Enlarging the no-control zone v2

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Choose a time scale such that $\gamma = 1$. The equations motion are simplified to

$$\dot{S} = -R_u S I$$
$$\dot{I} = (R_u S - 1) I$$

with

$$R_u = \begin{cases} R_0 & \text{if } u = u_0 \\ R_c & \text{if } u = u_{\max} \end{cases} .$$

Consider the change of coordinates

$$\mu = I - \frac{1}{R_0} \ln(S) + S$$
$$\nu = I - \frac{1}{R_c} \ln(S) + S$$

with inverse

$$S = \exp\left(\frac{R_0 R_c}{R_0 - R_c} (\mu - \nu) \right)$$
$$I = \frac{R_0}{R_0 - R_c} \mu - \frac{R_c}{R_0 - R_c} \nu - \exp\left(\frac{R_0 R_c}{R_0 - R_c} (\mu - \nu) \right) .$$

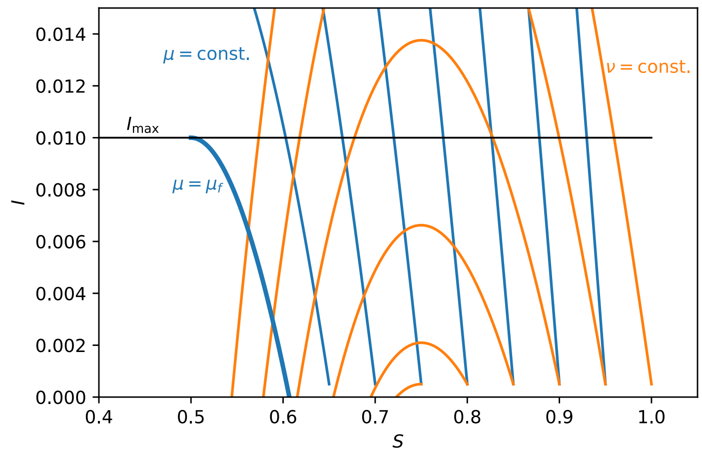
The equations of motion are now

$$\dot{\mu} = \frac{R_u - R_0}{R_0} I$$
$$\dot{\nu} = \frac{R_u - R_c}{R_c} I$$

Note that $\mu = \text{const}$ is an orbit when $R_u = R_0$ and $\nu = \text{const}$ is an orbit when $R_u = R_c$).

Given an initial condition (ν_0, μ_0), our goal is to reach the line

$$\mu = \mu_f \ , \quad \mu_f = I_{\max} + \frac{\ln(R_0) + 1}{R_0}$$



in minimal time. Green's theorem tells us that the optimal trajectory is the sequence

$$(\nu_0, \mu_0) \xrightarrow{R_0} (\nu_f, \mu_0) \xrightarrow{R_c} (\nu_f, \mu_f)$$

(as long as we don't violate the constraint $I \leq I_{\max}$), so our real goal is to find the optimal ν_f.

The first transition takes the time

$$T_0(\nu_f, \mu_0; \nu_0, \mu_0) = \int_{\nu_0}^{\nu_f} \frac{\mathrm{d}\nu}{\frac{R_0}{R_c}\mu_0 - \nu - \frac{R_0 - R_c}{R_c} \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu_0 - \nu)\right)}$$

(this is $\int \frac{\mathrm{d}\nu}{\dot{\nu}}$), while the second one takes the time

$$T_c(\nu_f, \mu_f; \nu_f, \mu_0) = \int_{\mu_0}^{\mu_f} \frac{\mathrm{d}\mu}{-\mu + \frac{R_c}{R_0}\nu_f + \frac{R_0 - R_c}{R_0} \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu - \nu_f)\right)}$$

(this is $\int \frac{\mathrm{d}\mu}{\dot{\mu}}$).

Problem 1. Find

$$\nu_f = \underset{\nu \geq \nu_0}{\arg\min} \left(T_0(\nu, \mu_0; \nu_0, \mu_0) + T_c(\nu, \mu_f; \nu, \mu_0)\right) . \tag{1}$$

Remark 1. We can easily formulate the minimal *intervention* problem by setting

$$\nu_f = \underset{\nu \geq \nu_0}{\arg\min}\, T_c(\nu, \mu_f; \nu, \mu_0) .$$

We have

$$\frac{\partial}{\partial \nu} T_0(\nu, \mu_0; \nu_0, \mu_0) = \frac{1}{\frac{R_0}{R_c}\mu_0 - \nu - \frac{R_0 - R_c}{R_c} \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu_0 - \nu)\right)} \tag{2}$$

and

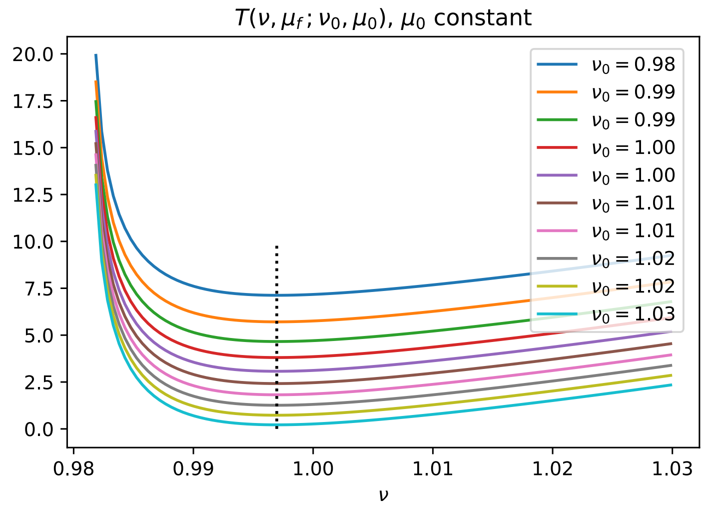
$$\frac{\partial}{\partial \nu} T_c(\nu, \mu_f; \nu, \mu_0) = \int_{\mu_0}^{\mu_f} \frac{1 + R_c \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu_0 - \nu)\right)}{\left(-\mu + \frac{R_c}{R_0}\nu + \frac{R_0 - R_c}{R_0} \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu - \nu)\right)\right)^2} \mathrm{d}\mu \quad (3)$$

We should be able to prove the following.

Assumption 1. *The function*

$$T(\nu, \mu_f; \nu_0, \mu_0) = T_0(\nu, \mu_0; \nu_0, \mu_0) + T_c(\nu, \mu_f; \nu, \mu_0)$$

is convex in ν (maybe in some region of interest).



Let ν^* be the solution to the equation

$$\frac{\partial}{\partial \nu} T(\nu^*, \mu_f; \nu_0, \mu_0) = 0 \,. \quad (4)$$

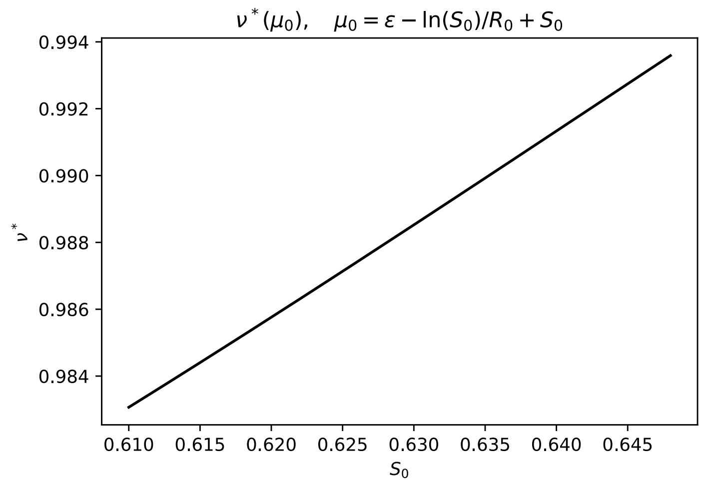
Remark 2. Neither the right-hand side of (2) nor (3) depend on ν_0. So the solution to (1) is of the form

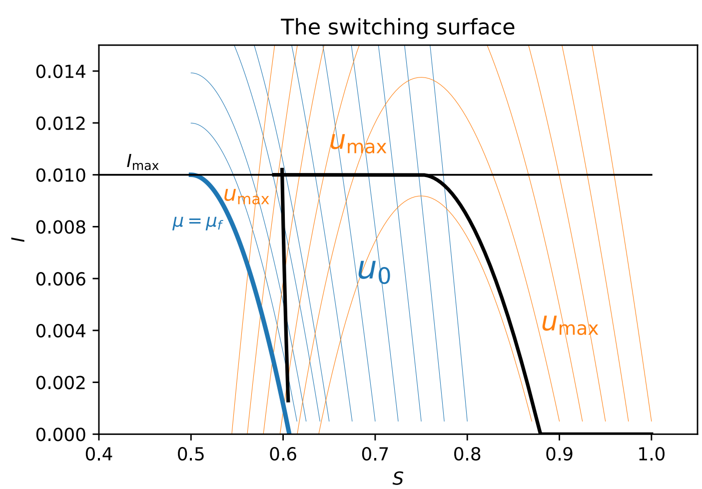
$$\nu_f = \begin{cases} \nu^* & \text{if } \nu_0 < \nu^* \\ \nu_0 & \text{if } \nu_0 \geq \nu^* \end{cases} .$$

We can write ν^* as a function of μ_0 only (the difference $\mu_0 - \mu_f$ measures the distance to our target set). This is one piece of our switching curve.

Note that, in the coordinates (S, ν), the switching curve is linear.

In the original coordinates (S, I), the switching curve is almost vertical (but not quite).


$\nu^*(\mu_0), \quad \mu_0 = \varepsilon - \ln(S_0)/R_0 + S_0$
0.994
0.992
0.990
0.988
0.986
0.984
ν^*
0.610
0.615
0.620
0.625
0.630
0.635
0.640
0.645
S_0


The switching surface
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
I
$I_{\max}$
$u_{\max}$
$u_{\max}$
$\mu = \mu_f$
u_0
$u_{\max}$
0.4
0.5
0.6
0.7
0.8
0.9
1.0
S

A On the convexity of T

Note that

$$\frac{\partial}{\partial \nu} I(\nu,\mu) = -\frac{R_c}{R_0 - R_c}\left(1 - R_0 S(\nu,\mu)\right)$$
$$\frac{\partial}{\partial \nu} S(\nu,\mu) = -\frac{R_0 R_c}{R_0 - R_c} S(\nu,\mu) ,$$

so that

$$\frac{\partial^2}{\partial \nu^2} I(\nu,\mu) = -\left(\frac{R_c R_0}{R_0 - R_c}\right)^2 S(\nu,\mu) .$$

We can rewrite

$$T_0(\nu_f,\mu_0;\nu_0,\mu_0) = \frac{R_0}{R_0 - R_c}\int_{\nu_0}^{\nu_f} \frac{\mathrm{d}\nu}{I(\nu,\mu_0)}$$

and compute

$$\frac{\partial}{\partial \nu} T_0(\nu,\mu_0;\nu_0,\mu_0) = \frac{R_0}{R_0 - R_c}\frac{1}{I(\nu,\mu_0)} > 0$$

(the function is *increasing*). Also,

$$\frac{\partial^2}{\partial \nu^2} T_0(\nu,\mu_0;\nu_0,\mu_0) = \frac{R_0 R_c}{(R_0 - R_c)^2}\frac{(1 - R_0 S(\nu,\mu_0))}{I^2(\nu,\mu_0)} .$$

In our region of interest ($1/R_0 < S \leq 1$) the function is *concave*.

We can write

$$T_c(\nu_f,\mu_f;\nu_f,\mu_0) = \frac{R_c}{R_0 - R_c}\int_{\mu_f}^{\mu_0} \frac{\mathrm{d}\mu}{I(\nu_f,\mu)}$$

(note that $\mu_f \leq \mu_0$) and compute

$$\frac{\partial}{\partial \nu} T_c(\nu,\mu_0;\nu,\mu_0) = \left(\frac{R_c}{R_0 - R_c}\right)^2 \int_{\mu_f}^{\mu_0} \frac{(1 - R_0 S(\nu,\mu))}{I^2(\nu,\mu)}\mathrm{d}\mu < 0 ,$$
$$1/R_0 < S \leq 1 .$$

In our region of interest, the function is *decreasing*. Also

$$\frac{\partial^2}{\partial \nu^2} T_c(\nu,\mu_0;\nu,\mu_0) = \left(\frac{R_c}{R_0 - R_c}\right)^3 \int_{\mu_f}^{\mu_0} \left[2\frac{(1 - R_0 S(\nu,\mu))^2}{I^3(\nu,\mu)} + R_0^2 \frac{S(\nu,\mu)}{I^2(\nu,\mu)}\right]\mathrm{d}\mu > 0 .$$

The function is *convex* everywhere.