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Radiation from a spherical source

Preparation: Read carefully Section 9.4 of the book, Ref. [1].

Theory

The sound field generated by a sphere with an axisymmetric vibrational velocity in free space is the solution to the axisymmetric Helmholtz equation in the spherical coordinate system. The solution should also satisfy the Sommerfeld radiation condition. The result is a sum of terms of the form

$$p(r,\theta) = A_m \mathrm{h}_m^{(2)}(kr)\mathrm{P}_m(\cos\theta)\mathrm{e}^{\mathrm{j}\omega t},$$

where r is the distance from the centre of the sphere, k is the wavenumber, θ is the polar angle, $\mathrm{h}_m^{(2)}$ is the spherical Hankel function of the second kind and order m, and P_m (cos θ) is the Legendre function of order m. The corresponding radial component of the particle velocity corresponding to the expression for the sound pressure given above is

$$u_r(r,\theta) = \frac{-1}{\mathrm{j}\omega\rho}\frac{\partial p(r,\theta)}{\partial r} = \frac{-1}{\mathrm{j}\rho c} A_m \mathrm{P}_m(\cos\theta)\mathrm{e}^{\mathrm{j}\omega t}\frac{\mathrm{dh}_m^{(2)}(kr)}{\mathrm{d}r},$$

Note that the derivative of the Hankel function can be written as

$$\frac{\mathrm{dh}_m^{(2)}(z)}{\mathrm{d}z} = \frac{1}{2m+1}\left(m\mathrm{h}_{m-1}^{(2)}(z) - (m+1)\mathrm{h}_{m+1}^{(2)}(z)\right),$$

and that the spherical Hankel functions can be expressed in terms of the cylindrical Hankel functions,

$$\mathrm{h}_m^{(2)}(z) = \sqrt{\frac{\pi}{2z}}\mathrm{H}_{m+½}^{(2)}(z).$$

You should use these relations when you construct your MATLAB program. The cylindrical Hankel function is called `besselh`. Note that you need the cylindrical Hankel function of the *second* kind. The MATLAB function `legendre` gives you the associated Legendre functions (you need only the first element in the resulting vector, to obtain the Legendre polynomials, corresponding to the axisymmetric solution. Please note that Matlab returns the Associated Legendre functions - not just the polynomials).

Description of the tasks

Construct a program that can calculate the sound pressure of the various terms of the expansion as a function of r, as a function of θ, and as a function of the frequency $f = kc/2\pi$. Plot the following results and discuss them:

- Plot the magnitude of the sound pressure as a function of θ for m = 0, 1 and 2. Plot each order separately (three curves), in dB (Note: polar plot in matlab uses only positive values; you might need to offset the plot). It is recommended to normalize by the axial pressure.
- Plot the magnitude of the pressure on the axis of symmetry (θ = 0) in dB as a function of r from a to 1000a for m = 0, 1 and 2 (separately) at a low frequency ka = 0.1. Use a logarithmic r-axis. You can normalize by the far-field pressure. Analyze the near- and far-field behavior.
- Plot the phase angle between the sound pressure and the radial component of the particle velocity as a function of r from a to 1000a for m = 0, 1 and 2 (separately) at a frequency of ka = 0.1. Analyze and discuss the results.

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The normalised sound pressure generated by a point source at $(r, \theta) = (a, 0)$ on a rigid sphere is

$$\frac{p(r,\theta)}{p(r,0)} = \frac{\sum_{m=0}^{\infty}(m+½)\mathrm{h}_m^{(2)}(kr)\mathrm{P}_m(\cos\theta)\left(\left.\frac{\mathrm{dh}_m^{(2)}(kr)}{\mathrm{d}r}\right|_{r=a}\right)^{-1}}{\sum_{m=0}^{\infty}(m+½)\mathrm{h}_m^{(2)}(kr)\left(\left.\frac{\mathrm{dh}_m^{(2)}(kr)}{\mathrm{d}r}\right|_{r=a}\right)^{-1}}.$$

- Plot the magnitude of the sound pressure generated by a point source on a rigid sphere in the far field (e.g., at a distance of $100a$) as a function of θ, for ka = 0.1, 1, 5, and 10. Normalise by the axial pressure $p(r, 0)$, and plot in dB. Infinite summations must be truncated, of course. Discuss the obtained results and the variations with frequency and angle.
- Instead of normalizing by the axial sound pressure, normalize by the pressure of a point source radiating in a free-field (i.e. the incident field, without the presence of the rigid sphere; Due to simplifications used in the equation above, the volume velocity of the source is assumed to be $\mathrm{j}\omega\rho Q = 2\pi a^2$). Discuss the results and relate them to the previous plot.

OPTIONAL: Can you construct the first three orders (*m = 0, 1 and 2*) of the full spherical harmonic functions $\mathrm{Y}_m^n(\theta,\phi)$? Note that in this case the Associated Legendre functions are needed. How do these relate to the first plot?

$$\mathrm{Y}_m^n(\theta,\phi) \equiv \sqrt{\frac{2m+1}{4\pi}\frac{(m-n)!}{(m+n)!}}\mathrm{P}_m^n(\cos\theta)\mathrm{e}^{\mathrm{j}n\phi},$$

Report

Discuss the results based on the relevant concepts that you have studied in the course. **Important:** This exercise does not require a typical written report. Instead, you will submit a recorded presentation video, of 5 to 8 minutes, where all members of the group present a part of it. The video should be submitted two weeks after the day of the exercise has been carried out. Please submit the video in a common format (preferably mp4) as well as the slides of the presentation as a pdf file. Please mind the size of the video, so that the file is not too large. It is expected that you cover briefly the introduction/motivation and theory, and place the focus of the presentation on the results and analysis.

References

[1] F. Jacobsen and P. M. Juhl, *Fundamentals of General Linear Acoustics*, John Wiley and Sons, 2013 (Corresponding to chapter 5 in the lecture note “Radiation of Sound”, 2011).