COGS 118A: Supervised Machine Learning Algorithms

Homework Assignment 4

1 (10 points) Conceptual Questions

(1.1)

A. $f(x) + g(x)$ is always monotonically increasing

D. $f(x^3)$ is always monotonically increasing

(1.2)

A. Matrix inverse could be expensive for computing the closed form solution in least square estimation when the dataset consists of samples in high dimension spaces.

(1.3)

$f(x) = x(10 - x) = -x^2 + 10x$
$f'(x) = 0$
$-2x + 10 = 0$
$x = 5$
$f(5) = 5(10 - 5) = 25$

A. $\text{argmax}_\text{x}\, f(x) = 5$

D. $\max_\text{x} f(x) = 25$

(1.4)

B. Given (2) and (3), we can prove that (1) holds.

Counterexample: an equation with two minima with x_0 as the global minimum. In this case there could also be another minimum that would make $f(x)$ not convex.

(1.5)

False.

2 (12 points) Convex

(2.a)

non-convex

(2.b)

convex

(2.c)

convex

(2.d)

non-convex

(2.e)

convex

(2.f)

non-convex

3 (7 points) Argmin and Argmax

$\text{argmin}_w G(w) = 15$
$\text{argmax}_w G(w) = 25$

$w^* = \text{argmin}_w[10 - 3 \times ln(G(w))] =$
$\text{argmin}_w[-ln(G(w))] =$
$\text{argmax}_w ln(G(w)) =$
$\text{argmax}_w G(w) = 25$

$w^* = 25$

4 (6 points) Error Metrics

(4.1)

C. 1/4

(4.2)

B. 3/10

(4.3)

B. 9/10

5 (30 points) Coding: Hyper-plane Estimation

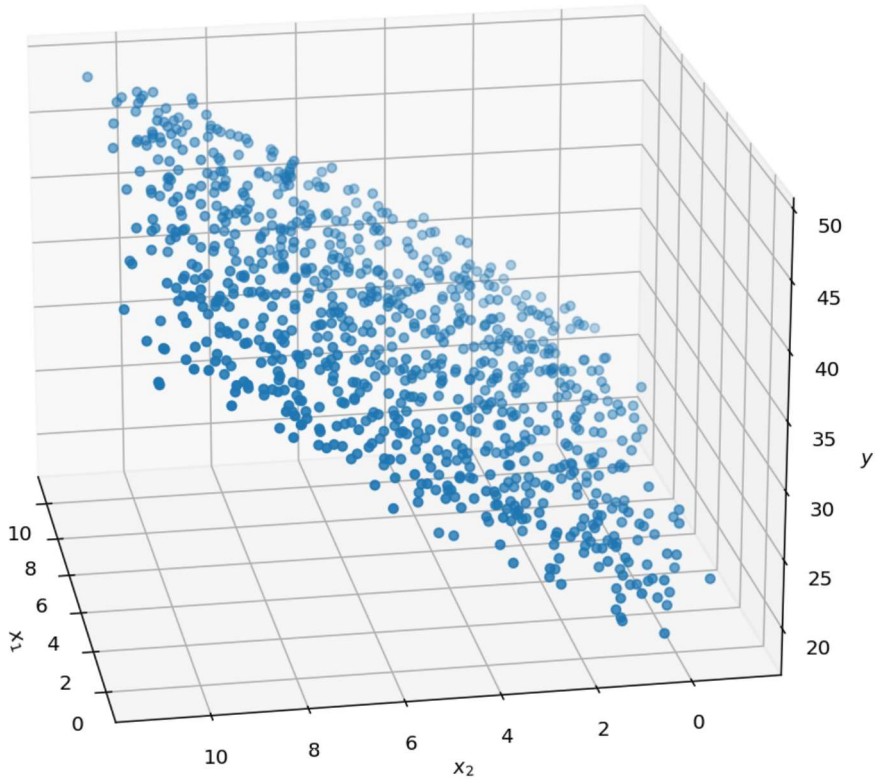
In [1]:
```
# Importing Important packages (nothing to add to this cell)

import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
%config InlineBackend.figure_format = 'retina'
```

In [2]:
```
# Loading the data (nothing to add to this cell)

X_and_Y = np.load('./hyperplane-estimation.npy')
X1 = X_and_Y[:, 0]    # Shape: (900,)
X2 = X_and_Y[:, 1]    # Shape: (900,)
Y  = X_and_Y[:, 2]    # Shape: (900,)
print(X1.shape, X2.shape, Y.shape)
```

```
(900,) (900,) (900,)
```

Original Data

In [3]:
```
# Visualization of the original datapoints (nothing to add to this cell).

def vis(w0, w1, w2):
    draw_plane = (w0 is not None) and (w1 is not None) and (w2 is not None)
    if draw_plane:
        num = 30
        X_plane_range = np.linspace(0,10,num)
        X_plane_range = np.linspace(0,10,num)
        X1_plane, X2_plane = np.meshgrid(X_plane_range, X_plane_range)
        Y_plane = w0 + w1 * X1_plane + w2 * X2_plane

    fig = plt.figure(figsize = (7, 7))
    ax = Axes3D(fig, elev = 20, azim = 170)
    ax.scatter(X1, X2, Y)
    if draw_plane:
        ax.scatter(X1_plane, X2_plane, Y_plane)
    ax.set_xlabel('$x_1$')
    ax.set_ylabel('$x_2$')
    ax.set_zlabel('$y$')
    plt.show()

vis(None, None, None)
```



Hyperplane Estimation Using the Closed Form Solution

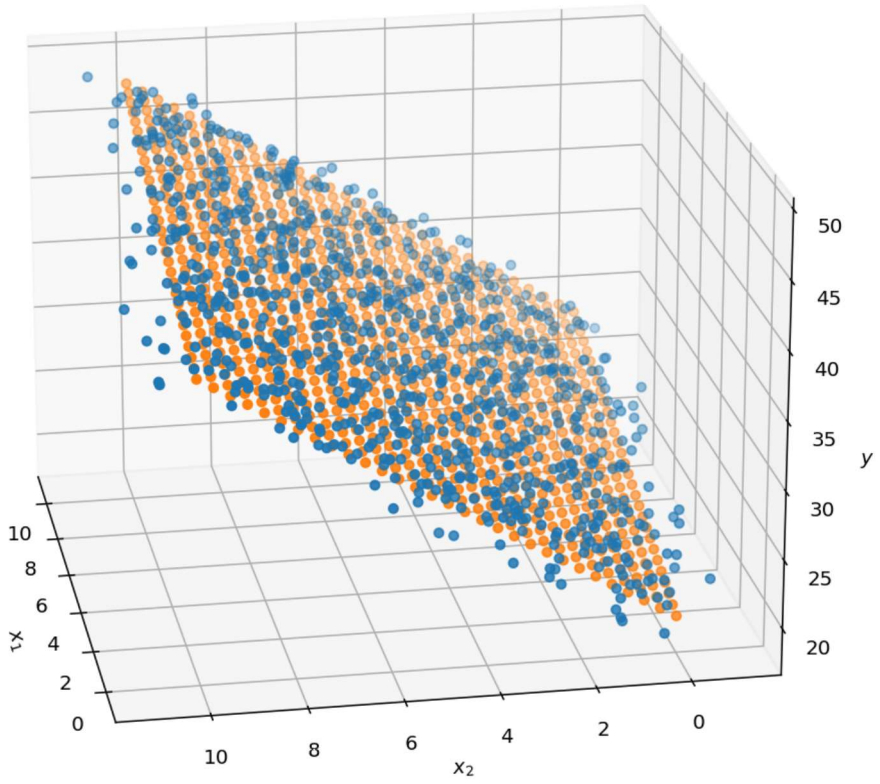
Assume data points are represented as matrices X and Y, please use the closed form solution to calculate the parameters W.

Hint: You may refer to HW3 Q4.

In [4]:
```
# Estimating W, which defines the hyperplane (you need to add code to this cell as indi
# y = w0 + w1*x1 + w2*x2

X = np.matrix(np.hstack((np.ones((len(X1),1)),
                         X1.reshape(-1,1),
                         X2.reshape(-1,1))))  # X contains 1, X1 and X2.
Y = Y
# Compute W using the closed form solution.
W = (X.T @ X).I @ X.T @ Y

w0, w1, w2 = np.array(W).reshape(-1)
print('y = {:.2f} + {:.2f}*x1 + {:.2f}*x2'.format(w0, w1, w2))
```

```
y = 19.30 + 0.98*x1 + 1.94*x2
```

In [5]:
```
# Visualization of the original datapoints and estimated plane (nothing to add to this

vis(w0, w1, w2)
```



Hyperplane Estimation Using Gradient Descent

In this problem, we would like to use the gradient descent to calculate the parameters W for the hyperplane. If we have an error function (a.k.a objective function or loss function), then a typical gradient descent algorithm contains the following steps:

Step 1. Initialize the parameters W.

for i = 1 to #iterations:

- **Step 2**. Compute the gradient $\nabla \mathcal{L}(W) = \frac{\partial \mathcal{L}(W)}{\partial W}$.
- **Step 3**. Update the parameters $W \leftarrow \mathcal{L}(W) = W - \eta \frac{\partial \mathcal{L}(W)}{\partial W}$ where η is the learning rate.

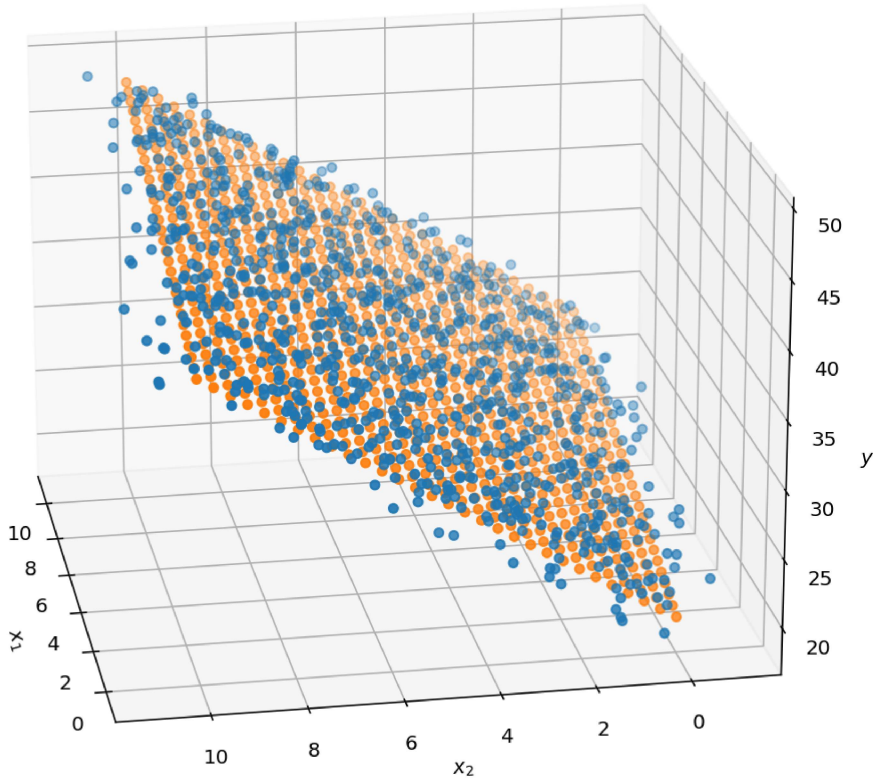
In [6]:
```
# Gradient of L(W) with respect to W (you need to add code to this cell as indicated be

def grad_L_W(X, Y, W):
    return 2 * X.T @ X @ W - 2 * X.T @ Y
```

In [7]:
```
# Estimating W, which defines the hyperplane (you need to add code to this cell as indi
# y = w0 + w1*x1 + w2*x2

# Some settings.
```

```
Y = Y.reshape(-1, 1)
print(X.shape, Y.shape, W.shape)
iterations    = 20000
learning_rate = 0.000001

# Gradient descent algorithm.
# Step 1. Initialize the parameters W.
W = np.matrix(np.zeros((3,1)))

for i in range(iterations):
    # Step 2. Calculate the gradient of L(W) w.r.t. W.
    grad = grad_L_W(X, Y, W)

    # Step 3. Update parameters W.
    W = W - learning_rate * grad

w0, w1, w2 = np.array(W).reshape(-1)
print('y = {:.2f} + {:.2f}*x1 + {:.2f}*x2'.format(w0, w1, w2))
```

```
(900, 3) (900, 1) (1, 3)
y = 19.22 + 0.99*x1 + 1.95*x2
```

In [8]:
```
# of the original datapoints and estimated plane (nothing to add to this cell).

vis(w0, w1, w2)
```



6 (30 points) Coding: Parabola Estimation

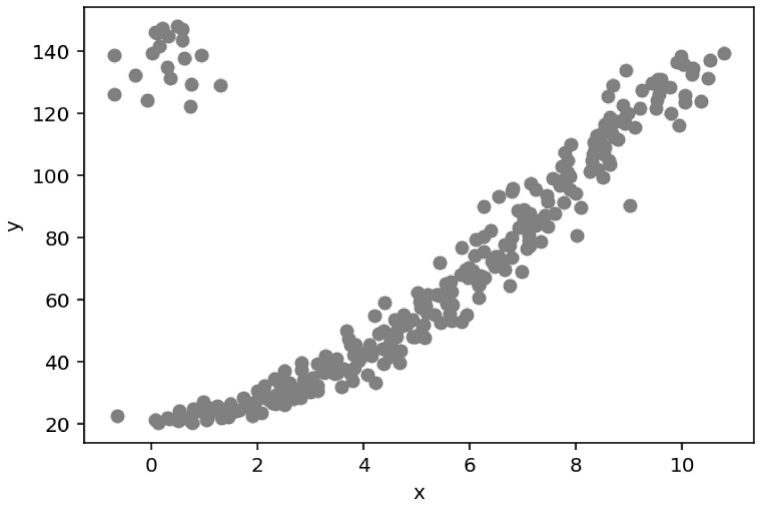
In [9]:
```
# Importing Important packages (nothing to add to this cell)

import numpy as np
import matplotlib.pyplot as plt
%config InlineBackend.figure_format = 'retina'
```

In [10]:
```
# Loading the data (nothing to add to this cell)

X_and_Y = np.load('./parabola-estimation.npy')
old_X = X_and_Y[:, 0]  # Shape: (300,)
Y = X_and_Y[:, 1]  # Shape: (300,)
old_X.shape
```

Out[10]:
```
(300,)
```

Original Data

In [11]:
```
# Visualization of the original datapoints (nothing to add to this cell).

def vis(w0, w1, w2):
    draw_plane = (w0 is not None) and (w1 is not None) and (w2 is not None)
    if draw_plane:
        X_line = np.linspace(0,10,300)
        Y_line = w0 + w1 * X_line + w2 * (X_line**2)
        plt.plot(X_line, Y_line, color='orange')

    plt.scatter(old_X, Y, color='gray')
    plt.xlabel('x')
    plt.ylabel('y')
    plt.show()

vis(None, None, None)
```



(6.1) Squared L_2 Norm (5 points)

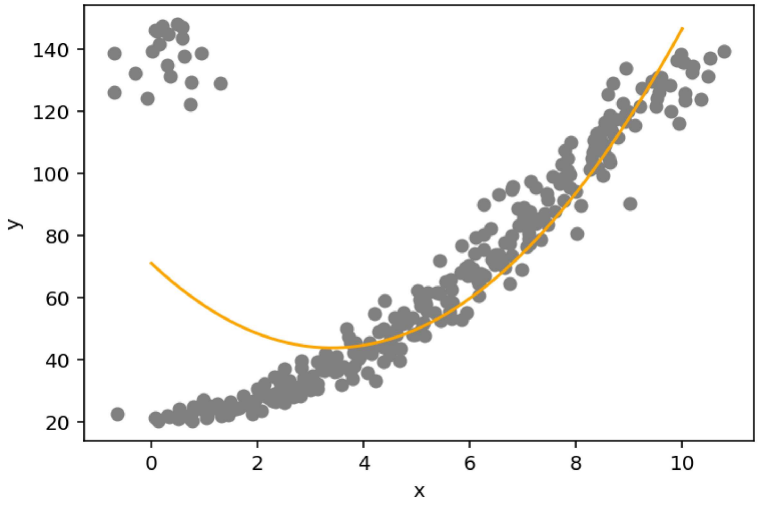
Assume data points are represented as matrices X and Y, please use the closed form solution to calculate the parameters W.

Hint: You may refer to HW3 Q4.

In [12]:
```
# Estimating W, which defines the parabola estimation (you need to add code to this cel
# y = w0 + w1*x + w2*x^2

X = np.matrix(np.hstack((np.ones((len(old_X),1)),
                         old_X.reshape(-1,1),
                         (old_X**2).reshape(-1,1))))

W = (X.T * X).I @ X.T @ Y

w0, w1, w2 = np.array(W).reshape(-1)
print('y = {:.2f} + {:.2f}*x + {:.2f}*x^2'.format(w0, w1, w2))
```

```
y = 71.07 + -16.06*x + 2.36*x^2
```

In [13]:
```
# Visualization of the original datapoints and estimated plane (nothing to add to this

vis(w0, w1, w2)
```



(6.2) L_1 Norm (10 points)

In this problem, we would like to use the gradient descent to calculate the parameters W for the parabola. If we have a loss function $\mathcal{L}(W)$, then a typical gradient descent algorithm contains the following steps:

Step 1. Initialize the parameters W.

for i = 1 to #iterations:

- **Step 2**. Compute the gradient $\nabla\mathcal{L}(W) = \frac{\partial\mathcal{L}(W)}{\partial W}$.
- **Step 3**. Update the parameters $W \leftarrow \mathcal{L}(W) = W - \eta\frac{\partial\mathcal{L}(W)}{\partial W}$ where η is the learning rate.

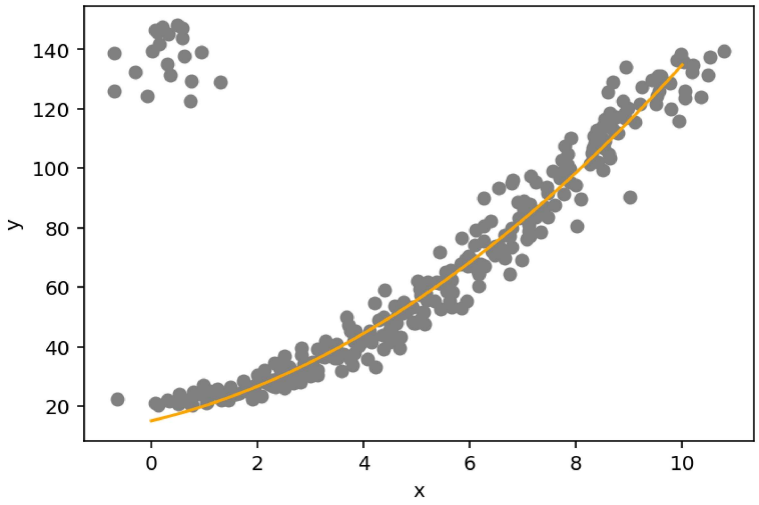
Hint: You may refer to HW3 Q5.

In [14]:
```
# Gradient of L(W) with respect to W (you need to add code to this cell as indicated be

def grad_L_W(X, Y, W):
    return (np.sign(X @ W - Y).T @ X).T
```

In [15]:
```
# Estimating W, which defines the hyperplane using gradient descent (you need to add co
```

```
# y = w0 + w1*x + w2*x^2

# Some settings.
Y = Y.reshape(-1, 1)
iterations    = 300000
learning_rate = 0.000001

# Gradient descent algorithm.
# Step 1. Initialize the parameters W.
W = np.array([[0], [0], [0]])

for i in range(iterations):
    # Step 2. Calculate the gradient of L(W) w.r.t. W.
    grad = grad_L_W(X, Y, W)

    # Step 3. Update parameters W.
    W = W - learning_rate * grad

# Print the parameters of the parabola.
w0, w1, w2 = np.array(W).reshape(-1)
print('y = {:.2f} + {:.2f}*x + {:.2f}*x^2'.format(w0, w1, w2))

# Visualization.
vis(w0, w1, w2)
```

```
y = 15.10 + 4.27*x + 0.77*x^2
```



(6.3) Squared L_2 Norm and L_1 Norm (12 points)

In this problem, we would like to use the gradient descent to calculate the parameters W for the parabola. The loss function $\mathcal{L}(W)$ now contains two parts: A squared L_2 norm and a L_1 norm. A coefficient α is used to control the ratio of these two norms:

$$\mathcal{L}(W) = \sum_{i=1}^{n} \left(\alpha \left(\mathbf{x}_i^T W - y_i \right)^2 + (1-\alpha) |\mathbf{x}_i^T W - y_i| \right) \tag{1}$$

$$= \alpha \|XW - Y\|_2^2 + (1-\alpha) \|XW - Y\|_1 \tag{2}$$

Note: It may take 2~3 mins to run the algorithm.

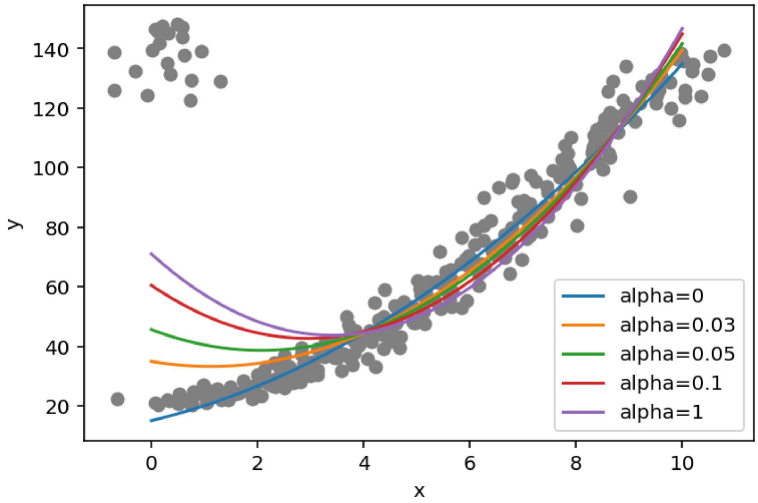
In [16]:
```
# Gradient of L(W) with respect to W (you need to add code to this cell as indicated be
```

```
def grad_L_W(X, Y, W, alpha):
    return alpha * (2 * X.T @ X @ W - 2 * X.T @ Y) + (1 - alpha) * ((np.sign(X @ W - Y)
```

In [17]:
```
# Function to use gradient descent to estimate parabola given a list of alphas
# (you need to add code to this cell as indicated below).
# Hint: For each alpha, you need to use gradient descent, hence, you need to write a lo

def parabola(alpha_list):
    # Some settings.
    global Y
    plt.scatter(old_X, Y, color='gray')
    Y = Y.reshape(-1, 1)
    iterations    = 300000
    learning_rate = 0.000001

    # Loop over alpha(s).
    for alpha in alpha_list:

        # Gradient descent algorithm.
        # Step 1. Initialize the parameters W.
        W = np.array([[0], [0], [0]])

        ####### To be filled. #######
        for i in range(iterations):
            # Step 2. Calculate the gradient of L(W) w.r.t. W.
            grad = grad_L_W(X, Y, W, alpha)

            # Step 3. Update parameters W.
            W = W - learning_rate * grad

        # Print the parameters of the parabola.
        w0, w1, w2 = np.array(W).reshape(-1)
        print('When alpha = {},'.format(alpha))
        print('y = {:.2f} + {:.2f}*x + {:.2f}*x^2'.format(w0, w1, w2))

        # Visualization.
        X_line = np.linspace(0,10,300)
        Y_line = w0 + w1 * X_line + w2 * (X_line**2)
        plt.plot(X_line, Y_line, label='alpha={}'.format(alpha))

    plt.legend()
    plt.xlabel('x')
    plt.ylabel('y')
    plt.show()
```

In [18]:
```
# Estimate the parabolas given the list of alpha(s).
parabola(alpha_list=[0,0.03,0.05,0.1,1])
```

```
When alpha = 0,
y = 15.10 + 4.27*x + 0.77*x^2
When alpha = 0.03,
y = 34.97 + -3.02*x + 1.35*x^2
When alpha = 0.05,
y = 45.68 + -6.78*x + 1.64*x^2
When alpha = 0.1,
y = 60.53 + -12.10*x + 2.05*x^2
When alpha = 1,
y = 71.07 + -16.06*x + 2.36*x^2
```



(6.4) Comparison (3 points)

At $\alpha = 0$, we are using L_1 norm, which is the least influenced by outiers. At $\alpha = 1$, we are using L_2 norm, which is the most influenced by outiers since a large distance squared becomes much larger than a small distance squared. As alpha increases, outliers influence the model less.