Time Optimal Control for COVID19

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Abstract

Here we just describe how to find the optimal orbit in a simple manner. We do not repeat many things we already did in the previous draft. Here, there are not yet proofs! Just a description of the optimal arc.

1 SI(R) Model

The model is given by

$$\begin{aligned}\dot{S} &= -\left(1-u\right)\beta SI \\ \dot{I} &= \left(1-u\right)\beta SI - \gamma I\,, \\ \dot{R} &= \gamma I\end{aligned}$$

where the constant parameters are positive $\beta > 0$, $\gamma > 0$. Since the total $N = S + I + R$ remains constant all the time, the model can be reduced to a second order system just using the states $(S,\, I)$, what we will do. The maximal (acceptable) value of I is $I_{\max}$ and the maximal achievable value of the control is $u_{\max}$. So the variables have to be in the following Feasible sets

$$\begin{aligned}X_F &= \left\{(S,\, I) \in \mathbb{R}^2_{\geq 0} | 0 \leq S_0 \leq S,\, 0 \leq I \leq I_{max}\right\} \\ U_F &= \left\{u \in \mathbb{R}_{\geq 0} | 0 \leq u \leq u_{\max} < 1\right\}\,.\end{aligned}$$

Sometimes it will be useful to write the differential equation in a compact form as

$$\begin{gathered}\dot{x} = f\left(x\right) + g\left(x\right) w,\; w = 1-u \\ \left[\begin{array}{c} S \\ I \end{array}\right]^{\cdot} = \left[\begin{array}{c} 0 \\ -\gamma I \end{array}\right] + \beta SI \left[\begin{array}{c} -1 \\ 1 \end{array}\right] w\,.\end{gathered}$$

$\phi\left(t,\, x_0,\, u\left(\cdot\right)\right)$ denotes a trajectory of this equation starting at the initial point $x_0 = (S_0,\, I_0)$.

Optimal control: find the control strategy u such that starting from the initial point $(S_0,\, I_0)$ the target set[1] $\mathcal{T} = \left\{(S,\, I) \in \mathbb{R}^2_{\geq 0} | S \leq \frac{\gamma}{\beta}\right\}$ is reached in

[1] This set is positively invariant without control ($u = 0$), and when the trajectory is in this set it will die out without control action.

the minimal time and so that the state restriction $I(t) \leq I_{\max}$ is satisfied for all the time.

In fact, what we want to minimize is not the total time, but the time during some control action has to be done. We will show (????) that this is equivalent to minimize the total time!

Now let us define the reachable set for an initial state x_0 as the set of points that can be reached with feasible control starting from the initial point x_0, i.e.

$$R(x_0) = \left\{x \in \mathbb{R}^2_{\geq 0} | x = \phi(t, x_0, u(\cdot)) \text{ for some finite } t \geq 0, \text{ for some } u(t) \in U_F\right\} .$$

Also, we define the controllable set of the target set $\mathcal{T}$ as the set of points from which some point in the target $\mathcal{T}$ can be reached with a feasible control, i.e.

$$C(\mathcal{T}) = \left\{x \in \mathbb{R}^2_{\geq 0} | \exists x_f \in \mathcal{T},\, x_f = \phi(t, x, u(\cdot)) \text{ for some finite } t \geq 0, \text{ for some } u(t) \in U_F\right\} .$$

The set $C(\mathcal{T})$ can be equivalently described as $R(\mathcal{T})$ for the system

$$\dot{x} = -f(x) - g(x)(1-u) ,$$

i.e. the set of points that can be reached from the set $\mathcal{T}$ for the dynamics with backward time. Now the optimal control problem has a solution if and only if

$$R(x_0) \cap C(\mathcal{T}) \cap X_F \neq \emptyset .$$

Since the set of points $(S, I) = (S, 0)$ is an equilibrium point for every control value, then $R((S, 0)) = (S, 0)$. So we exclude them from the initial conditions for which there is a solution (except if it is already in the target set). Now, since $\dot{S} < 0$ if $S > 0$, $I > 0$ then

$$R(x_0) \cap C(\mathcal{T}) \neq \emptyset$$

for every initial condition (except $(S, 0)$). It is also obvious that for the problem to be feasible the initial point has to be in the feasible set X_F, i.e.

$$R(x_0) \cap X_F \neq \emptyset .$$

1.1 Calculation of the orbits

Although it does not seem to be possible to find explicitely the trajectories of the system, it is easy to find its orbits. For this we write (we exclude the points for which $I = 0$ since they are equilibria)

$$\begin{aligned}\frac{dI}{dS} = \frac{\dot{I}}{\dot{S}} &= \frac{(1-u)\beta SI - \gamma I}{-(1-u)\beta SI} = \frac{(1-u)\beta S - \gamma}{-(1-u)\beta S} \\ &= \frac{\gamma}{(1-u)\beta S} - 1\end{aligned}$$

which is a separable DE. Integrating (we assume that u is constant!) we obtain

$$I - I_0 = \frac{\gamma}{(1-u)\beta} \ln\left(\frac{S}{S_0}\right) - (S - S_0) . \tag{1}$$

An interesting rewriting of (1) is

$$I(t)+S(t)-\frac{\gamma}{(1-u)\beta}\ln(S(t))=I_0+S_0-\frac{\gamma}{(1-u)\beta}\ln(S_0)\ .$$

This means that the quantity $I(t)+S(t)-\frac{\gamma}{(1-u)\beta}\ln(S(t))$ remains constant along the trajectory. Note that this constant depends on the control value used.

Given an initial condition $(S_0,\, I_0)$ this expression gives, for any $0<S<S_0$ the (unique) value of I that is reached (in future time)[2]. So, we can write $I(S;(S_0,\, I_0))$, since it gives us the value of I as a function of S and the initial condition. Moreover, from the first equation in the DE we obtain

$$\frac{dS}{(1-u)\beta SI}=-dt$$

and if we take the expression $I(S;(S_0,\, I_0))$ we obtain a separable DE that can be integrated

$$\begin{aligned} t=&-\int_{S_0}^{S}\frac{dS}{(1-u)\beta SI(S;(S_0,\, I_0))}\\ &-\frac{1}{(1-u)\beta}\int_{S_0}^{S}\frac{dS}{S\left(I_0+\frac{\gamma}{(1-u)\beta}\ln\left(\frac{S}{S_0}\right)-(S-S_0)\right)}\,, \end{aligned}$$

and that gives the time to reach the point $(S,\, I(S))$ from the initial point $(S_0,\, I_0)$ with the (constant) control u. Although it does not seem to be possible to give an explicit expression for this integral, it is clear that S parametrizes uniquely the solutions (since it is monotone).

1.2 The number of infected people

If we apply some constant control $0\le u\le u_{\max}$ the infection will die out, i.e. the value $I(\infty)=0$ will be reached (otherwise $R(t)$ would continue growing, what is impossible. We can therefore calculate $S(\infty)$ from (1) as

$$I(\infty)-I_0=\frac{\gamma}{(1-u)\beta}\ln\left(\frac{S(\infty)}{S_0}\right)-(S(\infty)-S_0)$$

and therefore

$$\frac{\gamma}{(1-u)\beta}\ln(S(\infty))-S(\infty)=\frac{\gamma}{(1-u)\beta}\ln(S_0)-S_0-I_0$$

that depends on the initial values but also on the control used!

If we asume that the model is normalized, and the initial value is $S_0=1$ and $I_0\approx 0$, then

$$S(\infty)-\frac{\gamma}{(1-u)\beta}\ln(S(\infty))=1\,.$$

Note that if $u\to 1^-$ then $S(\infty)\to 1^-$ and so, the larger is u the larger is also $S(\infty)$.

[2]If we select $S>S_0$ the obtained value of I is reached in a past time ($t<0$).

1.3 Reachable set from $(S_0,\, I_0)$

At each point in the state space the directions in which the vector field points for different values of the control are given by $F_u(x) = f(x) + g(x)(1-u)$. The extreme values are given by $F_0(x) = f(x) + g(x)$ and $F_{u_{\max}}(x) = f(x) + g(x)(1-u_{\max})$,

$$\begin{aligned} F_0(x) &= \begin{bmatrix} 0 \\ -\gamma I \end{bmatrix} + \beta SI \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ F_{u_{\max}}(x) &= \begin{bmatrix} 0 \\ -\gamma I \end{bmatrix} + \beta SI \begin{bmatrix} -1 \\ 1 \end{bmatrix} (1-u_{\max}) \ . \end{aligned}$$

In the phase plane $(S,\, I)$ both point to the "left", since the first component (in the direction of S) is always negative (if $SI > 0$). Since for the second components of the fields

$$-\gamma I + \beta SI > -\gamma I + \beta SI (1-u_{\max})$$

it follows that F_0 is "above" $F_{u_{\max}}$. And therefore[3], for the reachable set $R(x_0)$ is bounded by the two trajectories $\phi(t,\, x_0,\, u=0)$ and $\phi(t,\, x_0,\, u_{\max})$

Note that a natural question here is why not take the maximal action and "kill" the epidemie from the begining!

These two bounding orbits can be easily calculated using (1), that provides I as a function of $S < S_0$ for both extreme values of the control: $u = 0$, $u = u_{\max}$.

In particular we can calculate the maximal value achieved by I for every (constant) control action:

$$\begin{aligned} \frac{dI(S)}{dS} &= \frac{d}{dS}\left\{ I_0 + \frac{\gamma}{(1-u)\beta} \ln\left(\frac{S}{S_0}\right) - (S - S_0) \right\} \\ &= \frac{\gamma}{(1-u)\beta\left(\frac{S}{S_0}\right) S_0} - 1 = 0 \end{aligned}$$

that is achieved at

$$\bar{S} = \frac{\gamma}{(1-u)\beta} \tag{2}$$

and gives

$$\bar{I} = I(\bar{S}) = I_0 + \frac{\gamma}{(1-u)\beta} \ln\left(\frac{\gamma}{(1-u)\beta S_0}\right) - \left(\frac{\gamma}{(1-u)\beta} - S_0\right). \tag{3}$$

1.4 Comparing the cost of two different trajectories

In order to be able to find the optimal orbit (trajectory) solving the posed optimal control problem, it is necessary to be able to compare the value of two different trajectories that start at the same initial point and end at the same final point. So, consider two orbits (trajectories) $\omega_i(x_0,\, x_f,\, u_i)$, $i = 1,\, 2$,

[3] Here a more careful argumentation is probably needed.

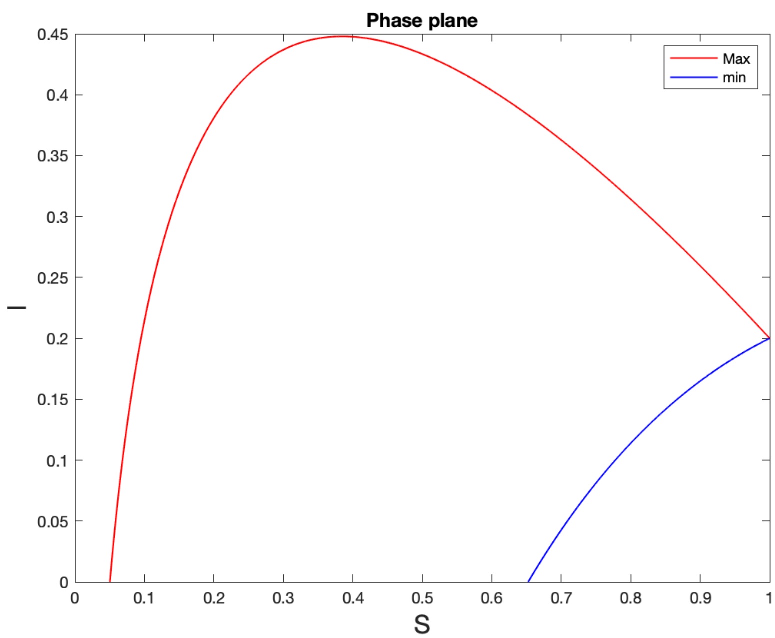

Figure 1: Phase Plane with maximal and minimal orbits bounding the reaching set $R(x_0)$. Max corresponds to the trajectory $\phi(t, x_0, u = 0)$ while Min to $\phi(t, x_0, u_{\max})$

joining the (same) points x_0 and x_f using two different control actions u_1 and u_2, respectively. The "cost" going through ω_i is (recall our cost is time)

$$J(u_i) = \int_0^{T_i} dt$$

along the trajectory. Given two such orbits, we want to compare the two costs. This can be done for example substracting them, i.e. if

$$J(u_1) - J(u_2) < 0$$

this means that the cost of ω_1 is lower than that of ω_2.

The cost $J(u_i)$ can be calculated as a line integral along the trajectory. We can see this in the following manner:

Calculate

$$\begin{aligned} \Delta(f(x), g(x)) &= -\det[f(x), g(x)] \\ &= -(f_1(x)g_2(x) - f_2(x)g_1(x)) \ . \end{aligned}$$

Now, by properties of the determinant this is also the same as

$$\begin{aligned} \Delta(f(x) + g(x)u_i, g(x)) &= -\det[\dot{x}, g(x)] \\ &= \dot{x}_2 g_1(x) - \dot{x}_1 g_2(x) \ . \end{aligned}$$

And therefore

$$\begin{aligned} J(u_i) &= \int_0^{T_i} dt = \int_0^{T_i} \frac{\dot{x}_2 g_1(x) - \dot{x}_1 g_2(x)}{\Delta(x)} dt \\ &= \int_{x_0}^{x_f} \left(\frac{g_1(x)}{\Delta(x)} dx_2 - \frac{g_2(x)}{\Delta(x)} dx_1 \right) \end{aligned}$$

which is a line integral along the orbit ω_i. Since the two paths have the same initial and final points, they form a closed curve, and calculating the line integral along the closed curve followed in the counterclockwise direction we obtain the diference of the costs, i.e.

$$J(u_1) - J(u_2) = \oint_\Gamma \left(\frac{g_1(x)}{\Delta(x)} dx_2 - \frac{g_2(x)}{\Delta(x)} dx_1 \right)$$

where Γ is the closed path of the two orbits followed in the counterclockwise direction.

For this we have to assume that: (1) the two paths (orbits) do not intersect at any points except the initial and final ones, and (ii) that $\Delta \neq 0$.

Using Green's theorem, the line integral can be calculated using a surface integral:

$$\oint_\Gamma (u dy + v dx) = \int\int_\mathcal{R} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy = \int\int_\mathcal{R} w(x, y) \, dx dy \, ,$$

where $\mathcal{R}$ is the region enclosed by the closed curve Γ. For our problem this becomes

$$\begin{aligned} J(u_1) - J(u_2) &= \int\int_{\mathcal{R}} w(x_1, x_2)\, dx_1 dx_2 \\ w(x_1, x_2) &= \frac{\partial}{\partial x_1}\left(\frac{g_1(x)}{\Delta(x)}\right) + \frac{\partial}{\partial x_2}\left(\frac{g_2(x)}{\Delta(x)}\right). \end{aligned}$$

In our case,

$$\begin{aligned} \Delta(x) &= -\left(f_1(x)\, g_2(x) - f_2(x)\, g_1(x)\right) \\ &= \gamma\beta S I^2 \\ w(x_1, x_2) &= \frac{\partial}{\partial S}\left(\frac{-\beta S I}{\gamma\beta S I^2}\right) + \frac{\partial}{\partial I}\left(\frac{\beta S I}{\gamma\beta S I^2}\right) \\ &= \frac{\partial}{\partial I}\left(\frac{1}{\gamma I}\right) = -\frac{1}{\gamma I^2} < 0\,. \end{aligned}$$

So, we see that $w < 0$ everywhere, and therefore the integral is negative implying that the "upper" orbit has a lower cost than the "lower" orbit (in the closed path traversed in the counterclockwise direction).

This observation allows us to find the optimal orbit, by comparing with others.

2 Determining the optimal orbit

From the previous results, the "upper" trajectory is the one with no control ($u = 0$) and this is better than any other joining the same two points. This is also intuitively clear. However, although this strategy allows us to go from every initial condition to the target set it is not always allowed, since the value of I can go over $I_{\max}$.

The study of the optimal control can be divided in two different approaches:

- We fix the initial condition x_0 and find for it the optimal orbit and then the optimal associated control.
- We study the optimal control problem for all possible initial conditions.

Although the second approach is obviously better it is more difficult. So we will consider first the first situation. In fact, both approaches should lead to the same conclusions.

Now we can divide the study of the optimal orbit in the following cases.

2.1 Trivial

This is the case if $S_0 \leq \frac{\gamma}{\beta}$ since we start in the target set.

2.2 Not feasible

This is the case if $I_0 > I_{\max}$.

2.3 $u = 0$ is optimal

This will be the case if and only if the curve (1) its maximum $\bar{I}$ given by (3) with $u = 0$ is such that

1. $\bar{I} = I_0 + \frac{\gamma}{\beta} \ln\left(\frac{\gamma}{\beta S_0^2}\right) - \left(\frac{\gamma}{\beta S_0} - S_0\right) \leq I_{\max}$, or
2. $\bar{I} = I_0 + \frac{\gamma}{\beta} \ln\left(\frac{\gamma}{\beta S_0^2}\right) - \left(\frac{\gamma}{\beta S_0} - S_0\right) \geq I_{\max}$ and $\bar{S}$ as given by (2) with $u = 0$ satisfies $\bar{S} = \frac{\gamma}{\beta S_0} \leq \frac{\gamma}{\beta}$ and $I\left(\frac{\gamma}{\beta}\right) \leq I_{\max}$, that is $1 \leq S_0$ and $I_0 + \frac{\gamma}{\beta} \ln\left(\frac{\gamma}{\beta S_0}\right) - \left(\frac{\gamma}{\beta} - S_0\right) \leq I_{\max}$.

2.4 Singular arc

If $\bar{I}$ given by (3) with $u = 0$ is larger than $I_{\max}$ and $\bar{S} = \frac{\gamma}{\beta S_0} > \frac{\gamma}{\beta}$ it is necessary to apply some control to mantain I below the maximal value $I_{\max}$.

Now we calculate the value of $S = S_c$ at which the orbit (first) touches $I_{\max}$. For this we solve (use (1))

$$I_{\max} = I_0 + \frac{\gamma}{\beta} \ln\left(\frac{S}{S_0}\right) - (S - S_0)$$

and we obtain two solutions S_1, S_2. $S_c = \max\{S_1, S_2\}$ is the largest.

Now we calculate the value of $S = S^*$ at which it is possible to achieve that $\dot{I} \leq 0$ (that is we can stop the growth of I. This value can be calculated from

$$\dot{I} = (1 - u_{\max})\,\beta S I - \gamma I \leq 0\,,$$

and it gives

$$S^* \leq \frac{\gamma}{(1 - u_{\max})\,\beta}\,.$$

There are two possible situations:

2.4.1 If $\frac{\gamma}{\beta} < S_c \leq S^*$: Bang plus singular arc

If $\frac{\gamma}{\beta} < S_c \leq S^*$ then the optimal control strategy consists in

$$u = \begin{cases} 0 & \text{from } t = 0 \text{ until } I = I_{\max} \\ u_{\text{sing}} & \text{until } S = \frac{\gamma}{\beta} \\ 0 & S < \frac{\gamma}{\beta} \end{cases}$$

where u_{sing} is the control required to maintain $I = I_{\max}$ constant, i.e. $\dot{I} = 0$

$$u_{\text{sing}} = 1 - \frac{\gamma}{\beta S}\,.$$

2.4.2 Bang-Bang plus singular arc

When $S_c > S^*$ then it is necessary to start (if possible) with the control strategy before reaching the maximal value of $I = I_{\max}$. Otherwise, this limit will be surpased.

However, this is only feasible, if moving backwards from the point $(S^*, I_{\max})$ with the maximal control $u_{\max}$ it is possible to reach a point (S_0, I_0^*) such that $I_0^* \geq I_0$. If $I_0^* < I_0$, then it is not possible to solve the optimal problem, since any strategy will either surpas the maximal value $I_{\max}$ or it will not reach the target.

The value of I_0^* can be calculated from (1) since it satisfies

$$I_{\max} - I_0^* = \frac{\gamma}{(1-u_{\max})\,\beta} \ln\left(\frac{S^*}{S_0}\right) - (S^* - S_0)\ ,$$

so that

$$I_0^* = (S^* - S_0) + I_{\max} - \frac{\gamma}{(1-u_{\max})\,\beta} \ln\left(\frac{S^*}{S_0}\right)\ .$$

If $I_0^* = I_0$ the optimal control is

$$u = \begin{cases} u_{\max} & \text{from } t = 0 \text{ until } I = I_{\max} \\ u_{\text{sing}} & \text{until } S = \frac{\gamma}{\beta} \\ 0 & S < \frac{\gamma}{\beta} \end{cases}$$

When $I_0^* > I_0$ then the control is given by

$$u = \begin{cases} 0 & \text{from } t = 0 \text{ until } S = S_s \\ u_{\max} & \text{from } S = S_s \text{ until } I = I_{\max} \\ u_{\text{sing}} & \text{until } S = \frac{\gamma}{\beta} \\ 0 & S < \frac{\gamma}{\beta} \end{cases}$$

where the value of (S_s, I_s) is a switching point: it is characterized because there the trajectory starting at (S_0, I_0), i.e. $\phi\left(t, (S_0, I_0), u = 0\right)$ intersects the trajectory that start at $(S^*, I_{\max})$ and go backwards in time, i.e. $\phi\left(-t, (S^*, I_{\max}), u_{\max}\right)$. This point (S_s, I_s) can calculated from (1) as

$$\begin{aligned} I_s - I_0 &= \frac{\gamma}{\beta} \ln\left(\frac{S_s}{S_0}\right) - (S_s - S_0) \\ I_{\max} - I_s &= \frac{\gamma}{(1-u_{\max})\,\beta} \ln\left(\frac{S^*}{S_s}\right) - (S^* - S_s)\ . \end{aligned}$$

Replacing from the first into the second we get

$$\begin{aligned} I_s &= I_0 + \frac{\gamma}{\beta} \ln\left(\frac{S_s}{S_0}\right) - (S_s - S_0) \\ I_{\max} &= I_0 + \frac{\gamma}{\beta} \ln\left(\frac{S_s}{S_0}\right) - (S_s - S_0) + \frac{\gamma}{(1-u_{\max})\,\beta} \ln\left(\frac{S^*}{S_s}\right) - (S^* - S_s)\ . \end{aligned}$$

And solving for S_s in the second we achieve

$$\begin{aligned}I_s =& I_0 + \frac{\gamma}{\beta}\ln\left(\frac{S_s}{S_0}\right) - (S_s - S_0)\\ \ln(S_s) =& \frac{(1-u_{\max})}{u_{\max}}\frac{\beta}{\gamma}\left\{-I_{\max} + I_0 + S_0 - S^* - \frac{\gamma}{\beta}\left(\ln(S_0) - \frac{1}{(1-u_{\max})}\ln(S^*)\right)\right\}.\end{aligned}$$

3 The optimal track

For calculating the optimal path it is useful to construct some trajectories of the system (not all of them are really required for the final calculation though). Let us define the initial point $x_0 = (S_0, I_0)$ and the final point as $x_f = \left(\bar{S}, I_{\max}\right) = \left(\frac{\gamma}{\beta}, I_{\max}\right)$. x_f is the point of the target set at the upper right corner. The trajectories (or orbits) we want to find are four:

1. $\phi(t, x_0, u = 0)$: in words is the trajectory without control starting at x_0.
2. $\phi(t, x_0, u_{\max})$: in words is the trajectory with maximal control starting at x_0.
3. $\phi(-t, x_f, u = 0)$: in words is the trajectory without control that ends in x_f.
4. $\phi(-t, x_f, u^*)$: in words is the trajectory with control

   $$u^* = \min\left\{1 - \frac{\gamma}{\beta S}, u_{\max}\right\}$$

   that ends in x_f.
   The control u^* is such that this trajectory does not violate the restriction $I \le I_{\max}$. For values of $S \ge S^*$ it is equal to $u_{\max}$ and for $S \le S^*$ it is the control for the singular arc, i.e. it maintains $I = I_{\max}$ until x_f is reached.

These trajectories are presented in the Figure 3 for the parameters: $\beta = 0.52$, $\gamma = \frac{1}{5}$, $I_{\max} = 0.1$, $S_0 = 1$, $I_0 = 0.01$ and $u_{max} = 0.4$. There are also some key points to find, besides the initial x_0 and final x_f ones. These are:

- The "blue" point $\left(\bar{S}, \bar{I}\right)$ in Figure 3, where $\phi(t, x_0, u = 0)$ attains its maximum value,

  $$\bar{S} = \frac{\gamma}{\beta}, \ \bar{I} = I_0 + \frac{\gamma}{\beta}\ln\left(\frac{\gamma}{\beta S_0}\right) - \left(\frac{\gamma}{\beta} - S_0\right).$$

  This is the same point at which this trajectory crosses the critical value $\bar{S} = \frac{\gamma}{\beta}$ after which $\dot{I} < 0$ with zero control.
  When $\bar{I} \le I_{\max}$ then the control strategy is simply "do nothing" all the time, i.e.

  $$u = 0.$$

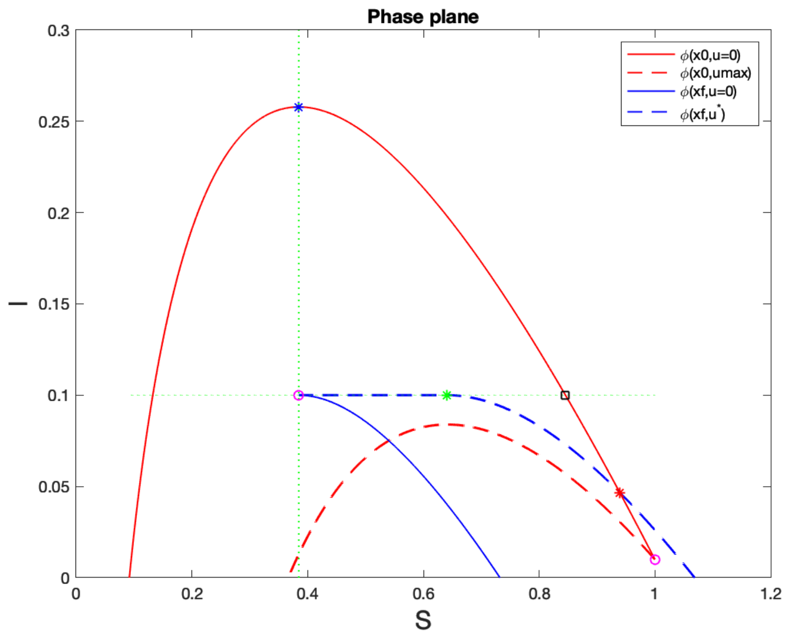

Figure 2: Phase Plane with the four key trajectories and points

- The "green" point $(S^*, I_{\max})$. S^* corresponds to the maximal value of S for which it is possible to keep $I_{\max}$ constant, given by

$$S^* = \min \left\{ \frac{\gamma}{(1 - u_{\max})\,\beta}, 1 \right\} .$$

We "saturate" the value of S^*, since $S^* > 1$ does not make sense. The control required to achieve this is the "singular" control

$$u_{\text{sing}} = 1 - \frac{\gamma}{\beta S} .$$

Note that if $S > S^*$ it is not possible to keep I at $I_{\max}$ and $\dot{I} > 0$.
If $S^* = 1$, then the optimal control is

$$u = \begin{cases} 0 & \text{from } t = 0 \text{ until } I = I_{\max} \\ u_{\text{sing}} & \text{until } S = \bar{S} = \frac{\gamma}{\beta} \\ 0 & S < \frac{\gamma}{\beta} \end{cases}$$

- The "black square" point $(S_{\max}, I_{\max})$ is the point at which the trajectory $\phi\,(t,\, x_0,\, u = 0)$ reaches the value $I_{\max}$ and can be calculated from the equation

$$I_{\max} - I_0 = \frac{\gamma}{\beta} \ln \left(\frac{S_{\max}}{S_0} \right) - (S_{\max} - S_0) .$$

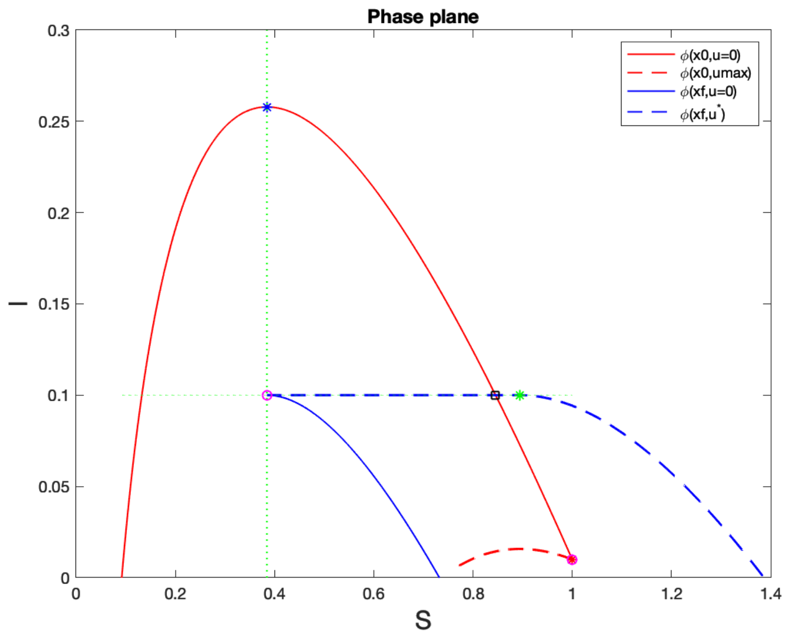

Figure 3: Phase Plane for $u_{\max} = 0.57$

This equation has a unique solution in the interval $\bar{S} < S_{\max} < S_0$ only if $I_0 < I_{\max} < \bar{I}$, $S_0 > \bar{S}$.
If $S^* \geq S_{\max}$, then the optimal control is also (as in the previous case)

$$u = \begin{cases} 0 & \text{from } t = 0 \text{ until } I = I_{\max} \\ u_{\text{sing}} & \text{until } S = \bar{S} = \frac{\gamma}{\beta} \\ 0 & S < \frac{\gamma}{\beta} \end{cases}$$

This is the situation in our example for $u_{\max} = 0.57$ (or any value larger than this) (see Figure 3).

- The "red" point (S_s, I_s). At this point the trajectory $\phi(-t, x_f, u^*)$ crosses the trajectory $\phi(t, x_0, u = 0)$. Note that if this crossing does not exist, then the optimal control problem is note feasible. We have three possible cases:
  - If $S^* \geq S_{\max}$ this point exists and it coincides with $(S_{\max}, I_{\max})$, i.e. $(S_s, I_s) = (S_{\max}, I_{\max})$. The optimal control is as in the previous case.
  - If $S^* < S_{\max}$ the point (S_s, I_s) may exist or not. We calculate

$$S_* = \exp\left\{ \frac{(1 - u_{\max})}{u_{\max}} \frac{\beta}{\gamma} \left[-I_{\max} + I_0 + S_0 - S^* - \frac{\gamma}{\beta} \left(\ln(S_0) - \frac{1}{(1 - u_{\max})} \ln(S^*) \right) \right] \right\}.$$

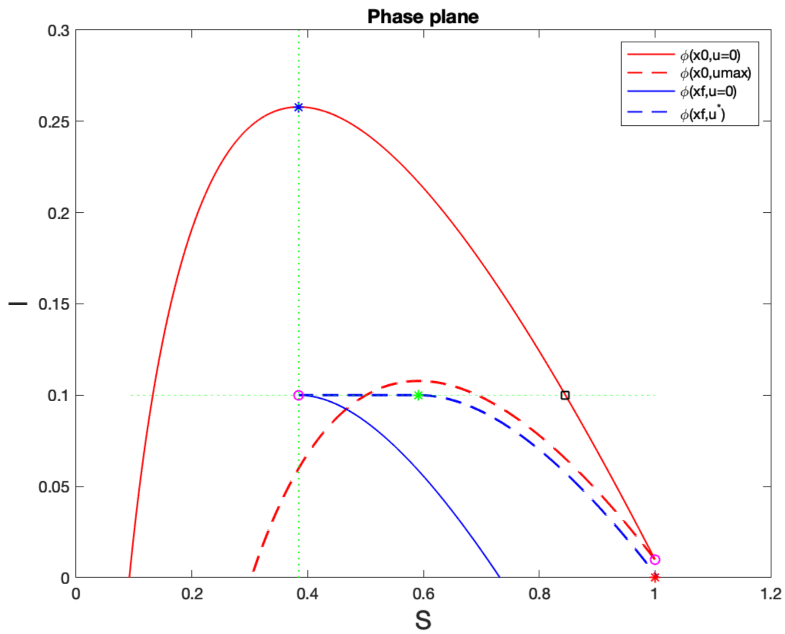

Figure 4: Phase Plane with $u_{\max} = 0.35$

* If $S^* \leq S_* \leq S_0$ then $(S_s,\, I_s)$ exists and $(S_s,\, I_s)$ is given by

$$S_s = S_*,\; I_s = I_0 + \frac{\gamma}{\beta} \ln\left(\frac{S_s}{S_0}\right) - (S_s - S_0)\ .$$

The optimal control is a bang-bang-singular arc-bang strategy:

$$u = \begin{cases} 0 & \text{from } t = 0 \text{ until } S = S_s \text{ (or } I = I_s \text{)} \\ u_{\max} & \text{from } S = S_s \text{ until } I = I_{\max} \\ u_{\text{sing}} & \text{until } S = \frac{\gamma}{\beta} \\ 0 & S < \frac{\gamma}{\beta} \end{cases}$$

This is the case of Figure 3.

* If $S_0 < S_*$ then $(S_s,\, I_s)$ does not exist. We fix its value (arbitrarily) as $(S_s,\, I_s) = (1,\, 0)$. In this case the optimal control problem is not feasible. This is the case if our example $u_{\max} \leq 0.35$. See Figure

4 A feedback control strategy

The previous "open loop" strategy can be implemented as a state feedback control. This strategy is rather simple, since there is basically only one switching curve: $\phi(-t, x_f, u^*)$ (this is the discontinuous blue line in the Figures). The second one is when the target region has been attained the control is switched off, but this happens in a "natural" manner.

The switching curve is defined as

$$I = \Phi(S) = \begin{cases} I_{\max} + \frac{\gamma}{(1-u_{\max})\beta} \ln\left(\frac{S}{S^*}\right) - (S - S^*) & \text{if } S^* \leq S \leq 1 \\ I_{\max} & \text{if } \bar{S} = \frac{\gamma}{\beta} \leq S \leq S^* \end{cases},$$

where

$$S^* = \min\left\{\frac{\gamma}{(1-u_{\max})\beta}, 1\right\}.$$

The optimal control feedback is thus given by

$$u(S, I) = \begin{cases} 0 & \text{if } I < \Phi(S) \vee S \leq \bar{S} = \frac{\gamma}{\beta} \\ u_{\max} & \text{if } I \geq \Phi(S) \wedge S \geq S^* \\ u_{\max} & \text{if } I > \Phi(S) \wedge S \leq S^* \\ u_{\text{sing}} = 1 - \frac{\gamma}{\beta S} & \text{if } I = \Phi(S) \wedge \bar{S} \leq S \leq S^* \end{cases}.$$

Alternatively, we can implement a pure switching control since the "equivalent control" will realize the singular control on the singular arc

$$u(S, I) = \begin{cases} 0 & \text{if } I < \Phi(S) \vee S \leq \bar{S} = \frac{\gamma}{\beta} \\ u_{\max} & \text{if } I \geq \Phi(S) \end{cases}.$$

Note that this control strategy extends the control action beyond the region where the optimal control is feasible. This extension is not strictly based on the value function, and therefore there is not a unique way to do so. In our case, for example, the zero control region is extended to the (non feasible) region with $I \geq I_{\max} \wedge S \leq \bar{S} = \frac{\gamma}{\beta}$. But another possible (and maybe better) extension is to define for this region $u = u_{\max}$ since then the limit $I_{\max}$ will be reached faster than without control action. The resulting controller is then given by

$$\tilde{u}(S, I) = \begin{cases} 0 & \text{if } I < \tilde{\Phi}(S) \vee \left(S \leq \bar{S} = \frac{\gamma}{\beta} \wedge I \geq I_{\max}\right) \\ u_{\max} & \text{if } I \geq \tilde{\Phi}(S) \end{cases},$$

$$\tilde{\Phi}(S) = \begin{cases} I_{\max} + \frac{\gamma}{(1-u_{\max})\beta} \ln\left(\frac{S}{S^*}\right) - (S - S^*) & \text{if } S^* \leq S \leq 1 \\ I_{\max} & \text{if } 0 \leq S \leq S^* \end{cases},$$

where we have slightly changed the switching function to include this region. Some results can be seen in the Figures 4 and 4.

From these figures one also observe, that an extra benefit of applying any control compared to not doing anything is that the total number of Infected people when the infection dies is larger if no action is taken than when some control action has been performed. We see this in the Figures by noting that $S(\infty)$ is larger with control than without it. This number can be further increased if instead of taking no control once $S < \frac{\gamma}{\beta}$ one still apply some control action (of course, the "best" is to use $u_{\max}$).

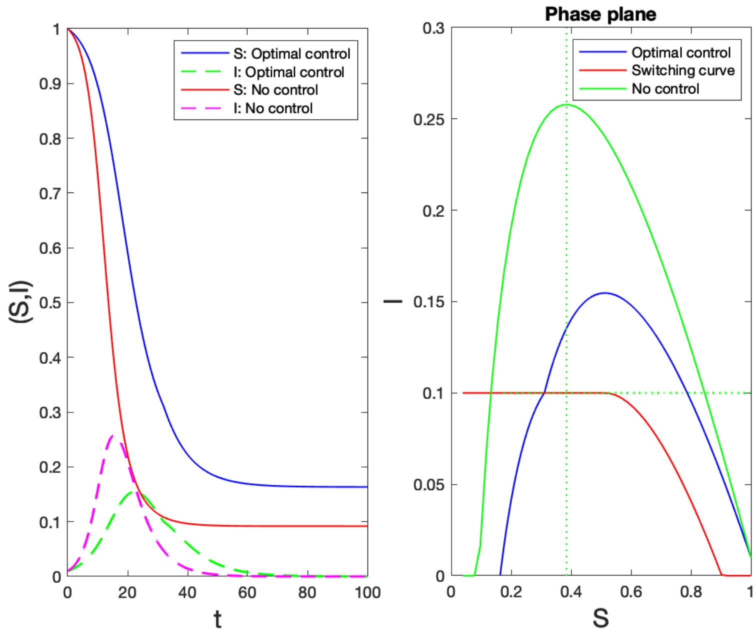

Figure 5: Optimal feedback control with $u_{\max} = 0.25$, so that the problem is unfeasible.

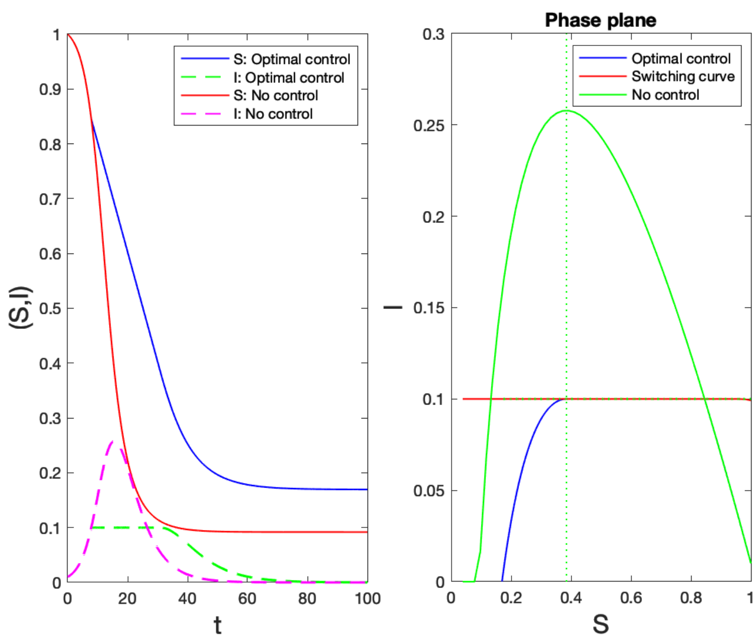

Figure 6: Optimal feedback control with $u_{\max} = 0.6$, so that the problem is feasible.