

Enlarging the no-control zone

v2

Fernando Castaños

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Choose a time scale such that $\gamma = 1$. The equations motion are simplified to

$$\begin{aligned}\dot{S} &= -R_u SI \\ \dot{I} &= (R_u S - 1)I\end{aligned}$$

with

$$R_u = \begin{cases} R_0 & \text{if } u = u_0 \\ R_c & \text{if } u = u_{\max} \end{cases}.$$

Consider the change of coordinates

$$\begin{aligned}\mu &= I - \frac{1}{R_0} \ln(S) + S \\ \nu &= I - \frac{1}{R_c} \ln(S) + S\end{aligned}$$

with inverse

$$\begin{aligned}S &= \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu - \nu)\right) \\ I &= \frac{R_0}{R_0 - R_c} \mu - \frac{R_c}{R_0 - R_c} \nu - \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu - \nu)\right).\end{aligned}$$

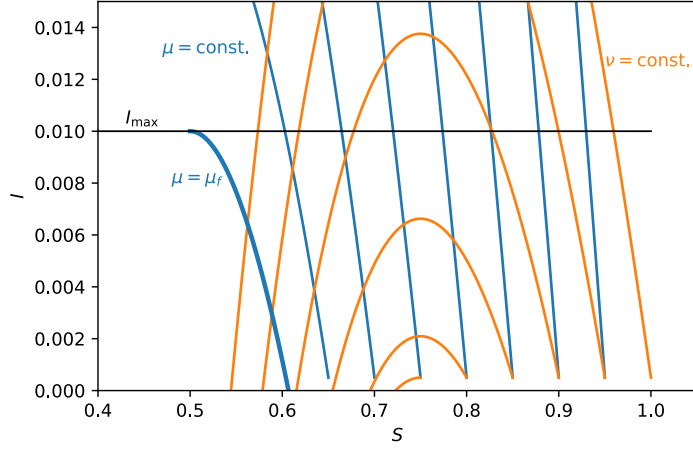
The equations of motion are now

$$\begin{aligned}\dot{\mu} &= \frac{R_u - R_0}{R_0} I \\ \dot{\nu} &= \frac{R_u - R_c}{R_c} I\end{aligned}$$

Note that $\mu = \text{const}$ is an orbit when $R_u = R_0$ and $\nu = \text{const}$ is an orbit when $R_u = R_c$.

Given an initial condition (ν_0, μ_0) , our goal is to reach the line

$$\mu = \mu_f, \quad \mu_f = I_{\max} + \frac{\ln(R_0) + 1}{R_0}$$



in minimal time. Green's theorem tells us that the optimal trajectory is the sequence

$$(\nu_0, \mu_0) \xrightarrow{R_0} (\nu_f, \mu_0) \xrightarrow{R_c} (\nu_f, \mu_f)$$

(as long as we don't violate the constraint $I \leq I_{\max}$), so our real goal is to find the optimal ν_f .

The first transition takes the time

$$T_0(\nu_f, \mu_0; \nu_0, \mu_0) = \int_{\nu_0}^{\nu_f} \frac{d\nu}{\frac{R_0}{R_c} \mu_0 - \nu - \frac{R_0 - R_c}{R_0} \exp\left(\frac{R_0 R_c}{R_0 - R_c} (\mu_0 - \nu)\right)}$$

(this is $\int \frac{d\nu}{\nu}$), while the second one takes the time

$$T_c(\nu_f, \mu_f; \nu_f, \mu_0) = \int_{\mu_0}^{\mu_f} \frac{d\mu}{-\mu + \frac{R_c}{R_0} \nu_f + \frac{R_0 - R_c}{R_c} \exp\left(\frac{R_0 R_c}{R_0 - R_c} (\mu - \nu_f)\right)}$$

(this is $\int \frac{d\mu}{\mu}$).

Problem 1. Find

$$\nu_f = \arg \min_{\nu \geq \nu_0} (T_0(\nu, \mu_0; \nu_0, \mu_0) + T_c(\nu, \mu_f; \nu, \mu_0)) . \quad (1)$$

Remark 1. We can easily formulate the minimal *intervention* problem by setting

$$\nu_f = \arg \min_{\nu \geq \nu_0} T_c(\nu, \mu_f; \nu, \mu_0) .$$

We have

$$\frac{\partial}{\partial \nu} T_0(\nu, \mu_0; \nu_0, \mu_0) = \frac{1}{\frac{R_0}{R_c} \mu_0 - \nu - \frac{R_0 - R_c}{R_0} \exp\left(\frac{R_0 R_c}{R_0 - R_c} (\mu_0 - \nu)\right)} \quad (2)$$

and

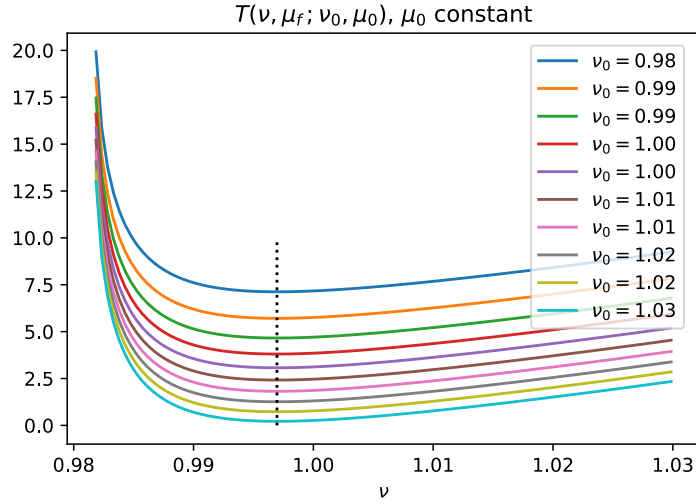
$$\frac{\partial}{\partial \nu} T_c(\nu, \mu_f; \nu, \mu_0) = \int_{\mu_0}^{\mu_f} \frac{1 + R_c \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu_0 - \nu)\right)}{\left(-\mu + \frac{R_c}{R_0} \nu + \frac{R_0 - R_c}{R_c} \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu - \nu)\right)\right)^2} d\mu \quad (3)$$

We should be able to prove the following.

Assumption 1. *The function*

$$T(\nu, \mu_f; \nu_0, \mu_0) = T_0(\nu, \mu_0; \nu_0, \mu_0) + T_c(\nu, \mu_f; \nu, \mu_0)$$

is convex in ν (maybe in some region of interest).



Let ν^* be the solution to the equation

$$\frac{\partial}{\partial \nu} T(\nu^*, \mu_f; \nu_0, \mu_0) = 0. \quad (4)$$

Remark 2. Neither the right-hand side of (2) nor (3) depend on ν_0 . So the solution to (1) is of the form

$$\nu_f = \begin{cases} \nu^* & \text{if } \nu_0 < \nu^* \\ \nu_0 & \text{if } \nu_0 \geq \nu^* \end{cases}.$$

We can write ν^* as a function of μ_0 only (the difference $\mu_0 - \mu_f$ measures the distance to our target set). This is one piece of our switching curve.

Note that, in the coordinates (S, ν) , the switching curve is linear.

In the original coordinates (S, I) , the switching curve is almost vertical (but not quite).

