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Massive higher spin fields in curved spacetime and necessity of non-minimal couplings

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Abstract

Free massive higher spin fields in weak background gravitational fields are discussed. Contrary to the spin one case, higher spin fields should have nontrivial non-minimal couplings to the curvature. A precise analysis is given for the spin 2 case, and it is shown that two conditions should be satisfied among five non-minimal coupling constants, which we derive both in the Hamiltonian and Lagrangian formalisms. It is checked that the linearized limit of the massive gravity theory indeed has the non-minimal couplings that satisfy the conditions. We also discuss the form of the non-minimal couplings for the spin 3 case.

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Contents

1 Introduction

Attempts to construct massive higher spin field theories showed up with the papers written by Fierz and Pauli, who formulated a free field theory of massive spin 2 particles in the Minkowski space [1][2]. In general, the natural object to describe a spin s particle is a rank-s traceless symmetric tensor field, but this has more independent components than necessary, because a spin s particle has only $2s + 1$ degrees of freedom (DOF). Therefore, the Lagrangian should give the equations of motion (EOM) that yield necessary and sufficient constraints to eliminate the redundant DOF. In fact, for the $s = 2$ case, Fierz and Pauli showed that an appropriate Lagrangian can be obtained if one introduces an auxiliary scalar field in addition to a rank-2 traceless tensor. These fields can actually be combined to form a single traceful symmetric tensor $h_{\mu\nu}$, which we call the Fierz-Pauli (FP) field.[1] For the case $s > 2$, the Lagrangian with the desired property was given by Singh-Hagen [4][5], which consist of traceless symmetric tensors of ranks s, $s-2$, $s-3$, $s-4$, ..., 0. These fields can be combined to form two traceful symmetric tensors of ranks s and $s - 3$.[2]

All the works above only consider the case where the background spacetime is flat. However, for curved backgrounds, it is non-trivial to formulate massive higher spin field theories.[3] In fact, as we will see in section 2, the mechanism to derive the constraints from the EOM breaks down because covariant derivatives do not commute with each other. There was also an argument that the transverse condition is not compatible with the wave equation

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[1]In [3] it was shown that the FP theory is the unique formulation of a spin 2 particle without ghosts or tachyons.

[2]The massless limit of that Lagrangian was studied by Fronsdal [6][7].

[3]For specific types of background, consistent EOM are obtained for massless fields by using the spacetime symmetry [8][9][10][11]. An attempt to generalize the theory to the massive case was made in [12].

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The coefficients (6.6) indeed satisfy (4.19), but depend only on a single parameter $\alpha_0 - \alpha_3$. We thus may conclude that the Lagrangian in sections 3 and 4 gives a more general description than the massive gravity theory, at least for the free FP field in weak gravitational backgrounds.

In this paper only the spin 2 and 3 cases have been discussed. However, it is natural to expect that massive particles with an arbitrary higher spin should also have nontrivial couplings to the curvatures, which we leave as a future work. Although we have not found a Lorentz covariant way to analyze the DOF, such formalism would help to investigate higher spin fields.

There are two concrete examples of higher-spin particles in the curved spacetime. One is string theory, where their couplings to gravity can be determined by the scattering amplitudes. The other is composite particles in a well-defined theory such as hadrons in quantum chromodynamics, where in principle we have a description based on the effective Lagrangian. It will be interesting to compare them with our results, and it might give a clue to the inevitability of string theory.[11]

Acknowledgments

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[11]See, e.g., [18][19] for an early study in this direction.

for arbitrary backgrounds [13]. It seems that currently there are no consistent massive higher spin theories for general backgrounds that reduces to the flat case smoothly.

On the other hand, we expect that such theories should exist for the following two reasons. One is that phenomenologically higher spin hadrons should exist in the gravitational field. The other is that string theory consistently contains higher spin modes interacting with gravitons. In this paper, as a first step to investigate higher spin theories, we give the quadratic Lagrangian for spin 2 particles in general gravitational backgrounds.

This paper is organized as follows. In section 2, we first show that the mechanism to eliminate the redundant DOF in the flat spacetime no longer works for general curved backgrounds. Then in section 3, we give a consistent quadratic Lagrangian of the massive spin 2 field in general backgrounds. To do that, we use the fact that the kinetic term of the FP field can be identified with the quadratic part in the perturbed Einstein-Hilbert action around the background metric. The analysis is based on the Hamiltonian formalism with the ADM decomposition. We find that a consistent theory can be constructed only when non-minimally coupled curvature terms are added to the Lagrangian with specific coefficients. In section 4 we reproduce the conditions on the coefficients within the Lagrangian formalism. In section 5 we apply our analysis to the spin 3 case and investigate the form of the non-minimal couplings. Finally in section 6, we discuss the relation between our results on the spin 2 case and the massive gravity theory [14][15][16][17].

[Note Added]

After the first manuscript of this paper was accepted for publication, we were informed that the main result in section 3 and 4 were already obtained in [18][19]. We thank I.L. Buchbinder, M. von Strauss and A. Waldron for their valuable comments. We were also informed of related works [20][21][22][23][24][25][26][27][28][29][30][31][32][33][34].

2 Breakdown of the transverse condition for curved backgrounds

In this section, we demonstrate that FP's original mechanism to eliminate the redundant DOF of a massive higher rank tensor field does not work for generic curved backgrounds.

We start by arguing that there is no such issue for massive spin 1 field A^μ (Proca field). The action of the Proca field in the flat Minkowski spacetime is given by

$$S = \int d^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^\mu A_\mu \right], \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the metric is chosen to be $\eta_{\mu\nu} = \text{diag}[-1, +1, +1, +1]$. Its

EOM are given by

$$\partial_\nu F^{\mu\nu} + m^2 A^\mu = 0\,. \tag{2.2}$$

The divergence of (2.2) gives the transverse condition $\partial_\mu A^\mu = 0$, and the substitution of this to the EOM in turn gives the wave equation, $(\Box - m^2)\, A^\mu = 0$. Thus, the action (2.1) gives the EOM which automatically include the constraint that eliminates the redundant DOF correctly. It is easy to see that this mechanism also works in general curved backgrounds. In fact, if we covariantize the action as

$$S = \int d^4x\, \sqrt{-g}\left[-\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\,m^2\,A^\mu A_\mu\right] \tag{2.3}$$

with $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu$, then the EOM are given by

$$\nabla_\nu F^{\mu\nu} + m^2 A^\mu = 0\,, \tag{2.4}$$

whose divergence again gives the transverse condition, $\nabla_\mu A^\mu = 0$, because $\nabla_\mu \nabla_\nu F^{\mu\nu} = [\nabla_\mu, \nabla_\nu]\, F^{\mu\nu} = R_{\mu\nu}{}^\mu{}_\alpha F^{\alpha\nu} + R_{\mu\nu}{}^\nu{}_\alpha F^{\mu\alpha} = -2R_{\mu\nu}F^{\mu\nu} = 0$.[4] Note that one could have added curvature terms to the action of the form $\int d^4x\, \sqrt{-g}\left[a\, R_{\mu\nu}A^\mu A^\nu + b\, R\, A^\mu A_\mu\right]$, where the coupling constants a and b are not determined only by requiring the action to become (2.1) in the flat limit. Such non-minimal couplings can be used to absorb the discrepancy that may arise when kinematic terms are covariantized in a different manner [e.g., a kinetic term $\partial_\mu A^\mu\, \partial_\nu A^\nu$ (up to total derivatives) can be covariantized in two ways: $\nabla_\mu A^\mu\, \nabla_\nu A^\nu$ or $\nabla_\mu A^\nu\, \nabla_\nu A^\mu$].

Now we discuss the spin 2 massive field (FP field). The Lagrangian in the flat spacetime is given by

$$\mathcal{L} = h_{\mu\nu}\, \mathcal{E}_0^{\mu\nu\rho\sigma} h_{\rho\sigma} - \frac{m^2}{2}\left(h_{\mu\nu}h^{\mu\nu} - h^2\right), \tag{2.5}$$

where $\mathcal{E}_0^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator for the flat spacetime:[5]

$$\mathcal{E}_0^{\mu\nu\rho\sigma} h_{\rho\sigma} \equiv \frac{1}{2}(\Box h^{\mu\nu} - \eta^{\mu\nu}\Box h) + \frac{1}{2}(\partial^\mu\partial^\nu h + \eta^{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma}) - \partial^{(\mu}\partial_\lambda h^{\nu)\lambda}\,. \tag{2.6}$$

The kinetic term $\mathcal{L}_{\mathcal{E}_0} = h_{\mu\nu}\, \mathcal{E}_0^{\mu\nu\rho\sigma} h_{\rho\sigma}$ can be formally obtained from the Einstein-Hilbert action[6]

$$S_{\mathrm{EH}}[\hat{g}] = \frac{1}{2}\int d^4x\sqrt{-\hat{g}}\hat{R} \tag{2.7}$$

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[4]The Riemann tensor is defined as $[\nabla_\mu, \nabla_\nu]\, v^\rho = R_{\mu\nu}{}^\rho{}_\sigma\, v^\sigma$. The Ricci tensor and the Ricci scalar are given by $R_{\mu\nu} \equiv R_{\rho\mu}{}^\rho{}_\nu$ and $R = g^{\mu\nu}\, R_{\mu\nu}$, respectively.

[5]We normalize the symmetrization as $X^{(\mu\nu)} \equiv (1/2)\,(X^{\mu\nu} + X^{\nu\mu})$.

[6]Throughout this paper quantities with turret should be understood to represent those associated with $\hat{g}_{\mu\nu}$.

The action of massive gravity is given by

$$S = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2} \hat{R} - m^2 \sum_{n=0}^{4} \alpha_n e_n(\mathbb{K}) \right] , \tag{6.1}$$

$$(\mathbb{K})^\mu_{\ \nu} \equiv (\sqrt{\hat{g}^{-1}f})^\mu_{\ \nu} - \delta^\mu_{\ \nu} . \tag{6.2}$$

Here $f_{\mu\nu}$ is the reference metric and not a dynamical variable. $\sqrt{\hat{g}^{-1}f}$ denotes the square root as a matrix: $((\sqrt{\hat{g}^{-1}f})^2)^\mu_{\ \nu} = \hat{g}^{\mu\lambda}f_{\lambda\nu}$. $e_n(\mathbb{K})$ is the elementary symmetric polynomial of degree n in the eigenvalues of $\mathbb{K}$. They are represented as follows ($[\mathbb{X}] \equiv \mathrm{tr}\mathbb{X}$):

$$e_0(\mathbb{K}) = 1 ,$$
$$e_1(\mathbb{K}) = [\mathbb{K}] ,$$
$$e_2(\mathbb{K}) = \frac{1}{2}([\mathbb{K}]^2 - [\mathbb{K}^2]) ,$$
$$e_3(\mathbb{K}) = \frac{1}{6}([\mathbb{K}]^3 - 3[\mathbb{K}][\mathbb{K}^2] + 2[\mathbb{K}^3]) ,$$
$$e_4(\mathbb{K}) = \frac{1}{24}([\mathbb{K}]^4 - 6[\mathbb{K}]^2[\mathbb{K}^2] + 3[\mathbb{K}^2]^2 + 8[\mathbb{K}][\mathbb{K}^3] - 6[\mathbb{K}^4]) . \tag{6.3}$$

Several conditions are imposed on the parameters α_n ($n = 0, \cdots, 4$) in order to satisfy the following requirements. We first set $\hat{g}_{\mu\nu} = g_{\mu\nu} + 2h_{\mu\nu}$, and expand the Lagrangian with respect to the fluctuation $h_{\mu\nu}$ around $g_{\mu\nu}$. We then require that the first-order terms in $h_{\mu\nu}$ vanish, and that the second-order terms involving m^2 take the same form as the FP mass term in the flat background. A straightforward calculation leads to the conditions $\alpha_1 = \alpha_0$, $\alpha_2 = \alpha_0 - 1$, and we find that the reference metric $f_{\mu\nu}$ is expressed by $g_{\mu\nu}$ as

$$f_{\mu\nu} = g_{\mu\nu} + \frac{2}{m^2}R_{\mu\nu} - \frac{1}{3m^2}g_{\mu\nu}R + O\left(\frac{R^2}{m^4}\right) . \tag{6.4}$$

Since $\mathbb{K}$ is of first or higher order both in $h_{\mu\nu}$ and in the curvature, α_4 does not contribute to the quadratic Lagrangian.

After some calculation, we obtain the Lagrangian for the fluctuation

$$\begin{aligned}
\mathcal{L} = \sqrt{-g} &\left[h_{\mu\nu}\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma} - \frac{m^2}{2}(h_{\mu\nu}h^{\mu\nu} - h^2) \right. \\
&+ \frac{2(\alpha_0 - \alpha_3) - 5}{2}R^{\mu\nu}h_{\mu\lambda}h^\lambda_\nu + \frac{-4(\alpha_0 - \alpha_3) + 11}{12}Rh_{\mu\nu}h^{\mu\nu} \\
&\left. + \frac{\alpha_0 - \alpha_3 - 2}{3}Rh^2 - (\alpha_0 - \alpha_3 + 2)R^{\mu\nu}h_{\mu\nu}h \right] , \tag{6.5}
\end{aligned}$$

which has the form of (3.1)–(3.3) with

$$a_1 = 0 , \quad a_2 = 2(\alpha_0 - \alpha_3) - 5 , \quad a_3 = -\frac{2(\alpha_0 - \alpha_3)}{3} + \frac{11}{6} ,$$

$$b_1 = \frac{2(\alpha_0 - \alpha_3)}{3} - \frac{4}{3} , \quad b_2 = -(\alpha_0 - \alpha_3) + 2 . \tag{6.6}$$

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straints,[9] and we find that the parameters in the non-minimal couplings must take the following values:

$$a = 3\,, \quad b_1 = -\frac{30}{37}\,, \quad b_2 = -\frac{51}{74}\,, \quad b_3 = \frac{30}{37}\,, \quad c_1 = \frac{119}{222}\,, \quad c_2 = -\frac{119}{74}\,. \qquad (5.14)$$

6 Discussion

In this paper we have obtained the Lagrangian that describes a free massive spin 2 or spin 3 particle propagating in the general gravitational background to the first order in the curvature. The Lagrangians contain non-minimal couplings. For the spin 2 case, the coefficients have three free parameters, and, in particular, the coupling constant associated with the Riemann tensor is arbitrary.

Actually, there is a well-known theory of massive spin 2 particles. That is the so-called massive gravity theory [14][15], whose consistency has been proven based on the analysis of the DOF [16][17] (for a review, see [35][36]). We now discuss its relation to our results.[10]

The massive gravity is a non-linear theory, which has a spin 2 massive field $\hat{g}_{\mu\nu}$ and a fixed *reference metric* $f_{\mu\nu}$. Here we will consider a classical solution and the fluctuation around it. In general, the classical solution $g_{\mu\nu}$ is determined after $f_{\mu\nu}$ and an initial condition are specified. However, because we are interested in the fluctuation around the classical solution, it is better to regard $f_{\mu\nu}$ as a function of the classical solution $g_{\mu\nu}$. Then the consistency of the EOM for the fluctuation field is automatically guaranteed due to that of the full non-linear theory. We will see that the quadratic Lagrangian for the fluctuation indeed satisfies the conditions (3.4) and (3.5). However, it has only one free parameter, although the massive gravity theory in general has two free parameters.

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[9] The primary and secondary constraints χ_1, χ_2 take the forms

$$\chi_1 = 2mP_D + 4g^{ij}\dot{g}_{ij}P_X + \sqrt{g}\,(2m^2 - \zeta)X\,,$$
$$\chi_2 = -4mg^{ij}\dot{g}_{ij}P_D + (4m^2 - \xi)P_X + \sqrt{g}\,(4m^3 + 2c_3mR)D - \sqrt{g}\,(5m^2g^{ij}\dot{g}_{ij} - \eta)X\,,$$

where ζ, ξ and η are functions of the curvature. In order for the constraints to give the tertiary and quaternary constraints, the Poisson bracket $\{\chi_1, \chi_2\}$ must vanish. However, apparently this does not hold at the next order $m^3 \times (R/m^2)$ for generic backgrounds. A detailed analysis on this issue will be reported elsewhere.

[10] They have developed the massive gravity theory further to construct a theory called bimetric gravity [37]. However, because our purpose is to discuss spin 2 particles in the gravitational background, it is more appropriate to consider its original form.

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by setting $\hat{g}_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$ and taking quadratic terms in $h_{\mu\nu}$. The EOM take the form

$$0 = 2\,\mathcal{E}_0^{\mu\nu\rho\sigma} h_{\rho\sigma} - m^2(h^{\mu\nu} - \eta^{\mu\nu} h)$$
$$= (\Box - m^2)(h^{\mu\nu} - \eta^{\mu\nu} h) + \eta^{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma} - 2\partial^{(\mu}\partial_\lambda h^{\nu)\lambda} + \partial^\mu\partial^\nu h\,. \tag{2.8}$$

A rank-2 symmetric tensor $h_{\mu\nu}$ has ten independent components, while a massive spin 2 particle has five DOF. In the flat background, the extra DOF are actually eliminated from the EOM as the Proca field. In fact, the divergence, double divergence, and trace of (2.8) respectively give

$$-m^2(\partial_\nu h^{\mu\nu} - \partial^\mu h) = 0\,, \tag{2.9}$$
$$-m^2(\partial_\mu\partial_\nu h^{\mu\nu} - \Box h) = 0\,, \tag{2.10}$$
$$2(\partial_\mu\partial_\nu h^{\mu\nu} - \Box h) + 3m^2 h = 0\,. \tag{2.11}$$

Thus, when $m \neq 0$, we obtain the traceless condition, $h = 0$, from (2.10) and (2.11). Then, substituting it to (2.9), we get the transverse condition, $\partial_\nu h^{\mu\nu} = 0$. Consequently, $h_{\mu\nu}$ is a rank-2 traceless symmetric, divergence-free tensor, which has five independent components. Note that the EOM (2.8) are then reduced to the Klein Gordon equations:

$$(\Box - m^2)h^{\mu\nu} = 0\,. \tag{2.12}$$

We thus see that the reduction mechanism works for a massive spin 2 field as long as the background is flat.

Next we show the breakdown of the reduction mechanism when the flat theory is naïvely lifted to curved backgrounds. A natural extension of (2.5) is obtained (a) by replacing the derivatives (2.5) with covariant derivatives, or (b) by substituting $\hat{g}_{\mu\nu} = g_{\mu\nu} + 2h_{\mu\nu}$ to (2.7) and taking only quadratic terms in $h_{\mu\nu}$. The discrepancy between (a) and (b) appears as the difference of non-minimal couplings (e.g., the difference of the coefficient of $R\,h^{\mu\nu}h_{\mu\nu}$). In this section we adopt the prescription (b).

The Lagrangian now takes the form

$$\mathcal{L} = \sqrt{-g}\Big[h_{\mu\nu}\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma} - \frac{m^2}{2}(h_{\mu\nu}h^{\mu\nu} - h^2)\Big]\,. \tag{2.13}$$

Here, $h = g^{\mu\nu}h_{\mu\nu}$, and $\mathcal{E}^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator acting on symmetric tensors in a curved spacetime:

$$\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma} = \frac{1}{2}(\Box h^{\mu\nu} - g^{\mu\nu}\Box h) + \frac{1}{2}(\nabla^\mu\nabla^\nu h + g^{\mu\nu}\nabla^\rho\nabla^\sigma h_{\rho\sigma}) - \nabla^{(\mu}\nabla_\lambda h^{\nu)\lambda}$$
$$+ R^{\mu\rho\nu\sigma}h_{\rho\sigma} + R^{\rho(\mu}h_\rho^{\nu)} - \frac{1}{2}(g^{\mu\nu}R^{\rho\sigma}h_{\rho\sigma} + R^{\mu\nu}h) - \frac{1}{2}Rh^{\mu\nu} + \frac{1}{4}Rg^{\mu\nu}h\,, \tag{2.14}$$

which reduces to (2.6) in the flat limit and enjoys the following properties:

$$\frac{1}{2}\sqrt{-\hat{g}}\,\hat{R} = \sqrt{-g}\left[\frac{1}{2}R - G^{\mu\nu}h_{\mu\nu} + h_{\mu\nu}\,\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma} + O(h^3)\right] \quad \left(\hat{g}_{\mu\nu} = g_{\mu\nu} + 2h_{\mu\nu}\right), \quad (2.15)$$

$$\nabla_\nu\left(\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma}\right) = \frac{1}{2}G^{\rho\sigma}\left(2\nabla_\rho h_\sigma^\mu - \nabla^\mu h_{\rho\sigma}\right), \tag{2.16}$$

$$g_{\mu\nu}\left(\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma}\right) = \nabla_\mu\nabla_\nu h^{\mu\nu} - \Box\,h, \tag{2.17}$$

where $G^{\mu\nu} = R^{\mu\nu} - (R/2)\,g^{\mu\nu}$ is the Einstein tensor. The EOM are given by

$$2\mathcal{E}^{\mu\nu\rho\sigma}h_{\rho\sigma} - m^2(h^{\mu\nu} - g^{\mu\nu}h) = 0. \tag{2.18}$$

The divergence, double divergence, and trace of (2.18) respectively give

$$G^{\rho\sigma}\left(2\nabla_\rho h_\sigma^\mu - \nabla^\mu h_{\rho\sigma}\right) - m^2(\nabla_\nu h^{\mu\nu} - \nabla^\mu h) = 0, \tag{2.19}$$

$$\nabla_\mu\left[G^{\rho\sigma}\left(2\nabla_\rho h_\sigma^\mu - \nabla^\mu h_{\rho\sigma}\right)\right] - m^2(\nabla_\mu\nabla_\nu h^{\mu\nu} - \Box\,h) = 0, \tag{2.20}$$

$$2(\nabla_\mu\nabla_\nu h^{\mu\nu} - \Box\,h) + 3m^2 h = 0. \tag{2.21}$$

Thus, if h vanished or at least could be expressed as a function of the traceless part of $h_{\mu\nu}$, (2.19) would give four constraints on the transverse component. However, (2.20) and (2.21) lead to

$$h = -\frac{2}{3m^4}\nabla_\mu\left[G^{\rho\sigma}\left(2\nabla_\rho h_\sigma^\mu - \nabla^\mu h_{\rho\sigma}\right)\right]. \tag{2.22}$$

This is, except for the vacuum case ($G_{\mu\nu} = 0$), a second-order differential equation for the trace h and the traceless part of $h_{\mu\nu}$, which cannot be regarded as a constraint eliminating unnecessary DOF. The situations are the same also for the cases of other spins, except for spin 1 (Proca field). In the spin 1 case, the divergence of the EOM always results in a first-order differential equation corresponding to the transverse condition, irrespective of how non-minimal couplings are introduced. For the case of higher spins, however, there is no choice of non-minimal couplings so as to cancel the RHS of (2.22). Another problem will emerge when formally substituting (2.22) to (2.18), since it results in fourth-order differential equations with respect to time. It is a singular perturbation, and yields an exponential growth of the amplitudes because the perturbation becomes much larger than the original kinetic term at short time scales. These facts seem to indicate that the Lagrangian above fails to describe a consistent FP field in a general background. In the following, we resolve this issue by giving up the attempt to express the constraint in a form that is directly related to the transverse condition and also by paying the cost of breaking the manifest covariance in the analysis.

3 Fierz-Pauli field in general curved backgrounds

In this section, we construct a consistent, linear field theory of a massive spin 2 field in a general curved spacetime.

(X, P_X, D, P_D, F) in our setup. We first discuss the subsystem (G_{ijk}, P^{ijk}, V_i). The primary and secondary constraints with respect to V_i are found to be

$$\kappa_1^i \equiv 3m^2 \delta_{jk} G^{ijk} = 0, \tag{5.6}$$

$$\kappa_2^i \equiv -\frac{3}{4} m^2 \delta_{jk} P^{ijk} = 0, \tag{5.7}$$

which have a nonvanishing Poisson bracket, $\{\kappa_1^i, \kappa_2^i\} = (15/4)m^4 \neq 0$. Thus, the multipliers V_i remove the DOF of the trace part of G_{ijk} and P^{ijk}, and V_i itself is determined by the equation $\dot{\kappa}_2^i = 0$. As for the subsystem (X, P_X, D, P_D, F), the multiplier F yields four constraints (primary, secondary, tertiary and quaternary), which are expressed as

$$\chi_1 \equiv 2m P_D + 2m^2 X = 0, \tag{5.8}$$

$$\chi_2 \equiv 4m^2 P_X + 4m^3 D = 0, \tag{5.9}$$

$$\chi_3 \equiv -12m^3 P_D - 2m^4 X = 0, \tag{5.10}$$

$$\chi_4 \equiv -4m^4 P_X - 24m^5 D = 0. \tag{5.11}$$

Their Poisson brackets take the form $\{\chi_1, \chi_2\} = \{\chi_1, \chi_3\} = 0$, $\{\chi_1, \chi_4\} = -40m^6 \neq 0$, and $\det\{\chi_a, \chi_b\} \neq 0$ $(a, b = 1, \ldots, 4)$. Thus, the multiplier F removes the DOF of (X, P_X, D, P_D), and F itself is determined by the equation $\dot{\chi}_4 = 0$.

We now require that the same mechanism also work for the background (5.4). One can easily show that the quadratic terms in V_i and F are given by

$$\mathcal{H}\big|_{V_i,F}^{\text{(quad)}} = \sqrt{g}\left[\left(-\frac{3}{4}g^{ij}\ddot{g}_{ij} - \frac{3}{8}\dot{g}^{ij}\dot{g}_{ij}\right)V_k V^k + \left(-\frac{3}{2}g^{ik}\ddot{g}_{kj} - \frac{3}{4}\dot{g}^{ik}\dot{g}_{kj}\right)V_i V^j \right.$$
$$\left. + \left(\frac{31}{6}g^{ij}\ddot{g}_{ij} + \frac{31}{12}\dot{g}^{ij}\dot{g}_{ij}\right)F^2\right] - \mathcal{L}_R\big|_{V_i,F}^{\text{(quad)}} \tag{5.12}$$

with

$$\mathcal{L}_R\big|_{V_i,F}^{\text{(quad)}}/\sqrt{g}$$
$$= \left[\frac{1}{2}(b_1 + b_2 + 3c_1 + c_2)g^{ij}\ddot{g}_{ij} + \frac{1}{8}(2b_1 + 2b_2 + 9c_1 + 3c_2)\dot{g}^{ij}\dot{g}_{ij} + \frac{1}{8}(3c_1 + c_2)(g^{ij}\dot{g}_{ij})^2\right]V_k V^k$$
$$+ \left[\frac{1}{4}(-2a + b_1 + b_3)g^{ik}\ddot{g}_{kj} + \frac{1}{4}(-a + b_1 + b_3)\dot{g}^{ik}\dot{g}_{kj} + \frac{1}{8}(b_1 + b_3)g^{kl}\dot{g}_{kl}g^{im}\dot{g}_{mj}\right]V_i V^j$$
$$+ \left[\left(\frac{5}{9}a - \frac{43}{18}b_1 - \frac{8}{3}b_2 - b_3 - 5c_1 - 2c_2\right)g^{ij}\ddot{g}_{ij}\right.$$
$$+ \left(\frac{19}{72}a - \frac{11}{9}b_1 - \frac{7}{6}b_2 - \frac{1}{2}b_3 - \frac{15}{4}c_1 - \frac{3}{2}c_2\right)\dot{g}^{ij}\dot{g}_{ij}$$
$$\left. + \left(-\frac{a}{72} - \frac{b_1}{36} + \frac{1}{6}b_2 - \frac{5}{4}c_1 - \frac{1}{2}c_2\right)(g^{ij}\dot{g}_{ij})^2\right]F^2. \tag{5.13}$$

These quadratic terms must vanish in order for the V_i and F to give four primary con-

The variables to describe a massive spin 3 field consist of a traceful, rank-3 symmetric tensor $G_{\mu\nu\lambda}$ and an auxiliary scalar D. Denoting the trace of $G_{\mu\nu\lambda}$ by $G_\mu \equiv g^{\nu\lambda}G_{\mu\nu\lambda}$, the Lagrangian can be written in the form

$$\mathcal{L} = \mathcal{L}_{\min} + \mathcal{L}_R \tag{5.1}$$

with

$$\begin{aligned}
\mathcal{L}_{\min} = \sqrt{-g}\Big[&-\frac{1}{2}\nabla_\mu G_{\nu\lambda\rho}\nabla^\mu G^{\nu\lambda\rho} + \frac{3}{2}\nabla^\alpha G_{\alpha\mu\nu}\nabla_\beta G^{\beta\mu\nu} - 3\nabla^\mu G_{\mu\nu\lambda}\nabla^\nu G^\lambda \\
&+ \frac{3}{2}\nabla_\mu G_\nu \nabla^\mu G^\nu + \frac{3}{4}\nabla^\mu G_\mu \nabla^\nu G_\nu + \frac{1}{4}\partial_\mu D \nabla^\mu D \\
&- \frac{m^2}{2}\big(G_{\mu\nu\lambda}G^{\mu\nu\lambda} - 3G_\mu G^\mu\big) + m^2 D^2 - \frac{m}{2}\nabla^\mu G_\mu D\Big],
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
\mathcal{L}_R = \sqrt{-g}\Big[&\frac{a}{2} R_{\mu\nu\lambda\rho} G^{\mu\lambda\alpha} G^{\nu\rho}{}_\alpha + \frac{b_1}{2} R_{\mu\nu} G^{\mu\alpha\beta} G^\nu{}_{\alpha\beta} + b_2 R_{\mu\nu} G^{\mu\nu\alpha} G_\alpha + \frac{b_3}{2} R_{\mu\nu} G^\mu G^\nu \\
&+ \frac{c_1}{2} R G_{\mu\nu\lambda} G^{\mu\nu\lambda} + \frac{c_2}{2} R G_\mu G^\mu + \frac{c_3}{2} R D^2\Big].
\end{aligned} \tag{5.3}$$

We will set the background metric to take the form

$$ds^2 = -dt^2 + g_{ij}(t)\,dx^i dx^j \tag{5.4}$$

and assume that all the fields depend only on time t. This setup greatly reduces the amount of necessary calculation, and, as we have observed in the preceding sections, should be sufficient for investigating how the DOF are removed due to constraints.

The coefficients in (5.2) are determined such that only the spatial, traceless part of the tensor $G_{\mu\nu\lambda}$ is dynamical in the flat Minkowski space. To confirm this, it is convenient to introduce the following parametrization for the temporal components of $G_{\mu\nu\lambda}$ in the background metric (5.4):

$$G_{000} = X + 3F, \quad G_{00i} = V_i, \quad G_{0ij} = \tilde{G}_{0ij} + \frac{1}{3}g_{ij}F, \tag{5.5}$$

where F is the trace of G_{0ij}, $F = g^{jk}G_{0jk}$, and $\tilde{G}_{0ij}$ is the traceless part of G_{0ij}. One can easily show that $\tilde{G}_{0ij}$ have a nonvanishing quadratic mass term and no kinetic terms, which means that $\tilde{G}_{0ij}$ can be removed from the Lagrangian algebraically (and thus are not dynamical variables). It is also easy to see for the case of flat Minkowski space, that the Legendre transformation from $\dot{G}_{ijk}$, $\dot{X}$, $\dot{D}$ to their conjugate momenta P^{ijk}, P_X, P_D yields only the linear terms for V_i and F, which means that V_i and F play the role of multiplier fields.

In the flat Minkowski case, the multipliers V_i and F actually yield the constraints that remove all the DOF except for the spatial, traceless part of the tensor $G_{\mu\nu\lambda}$. To see this, we note that the dynamics of (G_{ijk}, P^{ijk}, V_i) is totally decoupled from that of

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We start with the Lagrangian (2.13) with non-minimal couplings to the curvature:

$$S = \int d^4x \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_{\mathcal{E}} + \mathcal{L}_m + \mathcal{L}_R \tag{3.1}$$

with

$$\mathcal{L}_{\mathcal{E}} = \sqrt{-g}\, h_{\mu\nu} \mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma}, \quad \mathcal{L}_m = -\sqrt{-g}\, \frac{m^2}{2} (h_{\mu\nu} h^{\mu\nu} - h^2), \tag{3.2}$$

$$\mathcal{L}_R = \sqrt{-g}\left[\frac{a_1}{2} R_{\mu\nu\rho\sigma} h^{\mu\rho} h^{\nu\sigma} + \frac{a_2}{2} R_{\mu\nu} h^{\mu\rho} h^{\nu}_{\rho} + \frac{a_3}{2} R h_{\mu\nu} h^{\mu\nu} + \frac{b_1}{2} R h^2 + b_2 R_{\mu\nu} h^{\mu\nu} h \right]. \tag{3.3}$$

Here $\mathcal{L}_R$ expresses the non-minimal couplings, and the coupling constants a_1, a_2, a_3, b_1, b_2 cannot be determined *a priori* only by requiring the action to become the FP action in the flat limit. Note that such terms also exist in $\mathcal{L}_{\mathcal{E}}$. In the remaining of this section, we show that the action (3.1)–(3.3) describe a massive spin 2 field with correct DOF if and only if the constants in $\mathcal{L}_R$ satisfy the two conditions[7]

$$a_2 + 2b_2 = -1, \tag{3.4}$$

$$a_3 + b_1 = \frac{1}{2}. \tag{3.5}$$

The counting of DOF is usually easiest in the Hamiltonian formalism, and for this purpose we introduce the ADM decomposition of the metric:

$$(\hat{g}_{\mu\nu}) = \begin{pmatrix} -\hat{N}^2 + \hat{g}_{ij}\hat{N}^i\hat{N}^j & \hat{g}_{ij}\hat{N}^i \\ \hat{g}_{ij}\hat{N}^j & \hat{g}_{ij} \end{pmatrix}. \tag{3.6}$$

The functions $\hat{N}$ and $\vec{\hat{N}} = (\hat{N}^i)$ $(i = 1, 2, 3)$ are called the lapse and the shift, respectively, and $\hat{g}_{ij}$ describes the induced metric on a timeslice. The Einstein-Hilbert action then takes the following form up to surface integrals:

$$S_{EH} = \int d^4x \frac{1}{2}\hat{N}\sqrt{\hat{g}}\left[{}^{(3)}\hat{R} + \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2 \right] \quad (\hat{K} \equiv \hat{g}^{ij}\hat{K}_{ij}). \tag{3.7}$$

Here, ${}^{(3)}\hat{R}$ is the Ricci scalar associated with $\hat{g}_{ij}$, and $\hat{K}_{ij} \equiv (1/2\hat{N})\left[\dot{\hat{g}}_{ij} - \delta_{\vec{\hat{N}}}\hat{g}_{ij}\right]$ is the extrinsic curvature of the timeslice ($\delta_{\vec{\hat{N}}}$ is the Lie derivative with respect to the shift $\vec{\hat{N}}$). We now expand the action around a classical background metric. By using the diffeomorphism invariance of the Einstein-Hilbert action, we can set the background to the following form without loss of generality:

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 \\ 0 & g_{ij} \end{pmatrix}. \tag{3.8}$$

---

[7]The relations were first obtained in [19].

We then replace the metric in the action as

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \mathsf{h}_{\mu\nu} \quad (\mathsf{h}_{\mu\nu} \equiv 2h_{\mu\nu}), \tag{3.9}$$

or equivalently, rewrite the lapse and shifts in (3.7) as

$$\hat{N}^2 = 1 - \mathsf{h}_{00} + \hat{g}^{ij}\mathsf{h}_{0i}\mathsf{h}_{0j}, \tag{3.10}$$

$$\hat{g}_{ij}\hat{N}^j = \mathsf{h}_{0i}, \tag{3.11}$$

$$\hat{g}_{ij} = g_{ij} + \mathsf{h}_{ij}. \tag{3.12}$$

The quadratic terms in $\mathsf{h}_{\mu\nu}$ give $\mathcal{L}_{\mathcal{E}}$, whose explicit form is given by

$$
\begin{aligned}
\mathcal{L}_{\mathcal{E}} &= \left[\frac{1}{2}\hat{N}\sqrt{\hat{g}}\left[{}^{(3)}\hat{R} + \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2 \right] \right]_{(2)} \\
&= \left[\frac{\hat{N}\sqrt{\hat{g}}}{2}{}^{(3)}\hat{R} + \frac{1}{2}\hat{C}^{ijkl}\left(\dot{\hat{g}}_{ij} - \delta_{\vec{\hat{N}}}\hat{g}_{ij}\right)\left(\dot{\hat{g}}_{kl} - \delta_{\vec{\hat{N}}}\hat{g}_{kl}\right) \right]_{(2)} \\
&= \frac{1}{2}\hat{C}^{ijkl}_{(0)}\dot{\mathsf{h}}_{ij}\dot{\mathsf{h}}_{kl} + \hat{C}^{ijkl}_{(1)}\dot{\mathsf{h}}_{ij}\dot{g}_{kl} - \hat{C}^{ijkl}_{(0)}\dot{\mathsf{h}}_{ij}(\delta_{\vec{\hat{N}}}g_{kl})_{(1)} \\
&\quad + \left[\frac{\hat{N}\sqrt{\hat{g}}}{2}{}^{(3)}\hat{R} + \frac{1}{2}\hat{C}^{ijkl}\left(\dot{g}_{ij} - \delta_{\vec{\hat{N}}}g_{ij} - \delta_{\vec{\hat{N}}}\mathsf{h}_{ij}\right)\left(\dot{g}_{kl} - \delta_{\vec{\hat{N}}}g_{kl} - \delta_{\vec{\hat{N}}}\mathsf{h}_{kl}\right) \right]_{(2)}, \tag{3.13}
\end{aligned}
$$

where

$$\hat{C}^{ijkl} \equiv \frac{\sqrt{\hat{g}}}{4\hat{N}}\left[\frac{1}{2}\left(\hat{g}^{ik}\hat{g}^{jl} + \hat{g}^{il}\hat{g}^{jk}\right) - \hat{g}^{ij}\hat{g}^{kl} \right], \tag{3.14}$$

and a subscript in parenthesis denotes the order in $\mathsf{h}_{\mu\nu}$.

We now move on to the Hamiltonian formalism by making the Legendre transformation with respect to $\dot{\mathsf{h}}_{ij}$. Since $\dot{\mathsf{h}}_{ij}$ is contained only in $\mathcal{L}_{\mathcal{E}}$, the conjugate variable to h_{ij} is given by

$$
\begin{aligned}
\pi^{ij} &\equiv \frac{\partial\mathcal{L}}{\partial\dot{\mathsf{h}}_{ij}} = \frac{\partial\mathcal{L}_{\mathcal{E}}}{\partial\dot{\mathsf{h}}_{ij}} \\
&= \hat{C}^{ijkl}_{(0)}\dot{\mathsf{h}}_{kl} + \hat{C}^{ijkl}_{(1)}\dot{g}_{kl} - \hat{C}^{ijkl}_{(0)}(\delta_{\vec{\hat{N}}}g_{kl})_{(1)}, \tag{3.15}
\end{aligned}
$$

which can be solved for $\dot{\mathsf{h}}_{ij}$ as

$$\dot{\mathsf{h}}_{ij} = (\hat{C}^{-1}_{(0)})_{ijkl}\left(\pi^{kl} - \hat{C}^{klmn}_{(1)}\dot{g}_{mn} + \hat{C}^{klmn}_{(0)}(\delta_{\vec{\hat{N}}}g_{mn})_{(1)} \right). \tag{3.16}$$

The Hamiltonian is then obtained as

$$
\begin{aligned}
\mathcal{H} &= \pi^{ij}\dot{\mathsf{h}}_{ij} - \mathcal{L}_{\mathcal{E}} - \mathcal{L}_m - \mathcal{L}_R \\
&= \frac{1}{2}(\hat{C}^{-1}_{(0)})_{ijkl}\left(\pi^{ij} - \hat{C}^{ijmn}_{(1)}\dot{g}_{mn} + \hat{C}^{ijmn}_{(0)}(\delta_{\vec{\hat{N}}}g_{mn})_{(1)} \right)\left(\pi^{kl} - \hat{C}^{klpq}_{(1)}\dot{g}_{pq} + \hat{C}^{klpq}_{(0)}(\delta_{\vec{\hat{N}}}g_{pq})_{(1)} \right) \\
&\quad - \left[\frac{\hat{N}\sqrt{\hat{g}}}{2}{}^{(3)}\hat{R} + \frac{1}{2}\hat{C}^{ijkl}(\dot{g}_{ij} - \delta_{\vec{\hat{N}}}g_{ij} - \delta_{\vec{\hat{N}}}\mathsf{h}_{ij})(\dot{g}_{kl} - \delta_{\vec{\hat{N}}}g_{kl} - \delta_{\vec{\hat{N}}}\mathsf{h}_{kl}) \right]_{(2)} - \mathcal{L}_m - \mathcal{L}_R. \\
&\tag{3.17}
\end{aligned}
$$

where the parameters are related with those in the previous section, (3.3), as

$$\tilde{a}_1 = a_1 + 2, \quad \tilde{a}_2 = a_2 + 2, \quad \tilde{a}_3 = a_3 - 1,$$

$$\tilde{b}_1 = b_1 + \frac{1}{2}, \quad \tilde{b}_2 = b_2 - 1. \tag{4.8}$$

By substituting (4.6) to (4.7), the coefficients in (4.1) are expressed as

$$C^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - g^{ij}g^{kl}, \tag{4.9}$$

$$(C^{-1})_{ijkl} = \frac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk}) - \frac{1}{2}g_{ij}g_{kl}, \tag{4.10}$$

$$D^{ij} = \frac{1}{2}(g^{ik}\Gamma^j_{k0} + g^{jk}\Gamma^i_{k0}) - g^{ij}g^{kl}\Gamma^0_{kl}, \tag{4.11}$$

$$F^{ijk}h_{0k} = 2(g^{ij}g^{kl} - g^{ik}g^{jl})\partial_k h_{0l} + O(\Gamma^2), \tag{4.12}$$

$$H^i h_{0i} = \dot{g}^{ij}\partial_i h_{0j} + g^{ij}g^{kl}\dot{g}_{kl}\partial_i h_{0j} + O(\Gamma^2) \tag{4.13}$$

$$J = 2g^{ij}g^{kl}(\Gamma^0_{ik}\Gamma^0_{jl} - \Gamma^0_{ij}\Gamma^0_{kl}) + \frac{1}{2}g^{ij}\dot{g}_{ij}g^{kl}\Gamma^0_{kl}$$

$$+ \dot{g}^{ij}\Gamma^0_{ij} + g^{ij}\dot{\Gamma}^0_{ij} + 2\tilde{\alpha}R_{00} + 2\tilde{\beta}R, \tag{4.14}$$

where

$$\tilde{\alpha} = -\left(\frac{\tilde{a}_2}{2} + \tilde{b}_2\right) = -\left(\frac{a_2}{2} + b_2\right), \tag{4.15}$$

$$\tilde{\beta} = \frac{\tilde{a}_3}{2} + \frac{\tilde{b}_1}{2} = \frac{a_3}{2} + \frac{b_1}{2} - \frac{1}{4}. \tag{4.16}$$

One can easily check that the condition (4.4) is automatically satisfied (up to higher-order terms). On the other hand, the LHS of (4.3) can be rewritten to the form

$$J - DC^{-1}D = \frac{1}{2}(1 - 2\tilde{\alpha} + 4\tilde{\beta})g^{ij}\ddot{g}_{ij} + \frac{1}{4}(1 - 2\tilde{\alpha} + 6\tilde{\beta})\dot{g}^{ij}\dot{g}_{ij} + \frac{1}{2}\tilde{\beta}(g^{ij}\dot{g}_{ij})^2 + 2\tilde{\beta}\,^{(3)}R, \tag{4.17}$$

which vanishes only when $\tilde{\alpha} = 1/2$ and $\tilde{\beta} = 0$, i.e.,

$$\tilde{a}_2 + 2\tilde{b}_2 = -1 \quad \rightarrow \quad a_2 + 2b_2 = -1, \tag{4.18}$$

$$\tilde{a}_3 + \tilde{b}_1 = 0 \quad \rightarrow \quad a_3 + b_1 = \frac{1}{2}. \tag{4.19}$$

We thus have reproduced the conditions (3.4) and (3.5) without using the ADM formalism. The procedure in this section is a simpler algorithm, and might have some application to the analysis of higher spin theories.

5 Spin 3 case

In this section we discuss a massive spin 3 theory in the general background.

form $\dot{h}_{00}\dot{h}_{ij}$. Completing the square with respect to $\dot{h}_{ij}$ leads to

$$\mathcal{L} = \sqrt{g}\left[\frac{1}{2}C^{ijkl}\left(\dot{h}_{ij} + (C^{-1})_{ijmn}(D^{mn}h_{00} + F^{mnp}h_{0p})\right)\left(\dot{h}_{kl} + (C^{-1})_{klqr}(D^{qr}h_{00} + F^{qrs}h_{0s})\right)\right.$$

$$+ \frac{1}{2}M^{ijkl}h_{ij}h_{kl} + E^{ij}h_{ij}h_{00} + \frac{1}{2}J(h_{00})^2 + G^{ijk}h_{ij}h_{0k} + H^i h_{0i}h_{00} + \frac{1}{2}I^{ij}h_{0i}h_{0j}$$

$$\left. - \frac{1}{2}(C^{-1})_{ijkl}\left(D^{ij}h_{00} + F^{ijm}h_{0m}\right)\left(D^{kl}h_{00} + F^{kln}h_{0n}\right)\right]. \tag{4.2}$$

The condition for this Lagrangian to give the proper constraints is, as discussed in the previous section, that the terms of the form h_{00}^2 or $h_{00}h_{0i}$ do not survive after the Legendre transformation is made with respect to $\dot{h}_{ij}$. This is translated in the Lagrangian formalism as the condition that the second and third lines of (4.2) do not give terms of the form h_{00}^2 or $h_{00}h_{0i}$. This condition can be written as

$$J - DC^{-1}D = 0, \tag{4.3}$$

$$H^i - (DC^{-1}F)^i = 0. \tag{4.4}$$

In the following, we directly compute the LHS of (4.3) and (4.4), and show that (4.4) is always satisfied but (4.3) requires the conditions (3.4) and (3.5).[8]

With the metric (3.8), the connections are given by

$$\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^i_{00} = 0, \quad \Gamma^0_{ij} = \frac{1}{2}\dot{g}_{ij}, \quad \Gamma^i_{0j} = \frac{1}{2}g^{ik}\dot{g}_{kj}, \tag{4.5}$$

and Γ^i_{jk} agrees with the connection associated with g_{ij}. Accordingly, the covariant derivatives take the forms

$$\nabla_0 h_{00} = \dot{h}_{00},$$
$$\nabla_i h_{00} = \partial_i h_{00} - 2\Gamma^j_{i0}h_{0j},$$
$$\nabla_0 h_{0i} = \dot{h}_{0i} - \Gamma^j_{0i}h_{0j},$$
$$\nabla_j h_{0i} = \partial_j h_{0i} - \Gamma^k_{j0}h_{ki} - \Gamma^0_{ji}h_{00} - \Gamma^k_{ji}h_{0k},$$
$$\nabla_0 h_{ij} = \dot{h}_{ij} - \Gamma^k_{0i}h_{kj} - \Gamma^k_{0j}h_{ki},$$
$$\nabla_k h_{ij} = \partial_k h_{ij} - \Gamma^0_{ki}h_{0j} - \Gamma^0_{kj}h_{0i} - \Gamma^l_{ki}h_{lj} - \Gamma^l_{kj}h_{li}. \tag{4.6}$$

We now write the FP Lagrangian with non-minimal couplings in the following form:

$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{2}\nabla_\lambda h_{\mu\nu}\nabla^\lambda h^{\mu\nu} + \nabla^\mu h_{\mu\nu}\nabla_\lambda h^{\lambda\nu} - \nabla^\mu h_{\mu\nu}\nabla^\nu h + \frac{1}{2}\nabla_\mu h\nabla^\mu h\right.$$

$$- \frac{m^2}{2}(h_{\mu\nu}h^{\mu\nu} - h^2)$$

$$\left. + \frac{\tilde{a}_1}{2}R_{\mu\rho\nu\sigma}h^{\mu\nu}h^{\rho\sigma} + \frac{\tilde{a}_2}{2}R^\mu_{\ \lambda}h_{\mu\nu}h^{\lambda\nu} + \frac{\tilde{a}_3}{2}Rh_{\mu\nu}h^{\mu\nu} + \frac{\tilde{b}_1}{2}Rh^2 + \tilde{b}_2 R_{\mu\nu}h^{\mu\nu}h\right], \tag{4.7}$$

---

[8]After the first manuscript was accepted for publication, we found that a similar analysis was made in [19].

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Since h_{0i} is generically quadratic and has no kinetic terms, the corresponding DOF will drop out from the system by solving the EOM for h_{0i} and by substituting the obtained solution to the action. Then, if the resulting Hamiltonian has only linear terms in h_{00}, there will arise the primary constraint, from which will follow the secondary constraint as a condition for the primary constraint to be consistent under the time evolution. Furthermore, a further consistency condition will arise for the secondary constraint, which in turn will determine the form of h_{00}. Thus, if the Hamiltonian has only linear terms in h_{00} after the elimination of h_{0i}, the variables h_{00} and h_{0i} will disappear from the system, leaving two constraints. This means that the system has ten $(= 6 + 6 - 2)$ DOF, which agree with those of a massive spin 2 field. We are going to show that this is the case if and only if the conditions (3.4) and (3.5) are met.

There are actually two sources of h_{00}^2 terms. One is the h_{00}^2 terms that already exist in the Hamiltonian before solving the EOM for h_{0i}. The other is the h_{00}^2 terms that come out after h_{0i} is eliminated from the Hamiltonian.

First we point out that the latter source is absent, noticing that the mass term $\mathcal{L}_m$,

$$\mathcal{L}_m = -\sqrt{-g}\,\frac{m^2}{8}\left[-2g^{ij}h_{0i}h_{0j} + g^{ik}g^{jl}h_{ij}h_{kl} + 2h_{00}g^{ij}h_{ij} - (g^{ij}h_{ij})^2\right], \qquad (3.18)$$

contains quadratic terms in h_{0i} when $m \neq 0$. If the Lagrangian contains the terms of the form $h_{00}h_{0i}$, the EOM for h_{0i} take the form $h_{0i} = h_{00} \times A_{0i} + \cdots$ and give h_{00}^2 terms when substituted back to the Lagrangian. However, as we will see below, there are no such terms in the Lagrangian. Since there are no $h_{00}h_{0i}$ terms in $\mathcal{L}_m$, we only need to confirm the absence of such terms in the rest of the Hamiltonian (3.17). As for $\mathcal{L}_R$, we see that $a_2 R_{\mu\nu}h^{\mu\rho}h^{\nu}_{\rho}$ and $b_2 R_{\mu\nu}h^{\mu\nu}h$ actually give dangerous terms $-a_2 R_{0i}h_{00}h^{i}_{0}$ and $-2b_2 R_{0i}h^{i}_{0}h_{00}$. However, they can be ignored in our present approximation, because their contributions to the coefficients of h_{00}^2 will be $O(R^2/m^2)$ and can be neglected to the first order in the curvature. As for the remaining part of (3.17), we see from (3.10)–(3.12) that terms linear in h_{0i} appear only through $\delta_{\vec{N}}g_{ij}$. Thus, the possible terms containing $h_{00}h_{0i}$ are

$$-(\hat{C}^{-1}_{(0)})_{ijkl}\hat{C}^{ijmn}_{(1)}\dot{g}_{mn}\hat{C}^{klpq}_{(0)}(\delta_{\vec{N}}g_{pq})_{(1)} - \left(-\hat{C}^{ijkl}\dot{g}_{ij}\delta_{\vec{N}}g_{kl}\right)_{(2)}. \qquad (3.19)$$

However, the $h_{00}\,h_{0i}$ terms cancel out in (3.19), because it can be rewritten as

$$
\begin{aligned}
&- \hat{C}^{ijmn}_{(1)}\dot{g}_{mn}(\delta_{\vec{N}}g_{ij})_{(1)} + \hat{C}^{ijkl}_{(0)}\dot{g}_{ij}\left(\delta_{\vec{N}}g_{kl}\right)_{(2)} + \hat{C}^{ijkl}_{(1)}\dot{g}_{ij}\left(\delta_{\vec{N}}g_{kl}\right)_{(1)} + \hat{C}^{ijkl}_{(2)}\dot{g}_{ij}\left(\delta_{\vec{N}}g_{kl}\right)_{(0)} \\
&= \hat{C}^{ijkl}_{(0)}\dot{g}_{ij}\left(\delta_{\vec{N}}g_{kl}\right)_{(2)} + \hat{C}^{ijkl}_{(2)}\dot{g}_{ij}\left(\delta_{\vec{N}}g_{kl}\right)_{(0)},
\end{aligned} \qquad (3.20)
$$

which does not contain $h_{00}\,h_{0i}$.

We thus find that h_{0i} do not play any role in investigating the possible appearance of h_{00}^2 terms, so that we can safely set $h_{0i} = 0$ for further arguments. Since h_{00}^2 terms can appear

only through $\hat{N}$ in $\hat{C}^{ijkl}$, we only need to look at the h_{00}^2 terms in the reduced Hamiltonian

$$\mathcal{H} \sim \frac{1}{2}\left[(\hat{C}_{(0)}^{-1})_{ijkl}\hat{C}_{(1)}^{ijmn}\hat{C}_{(1)}^{klpq}\dot{g}_{mn}\dot{g}_{pq} - \hat{C}_{(2)}^{ijkl}\dot{g}_{ij}\dot{g}_{kl}\right] - \left[\frac{\hat{N}\sqrt{g}}{2}\,^{(3)}R\right]_{(2)} - \mathcal{L}_m - \mathcal{L}_R. \tag{3.21}$$

Here, the symbol $\sim$ stands for an equality that holds when $\hat{N}^i$ and h_{ij} are set to 0. $\hat{C}^{ijkl}$ now takes the form

$$\begin{aligned}
\hat{C}^{ijkl} &= \hat{C}_{(0)}^{ijkl} + \hat{C}_{(1)}^{ijkl} + \hat{C}_{(2)}^{ijkl} + \cdots \\
&\sim \frac{\sqrt{g}}{4\hat{N}}\left[\frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - g^{ij}g^{kl}\right] \\
&= \frac{1}{4}\sqrt{g}\left(1 + \frac{1}{2}\mathsf{h}_{00} + \frac{3}{8}\mathsf{h}_{00}^2 + \cdots\right)\left[\frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - g^{ij}g^{kl}\right]
\end{aligned} \tag{3.22}$$

with

$$\hat{C}_{(0)}^{ijkl} \sim \frac{\sqrt{g}}{4}\left[\frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - g^{ij}g^{kl}\right], \tag{3.23}$$

$$\hat{C}_{(1)}^{ijkl} \sim \frac{1}{2}\,\mathsf{h}_{00}\,\hat{C}_{(0)}^{ijkl}, \tag{3.24}$$

$$\hat{C}_{(2)}^{ijkl} \sim \frac{3}{8}\,\mathsf{h}_{00}^2\,\hat{C}_{(0)}^{ijkl}. \tag{3.25}$$

Because $\mathcal{L}_m$ does not include h_{00}^2 terms, we thus get

$$\begin{aligned}
\mathcal{H} &\sim \frac{1}{2}\left(\frac{1}{4} - \frac{3}{8}\right)\mathsf{h}_{00}^2\,\hat{C}_{(0)}^{ijkl}\dot{g}_{ij}\dot{g}_{kl} + \frac{\sqrt{g}}{16}\,^{(3)}R\,\mathsf{h}_{00}^2 - \mathcal{L}_R \\
&\sim \frac{1}{64}\sqrt{g}\left[\dot{g}_{ij}\dot{g}^{ij} + (g^{ij}\dot{g}_{ij})^2\right]\mathsf{h}_{00}^2 + \frac{\sqrt{g}}{16}\,^{(3)}R\,\mathsf{h}_{00}^2 - \mathcal{L}_R.
\end{aligned} \tag{3.26}$$

Finally, we substitute $\mathsf{h}_{\mu\nu} = 2h_{\mu\nu}$:

$$\mathcal{H} \sim \frac{1}{16}\sqrt{g}\left[\dot{g}_{ij}\dot{g}^{ij} + (g^{ij}\dot{g}_{ij})^2\right]h_{00}^2 + \frac{\sqrt{g}}{4}\,^{(3)}R\,h_{00}^2 - \mathcal{L}_R. \tag{3.27}$$

From this expression, we see that appropriate curvature terms must be supplied by $\mathcal{L}_R$ in order for the h_{00}^2 terms to disappear. To see that this is actually possible, we write down the explicit form of $\mathcal{L}_R$ for the background metric (3.8). Necessary formulae are

$$R = {}^{(3)}R + g^{ij}\ddot{g}_{ij} + \frac{3}{4}\dot{g}_{ij}\dot{g}^{ij} + \frac{1}{4}(g^{ij}\dot{g}_{ij})^2, \tag{3.28}$$

$$R_{00} = -\frac{1}{2}g^{ij}\ddot{g}_{ij} - \frac{1}{4}\dot{g}_{ij}\dot{g}^{ij}, \tag{3.29}$$

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from which the h_{00}^2 terms involved in (3.3) are obtained as

$$\frac{a_1}{2} R_{\mu\nu\rho\sigma} h^{\mu\rho} h^{\nu\sigma} \sim 0 \,, \tag{3.30}$$

$$\frac{a_2}{2} R_{\mu\nu} h^{\mu\rho} h^\nu_\rho \sim -\frac{a_2}{2} R_{00} h_{00}^2 = \frac{a_2}{2} \left[\frac{1}{2} g^{ij} \ddot{g}_{ij} + \frac{1}{4} \dot{g}_{ij} \dot{g}^{ij} \right] h_{00}^2 \,, \tag{3.31}$$

$$\frac{a_3}{2} R h_{\mu\nu} h^{\mu\nu} \sim \frac{a_3}{2} R h_{00}^2 = \frac{a_3}{2} \left[{}^{(3)}R + g^{ij} \ddot{g}_{ij} + \frac{3}{4} \dot{g}_{ij} \dot{g}^{ij} + \frac{1}{4} (g^{ij} \dot{g}_{ij})^2 \right] h_{00}^2 \,, \tag{3.32}$$

$$\frac{b_1}{2} R h^2 \sim \frac{b_1}{2} R h_{00}^2 = \frac{b_1}{2} \left[{}^{(3)}R + g^{ij} \ddot{g}_{ij} + \frac{3}{4} \dot{g}_{ij} \dot{g}^{ij} + \frac{1}{4} (g^{ij} \dot{g}_{ij})^2 \right] h_{00}^2 \,, \tag{3.33}$$

$$b_2 R_{\mu\nu} h^{\mu\nu} h \sim -b_2 R_{00} h_{00}^2 = b_2 \left[\frac{1}{2} g^{ij} \ddot{g}_{ij} + \frac{1}{4} \dot{g}_{ij} \dot{g}^{ij} \right] h_{00}^2 \,. \tag{3.34}$$

The reduced Hamiltonian is then expressed as

$$\mathcal{H} = \frac{\sqrt{g}}{16} \Big\{ 4 \, {}^{(3)}R + \dot{g}_{ij} \dot{g}^{ij} + (g^{ij} \dot{g}_{ij})^2 - 2(a_2 + 2b_2)(2 g^{ij} \ddot{g}_{ij} + \dot{g}_{ij} \dot{g}^{ij})$$
$$- 2(a_3 + b_1) \big[4 \, {}^{(3)}R + 4 g^{ij} \ddot{g}_{ij} + 3 \dot{g}_{ij} \dot{g}^{ij} + (g^{ij} \dot{g}_{ij})^2 \big] \Big\} h_{00}^2 \,, \tag{3.35}$$

and we find that the necessary and sufficient conditions for the coefficients of four inde-pendent terms ${}^{(3)}R$, $g^{ij} \ddot{g}_{ij}$, $\dot{g}_{ij} \dot{g}^{ij}$ and $(g^{ij} \dot{g}_{ij})^2$ to disappear are given by the conditions (3.4) and (3.5). They are the conditions we promised to show in the beginning of this section so that the action (3.1)–(3.3) describes a massive spin 2 field with correct DOF in an arbitrary curved background.

4 Analysis based on the Lagrangian

In this section we reproduce the results in the previous section directly from the Lagrangian without resort to the ADM decomposition. We again set the background metric to the form (3.8) by using the diffeomorphism invariance. Then the FP Lagrangian can be written in the following form, by decomposing $h_{\mu\nu}$ and their covariant derivatives to the temporal and spatial components and by integrating by parts appropriately:

$$\mathcal{L} = \sqrt{g} \Big[\frac{1}{2} C^{ijkl} \dot{h}_{ij} \dot{h}_{kl} + \frac{1}{2} M^{ijkl} h_{ij} h_{kl} + D^{ij} \dot{h}_{ij} h_{00} + E^{ij} h_{ij} h_{00}$$
$$+ F^{ijk} \dot{h}_{ij} h_{0k} + G^{ijk} h_{ij} h_{0k} + H^i h_{0i} h_{00} + \frac{1}{2} I^{ij} h_{0i} h_{0j} + \frac{1}{2} J (h_{00})^2 \Big] \,. \tag{4.1}$$

Here, dots denote derivatives with respect to t. C^{ijkl} does not include curvatures or spatial-derivative operators. I^{ij} does not include spatial-derivative operators but may include cur-vatures (as well as m^2). Note that the FP kinetic term $\mathcal{L}_\mathcal{E}$ does not contain terms of the