MATH230: Tutorial Seven

Peano Arithmetic

Key ideas

- Natural deduction practice,
- Proofs using the identity rules of inference,
- Prove first-order sentences in theories of arithmetic,
- Use the induction schema of Peano arithmetic, and
- Become exasperated enough to appreciate the help of proof assistants.

Relevant lectures: Lectures 14,15, and 16
Relevant reading: Natural Number Game

Hand in exercises: 1a, 2a, 2b, 2c, 3a, 3b, 3c
Due following Friday @ 5pm to the tutor, or to lecturer.

Discussion Questions

Proofs that make use of the axiom (schema) PA 7

$$[P(0) \land \forall x\ (P(x) \to P(s(x)))] \to \forall y(P(y))$$

will prove statements of the form $\forall y\ P(y)$ with the use of modus ponens. This requires proving the antecedent conjunction:

$$[P(0) \land \forall x\ (P(x) \to P(s(x)))]$$

This in turn requires proving each conjunct i.e. two proofs witnessing:

$$\text{PA} \vdash P(0) \qquad\qquad \text{PA} \ \vdash\ \forall x\ (P(x) \to P(s(x)))$$

If we piece this together then we see that all proofs by induction have the form:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_{BC}\end{array}}{P(0)} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_{IS}\end{array}}{\forall x\ (P(x) \to P(s(x)))}\ \forall I}{[P(0) \land \forall x\ (P(x) \to P(s(x)))]}\ \land I}{\forall y\ P(y)}\ \text{IND}$$

1. Identify the following steps involved in a proof by induction of the following sentence of Peano arithmetic:

$$\text{PA} \ \vdash\ \forall x\ (0 + x = x)$$

   (a) Identify the wff $P(x)$ to do induction on.

   (b) $\mathcal{D}_{BC}$: Write down the sequent PA $\vdash\ P(0)$.

   (c) $\mathcal{D}_{IS}$: Write down the sequent PA$, P(n) \ \vdash\ P(s(n))$.

Tutorial Exercises

1. Give natural deductions of the following theorems of identity.

   (a) $\vdash\ \forall x \forall y \forall z\ (x = y \land y = z) \rightarrow x = z$

   (b) $\vdash\ \forall x \forall y \forall z\ x \neq y \rightarrow (x \neq z \lor y \neq z)$ (RAA)

   (c) $\vdash\ \exists x\ (t = x)$

   Part (a) together with proofs from lectures imply that identity is reflexive, symmetric, and transitive. Thus behaving like an equivalence relation.

2. In this question PA denotes the first-order theory of Peano arithmetic which has signature PA: $\{0, s, +, \times\}$ and axioms:

   PA1 $\forall x \neg(s(x) = 0)$

   PA2 $\forall x\ \forall y((s(x) = s(y)) \rightarrow (x = y))$

   PA3 $\forall x\ (x + 0 = x)$

   PA4 $\forall x\ \forall y\ (x + s(y) = s(x + y))$

   PA5 $\forall x\ (x \times 0 = 0)$

   PA6 $\forall x\ \forall y\ (x \times s(y) = (x \times y) + x)$

   PA7 $[P(0) \land \forall x\ (P(x) \rightarrow P(s(x)))] \rightarrow \forall y(P(y))$

   Provide deductions to prove the following sequents.

   (a) PA $\vdash 1 + 1 = 2$

   (b) PA $\vdash 3 \neq 1$

   (c) PA $\vdash \forall x\ (x + 1 = s(x))$

   (d) PA $\vdash \forall x\ (x \times 1 = x)$

3. The followings sequents all require the use of the induction axiom schema. Recall that all proofs using the induction schema have the following form:

$$\dfrac{\dfrac{\dfrac{\vdots}{\mathcal{D}_{BC}}}{P(0)} \qquad \dfrac{\dfrac{\vdots}{\mathcal{D}_{IS}}}{\forall x\ (P(x) \to P(s(x)))}}{\dfrac{[P(0) \wedge \forall x\ (P(x) \to P(s(x)))]}{\forall y\ P(y)}\ IND}\ \wedge I$$

For this reason, once the wff $P(x)$ is identified, it suffices to provide the base case deduction $\mathcal{D}_{BC}$ and induction step $\mathcal{D}_{IS}$. The sequents are stated in such a way as to mean induction on the variable x will be the easiest approach. Always do induction on the variable x.

(a) PA $\vdash \forall x\ (0 + x = x)$

(b) PA $\vdash \forall x\ (0 \times x = 0)$

(c) PA $\vdash \forall x\ (1 \times x = x)$

(d) PA $\vdash \forall x\ (x = 0 \vee \exists y(x = s(y)))$ (Challenge!)

(e) PA $\vdash \forall x\ \forall y\ [s(y) + x = s(y + x)]$ (Challenge!)

(f) PA $\vdash \forall x\ \forall y\ \forall z\ [(y + z) + x = y + (z + x)]$ (Challenge!)

(g) PA $\vdash \forall x\ \forall y\ [y + x = x + y]$ (Challenge!)

4. Visit the Natural Number Game to write computer checked proofs of these theorems of Peano arithmetic as well as theorems of propositional logic from previous tutorials.

5. The first-order language of Peano Arithmetic is often presented with an extra binary relation symbol $<$ where $x < y$ is given the usual interpretation: x is strictly less than y. In fact it is not necessary to add anything extra, for this relation can be defined using a sentence in PA as stated.

   Write down a wff in PA which, when interpreted in the standard model, defines the binary relation $<$ of being "strictly less than". Use this to write down formulae that represent: less than or equal to, strictly greater than, and greater than or equal to.

6. Write down well-formed formulae in the first-order language of PA corresponding to the following statements.

   (a) Each natural number is either equal to 0 or greater than 0.

   (b) If x is not less than y, then x equals y or y is less than x.

   (c) If x is less than or equal to y and y is less than or equal to x, then $x = y$.