Time Optimal Control for COVID19: Correcting the previous results

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Abstract

Here we just describe how to find the optimal orbit in a simple manner. We do not repeat many things we already did in the previous draft. Here, there are not yet proofs! Just a description of the optimal arc. New: we try to generalize the approach to a larger class of dynamical models.

1 SI(R) Model

The model is given by (and assuming for simpicity that $\gamma = 1$, what is equivalent to a time-scaling)

$$\begin{aligned}\dot{S} &= -\left(1-u\right)\beta SI = -R_u SI, \quad R_u = \left(1-u\right)\beta \\ \dot{I} &= \left(1-u\right)\beta SI - \gamma I = \left(R_u S - 1\right) I\,.\end{aligned}$$

We write R_0 if $u = 0$ and R_c if $u = u_{\max}$. With the change of variables proposed by Fernando

$$\begin{aligned}\mu &= I - \frac{1}{R_0}\ln\left(S\right) + S \\ \nu &= I - \frac{1}{R_c}\ln\left(S\right) + S\,,\end{aligned}$$

having inverse tranformation

$$\begin{aligned}S &= \exp\left(\frac{R_0 R_c}{R_0 - R_c}\left(\mu - \nu\right)\right) = \exp\left(\frac{\beta\left(1-u_{\max}\right)}{u_{\max}}\left(\mu - \nu\right)\right) \\ I &= \frac{R_0}{R_0 - R_c}\mu - \frac{R_c}{R_0 - R_c}\nu - \exp\left(\frac{R_0 R_c}{R_0 - R_c}\left(\mu - \nu\right)\right) \\ &= \frac{1}{u_{\max}}\mu - \frac{\left(1-u_{\max}\right)}{u_{\max}}\nu - \exp\left(\frac{\beta\left(1-u_{\max}\right)}{u_{\max}}\left(\mu - \nu\right)\right)\,.\end{aligned}$$

The dynamics in these coordinates is given by

$$\begin{aligned}\dot{\mu} &= \frac{R_u - R_0}{R_0} I \\ \dot{\nu} &= \frac{R_u - R_c}{R_c} I\,.\end{aligned}$$

When $u = 0$, then $R_u - R_0 = 0$

$$\begin{aligned}\dot{\mu} &= 0 \Rightarrow \mu(t) = \mu_0 \\ \dot{\nu} &= \frac{R_0 - R_c}{R_c} I(\mu_0, \nu) = \frac{u_{\max}}{(1 - u_{\max})} I(\mu_0, \nu) > 0 \\ I(\mu_0, \nu) &= \frac{1}{u_{\max}}\mu_0 - \frac{(1 - u_{\max})}{u_{\max}}\nu - \exp\left(\frac{\beta(1 - u_{\max})}{u_{\max}}(\mu_0 - \nu)\right)\,.\end{aligned}$$

When $u = u_{\max}$, then $R_u - R_c = 0$

$$\begin{aligned}\dot{\mu} &= \frac{R_c - R_0}{R_0} I(\mu, \nu_0) = -u_{\max} I(\mu, \nu_0) \leq 0 \\ \dot{\nu} &= 0 \Rightarrow \nu(t) = \nu_0 \\ I(\mu, \nu_0) &= \frac{1}{u_{\max}}\mu - \frac{(1 - u_{\max})}{u_{\max}}\nu_0 - \exp\left(\frac{\beta(1 - u_{\max})}{u_{\max}}(\mu - \nu_0)\right)\,.\end{aligned}$$

Using $u = u_{\max}$ we can go from $\mu_0 \to \mu_f$ (with $\mu_0 > \mu_f$) along a constant ν_c and the time it takes can be calculated from

$$\dot{\mu} = -u_{\max} I(\mu, \nu_c) \Rightarrow T_c(\mu_0, \mu_f; \nu_c) = -\frac{1}{u_{\max}}\int_{\mu_0}^{\mu_f} \frac{d\mu}{I(\mu, \nu_c)} = \frac{1}{u_{\max}}\int_{\mu_f}^{\mu_0} \frac{d\mu}{I(\mu, \nu_c)}$$

or

$$\begin{aligned}T_c(\mu_0, \mu_f; \nu_c) &= \frac{1}{u_{\max}}\int_{\mu_f}^{\mu_0} \frac{d\mu}{I(\mu, \nu_c)} \\ I(\mu, \nu_c) &= \frac{1}{u_{\max}}\mu - \frac{(1 - u_{\max})}{u_{\max}}\nu_c - \exp\left(\frac{\beta(1 - u_{\max})}{u_{\max}}(\mu - \nu_c)\right)\,.\end{aligned}$$

Note that the integrand is positive $I(\mu, \nu_c) > 0$. Fixing $\mu_0 > \mu_f$ there are different transit times T_c depending on the value of ν_c. For an interval of values of $\nu_c \in [\nu_0, \nu_f]$ we look for a minimum of T_c. If in the interval $I(\mu, \nu_c) \geq \epsilon > 0$, then this minimum exists. If the minimum value is in the interior of the interval $\nu_{\min} \in [\nu_0, \nu_f]$, then it happens that

$$\left.\frac{\partial T_c(\mu_0, \mu_f; \nu)}{\partial \nu}\right|_{\nu = \nu_{\min}} = 0\,.$$

This happens for

$$\begin{aligned}\frac{\partial T_c(\mu_0, \mu_f; \nu)}{\partial \nu} &= -\frac{1}{u_{\max}}\int_{\mu_f}^{\mu_0} \frac{\frac{\partial I(\mu, \nu)}{\partial \nu} d\mu}{I^2(\mu, \nu)} \\ \frac{\partial I(\mu, \nu)}{\partial \nu} &= -\frac{(1 - u_{\max})}{u_{\max}}\left[1 - \beta\exp\left(\frac{\beta(1 - u_{\max})}{u_{\max}}(\mu - \nu)\right)\right]\,,\end{aligned}$$

and then

$$\frac{\partial T_c\left(\mu_0,\, \mu_f;\, \nu_{\min}\right)}{\partial \nu} = 0 \Leftrightarrow$$

$$\beta \exp\left(-\frac{\beta\left(1-u_{\max}\right)}{u_{\max}}\nu_{\min}\right)\int_{\mu_f}^{\mu_0}\frac{\exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}\mu\right)d\mu}{I^2\left(\mu,\, \nu_{\min}\right)} = \int_{\mu_f}^{\mu_0}\frac{d\mu}{I^2\left(\mu,\, \nu_{\min}\right)}\,. \tag{1}$$

Note that both sides of this equation are positive.

We calculate also the second partial derivative of $T_c\left(\mu_0,\, \mu_f;\, \nu\right)$.

$$\begin{aligned}
\frac{\partial^2 T_c\left(\mu_0,\, \mu_f;\, \nu\right)}{\partial \nu^2} &= -\frac{1}{u_{\max}}\int_{\mu_f}^{\mu_0}\frac{\partial}{\partial \nu}\frac{\frac{\partial I(\mu,\, \nu)}{\partial \nu}}{I^2\left(\mu,\, \nu\right)}d\mu \\
&= -\frac{1}{u_{\max}}\int_{\mu_f}^{\mu_0}\frac{I^2\left(\mu,\, \nu\right)\frac{\partial^2 I(\mu,\, \nu)}{\partial \nu^2} - 2I\left(\mu,\, \nu\right)\left(\frac{\partial I(\mu,\, \nu)}{\partial \nu}\right)^2}{I^4\left(\mu,\, \nu\right)}d\mu \\
&= \frac{1}{u_{\max}}\int_{\mu_f}^{\mu_0}\frac{2\left(\frac{\partial I(\mu,\, \nu)}{\partial \nu}\right)^2 - I\left(\mu,\, \nu\right)\frac{\partial^2 I(\mu,\, \nu)}{\partial \nu^2}}{I^3\left(\mu,\, \nu\right)}d\mu \\
\frac{\partial I\left(\mu,\, \nu\right)}{\partial \nu} &= -\frac{\left(1-u_{\max}\right)}{u_{\max}}\left[1-\beta \exp\left(\frac{\beta\left(1-u_{\max}\right)}{u_{\max}}\left(\mu-\nu\right)\right)\right] \\
\frac{\partial^2 I\left(\mu,\, \nu\right)}{\partial \nu^2} &= -\beta^2\left(\frac{\left(1-u_{\max}\right)}{u_{\max}}\right)^2 \exp\left(\frac{\beta\left(1-u_{\max}\right)}{u_{\max}}\left(\mu-\nu\right)\right)\,,
\end{aligned}$$

and

$$\begin{aligned}
2\left(\frac{\partial I\left(\mu,\, \nu\right)}{\partial \nu}\right)^2 - I\left(\mu,\, \nu\right)\frac{\partial^2 I\left(\mu,\, \nu\right)}{\partial \nu^2} &= 2\left(\frac{\partial I\left(\mu,\, \nu\right)}{\partial \nu}\right)^2 + \\
&\beta^2\left(\frac{\left(1-u_{\max}\right)}{u_{\max}}\right)^2 \exp\left(\frac{\beta\left(1-u_{\max}\right)}{u_{\max}}\left(\mu-\nu\right)\right) I\left(\mu,\, \nu\right) > 0\,.
\end{aligned}$$

So we conclude that $\frac{\partial^2 T_c(\mu_0,\, \mu_f;\, \nu)}{\partial \nu^2} > 0$ and therefore $T_c\left(\mu_0,\, \mu_f;\, \nu\right)$ is convex as a function of ν. Therefore the minimum is global, and if it is in the interior of the interval $\nu_c \in \left[\nu_0,\, \nu_f\right]$ then it is the only point satisfying (1).