MATH230: Tutorial Two

Propositional Logic: Natural Deductions

Key ideas

- Write natural deduction proofs using minimal logic.
- Write natural deduction proofs using the intuitionistic abusrdity rule.

Relevant lectures: Lectures 4,5, and 6
Relevant reading: L∃∀N Chapters 3,4

Hand in exercises: 1, 2a, 2b, 2d, 2f, 3a
Due following Friday @ 5pm to the tutor, or lecturer.
Email lecturer to report on topic and references for essay.

Discussion Questions

1. Show $A \vdash \neg\neg A$.

2. Show $A \to B \vdash A \to (A \wedge B)$.

3. Show $(A \wedge B) \vee C \ \vdash \ (A \vee C) \wedge (B \vee C)$.

Tutorial Exercises

1. This exercise breaks the proof of the sequent

$$\vdash\ (P \to Q) \to (\neg Q \to \neg P)$$

into steps to show what's happening when temporary hypotheses are used.

   (a) Using the deduction theorem (temporary hypotheses) move as many hypotheses as possible to the left of the turnstile $\vdash$ to get a new sequent. Proof of this new sequent will ultimately lead to the proof of the original sequent.
   (!) Remember $\neg A \equiv A \to \bot$.

   (b) Prove the following sequent

$$P \to Q, \neg Q, P\ \vdash\ \bot$$

   (c) Extend the proof above, through the use of implication introduction, to a proof of the original sequent.

2. **Minimal Logic.** Provide natural deduction proofs of the following sequents. These deductions require only the use of minimal logic. Some of these sequents have a double turnstile. This means you need to prove both directions.

   (a) $(A \wedge B) \to C \dashv\vdash A \to (B \to C)$
   (b) $\neg A \vee \neg B \vdash \neg(A \wedge B)$
   (c) $\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$
   (d) $A \to B,\ B \to C \vdash A \to C$
   (e) $A \vee B,\quad A \to C,\quad B \to D \vdash C \vee D$
   (f) $A \to C,\quad B \to D,\quad \neg C \vee \neg D \vdash \neg A \vee \neg B$
   (g) $A,\quad \neg A \vdash \neg B$
   (h) $A \to B,\ A \to \neg B \vdash \neg A$

3. **Intuitionistic derivations.** Provide natural deduction proofs of the following. You do not need to use the *classical* $\bot$ rule for these questions, but may find that the *intuitionistic* $\bot$ rule is necessary.

   (a) $A, \neg A \vdash\ B$
   (b) $\neg A \vee B \vdash\ A \to B$
   (c) $A \vee B,\quad \neg A \vdash\ B$
   (d) $\vdash\ \neg(Q \to P) \to (P \to Q)$