Exercise 7.1: Eigenvalue Analysis and Convergence Rate

A tube of length L_x is filled with a fluid with density ρ_0 and speed of sound c. Both ends of the tube are rigid walled (fixed pressure). The eigenvalue analysis of the acoustic tube is approximated using a finite element discretization, which consists of n linear elements of length $h = L_x/n$.

Implement a generalized eigenvalue analysis in `FEM_ex7_1_eigenmodes.m` (Download from Learn). The eigenvalue analysis part of the script should be capable of performing the following tasks.

a) Construct sparse mass and stiffness matrix using the MATLAB function `sparse`.

b) Call the sparse eigenvalue solver in MATLAB `eigs` and ask it to return the 6 smallest eigenvalues and eigenvectors.

c) Sort the eigenvalues (and vectors) in ascending order.

d) Plot the different modes for the acoustic tube and compare the eigenvalues to the analytic result.

Understand and test the code, then extend it to solve the tasks below.

7.1.1 Convergence Rate

Plot the convergence rate with respect to the number of elements. An indication of the convergence rate should be obtained by plotting the number of nodes ($n + 1$ for linear elements) of a certain FE discretization against the corresponding accuracy measures $\Delta\omega_i$ and Δp_i. Discuss the convergence rate.

$$\Delta\omega_i = \frac{|\omega_i - \omega_{\text{exact}}|^2}{|\omega_{\text{exact}}|^2}, \quad \Delta p_i = \frac{\int_0^L |p_i - p_{\text{exact}}|^2 \, \mathrm{d}x}{\int_0^L |p_{\text{exact}}|^2 \, \mathrm{d}x}, \quad i = 1, \ldots, n^{\text{th}} \text{ mode.} \tag{0.1}$$

7.1.2 Fluid Velocity

The fluid velocity v (or displacement) distributions in a finite element approximation is a derived secondary variable; plot the velocity and discuss the accuracy. (Hint: $v \propto \frac{\mathrm{d}p}{\mathrm{d}x}$)

7.1.3 Quadratic Elements

Implement quadratic elements using the local matrices for quadratic elements found in the lecture slides. Redo the above computations for the quadratic elements.

Exercise 7.2: Finite Element Procedure for Impedance Tube

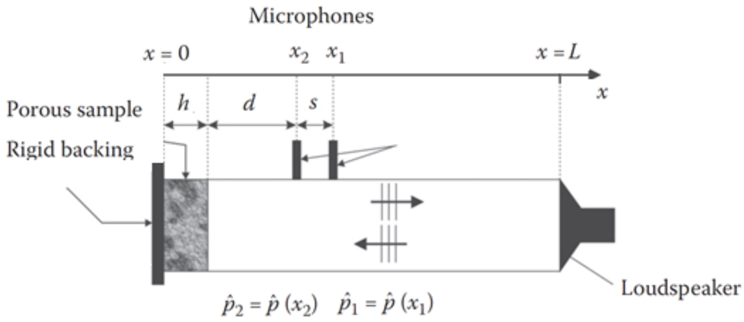
Implement a one-dimensional-FEM in `FEM_ex7_2_impedancetube.m` (Download from Learn). Some parts related with the FEM calculation are missing. Write some MATLAB code that calculates the acoustic pressure at the two microphones. Compare the estimation of (0.1) using FEM to its exact value.

The procedure is used in an impedance tube to measure the surface impedance and absorption coefficient of an acoustic material. The tube is of length L. A loudspeaker is installed at one end ($x = L$) and an acoustic material of thickness h and surface impedance $\hat{Z}$ is bonded onto the other rigid and impervious end ($x = 0$). Below the cut-off frequency of the tube, only plane waves propagate and the problem is suitable to a one-dimensional analysis.

In this case, the surface impedance of the material can be obtained from the measurement of the transfer function $\hat{H}_{12} = \hat{P}_2/\hat{P}_1$ between two microphones adequately placed in the tube (e.g., standard ASTM E-1050):

$$\frac{\hat{Z}}{\rho_0 c_0} = \frac{1 + R}{1 - R}, \quad R = \frac{\hat{H}_{12} - \exp(-ik_0 s)}{\exp(ik_0 s) - \hat{H}_{12}} \exp(2ik_0(d + s)), \tag{0.2}$$

where d is the distance from microphone 2 to the sample and s is the spacing between the two microphones. The normal incidence absorption coefficient α is directly obtained from the reflection coefficient R: $\alpha = 1 - |R|^2$.



- Cylindrical tube with diameter $D = 100$ mm (and thus cut-off frequency $0.59c_0/D = 2006$ Hz),
- Length: $L = 10D$,
- Microphone spacing: s = D/2
- Distance between microphone 2 and sample surface: $d = D/2$
- Sample thickness: $h = 2$ cm
- Characteristic impedance $\hat{Z}_c = \rho_0 c_0 \left[1 + 0.057X^{-0.754} - i0.189X^{-0.732}\right]$
- Wave number $\hat{k}_c = \frac{\omega}{c_0}\left[1 + 0.0978X^{-0.700} - i0.189X^{-0.595}\right]$
- $X = \frac{\rho_0 f}{\rho_p}$
- $\rho_0 = 1.2$ kgm^3, $c_0 = 340$ m/s
- Flow resistivity of the sample $\rho_p = 10000$ Rayls/m

Note that the impedance of a sample of thickness h bonded onto a rigid wall and excited by a plane wave with incidence angle θ is given by (Allard and Atalla, 2009).

$$\hat{Z} = -\frac{\hat{Z}_c \hat{k}_c}{\hat{k}_3} i \cot(\hat{k}_3 h), \quad \hat{k}_3 = \hat{k}_c \cos(\theta). \tag{0.3}$$

In our example $\theta = 0$ and thus $\hat{Z} = -i\hat{Z}_c \cot(\hat{k}_c h)$.