MATH230: Tutorial Two (Solutions)

Propositional Logic: Natural Deductions

Key ideas

- Write natural deduction proofs using minimal logic.
- Write natural deduction proofs using the intuitionistic abusrdity rule.

Relevant lectures: Lectures 4,5, and 6
Relevant reading: L∃∀N Chapters 3,4

Hand in exercises: 1, 2a, 2b, 2d, 2f, 3a
Due following Friday @ 5pm to the tutor, or lecturer.
Email lecturer to report on topic and references for essay.

Discussion Questions

1. Show $A \vdash \neg\neg A$.

   Solution

   Introduction rules: implication introduction as the conclusion consists of a single implication ($\neg\neg A \equiv (A \to \bot) \to \bot$) which this proof will need to introduce. This requires temporarily adding the antecedent $(A \to \bot)$ to our hypotheses.

   Elimination rules: implication elimination (aka modus ponens) as the hypotheses are A and $A \to \bot$.

   Hypotheses: A with the temporary hypothesis $\neg A \equiv A \to \bot$.

$$\dfrac{\dfrac{A \quad \dfrac{}{\cancel{A \to \bot}}\,1}{\bot}\,\text{MP}}{(A \to \bot) \to \bot} \to I, 1$$

2. Show $A \to B \vdash A \to (A \wedge B)$.

   Solution

   Introduction rules: implication introduction. Since the conclusion is an implication $A \to (A \wedge B)$ we temporarily add the antecedent A to our hypotheses. Proofs of hypotheticals/implications require use to assume the hypothesis.

   Elimination rules: implication elimination (aka modus ponens). The signle hypothesis is $A \to B$. We need A to use modus ponens on the assumption.

   Hypotheses: $A \to B$ with the temporary hypothesis A.

$$\dfrac{\dfrac{\dfrac{}{\cancel{A}}\,1 \quad \dfrac{A \to B \quad \dfrac{}{\cancel{A}}\,1}{B}}{A \wedge B}\,\wedge I}{A \to (A \wedge B)} \to I, 1$$

3. Show $(A \wedge B) \vee C \ \vdash \ (A \vee C) \wedge (B \vee C)$.

Solution

This proof requires the use of disjunction elimination; as the only hypothesis is a disjunction. Proofs with this rule of inference have three subproofs, all of which can be worked on separately and brought together at the end.

$$\dfrac{(A \wedge B) \vee C \qquad \dfrac{\vdots}{(A \wedge B) \to ?} \qquad \dfrac{\vdots}{C \to ?}}{?}\,\vee E$$

These subproofs are both derivations that introduce an implication; for this reason we can add the antecedent $A \wedge B$ (resp. C) to our hypotheses on each subproof.

In fact the disjunction elimination will be the final step, so the “?" is the conclusion to the argument: $(A \vee C) \wedge (B \vee C)$.

First subproof

On this branch we are trying to prove: $(A \wedge B) \to (A \vee C) \wedge (B \vee C)$. Our hypotheses are therefore $(A \wedge B) \vee C$ and $A \wedge B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{A \wedge B}}\,1}{A}\,\wedge E_l}{A \vee C}\,\vee I \qquad \dfrac{\dfrac{\dfrac{}{\cancel{A \wedge B}}\,1}{B}\,\wedge E_r}{B \vee C}\,\vee I}{(A \vee C) \wedge (B \vee C)}\,\wedge I}{(A \wedge B) \to (A \vee C) \wedge (B \vee C)}\,\to I, 1$$

Second subproof

On this branch we are trying to prove: $C \to (A \vee C) \wedge (B \vee C)$. Our hypotheses are therefore $(A \wedge B) \vee C$ and C.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{C}}\,2}{A \vee C}\,\vee I \qquad \dfrac{\dfrac{}{\cancel{C}}\,2}{B \vee C}\,\vee I}{(A \vee C) \wedge (B \vee C)}\,\wedge I}{C \to (A \vee C) \wedge (B \vee C)}\,\to I, 2$$

Combining the above subproofs yields

$$\boxed{\dfrac{(A \wedge B) \vee C \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{A \wedge B}}\,1}{A}\,\wedge E_l}{A \vee C}\,\vee I \qquad \dfrac{\dfrac{\dfrac{}{\cancel{A \wedge B}}\,1}{B}\,\wedge E_r}{B \vee C}\,\vee I}{(A \vee C) \wedge (B \vee C)}\,\wedge I}{(A \wedge B) \to (A \vee C) \wedge (B \vee C)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{C}}\,2}{A \vee C}\,\vee I \qquad \dfrac{\dfrac{}{\cancel{C}}\,2}{B \vee C}\,\vee I}{(A \vee C) \wedge (B \vee C)}\,\wedge I}{C \to (A \vee C) \wedge (B \vee C)}}{(A \vee C) \wedge (B \vee C)}\,\vee E}$$

Tutorial Exercises

1. This exercise breaks the proof of the sequent

$$\vdash\ (P \to Q) \to (\neg Q \to \neg P)$$

into steps to show what's happening when temporary hypotheses are used.

(a) Using the deduction theorem (temporary hypotheses) move as many hypotheses as possible to the left of the turnstile $\vdash$ to get a new sequent. Proof of this new sequent will ultimately lead to the proof of the original sequent.

(!) Remember $\neg A \equiv A \to \bot$.

Solution

The following sequence of sequents are obtained by assuming the antecedent on the outer-most implication of the consequent.

$$\begin{array}{rll}
 & \vdash\ (P \to Q) \to (\neg Q \to \neg P) & \\
P \to Q & \vdash\ \neg Q \to \neg P & \text{Apply Deduction Theorem} \\
P \to Q\ \neg Q & \vdash\ \neg P & \text{Apply Deduction Theorem} \\
P \to Q, \neg Q, P & \vdash\ \bot & \text{Apply Deduction Theorem}
\end{array}$$

(b) Prove the following sequent

$$P \to Q, \neg Q, P\ \vdash\ \bot$$

Solution

$$\cfrac{\cfrac{P \quad P \to Q}{Q}\ \mathsf{MP} \qquad \neg Q}{\bot}\ \mathsf{MP}$$

(c) Extend the proof above, through the use of implication introduction, to a proof of the original sequent.

Solution

To prove the original sequent, we have to undo each application of the deduction theorem i.e. introduce the implication back to the consequent.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{\cancel{P}}^{\,1} \quad \overline{\cancel{P \to Q}}^{\,3}}{Q}\ \mathsf{MP} \qquad \overline{\cancel{\neg Q}}^{\,2}}{\bot}\ \mathsf{MP}}{\neg P} \to I, 1}{\neg Q \to \neg P} \to I, 2}{(P \to Q) \to (\neg Q \to \neg P)} \to I, 3$$

2. **Minimal Logic.** Provide natural deduction proofs of the following sequents. These deductions require only the use of minimal logic. Some of these sequents have a double turnstile. This means you need to prove both directions.

   Hints and Tips: Read!

   The hypotheses of the arguments will suggest the elimination rules to use, while the conclusion will suggest the introduction rules. It's a good first step to look at these, as this will dictate which other (temporary) hypotheses are available in a proof. For example, an implication introduction allows for the (temporary) assumption of the antecedent.

   Always expand $\neg A \equiv A \to \perp$. Otherwise, it may not be clear that the implication introduction/elimination rules can be used.

   (a) $(A \wedge B) \to C \dashv\vdash A \to (B \to C)$

   Solution

   This double turnstile requires a proof in both directions.

   i. First consider the sequent $(A \wedge B) \to C \vdash A \to (B \to C)$.

      Introduction rules: two implications in the conclusion suggest the need for two implication introductions. This requires the temporary addition of A, B as hypotheses.

      Elimination rule: the implication in the hypothesis suggests one needs to use modus ponens with $A \wedge B$. Do we need to add $A \wedge B$ too?

      Hypotheses $(A \wedge B) \to C$ with temporary hypotheses A, B.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{A}}\,1 \quad \dfrac{}{\cancel{B}}\,2}{A \wedge B}\wedge I \quad (A \wedge B) \to C}{C}\text{MP}}{B \to C}\to I, 2}{A \to (B \to C)}\to I, 1$$

   ii. Now we may consider the sequent $A \to (B \to C) \vdash (A \wedge B) \to C$.

      Introduction rules: implication which suggests the temporary assumption $A \wedge B$ should be added.

      Elimination rules: implication which suggests the need for A, B in order to use modus ponens. Do we need to add these too?

      Hypothesis $A \to (B \to C)$ with temporary hypothesis $A \wedge B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{A \wedge B}}\,1}{A}\wedge E_l \quad A \to (B \to C)}{B \to C} \quad \dfrac{\dfrac{}{\cancel{A \wedge B}}\,1}{B}\wedge E_r}{C}\text{MP}}{(A \wedge B) \to C}\to I, 1$$

   Together these natural deductions complete the proof of the claim.

(b) $\neg A \vee \neg B \vdash \neg(A \wedge B)$

Solution

Unpack the negations!

Introduction rules: implication suggests the temporary assumption $A \wedge B$.

Elimination rules: disjunction suggests the temporary assumptions $\neg A, \neg B$.

Hypotheses in proof are: $\neg A \vee \neg B$, $A \wedge B$, $\neg A$, and $\neg B$.

$$
\begin{prooftree}
\AxiomC{$\neg A \vee \neg B$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\cancel{\neg A}$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\cancel{A \wedge B}$}
\RightLabel{$\wedge E_l$}
\UnaryInfC{A}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\to I, 2$}
\UnaryInfC{$\neg A \to \bot$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\cancel{\neg B}$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\cancel{A \wedge B}$}
\RightLabel{$\wedge E_r$}
\UnaryInfC{B}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\to I, 3$}
\UnaryInfC{$\neg B \to \bot$}
\RightLabel{$\vee E$}
\TrinaryInfC{$\bot$}
\RightLabel{$\to I, 1$}
\UnaryInfC{$\neg(A \wedge B)$}
\end{prooftree}
$$

(c) $\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$

Solution

i. First of the two sequents to prove

$$\neg(A \vee B) \vdash \neg A \wedge \neg B$$

Introduction rule: conjunction. This means the proof will be completed by bringing two separate proofs together. One of $\neg A$ and the other of $\neg B$. These can be worked on separately. Furthermore, proofs of negation use implication introduction, so this suggests the addition of temporary assumptions A, B.

Elimination rule: implication. This requires $A \vee B$ in order to use modus ponens. However, this does not need to be added to our hypotheses. Why?

Hypotheses in proof are: $\neg(A \vee B)$, A, and B.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{A}}\,1}{A \vee B}\,\vee I \qquad \neg(A \vee B)}{\bot}\,\text{MP}}{\neg A}\,\to I, 1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{B}}\,2}{A \vee B}\,\vee I \qquad \neg(A \vee B)}{\bot}\,\text{MP}}{\neg B}\,\to I, 2}{\neg A \wedge \neg B}\,\wedge I$$

ii. Second of two sequents to prove

$$\neg A \wedge \neg B \vdash \neg(A \vee B)$$

Introduction rule: negation — introducing negation is done by introducing implication. This suggests the temporary addition of the assumption $A \vee B$.

Elimination rule: conjunction and disjunction. The elimination rule for the temporary hypothesis $A \vee B$ will need to be used as well.

Hypotheses in proof are: $\neg A \wedge \neg B$, $A \vee B$.

$$\dfrac{\dfrac{\dfrac{}{\cancel{A \vee B}}\,1 \qquad \dfrac{\neg A \wedge \neg B}{A \to \bot}\,\wedge E_l \qquad \dfrac{\neg A \wedge \neg B}{B \to \bot}\,\wedge E_r}{\bot}\,\vee E}{\neg(A \vee B)}\,\wedge I$$

(d) $A \to B,\ B \to C \vdash A \to C$

Solution

Introduction rule: implication. Add A to hypotheses.

Elimination rule: implication. Proof needs B to use modus ponens, does it need to be added as well?

Hypotheses in proof are: $A \to B$, $B \to C$, A.

$$\dfrac{\dfrac{\dfrac{\cancel{A}^{\,1} \qquad A \to B}{B}\ \text{MP} \qquad B \to C}{C}\ \text{MP}}{A \to C}\ \to I, 1$$

(e) $A \lor B, \quad A \to C, \quad B \to D \vdash C \lor D$

Solution

Introduction rule: disjunction.

Elimination rule: disjunction and implication. Disjunction elimination requires two implication introduction subproofs. For this reason we add A, B to the hypotheses; one for each of the subproofs. Similarly, the elimination of $A \to C$ and $B \to D$ also suggest the addition of A, B to the hypotheses.

Hypotheses in proof are: $A \lor B$, $A \to C$, $B \to D$, A, B.

$$\dfrac{A \lor B \qquad \dfrac{\dfrac{\dfrac{\cancel{A}^{\,1} \qquad A \to C}{C}\ \text{MP}}{C \lor D}\ \lor I}{A \to (C \lor D)}\ \to I, 1 \qquad \dfrac{\dfrac{\dfrac{\cancel{B}^{\,1} \qquad B \to D}{D}\ \text{MP}}{C \lor D}\ \lor I}{B \to (C \lor D)}\ \to I, 2}{C \lor D}\ \lor E$$

(f) $A \to C, \quad B \to D, \quad \neg C \vee \neg D \vdash \neg A \vee \neg B$

Solution

Introduction rule: negation and disjunction. Add A, B to hypotheses.

Elimination rule: implication and disjunction. Introduction suggests adding A, B (already added) and the disjunction suggests adding $\neg C$ and $\neg D$.

Hypotheses in proof are: $A \to C, B \to D, \neg C \vee \neg D, A, B, \neg C$, and $\neg D$.

$$
\begin{prooftree}
\AxiomC{$\neg C \vee \neg D$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\cancel{\neg C}$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\cancel{A}$}
\AxiomC{$A \to C$}
\RightLabel{MP}
\BinaryInfC{C}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\to I, 1$}
\UnaryInfC{$\neg A$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg A \vee \neg B$}
\RightLabel{$\to I, 2$}
\UnaryInfC{$\neg C \to (\neg A \vee \neg B)$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\cancel{\neg D}$}
\AxiomC{}
\RightLabel{4}
\UnaryInfC{$\cancel{B}$}
\AxiomC{$B \to D$}
\RightLabel{MP}
\BinaryInfC{D}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\to I, 4$}
\UnaryInfC{$\neg B$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg A \vee \neg B$}
\RightLabel{$\to I, 3$}
\UnaryInfC{$\neg D \to (\neg A \vee \neg B)$}
\RightLabel{$\vee E$}
\TrinaryInfC{$\neg A \vee \neg B$}
\end{prooftree}
$$

(g) $A, \quad \neg A \vdash \neg B$

Solution

Introduction rule: negation.

Elimination rule: implication.

Hypotheses in proof are: $A, \neg A$, and B.

$$
\begin{prooftree}
\AxiomC{A}
\AxiomC{$\neg A$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\cancel{B}$}
\RightLabel{$\to I, 1$}
\BinaryInfC{$\neg B$}
\end{prooftree}
$$

(h) $A \to B, \ A \to \neg B \vdash \neg A$

Solution

Introduction rule: negation.

Elimination rule: implication.

Hypotheses in proof are: $A \to B, A \to \neg B$, and A.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\cancel{A}$}
\AxiomC{$A \to B$}
\RightLabel{MP}
\BinaryInfC{B}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\cancel{A}$}
\AxiomC{$A \to \neg B$}
\RightLabel{MP}
\BinaryInfC{$\neg B$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\to I, 1$}
\UnaryInfC{$\neg A$}
\end{prooftree}
$$

3. **Intuitionistic derivations.** Provide natural deduction proofs of the following. You do not need to use the *classical* RAA rule for these questions, but may find that the *intuitionistic* $\bot$ rule is necessary.

   (a) $A, \neg A \vdash B$

   Solution

$$\dfrac{\dfrac{A \quad \neg A}{\bot}\,\text{MP}}{B}\,XF$$

   (b) $\neg A \vee B \vdash A \to B$

   Solution

$$\dfrac{\dfrac{\neg A \vee B \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{\neg A}}\,1 \quad \dfrac{}{\cancel{A}}\,2}{\bot}\,\text{MP}}{B}\,XF}{\neg A \to B}\,\to I, 1 \qquad B \to B}{B}\,\vee E}{A \to B}\,\to I, 2$$

   (c) $A \vee B, \quad \neg A \vdash B$

   Solution

$$\dfrac{A \vee B \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{A}}\,1 \quad \neg A}{\bot}\,\text{MP}}{B}\,XF}{A \to B}\,\to I, 1 \qquad B \to B}{B}\,\vee E$$

   (d) $\vdash \neg(Q \to P) \to (P \to Q)$

   Solution

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{P}}\,1}{Q \to P}\,\to I \quad \dfrac{}{\cancel{\neg(Q \to P)}}\,2}{\bot}\,\text{MP}}{Q}\,\text{XF}}{P \to Q}\,\to I, 1}{\neg(Q \to P) \to (P \to Q)}\,\to I, 2$$