12,13
Relevant reading: L∃∀N Sections 7,8, and 10.

Hand in exercises: 1a, 1b, 1e, 1g, 1i
Due following Friday @ 5pm to the tutor, or to lecturer.

Discussion Questions

1. Consider the first order language with signature $\mathcal{L} : \{\emptyset, \in, \subset, =\}$ and the following well-formed formulae in this language

   - $\forall x\ \forall y\ [\forall z\ (z \in x \leftrightarrow z \in y) \to x = y]$
   - $\neg \exists x (x \in \emptyset)$
   - $\forall x\ \exists y\ \forall z\ (z \subset x) \to z \in y$

   Interpret these wff in a model of sets i.e. a model such that elements of the universe of discourse are sets. State an interpretation of the elements of the signature in this model. Translate each of the wff into English using that interpretation.

2. In the language above write down a first-order wff that can be interpreted (in a particular model) as defining what it means for x to be a subset of y.

3. $\forall x \neg Fx \dashv\vdash \neg \exists x Fx$

Tutorial Exercises

1. Prove the following in the predicate calculus.

   Propositional rules together with the rules for either $\forall$ *or* $\exists$ *alone:*

   (a) $\forall x(Fx \to Gx) \vdash \forall xFx \to \forall xGx$

   (b) $\forall x((Fx \vee Gx) \to Hx), \quad \forall x\neg Hx \vdash \forall x\neg Fx$

   (c) $\forall x(Fx \wedge Gx) \dashv\vdash \forall xFx \wedge \forall xGx$

   (d) $\forall x(P \to Fx) \dashv\vdash P \to \forall xFx$

   (e) $\exists x(P \to Fx) \dashv\vdash P \to \exists xFx$

   All propositional and predicate rules:

   (f) $\exists x\neg Fx \dashv\vdash \neg\forall xFx$

   (g) $\forall x\neg Fx \dashv\vdash \neg\exists xFx$

   (h) $\forall x(Fx \to P) \dashv\vdash \exists xFx \to P$

   (i) $\forall x(Fx \vee Gx) \vdash \forall xFx \vee \exists xGx$

   (j) $\exists x(Fx \to Gx) \dashv\vdash \forall xFx \to \exists xGx$

   (k) $\exists xFx \dashv\vdash \neg\forall x\, \neg Fx$

   (l) $\forall xFx \dashv\vdash \neg\exists x\, \neg Fx$

2. Presburger arithmetic has signature $\mathcal{P} : \{0, 1, +, =\}$ and axioms

   (a) $\forall x\ \neg(0 = x + 1)$

   (b) $\forall x\ \forall y\ ((x + 1 = y + 1) \to x = y)$

   (c) $\forall x\ (x + 0 = x)$

   (d) $\forall x\ \forall y\ (x + (y + 1) = (x + y) + 1)$

   (e) $(P(0) \wedge \forall x\ (P(x) \to P(x + 1))) \to \forall y(P(y))$

   The standard model for $\mathcal{P}$ is the natural numbers with (the logical symbol) 0 interpreted as (the number) 0, 1 as 1, + as addition of natural numbers, and = as equality as natural numbers. Translate these axioms into English using the standard model of arithmetic.

   For (e) assume P is a wff which can take one input from the universe of discourse.

3. Consider the first-order language with signature $\mathcal{L} : \{0, +, =\}$ such that 0 is a constant, + is a binary function, and = is a binary predicate.

   The following are three well-formed formulae in $\mathcal{L}$

   - $\forall x\ \forall y\ (x + y = y + x)$
   - $\forall x\ \exists y\ (x + y = 0)$
   - $\forall x\ (x + 0 = x)$

   (a) State a model in which all of these wff are *true*.

   (b) State a model in which at least one of these wff are *false*.