MATH230: Tutorial Three

Natural Deductions: Classical Logic

Key ideas

- Write some natural deductions using RAA,
- Prove LEM and DNE,
- Understand the induction step in the soundness theorem,

Relevant lectures: Lectures 6,7,8, and 9
Relevant reading: L∃∀N Chapters 3,4,5

Hand in exercises: 2a, 2b, 2e, 3a, 3b
Due following Friday @ 5pm to the tutor, or lecturer.

Discussion Questions

1. If $\Sigma_1 \models \alpha \to \beta$ and $\Sigma_2 \models \alpha$, then show $\Sigma_1 \cup \Sigma_2 \models \beta$.

2. $\neg(A \wedge B) \vdash \neg A \vee \neg B$

Tutorial Exercises

1. Make sure you have finished all of the minimal and intuitionistic natural deductions before doing this tutorial. It is more important that you understand those.

2. **Classical derivations.** Provide natural deduction proofs of the following. All rules *may* be required.

   (a) $\vdash A \vee \neg A$ (law of excluded middle - RAA)

   (b) $\neg\neg A \vdash A$ (double negation elimination - RAA)

   (c) $\neg(A \wedge B) \vdash \neg A \vee \neg B$ (De Morgan - RAA)

   (d) $A \to B \vdash \neg A \vee B$ (material implication - RAA)

   (e) $\vdash (A \to B) \vee (B \to C)$ (Challenge! - RAA)

   (f) $\vdash A \wedge \neg A \to B$ (the *paradox of entailment*)

   (g) $\vdash A \to B \vee \neg B$ (LEM)

   (h) $\vdash A \to (B \to A)$ (weakening)

   (i) $\vdash \neg A \to (A \to B)$ (a form of *ex falso*)

   (j) $\vdash (\neg A \to A) \to A$ (RAA)

   (k) $(A \to B) \wedge (C \to D) \vdash (A \to D) \vee (C \to B)$ (Challenge! - RAA)

   (l) $\neg(A \to B) \dashv\vdash A \wedge \neg B$ (Challenge! - RAA helps)

3. **Soundness Proof** Complete the induction step of the soundness theorem by answering the following.

   (a) Implication introduction
   If $\Gamma \models \beta$, then show $\Gamma \backslash \{\alpha\} \models \alpha \to \beta$.

   (b) Disjunction introduction
   If $\Gamma \models \alpha$, then show $\Gamma \models \alpha \vee \beta$.

   (c) Disjunction elimination
   If $\Gamma_1 \models \alpha \vee \beta$, $\Gamma_2 \models \alpha \to \gamma$, and $\Gamma_3 \models \beta \to \gamma$ then show the following is valid $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \models \gamma$.

   (d) Conjunction introduction
   If $\Gamma_1 \models \alpha$ and $\Gamma_2 \models \beta$, then we need to show that $\Gamma_1 \cup \Gamma_2 \models \alpha \wedge \beta$.

   (e) Conjunction elimination
   If $\Gamma \models \alpha \wedge \beta$, then show $\Gamma \models \alpha$.

   (f) Ex Falso Quodlibet
   If $\Gamma \models \bot$, then show $\Gamma \models \alpha$.

   (g) Reductio Ad Absurdum
   If $\Gamma, \neg\alpha \models \bot$, then show $\Gamma \models \alpha$.

4. Ex Falso Quodlibet (The Law of Explosion) states that, for any propositions P, Q we have the sequent $\{P \wedge \neg P\} \vdash Q$.

   Show that the rule of inference **XF** can be *derived* from **minimal logic + RAA**. In this sense we might say classical logic is more powerful the intuitionistic logic.

5. In class we discussed how classical logic can be obtained from intuitionistic logic by adding the following rule of inference *reductio ad absurdum*: If ${}^{\Sigma}_{\perp}\mathcal{D}$ is a deduction of $\perp$ from Σ, then

$$\begin{array}{c} \overline{\cancel{\neg\alpha}} \\ \Sigma \\ \mathcal{D} \\ \perp \\ \hline \alpha \end{array}\ RAA$$

is a derivation of α from the assumptions $\Sigma\backslash\{\neg\alpha\}$.

In this question we will explore this extension of logics in more detail. We will see that there are different methods for obtaining classical logic from intuitionistic logic.

(a) Show that adding the rule of inference *reductio ad absurdum* to intuitionistic propositional logic is equivalent to asserting that all the formulae $P \vee \neg P$ for each proposition P are theorems.

That is, if given a derivation $\mathcal{D}$ witnessing $\Sigma, \neg P \vdash \perp$, then show that it can be extended (without using RAA) using the assumption of LEM i.e. $\vdash P \vee \neg P$ to a derivation of P with the assumption $\neg P$ eliminated.

(b) Show that adding the rule of inference reductio ad absurdum to intuitionistic propositional logic is equivalent to adding the rule of inference of double negation elimination.

That is, if given a derivation $\mathcal{D}$ witnessing $\Sigma, \neg P \vdash \perp$, then show that it can be extended (without using RAA) using the assumption of DNE i.e. $\neg\neg P \ \vdash P$ to a derivation of P with the assumption $\neg P$ eliminated.

Together these exercises show that the logics:

(a) Intuitionistic + RAA

(b) Intuitionistic + LEM

(c) Intuitionistic + DNE

All have the same set of theorems. In this way, we see classical logic can be obtained from intuitionistic in a number of ways.