

Transient Dynamics from Quantum to Classical

- From the Developed Coherent State via Extreme Squeezing -

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We explore the transient dynamics associated with the emergence of the classical signal in the full quantum system. We start our study from the instability which promotes the squeezing of the quantum system. This is often interpreted as the particle production though being reversible in time. We associate this state a non-dissipative classical fluctuations and study their trigger to develop the coherent state which can be classical if sufficiently developed. The Schwinger-Keldysh in-in formalism yields the classical Langevin equation including the fluctuation force which faithfully reflects the particle production property of the original quantum system. This formalism is applied to some transient process; the initiation of the spontaneous symmetry breaking, appearance of the off-diagonal long-range order in Bose-Einstein condensation, a transient process of the classicalization of the quantum fluctuations in the inflationary cosmology,... and gives some implications on the origin of the irreversibility associated with the transition from quantum to classical.

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I. INTRODUCTION

Quantum mechanics is no doubt a complete theory and correctly describes nature in the most fields in laboratory science. On the other hand, in some situations when the operational control doesn't work, quantum mechanics seems to be lacking any autonomous description. This applies to the problem of the origin of inhomogeneity in the early Universe, the black hole information loss problem, and so on. All of them seem to be related to the emergence, in quantum mechanics, of any classical signal which spontaneously violates the original symmetry of the system.

We pickup three typical examples.

i) In the early Universe, the standard cosmology indicates that the ultimate origin of the spatially non-uniform density fluctuations $\delta\rho(t, \mathbf{x})$ is the quantum fluctuations of the inflaton field $\hat{\phi}(t, \mathbf{x})$ in the inflationary stage when the Universe transforms from microscopic to macroscopic scales through the exponential expansion. There is no explicit observer nor detector in the early Universe and therefore quantum mechanics cannot yield any definite outcome of the quantum fluctuations. In order to solve this problem, some authors introduce detectors [1][2] or modify the basic laws of quantum mechanics so that the wave function spontaneously collapses [3]. This situation is similar to the quantum-information-theory aspect of the black hole [4]. If one tries to describe the black hole and the Hawking radiation as a unitary evolution, one encounters the violation of the equivalence principle[5].

ii) A precise description of the spontaneous symmetry breaking (SSB) introduces the two limiting operations, large volume and small external field limits, which are carefully ordered. However, this is an explicit violation of the symmetry that the violation direction is artificially determined from the beginning. On the other hand, in the Bose-Einstein condensation(BEC), the classical order parameter φ_0 describes the amount of condensation of boson gas which accumulates into the lowest energy state. This is also the indicator how extent the phase symmetry is broken spontaneously. In the ordinary argument, this classical value is simply assumed from the beginning as simply dividing the total quantum field $\hat{\Phi}(t, \mathbf{x}) = \hat{\phi}(t, \mathbf{x}) + \varphi_0$ [6]. However, the transient time development from 0 to φ_0 is actually needed to describe the initiation of BEC process.

iii) Furthermore, we still do not have a full successful model of the quantum measurement apparatus, in which a definite classical signal $m(t)$ emerges that has a firm correlation

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The appearance of the classicality is the key feature in many important transient processes in Physics. They are the dynamics of the general spontaneous symmetry breaking, the initiation process of the Bose-Einstein condensation, the transient process of the quantum measurement and the appearance of the classical signal in the apparatus, the transient process that the arrow of time appears in the macroscopic system, the transient process of the inflation in the early Universe where the quantum fluctuations form the classical seeds of galaxies, and so on.

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with the further measurement of the same quantum system[7]. If we adopt the standard operational prescription of quantum mechanics, the measurement can be fully described by POVM. However, from the fundamental level, the measurement apparatus should be described by quantum mechanics as well as the system. In this case, the classical signal can have many values before the completion of the measurement. However, after the measurement it must choose one value among such possible values. This process is not simply the SSB above but must further include the back reaction process to the system[8].

All the above examples have three main characteristics, beside the appearance of the macroscopic classical signal.

1) *state selection*: The system chooses one state among many other possible states. If this all states can be transformed with each other by a local symmetry operation, then this is exactly the process of the spontaneous symmetry breaking.

2) *classical statistical probability*: At this stage, deterministic evolution is terminated but an intrinsic probability appears which governs the fate of the system.

3) *arrow of time*: This process is intrinsically irreversible. The system can evolve into any one state among many leaving a classical signal. However, this signal cannot be canceled and the system cannot return to the original state which allows another possibility to develop.

In order to describe the transient process of the development from quantum to classical yielding the classical signal, we follow the following two steps in this paper.

a) *squeezed state*: If the quantum state is unstable, say by the negative mass squared, the system tends to evolve into the squeezed state. This state allows classical statistical description, in the effective action method, which is well separated from the unitary dynamical evolution. However, at the free level without any interaction, this statistical fluctuation, being non-dissipative, never appears outside. At this stage, everything is still reversible and unitary whatever the squeezing is strong. Furthermore the original symmetry, if any, is not yet broken. The statistical fluctuations, all superposed, are always symmetric.

b) *coherent state*: If the above free system couples to the other state or the non linear interaction appears, then the above statistical fluctuations do affect the full system. If this coherent state develops either by the strong statistical fluctuations or by strong instability of the potential, the classical portion of the coherent state dominates the quantum portion. Thus the classical signal appears possibly with dissipation.

In this way, the classicality thus finally obtained in b) is a quantitative notion; there is a

continuous transition from quantum to classical. On the other hand, the classicality of the noise in a) is a qualitative notion; all the fluctuations are classical.

We follow these two steps in the subsequent sections. In section 2, we study the appearance of the squeezed state in the unstable system and show that this squeezing process can be identified as the particle production which can be interpreted as the classical statistical fluctuations without dissipation. In section 3, we consider the non-linearity of the system in which this fluctuation triggers the development of the coherent state. In section 4, we motivate the introduction of the Schwinger-Keldysh in-in formalism starting from the popular Langevin equation. In section 5, we introduce the in-in effective action method to describe the above two processes, a) and b), at once. In section 6, we briefly describe some relevant examples which are directly related with our formalism. In the last section 7, we summarize our study.

II. SQUEEZING - NON DISSIPATIVE DRY NOISE

An instability or the time-dependent classical source yield the squeezed state in general. We first consider a simple model which yields the squeezed state, the inverted harmonic oscillator[9]. The Hamiltonian is given by

$$\hat{H} = \frac{1}{2m}\hat{p}^2 - \frac{m\omega^2}{2}\hat{q}^2 = i\frac{\omega\hbar}{2}(\hat{a}^2 e^{-2i\phi} - h.c.), \quad (1)$$

where

$$\begin{aligned} \hat{a}^\dagger &= \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(q - i\frac{p}{m\omega}\right) \\ \hat{a} &= \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(q + i\frac{p}{m\omega}\right) \end{aligned} \quad (2)$$

and $\phi = -\pi/4$. The ordinary cross term $\hat{a}^\dagger \hat{a}$ disappears because the second term in the middle of Eq.(1) is negative. Then the wave function at time t becomes

$$|\Psi(t)\rangle = \exp\left[\frac{\omega t}{2}(\hat{a}^2 e^{-2i\phi} - h.c.)\right] |0\rangle \equiv S(t) |0\rangle \quad (3)$$

$$= \exp\left[-\frac{\omega t}{2}(\hat{a}^\dagger)^2 e^{2i\phi} - \left(\frac{\omega t}{2}\right)^2\right] |0\rangle. \quad (4)$$

Therefore, in the state $|\Psi(t)\rangle$, the particle pair is 'condensed'. The operator $S(t)$ defines the Bogolubov transformation from the canonical pair a, a^\dagger to the new pair :

yields the classical statistical power spectrum.

$$\langle \varphi_k \varphi_k \rangle_\xi \approx \lambda^2 \varphi_0^4 \frac{H^2}{k^3}. \quad (72)$$

This transient process from quantum to classical has been made possible both by the squeezed state and the interaction of the inflaton field.

VII. CONCLUSIONS

We have explored the transient emergent process of the classical degrees of freedom in the full quantum system. In this process, the existence of the squeezing state and the development of the coherent state are both essential. The former squeezed state may be triggered by an instability and yields tachyonic mode. The latter coherent state develops in the Langevin dynamics triggered by the interactions. Thus both processes are indispensable.

We first considered the generation of the squeezed state in the inverted harmonic oscillator model. The degree of squeezing increases unboundedly in time. Though the action and the expectation value of the particle number increases and therefore one may tend to think that many particles are actually produced. However, this is simply a squeezed state fully described by the quantum theory and can never be interpreted as classical nor random statistical. We have shown that some non-dissipative noise, *i.e.* dry noise, is associated to this state using the Schwinger-Keldysh in-in formalism. At this stage, the noise never destroys the original symmetry if any. Moreover, the generation process of this state is reversible and the squeezing can be canceled. This is always true for free quantum system.

Then, introducing the interaction, we considered the development of the coherent state triggered by the above dry noise. This process is described by the Langevin equation. The degree of the coherent state increases, firstly by the ever accumulating dry noise and secondly by the unstable potential. This means that the state is dominated by the classical part which determines the finite vacuum expectation values.

Thus we have considered two types of general classicality in our argument. The first one is associated to the squeezed state and is represented by the real exponent in the in-in path-integral formalism. The second one is associated to the development of the coherent state and is represented by the development of the vacuum expectation values. This latter one is the objective classicality we wanted to obtain.

D. Transient dynamics from quantum to classical fluctuations in the inflationary cosmology

In the cosmology, the hypothetical scalar field inflaton is supposed to cause the inflation in the early Universe and also to yield the ultimate seeds of density fluctuations [16]. This inflaton field and the space-time metric mix together to yield the gauge invariant variables. We consider such scalar variable $v_{\mathbf{k}}(\eta)$ where \mathbf{k} is the three momentum and $\eta(\in (-\infty, 0))$ is the conformal time variable. The Hamiltonian for $v_{\mathbf{k}}(\eta)$ in the inflation (de Sitter space) becomes [9],

$$H = \frac{1}{2} \int \frac{d^3k}{(2\pi)^{3/2}} \left[\hat{p}_{-\mathbf{k}} \hat{p}_{\mathbf{k}} + k^2 \hat{v}_{-\mathbf{k}} \hat{v}_{\mathbf{k}} + \frac{1}{\eta} (\hat{p}_{-\mathbf{k}} \hat{v}_{\mathbf{k}} + \hat{v}_{-\mathbf{k}} \hat{p}_{\mathbf{k}}) \right] \quad (64)$$

$$= \frac{1}{2} \int \frac{d^3k}{(2\pi)^{3/2}} \left[k \left(\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}} + 1 \right) + \frac{i}{\eta} \left(e^{-2i\vartheta} \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}} - h.c. \right) \right] \quad (65)$$

where $\vartheta = -\pi/2$ and the factor $1/|\eta|$ in this Eq.(65) infinitely increases and yields strong squeezed state. However this is a special quantum state and does not directly mean the appearance of the classical fluctuations as we have already studied. Introducing the non-linear interactions of v and constructing the effective action, we can decompose the action into the deterministic part and the stochastic part[13],

$$\exp[i\tilde{\Gamma}[\tilde{\varphi}]] = \int \mathcal{D}\xi P(\xi) \exp[iS_0[\varphi; \xi]], \quad (66)$$

$$\exp[iS_0[\varphi; \xi]] = \exp[iS_0[\tilde{\varphi}]] \int D\phi \exp i[(\lambda\varphi(x)^3)_\Delta G_R(x-y)(\lambda\varphi(y)^3)_C + \quad (67)$$

$$(\lambda\varphi(x)^3)_C G_A(x-y)(\lambda\varphi(y)^3)_\Delta + i(\lambda\varphi(x)^3)_\Delta G_C(x-y)(\lambda\varphi(y)^3)_\Delta] \quad (68)$$

$$P(\xi) = \exp[-\frac{1}{4} \int d^3k \xi(\vec{k}) G_C(\vec{k})^{-1} \xi(\vec{k})]. \quad (69)$$

reflecting the dry noise generated by the squeezed state from Eq.(65). The fluctuation kernel is given by

$$G_C(\vec{k}) = \frac{H^2}{k^3} ((1 + k^2 \eta \eta') \cos(k\eta) + k\eta \sin(k\eta)) \quad (70)$$

where H is the Hubble constant for the inflationary de Sitter space.

The Langevin equation for the field

$$3H\dot{\varphi}_k + (\lambda/2)\varphi_0^2\varphi_k = (\lambda/2)\varphi_0^2\xi_k \quad (71)$$

$$\begin{cases} b = S^\dagger a S = \hat{a} \cosh \omega t - \hat{a}^\dagger e^{2i\phi} \sinh \omega t, \\ b^\dagger = S^\dagger a^\dagger S = \hat{a}^\dagger \cosh \omega t - \hat{a} e^{-2i\phi} \sinh \omega t, \end{cases} \quad (5)$$

and $SS^\dagger = S^\dagger S = 1$. This state $|\Psi(t)\rangle$ is unlimitedly squeezed in time toward the direction $\phi = -\pi/4$ in phase space as,

$$\langle \Psi(t) | (\hat{p} \cos \phi \pm \hat{q} \sin \phi) | \Psi(t) \rangle = \begin{cases} 4e^{-2t} \\ 4e^{2t} \end{cases}. \quad (6)$$

This state is often regarded as the particle-creating state as

$$N \equiv \langle \Psi(t) | a^\dagger a | \Psi(t) \rangle = \langle 0 | b^\dagger b | 0 \rangle = (\sinh \omega t)^2. \quad (7)$$

However, $|\Psi(t)\rangle$ is simply a quantum mechanical state and the particles claimed to be created are clearly not the classical object before any observation. Actually, the state can be reversible to the original state $|0\rangle$ by the operation S^\dagger . This means that this inverted harmonic oscillator, if prepared in the symmetric state initially, never fall to some particular direction. The state is always in the symmetric neutral position $\langle \Psi(t) | \hat{x} | \Psi(t) \rangle = 0$ even if the quantum fluctuations develop infinitely. On the other hand, since the kinetic energy K and the negative of the potential energy $-V$ both ever increases, the action $A = K - V$ of the system would diverge. This may indicate some classical property of this system $A \gg \hbar$ [9]. Since all the one point function vanishes, unique property of the state appears in the two point functions:

$$\begin{aligned} \langle [x(t), x(t')] \rangle &= \frac{i}{2} \left(\frac{m\omega}{2\hbar} \right)^{-1} \sinh(\omega(t-t')), \\ \langle \{x(t), x(t')\} \rangle &= \frac{1}{2} \left(\frac{m\omega}{2\hbar} \right)^{-1} \cosh(\omega(t+t')), \end{aligned} \quad (8)$$

where the former is locally ($t \approx t'$) normal but the latter abnormally diverge in time.

More generally in quantum field theory, the Hamiltonian Eq.(1) is the infinite collection of the harmonic oscillator labeled by the three momentum \mathbf{k} . Since the momentum is conserved, the pair in the squeezed state must have exactly the opposite momentum. Thus the Bogolubov transformation Eq.(5) should now be the form,

$$\begin{cases} b_{\mathbf{k}} = \alpha_{\mathbf{k}}^* \hat{a}_{\mathbf{k}} - \beta \hat{a}_{-\mathbf{k}}^\dagger, \\ b_{-\mathbf{k}}^\dagger = \alpha_{\mathbf{k}} \hat{a}_{-\mathbf{k}}^\dagger - \beta_{\mathbf{k}}^* \hat{a}_{\mathbf{k}} \end{cases}. \quad (9)$$

Therefore the particle pair of momentum $\mathbf{k}, -\mathbf{k}$ is entangled with each other. Many cases of the particle production, Unruh effect, accelerated mirror, Hawking radiation from the black

hole,...[10] correspond to this type of Bogolubov transformation and the states are slightly generalized squeezed states. These are simply the pure quantum states represented by the wave functions. These states are also similar to the entangled spin pair of Silver atoms in the Stern–Gerlach experiments [11]. All the above quantum system, even under the external force, is free and therefore keeps quantum coherence, and is even reversible, until observed or disturbed by the quantum interaction.

III. COHERENT STATE -DEVELOPMENT OF CLASSICALITY

We have introduced the squeezed state in the previous section in relation with quantum-classical transition. We have an another popular state, the coherent state, which is worth consideration in our context. The coherent state is define to be the eigenstate of the annihilation operator,

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle \quad (10)$$

which has an explicit form,

$$|\alpha\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} |0\rangle \quad (11)$$

$$\equiv C(\alpha) |0\rangle \quad (12)$$

$$= e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} |0\rangle. \quad (13)$$

The operator $C(\alpha)$ shifts the creation and annihilation operators by C-number

$$\begin{cases} \hat{b} = C^\dagger \hat{a} C = \hat{a} + \alpha, \\ \hat{b}^\dagger = C^\dagger \hat{a}^\dagger C = \hat{a}^\dagger + \alpha^*. \end{cases} \quad (14)$$

The particle number expectation is

$$N = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2 \quad (15)$$

and the two developed coherent states, *i.e.* individually large α, β , have almost no superposition

$$|\langle \alpha | \beta \rangle|^2 = e^{-|\alpha - \beta|^2}, \quad (16)$$

in particular, for the infinite system, the volume factor V enhances this property.

$$\propto e^{-V|\alpha - \beta|^2}. \quad (17)$$

B. Transient dynamics of the quantum measurement

The appearance of the classicality is deeply related with the quantum measurement process in which a particular state is probabilistically selected among multiple possibilities. The situation is very similar to SSB above, however, there must be a back-reaction to the quantum system from the emerged classical degrees of freedom. This back-reaction guarantees the firm correlation between the quantum system and the measurement apparatus.

The prototype of the quantum measurement model has been analyzed in this line of thought introducing the external thermal bath in[8]. This model describes the transient dynamics of the detector field $\hat{\phi}$ measures the spin \hat{S} . The Lagrangian is given by

$$L = \frac{1}{2} (\nabla \hat{\phi})^2 - \frac{1}{2} m^2 \hat{\phi}^2 - \frac{1}{4!} \lambda \hat{\phi}^4 + \mu \hat{\phi} \hat{S} B + (bath). \quad (63)$$

From the present approach, the detector should be described by $X_C(t)$ in Eq.(54). On top of this dynamics, the back reaction of it to the spin \hat{S} is needed. The thermal bath degrees of freedom will not be needed. The fluctuation would be provided by the dry noise associated to the initial squeezed state triggered by the unstable potential ($m^2 < 0$ and $\mu > 0$).

C. Transient dynamics which shows macroscopic irreversibility

A classical degrees of freedom as a developed coherent state has appeared after the time evolution by the Langevin equation. This process cannot be canceled and is irreversible. This is true even if the energy dissipation does not exist. Actually, the Brownian motion of the classical degrees of freedom is described by the Langevin equation with random fluctuations. The recursion probability, the system comes back to the original position, would be vanishingly small after the Brownian motion described by the Langevin equation.

In the careful argument on the appearance of the arrow of time in quantum mechanics [15], the essence of the irreversibility is the appearance of the decaying and growing pair, as well as the natural boundary condition which picks up one from the pair. The first essence is similar to our tachyonic modes and the second part is automatically selected by the Schwinger-Keldysh in-in formalism. Actually we use it to put the retarded and advanced contribution together in the evolution equation(54). We have further discussed the appearance of the classical degrees of freedom.

A. Transient dynamics of the spontaneous symmetry breaking and the Bose-Einstein condensation

A standard method to describe the Spontaneous symmetry breaking (SSB) needs an infinitesimal explicit violation of the symmetry with the delicate order of the two limiting operations. For example in the case of ferromagnetic materials, the full order parameter is given by

$$M_{\pm} \equiv \lim_{B \rightarrow 0^{\pm}} \lim_{V \rightarrow \infty} m_V(B), \quad (58)$$

where $m_V(B)$ is the local average of the magnetization[6]. In the case of Bose-Einstein condensation (BEC), the argument starts from the assumption that the boson field can be separated as [14]

$$\hat{\phi} = \langle \hat{\phi} \rangle + \delta\hat{\phi}, \quad (59)$$

where the classical parameter $\langle \hat{\phi} \rangle$ represents the 0-momentum condensation.

On the other hand, based on our approach, general SSB can be described as follows. We suppose the unstable potential for the complex scalar field $\phi(x)$ such as

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \quad (60)$$

with $m^2 < 0$ and $\lambda > 0$. The initial tachyonic instability around $\phi \approx 0$ induces the squeezed state and thus dry noise. Through the interaction, this noise autonomously violates the $U(1)$ symmetry and allow the development of the classical degrees of freedom as a coherent state. Therefore, we do not need Eq(58) and SSB is genuinely spontaneous. Furthermore, the artificial separation is not needed as in Eq.(59) and the field $\langle \hat{\phi} \rangle$ does emerge as a developed coherent state.

In the case of BEC, the condensation is not only characterized by

$$\frac{\langle \hat{a}_0^* \hat{a}_0 \rangle}{V} > 0, \quad (61)$$

but also needs the condition[6]

$$\frac{|\langle \hat{a}_0 \rangle|^2}{V} > 0. \quad (62)$$

which guarantees the off-diagonal long-range order or the fact that the BEC as a phase transition accompanying the spontaneous symmetry breaking. In BEC, the Gross-Pitaevskii equation is generally given as Eq.(54). The fluctuation term disappears when the system moves out from the unstable transient region ($|\langle \hat{\phi} \rangle|^2 > -2m^2/\lambda$).

The coherent state is not special but is generally produced by the time dependent single variable interaction $\xi(t)\hat{x}$, where $\xi(t)$ is a classical external variable. This is contrasted with the squeezed state which is generally produced by the time dependent square variable interaction $m(t)\hat{x}^2$. For example, let us consider the quantum system is exerted by the classical field $\xi(t)$,

$$\ddot{\hat{x}}(t) = -\gamma\dot{\hat{x}}(t) - V'(\hat{x}(t)) + \xi(t). \quad (18)$$

Then the general solution would be

$$\hat{x}(t) = \hat{x}(0) + \int^t dt' \Delta(t-t') \xi(t') \quad (19)$$

where the second term on the RHS, including any appropriate green function $\Delta(t-t')$, represents the accumulation of the classical field. Thus, in this case,

$$\hat{x}(t) = C(t)^\dagger \hat{x}(0) C(t) \quad (20)$$

where

$$C(t) = \exp \left[\int^t dt' \Delta(t-t') \xi(t') \hat{x}(0) \right]. \quad (21)$$

Thus the time evolution induced by the single variable interaction $\xi(t)\hat{x}$ yields a coherent state. If the classical part gradually dominates in Eq.(19) due to the accumulation of the external force $\xi(t)$, then the state gradually becomes more classical.

It would be useful to comment that the external classical field $\xi(t)$ needs not to be a systematic force, but can be random with zero-mean. Even in this case, the classical component increases in time and develop classicality.

IV. LANGEVIN TO IN-IN FORMALISM - A MOTIVATION

In order to consider the transient process from quantum to classical, let us consider a simple example of the Langevin equation first. This gives us a natural motivation to the Schwinger-Keldysh in-in closed path time quantum theory (CTP)[12]. Starting from the classical Langevin equation, we derive the effective partition function. The quantization of this partition function almost deduces the CTP formalism in quantum mechanics.

The Langevin equation is a dynamical equation of a particle in the environment with the potential force $-V'$ and the random force ξ with friction γ :

$$\ddot{x}(t) = -\gamma\dot{x}(t) - V'(x(t)) + \xi(t). \quad (22)$$

The statistical average

$$\langle \dots \rangle_\xi = \int D[\xi] \dots P[\xi] \quad (23)$$

is determined by the Gaussian weight functional $P[\xi]$,

$$P[\xi] = e^{-\int \xi(t)^2 / (2\sigma^2)}. \quad (24)$$

We would like to know the action which drives this Langevin equation. We first try to construct the partition function of the system. The partition function is derived by summing all the possible motions in the whole phase space.

$$\begin{aligned} Z[J] &\equiv \langle \delta[\ddot{x}(t) + \gamma\dot{x}(t) + V'(x(t)) - \xi(t)] \rangle_\xi \\ &= \int D[\xi] P[\xi] \delta[\ddot{x}(t) + \gamma\dot{x}(t) + V'(x(t)) - \xi(t)] \\ &= \int D[\xi] D[x'] P[\xi] e^{i \int dt x'(t) \{ \ddot{x}(t) + \gamma\dot{x}(t) + V'(x(t)) - \xi(t) \}} \\ &= \int D[x'] e^{i\tilde{S}[x, x']} \end{aligned} \quad (25)$$

where the integral form of the delta functional is utilized introducing a fictitious variable $x'(t)$, and the action becomes

$$\tilde{S}[x, x'] \equiv \int dt \{ -\dot{x}'(t)\dot{x}(t) + \gamma x'(t)\dot{x}(t) + x'(t)V'(x(t)) + \frac{i}{2}\sigma^2 x'(t)^2 \} \quad (26)$$

where the boundary term is dropped. Note that the last term, which represents classical statistical fluctuations, is pure imaginary in the action. The rest of the terms describe the deterministic dynamics though including frictional term.

We now quantize this system because the most basic theory would be the quantum mechanics, from which classical dynamics eventually appear. The partition function for the quantized system simply becomes

$$Z[J] = \int D[x] D[x'] e^{i\tilde{S}[x, x']} \quad (27)$$

in the path integral form. The mixed expression of the two variables $x(t)$ and $x'(t)$ in Eq.(26) is dowdy. It is possible to rewrite the action more resemble to the ordinary dynamics. In order to do so, we rewrite the variables as

$$x_\pm = x \pm \frac{1}{2}x'. \quad (28)$$

Now the application of the variational principle for $S_{\text{eff}}[X]$

$$\frac{\delta S_{\text{eff}}[X]}{X_{\Delta}(t)}|_{X_{\Delta}=0} = 0, \quad (53)$$

yields the classical Langevin equation as

$$\ddot{X}_C(t) - \omega^2 X_C(t) + 2 \int dt' G_R(t, t') \left(1 + \lambda^2 G_C(t, t')^2\right) X_C(t') + \xi(t) = 0. \quad (54)$$

This equation describes the rapid evolution of the classical variable $X_C(t)$ under a) the strong fluctuations $\xi(t)$ with Eqs.(52,38) violently disturb the system, and b) the original classical instability $-\omega^2 X_C(t)$ which promotes the exponential development of the system. The retarded term $2 \int dt' G_R(t, t') \left(1 + \lambda^2 G_C(t, t')^2\right) X_C(t')$ sometimes shows the dissipative effects.

The above equation (54) describes the evolution of the classical variable $X_C(t)$ from 0, the symmetric state, to a finite value, the asymmetric state. Eventually in this evolution, other interaction terms in Eq.(45) gradually contribute. Some of them yield the new type of noise. Individual imaginary term seems to yield individual random noise.

$$\exp[-X_{\Delta} G_C X_{\Delta}] = \int D\xi_1 \exp[-\xi_1 G_C^{-1} \xi_1 + i\xi_1 X_{\Delta}], \quad (55)$$

and

$$\exp[-X_{\Delta} G_C^2 X_{\Delta}] = \int D\xi_2 \exp[-\xi_2 G_C^{-2} \xi_2 + i\xi_2 X_{\Delta}]. \quad (56)$$

However, all the perturbation terms have interference with each other and therefore should be treated at once:

$$\exp[-\lambda^2 X_{\Delta} (G_C + G_C^2 + G_C^3) X_{\Delta}] = \int D\xi \exp[-\xi [\lambda^2 (G_C + G_C^2 + G_C^3)]^{-1} \xi + i\xi X_{\Delta}]. \quad (57)$$

Since the fluctuation kernel $\lambda^2 X_{\Delta} (G_C + G_C^2 + G_C^3) X_{\Delta}$ is unique, the random field $\xi(t)$ is also unique; random fields do not appear separately.

VI. SOME APPLICATIONS AND IMPLICATIONS

Our argument is general and will have many applications and implications. Some of them are very briefly described below. Individual argument will be reported separately.

where we shifted the path-integration variable. Expanding $\tilde{S}[\tilde{X} + \tilde{x}]$ in the series of \tilde{x} , we have

$$\tilde{S}[\tilde{X} + \tilde{x}] = \tilde{S}[\tilde{X}] + \tilde{S}'[\tilde{X}]\tilde{x} + \frac{1}{2}\tilde{S}''[\tilde{X}]\tilde{x}^2 + \frac{1}{3!}\tilde{S}'''[\tilde{X}]\tilde{x}^3 + \dots \quad (45)$$

The first term represents the action for the classical field $\tilde{X}(t)$, the second term and the third term yield interaction for the quantum variable, and the second term a free action for \tilde{x} in the background of \tilde{X} .

If the interaction is quartic, $\lambda x(t)^4$, and the background $\tilde{X} = 0$ initially, then the term $\frac{1}{3!}\tilde{S}'''[\tilde{X}]\tilde{x}^3$ gives the dominant interaction term $\lambda\tilde{X}(t)\tilde{x}(t)^3$. The lowest contribution of this term yields the two-loop quantum effect for $\tilde{X}(t)$

$$\int dt dt' \lambda^2 \tilde{X}(t) \text{Tr} [T_C \rho \tilde{x}(t)^3 \tilde{x}(t')^3] \tilde{X}(t') \quad (46)$$

$$= \int dt dt' \lambda^2 \tilde{X}(t) \text{Tr} [T_C \rho \tilde{x}(t) \tilde{x}(t')]^3 \tilde{X}(t') \quad (47)$$

$$= \int dt dt' \lambda^2 \tilde{X}(t) G_C(t, t')^3 \tilde{X}(t') \quad (48)$$

$$= \int dt dt' \lambda^2 (X_C, X_\Delta)_t \begin{pmatrix} 0 & G_C^2 G_A \\ G_R G_C^2 & G_C^3 \end{pmatrix}_{t, t'} \begin{pmatrix} X_C \\ X_\Delta \end{pmatrix}_{t'} \quad (49)$$

Therefore, the imaginary term $X_\Delta(t) G_C(t, t')^3 X_\Delta(t)$ contributes to the fluctuation as before and can be separated from the real part of the action by the introduction of the auxiliary field $\xi(t)$,

We have, in the lowest order,

$$\exp[i\Gamma[X]] = \int D\xi P(\xi) \exp[iS_{\text{eff}}[X]], \quad (50)$$

where the real action is

$$S_{\text{eff}}[X] = S[X] + \int \int dt dt' X_C(t) (1 + \lambda^2 G_C(t, t')^2) G_A(t, t') X_\Delta(t') \\ + \int \int dt dt' X_\Delta(t) G_R(t, t') (1 + \lambda^2 G_C(t, t')^2) X_C(t') + \int dt \xi(t) X_\Delta(t), \quad (51)$$

and the fluctuation weight is given by

$$P(\xi) = \exp\left[-\frac{\lambda^2}{2} \int dt \xi(t) G_C(t, t')^3 \xi(t')\right]. \quad (52)$$

Note that the advanced term $\int \int dt dt' X_C(t) (1 + \lambda^2 G_C(t, t')^2) G_A(t, t') X_\Delta(t')$ yields the same retarded term if we exchange the variables $t \leftrightarrow t'$.

Then, Eq. (27) becomes

$$Z[J] = \int D[x_+] D[x_-] e^{i\tilde{S}[x_+, x_-]} \quad (29)$$

where

$$\tilde{S}[x_+, x_-] = \int dt \left\{ \begin{aligned} & ((\ddot{x}_+(t))^2 - V(x_+(t))) - ((\ddot{x}_-(t))^2 - V(x_-(t))) \\ & + \frac{\gamma}{2} (x_+(t)\dot{x}_-(t) - \dot{x}_+(t)x_-(t)) + \frac{i}{2}\sigma^2 (x_+(t) - x_-(t))^2 \end{aligned} \right\}. \quad (30)$$

The first line of the above represents the deterministic dynamics for the variables $x_{\pm}(t)$ separately, and the second line dissipation and fluctuation terms where the variables $x_{\pm}(t)$ mix up. It is a natural extension of this expression to introduce a closed time-path C : which starts from $-\infty$ to ∞ (+ branch) and then comes back from ∞ to $-\infty$ (− branch). We suppose the supports of the variables $x_{\pm}(t)$ are, respectively, the + and − branches. We denote the variable $\tilde{x}(t)$ on the countour C unifying the variables $x_{\pm}(t)$:

$$\tilde{x}(t) = \begin{cases} x_+(t) & t \in (+\text{branch}) \\ x_-(t) & t \in (-\text{branch}). \end{cases} \quad (31)$$

We use this unification for all the variables and denote them by tilde.

It is possible to reverse the above logic to get to the action $\tilde{S}[x_+, x_-]$; starting from the action $\tilde{S}[x_+, x_-]$ to get to the Langevin equation. If we find a complex action including an extra degrees of freedom like x' in the above, classical random field appears and the evolution equation becomes the Langevin equation. We study a typical case in the next section.

V. CTP TO LANGEVIN

The best way to describe the transient process from quantum to classical would be the Schwinger-Keldysh in-in formalism[12]. In this theory, the partition function for the system with the free action

$$S[x] = \int dt (\dot{x}^2 + \omega^2 x^2),$$

is given by

$$\begin{aligned} \tilde{Z}[\tilde{J}] &= \int_C D\tilde{x} \exp[iS[\tilde{x}] + i \int dt \tilde{J}(t)\tilde{x}(t)] \equiv \exp i\tilde{W} \\ &= \int D\tilde{x} \exp[iS[x_+] - iS[x_-] + i \int dt \tilde{J}(t)\tilde{x}(t)], \end{aligned}$$

where the tilde denotes the variables on the closed-path C as before. This reduces to

$$\tilde{Z}[\tilde{J}] = \exp\left[-\frac{1}{2} \int dt \tilde{J}(t) \tilde{G}_0(t, t') \tilde{J}(t')\right]. \quad (32)$$

where

$$\tilde{G}_0(t, t') = \begin{pmatrix} G_F(t, t') & G_+(t, t') \\ G_-(t, t') & G_{\overline{F}}(t, t') \end{pmatrix} \equiv \begin{pmatrix} \text{Tr}[Tx(t)x(t')\rho] & \text{Tr}[x(t')x(t)\rho] \\ \text{Tr}[x(t)x(t')\rho] & \text{Tr}[\overline{T}x(t)x(t')\rho] \end{pmatrix} \quad (33)$$

where T denotes the ordinary time-ordering operator and \overline{T} the anti time-ordering operator.

If we change the representation of the matrix by

$$J_{\pm}(t) = J_c \pm \frac{1}{2} J_{\Delta} \quad (34)$$

or

$$\tilde{J} = \begin{pmatrix} J_{\Delta} \\ J_C \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} J_+ \\ J_- \end{pmatrix}, \quad (35)$$

then, we have

$$\tilde{G}_0(t, t') = \begin{pmatrix} 0 & G_R(t, t') \\ G_A(t, t') & G_C(t, t') \end{pmatrix} \equiv \begin{pmatrix} 0 & \theta(t-t') \text{Tr}[x(t'), x(t)]\rho \\ \theta(t'-t) \text{Tr}[x(t'), x(t)]\rho & \text{Tr}\{x(t), x(t')\}\rho \end{pmatrix}. \quad (36)$$

In our case,

$$\begin{aligned} G_R(t, t') &= \theta(t-t') \langle [x(t), x(t')] \rangle \\ &= \frac{i}{2} \left(\frac{m\omega}{2\hbar}\right)^{-1} \sin(2\phi) \theta(t-t') \sinh(\omega(t-t')) \\ &= \frac{i}{2} \left(\frac{m\omega}{2\hbar}\right)^{-1} \theta(t-t') \sinh(\omega(t-t')), \end{aligned} \quad (37)$$

$$\begin{aligned} G_C(t, t') &= \langle \{x(t), x(t')\} \rangle \\ &= \frac{1}{2} \left(\frac{m\omega}{2\hbar}\right)^{-1} (\cosh(\omega(t+t')) - \cos(2\phi) \sinh(\omega(t+t'))) \\ &= \frac{1}{2} \left(\frac{m\omega}{2\hbar}\right)^{-1} \cosh(\omega(t+t')), \end{aligned} \quad (38)$$

where $\phi = -\pi/4$.

These two types of green functions sometimes show different infrared behavior. Actually, for low frequency $\omega(t-t') \ll 1$ and $\omega(t+t') \ll 1$,

$$\begin{aligned} G_R(t, t') &\propto \sinh(\omega(t-t')) \rightarrow \omega(t-t'), \\ G_C(t, t') &\propto \cosh(\omega(t+t')) \rightarrow 1. \end{aligned} \quad (39)$$

$G_R(t, t')$ has milder IR behavior than $G_C(t, t')$ by the factor ω . However, as we will see shortly, $G_C(t, t')$ can be separated from the action which describes the deterministic dynamics. In particular, this situation becomes prominent at the inflationary stage in the early Universe[13].

Comparing Eqs.(37,38), the symmetric term in Eq.(32)

$$-\frac{1}{2} \int dt J_\Delta(t) G_C(t, t') J_\Delta(t') \quad (40)$$

is real and positive. Therefore we can factor out this part as classical statistical fluctuations introducing an auxiliary field $\xi(t)$,

$$\tilde{Z}[\tilde{J}] = \int D\xi P(\xi) \exp\left[-\frac{1}{2} \int dt \tilde{J}(t) \tilde{G}'_0(t, t') \tilde{J}(t') + i \int dt J_\Delta(t) \xi(t)\right], \quad (41)$$

where

$$P(\xi) = \exp\left[-\frac{1}{2} \int dt \xi(t) G_C(t, t') \xi(t')\right], \quad (42)$$

and $\tilde{G}'_0(t, t')$ is thus separated green function. This separation procedure is just a reverse of the previous section where Eq.(30) yields Eq.(24).

As in the above, we have identified the classical statistical fluctuation, *i.e.* noise, in Eqs.(41, 42). However, this noise only couples to the field $J_\Delta(t)$, in Eq.(41), which is an external source term but not any dynamical variable. Therefore this noise never comes out as is. Nothing happens in free state before any measurement process or interaction according to the quantum mechanics theory. Furthermore, the noise here is dry and is not accompanied by dissipation.

This situation drastically changes if we introduce the interactions. Let us introduce the effective action, in the in-in formalism, which is the Legendre transformation of the partition function $\tilde{Z}[\tilde{J}] = \exp i[\tilde{W}[\tilde{J}]]$,

$$\exp[i\tilde{\Gamma}[\tilde{X}]] = \exp i[\tilde{W}[\tilde{J}] - \int dt \tilde{J}(t) \tilde{X}(t)]. \quad (43)$$

This becomes

$$\begin{aligned} \exp[i\tilde{\Gamma}[\tilde{X}]] &= \exp i[\tilde{W}[\tilde{J}] - \int dt \tilde{J}(t) \tilde{X}(t)] \\ &= \int_C D\tilde{x} \exp i[\tilde{S}[\tilde{x}] + \int d^4x \tilde{J}(x)(\tilde{x}(t) - \tilde{X}(t))] \\ &= \int_C D\tilde{x} \exp i[\tilde{S}[\tilde{X} + \tilde{x}] + \int dt \tilde{J}(t) \tilde{X}(t)], \end{aligned} \quad (44)$$