MATH230: Tutorial Eight

Curry Howard Correspondence

Key ideas

- Write well-typed programs simple type theory,
- Interpret programs as proofs of propositions.

Relevant lectures: Lectures x,y, and z
Relevant reading: L∃∀N Chapters 3,4

Hand in exercises:
Due Friday @ 5pm to the tutor, or to lecturer.

Discussion Questions

1. Show the following type is inhabited in the given local context:

$$f : P \to Q \vdash P \to (P \times Q)$$

2. Show the following type is inhabited in the given local context:

$$P \vdash \neg\neg P$$

3. Show the following type is inhabited in the given local context:

$$(P \times Q) + R \ \vdash \ (P + R) \times (Q + R)$$

Tutorial Exercises

1. This exercise breaks the proof that the following type is inhabited

$$\vdash\ (P \to Q) \to (\neg Q \to \neg P)$$

into steps to show what's happening when local variables are introduced.

   (a) Using local variables (much like temporary hypotheses) introduce as many terms as possible into the local context (left of the turnstile $\vdash$) to get a new sequent. Proof that this type is inhabited in the local context will ultimately lead to a proof that the original type is inhabited.

   (!) Remember $\neg P \equiv P \to \bot$.

   (b) Prove the following type is inhabited in the stated syntax

$$f : P \to Q,\ g : \neg Q,\ p : P\ \vdash\ \bot$$

   (c) Extend the typing derivation above, through the use of λ abstraction, to a proof that the original type is inhabited.

2. Each sequent below defines a local context (terms to the left of the $\vdash$ turnstile) and a goal (the type on the right of the $\vdash$ turnstile) - Using the terms of the local context, show that the goal is inhabited.

   (a) $f : (P \times Q) \to R \vdash P \to (Q \to R)$

   Solution:

$$\dfrac{\dfrac{\dfrac{f\ :\ (P \times Q) \to R \qquad \dfrac{\overline{a : P}^{\,1} \quad \overline{b : Q}^{\,2}}{(a,b) : P \times Q}\,\times}{f(a,b)\ :\ R}\,\mathsf{APP}}{\lambda y.\ f(a,y)\ :\ Q \to R}\,\lambda,2}{\lambda x.\lambda y.\ f(x,y)\ :\ P \to (Q \to R)}\,\lambda,1$$

   Compare this typing derivation to the natural deduction verifying the sequent:

$$(P \wedge Q) \to R\ \vdash\ P \to (Q \to R)$$

$$\dfrac{\dfrac{\dfrac{(P \wedge Q) \to R \qquad \dfrac{\overline{P}^{\,1} \quad \overline{Q}^{\,2}}{P \wedge Q}\,\wedge I}{R}\,\mathsf{MP}}{Q \to R}\,\to I,2}{P \to (Q \to R)}\,\to I,1$$

   The resulting λ-term (i.e. program) below

$$\lambda x.\lambda y.\ f(x,y)\ :\ P \to (Q \to R)$$

   is the proof-object witnessing the proof of the sequent.

(b) $f : P \to (Q \to R) \ \vdash \ (P \times Q) \to R$

Solution:

$$\dfrac{\dfrac{\dfrac{f : P \to (Q \to R) \qquad \dfrac{\overline{p : P \times Q}^{\;1}}{\mathsf{FST}\ p \ : \ P}\,\mathsf{FST}}{f\ (\mathsf{FST}\ p) \ : \ Q \to R}\,\mathsf{APP} \qquad \dfrac{\overline{p : P \times Q}^{\;1}}{\mathsf{SND}\ p : \ Q}\,\mathsf{SND}}{(f\ (\mathsf{FST}\ p))\ (\mathsf{SND}\ p) \ : \ R}\,\mathsf{APP}}{\lambda x.\ (f\ (\mathsf{FST}\ x))\ (\mathsf{SND}\ x) \ : \ (P \times Q) \to R}\,\lambda, 1$$

Compare this typing derivation to the natural deduction verifying the sequent:

$$P \to (Q \to R) \ \vdash \ (P \times Q) \to R$$

$$\dfrac{\dfrac{\dfrac{P \to (Q \to R) \qquad \dfrac{\overline{P \wedge Q}^{\;1}}{P}\,\wedge E_l}{Q \to R}\,\mathsf{MP} \qquad \dfrac{\overline{P \wedge Q}^{\;1}}{Q}\,\wedge E_r}{R}\,\mathsf{MP}}{(P \wedge Q) \to R}\,\to I, 1$$

The resulting λ-term (i.e. program) below

$$\lambda x.\ (f\ (\mathsf{FST}\ x))\ (\mathsf{SND}\ x) \ : \ (P \times Q) \to R$$

is the proof-object witnessing the proof of the sequent.

(c) $t : \neg P + \neg Q \ \vdash \ \neg(P \times Q)$

(d) $f : \neg(P + Q) \ \vdash \ \neg P \times \neg Q$

(e) $t : \neg P \times \neg Q \ \vdash \ \neg(P + Q)$

(f) $f : P \to Q,\ g : Q \to R \ \vdash \ P \to R$

Solution:

$$\dfrac{\dfrac{g \ : \ Q \to R \qquad \dfrac{f : P \to Q \quad \overline{a : P}^{\;1}}{f\ a \ : \ Q}\,\mathsf{APP}}{g(f\ a) \ : \ R}\,\mathsf{APP}}{\lambda x.\ g(f\ x) \ : \ P \to R}\,\lambda, 1$$

Compare this typing derivation to the natural deduction verifying the sequent:

$$P \to Q,\ Q \to R \vdash \ P \to R$$

$$\dfrac{\dfrac{Q \to R \qquad \dfrac{P \to Q \quad \overline{P}^{\;1}}{Q}\,\mathsf{MP}}{R}\,\mathsf{MP}}{P \to R}\,\to I, 1$$

The resulting λ-term (i.e. program) below

$$\lambda x.\ g(f\ x)\ :\ P \to R$$

is the proof-object witnessing the proof of the sequent.

(g) $t : P + Q,\quad f : P \to R,\quad g : Q \to S \vdash\ R + S$

Solution:

$$\dfrac{p : P+Q \qquad \dfrac{\dfrac{\dfrac{f : P \to R \quad \overline{a : P}^{\,1}}{f\ a : R}\,\mathsf{APP}}{\mathsf{inl}\ (f\ a) : R+S}\,\mathsf{inl}}{\lambda x.\ \mathsf{inl}\ (f\ x) : P \to R+S}\,\lambda,1 \qquad \dfrac{\dfrac{\dfrac{g : Q \to S \quad \overline{b : Q}^{\,2}}{g\ b : S}\,\mathsf{APP}}{\mathsf{inr}\ (g\ b) : R+S}\,\mathsf{inr}}{\lambda y.\ \mathsf{inl}\ (g\ y) : Q \to R+S}\,\lambda,2}{\mathsf{cases}\ p\ (\lambda x.\ \mathsf{inl}\ (f\ x))\ (\lambda y.\ \mathsf{inr}\ (g\ y))\ :\ R+S}\,\mathsf{cases}$$

Compare this typing derivation to the natural deduction verifying the sequent:

$$P \vee Q,\ P \to R,\ Q \to S \vdash\ R \vee S$$

$$\dfrac{P \vee Q \qquad \dfrac{\dfrac{\dfrac{P \to R \quad \overline{P}^{\,1}}{R}\,\mathsf{MP}}{R \vee S}\,\vee I}{P \to R \vee S}\,\to I,1 \qquad \dfrac{\dfrac{\dfrac{Q \to S \quad \overline{Q}^{\,2}}{S}\,\mathsf{MP}}{R \vee S}\,\vee I}{Q \to R \vee S}\,\to I,2}{R \vee S}\,\vee E$$

The resulting λ-term (i.e. program) below

$$\mathsf{cases}\ p\ (\lambda x.\ \mathsf{inl}\ (f\ x))\ (\lambda y.\ \mathsf{inr}\ (g\ y))\ :\ R + S$$

is the proof-object witnessing the proof of the sequent.

(h) $f : P \to R,\quad g : Q \to S,\quad t : \neg R + \neg S\ \vdash\ \neg P + \neg Q$

(i) $p : P,\quad f : \neg P\ \vdash\ \neg Q$

(j) $f : P \to Q,\ g : P \to \neg Q\ \vdash\ \neg P$

3. This exercise shows you an example of a general observation first made by William Tait, relating the simplifications of proofs and the process of computation in the λ-calculus.

   Consider the following proof of the theorem

$$\vdash \ A \wedge B \to B$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \wedge B}^{\,1}}{B}\wedge E_R \qquad \dfrac{\overline{A \wedge B}^{\,1}}{A}\wedge_L}{B \wedge A}\wedge I}{B}\wedge E_L}{A \wedge B \to B}\to, 1$$

   (a) Determine the corresponding proof-object for this proof.

   (b) Why does the proof-object have a redex in it?

   (c) Perform the β-reduction on the proof object from (a).

   (d) What proof does the reduced proof-object correspond to?

   Solution:

   The natural deduction proof stated in the question corresponds to the following type construction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{p \ : \ A \times B}^{\,1}}{B}\mathsf{SND} \qquad \dfrac{\overline{p \ : \ A \times B}^{\,1}}{A}\mathsf{FST}}{B \times A}\times}{B}\mathsf{FST}}{\lambda x : A \times B.\ \mathsf{FST}(\mathsf{SND}x, \mathsf{FST}x) \ : \ A \times B \to B}\lambda, 1$$

   The corresponding proof-object is

$$\lambda x : A \times B.\ \mathsf{FST}(\mathsf{SND}x, \mathsf{FST}x) \ : \ A \times B \to B$$

   Which can be B-reduced to

$$\lambda x : A \times B.\ \mathsf{SND}x \ : \ A \times B \to B$$

   This proof-object takes in a pair and returns the second of the pair.

   As a natural deduction this corresponds to the following:

$$\dfrac{\dfrac{\overline{A \wedge B}^{\,1}}{B}\wedge E_R}{A \wedge B \to B}\to, 1$$

   This simplified program corresponds to a shorter proof. In this sense B-reduction (i.e. computation!) is related to the simplification of proofs.