Numerical Acoustics -Spring 2025

Exercise 1: Analytical solution for the scattering by an infinite cylinder

In this exercise, you will write a program that calculates and displays the scattered sound pressure of an incident plane wave by an infinite cylinder. It is a two-dimensional problem, and the results will be used as a verification for numerical calculations in 2D later in the course.

The exercise has the objectives:

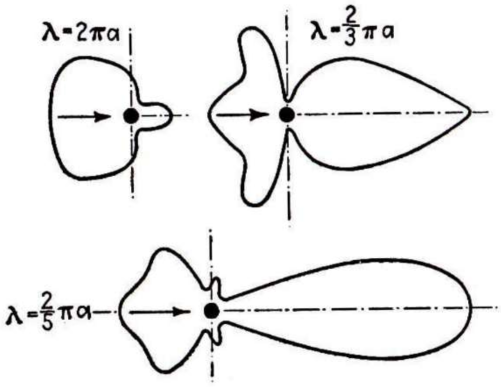
- Show how sound waves interact with objects at different frequencies.
- Train programming skills and good practice.
- Have an analytical solution that can be compared against numerical solutions.

Expressions for the scattering of a plane wave by an infinite cylinder are found in the references (find excerpts further down):

- *Vibration and Sound* (chapter 7) by P. M. Morse (equation 29.1 gives the incident pressure, and equation 29.2 gives the scattered pressure, $e^{-i\omega t}$ convention, plane wave from X axis).
- *Fundamentals of General Linear Acoustics* by Finn Jacobsen and Peter Juhl (equation 9.117 gives the sum of incident and scattered pressures, $e^{i\omega t}$ convention, plane wave from -X axis).

Exercise steps:

1. Implement the expressions into a Matlab function that can take a list of field point coordinates, the cylinder radius and the wavenumber, and produce the values of the incident and scattered waves in those field points. Document the function properly. Note that truncated series are needed. We only need to implement the sound pressure, not the particle velocity. If you use the equations in Morse, the value of P_0, the amplitude of the incident wave, should be made 1, as indicated with red ink in the book excerpt.
2. Write a Matlab script file to calculate the scattered pressure along the cylinder surface for several values of *k*a* (wavenumber*radius), calling the function above. Represent as a function of the angle.
3. Write a Matlab script (or continue with it) to calculate the total pressure and scattered pressure in a rectangular grid around the cylinder. Use the Matlab function *meshgrid* to construct the grid (check its description in Matlab help). Decide and justify how dense the grid should be as a function of the frequency (or k*a). Represent the result using the Matlab function *surf*.
4. Continue by calculating the sound pressure on a circle (or half circle, due to symmetry) centered in the cylinder and in the far field at several frequencies or *k*a* values, obtaining a polar scattering plot. Use the *polarplot* Matlab function. Compare this with the figure, which is based on the approximate equation 29.3 in the book by Morse. *How far is the far field?*
5. Draw conclusions from the results regarding the influence of the frequency and radius of the cylinder on the scattering.



Write comments along with your programming and structure the programs by grouping their different parts. Prepare your material for a brief presentation. Run the code, describe what you did and what you obtained. This and all remaining exercises are done in groups.

Vicente Cutanda Henríquez - 02-2025

angle ϑ. Directly ahead ($\vartheta = 0$) the pressure wave reproduces the piston *acceleration* exactly:

$$p = \left(\frac{\rho a^2}{4r}\right) A\left(t - \frac{r}{c}\right), \quad (\vartheta = 0)$$

The force on a microphone diaphragm directly ahead of the piston is therefore proportional to the piston acceleration; and if the diaphragm is mass controlled, its acceleration is proportional to the force, so that the accelerations, velocities, and displacements of piston and diaphragm are proportional. As ϑ is increased, however, the integral covers a larger and larger interval of time τ, so that more and more of the piston motion gets blurred together in the pressure wave arriving at P.

A simple example of this is for the case when the piston suddenly moves outward a distance Δ. In this case the displacement of the piston is $\Delta u(t)$, where $u(t)$ is the step function defined in Eq. (6.9); the velocity is $\Delta\delta(t)$, proportional to the impulse function; and the acceleration $\Delta\delta'(t)$ is formally proportional to the derivative of the impulse function, a "pathological" function going first to plus infinity and then to minus infinity in an infinitesimal period of time. Its integral properties are $\int_{-\infty}^{t} \delta'(\tau - a)f(\tau)\,d\tau = -[(df(a)/da)u(t - a)]$. The resulting pressure at (r,ϑ) is

$$p = \begin{cases} 0 & (ct < r - a\sin\vartheta) \\ \dfrac{\rho c^2\Delta}{2\pi r\sin^2\vartheta}\,\dfrac{r - ct}{\sqrt{a^2\sin^2\vartheta - (r - ct)^2}}, & (r - a\sin\vartheta < ct < r + a\sin\vartheta) \\ 0 & (ct > r + a\sin\vartheta) \end{cases}$$

This pulse, shown in Fig. 77, is a stretched out version of the δ' function; the greater the angle ϑ, the greater the stretch. Only directly ahead of the piston, at $\vartheta = 0$, is the pressure pulse as instantaneous as the piston pulse.

29. THE SCATTERING OF SOUND

When a sound wave encounters an obstacle, some of the wave is deflected from its original course. It is usual to define the difference between the actual wave and the undisturbed wave, which would be present if the obstacle were not there, as the *scattered* wave. When a plane wave, for instance, strikes a body in its path, in addition to the undisturbed plane wave there is a scattered wave, spreading out from the obstacle in all directions, distorting and interfering with the plane wave. If the obstacle is very large compared with the wavelength (as it usually is for light waves and very seldom is for sound), half of this scattered wave spreads out more or less uniformly in all directions from the scatterer, and the other half is concentrated behind the obstacle in such a manner as to interfere destructively with the unchanged plane wave behind the obstacle, creating a sharp-edged shadow there. This is the case of geometrical optics; in this case the half of the scattered wave spreading out uniformly is called the *reflected* wave, and the half responsible for the shadow is called the *interfering* wave. If the obstacle is very small compared with the wavelength (as it often is for sound waves), then all the scattered wave is sent out uniformly in all directions, and there is no sharp-edged shadow. In the intermediate cases, where the obstacle is about the same size as the wavelength, a variety of curious interference phenomena can occur.

In the present chapter, since we are studying sound waves, we shall be interested in the second and third cases, where the wavelength is longer or at least the same size as the obstacle. We shall not encounter or discuss sharply defined shadows. So much of the scattered wave will travel in a different direction from the plane wave that destructive interference will be unimportant, and we shall be able to separate all the scattered wave from the undisturbed plane wave. We shall be interested in the total amount of the wave that is scattered, in the distribution in angle of this wave, and in the effect of this scattered wave on the pressure at various points on the surface of the obstacle.

Scattering from a Cylinder.—Let us first compute the scattering, by a cylinder of radius a, of a plane wave traveling in a direction perpendicular to the cylinder's axis. If the plane wave has intensity Υ_0, the pressure wave, if the cylinder were not present, would be

$$p_p = P_0 e^{ik(x-ct)} = P_0 e^{ik(r\cos\phi - ct)}, \quad P_0 = \sqrt{2\rho c\Upsilon_0}, \quad k = \frac{2\pi}{\lambda}$$

where the direction of the plane wave has been taken along the positive x-axis.

In Eq. (19.13) we expressed this plane wave in terms of cylindrical waves:

$$p_p = P_0 e^{ik(r\cos\phi - ct)} = P_0\left[J_0(kr) + 2\sum_{m=1}^{\infty} i^m \cos(m\phi) J_m(kr)\right]e^{-2\pi i\nu t} \quad (29.1)$$

The radial velocity corresponding to this wave is

$$u_{pr} = \left(\frac{P_0}{\rho c}\right)\left\{iJ_1(kr) + \sum_{m=1}^{\infty} i^{m+1}[J_{m+1}(kr) - J_{m-1}(kr)]\cos(m\phi)\right\} e^{-2\pi i\nu t}$$

When the cylinder is present with its axis at $r = 0$, the wave cannot have the form given by the above series, for the cylinder distorts the wave. There is present, in addition to the plane wave, a scattered outgoing wave of such a size and shape as to make the radial velocity of the combination zero at $r = a$ the surface of the cylinder. We shall choose the form of this outgoing wave to be the general series

$$p_s = \sum_{m=0}^{\infty} A_m \cos(m\phi)[J_m(kr) + iN_m(kr)]e^{-2\pi i\nu t}$$

$$u_{sr} = \left(\frac{1}{\rho c}\right)\left\{iA_0[J_1(kr) + iN_1(kr)] + \frac{i}{2}\sum_{m=1}^{\infty} A_m \cos(m\phi)[J_{m+1}(kr) - J_{m-1}(kr) + iN_{m+1}(kr) - iN_{m-1}(kr)]\right\} e^{-2\pi i\nu t}$$

The combination $J + iN$ has been chosen because it ensures that all the scattered wave is outgoing.

Our first task is to find the values of the coefficients A which make the combination $u_{pr} + u_{sr}$ equal zero at $r = a$. Equating u_{sr} to $-u_{pr}$ at $r = a$ term by term, we obtain

$$\left.\begin{aligned} A_m &= -\epsilon_m P_0 i^{m+1} e^{-i\gamma_m}\sin(\gamma_m); \quad P_0 = \cancel{\sqrt{2\rho c\Upsilon_0}} = 1 \\ \tan\gamma_0 &= -\frac{J_1(ka)}{N_1(ka)}; \quad \tan\gamma_m = \frac{J_{m-1}(ka) - J_{m+1}(ka)}{N_{m+1}(ka) - N_{m-1}(ka)} \end{aligned}\right\} \tag{29.2}$$

where $\epsilon_0 = 1$ and $\epsilon_m = 2$ for all values of m larger than unity. These phase angles γ_m have already been defined in Eq. (26.6), in connection with the radiation of sound from a cylinder. Values of some of them are given in Table X at the back of the book. The behavior of these phase angles completely determines the behavior of the scattered wave. It is interesting to notice the close connection between the waves scattered by a cylinder and the waves radiated by the same cylinder when it is vibrating. The quantities needed to compute one are also needed to compute the other.

The pressure and radial velocity of the scattered wave at large distances from the cylinder are

$$p_s \simeq -\sqrt{\frac{4\rho c\Upsilon_0 a}{\pi r}}\,\psi_s(\phi)e^{ik(r-ct)}; \quad u_s \simeq \frac{P_s}{\rho c}$$

$$\psi_s(\phi) = \frac{1}{\sqrt{ka}}\sum_{m=0}^{\infty}\epsilon_m \sin(\gamma_m)e^{-i\gamma_m}\cos(m\phi)$$

The intensity of the scattered part, at the point $(r,\phi)(kr \gg 1)$, is, therefore,

$$\Upsilon_s \simeq \left(\frac{2\Upsilon_0 a}{\pi r}\right)|\psi_s(\phi)|^2 \tag{29.3}$$

$$|\psi_s|^2 = \frac{1}{ka}\sum_{m,n=0}^{\infty}\epsilon_m\epsilon_n \sin\gamma_m \sin\gamma_n \cos(\gamma_m - \gamma_n)\cos(m\phi)\cos(n\phi)$$

where $\epsilon_0 = 1$, $\epsilon_m = 2$ $(m > 0)$. This intensity is plotted as a function of ϕ on a polar plot in Fig. 78, for different values of $\mu = (2\pi\nu a/c) = (2\pi a/\lambda)$.

It is interesting to notice the change in directionality of the scattered wave as the wavelength is changed. For very long wavelengths (μ small) but little is scattered, and this is scattered almost uniformly in all the backward directions. As the frequency is increased, the distribution in angle becomes more and more complicated, diffraction peaks appearing and moving forward, until for very short wavelengths (much shorter than those shown in Fig. 78) one-half of the scattered wave is concentrated straight forward (the interfering beam), and the other half is spread more or less uniformly over all the other directions, giving a polar plot which is a cardioid, interrupted by a sharp very high peak in the forward direction, as will be shown in Eq. (29.4).

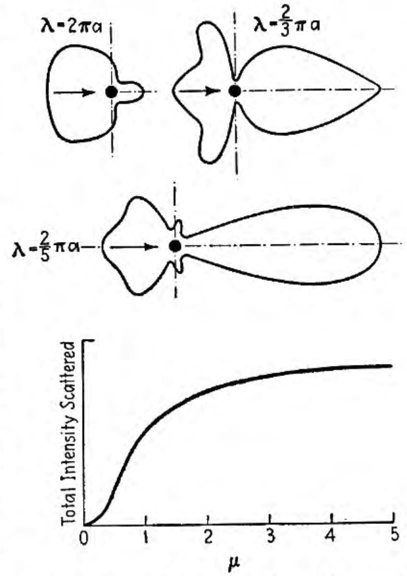


FIG. 78.—The scattering of sound waves from a rigid cylinder of radius a. Polar diagrams show the distribution in angle of the intensity of the scattered wave, and the lower graph shows the dependence of the total scattered intensity on $\mu = 2\pi a/\lambda$.

For very long wavelengths only the two cylindrical waves corre-

Inserting in Equation (9.101) yields:

$$U_m = \frac{1}{2\pi}\int_{-\Delta/2}^{\Delta/2} U e^{-jm\varphi}\,d\varphi, \tag{9.103}$$

and in the limit of $\Delta \to 0$, $U_m = \frac{U\Delta}{2\pi}$ is obtained for all m.

Using Equations (C.12) and (C.13) (in their Hankel function versions) along with Equation (9.99) results in

$$A_m = \begin{cases} j\rho c\frac{U\Delta}{2\pi}\frac{1}{H_1(ka)} & m = 0 \\ j\rho c\frac{U\Delta}{2\pi}\frac{2}{H_{m+1}(ka)-H_{m-1}(ka)} & m \neq 0. \end{cases} \tag{9.104}$$

Inserting Equation (9.104) into Equation (9.97) and making use of the fact that $A_{-m} = (-1)^m A_m$ along with the Hankel function version of Equation (C.10) results in

$$\hat{p}(r,\varphi) = j\rho c\frac{U\Delta}{2\pi}\left(\frac{H_0^{(2)}(kr)}{H_1^{(2)}(ka)} + \sum_{m=1}^{\infty}\frac{4H_m^{(2)}(kr)\cos\varphi}{H_{m+1}^{(2)}(ka) - H_{m-1}^{(2)}(ka)}\right)e^{j\omega t}. \tag{9.105}$$

Normalising Equation (9.105) by the sound field produced by a small cylindrical source of zeroth order and with the velocity $U_0 = \frac{u\Delta}{2\pi}$ (see Equation (9.65)) gives

$$\frac{\hat{p}(r,\varphi)}{\hat{p}_0(r)} = \frac{2j}{\pi ka}\left(\frac{1}{H_1^{(2)}(ka)} + \frac{1}{H_0^{(2)}(kr)}\sum_{m=1}^{\infty}\frac{4H_m^{(2)}(kr)\cos\varphi}{H_{m+1}^{(2)}(ka) - H_{m-1}^{(2)}(ka)}\right). \tag{9.106}$$

Figure 9.12 shows Equation (9.106) in the far field ($r \gg a$) for several values of ka. It is evident that at low frequencies, a line source on a cylinder is almost omni-directional – the radiated sound pressure of the source is approximately equal to the radiated sound pressure of a zeroth order source of the same strength. At higher frequencies the presence of the cylinder becomes important leading to an increased directivity of the response.

9.3.2 *Scattering by Cylinders*

It is possible to rewrite a scattering problem to an equivalent radiation problem. The approach makes use of linearity, which allows the total field outside a cylinder to be expressed as a sum of an incoming and a scattered field:

$$p_{tot} = p_{inc} + p_{sc}. \tag{9.107}$$

The incoming wave is the sound field in the absence of the cylinder – i.e., the undisturbed wave. It is worth noting that the incoming wave does not necessarily satisfy the Sommerfeld radiation condition, as is the case with a plane incoming wave. However, the scattered field, which represents the disturbance due to the cylinder, must satisfy the Sommerfeld radiation condition – the disturbance must 'wear off' at large distances. Therefore, the scattered field can be expressed as the sum given in Equation (9.97).

$$\hat{p}(r,\varphi) = \sum_{m=-\infty}^{\infty} A_m H_m^{(2)}(kr)e^{jm\varphi}e^{j\omega t}. \tag{9.108}$$

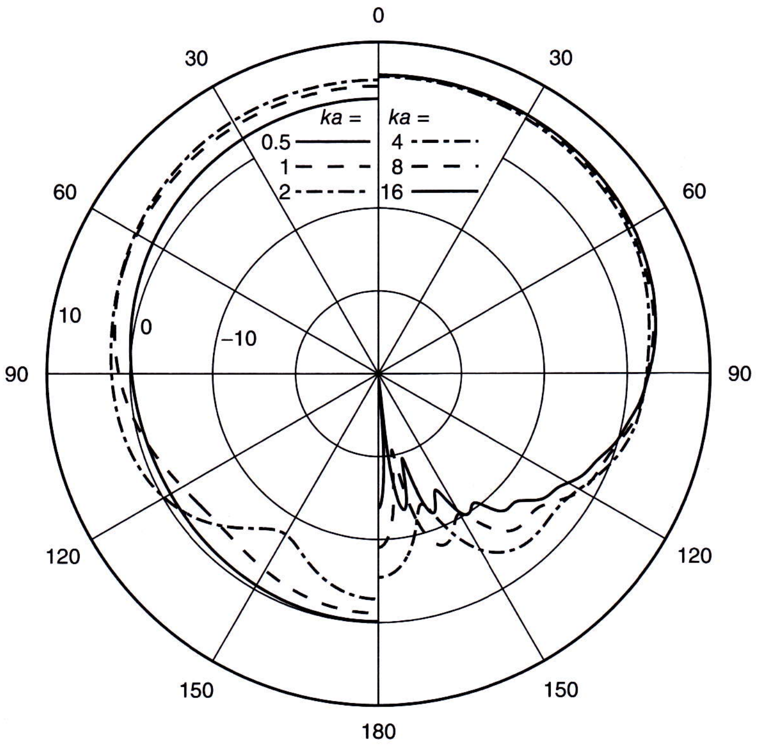


Figure 9.12 Normalised far-field pressure of a line source placed at $\varphi = 0$ on a cylinder at six frequencies

Hence, the scattered field corresponds to a radiation problem with a surface velocity in the radial direction given by Equation (9.98). If the scattered field can be constructed so that the total field satisfies the prescribed boundary condition on the surface of the scattering object, the total field must be the solution to the scattering problem due to the existence and uniqueness theorem [1].

In the following scattering of a normalised plane wave from a rigid infinite cylinder of radius a is considered. The boundary condition that must be fulfilled by the total field, is that the radial component of the velocity must vanish on the surface of the cylinder. Hence, the radial component of the scattered field must be equal in magnitude but opposite in phase to the incoming field,

$$\hat{u}_{\mathrm{sc},r}(a, \varphi) = -\hat{u}_{\mathrm{inc},r}(a, \varphi). \tag{9.109}$$

Without loss of generality the coordinate system can be aligned so that the plane wave travels along the negative x-axis in a Cartesian system. With this choice the problem is symmetric with respect to the x-axis, which has the consequence that the angular exponentials will appear in pairs of equal magnitude (see also Example 9.10) so that

Equations (9.97) and (9.98) specialise to

$$\hat{p}_{\mathrm{sc}}(r,\varphi) = \sum_{m=0}^{\infty} A_m \mathrm{H}_m^{(2)}(kr)\cos(m\varphi)\mathrm{e}^{\mathrm{j}\omega t}, \tag{9.110}$$

and

$$\hat{u}_{\mathrm{sc},r}(r,\varphi) = \sum_{m=0}^{\infty} \frac{\mathrm{j}A_m}{\rho c}\frac{\mathrm{d}\mathrm{H}_m^{(2)}(kr)}{\mathrm{d}(kr)}\cos(m\varphi)\mathrm{e}^{\mathrm{j}\omega t}. \tag{9.111}$$

In order to deal with the boundary condition on the surface of the cylinder, the incoming plane wave must be expressed in cylindrical coordinates. It is advantageous to express the incoming wave as a sum of Bessel functions (see Example 9.7), since the Neumann functions diverge for small arguments. The expansion must exist for all arguments, and it can immediately be concluded that only the expansion coefficients of the Bessel function are non-zero

$$\mathrm{e}^{\mathrm{j}kx} = \mathrm{e}^{\mathrm{j}kr\cos\varphi} = \sum_{m=0}^{\infty} B_m \mathrm{J}_m(kr)\cos(m\varphi). \tag{9.112}$$

The expansion coefficients are found by multiplying Equation (9.112) with $\cos(n\varphi)$ and integrating over φ

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{j}kr\cos\varphi}\cos(n\varphi)\mathrm{d}\varphi = \int_0^{2\pi}\sum_{m=0}^{\infty} B_m \mathrm{J}_m(kr)\cos(m\varphi)\cos(n\varphi)\mathrm{d}\varphi. \tag{9.113}$$

The left-hand side is rewritten using Equation (C.16), and as a consequence of the orthogonality of the trigonometric functions only one term ($n = m$) of the sum on the right-hand side remains. Therefore,

$$B_m = \varepsilon_m \mathrm{j}^m \tag{9.114}$$

is found, where $\varepsilon_m = 1$ for $m = 0$ and $\varepsilon_m = 2$ for $m > 0$.

The radial component of the particle velocity of the plane wave can now be expressed in terms of cylindrical components,

$$\hat{u}_{\mathrm{inc},r}(r,\varphi) = \sum_{m=0}^{\infty} \frac{\mathrm{j}B_m}{\rho c}\frac{\mathrm{d}\mathrm{J}_m(kr)}{\mathrm{d}(kr)}\cos(m\varphi)\mathrm{e}^{\mathrm{j}\omega t} = \sum_{m=0}^{\infty} \frac{\mathrm{j}^{m+1}\varepsilon_m}{\rho c}\frac{\mathrm{d}\mathrm{J}_m(kr)}{\mathrm{d}(kr)}\cos(m\varphi)\mathrm{e}^{\mathrm{j}\omega t}. \tag{9.115}$$

Due to orthogonality Equation (9.109) can be enforced term by term leading to an expression for A_m

$$\begin{aligned} A_m &= -\varepsilon_m \mathrm{j}^m \left.\frac{\mathrm{d}\mathrm{J}_m(kr)}{\mathrm{d}(kr)}\right|_{r=a} \left.\frac{\mathrm{d}\mathrm{H}_m^{(2)}(kr)}{\mathrm{d}(kr)}\right|_{r=a} \\ &= -\varepsilon_m \mathrm{j}^m \frac{\mathrm{J}_{m-1}(ka) - \mathrm{J}_{m+1}(ka)}{\mathrm{H}_{m-1}^{(2)}(ka) - \mathrm{H}_{m+1}^{(2)}(ka)}, \end{aligned} \tag{9.116}$$

where Equation (C.12) has been used in order to obtain the last result.

The total sound pressure becomes

$$\hat{p}(r, \varphi) = \sum_{m=0}^{\infty} (B_m \mathrm{J}_m(kr) + A_m \mathrm{H}_m^{(2)}(kr)) \cos(m\varphi) \mathrm{e}^{\mathrm{j}\omega t}, \quad (9.117)$$

and a non-normalised expression can be found by simply multiplying Equation (9.117) with the amplitude of the incoming wave.

Due to reciprocity Figure 9.12 can also be interpreted as the relative amplitude of the sound pressure on the surface of a cylinder when exposed to a plane wave coming from 0° degrees.

The sound field in the vicinity of the cylinder can be seen on Figure 9.13 for a few frequencies. The disturbance is modest at low frequencies, at high frequencies zones of shadow appear and a standing wave builds up in front of the cylinder.

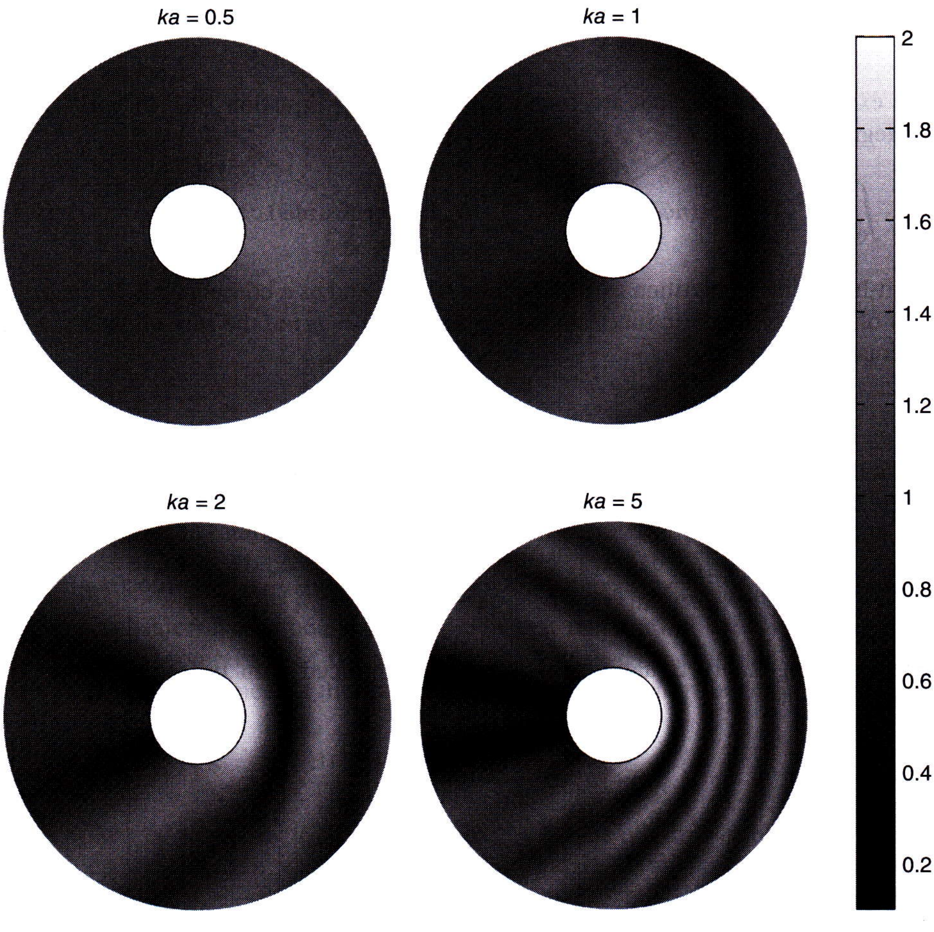


Figure 9.13 Normalised total sound pressure for scattering by a rigid cylinder, calculated for various values of *ka*. Sound incidence from the right