Propositional Logic

MATH230

Te Kura Pāngarau
Te Whare Wānanga o Waitaha

Outline

Theorems

Arguments and Proofs

Analysis of the correctness of a theorem relies as much on the connective words as it does the technical definitions.

Example:

If p divides ab, then p divides a or p divides b.

Arguments and Proofs

Analysis of the correctness of a theorem relies as much on the connective words as it does the technical definitions.

Example

If a divides b and b divides a, then $a = b$ or $a = -b$.

Arguments and Proofs

Analysis of the correctness of a theorem relies as much on the connective words as it does the technical definitions.

Example

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Connective Tissue

It is the connective words, as much as the mathematical content words, that we have to analyse when deciding whether these statements are correct; whether these are theorems.

Propositional logic formalises the structure of these connective words.

Example: Natural Language

- If Watson moves in with Holmes, then Holmes will be forever annoyed. Watson moved in with Holmes. Therefore, Holmes will be forever annoyed.

- If Watson can trap Moriarty, then Holmes can. Holmes can't trap Moriarty. Therefore, Watson can't.

- Either Holmes catches Moriarty or the world will fall into chaos. The world has fallen into chaos. Therefore, Holmes did not catch Moriarty.

Argument

Definition

An argument is a finite collection of declarative sentences (propositions), one of which is singled out as the conclusion, while the others are considered premises.

Premises are the evidence claiming to support the conclusion.

Example: Natural Language

- If Watson moves in with Holmes, then Holmes will be forever annoyed. Watson moved in with Holmes. Therefore, Holmes will be forever annoyed.

  Let's break this up into premises and conclusion:

Propositional Structures

Definition

An atomic proposition has no propositional substructure.

We saw above that some propositions do have extra structure: "If... , then...." and "Either .... or ... " and "can't" are important to the nature of the argument.

Such connectives are used to join atomic propositions into compound propositions.

Example: Natural Language

- Either Holmes catches Moriarty or the world will fall into chaos. The world has fallen into chaos. Therefore, Holmes did not catch Moriarty.

  Let's break this up into premises and conclusion and determine the atomic propositions.

Moving Away from Natural Language

It was hoped that mathematics could be written in such a precise manner that it could be routinely checked. Furthermore, it was thought that once mathematics was so formalised, that it could be shown consistent and complete; that is, not able to prove non-sense and able to prove (or refute) every statement.

Toward this end mathematicians (Frege and those that followed him) chose to write mathematics in the language of logic:

- Propositional Logic.
- First Order Predicate Logic.

Propositional Connectives

To express the same syntactic structure of an argument without the ambiguities of a natural language we use capital (English) letters to denote atomic propositions, called *propositional variables*. We use the following symbols to construct compound propositions:

- $\neg$: “It is not the case that... ” or “Not... ”
- $\wedge$: “Both... and ... ”
- $\vee$: “Either... or ... ”
- $\rightarrow$: “If... , then ... ”
- $\leftrightarrow$: “ ... if and only if ... ”

These symbols, the propositional connectives, play the role of the connective tissue in the statements given on previous slides.

Example: NL to PL

- If Watson can trap Moriarty, then Holmes can. Holmes Can't trap Moriarty. Therefore Watson can't.

Grammar

Our language is further made up of *well-formed formulae* which we define inductively as follows:

Definition (Well-Formed Formulae)

- **Atomic Formulae:** *If* α *is a single propositional variable, then* α *is a wff.*
- **Negation:** *If* α *is a wff, then* $\neg\alpha$ *is a wff.*
- **Binary Connective:** *If* α *and* β *are wff and* $*$ *is a binary connective, then* $(\alpha * \beta)$ *is a wff.*

Notation: We will refer to the totality of well-formed propositions as "Prop" and we will write "$\alpha : \mathrm{Prop}$" to denote the fact that α is a well-formed proposition.

Examples

Which of the following are wff in propositional logic?

1. A
2. AB
3. $(A \rightarrow B)$
4. $A \rightarrow B \rightarrow C$
5. $((A \rightarrow B) \rightarrow C)$
6. $\neg Q$
7. $A \vee Q$
8. $A \rightarrow \neg B \vee C$

Binding Conventions

Binding Conventions:

- $\neg$ binds most tightly,
- $\vee$ and $\wedge$ bind more tightly than $\rightarrow$,
- $\rightarrow$ binds more tightly than $\leftrightarrow$.

Example: Parse the wff $A \rightarrow \neg B \vee C$.

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To disambiguate $\wedge$ and $\vee$ we group terms from the left. In this way, we say that $\wedge$ and $\vee$ associate to the left.

Example: Parse the wff $A \wedge B \wedge C$.

Syntax Trees

Using the binding convention above allows for each well-formed formula to be parsed into a syntax tree.

$A \wedge (B \vee C)$	$A \wedge \neg B \wedge C$	$A \to B \vee C \wedge D$

Hiding the Goods

By packing the statements:

"p divides ab" or "f is continuous on $[a, b]$"

into a propositional variable A, we have lost a lot of information from the statement that we're trying to analyse. It's no longer about primes or continuous functions.

For now, we will focus on the connectives alone. Studying the structure of the argument, rather than the mathematics.

Later we will introduce more structure to our logic which will allow us to bring back the mathematical content.

Example

"Thin is guilty," observed Watson, "because either Holmes is right and the vile Moriarty is guilty, or he (Holmes) is wrong and Thin did the job; but those scoundrels are either both guilty or both innocent; and, as usual, Holmes is correct".

Example from Richard Jeffrey, *Formal Logic: Its Scope and Limits*

Argument Structure

$$\frac{\begin{array}{c}\text{Proposition 1}\\ \text{Proposition 2}\\ \vdots\\ \text{Proposition n}\end{array}}{\text{Conclusion}}$$

Question: What makes for a "good argument"? What might we mean by a "good argument"? What dos it mean for the conclusion to follow from the hypotheses?

Truth and Proof

Semantic — If the conclusion is "true" whenever all hypotheses are "true", then the conclusion is said to be a semantic consequence of the hypotheses.

Syntactic — If there is a "proof" that the hypotheses "unfold" and "combine" towards the conclusion, then the conclusion is said to be a syntactic consequence of the hypotheses.

Theorem: γ is a semantic consequence of Σ if and only if γ is a syntactic consequence of Σ [**vDalen**].

Example

Provide a proof to show

$$P \wedge (Q \wedge R),\ R \to T\ \vdash\ P \wedge T$$

- How are we to “unfold” or make use of hypotheses?
- How are to obtain the conclusion?
- What should such a proof look like?

BHK Intepretation

Brouwer, Heyting, and Kolmogorov proposed the following (inductive) interpretation of what it should mean to prove statements involving propositional connectives:

$P \wedge Q$	to prove a conjunction we must provide both a proof of P and a proof of Q.
$P \rightarrow Q$	to prove an implication we must provide an algorithm for turning a proof of P into a proof of Q.
$P \vee Q$	to prove a disjunction we must provide either a proof of P or a proof of Q.
$\neg P$	to prove a negation we must provide an algorithm that turns a proof of P into a proof of $\perp$.

This presentation is taken from the Standford Encyclopedia of Philosophy article by Bridges, Palmgren, and Ishihara [**sep-mathematics-constructive**].

Natural Deduction Calculus

We are going to develop a proof method that is inline with the BHK interpretation of the logical connectives. This proof method was first presented by Gerhard Gentzen. As such it is often referred to as the Gentzen Calculus.

This method will develop proofs by unfolding their hypotheses in a manner consistent with the BHK.

Example

Provide a proof to show

$$P \wedge (Q \wedge R),\ R \rightarrow T \ \vdash\ P \wedge T$$

Natural Deduction Calculus

Example: What would it require to deduce $A \wedge B$ in the course of a proof? Use the BHK!

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$$\dfrac{\begin{matrix}\Sigma_1 & \Sigma_2\\ \mathcal{D}_1 & \mathcal{D}_2\\ A & B\end{matrix}}{A \wedge B}\ \wedge I$$

If we have a deduction for A and a deduction for B, then together we should consider those deductions a proof for $A \wedge B$.

$\wedge$ Elimination

Example: Suppose $A \wedge B$ were a premise in a proof. What can we conclude from such a premise?

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$$\cfrac{\begin{array}{c}\Sigma \\ \mathcal{D} \\ A \wedge B\end{array}}{A}\ \wedge E_L \qquad\qquad \cfrac{\begin{array}{c}\Sigma \\ \mathcal{D} \\ A \wedge B\end{array}}{B}\ \wedge E_R$$

Example: Commutativity of $\wedge$

Provide a proof to show

$$A \wedge B \vdash B \wedge A$$

Example: Idempotence of $\wedge$

Provide a proof to show

$$A \wedge A \vdash A$$

Deductions

Definition

We define deductions (or derivations, or proofs) inductively according to the following rules:

- *For each formula α,*

  $$\alpha$$

  is a deduction with conclusion α and premises $\{\alpha\}$.
- *From a given deduction, an application of a rule of inference yields a new deduction.*
- *Anything that is not a deduction by virtue of the above is not a deduction.*

If there exists a deduction ${}^{\Sigma}_{\alpha}\mathcal{D}$ of α from Σ, then we say α is a *syntactic consequence of* (*derivable from*, or *provable from*) Σ and denote this $\Sigma \vdash \alpha$.

Propositional Calculus

If we follow this idea for all of the logical connectives in propositional logic, then we can develop a method for writing proofs based on the *syntactic* structre of the logical connectives alone. We call this method of proof *Natural Deduction* and we are following Gerhard Gentzen's notation [**vDalen**, **thompson**].

We need to know how to (i) deduce and (ii) conclude from, each logical connective. In other words, for each logical connective we need to develop rules for introducing the logical connective and eliminating the logical connective.

So we will spend some time writing down the **Rules of Inference** for our logical connectives.

Hypothetical Reasoning

In mathematics we often prove statements of the following hypothetical form: "If ..., then ..."

Example: If f is differentiable at x_0, then it is continuous at x_0.

Proof: "Let f be a differentiable function..."

The proof of an implication will assume the hypothesis of differentiability and show that it implies continuity. In order to prove an implication $P \rightarrow Q$ the proof starts by assuming we know P and then using that to tell us about Q.

The conclusion is the entire implication, not just continuity.

Example: Hypothetical Reasoning

If $p|a$ and $p|(a+b)$, then $p|b$.

Example: Hypothetical Reasoning

$$\dfrac{\dfrac{\overline{A \wedge (B \wedge C)}^{\;1}}{B \wedge C}\,\wedge E_L}{B}\,\wedge E_L$$

Question: What does the proof prove?

$\rightarrow$ Introduction

If ${}^{\Sigma}_{\beta}\mathcal{D}$ is a deduction of β from Σ, then

$$\dfrac{\begin{array}{c}\Sigma \cup \{\cancel{\alpha}\} \\ \mathcal{D} \\ \beta\end{array}}{\alpha \to \beta} \to I$$

is a deduction of $\alpha \to \beta$ from hypotheses $\Sigma \backslash \{\alpha\}$.

Note: As the assumption α is struck out after this deduction, we are free to use α *even if it is not in* Σ when using implication introduction.

$\rightarrow$ Elimination (MP)

If ${}^{\Sigma_1}_{\alpha\to\beta}\mathcal{D}_1$ and ${}^{\Sigma_2}_{\alpha}\mathcal{D}_2$ are deductions, then

$$\dfrac{\begin{array}{c}\Sigma_1\\ \mathcal{D}_1\\ \alpha \rightarrow \beta\end{array} \qquad\qquad \begin{array}{c}\Sigma_2\\ \mathcal{D}_2\\ \alpha\end{array}}{\beta} \rightarrow E$$

is a deduction of β from $\Sigma_1 \cup \Sigma_2$.

Example

Show $P \to Q,\ Q \to R\ \vdash\ P \to R$

Deduction Theorem

Theorem: $\Sigma \vdash \alpha \to \beta$ if and only if $\Sigma \cup \{\alpha\} \vdash \beta$

Proof

Example

Show that $P \to (\neg S \to L), \quad P \to \neg S, \quad P \vdash L$

Currying

Show $(A \land B) \to C \vdash A \to (B \to C)$

Example

Show $A \to B, A \to C, A \vdash B \wedge C$

$\vee$ Introduction

If ${}^{\Sigma}_{\alpha}\mathcal{D}$ is a derivation of α from Σ, then

$$\dfrac{\begin{array}{c}\Sigma\\ \mathcal{D}\\ \alpha\end{array}}{\alpha \vee \beta}\ \vee I_R \qquad\qquad \dfrac{\begin{array}{c}\Sigma\\ \mathcal{D}\\ \alpha\end{array}}{\beta \vee \alpha}\ \vee I_R$$

are derivations of $\alpha \vee \beta$ and $\beta \vee \alpha$ from Σ.

Note: We are free to choose β as, if we know α to be the case, then $\alpha \vee \beta$ is necessarily the case *for any* β.

$\vee$ Elimination

If ${}^{\Sigma_1}_{\alpha\vee\beta}\mathcal{D}_1$, ${}^{\Sigma_2}_{\alpha\to\gamma}\mathcal{D}_2$, and ${}^{\Sigma_2}_{\beta\to\gamma}\mathcal{D}_3$ are derivations, then

$$\dfrac{\begin{array}{c}\Sigma_1\\ \mathcal{D}_1\\ \alpha\vee\beta\end{array}\qquad \begin{array}{c}\Sigma_2\\ \mathcal{D}_2\\ \alpha\to\gamma\end{array}\qquad \begin{array}{c}\Sigma_2\\ \mathcal{D}_2\\ \beta\to\gamma\end{array}}{\gamma}\ \vee_E$$

is a derivation of γ from $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$.

Note: You can't remove one of the arguments from a disjunction. Knowledge of $\alpha \vee \beta$ is not sufficient to conclude either α or β alone. Following the BHK, a proof of $\alpha \vee \beta$ is either a proof of α or a proof of β, but without a record of which case we are in we can't assume either way. This means we need to account for both possibilities.

Example

Show $A \vee B,\ (A \vee C) \rightarrow D,\ B \rightarrow D\ \vdash D$

Example

Show $A \vee (B \wedge C) \;\vdash (A \vee B) \wedge (A \vee C)$

Common Mistake (!)

Positive Minimal Logic

The calculus developed so far with the introduction and elimination rules of the three connectives $\wedge, \vee$, and $\rightarrow$ is called positive minimal logic.

Many theorems can be (stated and) proved with these rules of inference alone.

But there are theorems that can't be proved using these rules alone.

Falsum

We introduce the logical constant $\perp$ (falsum or absurdity) to define the syntactic form of the $\neg$ connective. We make the following definition:

$$\neg\alpha := \alpha \to \perp$$

$$\frac{\alpha \qquad \alpha \to \perp}{\perp}\ MP \qquad\qquad \frac{\begin{array}{c}\overline{\cancel{\alpha}}\\ \mathcal{D}\\ \perp\end{array}}{\alpha \to \perp} \to I$$

Falsum $\perp$ is an atomic proposition which is to be thought of as denoting "absurdity" or "contradiction".

Example: Modus Tollens

Show $A \rightarrow B,\ \neg B\ \vdash \neg A$

Contradiction Implies Absurdity

Show $A \wedge \neg A \;\vdash\; \bot$

Minimal Logic

Together the rules of inference that we've given so far define *minimal* logic. They include much, but not all, of the logical inferences that practising mathematicians might use in a proof. What they do include is uncontroversial.

However, it is not universally agreed as to how minimal logic should be extended. There are philosophical differences among mathematicians and logicians about what other rules of inference should be included.

- Intuitionistic logic
- Classical logic
- Modal logic

What's Missing?

What should be done if the hypotheses yield a contradiction?

$$\cfrac{\begin{array}{c}\Sigma \\ \mathcal{D} \\ \bot\end{array}}{?}\ ?$$

Should we be able to prove the following?

$\vdash\ P \vee \neg P$

$\neg\neg P\ \vdash\ P$

$\vdash\ (P \to Q) \vee (Q \to P)$

Affirmative answers to these questions require further rules of inference.

Ex Falso Sequitur Quadlibet

So far, we have not made much mention of how to deal with the derivation of $\perp$ absurdity. Indeed, it has no introduction rule.

If ${}^{\Sigma}_{\perp}\mathcal{D}$ is a deduction of $\perp$ from Σ, then

$$\dfrac{\begin{array}{c}\Sigma\\ \mathcal{D}\\ \perp\end{array}}{\alpha}\ \mathsf{XF}$$

is a derivation of α from the assumptions Σ.

Anything you want follows from a falsehood.

Disjunctive Syllogism

Show $A \vee B, \neg B \vdash A$

XF as Null Disjunction

Disjunction has two introduction rules; both of which need to be taken into account when eliminating a disjunction.

Following this one can argue that since $\perp$ has no introduction rules, there is nothing to take into account of when eliminating $\perp$ and hence XF allows one to conclude anything. All possible cases (i.e. all zero of them) lead to P, therefore we may conclude P follows from Falsum.

Intuitionistic Logic

Ex Falso Quodlibet extends the class of theorems provable in the natural deduction calculus. It is the logic of intuitionists and constructivists; mathematicians who believe proofs should have computational content.

$$\text{Minimal Logic} + \text{Ex Falso} = \text{Intitionistic Logic}$$

However, there are classically valid sequents, such as the LEM, which are not derivable in the intuitionistic calculus.

Double Negation Elimination

Show $\neg\neg A \vdash A$

Double Negation Elimination

Show $\neg\neg A \vdash A$

Ex falso does not give us a proof. In fact we have shown the following: $\{\neg\neg A, \neg A\} \vdash A$.

Summary

We have built the propositional calculus up in steps:

- Positive minimal logic,
- Minimal logic,
- Intuitionistic logic.

One can show, using non-classical semantics, that these logics are unable to prove some theorems that are fundamental to much of mathematics.

We will consider one more mode of reasoning.

Example

If $x, y > 0$, then $x + y > 0$.

Reductio Ad Absurdum

If $\begin{smallmatrix}\Sigma\\ \mathcal{D}\\ \bot\end{smallmatrix}$ is a deduction of $\bot$ from Σ, then

$$\dfrac{\begin{array}{c}\Sigma \cup \{\cancel{\neg\alpha}\}\\ \mathcal{D}\\ \bot\end{array}}{\alpha}\ \mathsf{RAA}$$

is a derivation of α from the assumptions $\Sigma \backslash \{\neg\alpha\}$.

If absurdity follows from $\neg\alpha$, then we may conclude α **and discharge $\neg\alpha$ from our assumptions.**

Double Negation

Show $\neg\neg A \vdash A$

Law of Excluded Middle

Show $\vdash A \vee \neg A$

Classical Logic

The class of theorems one can prove increases with the addition of RAA. In fact, the class of theorems provable in classical logic includes all theorems of intuitionistic logic.

$$\text{Minimal Logic} + \text{RAA} = \text{Classical Logic}$$

One can derive the Ex Falso rule of inference using RAA.

Common Misconception

Many proofs that claim to use reductio ad absurdum are really *refutations by contradiction.*

- Irrationality of $\sqrt{2}$
- Infinitude of primes
- No smallest positive rational number

These proofs just use implication introduction to prove a negation.

Example

Show $A \to B, \;\; A \to \neg B \; \vdash \neg A$

Derived Rules of Inference

Proofs can be simplified by using results already proved. You may, in the course of a proof, use any result that has been proven in class or previously in a tutorial. However, when substituting previous proofs, you must bring all of the premises with the conclusion.

We have already seen this with the use of *modus tollens* (MT) in some examples.

Making use of (an instance of) LEM instead of RAA can make proofs more straight forward.

This can help keep proofs manageable and neat.

Example: Substituting LEM

Show $A \to B \vdash \neg A \vee B$

Intuitionistic to Classical

$$\text{Classical} = \text{IL} + \text{RAA} = \text{IL} + \text{LEM} = \text{IL} + \text{DNE}$$

We described the passage from intuitionistic logic to classical by the addition of the RAA rule of inference. We can get a logic of equivalent power in a number of ways.

One could declare for each P, $P \vee \neg P$ as a theorem.

One could add a double negation elimination rule of inference.

Adding any of these to minimal logic gives you the same set of theorems as classical logic as we defined it above.

See tutorial to prove this.

Departure from BHK

Notice that the addition of RAA has forced us to lose the BHK interpretation of our proofs. For each proposition P LEM is a theorem:

$$\vdash P \vee \neg P$$

The BHK asserts a proof of $A \vee B$ must consist either of a proof of A or a proof of B. But the classical proof of $\vdash P \vee \neg P$ does not contain that information; it does not tell us which of P or $\neg P$ is provable.

The inclusion of RAA allows for proofs of some apparently harmless theorems like DNE and RAA. However this power is not without its consequences...

Example

Provide a proof to show

$$\vdash (A \to B) \vee (B \to A)$$

Dealer's Choice

Choice of logic (i.e. rules of inference) is upto the logician/mathematician. Therefore, we should be more specific when we assert one Prop is a syntactic consequence of a set Σ. This is a *relative* notion and so one should quote the logic used in the derivation.

Logical Equivalence

Definition

We say well-formed formulae are syntactically equivalent if both

$$\alpha \vdash \beta \quad \text{and} \quad \beta \vdash \alpha$$

Examples

- $A \vee B \dashv\vdash B \vee A$
- $A \rightarrow B \;\; \dashv\vdash \;\; \neg A \vee B$

Logical equivalences may be stated *with respect to a logic*.

Theorems

Definition

We say a well-formed formula α is a theorem if there exists a natural deduction $\mathcal{D}$ from no assumptions i.e. $\Sigma = \emptyset$ and we denote this as $\vdash \alpha$.

Example: Law of the Excluded Middle

Example: $\vdash A \to (B \to A)$

Note: This should be stated *with respect to a logic.*

Different Logics

When we say one logic is weaker relative to another, what we are saying is that the set of theorems is a subset of the other.

Reference to "classical logic" as a whole might be referring to the collection of the rules of inference, or it might be referring to the collection of all theorems of classical logic.

$$\text{Positive Minimal} \subset \text{Minimal} \subset \text{Intuitionistic} \subset \text{Classical}$$

Example of Equivalence

If α and β are syntactically equivalent, then $\vdash \alpha \leftrightarrow \beta$.

Equivalence of Theorems

If $\vdash \alpha$ and $\vdash \beta$, then $\vdash \alpha \leftrightarrow \beta$

Further Reading