MATH230: Tutorial Eight

Curry Howard Correspondence

Key ideas

- Write well-typed programs simple type theory,
- Interpret programs as proofs of propositions,
- Compare computation and proof-simplification.

Relevant lectures: Lectures x,y, and z
Relevant reading: L∃∀N Chapters 3,4

Hand in exercises:
Due Friday @ 5pm to the tutor, or to lecturer.

Discussion Questions

1. Show the following type is inhabited in the given local context:

$$f : P \to Q \vdash P \to (P \times Q)$$

2. Show the following type is inhabited in the given local context:

$$P \vdash \neg\neg P$$

3. Show the following type is inhabited in the given local context:

$$(P \times Q) + R \ \vdash \ (P + R) \times (Q + R)$$

Tutorial Exercises

1. This exercise breaks the proof that the following type is inhabited

$$\vdash\ (P \to Q) \to (\neg Q \to \neg P)$$

into steps to show what's happening when local variables are introduced.

   (a) Using local variables (much like temporary hypotheses) introduce as many terms as possible into the local context (left of the turnstile $\vdash$) to get a new sequent. Proof that this type is inhabited in the local context will ultimately lead to a proof that the original type is inhabited.

   (!) Remember $\neg P \equiv P \to \bot$.

   (b) Prove the following type is inhabited in the stated syntax

$$f : P \to Q,\ g : \neg Q,\ p : P\ \vdash\ \bot$$

   (c) Extend the typing derivation above, through the use of λ abstraction, to a proof that the original type is inhabited.

2. Each sequent below defines a local context (terms to the left of the $\vdash$ turnstile) and a goal (the type on the right of the $\vdash$ turnstile) - Using the terms of the local context, show that the goal is inhabited.

   (a) $f : (P \times Q) \to R \vdash P \to (Q \to R)$

   (b) $f : P \to (Q \to R)\ \vdash\ (P \times Q) \to R$

   (c) $t : \neg P + \neg Q\ \vdash\ \neg(P \times Q)$

   (d) $f : \neg(P + Q)\ \vdash\ \neg P \times \neg Q$

   (e) $t : \neg P \times \neg Q\ \vdash\ \neg(P + Q)$

   (f) $f : P \to Q,\ g : Q \to R\ \vdash\ P \to R$

   (g) $t : P + Q,\quad f : P \to R,\quad g : Q \to S \vdash\ R + S$

   (h) $f : P \to R,\quad g : Q \to S,\quad t : \neg R + \neg S\ \vdash\ \neg P + \neg Q$

   (i) $p : P,\quad f : \neg P\ \vdash\ \neg Q$

   (j) $f : P \to Q,\ g : P \to \neg Q\ \vdash\ \neg P$

3. This exercise shows you an example of a general observation first made by William Tait, relating the simplifications of proofs and the process of computation in the λ-calculus.

Consider the following proof of the theorem

$$\vdash\ A \land B \to B$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \land B}^{\,1}}{B}\land E_R \qquad \dfrac{\overline{A \land B}^{\,1}}{A}\land_L}{B \land A}\land I}{B}\land E_L}{A \land B \to B}\to, 1$$

   (a) Determine the corresponding proof-object for this proof.

   (b) Why does the proof-object have a redex in it?

(c) Perform the β-reduction on the proof object from (a).

(d) What proof does the reduced proof-object correspond to?