The diffuse sound field

Preparation: Read chapter 8 in the book carefully. If you are not familiar with fundamentals of statistics you should also read the appendix of this note.

The sound field in a reverberation room driven with a pure tone above the Schroeder frequency can be modeled as a sum of plane waves with uniformly distributed random phases φ_i and random amplitudes $|A_i|$, coming from random directions,

$$\hat{p}(\mathbf{r_1}) = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} A_i \mathrm{e}^{\mathrm{j}(\omega t - \mathbf{k_i}.\mathbf{r_1})} = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |A_i| \mathrm{e}^{\mathrm{j}(\omega t + \varphi_i - \mathbf{k_i}.\mathbf{r_1})}$$

where $\mathbf{k_i}$ is the wavenumber vector of the ith wave. This is a vector with the length $k(=\omega/c)$ and a direction that is uniformly distributed over a solid angle of 4π. If we move the observation point from $\mathbf{r_1}$ to $\mathbf{r_2}$, each wave will arrive with a phase shift that depends on the vector that separates the two positions, $\mathbf{r_2} - \mathbf{r_1}$, and the direction of the wave, $\mathbf{k_i}$. Unless the separation vector is short compared with the wavelength the phase shift, $\mathbf{k_i}.(\mathbf{r_2} - \mathbf{r_1})$ is also completely random. It follows that the spatial statistics can be studied either by varying the observation point at random or by studying the ensemble statistics – the statistics with respect to sets of random complex amplitudes A_i and wavenumber vectors $\mathbf{k_i}$.

It is possible to estimate statistical results with computer simulations employing random numbers; this is called 'Monte Carlo simulation'. In a practical implementation of the model described above the number of waves must obviously be finite. Moreover, it can be shown that distribution of A_i is not important, and therefore all the values can be set to 1. The mean square pressure is thus modeled as the square of the absolute value of a sum of complex exponentials,

$$\frac{|\hat{p}|^2}{2} = \frac{1}{2N} \left| \sum_{i=1}^{N} \mathrm{e}^{\mathrm{j}\varphi_i} \right|^2$$

where φ_i is a random variable with a rectangular probability density over the interval from 0 to 2π. The spatial correlation of the pressure is modeled by combining two sums of complex exponentials, one of which with phase shifts introduced in each term as described above,

$$\frac{\hat{p}_1 \hat{p}_2^*}{2} = \frac{1}{2N} \left(\sum_{i=1}^{N} \mathrm{e}^{\mathrm{j}\varphi_i} \right) \left(\sum_{i=1}^{N} \mathrm{e}^{\mathrm{j}(\varphi_i - kr\cos\theta_i)} \right)^*$$

The polar angle of incidence of the ith wave, θ_i, should have a sinusoidal distribution from 0 to π. This arrangement ensures a uniform distribution of N plane waves over all angles of incidence. If RAND is a random variable uniformly distributed over the interval from 0 to 1, then the random phases φ_i can be generated as RAND×2π, and the polar angles θ_i can be calculated as Arcsin (2(RAND - 0.5)) + π/2. NB: use independent random numbers for the angle and the phase!

Construct a MATLAB program that implements the expression for the mean square pressure, $|\hat{p}|^2$, and show using Monte Carlo simulation that this is a random variable with a relative standard deviation of 1. This can be done by repeating calculations of $|\hat{p}|^2$ many times using different sets of random phases and determining the sample average and standard deviation of the resulting values. Alternatively (or as a supplement), show that the corresponding sound pressure amplitude, $|\hat{p}|$, is a random value with a relative standard deviation of 0.52. Also show that the standard deviation (not the relative standard deviation) of the corresponding sound pressure level is 5.6 dB. You can include the obtained values in a table.

Construct another MATLAB program that implements the expression for the spatial correlation $\hat{p}_1\hat{p}_2^*/2$, and show that the average value of this random variable normalized by the average value of $|\hat{p}|^2/2$ equals $\sin kr/kr$. You can do that by plotting $E\{\hat{p}_1\hat{p}_2^*\}/E\{|\hat{p}|^2\}$ and $\sin kr/kr$ in the same figure as functions of kr. Consider what happens when you vary the number of waves in each realization, and when you vary the number of realizations in an ensemble.

Finally, consider the mean square pressure as a function of position in an arbitrary room; For simplicity we can assume that we are in the line normal to the *z=0* plane, i.e. *(x, y, z)=(0, 0, r)*. The mean square pressure can be described as

$$\frac{|\hat{p}(r)|^2}{2} = \frac{1}{2N}\left|\sum_{i=1}^{N} \mathrm{e}^{-\mathrm{j}(kr\cos\theta_i+\varphi_i)}\right|^2,$$

where θ_i is the polar angle of incidence of the ith wave, as before. Plot the mean square pressure and the sound pressure level at 700 Hz as a function of distance, in a room of length l_z = 7 m, and considering a sufficiently high number of waves, at least $N=10^2$. Calculate the histogram of the data (you can use the MATLAB function "hist(data, N)", where N is the number of bins used in the histogram). Keep in mind that now you are considering spatial statistics of a single sound field (and not an ensemble).

Sound Field Statistics in an Actual Room (using Green's function from exercise 2)

You will examine the statistical properties of a particular room: the small rectangular room that you modeled and measured in the previous laboratory exercise, namely room 010 in building 355, which has dimensions `[x,y,z] = [3.28, 4.40, 3.28] m`.
Use your implementation of the Green's function in a room to evaluate the sound pressure at the Schroeder frequency (`T_rev = 3.5s`). Consider a sufficient number of modes, e.g. up to `n_max = 10` in any dimension. Sample the sound pressure at 1000 random points drawn from a uniform distribution, and preferably excluding points that are closer than half a wavelength to any wall.

Depending on the implementation the calculation can take a while, so consider debug prints and testing the function before running the whole set or even saving the variables for later use.

Plot the histogram of the mean square pressure and of the sound pressure level in the room. Compare the results to those from the plane wave expansion simulation. Estimate the mean, standard deviation and relative standard deviation.

Report

Describe the program you have constructed and present and discuss the results. Keep in mind that you are simulating an acoustic process, specifically the sound field in a room (or an ensemble of rooms) at high frequencies, where a large number of waves are present. It is important to describe the results that you obtain under this optic.

The report should be submitted as a combined report with the simulation exercise of *the Green´s function in a room*. The report should be submitted two weeks after this exercise. For guidance: expected total length of the report (both exercises) is approx. 8-11 pages (Intro 0.5p; Theory 1-2 pp; Setup 1 p. Results & discussion 4-7 pp.; Conclusion 0.5 p.). The report should be *complete* and concisely written. The two parts of the report should be presented fairly independently.

APPENDIX

Random (or stochastic) variables can be described in terms of their probability density functions. By definition the integral of such a function from minus infinity to infinity is unity, indicating that the variable must always assume some value. The integral of the probability density over a certain interval gives the probability of the variable being in that interval. Thus

$$P\{a \le x \le b\} = \int_a^b f_x(u)\mathrm{d}u$$

is the probability that the random variable x with the probability density $f_x(u)$ takes a value in the interval from a to b.

It is sometimes sufficient to characterize a random variable in terms of its average (or expected) value and its variance. The average value of x is

$$E\{x\} = \int_{-\infty}^{\infty} u f_x(u)\mathrm{d}u$$

One can estimate the average value by averaging an adequate number of outcomes of x,

$$\hat{E}\{x\} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The variance is

$$\sigma^2\{x\} = E\{(x - E\{x\})^2\} = E\{x^2\} - E^2\{x\}$$

One can estimate this quantity from an appropriate number of outcomes as follows:

$$\widehat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \hat{E}\{x\}\right)^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right)$$

This is the sample variance. The square root of the sample variance is called the sample standard deviation. It is useful to describe random variables in terms of their average value and relative standard deviation,

$$\varepsilon\{x\} = \sigma\{x\}/E\{x\}$$

A Gaussian (or normal) random variable x with the average value μ and the variance σ^2 has the probability density

$$f_x(u) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(u-\mu)^2}{2\sigma^2}\right)$$

A sum of two squared independent normally distributed random variables with zero mean and the same standard deviation is a random variable with an exponential distribution. An exponentially distributed random variable y with the average value μ has the probability density

$$f_y(u) = \begin{cases} \frac{1}{\mu}\exp\left(-\frac{u}{\mu}\right) & \text{for } u > 0 \\ 0 & \text{elsewhere} \end{cases}$$

The variance of this variable is

$$\sigma^2\{y\} = \int_0^{\infty} (u-\mu)^2 \frac{1}{\mu}\exp\left(-\frac{u}{\mu}\right)\mathrm{d}u = \mu^2$$

from which it can be seen that the relative standard deviation of an exponentially distributed random variable is unity:

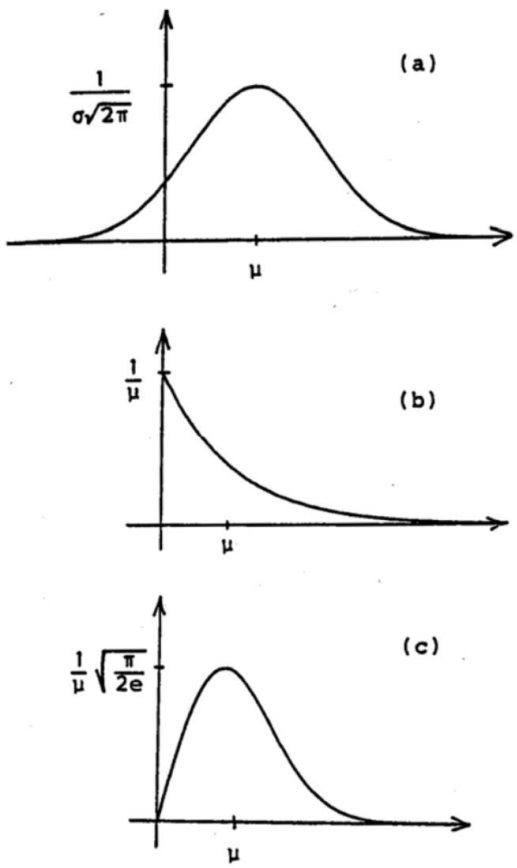
$$\varepsilon\{y\} = 1$$

The square root of an exponentially distributed random variable is yet another random variable with a Rayleigh probability distribution. A Rayleigh distributed random variable z with the average value μ has the probability density

$$f_y(u) = \begin{cases} \frac{u\pi}{2\mu^2} \exp\left(-u^2 \frac{\pi}{4\mu^2}\right) & \text{for } u > 0 \\ 0 & \text{elsewhere} \end{cases}$$

This variable has the relative standard deviation

$$\varepsilon\{z\} = \sqrt{(4-\pi)/\pi} \approx 0.52$$



Probability density functions: (a) Gauss distribution, (b) exponential distribution, and (c) Rayleigh distribution.