Clifford Quantum Cellular Automata

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Cellular Automata

Cellular automaton:

- Lattice of cells
- Each cell is in one of a finite set of states
- State of each cell changes over time by applying a global ruleset

This concept can also be applied to quantum systems: Quantum cellular automata

Example: Conway's Game of Life

- 2D lattice
- Cell either alive (black) or dead (white)
- Rules [2]:
  1. A live cell with fewer than two live neighbours dies
  2. A live cell with more than three live neighbours dies
  3. A live cell with two or three live neighbours lives
  4. A dead cell with exactly three live neighbours comes to life
- The same rules can also be seen from the perspective of the neighbors: Each cell affects its surroundings

Figure 1: Conway's Game of Life - Glider Gun [3]

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Clifford Quantum Cellular Automata

- Infinite Lattice of cells with Clifford operators (Pauli X/Y/Z or identity gate)
- Most simple case: 1D spin chains
- A Clifford Quantum Cellular Automaton (CQCA) is a globally unique ruleset
  - Maps Clifford operators to sets of Clifford operators
  - Applied to every cell at each time step

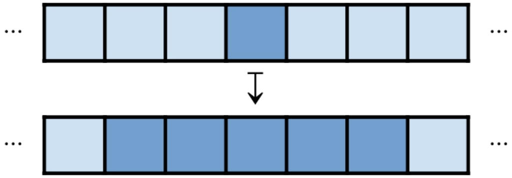

Figure 2: Application of CQCA to a single cell

Operator Application

Clifford operator application rules:

1. $\mathrm{i} \odot \mathrm{i} = \mathrm{I}, \mathrm{i} \in \{\mathrm{I}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$ (Gates are unitary)
2. $\mathrm{i} \odot \mathrm{I} = \mathrm{I} \odot \mathrm{i} = \mathrm{i}, \mathrm{i} \in \{\mathrm{I}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$ (Identity is neutral)
3. $\mathrm{i} \odot \mathrm{j} = \mathrm{k}, \mathrm{i} \neq \mathrm{j} \neq \mathrm{k} \in \{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$

Rulesets

- Goal: Map Clifford operators to sets of Clifford operators
- Each rule is relative to the origin cell (underlined operator)
- Example:
  - $\mathrm{X} \rightarrow \underline{\mathrm{Z}}$
  - $\mathrm{Z} \rightarrow \mathrm{Z}\underline{\mathrm{X}}\mathrm{Z}$

Rulesets

- Rules:
  - Mapping from an X gate to X and Z gates
  - Mapping from a Z gate to X and Z gates
  - Mapping from a Y gate is implicit: $\mathrm{Y} = \mathrm{X} \odot \mathrm{Z}$
- Possible to define the rules by the indices (relative to the origin) for X and Z gates: $\mathrm{M} = \begin{pmatrix} \mathrm{m}_{\mathrm{X}\to\mathrm{X}} & \mathrm{m}_{\mathrm{Z}\to\mathrm{X}} \\ \mathrm{m}_{\mathrm{X}\to\mathrm{Z}} & \mathrm{m}_{\mathrm{Z}\to\mathrm{Z}} \end{pmatrix}$
- Previous example ("Glider CQCA"):
  - $\mathrm{X} \to \underline{\mathrm{Z}}$
  - $\mathrm{Z} \to \mathrm{Z}\underline{\mathrm{X}}\mathrm{Z}$
  - Implicit rule: $\mathrm{Y} \to \mathrm{Z}\underline{\mathrm{Y}}\mathrm{Z}$

$$\mathrm{M_G} = \begin{pmatrix} \emptyset & \{0\} \\ \{0\} & \{-1, 1\} \end{pmatrix}$$

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Classes of CQCAs

CQCAs can be sorted into 3 classes:

1. Glider CQCAs
2. Fractal CQCAs
3. Periodic CQCAs

Glider CQCAs

Glider rules:

- $\mathrm{X} \rightarrow \underline{\mathrm{Z}}$
- $\mathrm{Z} \rightarrow \mathrm{Z}\underline{\mathrm{X}}\mathrm{Z}$
- Implicit rule: $\mathrm{Y} \rightarrow \mathrm{Z}\underline{\mathrm{Y}}\mathrm{Z}$

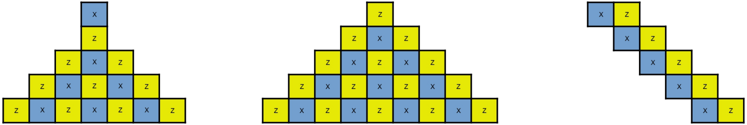

Figure 3: Evolution of $\mathrm{M_G}$

Fractal CQCAs

The ruleset $M_F = \begin{pmatrix} \{-1,0,1\} & \{0\} \\ \{0\} & \emptyset \end{pmatrix}$ produces fractal behavior

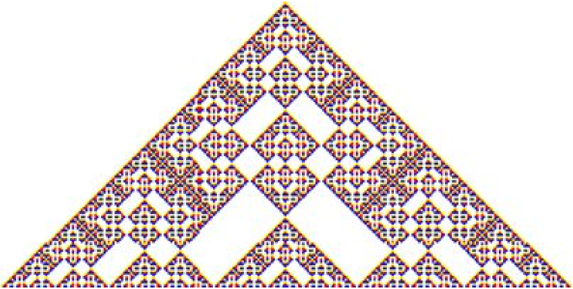

Figure 4: Evolution of M_F [1]

Periodic CQCAs

The ruleset $\mathrm{M_P} = \begin{pmatrix} \{0\} & \emptyset \\ \{-1, 1\} & \{0\} \end{pmatrix}$ produces periodic behavior

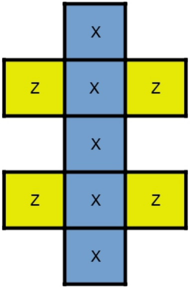

Figure 5: Evolution of $\mathrm{M_P}$

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Quantum Entanglement

- CQCAs are designed with quantum systems in mind
- Quantum systems allow quantum entanglement
- How to measure the entanglement in CQCAs?

Translation Invariant Stabilizer States

- A translation invariant stabilizer state ω (for 1-dimensional lattices) is a chain of quantum states (qubits)
- Returns to the same state after application of a set of Clifford operator chains
- For CQCAs: Generators $\mathbb{S}$
- Example: $\mathbb{S} = \{(\cdots \mathrm{I}_{i-1}\mathrm{Z}_i\mathrm{I}_{i+1}\cdots), \forall i \in \mathbb{Z}\}$
- Important aspects:
  - For every such set of generators there exists a stabilizer state
  - The application of a CQCA results in a different set of operator chains with their own stabilizer state

Translation Invariant Stabilizer States

- The stabilizer state changes from ω to ω' by CQCA application
- Example: CQCA with rule Z $\rightarrow$ $\underline{\text{X}}$Z

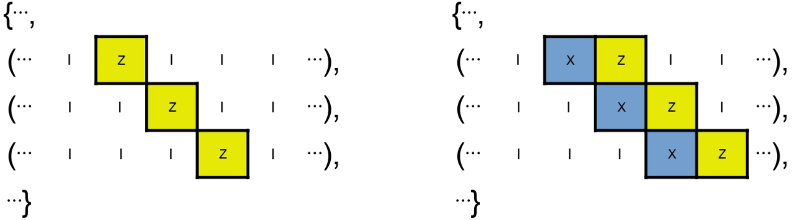

Figure 6: Generators $\mathbb{S}$ before and after CQCA application

Bipartite Cuts

- The entanglement E(t) is the highest number of entangled qubit pairs with respect to any bipartite cut in $\mathbb{S}$ at time step t
- Chain length with the most entanglements: Take the cell c_{max} with the highest entanglement number n_{max} and add 1
- $E(t) = \lfloor \frac{1}{2}(n_{max} + 1) \rfloor$

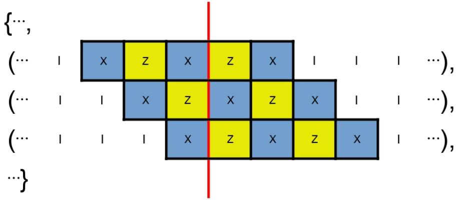

Figure 7: Bipartite cut in $\mathbb{S}$

Entanglement Measurement

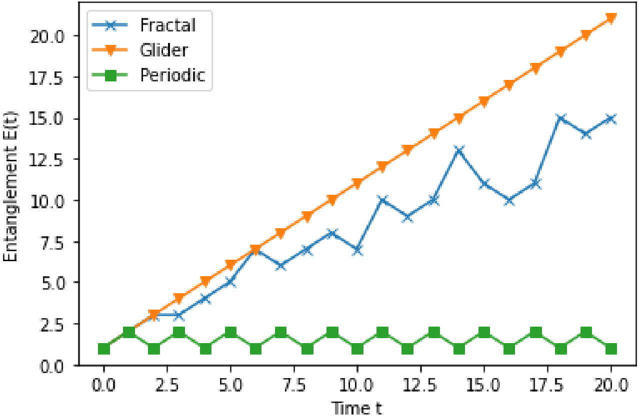

Figure 8: Entanglements E(t) of different CQCAs for initial configuration $(\cdots \mathrm{IYXYI} \cdots)$

Entanglement Measurement

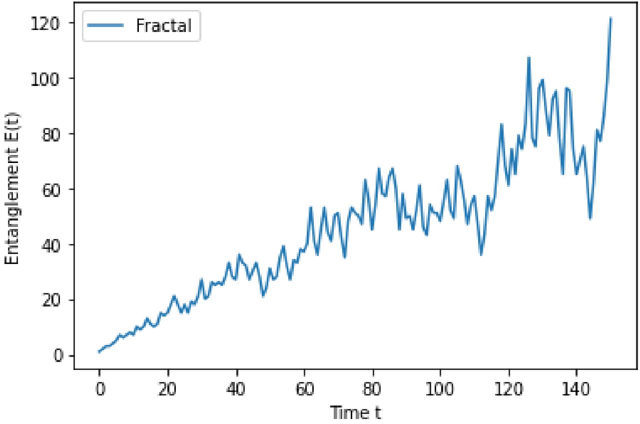

Figure 9: Continuation of E(t) of M_F

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