OR Prelim Solutions

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1. (a) $\mathcal{L}(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) + \lambda^\top \boldsymbol{g}(\boldsymbol{x})$

   (i) $\nabla_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \lambda) = 0 : \begin{cases} 2(1+\lambda_1+\lambda_2)x_1 - 16 + 5\lambda_3 &= 0 \\ 2x_2 - 20 - \lambda_1 + \lambda_2 + \lambda_3 &= 0 \end{cases}$

   (ii) $\lambda \boldsymbol{g}(\boldsymbol{x}) = 0 : \begin{cases} \lambda_1(x_1^2 - x_2) &= 0 \\ \lambda_2(x_1^2 + x_2 - 18) &= 0 \\ \lambda_3(5x_1 + x_2 - 24) &= 0 \end{cases}$

   (iii) $\boldsymbol{g}(\boldsymbol{x}) \leq 0 : \begin{cases} x_1^2 - x_2 &\leq 0 \\ x_1^2 + x_2 - 18 &\leq 0 \\ 5x_1 + x_2 - 24 &\leq 0 \end{cases}$

   (iv) $\lambda \geq 0$

   (b) (ii) and (iii) in KKT conditions are always true for $\bar{\boldsymbol{x}}$. In other words, they don't provide us useful information about λ.

   (i) $\nabla_{\boldsymbol{x}} \mathcal{L}(\bar{\boldsymbol{x}}, \lambda) = 0 : \begin{cases} 6\lambda_1 + 6\lambda_2 + 5\lambda_3 &= 10 \\ -\lambda_1 + \lambda_2 + \lambda_3 &= 2 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -\dfrac{1}{12}(2-\lambda_3) \\ \lambda_2 = \dfrac{11}{12}(2-\lambda_3) \end{cases}$

   (iv) $\lambda \geq 0$

   Observe that if $\lambda_3 \neq 2$, then $\lambda_1\lambda_2 < 0$. Since $\lambda_1, \lambda_2 \geq 0$, we must have

   $$\lambda_3 = 2, \lambda_1 = \lambda_2 = 0$$

   (c) Obviously, all $g_i(\boldsymbol{x})$ are convex and so is $f(\boldsymbol{x})$. Let $\boldsymbol{y} = (0, 1)$, then $g_1(\boldsymbol{y}), g_2(\boldsymbol{y}), g_3(\boldsymbol{y}) < 0$. By Slater's condition, KKT FONC holds at $\bar{\boldsymbol{x}}$.

   (d) Since the problem is minimizing a convex function $f(\boldsymbol{x})$, KKT point $\bar{\boldsymbol{x}}$ is an optimizer.

2. Suppose that P is a nonempty polyhedron with expression $P = \{\boldsymbol{x} \in \mathbb{R}^n | A\boldsymbol{x} \geq \boldsymbol{b}\}$, and that $\boldsymbol{w} \in P$ is an extreme point of P. Prove that if there exists $\boldsymbol{d}$ such that for some $\boldsymbol{x} \in P$, $\boldsymbol{x} + \lambda\boldsymbol{d} \in P$ for all $\lambda \in \mathbb{R}$, then $\boldsymbol{d}$ must be $\boldsymbol{0}$.

Proof.

$$\forall\lambda,\ \boldsymbol{x} + \lambda\boldsymbol{d} \in P \ \Leftrightarrow\ \forall\lambda,\ A(\boldsymbol{x} + \lambda\boldsymbol{d}) \geq \boldsymbol{b}$$

Then we must have $A\boldsymbol{d} = \boldsymbol{0}$. Otherwise, $\exists i$ such that $(A\boldsymbol{d})_i \neq 0$. If $(A\boldsymbol{d})_i > 0$, then $\lim_{\lambda\to-\infty} \lambda(A\boldsymbol{d})_i = -\infty$ which violates

$$\lim_{\lambda\to-\infty} (A\boldsymbol{x} + \lambda A\boldsymbol{d})_i \geq b_i > -\infty$$

Similarly, if $(A\boldsymbol{d})_i < 0$¡ then $\lim_{\lambda\to\infty} \lambda(A\boldsymbol{d})_i = -\infty$ conflicting

$$\lim_{\lambda\to\infty} (A\boldsymbol{x} + \lambda A\boldsymbol{d})_i \geq b_i > -\infty$$

Note that the existence of extreme point $\boldsymbol{w}$ implies that $A\boldsymbol{d} = \boldsymbol{0}$ if and only if $\boldsymbol{d} = \boldsymbol{0}$. Otherwise, suppose $\boldsymbol{y} \neq \boldsymbol{0}$ and $A\boldsymbol{y} = \boldsymbol{0}$. Then, obviously, $\boldsymbol{w} + \boldsymbol{y}, \boldsymbol{w} - \boldsymbol{y} \in P$. Moreover,

$$\boldsymbol{w} = \frac{1}{2}(\boldsymbol{w} + \boldsymbol{y}) + \frac{1}{2}(\boldsymbol{w} - \boldsymbol{y})$$

which is contradictory to the fact $\boldsymbol{w}$ is an extreme point.

$\square$

3. For any $A \in \mathbb{R}^{m\times n}$, consider problems

$$p^* := \min_{\boldsymbol{x}\in\mathbb{R}^n} \|A\boldsymbol{x} - \boldsymbol{b}\|_1$$

and

$$\begin{aligned} d^* := \max_{\boldsymbol{v}\in\mathbb{R}^m} &\ \boldsymbol{b}^T\boldsymbol{v} \\ \text{subject to } &\ \|\boldsymbol{v}\|_\infty \leq 1,\ A^T\boldsymbol{v} = \boldsymbol{0} \end{aligned}$$

Show that $p^* \geq d^*$.

Proof. Observe that, for two vectors $\boldsymbol{a}, \boldsymbol{c} \in \mathbb{R}^m$,

$$\|\boldsymbol{a}^T\boldsymbol{c}\|_1 = |\boldsymbol{a}^T\boldsymbol{c}| = \left|\sum_{i=1}^m a_i c_i\right| \leq \sum_{i=1}^m |a_i c_i| \leq \|\boldsymbol{a}\|_\infty \sum_{i=1}^m |c_i| = \|\boldsymbol{a}\|_\infty \|\boldsymbol{c}\|_1$$

Fix $\boldsymbol{x} \in \mathbb{R}^n$,

$$\forall \boldsymbol{v} \in \mathbb{R}^m,\ |\boldsymbol{v}^T\boldsymbol{b}| = \|\boldsymbol{v}^T A\boldsymbol{x} - \boldsymbol{v}^T\boldsymbol{b}\|_1 \leq \|\boldsymbol{v}\|_\infty \|A\boldsymbol{x} - \boldsymbol{b}\|_1 \leq \|A\boldsymbol{x} - \boldsymbol{b}\|_1$$

Therefore, $\|A\boldsymbol{x} - \boldsymbol{b}\|_1 \geq d^*$. Consequently,

$$p^* = \min_{\boldsymbol{x}\in\mathbb{R}^n} \|A\boldsymbol{x} - \boldsymbol{b}\|_1 \geq \min_{\boldsymbol{x}\in\mathbb{R}^n} d^* = d^*$$

$\square$

3. *Alternative proof:*

   Rewrite $p^* := \min_{\boldsymbol{x}\in\mathbb{R}^n} \|A\boldsymbol{x} - \boldsymbol{b}\|_1$ as a linear optimization problem.

$$
\begin{aligned}
p^* = \quad \min \quad & \sum_{i=1}^{n} \left(y_i^+ + y_i^-\right) \\
s.t. \quad & y_i^+ \geq (A\boldsymbol{x})_i - b_i, \quad \forall i = 1, \ldots, n \\
& y_i^- \geq b_i - (A\boldsymbol{x})_i, \quad \forall i = 1, \ldots, n \\
& y_i^+, y_i^- \geq 0, \qquad\quad \forall i = 1, \ldots, n
\end{aligned}
$$

   Its dual problem is

$$
\begin{aligned}
q^* = \quad \max \quad & \boldsymbol{b}^\top(\boldsymbol{z}^+ - \boldsymbol{z}^-) \\
s.t. \quad & \boldsymbol{z}^+ \leq \mathbf{1} \\
& \boldsymbol{z}^- \leq \mathbf{1} \\
& A^\top(\boldsymbol{z}^+ - \boldsymbol{z}^-) = \mathbf{0} \\
& z_i^+, z_i^- \geq 0, \quad \forall i = 1, \ldots, n
\end{aligned}
$$

   Let $\boldsymbol{w} = \boldsymbol{z}^+ - \boldsymbol{z}^-$. Then we can rewrite dual problem as

$$
\begin{aligned}
q^* = \quad \max \quad & \boldsymbol{b}^\top \boldsymbol{w} \\
s.t. \quad & \|\boldsymbol{w}\|_\infty \leq 1 \\
& A^\top \boldsymbol{w} = \mathbf{0}
\end{aligned}
$$

   Hence $q^* = d^*$. By weak duality,

$$
q^* \geq q^* = d^*
$$

4. Consider the feasible and bounded linear programming problem

$$z = \min\{\boldsymbol{c}^\top \boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \in X\}$$

where A is an $m \times n$ matrix and X is a polyhedral set.

(a) Prove or disprove the following statement:

Given any nonnegative $\pi \in \mathbb{R}^m$,

$$v(\pi) = \boldsymbol{b}^\top \pi + \min\{(\boldsymbol{c}^\top - \pi^\top A)\boldsymbol{x} : \boldsymbol{x} \in X\}$$

has $v(\pi) \leq z$.

(b) Prove or disprove the following statement. There exists a nonnegative π so that $v(\pi) = z$.

(c) Does your answer to either part 4(a) or part 4(b) change if the set X is not polyhedral? Explain.

Proof. (a) Since π is non-negative, $-\pi^\top A\boldsymbol{x} \leq -\pi^\top \boldsymbol{b}$ if $A\boldsymbol{x} \geq \boldsymbol{b}$.

$$\begin{aligned}
v(\pi) &\leq \boldsymbol{b}^\top \pi + \min\{(\boldsymbol{c}^\top - \pi^\top A)\boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \in X\} \\
&\leq \boldsymbol{b}^\top \pi + \min\{\boldsymbol{c}^T \boldsymbol{x} - \pi^\top \boldsymbol{b} : A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \in X\} \\
&= \min\{\boldsymbol{c}^\top \boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \in X\} = z
\end{aligned}$$

(b) Since X is polyhedral, there exists a $k \times n$ matrix D and a vector $\boldsymbol{h} \in \mathbb{R}^k$ such that $X = \{\boldsymbol{x} \in \mathbb{R}^n : D\boldsymbol{x} \geq \boldsymbol{h}\}$. Then, we construct the dual problem of z.

$$\begin{aligned}
z = \quad \min \quad & \boldsymbol{c}^\top \boldsymbol{x} & (P) \qquad\qquad & \max \quad \boldsymbol{b}^\top \boldsymbol{y} + \boldsymbol{h}^\top \boldsymbol{w} & (D) \\
s.t. \quad & A\boldsymbol{x} \geq \boldsymbol{b} & & s.t. \quad A^\top \boldsymbol{y} + D^\top \boldsymbol{w} = \boldsymbol{c}^\top & \\
& D\boldsymbol{x} \geq \boldsymbol{h} & & \boldsymbol{y}, \boldsymbol{w} \geq 0 &
\end{aligned}$$

Since the primal problem is feasible and bounded, (D) has an optimizer $(\boldsymbol{y}^*, \boldsymbol{w}^*)$ and $\boldsymbol{b}^\top \boldsymbol{y}^* + \boldsymbol{h}^\top \boldsymbol{w}^* = z$. On the other side, we can find the dual problem of $\min\{(\boldsymbol{c}^\top - \pi^\top A)\boldsymbol{x} : \boldsymbol{x} \in X\}$. By weak duality,

$$\begin{aligned}
v(\boldsymbol{y}^*) \geq \boldsymbol{b}^\top \boldsymbol{y}^* + \quad \max \quad & \boldsymbol{h}^\top \boldsymbol{w} & (D') \\
s.t. \quad & A^\top \boldsymbol{y}^* + D^\top \boldsymbol{w} = \boldsymbol{c}^\top & \\
& \boldsymbol{w} \geq 0 &
\end{aligned}$$

Note that (D') is feasible and hence weak duality can be applied here. Then,

$$\max_{\pi \geq \boldsymbol{0}} v(\pi) \geq v(\boldsymbol{y}^*) \geq \boldsymbol{b}^\top \boldsymbol{y}^* + \boldsymbol{h}^\top \boldsymbol{w}^* = z \geq \max_{\pi \geq \boldsymbol{0}} v(\pi)$$

Hence $v(\boldsymbol{y}^*) = z$.

(c) Only the answer of part 4(b) will change. Because the result of part 4(b) is strongly relied on the strong and weak duality which are not true in general. Note that $v(\pi)$ is Lagrangian of original problem defining z. The following example shows that part 4(b) may fail when X is not polyhedral.

$$\begin{aligned} z = \quad \min \quad & -2x_1 + x_2 \\ s.t. \quad & x_1 + x_2 = 3 \\ & x \in X \end{aligned}$$

where $X = \{(0,0),(0,4),(4,4),(4,0),(1,2),(2,1)\}$. One can verify that $z = -3$ and $\max_{\pi \geq \mathbf{0}} v(\pi) = -6$.

$\square$

5. Consider the problem of finding a shortest path in a directed network $G = (N, A)$ with nodes $N = \{1, 2, ..., n\}$ and arc distances $d_{ij} \in \mathbb{R}$, from a specified origin node s to a specified destination node t. We create an assignment problem from the network data as follows:

   > Create nodes i and i' for each $i \in N$. For each arc $(i, j) \in A$, create arc (i, j') with length d_{ij} in the assignment network. Also create arcs (i, i') with weight zero.

   (a) Prove that the original network contains a negative-length cycle if and only if the minimum-cost assignment has negative total weight.

   (b) If the original network contains no negative cycle, then remove nodes s' and t and all incidence arcs from the assignment network. Prove that the minimum-weight assignment in this modified problem corresponds to a shortest $s - t$ path in the original network.

Proof.

Two theorems related to graph theory are heavily used in this proof.

Theorem 1.1. *A connected graph is a cycle if and only if every node has exactly two connected edges.*

Theorem 1.2. *A connected graph is a path if and only if exactly two nodes has only one connected edge and all other nodes has exactly two connected edges.*

Observation: Each cycle corresponds to a feasible solution to the assignment problem. For example,

$$\text{Cycle } 2 \to 4 \to 2 \;\Leftrightarrow\; \{(2, 4'), (4, 2'), (1, 1'), (3, 3'), (5, 5')\}$$

(a) We first formulate this problem. For each arc (i, j'), define a variable

$$x_{ij} = \begin{cases} 1, & \text{arc } (i, j') \text{ is chosen} \\ 0, & o.w. \end{cases}$$

It's well-known that, for the assignment problem, we can relax the integrity constraint on x_{ij}.

$$\begin{aligned} \min \quad & \sum_{(i,j)\in A} d_{ij} x_{ij} && (P) \\ s.t. \quad & \sum_{j=1}^{n} x_{ij} = 1, \quad i \in N \\ & \sum_{j=1}^{n} x_{ji} = 1, \quad i \in N \\ & 0 \le x_{ij} \le 1, \quad \forall (i,j) \in A \\ & 0 \le x_{ii} \le 1, \quad \forall i \in N \end{aligned}$$

If there exists a negative-length cycle, say $i_1 \to i_2 \to \cdots \to i_n \to i_1$, in the graph, then we let $x_{i_1 i_2} = x_{i_2 i_3} = \cdots = x_{i_{n-1} i_n} = x_{i_n i_1} = 1$. For all other nodes $j \in N \setminus \{i_1, i_2, \ldots, i_n\}$, we let $x_{jj} = 1$. Obviously, this is a feasible solution for (P). Moreover,

$$\sum_{(i,j)\in A} d_{ij} x_{ij} = d_{i_1 i_2} + d_{i_2 i_3} + \ldots + d_{i_{n-1} i_n} + d_{i_n i_1} = \text{ length of cycle } < 0$$

So the optimal objective value of (P) must be negative.
Conversely, assume (P) has a negative optimal value and the optimal solution is $\boldsymbol{x}^*$. Let $B = \{(i,j) \in A : x^*_{ij} = 1\}$. Note that $\sum_{(i,j)\in B} d_{ij} = \sum_{(i,j)\in A} d_{ij} x^*_{ij} < 0$. Let N_B be the set of endpoints of arcs in B, i.e. $N_B = \{i \in N : \exists j \in N \ s.t. \ (i,j) \text{ or } (j,i) \in B\}$. We need to show there exists a cycle with negative length in $G(N_B, B)$. For each node $i \in N_B$, there exist exactly two edges $(i,k), (j,i) \in B$ where k can be the same as j. Therefore, $G(N_B, B)$ consists of cycles $C_1, \ldots, C_m$.

$$\ell(G(N_B, B)) = \ell(C_1) + \ldots + \ell(C_m) < 0$$

where $\ell(C)$ is the length of C. The inequality above asserts that there exists $i \in \{1, 2, \ldots, m\}$ such that $\ell(C_i) < 0$.

(b) Obviously, the minimum-weight assignment is no more than the length of the shortest $s-t$ path in the original network since we can convert every path to a feasible solution to the modified problem such that the length of path is equal to the objective value of corresponding solution.

To prove the minimum-weight is no less than the length of the shortest $s-t$ path, let $\boldsymbol{x}^*$ be the optimizer for the modified problem and let $B = \{(i,j) \in A : x^*_{ij} = 1\}$ and $N_B = \{i \in N : \exists j \in N \ s.t. \ (i,j) \text{ or } (j,i) \in B\}$. In the graph $G(N_B, B)$, each node $i \in N \setminus \{s,t\}$ has two connected edges while s,t only have 1 connected edge. Therefore, $G(N_B, B)$ consists of cycles $C_1, \ldots, C_m$ and a path P. In fact, P is a $s-t$ path.

$$\ell(G(N_B, B)) = \ell(C_1) + \cdots + \ell(C_m) + \ell(P)$$

Since $\boldsymbol{x}^*$ is the minimizer and no cycle has negative length, we claim that $\ell(C_i) = 0$. Otherwise, if $\ell(C_i) > 0$ for all $i = 1, \ldots, m$, then $\boldsymbol{x}^*$ will not be optimal since we can delete C_i from $G(N_B, B)$ to derive a new solution with smaller length. Consequently,

$$\text{The shortest } s-t \text{ path has length } \leq \ell(P) = \sum_{(i,j) \in A} d_{ij} x^*_{ij}.$$

$\square$

6. (a) There are totally 8 vertices of X, say

$$\alpha_1 = \begin{bmatrix}0\\0\\0\end{bmatrix}, \alpha_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \alpha_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \alpha_4 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \alpha_5 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \alpha_6 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \alpha_7 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \alpha_8 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

To construct DW-master problem, we replace $\boldsymbol{x}$ by the convex combination of $\{\alpha_i\}_{i=1}^8$.

$$\boldsymbol{x} = \sum_{i=1}^{8} \lambda_i \alpha_i \;\Rightarrow\; \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}\lambda_2+\lambda_5+\lambda_6+\lambda_8\\ \lambda_3+\lambda_5+\lambda_7+\lambda_8\\ \lambda_4+\lambda_6+\lambda_7+\lambda_8\end{bmatrix}$$

DW-master problem:

$$\begin{aligned} \max \quad & 2\lambda_2 - \lambda_3 + 5\lambda_4 + \lambda_5 + 7\lambda_6 + 4\lambda_7 + 6\lambda_8 \\ s.t. \quad & 6\lambda_2 - \lambda_3 + 3\lambda_4 + 5\lambda_5 + 9\lambda_6 + 2\lambda_7 + 8\lambda_8 \le 4 \\ & \sum_{i=1}^{8} \lambda_i = 1 \\ & \lambda_i \ge 0 \qquad\qquad \forall i = 1, 2, \ldots, 8 \end{aligned}$$

(b) Restricted master problem:

$$\begin{aligned} \max \quad & 0 \\ s.t. \quad & s = 4 \\ & \lambda_1 = 1 \end{aligned}$$

The solution is $\begin{bmatrix}\lambda_1^*\\ s^*\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$.

(c) The associated dual solution of $\begin{bmatrix}\lambda_1^*\\ s^*\end{bmatrix}$ is $\begin{bmatrix}\pi\\ \pi_0\end{bmatrix}^\top = \begin{bmatrix}0\\0\end{bmatrix}^\top \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}0\\0\end{bmatrix}^\top$. Based on the original formulation given in the problem, we introduce two vectors.

$$\boldsymbol{c} = \begin{bmatrix}2\\-1\\5\end{bmatrix}, \; A = \begin{bmatrix}6 & -1 & 3\end{bmatrix}$$

The corresponding subproblem:

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} - \pi^\top A\boldsymbol{x} - \pi_0 \\ s.t. \quad & \boldsymbol{x} \in X \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \max \quad & 2x_1 - x_2 + 5x_3 \\ s.t. \quad & 0 \le x_i \le 1 \qquad \forall i = 1, 2, 3 \end{aligned}$$

Obviously, the optimal solution is $\alpha_6 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. And the updated restricted problem is

$$\begin{aligned} \max \quad & 7\lambda_6 \\ s.t. \quad & 9\lambda_6 + s = 4 \\ & \lambda_1 + \lambda_6 = 1 \\ & \lambda_1, \lambda_6, s \geq 0 \end{aligned}$$

The optimal solution is

$$\begin{bmatrix} \lambda_1^* \\ \lambda_6^* \\ s^* \end{bmatrix} = \begin{bmatrix} 5/9 \\ 4/9 \\ 0 \end{bmatrix}$$

(d) The dual solution of $\begin{bmatrix} \lambda_1^* \\ \lambda_6^* \\ s^* \end{bmatrix}$ is $\begin{bmatrix} \pi \\ \pi_0 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 7 \end{bmatrix}^\top \begin{bmatrix} 0 & 9 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 7/9 \\ 0 \end{bmatrix}^\top$

The corresponding subproblem:

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} - \pi^\top A\boldsymbol{x} - \pi_0 \\ s.t. \quad & \boldsymbol{x} \in X \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \max \quad & \frac{2}{9}(-12x_1 - x_2 + 12x_3) \\ s.t. \quad & 0 \leq x_i \leq 1 \qquad \forall i = 1, 2, 3 \end{aligned}$$

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1. (a) Suppose $B(\xi, \xi_0) \subseteq P$. Obviously, $\xi \in P$. Assume $\xi_0 > \mathrm{dist}_{(a_j,b_j)}(\xi)$ for some $j \in \{1, \dots, m\}$. In other words, we can find a number $\epsilon > 0$ such that $\xi_0 \geq (1+\epsilon)\,\mathrm{dist}_{(a_j,b_j)}(\xi)$. There exists an unique ξ^* such that $\|\xi - \xi^*\|_2 = \mathrm{dist}_{(a_j,b_j)}(\xi)$. Furthermore, since $a_j \neq \mathbf{0}$, there exists $z \in \mathbb{R}^n$ such that $a_j^\top z = 1$. Let $\xi' = \frac{\mathrm{dist}_{(a_j,b_j)}(\xi)}{\|z\|_2} z$. Consider

$$\widetilde{\xi} = \xi^* + \epsilon \xi'$$

Observe that

$$\|\xi - \widetilde{\xi}\|_2 \leq \|\xi - \xi^*\|_2 + \epsilon\|\xi'\| = (1+\epsilon)\,\mathrm{dist}_{(a_j,b_j)}(\xi) \leq \xi_0$$

Therefore, $\widetilde{\xi} \in B(\xi, \xi_0) \subseteq P$. However,

$$a_j^\top \widetilde{\xi} = a_j^\top(\xi^* + \epsilon\xi') = b_j + \epsilon a_j^\top \xi' > b_j$$

This contradiction leads to that $\xi_0 \leq \mathrm{dist}_{(a_i,b_i)}(\xi)$ for all i.

Conversely, assume $\xi \in P$ and $\xi_0 \leq \mathrm{dist}_{(a_i,b_i)}(\xi)$ for all i. Suppose there is a point $z \in B(\xi, \xi_0)$ such that $z \notin P$. Then $a_j^\top z > b_j$ for some j. Since $a_j^\top \xi \leq b_j$ and ball is convex, we can find $\beta \in (0,1)$ such that $a_j^\top(\beta z + (1-\beta)\xi) = b_j$.

$$\|\xi - (\beta z + (1-\beta)\xi)\|_2 = \beta\|\xi - z\|_2 \leq \beta\xi_0 < \xi_0$$

On the other side,

$$\mathrm{dist}_{(a_j,b_j)}(\xi) \leq \|\xi - (\beta z + (1-\beta)\xi)\|_2 \;\Rightarrow\; \mathrm{dist}_{(a_j,b_j)}(\xi) < \xi_0$$

We have derived a contradiction.

(b) Let r denote the radius.

$$\begin{aligned}
\max \quad & r \\
s.t. \quad & \xi \in P \\
& r \leq \operatorname{dist}_{(a_i,b_i)}(\xi) = \frac{|a_i^\top \xi - b_i|}{\|a_i\|_2}, \quad \forall i = 1, \ldots, m \\
& r \geq 0
\end{aligned}$$

Observe that $|a_i^\top \xi - b_i| = x_i^+ - x_i^-$ where $x_i^+ = \max\{a_i^\top \xi - b_i, 0\}$ and $x_i^- = \min\{a_i^\top \xi - b_i, 0\}$. We can construct an equivalent linear formulation:

$$\begin{aligned}
\max \quad & r - M \sum_{i=1}^{n} \left(x_i^+ - x_i^-\right) & \\
s.t. \quad & a_i^\top \xi \leq b_i & \forall i = 1, \ldots, m \\
& \|a_i\|_2 r \leq x_i^+ - x_i^- & \forall i = 1, \ldots, m \\
& r \geq 0 & \\
& x_i^+ \geq a_i^\top - b_i & \forall i = 1, \ldots, m \\
& x_i^+ \geq 0 & \forall i = 1, \ldots, m \\
& x_i^- \leq a_i^\top \xi - b_i & \forall i = 1, \ldots, m \\
& x_i^- \leq 0 & \forall i = 1, \ldots, m
\end{aligned}$$

where M is a sufficiently large positive number such that all $x_i^+ - x_i^-$ are enforced to attain its minimum $|a_i^\top \xi - b_i|$ in the optimal solution.

(c) Linear and convex.

2. (a) Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be two points in X. Furthermore, let $\lambda \in (0,1)$ and $(z_1, z_2, z_3) = \lambda(x_1, x_2, x_3) + (1-\lambda)(y_1, y_2, y_3)$. We need to show $(z_1, z_2, z_3) \in X$.

$$z_1 + z_3 = \lambda(x_1 + x_3) + (1-\lambda)(y_1 + y_3) = \lambda + 1 - \lambda = 1$$

In order to check the second constraint, we need compute two terms separately

$$z_2^2 = (\lambda x_2 + (1-\lambda)y_2)^2 = \lambda^2 x_2^2 + (1-\lambda)^2 y_2^2 + 2\lambda(1-\lambda)x_2 y_2$$

$$z_1 z_3 = (\lambda x_1 + (1-\lambda)y_1)(\lambda x_3 + (1-\lambda)y_3) = \lambda^2 x_1 x_3 + (1-\lambda)y_1 y_3 + \lambda(1-\lambda)(x_1 y_3 + x_3 y_1)$$

Note that $x_2^2 \leq x_1 x_3$ and $y_2^2 \leq y_1 y_3$. Also, observe

$$2\lambda(1-\lambda)x_2 y_2 \leq \lambda(1-\lambda)\left(x_1 y_3 + x_3 y_1 - (\sqrt{x_1 y_3} - \sqrt{x_3 y_1})^2\right) \leq \lambda(1-\lambda)(x_1 y_3 + x_3 y_1)$$

We conclude $z_2^2 \leq z_1 z_3$.

(b) No. The determinant of its hessian is negative.

$$\nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \Rightarrow \det\left(\nabla^2 f(\boldsymbol{x})\right) = -2 < 0$$

(c) The optimization problem is

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 \\ s.t. \quad & x_1 + x_3 - 1 = 0 \\ & x_2^2 - x_1 x_3 \leq 0 \end{aligned}$$

$\mathcal{L}(x_1, x_2, x_3, \mu, \lambda) = x_1 + x_2 + x_3 - \mu(x_1 + x_3 - 1) - \lambda(x_2^2 - x_1 x_3)$.

KKT conditions:

$$1 - \mu + \lambda x_3 = 0 \tag{1}$$
$$1 - 2\lambda x_2 = 0 \tag{2}$$
$$1 - \mu + \lambda x_1 = 0 \tag{3}$$
$$\lambda(x_2^2 - x_1 x_3) = 0 \tag{4}$$
$$x_1 + x_3 - 1 = 0 \tag{5}$$
$$x_2^2 - x_1 x_3 \leq 0 \tag{6}$$
$$\lambda \geq 0 \tag{7}$$

Equation (3) implies that $\lambda \neq 0$ and $x_2 \neq 0$. Moreover, (2) and (7) show that $\lambda, x_2 > 0$. Therefore, (1) $-$ (3) $= 0$ implies $x_1 = x_3$. Then $x_1 = x_3 = 1/2$ follows from (5). Equation (4) enforces $x_2^2 = x_1 x_3 = 1/4 \;\Rightarrow\; x_2 = 1/2$. Solve (1) and (2) for μ, λ to get $\mu = 3/2, \lambda = 1$. In a summary, there is only one solution to the KKT conditions.

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{\mu}, \bar{\lambda}) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1\right)$$

i.e. $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is the only KKT point.

(d) Let $g(\boldsymbol{x}) = x_2^2 - x_1 x_3$ and $h(\boldsymbol{x}) = x_1 + x_3 - 1$. Then

$$\nabla g(\bar{\boldsymbol{x}}) = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}; \quad \nabla h(\bar{\boldsymbol{x}}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Since $\nabla g(\bar{\boldsymbol{x}}), \nabla h(\bar{\boldsymbol{x}})$ are linearly independent, KKT conditions are necesssary. As $g(\boldsymbol{x})$ is not convex, KKT conditions may not be sufficient.

Let $\boldsymbol{x}^*$ be the optimizer of the problem in part (c). Since KKT conditions are necessary, $\boldsymbol{x}^*$ is a KKT point. Together with the fact $\bar{\boldsymbol{x}}$ is the only KKT point, we have $\bar{\boldsymbol{x}} = \boldsymbol{x}^*$ i.e. $\bar{\boldsymbol{x}}$ is the optimizer.

3. Let $A \in \mathbb{R}^{m\times n}$ be the matrix of which $i-$th *column* is a_i. Then

$$(a) \ \Leftrightarrow \ P := \left\{ \lambda \in \mathbb{R}^m : \begin{bmatrix} A \\ -A \\ I \end{bmatrix} \lambda \geq \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \ -\mathbf{1}^\top \lambda < 0 \right\} \neq \emptyset$$

Note that the first inequality guarantees that $\sum_{i=1}^n \lambda_i a_i = A\lambda = \mathbf{0}$ and $\lambda \geq \mathbf{0}$. The second inequality is $\sum_{i=1}^n \lambda_i > 0$. Since $\lambda \geq \mathbf{0}$, it is equivalent to $\lambda \neq \mathbf{0}$.

By Farkas' lemma, we can construct the dual system of P.

$$D := \left\{ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} \in \mathbb{R}^{3n} : \begin{bmatrix} A^\top & -A^\top & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = -\mathbf{1}, \ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} \geq \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \right\}$$

The first inequality is $\boldsymbol{z} + \mathbf{1} = A^\top(\boldsymbol{y} - \boldsymbol{x})$. We want to show $D \neq \emptyset$ is equivalent to (b). Suppose $D \neq \emptyset$, then let $\boldsymbol{d} = \boldsymbol{y} - \boldsymbol{x}$.

$$A^\top \boldsymbol{d} = \boldsymbol{z} + \mathbf{1} \geq \mathbf{0} \ \Rightarrow \ a_i^\top \boldsymbol{d} \geq \mathbf{0}, \ \forall i = 1, 2, \ldots, m \ \Rightarrow \ (b) \text{ is true}$$

Conversely, if (b) is true, then $\exists\ w \geq 0$ such that $w a_i^\top \boldsymbol{d} \geq 1$ for all $i = 1, \ldots, m$. Let

$$y_i = \max\{w d_i, 0\}, \ x_i = \max\{-w d_i, 0\}, \ \forall i = 1, \ldots, m$$

Then $\boldsymbol{y}, \boldsymbol{x} \geq \mathbf{0}$ and $\boldsymbol{y} - \boldsymbol{x} = w\boldsymbol{d}$. Moreover, let $\boldsymbol{z} = A^\top(\boldsymbol{y} - \boldsymbol{x}) - \mathbf{1} = wA^\top \boldsymbol{d} - \mathbf{1} \geq \mathbf{0}$. It's obvious that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in D$. Consequently,

$$(a) \ \Leftrightarrow \ P \neq \emptyset; \ \ (b) \ \Leftrightarrow \ D \neq \emptyset$$

Farkas' lemma says exactly one of P and D is nonempty. It follows that exactly one of (a) and (b) holds.

4. Not required.

5. (a) Let $E_0 = \{e \in E : w(e) = 0\}$ be the set of edges with weight 0 and $E_1 = E \setminus E_0$. It's sufficient to show there exists a assignment in which all edges come from E_0. Equivalently, the following optimization problem is feasible.

$$\begin{aligned} \min \quad & 0 && \text{(P)} \\ s.t. \quad & \sum_{i:(i,j)\in E_0} x_{ij} = 1, \quad \forall j = 1,\ldots,n \\ & \sum_{j:(i,j)\in E_0} x_{ij} = 1, \quad \forall i = 1,\ldots,n \\ & 0 \le x_{ij} \le 1, \quad \forall (i,j) \in E_0 \end{aligned}$$

The dual problem is

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} y_j + \sum_{i=1}^{n} z_i + \sum_{(i,j)\in E_0} t_{ij} && \text{(D)} \\ s.t. \quad & y_j + z_i + t_{ij} \le 0, \quad \forall (i,j) \in E_0 \\ & t_{ij} \le 0, \quad \forall (i,j) \in E_0 \end{aligned}$$

Note that (D) is always feasible because $(y,z,t) = (0,0,0)$ is a feasible solution. Moreover, each vertex i has at least 1 edge in E_0. So

$$y_j \le 0, \ \forall j = 1,\ldots,n; \quad z_i \le 0, \ \forall i = 1,\ldots,n$$

Hence (D) has an upper bound 0. It follows that $(0,0,0)$ is the optimal solution. By strong duality theorem, (P) is also feasible.

(b) Suppose $\ell_1 \subseteq E_0$ is an assignment with weight 0. If we remove ℓ_1 from the graph, then each vertex is incident to exactly $k-1$ edges with weight 0. By part (a), there exists an assignment $\ell_2 \subseteq E_0$ with weight 0. Then, after removing ℓ_2, each vertex in new graph is incident to exactly $k-2$ edges with weight 0.
Continue this procedure, we can find k pairwise disjoint assignments $\ell_1, \ell_2, \ldots, \ell_k$. Furthermore, after deleting ℓ_k, each vertex is incident to exactly $k-k=0$ edges with weight 0. These assignments form a partition of E_0.

$$\ell_1 \bigcup \ell_2 \bigcup \cdots \bigcup \ell_k = E_0$$

3 Summer 18

1. (a) Infeasible: (x_1, x_2^+). Feasible but not optimal: (x_1, s_1). Non-basic: (x_2^+, x_2^-).

   (b) The optimization problem in standard form is

$$\begin{aligned} \min \quad & c_1x_1 + c_2x_2^+ - c_2x_2^- \\ s.t. \quad & a_{11}x_1 + a_{12}x_2^+ - a_{12}x_2^- + s_1 \qquad = b_1 \\ & a_{21}x_1 + a_{22}x_2^+ - a_{22}x_2^- \qquad +s_2 = b_2 \\ & x_1, x_2^+, x_2^-, s_1, s_2 \geq 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min \quad & c^\top x \\ s.t. \quad & Ax + Is = b \\ & x, s \geq 0 \end{aligned}$$

   In the optimal tableau, $\{x_1, x_2^-\}$ are basic variables. The corresponding constraint matrix is

$$B = \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ is the constraint matrix of } s \text{ in the tableau}$$

   Furthermore, the constraint matrix of x in the tableau is

$$B^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } B^{-1}b = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

   In order to find A, we need compute B first.

$$B = \left(B^{-1}\right)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow A = B(B^{-1}A) = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix} \text{ and } b = B(B^{-1}b) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

   The first row in the tableau is the reduced cost vector.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}^\top = \begin{bmatrix} c_1 \\ c_2 \\ -c_2 \\ 0 \\ 0 \end{bmatrix}^\top - \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix}^\top \begin{bmatrix} B^{-1}A & B^{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2c_1 + c_2 \\ -c_1 + c_2 \end{bmatrix}$$

   Solve for c_1, c_2 to get $c_1 = -2$ and $c_2 = -1$. The original optimization problem is

$$\begin{aligned} \min \quad & -2x_1 - x_2 \\ s.t. \quad & x_1 + x_2 \leq 1 \\ & -x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{aligned}$$

   And $(x_1^*, x_2^*) = (6, -5)$.

(c) (x_1^*, x_2^*) remains optimal if and only if all reduced costs of nonbasic variables are non-negative.

$$\begin{cases} -2c_1 - 1 \geq 0 \\ -c_1 - 1 \geq 0 \end{cases} \Rightarrow \; c_1 \leq -1$$

(d) The Lagrangian is

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = -2x_1 - x_2 + \lambda_1(x_1 + x_2 - 1) + \lambda_2(-x_1 - 2x_2 - 4) + \lambda_3(-x_1)$$

KKT conditions:

$$\begin{aligned} -2 + \lambda_1 - \lambda_2 - \lambda_3 &= 0 \\ -1 + \lambda_1 - 2\lambda_2 &= 0 \\ \lambda_1(x_1 + x_2 - 1) &= 0 \\ \lambda_2(-x_1 - 2x_2 - 4) &= 0 \\ \lambda_3(-x_1) &= 0 \\ x_1 + x_2 - 1 &\leq 0 \\ -x_1 - 2x_2 - 4 &\leq 0 \\ -x_1 &\leq 0 \\ \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned}$$

KKT conditions at $(x_1^*, x_2^*) = (6, -5)$.

$$\begin{aligned} -2 + \lambda_1 - \lambda_2 - \lambda_3 &= 0 \\ -1 + \lambda_1 - 2\lambda_2 &= 0 \\ \lambda_3 &= 0 \end{aligned} \Rightarrow \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 1 \\ \lambda_3 = 0 \end{cases}$$

(e) Compared to KKT conditions in part (d), only stationarity constraints are different.

New stationarity:

$$\begin{aligned} -2(x_1 - 5) + \lambda_1 - \lambda_2 - \lambda_3 &= 0 \\ -1 + \lambda_1 - 2\lambda_2 &= 0 \end{aligned}$$

Evaluate new KKT conditions at $(x_1^*, x_2^*) = (6, -5)$:

$$\begin{aligned} -2 + \lambda_1 - \lambda_2 - \lambda_3 &= 0 \\ -1 + \lambda_1 - 2\lambda_2 &= 0 \\ \lambda_3 &= 0 \end{aligned} \Rightarrow \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 1 \\ \lambda_3 = 0 \end{cases}$$

$(x_1^*, x_2^*) = (6, -5)$ is still a KKT point but not an optimizer to the new problem since the objective function $-(x_1 - 5)^2 - x_2$ attains a smaller value at $(0, 0)$, which is -25.

2. Observe that $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top Q\boldsymbol{x}$ is convex and differentiable. $\nabla f(\boldsymbol{x}) = Q\boldsymbol{x}$. Let

$$g(\lambda) := f(\boldsymbol{x}_p - \lambda\nabla f(\boldsymbol{x}_p)) = \frac{1}{2}\left(\boldsymbol{x}_p^\top Q\boldsymbol{x}_p - 2\lambda\boldsymbol{x}_p^\top Q^\top Q\boldsymbol{x}_p + \lambda^2\boldsymbol{x}_p^\top Q^\top Q^2\boldsymbol{x}_p\right)$$

Note that $g(\lambda)$ is a quadratic function with positive leading coefficient and thus $g(\lambda)$ is convex and differentiable. The minimum value of $g(\lambda)$ attains at its stationary point λ^*

$$\nabla_\lambda g(\lambda^*) = -\boldsymbol{x}_p^\top Q^\top Q\boldsymbol{x}_p + \lambda^*\boldsymbol{x}_p^\top Q^\top Q^2\boldsymbol{x}_p = 0$$

Conclude that

$$\lambda^* = \frac{\boldsymbol{x}_p^\top Q^\top Q\boldsymbol{x}_p}{\boldsymbol{x}_p^\top Q^\top Q^2\boldsymbol{x}_p}$$

3. (a) Let $(P^\top \boldsymbol{x})_h$ be the h-th element of $P^\top \boldsymbol{x}$.

$$(P^\top \boldsymbol{x})_h = \sum_{j=1}^{n} P_{hj} x_j \le \left(\max_{i=1,\dots,n} x_i\right) \sum_{j=1}^{n} P_{ij} = \left(\max_{i=1,\dots,n} x_i\right) (P^\top \boldsymbol{e})_h = \left(\max_{i=1,\dots,n} x_i\right) e_h$$

Thus

$$P^\top \boldsymbol{x} \le \left(\max_{i=1,\dots,n} x_i\right) \boldsymbol{e}$$

(b) Rewrite the linear system $\mathcal{P}$ as follows:

$$\begin{bmatrix} P - I \\ \boldsymbol{e}^\top \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}, \ \boldsymbol{x} \ge \boldsymbol{0}$$

Its dual system $\mathcal{D}$ is

$$\begin{bmatrix} P^\top - I & \boldsymbol{e} \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ z \end{bmatrix} \ge \boldsymbol{0}, \ \begin{bmatrix} \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ z \end{bmatrix} < 0$$

Or equivalently,

$$P^\top \boldsymbol{y} \ge \boldsymbol{y} - z\boldsymbol{e}, \ z < 0$$

We need to show $\mathcal{D}$ is infeasible. Suppose not, there exists $(\boldsymbol{y}, z)$ satisfies above inequalities. For some $r \in \{1, \dots, b\}$, we have $y_r = \max_{i=1,\dots,n} y_i$. Then

$$(P^\top \boldsymbol{y})_r \ge y_r - z > y_r$$

However,

$$(P^\top \boldsymbol{y})_r \le \left(\max_{i=1,\dots,n} y_i\right) \boldsymbol{e}_r = \max_{i=1,\dots,n} y_i = y_r$$

Since $\mathcal{D}$ is infeasible, Farkas' lemma says $\mathcal{P}$ must be feasible.

4. (a) Let x_{ij} be the amount of flow on the arc $i \to j$.

$$\begin{aligned} \min \quad & c_{12}x_{12} + c_{13}x_{13} + c_{23}x_{23} + c_{24}x_{24} + c_{34}x_{34} \qquad (\mathcal{P}) \\ s.t. \quad & x_{12} + x_{13} = 1 \\ & -x_{12} + x_{23} + x_{24} = 1 \\ & -x_{13} - x_{23} + x_{34} = 1 \\ & -x_{24} - x_{34} = -3 \\ & x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} \max \quad & p_1 + p_2 + p_3 - 3p_4 \quad (\mathcal{D}) \\ s.t. \quad & p_1 - p_2 \leq c_{12} \\ & p_1 - p_3 \leq c_{13} \\ & p_2 - p_3 \leq c_{23} \\ & p_2 - p_4 \leq c_{24} \\ & p_3 - p_4 \leq c_{34} \end{aligned}$$

(c) Suppose $\boldsymbol{x}^*$ and $\boldsymbol{p}^*$ are optimal solutions to primal and dual problems respectively.

$$\begin{aligned} x_{12}^*(p_1^* - p_2^* - c_{12}) = 0 \\ x_{13}^*(p_1^* - p_3^* - c_{13}) = 0 \\ x_{23}^*(p_2^* - p_3^* - c_{23}) = 0 \\ x_{24}^*(p_2^* - p_4^* - c_{24}) = 0 \\ x_{34}^*(p_3^* - p_4^* - c_{34}) = 0 \end{aligned}$$

(d) It is easy to find the primal optimal solution and then the dual solution can also be derived easily

$$\begin{cases} x_{12}^* = 1 \\ x_{13}^* = 0 \\ x_{23}^* = 2 \\ x_{24}^* = 0 \\ x_{34}^* = 3 \end{cases} \Rightarrow \begin{cases} p_1^* = 4 \\ p_2^* = 3 \\ p_3^* = 2 \\ p_4^* = 0 \end{cases}$$

The shortest directed paths to node 4 from other nodes are

$$\text{node 1: } 1 \to 2 \to 3 \to 4; \quad \text{node 2: } 2 \to 3 \to 4; \quad \text{node 3: } 3 \to 4$$

(e) Let's analyze the constrains of $\mathcal{P}$ first,

$$\left.\begin{aligned} x_{12}+x_{13} &= 1 \;\Rightarrow\; \text{at least one of } \{x_{12},x_{13}\}>0 \\ -x_{12}+x_{23}+x_{24} &= 1 \;\Rightarrow\; \text{at least one of } \{x_{23},x_{24}\}>0 \\ -x_{13}-x_{23}+x_{34} &= 1 \;\Rightarrow\; x_{34}>0 \end{aligned}\right\} \text{ at least three } x_{ij}>0$$

Since the rank of constraint matrix of $\mathcal{P}$ is 3, there are *exactly* three x_{ij} are positive. By complementary slackness, every dual solution satisfying three active constraints is optimal. Note that the number of dual variables is more than 3. So there always exist a solution $(p_1^*, p_2^*, p_3^*, p_4^*)$ such that $p_4^*=0$.

Suppose ℓ_i is the length of the shortest path from node i to node 4. Then, obviously,

$$\begin{aligned} \ell_1 &= \min\{\ c_{12}+\ell_2,\ c_{13}+\ell_3\ \} \\ \ell_2 &= \min\{\ c_{23}+\ell_3,\ c_{24}\ \} \\ \ell_3 &= c_{34} \end{aligned}$$

Therefore, we can formulate a linear programming $\mathcal{Q}$ to find all ℓ_i.

$$\begin{array}{rll} \max & l_1+l_2+l_3 & (\mathcal{Q}) \\ s.t. & l_1 \le c_{12}+l_2 & \\ & l_1 \le c_{13}+l_3 & \\ & l_2 \le c_{23}+l_3 & \\ & l_2 \le c_{24} & \\ & l_3 \le c_{34} & \end{array} \quad \text{vs.} \quad \begin{array}{rll} \max & p_1+p_2+p_3-3p_4 & (\mathcal{D}) \\ s.t. & p_1 \le c_{12}+p_2 & \\ & p_1 \le c_{13}+p_3 & \\ & p_2 \le c_{23}+p_3 & \\ & p_2 \le c_{24}+p_4 & \\ & p_3 \le c_{34}+p_4 & \end{array}$$

Note that (ℓ_1,ℓ_2,ℓ_3) is an optimal solution to $\mathcal{Q}$ and $(\ell_1,\ell_2,\ell_3,0)$ is a feasible solution to $\mathcal{D}$. Thus $\ell_1+\ell_2+\ell_3 \le p_1^*+p_2^*+p_3^*-3p_4^* = p_1^*+p_2^*+p_3^*$. Conversely, since $p_4^*=0$, (p_1^*,p_2^*,p_3^*) is a feasible solution to $\mathcal{Q}$. $p_1^*+p_2^*+p_3^* \le \ell_1+\ell_2+\ell_3$. Therefore,

$$p_1^*+p_2^*+p_3^* = \ell_1+\ell_2+\ell_3$$

In order to conclude $p_i^*=\ell_i$ for all $i=1,2,3$, we need to show $p_1^* \ge p_2^*$. If $p_1^*<p_2^*$, then one can show $(p_2^*,p_2^*,p_3^*,0)$ is a feasible solution to $\mathcal{D}$ with greater objective value, which violates the optimality of $(p_1^*,p_2^*,p_3^*,0)$.

$$\begin{cases} p_1^*+p_2^*+p_3^* = \ell_1+\ell_2+\ell_3 \\ p_3^* = \ell_3 = c_{34} \\ p_1^* \ge p_2^* \\ \ell_1 \ge \ell_2 \end{cases} \quad\Rightarrow\quad \begin{cases} p_1^* = \ell_1 \\ p_2^* = \ell_2 \\ p_3^* = \ell_3 \end{cases}$$

5. (a) Let $\{\alpha_{ij}\}_{j=1}^{n_j}$ be the set of extreme points of X_i. Then

$$X_i = \left\{ \sum_{j=1}^{n_j} \lambda_{ij}\alpha_{ij} \ : \quad \sum_{j=1}^{n_j} \lambda_{ij} = 1,\ \lambda_{ij} \geq 0, \forall j = 1, \ldots, n_j \right\},\ \forall i = 1, \ldots, T$$

DW master problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^{T}\sum_{j=1}^{n_j} \lambda_{ij}\, \boldsymbol{c}_i^\top \alpha_{ij} \\ s.t. \quad & \sum_{i=1}^{T}\sum_{j=1}^{n_j} \lambda_{ij} A_i \alpha_{ij} \leq \boldsymbol{b} \\ & \sum_{j=1}^{n_j} \lambda_{ij} = 1, \quad \forall i = 1, \ldots, T \\ & \lambda_{ij} \geq 0, \quad \forall j = 1, \ldots, n_j,\ \forall i = 1, \ldots, T \end{aligned}$$

i-th subproblem:

$$\begin{aligned} \min \quad & (c_i^\top - q_i^\top A_i)x - r_i \\ s.t. \quad & x \in X_i \end{aligned}$$

where $\begin{bmatrix} q_r & r_i \end{bmatrix} = (c_i)_B^\top (A_i)_B^{-1}$

(b) Suppose $\{\lambda_{ij}\}$ is a feasible solution to the DW master problem. For each $i = 1, \ldots, T$,

$$\sum_{j=1}^{n_j} \lambda_{ij} = 1 \text{ and } \lambda_{ij} \geq 0, \quad \forall j = 1, \ldots, n_j$$

Therefore, at least of $\{\lambda_{ij}\}_{j=1}^{n_j}$ is positive, i.e. $\{\lambda_{ij}\}_{j=1}^{n_j}$ contains at least one basic variable.

(c) i-th subproblem:

$$\begin{aligned} \max \quad & \sum_j c_{ij} x_{ij} \\ s.t. \quad & \omega_{ij} x_{ij} \leq b_j,\ \forall j \\ & \sum_j x_{ij} = 1 \\ & x_{ij} \geq 0, \quad \forall j \end{aligned}$$

4 Winter 19

1. (a) $b = -4$. The line $x_1 + 4x_2 + b = 0$ should pass through $\bar{\boldsymbol{x}} = (0, 1)$.

   (b) Let $\mathcal{L}(x_1, x_2, \mu, \lambda) = (x_1 + 4)^2 + (x_2 - 2)^2 + \mu(x_1^2 + 16x_2^2 - 16) - \lambda(x_1 + 4x_2 - 5)$. Then KKT conditions are

$$\begin{aligned} 2(x_1 + 4) + 2\mu x_1 - \lambda &= 0 \\ 2(x_2 - 2) + 32\mu x_2 - 4\lambda &= 0 \\ \lambda(x_1 + 4x_2 - 4) &= 0 \\ x_1^2 + 16x_2^2 - 16 &= 0 \\ -x_1 - 4x_2 + 4 &\leq 0 \\ \lambda &\geq 0 \end{aligned}$$

   At $\bar{\boldsymbol{x}} = (0, 1)$, KKT conditions become

$$\begin{aligned} 8 - \lambda &= 0 \\ -2 + 32\mu - 4\lambda &= 0 \\ \lambda &\geq 0 \end{aligned}$$

   Solve for μ, λ to get $\begin{bmatrix} \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} 17/16 \\ 8 \end{bmatrix}$. So $\bar{\boldsymbol{x}}$ is a KKT point. Moreover, the gradients of active constraints are

$$\nabla h(\bar{\boldsymbol{x}}) = \begin{bmatrix} 0 \\ 32 \end{bmatrix}; \quad \nabla g(\bar{\boldsymbol{x}}) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

   Obviously, these two vectors are linearly independent and thus $\bar{\boldsymbol{x}}$ satisfies the KKT first-order KKT necessary conditions.

2. (a) There are totally 8 vertices of $X = \{\boldsymbol{x} \in \mathbb{R}^3 : 0 \leq x_1, x_2, x_2 \leq 1\}$, say

$$\alpha_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \alpha_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha_6 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \alpha_7 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \alpha_8 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

DW-master problem:

$$\begin{aligned} \min \quad & 3\lambda_2 + 6\lambda_3 + 4\lambda_4 + 9\lambda_5 + 7\lambda_6 + 10\lambda_7 + 13\lambda_8 \\ s.t. \quad & 6\lambda_2 + 4\lambda_3 + 3\lambda_4 + 10\lambda_5 + 9\lambda_6 + 7\lambda_7 + 13\lambda_8 \geq 8 \\ & \sum_{i=1}^{8} \lambda_i = 1 \\ & \lambda_i \geq 0 \qquad \forall i = 1, 2, \ldots, 8 \end{aligned}$$

(b) Restricted master problem: (introduce a slack variable s)

$$\begin{aligned} \min \quad & 7\lambda_6 \\ s.t. \quad & 9\lambda_6 - s = 8 \\ & \lambda_1 + \lambda_6 = 1 \\ & \lambda_1, \lambda_6, s \geq 0 \end{aligned}$$

The optimal solution is $\begin{bmatrix} \lambda_1^* \\ \lambda_6^* \\ s^* \end{bmatrix} = \begin{bmatrix} 1/9 \\ 8/9 \\ 0 \end{bmatrix}$. The corresponding dual solution is

$$\begin{bmatrix} \pi \\ \pi_0 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 7 \end{bmatrix}^\top \begin{bmatrix} 0 & 9 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 7/9 \\ 0 \end{bmatrix}^\top$$

(c) From the original problem,

$$\boldsymbol{c} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} ;\ A = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}^\top$$

The corresponding subproblem:

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} - \pi^\top A\boldsymbol{x} - \pi_0 \\ s.t. \quad & \boldsymbol{x} \in X \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min \quad & \frac{1}{9}(-15x_1 + 26x_2 + 15x_3) \\ s.t. \quad & 0 \le x_i \le 1 \qquad\qquad \forall i = 1,2,3 \end{aligned}$$

The optimal solution is $\alpha_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(d) The new restricted master problem after adding α_2:

$$\begin{aligned} \min \quad & 3\lambda_2 + 7\lambda_6 \\ s.t. \quad & 6\lambda_2 + 9\lambda_6 \ge 8 \\ & \lambda_1 + \lambda_2 + \lambda_6 = 1 \\ & \lambda_1, \lambda_2, \lambda_6 \ge 0 \end{aligned}$$

3. (a) Let x_{ij} be the amount of flow passing through arc $i \to j$.

$$\begin{aligned} \min \quad & 2x_{12} + 6x_{13} + x_{23} + 5x_{24} + 2x_{34} \\ s.t \quad & x_{12} + x_{13} = 8 \\ & x_{24} + x_{23} = x_{12} + 6 \\ & x_{34} + 3 = x_{13} + x_{23} \\ & 11 = x_{24} + x_{34} \\ & x_{12}, x_{13}, x_{24}, x_{23}, x_{34} \geq 0 \end{aligned}$$

(b) Let $x_{23} = x_{34} = 0$. We derive $\begin{bmatrix} x_{12} \\ x_{13} \\ x_{24} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 11 \end{bmatrix}$. Solve $\begin{cases} p_2 - p_1 = 2 \\ p_3 - p_1 = 6 \\ p_4 - p_2 = 5 \\ p_4 = 0 \end{cases}$ to get $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \\ -1 \\ 0 \end{bmatrix}$.

The reduced costs are

$$\begin{bmatrix} \bar{c}_{23} \\ \bar{c}_{34} \end{bmatrix} = \begin{bmatrix} c_{23} + p_2 - p_3 \\ c_{34} + p_3 - p_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Therefore, we move x_{23} into basis and remove x_{13} from the basis. Recompute the solution, we get $\begin{bmatrix} x_{12} \\ x_{23} \\ x_{24} \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 11 \end{bmatrix}$. Similarly, solve $\begin{cases} p_2 - p_1 = 2 \\ p_3 - p_2 = 1 \\ p_4 - p_2 = 5 \\ p_4 = 0 \end{cases}$ to get $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \\ -4 \\ 0 \end{bmatrix}$. The reduced costs are

$$\begin{bmatrix} \bar{c}_{13} \\ \bar{c}_{34} \end{bmatrix} = \begin{bmatrix} c_{13} + p_1 - p_3 \\ c_{34} + p_3 - p_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Move x_{34} into basis and remove x_{24}. The new solution is $\begin{bmatrix} x_{12} \\ x_{23} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \\ 11 \end{bmatrix}$. Solve $\begin{cases} p_2 - p_1 = 2 \\ p_3 - p_2 = 1 \\ p_4 - p_3 = 2 \\ p_4 = 0 \end{cases}$ to get $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -2 \\ 0 \end{bmatrix}$. The reduced costs of nonbasic variables are

$$\begin{bmatrix} \bar{c}_{13} \\ \bar{c}_{24} \end{bmatrix} = \begin{bmatrix} c_{13} + p_1 - p_3 \\ c_{24} + p_2 - p_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

All entries are positve, so current solution is optimal.

(c) $c_{13} + p_1 - p_3 \geq 0 \;\Leftrightarrow\; c_{13} \geq 3.$

(d) The reduced costs now are

$$\begin{bmatrix} \bar{c}_{13} \\ \bar{c}_{24} \end{bmatrix} = \begin{bmatrix} c_{13} + p_1 - p_3 \\ c_{24} + p_2 - p_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

We need to move x_{24} into basis and remove x_{34}. The new solution is $\begin{bmatrix} x_{12} \\ x_{23} \\ x_{24} \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 11 \end{bmatrix}$.

Solve $\begin{cases} p_2 - p_1 = 2 \\ p_3 - p_2 = 1 \\ p_4 - p_2 = 2 \\ \qquad p_4 = 0 \end{cases}$ to get $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -1 \\ 0 \end{bmatrix}$. The reduced costs are

$$\begin{bmatrix} \bar{c}_{13} \\ \bar{c}_{24} \end{bmatrix} = \begin{bmatrix} c_{13} + p_1 - p_3 \\ c_{34} + p_3 - p_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The new solution is the optimizer since reduced costs are all nonnegative.

(e) With the optimal solution derived in part (a), we can compute the reduced cost of x_{14}.

$$\bar{c}_{14} = c_{14} + p_1 - p_4 = -1$$

So we should move x_{14} into basis and remove x_{12}. The new basic solution is $\begin{bmatrix} x_{14} \\ x_{23} \\ x_{24} \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 3 \end{bmatrix}$.

Solve $\begin{cases} p_4 - p_1 = 4 \\ p_3 - p_2 = 1 \\ p_4 - p_2 = 2 \\ \qquad p_4 = 0 \end{cases}$ to get $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -1 \\ 0 \end{bmatrix}$. The reduced costs are

$$\begin{bmatrix} \bar{c}_{13} \\ \bar{c}_{24} \end{bmatrix} = \begin{bmatrix} c_{13} + p_1 - p_3 \\ c_{34} + p_3 - p_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Since all nonbasic variables have positive reduced costs, current solution is optimal.

4. (a) First, we need to show C^* is convex. $\forall \lambda \in (0,1)$ and $\boldsymbol{y}_1, \boldsymbol{y}_2 \in C^*$,

$$(\lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2)^\top \boldsymbol{x} = \lambda \boldsymbol{y}_1^\top \boldsymbol{x} + (1-\lambda)\boldsymbol{y}_2^\top \boldsymbol{x} \leq 0, \ \forall \boldsymbol{x} \in C$$

So $\lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2 \in C^*$ and C^* is therefore convex. To prove C^* is a cone, we observe that, for all $\gamma \geq 0$,

$$\forall \boldsymbol{y} \in C^*, \ \gamma \boldsymbol{y}^\top \boldsymbol{x} \leq 0, \ \forall \boldsymbol{x} \in C$$

Hence $\gamma \boldsymbol{y} \in C^*$ which implies C^* is a cone.

(b) Let's show $\{A^\top \lambda : \lambda \geq \mathbf{0}\} \subseteq C^*$ first. It follows directly from definitions of C and C^*.

$$\forall \boldsymbol{x} \in C, \ \left(A^\top \lambda\right)^\top \boldsymbol{x} = \lambda^\top A \boldsymbol{x} \leq 0 \ \Rightarrow \ A^\top \lambda \in C^*, \ \forall \lambda \geq \mathbf{0}$$

Conversely, we need to show $C^* \subseteq \{A^\top \lambda : \lambda \geq \mathbf{0}\}$. Suppose not, there exists $\boldsymbol{y} \in C^*$ but $\boldsymbol{y} \notin \{A^\top \lambda : \lambda \geq \mathbf{0}\}$. In other words,

$$\{\lambda \in \mathbb{R}^n : A^\top \lambda = \boldsymbol{y}, \lambda \geq \mathbf{0}\} = \emptyset$$

By Farkas' lemma, there exists $\boldsymbol{x}$ such that $A\boldsymbol{x} \leq \mathbf{0}$ and $\boldsymbol{y}^\top \boldsymbol{x} > 0$. However, this is impossible. By assumption $\boldsymbol{y} \in C^*$, we know if $\boldsymbol{x} \in C$, then $\boldsymbol{y}^\top \boldsymbol{x} \leq 0$. The contraposition is, if $\boldsymbol{y}^\top \boldsymbol{x} > 0$, then $\boldsymbol{x} \notin C$ which disproves $A\boldsymbol{x} \leq \mathbf{0}$.

(c) Draw a picture of P, then it's easy to derive

$$0^+P = \{\alpha \boldsymbol{x} + \beta \boldsymbol{y} : \boldsymbol{x} = (2,6), \boldsymbol{y} = (4,2), \alpha, \beta \geq 0\}$$

Note that $(-3,1) \perp (2,6)$ and $(-1,2) \perp (4,2)$.

$$(0^+P)^* = \left\{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 + 3x_2 \leq 0, \ 2x_1 + x_2 \leq 0\right\}$$

5. (a) Since P is a polyhedron, there exist a matrix $B \in \mathbb{R}^{h\times m}$ and vector $\boldsymbol{d} \in \mathbb{R}^h$ such that $P = \{\boldsymbol{y} \in \mathbb{R}^m : B\boldsymbol{y} \leq \boldsymbol{d}\}$. Then we can rewrite $\mathcal{A}(P)$ as follows:

$$\mathcal{A}(P) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \exists \boldsymbol{y} \in \mathbb{R}^m \text{ such that } \begin{bmatrix} I & A \\ I & -A \\ 0 & B \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{d} \end{bmatrix} \right\}$$

Let $T = \left\{ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \in \mathbb{R}^{n+m} \;\middle|\; \begin{bmatrix} I & A \\ I & -A \\ \mathcal{O} & B \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{d} \end{bmatrix} \right\}$. Then T is a polyhedron and

$$\mathcal{A}(P) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \exists \boldsymbol{y} \in \mathbb{R}^m \text{ such that } (\boldsymbol{x}, \boldsymbol{y}) \in T\}$$

By the proposition given in the problem, $\mathcal{A}(P)$ is also a polyhedron.

(b) We use the same trick as in part (a). Since P and Q are two polyhedrons, there exist $B \in \mathbb{R}^{h\times n}$, $C \in \mathbb{R}^{r\times n}$, $\boldsymbol{d} \in \mathbb{R}^h$, and $\boldsymbol{g} \in \mathbb{R}^r$ such that

$$P = \{\boldsymbol{x} \in \mathbb{R}^n : B\boldsymbol{x} \leq \boldsymbol{d}\};\; Q = \{\boldsymbol{y} \in \mathbb{R}^n : C\boldsymbol{y} \leq \boldsymbol{g}\}$$

Let $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. Then we rewrite $P + Q$ as

$$P + Q = \left\{ \boldsymbol{z} \in \mathbb{R}^n \;\middle|\; \exists \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \in \mathbb{R}^{2n} \text{ such that } \begin{bmatrix} B & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & C & \mathcal{O} \\ I & I & -I \\ -I & -I & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{g} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \right\}$$

Let

$$T = \left\{ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} \in \mathbb{R}^{3n} \;\middle|\; \begin{bmatrix} B & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & C & \mathcal{O} \\ I & I & -I \\ -I & -I & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{g} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \right\}$$

Then T is a polyhedron and

$$P + Q = \left\{ \boldsymbol{z} \in \mathbb{R}^n \;\middle|\; \exists \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \in \mathbb{R}^{2n} \text{ such that } (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in T \right\}$$

Therefore, $P + Q$ is a polyhedron.

(c) Let $\boldsymbol{z}^*$ be an extreme point of $P + Q$. By definition, there exist $\boldsymbol{a} \in P, \boldsymbol{b} \in Q$ such that $\boldsymbol{z}^* = \boldsymbol{a} + \boldsymbol{b}$. Now let's prove $\boldsymbol{a}$ is an extreme point of P by contradiction. Suppose not, then there exist two different elements $\boldsymbol{x}_1, \boldsymbol{x}_2 \in P$ and a scalar $\lambda \in (0, 1)$ such that $\boldsymbol{a} = \lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2$. Observe that

$$\boldsymbol{x}_1 + \boldsymbol{b}, \boldsymbol{x}_2 + \boldsymbol{b} \in P + Q \;\Rightarrow\; \lambda(\boldsymbol{x}_1 + \boldsymbol{b}) + (1 - \lambda)(\boldsymbol{x}_2 + \boldsymbol{b}) = \boldsymbol{z}^*$$

Then $\boldsymbol{z}^*$ is not an extreme point which conflicts to our choice of $\boldsymbol{z}^*$. Hence $\boldsymbol{a}$ is an extreme point of P. Similarly, one can show $\boldsymbol{b}$ is an extreme point of Q. Consequently, $\boldsymbol{z}^*$ is the sum of two extreme points.