New box

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1 The Box

The SIR model is a keystone in our understanding of infectious diseases, capturing the most essential features of the epidemiological dynamics for the mitigation or eradication of epidemic outbreaks [**?**, **?**].

The SIR model with interventions $u(t) \in [0, u_{\max}]$ reducing disease transmission takes the form

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -(1-u)\beta\, SI, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = (1-u)\beta\, SI - \gamma I, \quad \frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I. \tag{1}$$

Here, $S(t)$, $I(t)$, and $R(t)$ are the proportion of the population that is susceptible, infected, or removed (i.e., recovered or dead) at time $t \geq 0$, respectively. Because $S(t) + I(t) + R(t) = 1$ for all $t \geq 0$, the epidemic state can be characterized by $(S, I) \in [0,1]^2$. We denote by (S_0, I_0) the initial state of the model at $t = 0$. The parameters of the model are the *contact rate* $\beta \geq 0$ and the the mean *residence time* of infected individuals $\gamma \geq 0$ (in units of day^{-1}). By assuming $S_0 \approx 1$, these two parameters yield the *basic reproduction number* $R_0 = \beta/\gamma$.

We are interested in reaching the set

$$\mathcal{T}_0 = \{(S, I) \mid I \leq \Phi_{R_0}(S)\} \ ,$$

which is the largest set with the following property: If, for any given time t_1, the state (S_1, I_1) belongs to $\mathcal{T}_0$, we can set $u = 0$ henceforth and still have $I(t) \leq I_{\max}$ for all $t \geq t_1$. That is, when $\mathcal{T}_0$ is reached, we can terminate the intervention with the assurance that a possible rebound will not overwhelm the healthcare system.

Our goal is to steer an arbitrary initial state (S_0, I_0) to the *safe set* $\mathcal{T}_0$ in minimal time without infringing the constraint $I(t) \leq I_{\max}$. We will say that an intervention achieving this goal is an *optimal intervention.*

In Supplementary Note S1, we define Φ_R explicitly and prove that the existence of an optimal intervention is characterized by the *separating curve* Φ_{R_c}:

(1) An optimal intervention exists if and only if the initial state (S_0, I_0) lies below this separating curve (i.e., $I_0 \leq \Phi_{R_c}(S_0)$).

Moreover:

(2) If it exists, the optimal intervention u^* at the state (S, I) is

$$u^*(S,I) = \begin{cases} 0 & \text{if } (S,I) \in \mathcal{T}_0 \cup \mathcal{T}_1 \\ 1 - 1/R_c S & \text{if } I = \Phi_{R_c}(S) \text{ and } S^* < S < R_c^{-1} \\ u_{\max} & \text{otherwise} \end{cases} \tag{2}$$

with

$$\mathcal{T}_1 = \{(S,I) \mid I < \Phi_{R_c}(S),\, S > \Psi(I)\} \ .$$

The curve $S = \Psi(I)$ is also defined in Supplementary Note S1, while S^* takes place at the intersection of $S = \Psi(I)$ and $I = \Phi_{R_c}(S)$. **(I'm redefining S^*)**

2 For the Supplementary Note S1

Let us start by defining

$$\Phi_R(S) = \begin{cases} I_{\max} & \text{if } S < R^{-1} \\ I_{\max} + R^{-1}\left(\ln(RS) + 1 - RS\right) & \text{otherwise} \end{cases} \ .$$

Now, the curve $S = \Psi(I)$ corresponds to the optimal switching curve for the optimization problem without the constraint $I(t) \leq I_{\max}$. It is best described using the change of coordinates

$$\mu(S,I) = I - \frac{1}{R_0}\ln(S) + S$$
$$\nu(S,I) = I - \frac{1}{R_c}\ln(S) + S$$

with inverse

$$S(\nu,\mu) = \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu - \nu)\right) \tag{3a}$$
$$I(\nu,\mu) = \frac{R_0}{R_0 - R_c}\mu - \frac{R_c}{R_0 - R_c}\nu - \exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu - \nu)\right) \ . \tag{3b}$$

Note that, given $I = 0$, we can uniquely map μ to $S \geq R_0^{-1}$, which we will denote by $S = \hat{S}_\mu(\mu)$. Likewise, we can uniquely map ν to $S \geq R_c^{-1}$, and we will denote it by $S = \hat{S}_\nu(\nu)$.

Our region of interest is the rectangle $[\nu_{\min}, \nu_{\max}] \times [\mu_f, \mu_{\max}]$ with

$$\nu_{\min} = -\frac{1}{R_c}\ln(\hat{S}_\mu(\mu_f)) + \hat{S}_\mu(\mu_f) \qquad \mu_f = I_{\max} + \frac{\ln(R_0) + 1}{R_0}$$
$$\nu_{\max} = I_{\max} + \frac{\ln(R_0)}{R_c} + \frac{1}{R_0} \qquad \mu_{\max} = -\frac{1}{R_0}\ln(\hat{S}_\nu(\nu_{\min})) + \hat{S}_\nu(\nu_{\min})$$

(see Fig. 1).

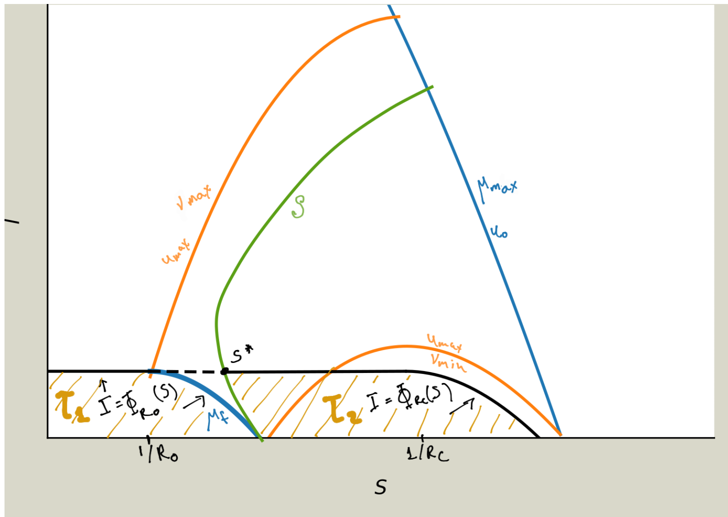

Figure 1: Phase plane

It follows from Green's theorem that the solutions of the unconstrained problem are arcs of the form

$$(\nu_0, \mu_0) \xrightarrow{R_0} (\nu_f, \mu_0) \xrightarrow{R_c} (\nu_f, \mu_f)$$

The first transition takes the time

$$T_0(\nu_f, \mu_0; \nu_0, \mu_0) = \int_{\nu_0}^{\nu_f} \frac{\mathrm{d}\nu}{\frac{R_0}{R_c}\mu_0 - \nu - \frac{R_0 - R_c}{R_c}\exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu_0 - \nu)\right)} \, ,$$

while the second one takes the time

$$T_c(\nu_f, \mu_f; \nu_f, \mu_0) = \int_{\mu_0}^{\mu_f} \frac{\mathrm{d}\mu}{-\mu + \frac{R_c}{R_0}\nu_f + \frac{R_0 - R_c}{R_0}\exp\left(\frac{R_0 R_c}{R_0 - R_c}(\mu - \nu_f)\right)} \, .$$

The total time is thus

$$T(\nu_f, \mu_f; \nu_0, \mu_0) = T_0(\nu_f, \mu_0; \nu_0, \mu_0) + T_c(\nu_f, \mu_f; \nu_f, \mu_0) \, .$$

Lemma 1. *Fix $\mu_0 \in [\mu_f, \mu_{\max}]$ and consider the map*

$$\begin{aligned} \varphi_{\mu_0} : [\nu_{\min}, \nu_{\max}] &\to \mathbb{R} \\ \nu_f &\mapsto T(\nu_f, \mu_f; \nu_{\min}, \mu_0) \, . \end{aligned}$$

Every local maxima of φ_{μ_0} is a global one.

In other words, maxima can only occur either at $\nu_f = \nu_{\min}$ or at $\nu_f = \nu_{\max}$. The lemma also implies that every global minima is unique. Thus, the following function is well defined:

$$\begin{aligned}\nu^* : [\mu_f, \mu_{\max}] &\to [\nu_{\min}, \nu_{\max}] \\ \mu_0 &\mapsto \arg\min_{\nu_f} \varphi_{\mu_0}(\nu_f) \ .\end{aligned}$$

Simply put, an initial state $(\nu_{\min}, \mu_0)$ is driven to the line $\mu = \mu_f$ in minimal time by setting first $u = 0$ and then switching to $u = u_{\max}$ at the point $(\nu^*(\mu_0), \mu_0)$. This defines a curve parameterized by μ_0. In the original coordinates (3), the switching curve takes the form

$$\mathcal{S} = \{(S, I) \mid S = S(\nu^*(\mu_0), \mu_0),\ I = I(\nu^*(\mu_0), \mu_0),\ \mu_0 \in [\mu_f, \mu_{\max}]\} \ .$$

Lemma 2. *If*

$$(S', I) \in \mathcal{S} \quad \textit{and} \quad (S'', I) \in \mathcal{S} \ ,$$

then $S' = S''$.

Finally, let

$$\bar{I} = \max_{(S,I)\in\mathcal{S}} I \ .$$

The lemma implies the existence of a function Ψ, defined on $[0, \bar{I}]$, such that

$$\mathcal{S} = \left\{(S, I) \mid S = \Psi(I), I \in [0, \bar{I}]\right\} \ .$$