Time Optimal Control for COVID19: Correcting the previous results

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Abstract

Here we just describe how to find the optimal orbit in a simple manner. We do not repeat many things we already did in the previous draft. Here, there are not yet proofs! Just a description of the optimal arc. New: we try to generalize the approach to a larger class of dynamical models.

1 SI(R) Model

The model is given by (and assuming for simpicity that $\gamma = 1$, what is equivalent to a time-scaling)

$$\begin{aligned}\dot{S} &= -\left(1-u\right)\beta S I = -R_u S I, \quad R_u = \left(1-u\right)\beta \\ \dot{I} &= \left(1-u\right)\beta S I - \gamma I = \left(R_u S - 1\right) I\,.\end{aligned}$$

We write R_0 if $u = 0$ and R_c if $u = u_{\max}$. With the change of variables proposed by Fernando

$$\begin{aligned}\mu &= I - \frac{1}{R_0}\ln\left(S\right) + S \\ \nu &= I - \frac{1}{R_c}\ln\left(S\right) + S\,,\end{aligned}$$

having inverse tranformation

$$\begin{aligned}S &= \exp\left(\frac{R_0 R_c}{R_0 - R_c}\left(\mu - \nu\right)\right) = \exp\left(\frac{\beta\left(1-u_{\max}\right)}{u_{\max}}\left(\mu - \nu\right)\right) \\ I &= \frac{R_0}{R_0 - R_c}\mu - \frac{R_c}{R_0 - R_c}\nu - \exp\left(\frac{R_0 R_c}{R_0 - R_c}\left(\mu - \nu\right)\right) \\ &= \frac{1}{u_{\max}}\mu - \frac{\left(1-u_{\max}\right)}{u_{\max}}\nu - \exp\left(\frac{\beta\left(1-u_{\max}\right)}{u_{\max}}\left(\mu - \nu\right)\right)\,.\end{aligned}$$

The dynamics in these coordinates is given by

$$\dot{\mu} = \frac{R_u - R_0}{R_0} I$$
$$\dot{\nu} = \frac{R_u - R_c}{R_c} I \,.$$

When $u = 0$, then $R_u - R_0 = 0$

$$\begin{aligned}
\dot{\mu} &= 0 \Rightarrow \mu\left(t\right) = \mu_0 \\
\dot{\nu} &= \frac{R_0 - R_c}{R_c} I\left(\mu_0,\, \nu\right) = \frac{u_{\max}}{\left(1 - u_{\max}\right)} I\left(\mu_0,\, \nu\right) > 0 \\
I\left(\mu_0,\, \nu\right) &= \frac{1}{u_{\max}} \mu_0 - \frac{\left(1 - u_{\max}\right)}{u_{\max}} \nu - \exp\left(\frac{\beta\left(1 - u_{\max}\right)}{u_{\max}}\left(\mu_0 - \nu\right)\right) \,.
\end{aligned}$$

When $u = u_{\max}$, then $R_u - R_c = 0$

$$\begin{aligned}
\dot{\mu} &= \frac{R_c - R_0}{R_0} I\left(\mu,\, \nu_0\right) = -u_{\max} I\left(\mu,\, \nu_0\right) \leq 0 \\
\dot{\nu} &= 0 \Rightarrow \nu\left(t\right) = \nu_0 \\
I\left(\mu,\, \nu_0\right) &= \frac{1}{u_{\max}} \mu - \frac{\left(1 - u_{\max}\right)}{u_{\max}} \nu_0 - \exp\left(\frac{\beta\left(1 - u_{\max}\right)}{u_{\max}}\left(\mu - \nu_0\right)\right) \,.
\end{aligned}$$

1.1 Time for the path $\mu_0 \to \mu_f$

Using $u = u_{\max}$ we can go from $\mu_0 \to \mu_f$ (with $\mu_0 > \mu_f$) along a constant ν_c and the time it takes can be calculated from

$$\dot{\mu} = -u_{\max} I\left(\mu,\, \nu_c\right) \Rightarrow T_c\left(\mu_0,\, \mu_f;\, \nu_c\right) = -\frac{1}{u_{\max}} \int_{\mu_0}^{\mu_f} \frac{d\mu}{I\left(\mu,\, \nu_c\right)} = \frac{1}{u_{\max}} \int_{\mu_f}^{\mu_0} \frac{d\mu}{I\left(\mu,\, \nu_c\right)}$$

or

$$\begin{aligned}
T_c\left(\mu_0,\, \mu_f;\, \nu_c\right) &= \frac{1}{u_{\max}} \int_{\mu_f}^{\mu_0} \frac{d\mu}{I\left(\mu,\, \nu_c\right)} \\
I\left(\mu,\, \nu_c\right) &= \frac{1}{u_{\max}} \mu - \frac{\left(1 - u_{\max}\right)}{u_{\max}} \nu_c - \exp\left(\frac{\beta\left(1 - u_{\max}\right)}{u_{\max}}\left(\mu - \nu_c\right)\right) \,.
\end{aligned}$$

Note that the integrand is positive $I\left(\mu,\, \nu_c\right) > 0$. Fixing $\mu_0 > \mu_f$ there are different transit times T_c depending on the value of ν_c. For an interval of values of $\nu_c \in \left[\nu_0,\, \nu_f\right]$ we look for a minimum of T_c. If in the interval $I\left(\mu,\, \nu_c\right) \geq \epsilon > 0$, then this minimum exists. If the minimum value is in the interior of the interval $\nu_{\min} \in \left[\nu_0,\, \nu_f\right]$, then it happens that

$$\left.\frac{\partial T_c\left(\mu_0,\, \mu_f;\, \nu\right)}{\partial \nu}\right|_{\nu = \nu_{\min}} = 0 \,.$$

This happens for

$$
\begin{aligned}
\frac{\partial T_c(\mu_0,\, \mu_f;\, \nu)}{\partial \nu} &= -\frac{1}{u_{\max}} \int_{\mu_f}^{\mu_0} \frac{\frac{\partial I(\mu,\, \nu)}{\partial \nu} d\mu}{I^2(\mu,\, \nu)} \\
\frac{\partial I(\mu,\, \nu)}{\partial \nu} &= -\frac{(1-u_{\max})}{u_{\max}} \left[1 - \beta \exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}(\mu-\nu)\right)\right],
\end{aligned}
$$

and then

$$
\frac{\partial T_c(\mu_0,\, \mu_f;\, \nu_{\min})}{\partial \nu} = 0 \Leftrightarrow
$$

$$
\beta \exp\left(-\frac{\beta(1-u_{\max})}{u_{\max}}\nu_{\min}\right) \int_{\mu_f}^{\mu_0} \frac{\exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}\mu\right) d\mu}{I^2(\mu,\, \nu_{\min})} = \int_{\mu_f}^{\mu_0} \frac{d\mu}{I^2(\mu,\, \nu_{\min})}. \tag{1}
$$

Note that both sides of this equation are positive.

We calculate also the second partial derivative of $T_c(\mu_0,\, \mu_f;\, \nu)$.

$$
\begin{aligned}
\frac{\partial^2 T_c(\mu_0,\, \mu_f;\, \nu)}{\partial \nu^2} &= -\frac{1}{u_{\max}} \int_{\mu_f}^{\mu_0} \frac{\partial}{\partial \nu} \frac{\frac{\partial I(\mu,\, \nu)}{\partial \nu}}{I^2(\mu,\, \nu)} d\mu \\
&= -\frac{1}{u_{\max}} \int_{\mu_f}^{\mu_0} \frac{I^2(\mu,\, \nu)\frac{\partial^2 I(\mu,\, \nu)}{\partial \nu^2} - 2I(\mu,\, \nu)\left(\frac{\partial I(\mu,\, \nu)}{\partial \nu}\right)^2}{I^4(\mu,\, \nu)} d\mu \\
&= \frac{1}{u_{\max}} \int_{\mu_f}^{\mu_0} \frac{2\left(\frac{\partial I(\mu,\, \nu)}{\partial \nu}\right)^2 - I(\mu,\, \nu)\frac{\partial^2 I(\mu,\, \nu)}{\partial \nu^2}}{I^3(\mu,\, \nu)} d\mu \\
\frac{\partial I(\mu,\, \nu)}{\partial \nu} &= -\frac{(1-u_{\max})}{u_{\max}} \left[1 - \beta \exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}(\mu-\nu)\right)\right] \\
\frac{\partial^2 I(\mu,\, \nu)}{\partial \nu^2} &= -\beta^2 \left(\frac{(1-u_{\max})}{u_{\max}}\right)^2 \exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}(\mu-\nu)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
2\left(\frac{\partial I(\mu,\, \nu)}{\partial \nu}\right)^2 - I(\mu,\, \nu)\frac{\partial^2 I(\mu,\, \nu)}{\partial \nu^2} &= 2\left(\frac{\partial I(\mu,\, \nu)}{\partial \nu}\right)^2 + \\
&\quad \beta^2 \left(\frac{(1-u_{\max})}{u_{\max}}\right)^2 \exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}(\mu-\nu)\right) I(\mu,\, \nu) > 0.
\end{aligned}
$$

So we conclude that $\frac{\partial^2 T_c(\mu_0,\, \mu_f;\, \nu)}{\partial \nu^2} > 0$ and therefore $T_c(\mu_0,\, \mu_f;\, \nu)$ is convex as a function of ν. Therefore the minimum is global, and if it is in the interior of the interval $\nu_c \in [\nu_0,\, \nu_f]$ then it is the only point satisfying (1).

1.2 Total time

Using $u = 0$ we can go from $\nu_0 \to \nu_f$ (with $\nu_0 < \nu_f$) along a constant μ_c and the time it takes can be calculated from

$$\dot{\nu} = \frac{u_{\max}}{(1-u_{\max})} I(\mu_c, \nu) \Rightarrow T_0(\nu_0, \nu_f; \mu_c) = \frac{(1-u_{\max})}{u_{\max}} \int_{\nu_0}^{\nu_f} \frac{d\nu}{I(\mu_c, \nu)}$$
$$I(\mu_0, \nu) = \frac{1}{u_{\max}}\mu_0 - \frac{(1-u_{\max})}{u_{\max}}\nu - \exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}(\mu_0 - \nu)\right).$$

or

$$\begin{aligned} T_0(\nu_0, \nu_f; \mu_c) &= \frac{(1-u_{\max})}{u_{\max}} \int_{\nu_0}^{\nu_f} \frac{d\nu}{I(\mu_c, \nu)} \\ I(\mu_c, \nu) &= \frac{1}{u_{\max}}\mu_c - \frac{(1-u_{\max})}{u_{\max}}\nu - \exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}(\mu_c - \nu)\right). \end{aligned}$$

Now, joining the paths $\nu_0 \to \nu_c$ and $\mu_0 \to \mu_f$: use $u = 0$ to go from $\nu_0 \to \nu_c$ and then use $u = u_{\max}$ to go from $\mu_0 \to \mu_f$ along constant ν_c, the total time is given by

$$\begin{aligned} T(\nu_0, \nu_c; \mu_0, \mu_f) &= T_0(\nu_0, \nu_c; \mu_0) + T_c(\mu_0, \mu_f; \nu_c) \\ &= \frac{(1-u_{\max})}{u_{\max}} \int_{\nu_0}^{\nu_c} \frac{d\nu}{I_a(\mu_0, \nu)} + \frac{1}{u_{\max}} \int_{\mu_f}^{\mu_0} \frac{d\mu}{I_b(\mu, \nu_c)} \\ I_a(\mu_0, \nu) &= \frac{1}{u_{\max}}\mu_0 - \frac{(1-u_{\max})}{u_{\max}}\nu - \exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}(\mu_0 - \nu)\right) \\ I_b(\mu, \nu_c) &= \frac{1}{u_{\max}}\mu - \frac{(1-u_{\max})}{u_{\max}}\nu_c - \exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}(\mu - \nu_c)\right). \end{aligned}$$

We fix (ν_0, μ_0, μ_f) and leave ν_c free. And want to find the minimum of $T(\nu_0, \nu_c; \mu_0, \mu_f)$ with respect to ν_c. If this happens in the interior of the interval of interest (and assuming the T is differentiable!), then

$$\left.\frac{\partial T(\nu_0, \nu; \mu_0, \mu_f)}{\partial \nu}\right|_{\nu_{\min}} = 0.$$

This happens if

$$\begin{aligned} \frac{\partial T(\nu_0, \nu; \mu_0, \mu_f)}{\partial \nu} &= \frac{(1-u_{\max})}{u_{\max}} \frac{1}{I_a(\mu_0, \nu)} - \frac{1}{u_{\max}} \int_{\mu_f}^{\mu_0} \frac{\frac{\partial I_b(\mu, \nu)}{\partial \nu} d\mu}{I_b^2(\mu, \nu)} \\ \frac{\partial I_b(\mu, \nu)}{\partial \nu} &= -\frac{(1-u_{\max})}{u_{\max}} \left[1 - \beta \exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}(\mu - \nu)\right)\right]. \end{aligned}$$

The second derivative of T is

$$
\begin{aligned}
\frac{\partial^2 T\left(\nu_0,\,\nu;\,\mu_0,\,\mu_f\right)}{\partial\nu^2} &= -\frac{\left(1-u_{\max}\right)}{u_{\max}}\frac{\frac{\partial I_a(\mu_0,\,\nu)}{\partial\nu}}{I_a^2\left(\mu_0,\,\nu\right)} - \frac{1}{u_{\max}}\int_{\mu_f}^{\mu_0}\frac{\partial}{\partial\nu}\frac{\frac{\partial I_b(\mu,\,\nu)}{\partial\nu}}{I_b^2\left(\mu,\,\nu\right)}d\mu \\
&= -\frac{\left(1-u_{\max}\right)}{u_{\max}}\frac{\frac{\partial I_a(\mu_0,\,\nu)}{\partial\nu}}{I_a^2\left(\mu_0,\,\nu\right)} \\
&\quad + \frac{1}{u_{\max}}\int_{\mu_f}^{\mu_0}\frac{2\left(\frac{\partial I_b(\mu,\,\nu)}{\partial\nu}\right)^2 - I_b\left(\mu,\,\nu\right)\frac{\partial^2 I_b(\mu,\,\nu)}{\partial\nu^2}}{I_b^3\left(\mu,\,\nu\right)}d\mu \\
\frac{\partial I_a\left(\mu_0,\,\nu\right)}{\partial\nu} &= -\frac{\left(1-u_{\max}\right)}{u_{\max}}\left[1-\beta\exp\left(\frac{\beta\left(1-u_{\max}\right)}{u_{\max}}\left(\mu_0-\nu\right)\right)\right].
\end{aligned}
$$

And therefore

$$
\begin{aligned}
\frac{\partial^2 T\left(\nu_0,\,\nu;\,\mu_0,\,\mu_f\right)}{\partial\nu^2} &= \left(\frac{\left(1-u_{\max}\right)}{u_{\max}}\right)^2\frac{\left[1-\beta\exp\left(\frac{\beta(1-u_{\max})}{u_{\max}}\left(\mu_0-\nu\right)\right)\right]}{I_a^2\left(\mu_0,\,\nu\right)} \\
&\quad + \frac{1}{u_{\max}}\int_{\mu_f}^{\mu_0}\frac{2\left(\frac{\partial I_b(\mu,\,\nu)}{\partial\nu}\right)^2 - I_b\left(\mu,\,\nu\right)\frac{\partial^2 I_b(\mu,\,\nu)}{\partial\nu^2}}{I_b^3\left(\mu,\,\nu\right)}d\mu\,.
\end{aligned}
$$

The second term is positive (as we have shown before), but the first term has no definite sign, since

$$
1-\beta\exp\left(\frac{\beta\left(1-u_{\max}\right)}{u_{\max}}\left(\mu_0-\nu\right)\right)
$$

can be positive or negative for diferent values of ν. So, we cannot conclude a priori that $\frac{\partial^2 T(\nu_0,\,\nu;\,\mu_0,\,\mu_f)}{\partial\nu^2}$ is positive or negative!