Enlarging the no-control zone

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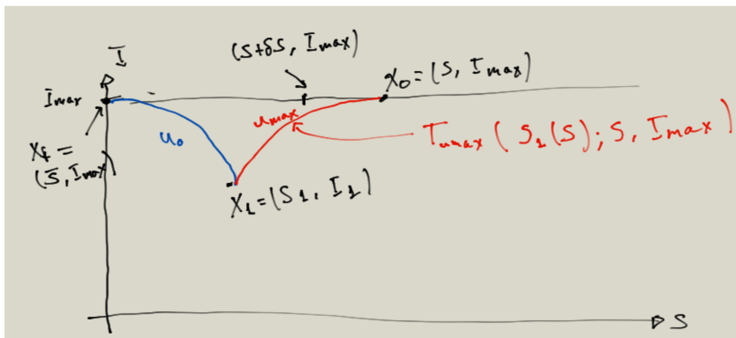


The orbits satisfy

$$I - I_0 = \frac{1}{R_u} \ln\left(\frac{S}{S_0}\right) - (S - S_0) \; , \quad R_u = \frac{\gamma}{\beta(1-u)} \; . \tag{1}$$

Given a constant u, the time it takes for the transition $x_0 \xrightarrow{u} x$ to happen is

$$\begin{aligned} T_u(x, x_0) &= -\frac{R_u}{\gamma} \int_{S_0}^{S} \frac{\mathrm{d}\sigma}{\sigma I(\sigma; S_0, I_0)} \\ &= -\frac{R_u}{\gamma} \int_{S_0}^{S} \frac{\mathrm{d}\sigma}{\sigma \left(I_0 + \frac{1}{R_u} \ln\left(\frac{\sigma}{S_0}\right) - (\sigma - S_0) \right)} \; . \end{aligned} \tag{2}$$

For generating concrete graphs we consider the parameters

$$\begin{aligned} I_{\max} &= 12.63e - 3 \\ \gamma &= 1/7 \\ R_0 &= 1.9 \\ R_c &= 1.9 \cdot 0.61 \; . \end{aligned}$$

(to be consistent with the paper, I should have written $R_c = 1.9 \cdot (1 - 0.61)$).

Problem 1. Consider the transitions $x_0 \xrightarrow{u_{\max}} x_1 \xrightarrow{u_0} x_f$. Given

$$x_0 = (S, I_{\max}) \quad \text{and} \quad x_f = (\bar{S}, I_{\max}) \ ,$$

we wish to find (S_1, I_1).

Using (1) directly,

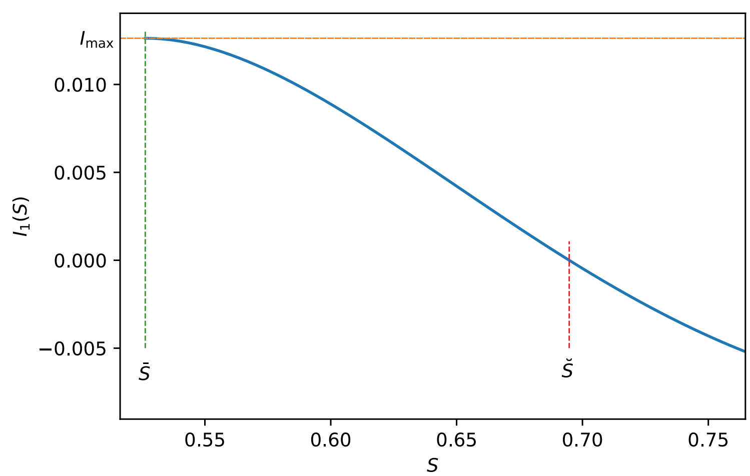
$$\begin{aligned} I_1 - I_{\max} &= \frac{1}{R_c} \ln\left(\frac{S_1}{S}\right) - (S_1 - S) \\ I_{\max} - I_1 &= \frac{1}{R_0} \ln\left(\frac{\bar{S}}{S_1}\right) - (\bar{S} - S_1) \ , \end{aligned}$$

we can solve for S_1,

$$\begin{aligned} \ln S_1^{R_0 - R_c} &= R_0 R_c(\bar{S} - S) + \ln S^{R_0} - \ln \bar{S}^{R_c} \\ S_1^{R_0 - R_c} &= \exp\left(R_0 R_c(\bar{S} - S) + R_0 \ln S - R_c \ln \bar{S}\right) \\ S_1 &= \exp\left(\frac{R_0 R_c(\bar{S} - S) + R_0 \ln S - R_c \ln \bar{S}}{R_0 - R_c}\right) \ . \end{aligned}$$

Now we can find

$$I_1 = I_{\max} + \frac{1}{R_c} \ln\left(\frac{S_1}{S}\right) - (S_1 - S) \ .$$



Remark 1. The condition $I_1 \geq 0$ imposes the restriction $S \leq \breve{S}$ with the upper bound implicitely defined by

$$I_{\max} + \frac{1}{R_c} \ln\left(\frac{S_1(\breve{S})}{\breve{S}}\right) = (S_1(\breve{S}) - \breve{S}) \ .$$

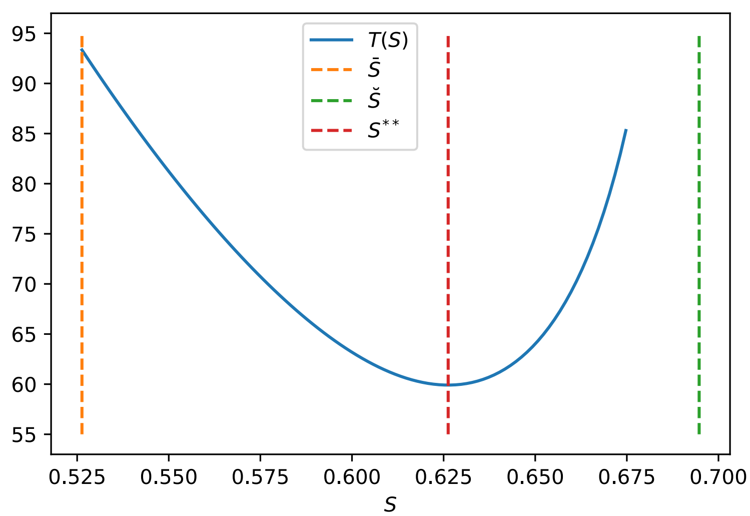
Problem 2. Consider the transition $(\breve{S}, I_{\max}) \xrightarrow{u_s} (S, I_{\max})$. Compute the time $T_{u_s}(S, I_{\max}; \breve{S}, I_{\max})$.

We have
$$u_s = \frac{\gamma}{\beta S}$$
and
$$\dot{S} = -\gamma I_{\max} \ ,$$
so that
$$T_{u_s}(S, I_{\max}; \breve{S}, I_{\max}) = \frac{1}{\gamma I_{\max}}(\breve{S} - S) \ .$$

Problem 3 (The main one)**.** Suppose that we start at $\breve{S}$ and slide along $I_{\max}$. At S, we switch to the arc defined by $u_{\max}$. The total time to reach the 'golden set' is
$$T(S) = T_{u_s}(S, I_{\max}; \breve{S}, I_{\max}) + T_{u_{\max}}(S_1(S); S, I_{\max}) \ . \tag{3}$$
Find
$$S^{**} = \underset{\bar{S} < S < \breve{S}}{\arg\min}\, T(S) \ .$$

Numerical approach

We simply compute (3) numerically and search for the minimum. It only takes one line of code.



Pseudo-analytical approach

We don't really gain much, but we can formulate the problem pseudo-analytically.

Problem 4. Compute the derivative $S_1'(S)$.

We have

$$S_1'(S) = \frac{R_0}{R_0 - R_c} \left(\frac{1}{S} - R_c \right) S_1(S)$$

Problem 5. Compute the derivative of the transition time

$$T_{u_{\max}}(S_1(S); S, I_{\max}) \ .$$

For

$$f(\sigma) = \frac{1}{\sigma \left(I_{\max} + \frac{1}{R_c} \ln\left(\frac{\sigma}{S}\right) - (\sigma - S) \right)}$$

we have

$$\begin{aligned} f'(\sigma) &= - \frac{\left(I_{\max} + \frac{1}{R_c} \ln\left(\frac{\sigma}{S}\right) - (\sigma - S) \right) + \sigma \left(\frac{1}{R_c S \sigma} - 1 \right)}{\sigma^2 \left(I_{\max} + \frac{1}{R_c} \ln\left(\frac{\sigma}{S}\right) - (\sigma - S) \right)^2} \\ &= - \frac{1}{\sigma^2 \left(I_{\max} + \frac{1}{R_c} \ln\left(\frac{\sigma}{S}\right) - (\sigma - S) \right)} \left[1 + \sigma \frac{\frac{1}{R_c S \sigma} - 1}{I_{\max} + \frac{1}{R_c} \ln\left(\frac{\sigma}{S}\right) - (\sigma - S)} \right] \end{aligned}$$

Applying the rule of 'derivation under the integral sign' to (2), we have

$$T'(S_1(S); S, I_{\max}) = -\frac{R_c}{\gamma} \left[f(S_1(S)) S_1'(S) - f(S) + \int_S^{S_1(S)} f'(\sigma) \mathrm{d}\sigma \right] .$$

Note that

$$f(S) = \frac{1}{S I_{\max}}$$

and

$$f(S_1(S)) = \frac{1}{S_1 I_1} \ .$$

Problem 6. Solve the equation

$$T'(S_1(S^{**}); S^{**}, I_{\max}) = \frac{1}{\gamma I_{\max}} \ .$$

We still have to compute an integral numerically. I don't feel it's worth it...

The whole picture

Here are some simulations using the numerical solution.