MATH230: Tutorial Three (Solutions)

Natural Deductions: Classical Logic

Key ideas

- Write some natural deductions using RAA,
- Prove LEM and DNE,
- Understand the induction step in the soundness theorem,
- Appreciate the oddities of classical theorems.

Relevant lectures: Lectures 6,7,8, and 9
Relevant reading: L∃∀N Chapters 3,4,5

Hand in exercises: 1a, 1b, 1e, 2a, 2b
Due following Friday @ 5pm to the tutor, or lecturer.

Discussion Questions

1. If $\Sigma_1 \models \alpha \to \beta$ and $\Sigma_2 \models \alpha$, then show $\Sigma_1 \cup \Sigma_2 \models \beta$.

   Solution

   Let v be a valuation that satisfies $\Sigma_1 \cup \Sigma_2$. It follows that v satisfies Σ_1 and hence $\alpha \to \beta$. Therefore $v(\alpha) = v(\alpha)v(\beta)$. Similarly v satisfies Σ_2 and hence $v(\alpha) = 1$.

   Together these imply $v(\beta) = 1$. Therefore any valuation satisfying $\Sigma_1 \cup \Sigma_2$ must also satisfy β. It follows that modus ponens is a truth preserving rule of inference.

2. $\neg(A \land B) \vdash \neg A \lor \neg B$

Solution

Proofs that use RAA have a different approach to minimal and intuitionistic proofs. They require more than just unpacking the logical connectives in the hypotheses and conclusion.

However, there are common elements to all the RAA proofs in this tutorial. All RAA proofs require adding the negation of the conclusion to the hypotheses. Once you have identified that you're to write an RAA proof, you can immediately add the negation of the conclusion to the hypotheses - it will be discharged at the RAA step.

The RAA step only occurs once $\perp$ has been derived. Where is $\perp$ going to come from? Typically from the negated conclusion added to the hypotheses, via modus ponens. Counterintuitively, this means we need to now derive the statement we are trying to prove - but we do so with extra assumptions that will be discharged along the way. NOTE: This step may need to be done twice in order to remove all of the hypotheses added to make the proof. That is, you will arrive at the final conclusion multiple times in the proof; but you will still have hypotheses to discharge. Be mindful of the hypotheses that are still in use at any stage in the proof.

In summary, when writing an RAA proof add the negation of the conclusion to the hypotheses, together with assumptions that allow the proof of the conclusion in the context of the question. All of these extra assumptions must be discharged with either implication introduction or RAA.

Hypotheses: $\neg(A \land B)$ with $\neg(\neg A \lor \neg B)$ added for RAA and then A, B added in order to obtain $\neg A \lor \neg B$ to derive $\perp$ for the RAA step.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\cancel{\neg(\neg A \lor \neg B)}$}
\AxiomC{$\neg(A \land B)$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\cancel{A}$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\cancel{B}$}
\RightLabel{$\land I$}
\BinaryInfC{$A \land B$}
\RightLabel{MP}
\BinaryInfC{$\perp$}
\RightLabel{$\to I, 2$}
\UnaryInfC{$\neg B$}
\RightLabel{$\lor I$}
\UnaryInfC{$\neg A \lor \neg B$}
\RightLabel{MP}
\BinaryInfC{$\perp$}
\RightLabel{$\to I, 1$}
\UnaryInfC{$\neg A$}
\RightLabel{$\lor I$}
\UnaryInfC{$\neg A \lor \neg B$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\cancel{\neg(\neg A \lor \neg B)}$}
\BinaryInfC{$\perp$}
\RightLabel{$RAA, 3$}
\UnaryInfC{$\neg A \lor \neg B$}
\end{prooftree}
$$

Tutorial Exercises

1. Make sure you have finished all of the minimal and intuitionistic natural deductions before doing this tutorial. It is more important that you understand those.

   Solution

   See the solutions for Tutorial Two.

2. **Classical derivations.** Provide natural deduction proofs of the following. All rules may be required.

   (a) $\vdash A \vee \neg A$ (law of excluded middle - RAA)

   Solution

   Hypotheses: $\neg(A \vee \neg A)$ for RAA and A to obtain $A \vee \neg A$ for the derivation of $\bot$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{A}}\,1}{A \vee \neg A} \quad \dfrac{}{\cancel{\neg(A \vee \neg A)}}\,2}{\bot}\,MP}{\neg A}\,\to I, 1}{A \vee \neg A}\,\vee I \quad \dfrac{}{\cancel{\neg(A \vee \neg A)}}\,2}{\bot}\,MP}{A \vee \neg A}\,RAA, 2$$

   Notice the final conclusion $A \vee \neg A$ is arrived at three times in the course of this proof. Only the last time "counts" as only then have all the hypotheses been discharged. This will happen for each RAA proof below.

   (b) $\neg\neg A \vdash A$ (double negation elimination - RAA)

   Solution

   Hypotheses: RAA comes in the proof below through the use of $A \vee \neg A$. This might be called a *proof by cases.* It still relies on RAA, because the proof relies on $A \vee \neg A$ being provable from no hypotheses; which is a classical theorem.

$$\dfrac{A \vee \neg A \quad A \to A \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{\neg A}}\,1 \quad \neg\neg A}{\bot}\,MP}{A}\,XF}{\neg A \to A}\,\to I, 1}{A}\,\vee E$$

(c) $\neg(A \wedge B) \vdash \neg A \vee \neg B$ (De Morgan - RAA)

Solution

Hypotheses: $\neg(A \wedge B)$ with $\neg(\neg A \vee \neg B)$ added for RAA and then A, B added in order to obtain $\neg A \vee \neg B$ to derive $\bot$ for the RAA step.

$$
\begin{prooftree}
\AxiomC{$\cancel{\neg(\neg A \vee \neg B)}^{\,3}$}
\AxiomC{$\neg(A \wedge B)$}
\AxiomC{$\cancel{A}^{\,1}$}
\AxiomC{$\cancel{B}^{\,2}$}
\RightLabel{$\wedge I$}
\BinaryInfC{$A \wedge B$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\to I, 2$}
\UnaryInfC{$\neg B$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg A \vee \neg B$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\to I, 1$}
\UnaryInfC{$\neg A$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg A \vee \neg B$}
\AxiomC{$\cancel{\neg(\neg A \vee \neg B)}^{\,3}$}
\BinaryInfC{$\bot$}
\RightLabel{$RAA, 3$}
\UnaryInfC{$\neg A \vee \neg B$}
\end{prooftree}
$$

(d) $A \to B \vdash \neg A \vee B$ (material implication - RAA)

Solution

Hypotheses: $\neg(\neg A \vee B)$ for the RAA step. Also add A to obtain $\neg A \vee B$ for the derivation of $\bot$.

$$
\begin{prooftree}
\AxiomC{$\cancel{\neg(\neg A \vee B)}^{\,2}$}
\AxiomC{$\cancel{A}^{\,1}$}
\AxiomC{$A \to B$}
\RightLabel{MP}
\BinaryInfC{B}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg A \vee B$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\to I, 1$}
\UnaryInfC{$\neg A$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg A \vee B$}
\AxiomC{$\cancel{\neg(\neg A \vee B)}^{\,2}$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$RAA, 2$}
\UnaryInfC{$\neg A \vee B$}
\end{prooftree}
$$

(e) $\vdash (A \to B) \vee (B \to C)$ (Challenge! - RAA)

Hypotheses: $\neg((A \to B) \vee (B \to C))$ for RAA and just B to obtain $\bot$ in the course of the proof.

Solution

$$
\begin{prooftree}
\AxiomC{$\cancel{\neg[(A \to B) \vee (B \to C)]}^{\,2}$}
\AxiomC{$\cancel{B}^{\,1}$}
\RightLabel{$\to I$}
\UnaryInfC{$A \to B$}
\RightLabel{$\vee I$}
\UnaryInfC{$(A \to B) \vee (B \to C)$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{XF}
\UnaryInfC{C}
\RightLabel{$\to I, 1$}
\UnaryInfC{$B \to C$}
\RightLabel{$\vee I$}
\UnaryInfC{$(A \to B) \vee (B \to C)$}
\AxiomC{$\cancel{\neg[(A \to B) \vee (B \to C)]}^{\,2}$}
\BinaryInfC{$\bot$}
\RightLabel{$RAA, 2$}
\UnaryInfC{$(A \to B) \vee (B \to C)$}
\end{prooftree}
$$

(f) $\vdash (A \wedge \neg A) \to B$ (the *paradox of entailment*)

Solution

RAA not needed.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{A \wedge \neg A}}\,1}{A}\,\wedge E_l \qquad \dfrac{\dfrac{}{\cancel{A \wedge \neg A}}\,1}{\neg A}\,\wedge E_r}{\bot}\,MP}{B}\,XF}{(A \wedge \neg A) \to B}\,\to I, 1$$

(g) $\vdash A \to B \vee \neg B$ (LEM)

Solution

Since $B \vee \neg B$ is a classical theorem, we can extend the proof of LEM (Exercise 2 a) by an implication introduction to get this theorem.

(h) $\vdash A \to (B \to A)$ (weakening)

Solution

$$\dfrac{\dfrac{\dfrac{}{\cancel{A}}\,1 \quad \dfrac{}{\cancel{B}}\,2}{B \to A}\,\to I, 2}{A \to (B \to A)}\,\to I, 1$$

(i) $\vdash \neg A \to (A \to B)$ (a form of *ex falso*)

Solution

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{\neg A}}\,1 \quad \dfrac{}{\cancel{A}}\,2}{\bot}\,MP}{B}\,XF}{A \to B}\,\to I, 2}{\neg A \to (A \to B)}\,\to I, 1$$

(j) $\vdash (\neg A \to A) \to A$ (RAA)

Solution

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{\neg A \to A}}\,1 \quad \dfrac{}{\cancel{\neg A}}\,2}{A}\,MP \qquad \dfrac{}{\cancel{\neg A}}\,2}{\bot}\,MP}{A}\,RAA, 2}{(\neg A \to A) \to A}\,\to I, 1$$

(k) $(A \to B) \land (C \to D) \vdash (A \to D) \lor (C \to B)$ (Challenge! - RAA)

Solution

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cancel{\neg[(A \to D) \lor (C \to B)]}^{\,3} \quad \dfrac{\dfrac{\dfrac{\cancel{C}^{\,1} \quad \dfrac{(A \to B) \land (C \to D)}{C \to D}\,\land E_r}{D}\,MP}{A \to D}\,\to I}{(A \to D) \lor (C \to B)}\,\lor I}{\bot}\,MP}{B}\,XF}{C \to B}\,\to I, 1}{(A \to D) \lor (C \to B)}\,\lor I \quad \cancel{\neg[(A \to D) \lor (C \to B)]}^{\,3}}{\dfrac{\bot}{(A \to D) \lor (C \to B)}\,RAA, 3}\,MP
$$

(l) $\neg(A \to B) \dashv\vdash A \land \neg B$ (Challenge! - RAA helps)

Solution

The first proof below makes use of the theorem $\vdash \neg A \to (A \to B)$.

First prove the sequent $\neg(A \to B) \vdash A \land \neg B$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\cancel{\neg A}^{\,1} \quad \neg A \to (A \to B)}{A \to B}\,MP \quad \neg(A \to B)}{\bot}\,MP}{A}\,RAA, 1 \qquad \dfrac{\dfrac{\dfrac{\cancel{B}^{\,2}}{A \to B}\,\to I \quad \neg(A \to B)}{\bot}\,MP}{\neg B}\,\to I, 2}{A \land \neg B}\,\land I
$$

Now prove the other sequent $A \land \neg B \vdash \neg(A \to B)$

$$
\boxed{\dfrac{\dfrac{\dfrac{\dfrac{A \land \neg B}{A}\,\land E_l \quad \cancel{A \to B}^{\,1}}{B}\,MP \quad \dfrac{A \land \neg B}{\neg B}\,\land E_r}{\bot}\,MP}{\neg(A \to B)}\,\to I, 1}
$$

3. **Soundness Proof**

Complete the induction step of the soundness theorem by answering the following.

(a) Implication introduction

If $\Gamma \models \beta$, then show $\Gamma \backslash \{\alpha\} \models \alpha \to \beta$.

Solution

It is sufficient to show any valuation satisfying $\Gamma \backslash \{\alpha\}$ must also satisfy $\alpha \to \beta$.

To this end, let v be a valuation that satisfies $\Gamma \backslash \{\alpha\}$.

There are two cases: either v satisfies α or it does not.

If v also satisfies α, then v satisfies all of Γ and hence ($\Gamma \models \beta$) $v(\beta) = 1$. In this case

$$v(\alpha \to \beta) = 1 - v(\alpha) + v(\alpha)v(\beta) = 1 - 1 + 1 = 1$$

In the other case, $v(\alpha) = 0$ and hence

$$v(\alpha \to \beta) = 1 - v(\alpha) + v(\alpha)v(\beta) = 1 - 0 + 0 = 1$$

So it follows, in both cases, that if $\Gamma \models \beta$ then any valuation satisfying $\Gamma \backslash \{\alpha\}$ will also satisfy $\alpha \to \beta$. Therefore implication introduction is truth preserving rule of inference.

(b) Disjunction introduction

If $\Gamma \models \alpha$, then show $\Gamma \models \alpha \vee \beta$.

Solution

Suppose v is a valuation satisfying Γ.

$$\begin{aligned} v(\alpha \vee \beta) &= v(\alpha) + v(\beta) - v(\alpha)v(\beta) \\ &= 1 + v(\beta) - 1 \times v(\beta) && \Gamma \models \alpha \\ &= 1 \end{aligned}$$

This shows that, if $\Gamma \models \alpha$ and v satisfies Γ, then v necessarily satisfies $\alpha \vee \beta$. Therefore disjunction introduction is a truth preserving rule of inference.

(c) Disjunction elimination

If $\Gamma_1 \models \alpha \vee \beta$, $\Gamma_2 \models \alpha \rightarrow \gamma$, and $\Gamma_3 \models \beta \rightarrow \gamma$ then show the following is valid $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \models \gamma$.

Solution

Suppose v satisfies $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

Certainly v satisfies Γ_1 and hence $v(\alpha \vee \beta) = 1$.

$$1 = v(\alpha) + v(\beta) - v(\alpha)v(\beta) \tag{1}$$

Also, since v satisfies Γ_2 and hence $v(\alpha \rightarrow \gamma) = 1$

$$\begin{aligned} 1 &= v(\alpha \rightarrow \gamma) \\ &= 1 - v(\alpha) + v(\alpha)v(\gamma) \\ \text{Therefore } v(\alpha) &= v(\alpha)v(\gamma) \end{aligned} \tag{2}$$

Finally, since v satisfies Γ_3 and hence $v(\beta \rightarrow \gamma) = 1$

$$\begin{aligned} 1 &= v(\beta \rightarrow \gamma) \\ &= 1 - v(\beta) + v(\beta)v(\gamma) \\ \text{Therefore } v(\beta) &= v(\beta)v(\gamma) \end{aligned} \tag{3}$$

To see that $v(\gamma) = 1$ one can multiply (1) on both sides by $v(\gamma)$ and use (2) and (3) to simplify. This shows that disjunction elimination is a truth preserving rule of inference.

(d) Conjunction introduction

If $\Gamma_1 \models \alpha$ and $\Gamma_2 \models \beta$, then we need to show that $\Gamma_1 \cup \Gamma_2 \models \alpha \wedge \beta$.

Solution

If v satisfies $\Gamma_1 \cup \Gamma_2$, then v satisfies Γ_1 and Γ_2 separately. Therefore v satisfies α and β. So $v(\alpha \wedge \beta) = v(\alpha)v(\beta) = 1$.

Conjunction introduction is therefore seen to be a truth preserving rule of inference.

(e) Conjunction elimination

If $\Gamma \models \alpha \wedge \beta$, then show $\Gamma \models \alpha$.

Solution

If v satisfies Γ, then

$$v(\alpha \wedge \beta) = v(\alpha)v(\beta) = 1$$

Therefore $v(\alpha) = 1$.

Conjunction elimination (left and right instances are basically the same) is therefore seen to be truth preserving.

(f) Ex Falso Quodlibet

If $\Gamma \models \bot$, then show $\Gamma \models \alpha$.

Solution

Let v be a valuation that satisfies Γ. It follows from $\Gamma \models \bot$, that $v(\bot) = 1$. This is impossible! Therefore there can be no such v and hence no counterexample to $\Gamma \models \alpha$.

(g) Reductio Ad Absurdum

If $\Gamma, \neg\alpha \models \bot$, then show $\Gamma \models \alpha$.

Solution

Let v be a valuation that satisfies Γ. We want to show that this valuation must also satisfy α.

Such a valuation either satisfies α or satisfies $\neg\alpha$. We need to consider each case in turn.

If v satisfies $\neg\alpha$, then by $\Gamma, \neg\alpha \models \bot$, we conclude that $v(\bot) = 1$. This is impossible, so there can be no such valuation.

Since all such v don't satisfy $\neg\alpha$, they must satisfy α. For either $v(\alpha) = 1$ or $v(\neg\alpha) = 1$.

Therefore, given $\Gamma, \neg\alpha \models \bot$, we may conclude that any valuation satisfying Γ must also satisfy α i.e. $\Gamma \models \alpha$.

4. Ex Falso Quodlibet (The Law of Explosion) states that, for any propositions P, Q we have the sequent $\{P \wedge \neg P\} \vdash Q$.

   Show that the rule of inference **XF** can be *derived* from **minimal logic + RAA**. In this sense we might say classical logic is more powerful the intuitionistic logic.

   Solution

   The rules are the same except for the discharge of a hypothesis. One can *always* swap an instance of XF for RAA — whether or not there is a $\neg P$ to discharge.

5. In class we discussed how Classical Logic can be obtained from intuitionistic logic by adding the rule of inference *Reductio Ad Absurdum.*

   If $\overset{\Sigma}{\underset{\bot}{\mathcal{D}}}$ is a deduction of $\bot$ from Σ, then

$$\dfrac{\begin{array}{c}\overline{\cancel{\neg\alpha}}\\ \Sigma\\ \mathcal{D}\\ \bot\end{array}}{\alpha}\,RAA$$

   is a derivation of α from the assumptions $\Sigma\backslash\{\neg\alpha\}$.

   In this question we will explore this extension of logics in more detail. We will see that there are different methods for obtaining classical logic from intuitionistic logic.

   (a) Show that adding the rule of inference *reductio ad absurdum* to intuitionistic propositional logic is equivalent to asserting that all the formulae $P \lor \neg P$ for each proposition P are theorems.

   That is, if given a derivation $\mathcal{D}$ witnessing $\Sigma, \neg P \vdash \bot$, then show that it can be extended (without using RAA) using the assumption of LEM i.e. $\vdash P \lor \neg P$ to a derivation of P with the assumption $\neg P$ eliminated.

   Solution

   Suppose we have a derivation $\mathcal{D}$ witnessing $\Sigma, \neg P \vdash \bot$. Extend this derivation using the assumption of LEM i.e. $P \lor \neg P$ in a manner equivalent to RAA.

   That is to say, derive P and eliminate $\neg P$ without using RAA.

$$\dfrac{P \lor \neg P \qquad P \to P \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Sigma \quad \overline{\cancel{\neg P}}^{\,1}}{\mathcal{D}}}{\bot}}{P}\,XF}{\neg P \to P}\to I, 1}{P}\,\lor E$$

   This shows RAA = LEM in the presence of XF i.e. adding either to intuitionistic logic yields the same theorems.

(b) Show that adding the rule of inference reductio ad absurdum to intuitionistic propositional logic is equivalent to adding the rule of inference of double negation elimination.

That is, if given a derivation $\mathcal{D}$ witnessing $\Sigma, \neg P \vdash \perp$, then show that it can be extended (without using RAA) using the assumption of DNE i.e. $\neg\neg P \ \vdash P$ to a derivation of P with the assumption $\neg P$ eliminated.

Solution

Suppose we have a derivation $\mathcal{D}$ witnessing $\Sigma, \neg P \vdash \perp$. Extend this derivation using the assumption of DNE i.e. $\neg\neg P \vdash P$ in a manner equivalent to RAA.

That is to say, derive P and eliminate $\neg P$ without using RAA.

$$\dfrac{\dfrac{\begin{array}{c}\Sigma \quad \overline{\cancel{\neg P}}\;1 \\ \hline \mathcal{D} \\ \hline \perp\end{array}}{\neg\neg P}\to I, 1}{P}\ DNE$$

This shows RAA = DNE in the presence of XF i.e. intuitionistic logic.

Note that although this direction only requires minimal logic, the proof we gave of RAA implies DNE (Tutorial Exercise 2 b) used XF.