MATH230: Tutorial Six (Solutions)

Peano Arithmetic

Key ideas

- Natural deduction practice,
- Proofs using the identity rules of inference,
- Prove first-order sentences in theories of arithmetic,
- Use the induction schema of Peano arithmetic, and
- Become exasperated enough to appreciate the help of proof assistants.

Relevant lectures: Lectures 14,15, and 16
Relevant reading: Natural Number Game

Hand in exercises: 1a, 2a, 2b, 2c, 3a, 3b, 3c
Due following Friday @ 5pm to the tutor, or to lecturer.

Discussion Questions

Proofs that make use of the axiom (schema) PA 7

$$[P(0) \wedge \forall x\ (P(x) \to P(s(x)))] \to \forall y(P(y))$$

will prove statements of the form $\forall y\ P(y)$ with the use of modus ponens. This requires proving the antecedent conjunction:

$$[P(0) \wedge \forall x\ (P(x) \to P(s(x)))]$$

This in turn requires proving each conjunct i.e. two proofs witnessing:

$$\text{PA} \vdash P(0) \qquad\qquad \text{PA} \vdash \forall x\ (P(x) \to P(s(x)))$$

If we piece this together then we see that all proofs by induction have the form:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \mathcal{D}_{BC}\end{matrix}}{P(0)} \qquad \dfrac{\begin{matrix}\vdots\\ \mathcal{D}_{IS}\end{matrix}}{\forall x\ (P(x) \to P(s(x)))}\ \forall I}{[P(0) \wedge \forall x\ (P(x) \to P(s(x)))]}\ \wedge I}{\forall y\ P(y)}\ \text{IND}$$

1. Identify the following steps involved in a proof by induction of the following sentence of Peano arithmetic:

$$\mathsf{PA} \vdash \forall x \ (0 + x = x)$$

   (a) Identify the wff $P(x)$ to do induction on.

   Solution: $P(x) \equiv 0 + x = x$.

   (b) $\mathcal{D}_{BC}$: Write down the sequent PA $\vdash P(0)$.

   Solution: PA $\vdash 0 + 0 = 0$

   (c) $\mathcal{D}_{IS}$: Write down the sequent PA$, P(n) \vdash P(s(n))$.

   Solution: PA, $0 + n = n \vdash 0 + s(n) = s(n)$

   Ultimately the induction step requires $\forall$ introduction. But the sequent stated above will lead to that. Can you see why?

Tutorial Exercises

1. Give natural deductions of the following theorems of identity.

   (a) $\vdash\ \forall x \forall y \forall z\ (x = y \wedge y = z) \to x = z$

   Solution: To introduce the $\forall$ we argue from a general case. Furthermore, we are to prove an implication, so we add a temporary hypothesis to discharge later. So the crux of the proof comes down to showing:

   $$(a = b) \wedge (b = c)\ \vdash\ a = c$$

   $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{a = b \wedge b = c}}\,1}{b = c}\,\wedge E_r \qquad \dfrac{\dfrac{}{\cancel{a = b \wedge b = c}}\,1}{a = b}\,\wedge E_l}{a = c}\,= E}{(a = b \wedge b = c) \to a = c}\,\to I, 1}{\forall x\ \forall y\ \forall z\ (x = y \wedge y = z) \to x = z}\,\forall I$$

   (b) $\vdash\ \forall x \forall y \forall z\ x \neq y \to (x \neq z \vee y \neq z)$ (RAA)

   Solution: As above we argue from a general case and add a temporary hypothesis. The crux of the proof comes down to a proof witnessing

   $$a \neq b\ \vdash\ (a \neq c) \vee (b \neq c)$$

   Since this is a disjunction, one might suspect RAA is needed. Indeed, the proof below uses RAA. In the end one is really proving:

   $$a \neq b,\ \neg[(a \neq c) \vee (b \neq c)]\ \vdash\ \bot$$

   and then using RAA, $\to I$, and $\forall I$ to tidy up the end steps.

   $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{\neg(a = c \vee b = c)}}\,1}{\neg(a \neq c) \wedge \neg(b \neq c)}\,\text{DM}}{\neg(a \neq c)}\,\wedge E_l}{a = c}\,\text{DNE} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{\neg(a = c \vee b = c)}}\,1}{\neg(a \neq c) \wedge \neg(b \neq c)}\,\text{DM}}{\neg(b \neq c)}\,\wedge E_r}{b = c}\,\text{DNE}}{a = b}\,= E \qquad \dfrac{}{\cancel{a \neq b}}\,2}{\bot}\,\text{MP}}{(a \neq c \vee b \neq c)}\,\text{RAA}, 1}{a \neq b \to (a \neq c \vee b \neq c)}\,\to I, 2}{\forall x \forall y \forall z\ x \neq y \to (x \neq z \vee y \neq z)}\,\forall I$$

   (c) $\vdash\ \exists x\ (t = x)$

   Solution:

   $$\dfrac{\dfrac{}{t = t}\,= I}{\exists x\ (t = x)}\,\exists I$$

2. In this question PA denotes the first-order theory of Peano arithmetic which has signature PA: $\{0, s, +, \times\}$ and axioms:

   PA1 $\forall x \neg(s(x) = 0)$

   PA2 $\forall x\ \forall y((s(x) = s(y)) \to (x = y))$

   PA3 $\forall x\ (x + 0 = x)$

   PA4 $\forall x\ \forall y\ (x + s(y) = s(x + y))$

   PA5 $\forall x\ (x \times 0 = 0)$

   PA6 $\forall x\ \forall y\ (x \times s(y) = (x \times y) + x)$

   PA7 $[P(0) \land \forall x\ (P(x) \to P(s(x)))] \to \forall y(P(y))$

   Provide deductions to prove the following sequents.

   Comments: Axioms PA1 - PA6 are each of the form $\forall$... which means that $\forall E$ must be used to start these proofs. A lot of these proofs turn on the right choice of term to substitute in for x, y, z in the $\forall E$ step. Look at the statement you're trying to prove and find the correct substitution into an axiom to make it come out.

   Theorems involving the binary function $+$ will necessarily make use of PA3 and/or PA4 as those are the axioms defining the properties of addition.

   Theorems involving the binary function $\times$ will necessarily make use of PA5 and/or PA6 as those are the axioms defining the properties of multiplication.

   PA 1 is the only statement of the form "this is not equal to that". This means that showing any two things are not equal must ultimately boil down to showing that if they were equal, then (something like) 0=1 would follow.

(a) PA $\vdash 1+1=2$

Solution: First the sequent should be desugared as 1,2 are not terms in PA. So the sequent in PA is the following:

$$\text{PA} \ \vdash \ s(0)+s(0)=s(s(0))$$

$$\dfrac{\dfrac{\text{PA3}}{s(0)+0=s(0)} \qquad \dfrac{\text{PA4}}{s(0)+s(0)=s(s(0)+0)}\,\forall E}{s(0)+s(0)=s(s(0))}\,{=}E$$

(b) PA $\vdash 3 \neq 1$

Solution: Desugaring into PA yields the following sequent:

$$\text{PA} \ \vdash \ \neg[s(s(s(0)))=s(0)]$$

Recall that introducing a negation ultimately comes down to showing

$$\text{PA}, \ s(s(s(0)))=s(0) \ \vdash \ \bot$$

and then using $\to I$ to tidy up at the end.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{s(s(s(0)))=s(0)}}\,1 \qquad \dfrac{\text{PA2}}{[s(s(s(0)))=s(0)]\to[s(s(0))=0]}\,\forall E}{s(s(0))=0}\,\text{MP} \qquad \dfrac{\text{PA1}}{\neg[s(s(0))=0]}\,\forall E}{\bot}\,\text{MP}}{\neg[s(s(s(0)))=s(0)]}\,\to I,1$$

(c) PA $\vdash \forall x\ [x+1=s(x)]$

Solution: Desugaring into PA yields the following sequent:

$$\text{PA} \vdash \forall x\ [x+s(0)=s(x)]$$

$$\cfrac{\cfrac{\dfrac{\text{PA3}}{a+0=a}\ \forall E \qquad \dfrac{\text{PA4}}{a+s(0)=s(a+0)}\ \forall E}{a+s(0)=s(a)}\ {=}E}{\forall x\ [x+s(0)=s(x)]}\ \forall I$$

(d) PA $\vdash \forall x\ (x \times 1 = x)$

Solution: Desugaring into PA yields the following sequent:

$$\text{PA} \ \vdash \ \forall x\ [x \times s(0) = x]$$

This proof will make use of Exercise 3a of this tutorial.

$$\cfrac{\cfrac{\cfrac{\dfrac{\text{PA6}}{s(a)\times s(0)=(s(a)\times 0)+s(a)}\ \forall E \qquad \dfrac{\text{PA5}}{s(a)\times 0=0}\ \forall E}{s(a)\times s(0)=0+s(a)} \qquad \dfrac{\text{PA}}{0+s(a)=s(a)}\ \text{THM}}{s(a)\times s(0)=s(a)}\ {=}E}{\forall x\ [x\times s(0)=x]}\ \forall I$$

3. The followings sequents all require the use of the induction axiom schema. Recall that all proofs using the induction schema have the following form:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_{BC}\end{array}}{P(0)} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_{IS}\end{array}}{\forall x\ (P(x) \to P(s(x)))}\ \forall I}{[P(0) \land \forall x\ (P(x) \to P(s(x)))]}\ \land I}{\forall y\ P(y)}\ \text{IND}$$

For this reason, once the wff $P(x)$ is identified, it suffices to provide the base case deduction $\mathcal{D}_{BC}$ and induction step $\mathcal{D}_{IS}$. The sequents are stated in such a way as to mean induction on the variable x will be the easiest approach. Always do induction on the variable x.

(a) PA $\vdash \forall x\ (0 + x = x)$

Solution:

We prove this using induction on the wff

$$P(x) :\ (0 + x = x)$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash (0 + 0 = 0)$$

$$\dfrac{\text{PA3}}{0 + 0 = 0}\ \forall E$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ (0 + n = n)\ \vdash\ (0 + s(n) = s(n))$$

$$\dfrac{\dfrac{\dfrac{\text{PA4}}{0 + s(n) = s(0 + n)}\ \forall E \qquad \dfrac{}{\cancel{0 + n = n}}\ \text{IH}}{0 + s(n) = s(n)}\ = E}{(0 + n = n) \to (0 + s(n) = s(n))}\ \to I, \text{IH}$$

(b) PA $\vdash \forall x\ (0 \times x = 0)$

Solution:

This proof is by induction on the wff

$$P(x):\ (0 \times x = 0)$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash 0 \times 0 = 0$$

$$\dfrac{\text{PA5}}{0 \times 0 = 0}\ \forall E$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ 0 \times n = 0\ \vdash\ 0 \times s(n) = 0$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{PA6}}{0 \times s(n) = 0 \times n + 0}\ \forall E \qquad \dfrac{\text{PA3}}{0 \times n + 0 = 0 \times n}\ \forall E}{0 \times s(n) = 0 \times n}\ {=}E \qquad \dfrac{}{\cancel{0 \times n = 0}}\ \text{IH}}{0 \times s(n) = 0}\ {=}E}{(0 \times n = 0) \to (0 \times s(n) = 0)}\ \to I, \text{IH}$$

(c) PA $\vdash \forall x\ (1 \times x = x)$

Solution:

This proof is by induction on the wff

$$P(x) :\ s(0) \times x = x$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash s(0) \times 0 = 0$$

$$\dfrac{\text{PA5}}{s(0) \times 0 = 0}\ \forall E$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ s(0) \times n = n\ \vdash\ s(0) \times s(n) = s(n)$$

The deduction below makes use of Exercise 2c above.

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{PA6}}{s(0) \times s(n) = (s(0) \times n) + s(0)}\ \forall E \qquad \dfrac{}{\cancel{s(0) \times n = n}}\ \text{IH}}{s(0) \times s(n) = n + s(0)}\ {=}E \qquad \dfrac{\text{PA}}{n + s(0) = s(n)}\ \text{THM}}{s(0) \times s(n) = s(n)}\ {=}E}{(s(0) \times n = n) \to (s(0) \times s(n) = s(n))}\ \to I, \text{IH}$$

(d)

$\mathsf{PA} \vdash \forall x\ (x = 0 \vee \exists y (x = s(y)))$

Solution:

This proof is by induction on the wff

$$P(x): \ [x = 0 \vee \exists y (x = s(y))]$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\mathsf{PA} \vdash [0 = 0 \vee \exists y (0 = s(y))]$$

$$\dfrac{\dfrac{}{0 = 0}\; = I}{0 = 0 \vee \exists y\ (0 = s(y))}\; \vee I$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\mathsf{PA},\ [n = 0 \vee \exists y (n = s(y))]\ \vdash\ [s(n) = 0 \vee \exists y (s(n) = s(y))]$$

Since the induction hypothesis is a disjunction, the proof will finish with a disjunction elimination step. This requires proving the following sequents along the way:

$$\mathsf{PA}\ \vdash\ (n = 0) \rightarrow [s(n) = 0 \vee \exists y (s(n) = s(y))]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{n = 0 \rightarrow s(n) = s(0)}\;\mathsf{THM} \qquad \dfrac{}{\cancel{n = 0}}\;1}{s(n) = s(0)}\;\mathsf{MP}}{\exists y\ (s(n) = s(y))}\;\exists I}{(s(n) = 0) \vee \exists y\ (s(n) = s(y))}\;\vee I}{(n = 0) \rightarrow [(s(n) = 0) \vee \exists y\ (s(n) = s(y))]}\;\rightarrow I, 1$$

$$\mathsf{PA}\ \vdash\ \exists y\ (n = s(y)) \rightarrow [s(n) = 0 \vee \exists y (s(n) = s(y))]$$

$$\dfrac{\dfrac{\dfrac{}{\cancel{\exists y\ (n = s(y))}}\;2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{n = s(w)}}\;3 \qquad \dfrac{}{(n = s(w)) \rightarrow (s(n) = s(s(w)))}\;\mathsf{THM}}{s(n) = s(s(w))}\;\mathsf{MP}}{\exists y\ (s(n) = s(y))}\;\exists I}{(s(n) = 0) \vee \exists y\ (s(n) = s(y))}\;\vee I}{n = s(w) \rightarrow [(s(n) = 0) \vee \exists y\ (s(n) = s(y))]}\;\rightarrow I, 3}{(s(n) = 0) \vee \exists y\ (s(n) = s(y))}\;\exists E}{\exists y\ (n = s(y)) \rightarrow [(s(n) = 0) \vee \exists y\ (s(n) = s(y))]}\;\rightarrow I, 2$$

See over the page for these steps combined with the disjunction elimination to complete the proof of the induction step.

PA, $[n = 0 \lor \exists y(n = s(y))] \;\vdash\; [s(n) = 0 \lor \exists y(s(n) = s(y))]$

$$
\dfrac{
\dfrac{}{n = 0 \lor \exists y\ (n = s(y))}\ \mathsf{IH}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{n = 0 \to s(n) = s(0)}\ \mathsf{THM} \quad \dfrac{}{\cancel{n = 0}}\ 1}{s(n) = s(0)}\ \mathsf{MP}}{\exists y\ (s(n) = s(y))}\ \exists I}{(s(n) = 0) \lor \exists y\ (s(n) = s(y))}\ \lor I}{(n = 0) \to [(s(n) = 0) \lor \exists y\ (s(n) = s(y))]}\ \to I, 1
\qquad
\dfrac{\dfrac{\dfrac{}{\cancel{\exists y\ (n = s(y))}}\ 2 \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{n = s(w)}}\ 3 \quad \dfrac{}{(n = s(w)) \to (s(n) = s(s(w)))}\ \mathsf{THM}}{s(n) = s(s(w))}\ \mathsf{MP}}{\exists y\ (s(n) = s(y))}\ \exists I}{(s(n) = 0) \lor \exists y\ (s(n) = s(y))}\ \lor I}{n = s(w) \to [(s(n) = 0) \lor \exists y\ (s(n) = s(y))]}\ \to I, 3}{(s(n) = 0) \lor \exists y\ (s(n) = s(y))}\ \exists E}{\exists y\ (n = s(y)) \to [(s(n) = 0) \lor \exists y\ (s(n) = s(y))]}\ \to I, 2
}{[(s(n) = 0) \lor \exists y\ (s(n) = s(y))]}\ \lor E
$$

(e) PA $\vdash \forall x\ \forall y\ [s(y) + x = s(y + x)]$ (Challenge!)

Solution:

This proof is by induction on the wff

$$P(x) :\ \forall y\ [s(y) + x = s(y + x)]$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash \forall y\ [s(y) + 0 = s(y + 0)]$$

$$\dfrac{\dfrac{\dfrac{}{s(a) = s(a)}{=}I \quad \dfrac{\text{PA3}}{s(a) + 0 = s(a)}\forall E}{s(a) + 0 = s(a)}{=}E \quad \dfrac{\text{PA3}}{a = a + 0}\forall E}{s(a) + 0 = s(a + 0)}{=}E$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ \forall y\ [s(y) + n = s(y + n)]\ \vdash\ \forall y\ [s(y) + s(n) = s(y + s(n))]$$

$$\dfrac{\dfrac{\dfrac{\text{PA4}}{s(a) + s(n) = s(s(a) + n)}\forall E \quad \dfrac{}{s(a) + n = s(a + n)}\text{IH}}{s(a) + s(n) = s(s(a + n))} \quad \dfrac{\text{PA4}}{a + s(n) = s(a + n)}\forall E}{s(a) + s(n) = s(a + s(n))}{=}E$$

(f) PA $\vdash \forall x\, \forall y\, \forall z\, [(y+z)+x = y+(z+x)]$ (Challenge!)

Solution:

This proof is by induction on the wff

$$P(x):\ \forall y\, \forall z\, [(y+z)+x = y+(z+x)]$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash \forall y\, \forall z\, [(y+z)+0 = y+(z+0)]$$

$$\dfrac{\dfrac{\dfrac{\text{PA3}}{(b+c)+0 = b+c}\ \forall E \qquad \dfrac{\text{PA3}}{c+0=c}\ \forall E}{(b+c)+0 = b+(c+0)}\ =E}{\forall y\, \forall z\, [(y+z)+0 = y+(z+0)]}\ \forall I$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ \forall y\ \forall z\ [(y+z)+n=y+(z+n)]\ \vdash\ \forall y\ \forall z\ [(y+z)+s(n)=y+(z+s(n))]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{PA4}}{(b+c)+s(n)=s((b+c)+n)}\,\forall E \qquad \dfrac{}{(b+c)+n=b+(c+n)}\,\text{IH}}{(b+c)+s(n)=s(b+(c+n))}\,{=}E \qquad \dfrac{\text{PA4}}{b+s(c+n)=s(b+(c+n))}\,\forall E}{(b+c)+s(n)=b+s(c+n)}\,{=}E \qquad \dfrac{\text{PA4}}{c+s(n)=s(c+n)}\,\forall E}{(b+c)+s(n)=b+(c+s(n))}\,{=}E$$

(g) PA $\vdash \forall x\ \forall y\ [y + x = x + y]$ (Challenge!)

Solution:

This proof is by induction on the wff

$$P(x):\ \forall y\ [y + x = x + y]$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash \forall y\ (y + 0 = 0 + y)$$

$$\dfrac{\dfrac{\dfrac{}{a = a}\,{=}I \qquad \dfrac{\text{PA3}}{a + 0 = a}\,\forall E}{a + 0 = a}\,{=}E \qquad \dfrac{\text{PA}}{0 + a = a}\,\text{THM}}{a + 0 = 0 + a}\,{=}E$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ \forall y\ (y + n = n + y)\ \vdash\ \forall y\ (y + s(n) = s(n) + y)$$

$$\dfrac{\dfrac{\dfrac{\text{PA4}}{a + s(n) = s(a + n)}\,\forall E \qquad \dfrac{}{a + n = n + a}\,\text{IH}}{a + s(n) = s(n + a)}\,{=}E \qquad \dfrac{\text{PA}}{s(n) + a = s(n + a)}\,\text{THM}}{a + s(n) = s(n) + a}\,{=}E$$