Applications to string theory:
Dq-brane contains $d = q + 1$-dim. gauginos χ.
$N = 8$ (determined by supersymmetry).

$$\eta(\mathbb{RP}^{q+2}) = 8/2^{(q+1)+1} = 2^{-q+1} \tag{7.48}$$

Op-plane background

$$\mathbb{R}^{9-p}/\mathbb{Z}_2 \times \mathbb{R}^{p+1}. \tag{7.49}$$

The anomaly of Dq-brane fermion around Op with $p + q = 6$ (Dirac quantization pair)
$\Longrightarrow 9 - p = q + 3$

$$\eta(\mathbb{RP}^{q+2}) = 2^{-q+1} = 2^{p-5}. \tag{7.50}$$

Coincides with Op-plane RR charge. (Sign neglected)

Fermion anomalies

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ABSTRACT: Lecture notes on fermion anomalies.

Contents

1 Introduction

Anomalies are important in two ways.

1. Gauge symmetries : anomalies must be cancelled for the consistency of a theory.
2. Global symmetries : ’t Hooft anomaly matching.

(Orientation neglected)
APS index theorem :

$$\mathrm{index}(\not{\partial}) = 2^{d+2}\eta(\mathbb{RP}^{d+1}). \tag{7.36}$$

(No curvature term since T^{d+2} is flat.)

$$\frac{K}{2} = 2^{d+2}\eta(\mathbb{RP}^{d+1}) \tag{7.37}$$

$$\eta(\mathbb{RP}^{d+1}) = \frac{K}{2^{d+3}} \tag{7.38}$$

The number of components:

$$\begin{aligned} &\text{Def : } N \text{ in } d\text{-dim.} && (\text{ chiral } \chi) && (7.39)\\ &\longrightarrow\ 2N \text{ in } (d+1)\text{-dim.} && (\Psi = (\chi_+, \chi_-)) && (7.40)\\ &\longrightarrow\ 4N \text{ in } (d+2)\text{-dim.} && (\text{nonchiral } \Phi = (\Psi_+, \Psi_-)) && (7.41)\end{aligned}$$

$K = 4N$.

$$\eta(\mathbb{RP}^{d+1}) = \frac{N}{2^{d+1}}. \tag{7.42}$$

N depends on the rep. of the symmetry H_d.

Example:
$d = 1$ fermions with N components

$$-\frac{1}{2}\int d\sigma\, \psi_i \frac{d}{d\sigma}\psi_i \tag{7.43}$$

ψ_i : majorana

$$\exp(-\pi i \eta(\mathbb{RP}^2)) = \exp\left(-2\pi i \cdot \frac{N}{8}\right). \tag{7.44}$$

$$\begin{aligned} &\mathbb{RP}^2 \text{ non-orientable manifold} && (7.45)\\ &\Longrightarrow \exp(-\pi i \eta(\mathbb{RP}^2)) = \text{ time reversal anomaly} && (7.46)\end{aligned}$$

It turns out the above computation is for $H_d = \mathrm{Pin}^-(d)$ in $d = 1$.

$$\mathsf{T}(\psi(t)) = \psi(-t). \tag{7.47}$$

The anomaly classified by $\mathbb{Z}_8$. The bulk $d + 1 = 2$ system: Kitaev Majorana chain.

Similar thing for $d + 1 = 4$ ($\mathbb{RP}^4$ non-orientable) : topological superconductors.

$\overline{\gamma}_X$: some "chirality" operator in $d+2$-dim.
Properties of the transf.
(1) $g^2 = 1$.
(2) $g \cdot (\partial\!\!\!/ \Phi) = \partial\!\!\!/ (g \cdot \Phi)$. $\quad (\overline{\gamma}_X \gamma^\mu = -\gamma^\mu \overline{\gamma}_X,\ \partial_\mu \to -\partial_\mu)$

Spinor on $X = T^{d+2}/\mathbb{Z}_2$ = Spinor on T^{d+2} with $g \cdot \Phi = \Phi$.

Index :

$$\mathrm{index}(\partial\!\!\!/) = n_+ - n_-. \tag{7.29}$$

$n_\pm$: # of zero modes with $\overline{\gamma}_X = \pm 1$.
Zero modes: $\partial\!\!\!/^2 \Phi = -\partial^2 \Phi = 0 \Longrightarrow \partial_\mu \Phi = 0 \Longrightarrow \Phi = \mathrm{const.}$
On $T^{d+2}/\mathbb{Z}_2$,

$$g \cdot \Phi = \Phi \Longrightarrow \overline{\gamma}_X \Phi = \Phi. \tag{7.30}$$

$$n_+ = \frac{1}{2}\ (\#\text{ of components of }\Phi) := \frac{K}{2}, \tag{7.31}$$

$$n_- = 0. \tag{7.32}$$

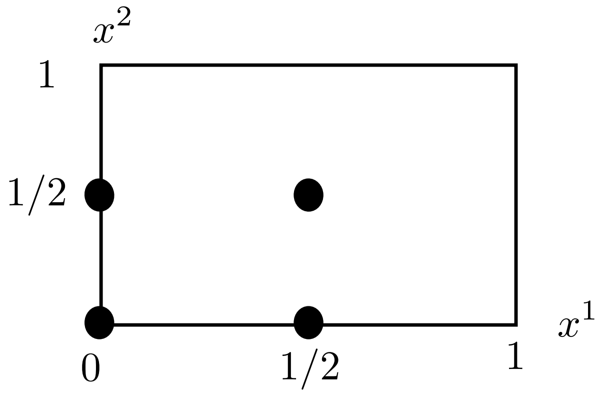
X singular manifold: Remove singular points of $\mathbb{Z}_2$ action:

$$x^i = 0 \text{ or } 1/2 \text{ for each } x^i. \tag{7.33}$$

2^{d+2} singular points.

$$X' = T^{d+2}/\mathbb{Z}_2 - \bigsqcup_{i=1}^{2^{d+2}} B_i \tag{7.34}$$

B_i : small ball around singular points such that $\partial B_i = S^{d+1}/\mathbb{Z}_2 = \mathbb{RP}^{d+1}$.



$$\partial X' = 2^{d+2} \text{ copies of } \mathbb{PR}^{d+1}. \tag{7.35}$$

Gauge. Perturbative anomaly cancellation in the standard model, string theory, etc. More nontrivial at the nonperturbative level.
(I do not know the complete answer for string theory.)

Q : We are confident that string theory is consistent. Why do we care?
A : studies of anomalies = studies of possible topology

Example: most branes are anomalous. The consistency requires nontrivial fluxes.

$$\text{E.g. } F_5 \text{ flux around O3-plane : } \quad \int F_5 = n - \frac{1}{4}, \quad n \in \mathbb{Z}. \tag{1.1}$$

$-1/4$ seems to violate Dirac quantization, but is required by anomaly cancellation for D3.

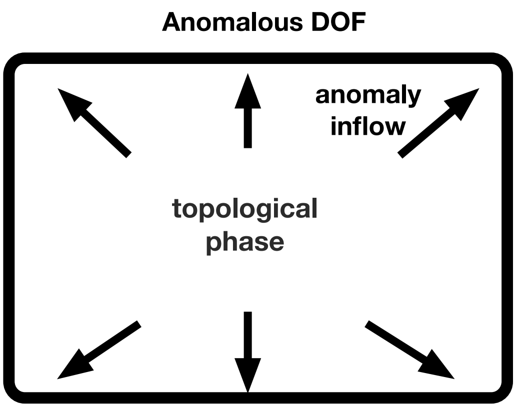
Global. 't Hooft anomalies : anomalies of global symmetries
Conserved in renormalization group (RG) flows.

$$\text{UV theory} \xrightarrow{\text{RG flow}} \text{IR theory} \tag{1.2}$$

Either UV or IR may be difficult due to strong coupling.
UV anomaly = IR anomaly : useful constraints on dynamics

In cond.mat.phys.,

$$\text{Properties of a bulk topological phase} = \text{Anomalies of boundary degrees of freedom} \tag{1.3}$$



Equality by anomaly inflow. This turns out to be essential for nonperturbative formulation of the concept of anomalies itself, even without thinking cond.mat.

Both gauge and global. They can be treated in the same way.
Couple the theory to background gauge fields of the symmetry.
Gauge : it is before the path integral of gauge fields.
Global : Source fields of the symmetry. (e.g. $A_\mu J^\mu$ for continuous symmetries)

2 Preliminaries

Euclidean $g_{\mu\nu} = (+, \cdots, +)$ unless otherwise stated.
Spacetime d-dim. (or $d+1$, $d+2$ depending on the context)
γ^μ: gamma matrices

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{2.1}$$

Covariant derivative on fermions

$$D_\mu \Psi = (\partial_\mu + A_\mu + \omega_\mu)\Psi \tag{2.2}$$
$$A_\mu = -A_\mu^\dagger : \text{ gauge fields} \tag{2.3}$$
$$\omega_\mu = \omega_{\mu ab}\frac{1}{4}\gamma^a\gamma^b : \text{ spin connection} \tag{2.4}$$
$$\not{D} = \gamma^\mu D_\mu \tag{2.5}$$

Differential forms. Differential form : a convenient way to treat antisymmetric tensors $\omega_{\mu_1\cdots\mu_p}$.

$$\omega_{\mu_1\cdots\mu_p} \to \omega = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{2.6}$$

$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ antisymmetric under exchange of indices $\mu_1, \cdots, \mu_p$.

For general (not antisymmetric) tensors, define

$$A_{[\mu_1\cdots\mu_p]} = \frac{1}{p!}\sum_\sigma \mathrm{sign}(\sigma) A_{\mu_{\sigma(1)}\cdots\mu_{\sigma(p)}} \tag{2.7}$$

Sum is over all permutations σ.
Rules :

$$d\omega = \frac{1}{p!}\partial_{[\nu}\omega_{\mu_1\cdots\mu_p]} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{2.8}$$
$$\omega \wedge \eta = \frac{1}{p!q!}\omega_{[\mu_1\cdots\mu_p}\eta_{\nu_1\cdots\mu_q]} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_q}. \tag{2.9}$$

Example: gauge field $A_\mu = A_\mu^a T_a$.

$$dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu \tag{2.10}$$
$$A \wedge A = \frac{1}{2}[A_\mu, A_\nu] dx^\mu \wedge dx^\nu \tag{2.11}$$
$$F = \frac{1}{2}F_{\mu\nu} dx^\mu \wedge dx^\nu = dA + A \wedge A. \tag{2.12}$$

Wedge $\wedge$ is sometimes omitted.
From the rules we also get

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d\eta. \tag{2.13}$$

etc.

$$n_+ \mod 2 : \text{called mod 2 index.} \tag{7.15}$$

Example 1:

$$Y = T^2 \tag{7.16}$$

$$g = \mathrm{diag}(-1, 1, \cdots, 1) \in \mathrm{O}(N). \tag{7.17}$$

$$\Psi(\tau+1, \sigma) = -g\Psi(\tau, \sigma), \qquad \Psi(\tau, \sigma+1) = \Psi(\tau, \sigma). \tag{7.18}$$

$$\not{D} = \gamma^\mu \partial_\mu. \tag{7.19}$$

Zero modes:

$$\Psi_1(\tau, \sigma) = \text{const.}, \qquad \Psi_j(\tau, \sigma) = 0 \quad (j \geq 2). \tag{7.20}$$

Ψ_1 has two components $\overline{\gamma} = \pm 1 \Longrightarrow n_+ = n_- = 1$.

$$(-1)^{n_+} = -1 : \text{anomaly of O}(N) \tag{7.21}$$

Example 2:

$$Y = S^2 \tag{7.22}$$

$$\mathrm{U}(1) = \mathrm{SO}(2) \subset \mathrm{SO}(N), \quad \Psi_\pm := \Psi_1 \pm i\Psi_2 : \text{charge } \pm 1, \text{ others } 0. \tag{7.23}$$

$$\mathrm{U}(1) \text{ gauge field } A = \frac{1}{2} i(\cos\theta \pm 1) d\phi, \qquad \int_{S^2} \frac{iF}{2\pi} = 1 \tag{7.24}$$

$$\tag{7.25}$$

Ψ_+ has one zero mode with $\overline{\gamma} = +1$, Ψ_- has one zero mode with $\overline{\gamma} = -1$.
$\Longrightarrow n_+ = n_- = 1$.

$$(-1)^{n_+} = -1 : \text{anomaly of SO}(N) \tag{7.26}$$

7.2 η on real projective space

Examples of global anomalies so far: $\mathbb{Z}_2$.
This is not generally true.

We want to compute $\eta(\mathbb{RP}^{d+1})$.
(I omit many important details.)
Stragegy: Use APS index theorem to relate η to some index in $d+2$-dim space X.

Consider $d+2$-manifold $X = T^{d+2}/\mathbb{Z}_2$.
A spinor Φ on T^{d+2} transforming under nontrivial $g \in \mathbb{Z}_2$ as

$$\Phi(x) \to g \cdot \Phi(x) = \overline{\gamma}_X \Phi(-x) \tag{7.27}$$

$$x \in T^{d+2} = \{x = (x^1, \cdots, x^{d+2}); \quad x^i \sim x^i + 1\}. \tag{7.28}$$

7 Some examples of fermion global anomalies

7.1 $d = 1$ fermions again

$$\text{boundary:} \quad S \to -\frac{1}{2}\int d\sigma\, \psi_i \frac{d}{d\sigma}\psi_i. \tag{7.1}$$

$$\text{bulk:} \quad S = -\frac{1}{2}\int d\tau d\sigma \sum_i \Psi_i^T \epsilon(\gamma^\tau \partial_\tau + \gamma^\sigma \partial_\sigma + m)\Psi_i, \tag{7.2}$$

Let's compute bulk η on orientable spin manifolds.

$$i\not{D} = i\gamma^\mu D_\mu \qquad (x^1 = \tau, x^2 = \sigma). \tag{7.3}$$

Define

$$\overline{\gamma} = i^{-1}\gamma^1\gamma^2. \tag{7.4}$$

$$i\not{D}\Psi = \lambda\Psi \Longrightarrow i\not{D}\overline{\gamma}\Psi = -\lambda\Psi : \ (\lambda, -\lambda) \text{ pair} \tag{7.5}$$

$$\eta = \frac{1}{2}\sum_\lambda \operatorname{sign}(\lambda) \qquad (\operatorname{sign}(0) = +1) \tag{7.6}$$

$$= \frac{1}{2}(n_+ + n_-) \qquad (n_\pm : \text{ the numbers of positive, negative zero modes}) \tag{7.7}$$

Real basis:

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.8}$$

$$D_\mu : \text{covariant derivative for } \mathrm{O}(N) \text{ and spin connection, real } (D_\mu)^* = D_\mu \tag{7.9}$$

$$i\not{D}\Psi = 0 \Longrightarrow i\not{D}\Psi^* = 0 \tag{7.10}$$

$$\overline{\gamma}\Psi^* = (\overline{\gamma}^*\Psi)^* = -(\overline{\gamma}\Psi)^* \qquad (\overline{\gamma} = i^{-1}\gamma^1\gamma^2 : \text{ pure imaginary}) \tag{7.11}$$

$$\Psi \text{ positive chirality} \Longleftrightarrow \Psi^* \text{ negative chirality} \tag{7.12}$$

$n_+ = n_-$. Thus

$$\eta = n_+ \tag{7.13}$$

Anomaly for majorana:

$$\exp(-\pi i\eta) = (-1)^{n_+}. \tag{7.14}$$

Fiber bundle. G : group. V : representation of G (i.e. $g \in G$ acts on V).
A fiber bundle E with a fiber V and structure group G on a manifold (spacetime) X is:

- On a small enough region $U_\alpha \subset X$, elements of E are pairs $(x, v_\alpha) \in U_\alpha \times V$.
- Between two regions (patches) $U_\alpha, U_\beta \subset X$, there is a transition function $g_{\alpha\beta}(x)$ such that $v_\alpha = g_{\alpha\beta}(x) v_\beta$. Points $(x, v_\alpha) \in U_\alpha \times V$ and $(x, v_\beta) \in U_\beta \times V$ are identified in this way.
- Gauge field (connection) on U_α and U_β, A_α and A_β are related as

$$d + A_\beta = g_{\alpha\beta}^{-1}(d + A_\alpha) g_{\alpha\beta}. \tag{2.14}$$

  More explicitly $A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d g_{\alpha\beta}$.

Spin connection. Take orthogonal frame e_a^μ $(a = 1, \cdots, d)$

$$g_{\mu\nu} e_a^\mu e_b^\nu = \delta_{ab}. \tag{2.15}$$

Levi-Civita connection in Riemann geometry $\Rightarrow$ O(d) connection $\omega_{\mu ab} = -\omega_{\mu ba}$.
Vector $v^\mu = v_a e_a^\mu$,

$$D_\mu v_a = (\delta_{ab}\partial_\mu + \omega_{\mu ab}) v_b. \tag{2.16}$$

$\omega_{\mu ab}$ can be defined to satisfy

$$e_a^\mu (D_\nu v_a) = \partial_\nu v^\mu + \Gamma^\mu_{\nu\rho} v^\rho. \tag{2.17}$$

Fermions on general manifolds: formulated as fiber bundles of the Lorentz group in the spin rep.

3 Axial anomaly and Atiyah-Singer index theorem

Index theorems are important in many aspects of anomalies.
We motivate it by axial anomaly.

$d = 2n$ dim. Dirac fermion. Chirality (called γ_5 in 4d) denoted as $\overline{\gamma}$.
In this section we define

$$\overline{\gamma} = \frac{1}{i^n}\gamma^1 \cdots \gamma^{2n} = (i^{-1}\gamma^1\gamma^2)(i^{-1}\gamma^3\gamma^4)\cdots \tag{3.1}$$

$$\{\overline{\gamma}, \gamma_\mu\} = \overline{\gamma}\gamma_\mu + \gamma_\mu\overline{\gamma} = 0, \qquad \overline{\gamma}^\dagger = \overline{\gamma}, \qquad \overline{\gamma}^2 = 1. \tag{3.2}$$

Eigenvalues $\overline{\gamma} = \pm 1$.

$$\overline{\gamma} = +1\ (-1): \text{ positive (negative) chirality} \tag{3.3}$$

$$\tag{3.4}$$

Massless Dirac Lagrangian

$$\mathcal{L} = -\overline{\Psi} \not{D} \Psi \tag{3.5}$$

Clifford algebra. $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. Define “creation and annihilation operators”

$$a_i = \frac{1}{2}(\gamma_{2i-i} - i\gamma_{2i}), \qquad a_i^\dagger = \frac{1}{2}(\gamma_{2i-i} + i\gamma_{2i}). \tag{3.6}$$

$i = 1, \cdots, n$. Anti-commutation

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \qquad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0. \tag{3.7}$$

The irreducible representation

$$|s_1, s_2, \cdots, s_n\rangle \qquad (s_i = \pm\frac{1}{2}) \tag{3.8}$$

$$(a_i^\dagger a_i - \frac{1}{2})|s_1, s_2, \cdots, s_n\rangle = s_i|s_1, s_2, \cdots, s_n\rangle. \tag{3.9}$$

s_i : spin in the $x_{2i-1}x_{2i}$ plane.
2^n-dimensional rep.

$$\overline{\gamma} = (2a_1^\dagger a_1 - 1)(2a_2^\dagger a_2 - 1)\cdots \tag{3.10}$$

3.1 Axial rotation

Classical: axial symmetry

$$\Psi \to \exp(i\alpha\overline{\gamma})\Psi \tag{3.11}$$

$$\overline{\Psi} \to \overline{\Psi}\exp(i\alpha\overline{\gamma}) \tag{3.12}$$

$$\overline{\Psi}\not{D}\Psi \to \overline{\Psi}\exp(i\alpha\overline{\gamma})\not{D}\exp(i\alpha\overline{\gamma})\Psi = \overline{\Psi}\not{D}\exp(-i\alpha\overline{\gamma})\exp(i\alpha\overline{\gamma})\Psi = \overline{\Psi}\not{D}\Psi. \tag{3.13}$$

Quantum: violated.

$$\text{Eigenmodes}: \qquad i\not{D}\Psi = \lambda\Psi, \tag{3.14}$$

(1) $\lambda \neq 0$: nonzero modes

$$i\not{D}\overline{\gamma}\Psi = -\lambda\overline{\gamma}\Psi, \qquad (\because \{\not{D}, \overline{\gamma}\} = 0) \tag{3.15}$$

Always pair (Ψ_+, Ψ_-)

$$\Psi_\pm = \left(\frac{1 \pm \overline{\gamma}}{2}\right)\Psi \tag{3.16}$$

with

$$\overline{\gamma}\Psi_\pm = \pm\Psi_\pm, \qquad i\not{D}\Psi_\pm = \lambda\Psi_\mp, \tag{3.17}$$

(2) $\lambda = 0$: zero modes
Projector

$$P_\pm = \frac{1 \pm \overline{\gamma}}{2}, \tag{3.18}$$

Abelian group structure:

$$[Y_1] + [Y_2] = [Y_1 \sqcup Y_2], \quad [\varnothing] = 0, \quad [\overline{Y}] = -[Y]. \tag{6.11}$$

Def
Cobordism group with U(1) coefficient

$$\mathrm{Hom}(\Omega^H_D \to \mathrm{U}(1)) \tag{6.12}$$

Meaning:
(1) Given a Y, we have a partition function $Z(Y) \in \mathrm{U}(1)$.
(2) If $Y' \sim Y$, we get $Z(Y') = Z(Y)$.
Such a quantity is called cobordism invariant.

$Z(Y) = \exp(-2\pi i\eta(Y))$ is cobordism invariant if $I_{d+2} = 0$.

Theorem
Topologically invariant invertible theories are classified by cobordism invariants.
Not restricted to fermions, but more generally true.

Example : $H_D = \mathrm{Spin}(D) \times \mathrm{SU}(N)$

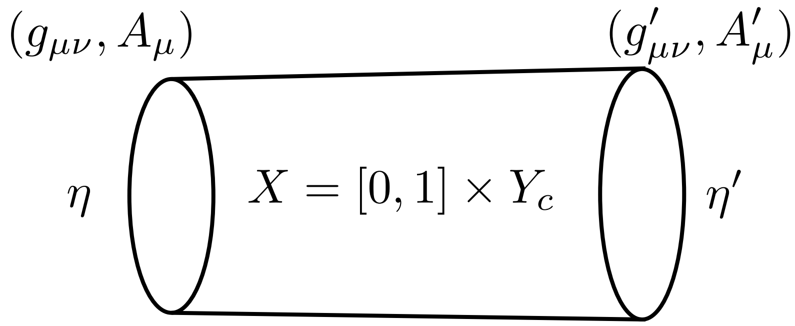
Table of Ω^H_D : (6.13)

$D = d+1$	0	1	2	3	4	5
No SU(N)	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
SU(2)	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}^2$	$\mathbb{Z}_2$
SU($N \geq 3$)	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}^2$	0

If you are interested in $d = 4$ physics:
$\Omega^H_5 = \mathbb{Z}_2$ for SU(2) : Witten SU(2) anomaly $\exp(-2\pi\eta) = (-1)^{N_R}$
$\Omega^H_5 = 0$ for SU($N \geq 3$): no global anomaly if $I_6 = 0$. (E.g. SU(5) GUT anomaly free).
$\Omega^H_4 = \mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$: gauge and gravitational instantons

6.4 Summary

- General anomaly controlled by $\exp(-2\pi i\eta)$ on $(d+1)$-dim. closed Y.
- If $Y = \partial X$, $\exp(-2\pi i\eta(Y)) = \exp(2\pi i \int_X I_{d+2})$.
  I_{d+2} = anomaly polynomial for perturbative anomalies
- If $I_{d+2} = 0$, global anomalies are cobordism invariant.



$$\exp(-2\pi i(\eta' - \eta)) = \exp(2\pi i \int_X I_{d+2}) = 1. \tag{6.7}$$

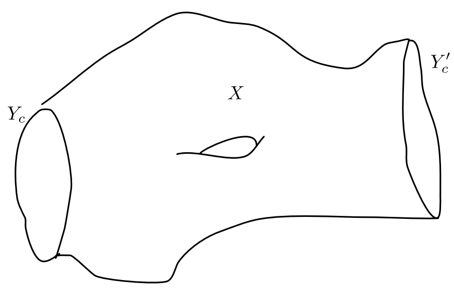
$$\therefore \exp(-2\pi i\eta') = \exp(-2\pi i\eta). \tag{6.8}$$

A property stronger than topological invariance:

$$\exp(-2\pi i\eta(Y)) = 1 \text{ if } Y = \partial X. \tag{6.9}$$

Notation:
$\overline{Y}$: orientation reversal of Y. (More precisely, opposite H-structure. Detail omitted.)
$Y_1 \sqcup Y_2$: disjoint union of Y_1 and Y_2.

Define the bordism group Ω_D^H as follows:
Y : Closed $D(= d+1)$-dm. manifold ($\partial Y = 0$) with H-structure.
Equivalence relation $Y \sim Y'$ if and only if $Y' \sqcup \overline{Y} = \partial X$ for some X to which the H-structure is extended.



Equivalence class denoted as $[Y]$. ($[Y'] = [Y]$ if $Y' \sim Y$).
Def

$$\Omega_D^H = \{[Y]; \quad Y\text{: all closed } D\text{-manifold with } H \text{ structure } \} \tag{6.10}$$

$$(P_\pm)^2 = P_\pm, \quad P_+ P_- = 0 \tag{3.19}$$

$$\not{D}\Psi = 0 \Rightarrow \not{D} P_\pm \Psi = 0 \tag{3.20}$$

$$\not{D}\Psi = 0, \quad \overline{\gamma}\Psi = +\Psi : \text{ zero mode with positive chirality} \tag{3.21}$$
$$\not{D}\Psi = 0, \quad \overline{\gamma}\Psi = -\Psi : \text{ zero mode with negative chirality} \tag{3.22}$$

Mode expansion

$$\Psi = \sum_a A_{+,a}\Psi_{+,a} + \sum_b A_{-,b}\Psi_{-,b}, \tag{3.23}$$

$$\overline{\gamma}\Psi_{\pm,a} = \pm\Psi_{\pm,a}, \tag{3.24}$$

Path integral measure

$$[D\Psi] = \prod_a dA_{+,a} \prod_b dA_{-,b} \tag{3.25}$$

Axial rotation $\Psi \to \exp(i\overline{\gamma}\alpha)\Psi$

$$A_{a,\pm} \to e^{\pm i\alpha} A_{a,\pm} \tag{3.26}$$
$$[D\Psi] \to \exp(-i\alpha(N_+ - N_-))[D\Psi] \tag{3.27}$$
$$(N_\pm : \text{ number of modes with } \gamma\psi = \pm\psi.) \tag{3.28}$$
$$= \exp(-i\alpha(n_+ - n_-))[D\Psi] \tag{3.29}$$
$$(n_\pm : \text{ number of zero modes with } \gamma\psi = \pm\psi.) \tag{3.30}$$

Nonzero modes cancel in each pair.
$\overline{\Psi}$ also contributes (exercise)

$$\overline{\Psi} \to \overline{\Psi}\exp(i\alpha\overline{\gamma}), \tag{3.31}$$
$$[D\overline{\Psi}] \to \exp(-i\alpha(n_+ - n_-))[D\overline{\Psi}]. \tag{3.32}$$

$$[D\Psi D\overline{\Psi}] \to \exp(-2i\alpha(n_+ - n_-))[D\Psi D\overline{\Psi}]. \tag{3.33}$$

$[D\Psi]$ is not invariant under the axial U(1).

3.2 Atiyah-Singer (AS) index theorem

Let's compute $\mathrm{index}(\not{D}) := n_+ - n_-$ when $g_{\mu\nu} = \delta_{\mu\nu}$ (flat metric).
Heat kernel method

$$\mathrm{index}(\not{D}) = n_+ - n_- = \mathrm{Tr}\,\overline{\gamma}\exp(t\not{D}^2). \tag{3.34}$$

$t > 0$: arbitrary. The trace is over all modes.
Nonzero modes cancel between $\Psi_\pm$, since

$$\not{D}^2 \Psi_\pm = -\lambda^2 \Psi, \qquad \overline{\gamma}\Psi_\pm = \pm\Psi_\pm. \tag{3.35}$$

We will take $t \to 0$.

$$\not{D}^2 = D_\mu D_\nu \frac{1}{2}\left(\{\gamma_\mu, \gamma_\nu\} + [\gamma_\mu, \gamma_\nu]\right) \tag{3.36}$$
$$= D^2 + \frac{1}{2} F_{\mu\nu}\gamma^{\mu\nu} \tag{3.37}$$

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\mu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{3.38}$$
$$\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{3.39}$$

If periodic boundary condition $x^\mu \sim x^\mu + L$, Fourier modes

$$\frac{1}{L^{d/2}} \exp(2\pi i x^\mu n_\mu / L), \qquad n_\mu \in \mathbb{Z}. \tag{3.40}$$

Trace of the operator $O = \overline{\gamma}\exp(t\not{D}^2)$:

$$\operatorname{Tr} O = \sum_{n_\mu} \operatorname{tr}_s \operatorname{tr}_g O_{n,n}, \qquad O_{n,m} = \int \frac{d^d x}{L^d} e^{-2\pi i n\cdot x} O e^{2\pi i m\cdot x} \tag{3.41}$$

Set $k_\mu = 2\pi n_\mu / L$.

$$\sum_{n_\mu} \int \frac{d^d x}{L^d} \longrightarrow \int \frac{d^d x d^d k}{(2\pi)^d} \tag{3.42}$$

$$\operatorname{Tr} O = \int \frac{d^d x d^d k}{(2\pi)^d} \operatorname{tr}_s \operatorname{tr}_g e^{-ikx} O e^{ikx}. \tag{3.43}$$

tr_s: spin index, tr_g: gauge index, $d = 2n$.
Use

$$\partial_\mu e^{ikx} = e^{ikx}(\partial_\mu + ik_\mu). \tag{3.44}$$

$$\operatorname{Tr} \overline{\gamma}\exp(t\not{D}^2) = \int \frac{d^d x d^d k}{(2\pi)^d} \operatorname{tr}_s \operatorname{tr}_g \overline{\gamma} \exp\left(t(D_\mu + ik_\mu)^2 + \frac{t}{2} F_{\mu\nu}\gamma^{\mu\nu}\right) \tag{3.45}$$

D_μ : acts on $F_{\mu\nu}$ etc.
Take limit $t \to +0$.

$$t(D + ik)^2 = (\sqrt{t}D + iK)^2 \quad (K = \sqrt{t}k) \tag{3.46}$$
$$\xrightarrow{t\to 0} -K^2. \tag{3.47}$$

6 Structure of anomaly

Omit subscript c of Y_c and write just as Y. $\partial Y = 0$.

6.1 Atiyah-Patodi-Singer (APS) index theorem

Suppose X: $(d+2)$-dim. manifold with

$$\partial X = Y. \tag{6.1}$$

APS index theorem (proof omitted):

$$\text{index}\, \not{D}_X = \int_X I_{d+2} + \eta(Y) \tag{6.2}$$

$$I_{d+2} = \hat{A}(X)\, \mathrm{tr}_{\mathfrak{g}} \exp\left(\frac{iF}{2\pi}\right)\Big|_{d+2} \tag{6.3}$$

At the level of Lie algebra $\mathfrak{h}_d$ of H_d,

$$\mathfrak{h}_d = \mathfrak{so}(d) \oplus \mathfrak{g}, \qquad \mathfrak{g} : \text{internal symmetry} \tag{6.4}$$

The trace is over the internal part $\mathfrak{g}$ in the rep. of the fermion.

6.2 Anomaly polynomial

Anomaly: $\exp(-2\pi i\eta(Y))$. If $Y = \partial X$ for some X,

$$\exp(-2\pi i\eta(Y)) = \exp(2\pi i \int_X I_{d+2}) \tag{6.5}$$

I_{d+2} : anomaly polynomial.
The modern version of the anomaly descent equation:
perturbative anomaly captured by I_{d+2}.
Majorana : 1/2.

Meaning of "Perturbative":
If $I_{d+2} \neq 0$, small change of the metric and gauge fields give nonzero change of η.
η is sensitive to infinitesimal change.
Perturbatively,

$$\eta \sim \text{Chern-Simons} \tag{6.6}$$

6.3 Global anomalies and cobordism

Suppose $I_{d+2} = 0$ (perturbative anomaly cancellation).
$\exp(-2\pi i\eta)$: topological invariant.
Proof

For Majorana fermion

$$Z(Y_c) = \frac{\mathrm{Pf}(\not{D} - M)}{\mathrm{Pf}(\not{D} + M)} = \exp(-\pi i \eta) \tag{5.40}$$

one Dirac = two Majorana.
For simplicity let us focus on Dirac. (Majorana is more general).

A freedom to modify $Z(Y)$:

$$Z(Y) \to Z(Y) \exp(-\int \mathcal{L}_{\text{c.t.}}) \tag{5.41}$$

$$\mathcal{L}_{\text{c.t.}} : \text{manifestly gauge invariant counterterm} \tag{5.42}$$

Chern-Simons is not gauge invariant on manifolds with boundary, and should not be included in $\mathcal{L}_{\text{c.t.}}$.

Invertible topological phase up to counterterms
= symmetry protected topological (SPT) phase with symmetry H_d
= anomaly of the boundary theory.

5.4 Example: $d = 4$, SU(2) anomaly

$d = 4$ SU(2), Weyl in rep. R
No perturbative anomaly (exercise)

Take

$$Y_c = S^1 \times S^4 \tag{5.43}$$

$$S^4 : \text{ contain an SU(2) instanton} \tag{5.44}$$

$$S^1 : \text{ periodic b.c.} \tag{5.45}$$

$$\eta = \frac{1}{2} \sum \mathrm{sign}(\lambda) \qquad (\mathrm{sign}(0) = +1) \tag{5.46}$$

$$= \text{ zero modes } \frac{1}{2}(n_+ + n_-) \text{ on } S^4 \tag{5.47}$$

$$\text{(Nonzero modes do not contribute in this case: Exercise)} \tag{5.48}$$

$$= \frac{1}{2}(n_+ - n_-) \quad \text{mod } 1 \tag{5.49}$$

$$= \frac{1}{2} N_R \qquad N_R : \text{AS index of rep. } R \text{ on an instanton} \tag{5.50}$$

$$\exp(-2\pi i \eta) = (-1)^{N_R} : \text{Witten SU(2) anomaly} \tag{5.51}$$

For $R = n$-dim. rep.

$$N_R = \frac{1}{6} n(n^2 - 1). \tag{5.52}$$

2-dim. rep. (fundamental of SU(2)) $N_R = 1$: anomalous.

Drop D_μ. On the other hand, (exercise)

$$\mathrm{tr}_s(\overline{\gamma}\gamma^{\mu_1}\cdots\gamma^{\mu_k}) = \begin{cases} 0 & k < d = 2n \\ i^n 2^n \epsilon^{\mu_1\cdots\mu_d} & k = d = 2n \end{cases} \tag{3.48}$$

We need multiple γ^μ's to get nonzero value $\Longrightarrow$ Keep $\frac{t}{2}F_{\mu\nu}\gamma^{\mu\nu}$.

$$\begin{aligned} \mathrm{Tr}\,\overline{\gamma}\exp(t\not{D}^2) &= \int \frac{d^dx d^dk}{(2\pi)^d}\,\mathrm{tr}_s\,\mathrm{tr}_g\,\overline{\gamma}\exp\left(-tk^2 + \frac{t}{2}F_{\mu\nu}\gamma^{\mu\nu}\right) && (3.49) \\ &= \int \frac{d^dx}{(2\pi)^d}\frac{\pi^n}{t^n}\,\mathrm{tr}_s\,\mathrm{tr}_g\,\overline{\gamma}\frac{1}{n!}\left(\frac{t}{2}F_{\mu\nu}\gamma^{\mu\nu}\right)^n && (3.50) \\ &= \int d^dx\,\frac{1}{n!}\left(\frac{i}{4\pi}\right)^n \mathrm{tr}_g(F_{\mu_1\mu_2}\cdots F_{\mu_{2n-1}\mu_{2n}})\epsilon^{\mu_1\cdots\mu_{2n}} && (3.51) \end{aligned}$$

Differential form notation

$$A = A_\mu dx^\mu \tag{3.52}$$

$$F = dA + A\wedge A = \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu \tag{3.53}$$

Chern character

$$\mathrm{ch}_k(F) = \frac{1}{k!}\,\mathrm{tr}\left(\frac{iF}{2\pi}\right)^k \quad : 2k\text{-form} \tag{3.54}$$

In $d = 2n$-dim.

$$\mathrm{ch}_n(F) = \left(\frac{i}{4\pi}\right)^n \mathrm{tr}_g(F_{\mu_1\mu_2}\cdots F_{\mu_{2n-1}\mu_{2n}})\epsilon^{\mu_1\cdots\mu_{2n}}d^dx \tag{3.55}$$

$$d^dx = dx^1\wedge\cdots\wedge dx^d \tag{3.56}$$

AS index theorem ($g_{\mu\nu} = \delta_{\mu\nu}$):

$$\mathrm{index}\,\not{D} = \mathrm{Tr}\,\overline{\gamma}\exp(t\not{D}^2) = \int \mathrm{ch}_n(F) \tag{3.57}$$

Remark.
In $t\to 0$, only $k^\mu\to\infty$ (short distance) contributes. Thus

$$\text{index} = \int (\text{local quantity}) \tag{3.58}$$

3.3 AS index theorem with general metric

$$\omega = (\omega_{\mu ab}dx^\mu) \tag{3.59}$$

$$\begin{aligned} R &= \left(\frac{1}{2}R_{ab\mu\nu}dx^\mu\wedge dx^\nu\right)_{1\le a,b\le d} \quad : \text{Riemann curvature 2-form} && (3.60) \\ &= d\omega + \omega\wedge\omega && (3.61) \end{aligned}$$

Guess the index theorem (Detail: Atiyah-Bott-Patodi)

(1) Heat kernel method suggests

$$\text{index}\, \not{D} = \int (\text{gauge invariant polynomial of } F \text{ and } R) \tag{3.62}$$

Invariant polynomials of R:

$$\operatorname{tr} R^2, \quad \operatorname{tr} R^4, \quad \operatorname{tr} R^6, \cdots, \text{ products of them} \tag{3.63}$$

$4k$-forms.
When $F = 0$, denote

$$\text{index}\, \not{D} = \int \hat{A}_{d/4}(R) \tag{3.64}$$

$$\hat{A}_k(R) : \text{a polynomial of } R,\ 4k\text{-form} \tag{3.65}$$

(2) Suppose spacetime manifold X and gauge fields on it factorize

$$X = X_1 \times X_2 \tag{3.66}$$

Then

$$\text{index}\, \not{D} = (\text{index}\, \not{D}_1)(\text{index}\, \not{D}_2) \tag{3.67}$$

Proof : $i\not{D}$: Hermite (self-adjoint). So $i\not{D}\Psi = 0 \Longleftrightarrow (i\not{D}\Psi)^2 = 0$.

$$\begin{aligned}(i\not{D})^2 &= (i\not{D}_1 + i\not{D}_2)^2 &(3.68)\\ &= (i\not{D}_1)^2 + (i\not{D}_2)^2 - \{\not{D}_1, \not{D}_2\} &(3.69)\\ &= (i\not{D}_1)^2 + (i\not{D}_2)^2 &(3.70)\end{aligned}$$

$(i\not{D}_1)^2$, $(i\not{D}_2)^2$ non-negative ≥ 0.

$$\begin{aligned}&[(i\not{D}_1)^2 + (i\not{D}_2)^2]\Psi = 0 &(3.71)\\ \Longleftrightarrow\ &(i\not{D}_1)^2\Psi = 0, \quad (i\not{D}_2)^2\Psi = 0 &(3.72)\\ \Longleftrightarrow\ &i\not{D}_1\Psi = 0, \quad i\not{D}_2\Psi = 0 &(3.73)\end{aligned}$$

$$\not{D}\Psi = 0 \Longleftrightarrow \not{D}_1\Psi = \not{D}_2\Psi = 0 \tag{3.74}$$

(3) Suppose

$$X_1 : \text{ purely } R \qquad \text{index}\, \not{D}_1 = \int_{X_1} \hat{A}_{d_1/4}(R) \tag{3.75}$$

$$X_2 : \text{ purely } F \qquad \text{index}\, \not{D}_2 = \int_{X_2} \operatorname{ch}_{d_2/2}(F) \tag{3.76}$$

5.3 Bulk theory: invertible topological phase, SPT phase

Invertible topological phase :

1. The Hilbert spaces are one-dimensional (only the ground state) in the low energy limit on closed manifolds W.
2. But the partition function may be nontrivial.

Massive fermion $\to$ topological phase when $|m| \to \infty$.

Compute the bulk $d+1$-dim. partition function in the case $\partial Y_c = \varnothing$.
For simplicity take $m = -M$ and $M \to +\infty$. Dirac fermion.

$$Z(Y_c) = \frac{\det(\not{D} + m)}{\det(\not{D} + M)} \tag{5.30}$$
$$= \prod_{\lambda \in \text{eigenvalue}(i\not{D})} \frac{(-i\lambda - M)}{(-i\lambda + M)} \tag{5.31}$$
$$= \prod_{\lambda} e^{-2\pi i s(\lambda)} \tag{5.32}$$

Here

$$s(\lambda) = -\frac{1}{2\pi} \arg\left(\frac{-i\lambda - M}{-i\lambda + M}\right) \tag{5.33}$$
$$-\pi \leq \arg < \pi. \tag{5.34}$$

Note

$$\lim_{M\to\infty} s(\lambda) = \frac{1}{2}\text{sign}(\lambda). \qquad (\text{sign}(\lambda = 0) \overset{\text{def}}{=} +1) \tag{5.35}$$

Atiyah-Patodi-Singer (APS) η-invariant

$$\eta = \left(\frac{1}{2}\sum_{\lambda} \text{sign}(\lambda)\right)_{\text{reg}} = \lim_{M\to\infty}\left(\frac{1}{2}\sum_{\lambda} s(\lambda)\right) \tag{5.36}$$

The theory with $m < 0$ is nontrivial.

$$Z(Y_c) = \exp(-2\pi i\eta) \tag{5.37}$$

The theory with $m > 0$: trivial. Take $m = M$,

$$Z(Y_c) = \frac{\det(\not{D} + M)}{\det(\not{D} + M)} = 1. \tag{5.38}$$

	$m < 0$	$m > 0$
edge	localized χ	no localized mode
bulk	$Z(Y_c) = \exp(-2\pi i\eta)$	$Z(Y_c) = 1$

(5.39)

Hence

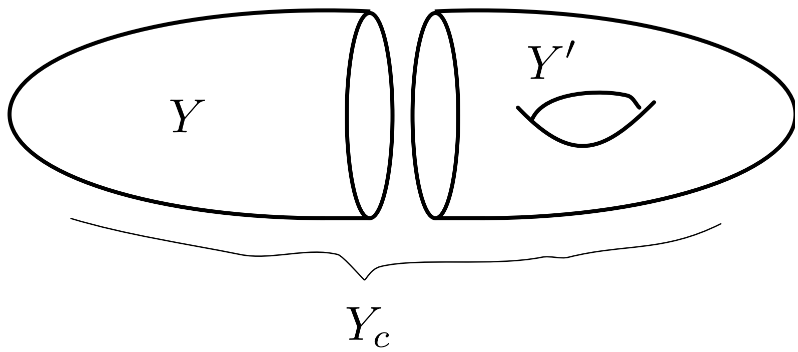
$$Z(Y, \mathsf{L}) = \langle \mathsf{L}|Y\rangle = Z(Y, \mathsf{L}) = \langle \mathsf{L}|\Omega\rangle\langle\Omega|Y\rangle \tag{5.22}$$

Let us assume $|\langle\Omega|Y\rangle| = 1$. (Detail omitted).

Y' : another manifold with $\partial Y' = W$.

$$\frac{Z(Y, \mathsf{L})}{Z(Y', \mathsf{L})} = \frac{\langle\Omega|Y\rangle}{\langle\Omega|Y'\rangle} \tag{5.23}$$
$$= \langle Y'|\Omega\rangle\langle\Omega|Y\rangle(\because |\langle\Omega|Y\rangle| = 1) \tag{5.24}$$
$$= \langle Y'|Y\rangle \qquad (\because |Y\rangle \propto |\Omega\rangle) \tag{5.25}$$
$$= Z(Y_c) \tag{5.26}$$

$$Y_c = Y \text{ and } Y' \text{ glued along the common boundary } W\text{, with orientation reversal of } Y' \tag{5.27}$$



General anomaly formula:

$$\frac{Z(Y, \mathsf{L})}{Z(Y', \mathsf{L})} = Z(Y_c) \tag{5.28}$$

If $Z(Y_c) = 1$ for all closed manifolds, $Z(Y, \mathsf{L})$ can be used as a definition of the partition function of χ,

$$Z_\chi(W) := Z(Y, \mathsf{L}) \quad \text{if } Z(Y_c) = 1 \text{ for any } Y_c. \tag{5.29}$$

(Some important details omitted.)

$$\begin{aligned}\text{index}\, \not{D} &= (\text{index}\, \not{D}_1)(\text{index}\, \not{D}_2) && (3.77)\\ &= \int_{X=X_1\times X_2} \hat{A}_{d_1/4}(R)\mathrm{ch}_{d_2/2}(F) && (3.78)\end{aligned}$$

(4) General X: to reproduce (3) as a special case,

$$\text{index}\, \not{D} = \int \sum_k \hat{A}_{(d-2k)/4}(R)\mathrm{ch}_k(F). \tag{3.79}$$

Define

$$\mathrm{ch} = \sum_k \mathrm{ch}_k \tag{3.80}$$

$$\hat{A} = \sum_k \hat{A}_k \tag{3.81}$$

$$\text{index}\, \not{D} = \int \hat{A}(R)\mathrm{ch}(F)|_{d\text{-form}} \tag{3.82}$$

(4) Explicit form of $\hat{A}(R)$?
R : Formally anti-symmetrix matrix (whose matrix elements are 2-forms).
Normal form

$$\frac{R}{2\pi} = \left(\begin{array}{cc|cc|c} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ \hline & & 0 & x_2 & \\ & & -x_2 & 0 & \\ \hline & & & & \ddots \\ & & & & \quad\ddots \end{array}\right) \tag{3.83}$$

Answer:

$$\hat{A}(R) = \prod_i \frac{x_i/2}{\sinh(x_i/2)} \tag{3.84}$$

Consistent with factorization index $\not{D} = (\text{index}\, \not{D}_1)(\text{index}\, \not{D}_2)$.
How to use: Expand it in terms of x's and rewrite it by $\mathrm{tr}\, R^2, \mathrm{tr}\, R^4, \cdots$

$$\begin{aligned}\hat{A}(R) &= 1 - \frac{1}{24}\sum_i x_i^2 + \cdots && (3.85)\\ &= 1 - \frac{1}{24}\cdot\left(-\frac{1}{2}\,\mathrm{tr}(\frac{R}{2\pi})^2\right) + \cdots && (3.86)\end{aligned}$$

$$\hat{A}_0 = 1, \qquad \hat{A}_1 = -\frac{1}{24}\cdot\left(-\frac{1}{2}\,\mathrm{tr}(\frac{R}{2\pi})^2\right), \qquad \cdots \tag{3.87}$$

Derivation of $\hat{A}$:

- Check it for as many examples as necessary to fix coefficients. (Atiyah-Bott-Patodi)
- K-theory manipulation (Atiyah-Hirzebruch, AS)
- Supersymmetric quantum mechanics. (Alvarez-Gaume)

3.4 Characteristic classes

(Not used later)

$$\frac{iF}{2\pi} = \begin{pmatrix} y_1 & & \\ & y_2 & \\ & & \ddots \end{pmatrix}, \qquad \frac{R}{2\pi} = \left(\begin{array}{cc|cc|c} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ \hline & & 0 & x_2 & \\ & & -x_2 & 0 & \\ \hline & & & & \ddots \\ & & & & \quad\ddots \end{array}\right) \tag{3.88}$$

Chern class

$$\mathrm{c}(F) = \sum_k \mathrm{c}_k(F) = \prod_i (1+y_i). \tag{3.89}$$

Chern character

$$\mathrm{ch}(F) = \sum_k \mathrm{ch}_k(F) = \sum_k \frac{1}{k!} \sum_i y_i^k = \sum_i e^{y_i} \tag{3.90}$$

Pontryagin class

$$\mathrm{p} = \sum_k \mathrm{p}_k = \prod_i (1+x_i^2). \tag{3.91}$$

$$\hat{A} = 1 - \frac{1}{24}\mathrm{p}_1 + \frac{7\mathrm{p}_1^2 - 4\mathrm{p}_2}{5760} + \cdots \tag{3.92}$$

3.5 Example

$d = 2n = 2$, U(1) gauge field A, $X = S^2$. Polar coordinates

$$(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \qquad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi. \tag{3.93}$$

Dirac monopole configuration

$$A = \begin{cases} A_N = \frac{k}{2} i(\cos\theta + 1) d\phi & \theta \le \pi/2 \\ A_S = \frac{k}{2} i(\cos\theta - 1) d\phi & \theta \ge \pi/2. \end{cases} \tag{3.94}$$

k : parameter. A_N, A_S non-singular.
For example, $\cos\theta - 1 \sim -\frac{1}{2}\theta^2$ near $\theta \sim 0$, cancelling the singularity of $d\phi$.
(Exercise : take coordinates $(x, y, \sqrt{1 - x^2 - y^2})$ near $\theta \sim 0$ and check that A_N is non-singular.)

5.2 General anomaly

Partition function of $d+1$-massive fermion on Y with boundary $\partial Y = W$ and the local boundary condition L:

$$Z(Y, \mathsf{L}) = \int [D\Psi] e^{-S(\Psi)}. \tag{5.17}$$

For $m < 0$,

- Bulk: large mass gap in the limit $|m| \to \infty$
- Edge: chiral fermion χ

We want to think

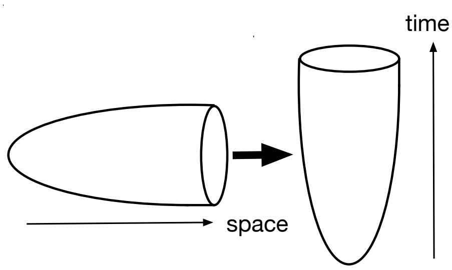
$$Z(Y, \mathsf{L}) \sim Z_\chi(W). \qquad \text{(chiral fermion partition function)} \tag{5.18}$$

Left hand side : completely gauge invariant, but may depend on Y.

Modern formulation of anomalies

- Everything is formulated in a completely gauge invariant way.
- Definition of the d-dim. theory depends on $(d+1)$-dim. : anomaly

Computation of $Z(Y, \mathsf{L})$.
Wick rotation



Path integral on Y gives a state vector $|Y\rangle \in \mathcal{H}_W$. $\mathcal{H}_W$: Hilbert space on W.
The local boundary condition L also gives a state vector $|\mathsf{L}\rangle$. (Compare $|x = x_0\rangle$ of QM.)

$$Z(Y, \mathsf{L}) = \langle \mathsf{L} | Y \rangle. \tag{5.19}$$

Time evolution near the boundary $(-\epsilon, 0] \times W$:

$$e^{-\epsilon H} \to |\Omega\rangle\langle\Omega| \qquad (|m|\epsilon \gg 1) \tag{5.20}$$

So

$$|Y\rangle \propto |\Omega\rangle \tag{5.21}$$

$$(1-\gamma^\tau)\Psi|_{\tau=0} = (1-\gamma^\tau)\chi = 0. \tag{5.9}$$

$$(\partial_\tau + \mathcal{D}_W + m\gamma^\tau)\Psi = (-m\chi + \mathcal{D}_W\chi + m\gamma^\tau\chi)\exp(-m\tau) = 0. \tag{5.10}$$

$\tau < 0$ in our convention.
$\exp(-m\tau)$ localized near $\tau \sim 0$ if and only if $m < 0$.
No such mode for $m > 0$.

γ^τ : generalized "chirality" operator of the boundary W.

$$\gamma^\tau \mathcal{D}_W + \mathcal{D}_W \gamma^\tau = 0 \tag{5.11}$$

χ : Massless "chiral" fermion on the boundary

$$\gamma^\tau \chi = +\chi, \qquad \mathcal{D}_W \chi = 0 \tag{5.12}$$

Example: $d+1=5$, $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5$.
Take $\gamma^\tau = \gamma^5$: chirality in $d=4$.

Take Pauli-Villars mass $M > 0$: no localized massless mode from unphysical PV field.

Remark:
Current framework is completely general : any d and H_d. Any chiral fermion with spin 1/2 is realized in the above way.

Example: $d+1=2$, Majorana massive fermion Ψ_i with O(N) symmetry $\Psi_i \to M_{ij}\Psi_j$

$$S = -\frac{1}{2}\int_{\tau\leq 0} d\tau d\sigma \sum_i \Psi_i^T \epsilon(\gamma^\tau\partial_\tau + \gamma^\sigma\partial_\sigma + m)\Psi_i, \tag{5.13}$$

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.14}$$

Localized solution

$$\Psi(\tau,\sigma) = \frac{e^{-m\tau}}{\sqrt{2|m|}} \begin{pmatrix} \psi_i(\sigma) \\ 0 \end{pmatrix}. \tag{5.15}$$

$$S \to -\frac{1}{2}\int d\sigma\, \psi_i \frac{d}{d\sigma}\psi_i. \tag{5.16}$$

$d=1$ anomalous majorana fermions studied before.

Difference

$$A_N - A_S = g^{-1}dg, \qquad g = \exp(ik\phi). \tag{3.95}$$

g : U(1) gauge transformation (transition function) between A_N and A_S.
g must be single-valued under $\phi \sim \phi + 2\pi \Longrightarrow k$ is integer: Dirac quantization.

$$F = dA = -\frac{k}{2} i \sin\theta d\theta \wedge d\phi. \tag{3.96}$$

$$\text{index} = \int_{S^2} \frac{i}{2\pi} F = \int \frac{k}{4\pi} \sin\theta d\theta d\phi = k. \tag{3.97}$$

Dirac quantization guarantees that the index is integer.

4 Symmetry, topology, and anomaly

Recent development : global anomalies (nonperturbative)
Discrete symmetries can have global anomalies.
Global anomalies depend on topology of symmetry groups.

4.1 Topology of groups

We consider topology of a group G.
Basic example:

$$\mathrm{SU}(2) \neq \mathrm{SO}(3) \neq \mathrm{O}(3) \tag{4.1}$$

Their Lie algebras are the same.

$$J_1, J_2, J_3 : \text{Standard Lie algebra generators}, \quad [J_i, J_j] = i\epsilon_{ijk}J_k. \tag{4.2}$$

Difference between SU(2) and SO(3):

- "spin" odd representations are allowed in SU(2) but forbidden in SO(3).
- Some fiber bundles which are impossible in SU(2) are possible in SO(3).

Example:
Take $\mathrm{U}(1) \subset \mathrm{SO}(3)$.

$$\mathrm{U}(1) = \{\exp(i\phi J_3)\}. \tag{4.3}$$

$\exp(2\pi i J_3) = 1$ in SO(3). This excludes $J_3 = \frac{1}{2} + \mathbb{Z}$.
Fiber bundle on S^2 (polar coordinates (θ, ϕ))

$$A = \begin{cases} A_N = \frac{i}{2}J_3(\cos\theta + 1)d\phi & \theta \leq \pi/2 \\ A_S = \frac{i}{2}J_3(\cos\theta - 1)d\phi & \theta \geq \pi/2. \end{cases} \tag{4.4}$$

$$A_N - A_S = g^{-1}dg, \qquad g = \exp(iJ_3\phi) \tag{4.5}$$

g: Single valued for SO(3) since $J_3 \in \mathbb{Z}$. But not for SU(2).
Topological characterization:

$$g : S^1 \to \mathrm{SO}(3). \tag{4.6}$$

It gives an element $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$.
Topologically nontrivial bundle.

Difference between SO(3) and O(3):
some new configuration is possible in O(3).
Example:
$S^1 = \{[0,1] \text{ with two ends glued}\}$. O(3) transition function g;

$$(0, v_0),\ (1, v_1) \in [0,1] \times V, \qquad v_1 = gv_0 \tag{4.7}$$

V: 3-dim. vector space. Take $\det g = -1$. Topologically nontrivial bundle.
g gives an element of $\pi_0(\mathrm{O}(3)) \cong \mathbb{Z}_2$.

5 General chiral fermion anomalies

Systematic description of chiral fermions and their anomalies.

5.1 Chiral fermion as edge modes

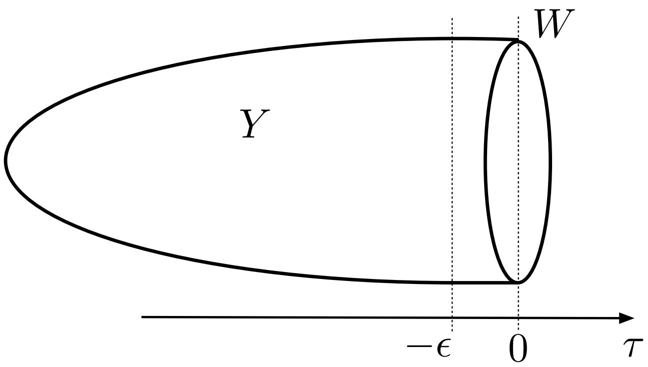
Setup:

$$W : d\text{-dim. manifold} \tag{5.1}$$
$$Y : d+1\text{-dim manifold with } \partial Y = W \tag{5.2}$$

Near the boundary ∂Y, assume

$$(-\epsilon, 0] \times W \subset Y \tag{5.3}$$

Coordinate normal to the boundary: $\tau \in (-\epsilon, 0]$.



Massive fermion in $(d+1)$-dim. Y

$$\mathcal{L} = -\overline{\Psi}(\not{D}_Y + m)\Psi \tag{5.4}$$

Remark: Majorana fermion is also OK, $\overline{\Psi} \sim \Psi$.

γ^τ : gamma matrix in the direction τ. Eigenvalues $\gamma^\tau = \pm 1$.
Local boundary condition

$$\mathsf{L} : \ (1 - \gamma^\tau)\Psi|_{\tau=0} = 0 \tag{5.5}$$

Near ∂Y,

$$(\not{D}_Y + m)\Psi = \gamma^\tau(\partial_\tau + \mathcal{D}_W + m\gamma^\tau)\Psi \tag{5.6}$$
$$\mathcal{D}_W = \sum_{\mu \neq \tau} \gamma^\tau \gamma^\mu D_\mu. \tag{5.7}$$

Take $|m|$ very large. Localized solution of $(\not{D}_Y + m)\Psi = 0$ near ∂Y if $m < 0$:

$$\Psi = \chi \exp(-m\tau), \qquad (1 - \gamma^\tau)\chi = 0, \qquad \mathcal{D}_W \chi = 0. \tag{5.8}$$

Groups $\mathrm{Pin}^{\pm}(d)$:

$$\mathrm{Pin}^{+}(d) = \{\text{Generated by Spin}(d) \text{ and } \widetilde{\mathsf{R}}_{+}.\} \tag{4.41}$$

$$\mathrm{Pin}^{-}(d) = \{\text{Generated by Spin}(d) \text{ and } \widetilde{\mathsf{R}}_{-}.\} \tag{4.42}$$

In Lorentz signature, time reversal T is

$$\mathsf{T}^2 = \begin{cases} (-1)^F & \text{for Pin}^{+} \text{ symmetry} \\ 1 & \text{for Pin}^{-} \text{ symmetry} \end{cases} \tag{4.43}$$

The reason for opposite behavior for T and $\widetilde{\mathsf{R}}$: Wick rotation $\gamma^0_E = i\gamma^0_M$.

Example : M-theory has $\mathrm{Pin}^{+}(11)$.

4.4 More general symmetries

Generally, Lorenz and internal symmetries cannot be distinguished.
Example:

$$\frac{\mathrm{Spin}(d) \times \mathrm{U}(1)}{\mathbb{Z}_2}. \tag{4.44}$$

$\mathbb{Z}_2$ is embedded by $(-1,-1) \in \mathrm{Spin}(d) \times \mathrm{U}(1)$.

- Lie algebra is factorized, $\mathfrak{so}(d) \times \mathfrak{u}(1)$.
- Global topology is different from $\mathrm{Spin}(d) \times \mathrm{U}(1)$.

Abstract description of a large class of symmetries:

$$\text{A group } H_d \text{ with a homomorphism} \tag{4.45}$$

$$\rho : H_d \to \mathrm{O}(d) \text{ with } \rho(H_d) \supset \mathrm{SO}(d). \tag{4.46}$$

$\mathrm{O}(d)$: Lorentz group.
Fermions : in some representation r of H_d such that for $h \in H_d$,

$$r(h)^{-1}\gamma_a r(h) = \rho(h)_{ab}\gamma_a \qquad (\gamma_a : \text{ gamma matrix}) \tag{4.47}$$

Background fields for H_d : a bundle with connection (gauge field) B_μ such that

$$\rho(B_\mu)_{ab} = \omega_{\mu ab} \text{ (spin connection)} \tag{4.48}$$

Such a H_d-bundle : H-structure of the manifold.
Example:

H_d	$\mathrm{SO}(d)$	$\mathrm{O}(d)$	$\mathrm{Spin}(d)$	$\mathrm{Pin}^{\pm}$	$[\mathrm{Spin}(d) \times \mathrm{U}(1)]/\mathbb{Z}_2$
structure	orientation	no orientation	spin structure	$\mathrm{pin}^{\pm}$ structure	spin^c structure

(4.49)

4.2 Simple examples of global anomalies

Let's see anomalies associated to $\pi_0(\mathrm{O}(N))$, $\pi_1(\mathrm{SO}(N))$.

$d = 1$ majorana fermions ψ_i $(i = 1, 2, \cdots N)$.

$$\mathcal{L} = \sum_{i=1}^{N} \frac{i}{2}\psi_i \frac{d}{dt}\psi_i. \tag{4.8}$$

Flavor $\mathrm{O}(N)$ symmetry

$$\psi_i \to M_{ij}\psi_j, \qquad M \in \mathrm{O}(N). \tag{4.9}$$

Partition function

$$\mathrm{Tr}\, e^{-\beta H} = \text{The path integral on } S^1. \tag{4.10}$$

$S^1 = [0, \beta]$ with 0 and β glued. Thermal boundary condition

$$\psi_i(\beta) = -\psi_i(0). \tag{4.11}$$

Consider $g \in \mathrm{O}(N)$ with $\det g = -1$.

$$g = \mathrm{diag}(-1, +1, \cdots, +1). \tag{4.12}$$

$$\mathrm{Tr}(e^{-\beta H} g) = \text{The path integral on } S^1 \tag{4.13}$$

with

$$\psi_1(\beta) = +\psi_1(0), \qquad \psi_j(\beta) = -\psi_j(0) \ (j \neq 1). \tag{4.14}$$

Mode expansion

$$\psi_1(\tau) = A_0^{(1)} + \sum_{n\geq 1}(B_n^{(1)} e^{2\pi n i\tau/\beta} + C_n^{(1)} e^{-2\pi n i\tau/\beta}), \tag{4.15}$$

$$\psi_j(\tau) = \sum_{n\geq 1}(B_n^{(j)} e^{2\pi i(n-1/2)\tau/\beta} + C_n^{(j)} e^{2\pi i(n-1/2)\tau/\beta}) \tag{4.16}$$

Nonzero modes appears in pairs (B, C). $A_0^{(1)}$: zero mode.
The path integral measure

$$[D\psi] = dA_0^{(1)} \prod dB_n^{(i)} dC_n^{(i)} \tag{4.17}$$

$(-1)^F$ transformation

$$\psi_i(\tau) \to -\psi_i(\tau) \tag{4.18}$$

$$[D\psi] \to -[D\psi]. \tag{4.19}$$

The sign change due to the zero mode $A_0^{(1)}$.
$[D\psi]$ is not invariant under $(-1)^F$. An anomaly associated to $\pi_0(\mathrm{O}(N))$ $(\det g = -1)$.

Next consider

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} + \mathrm{diag}(1, \cdots, 1). \tag{4.20}$$

We set

$$\psi = \frac{\psi_1 - i\psi_2}{\sqrt{2}}, \qquad \overline{\psi} = \frac{\psi_1 + i\psi_2}{\sqrt{2}}, \qquad \text{neglect } \psi_i \ (i \geq 3). \tag{4.21}$$

$$\mathcal{L} = i\overline{\psi}\frac{d}{dt}\psi \tag{4.22}$$

$$\psi(\beta) = -e^{i\theta}\psi(0) \tag{4.23}$$

$$\psi(\tau) = \sum_{n\in\mathbb{Z}} A_n \exp\left(2\pi i(n - \frac{1}{2} + \frac{\theta}{2\pi})\right), \qquad \overline{\psi}(\tau) = \cdots . \tag{4.24}$$

$$\mathrm{Tr}(e^{-\beta H} g(\theta)) = \int [D\psi] e^{-S} \tag{4.25}$$

$$\propto \prod_{n\in\mathbb{Z}} (n - \frac{1}{2} + \frac{\theta}{2\pi}) \tag{4.26}$$

Sign choice : $\mathrm{Tr}(e^{-\beta H} g) > 0$ at $\theta = 0$.

$$\int [D\psi] e^{-S} = C \prod_{n\geq 1} (n - \frac{1}{2} + \frac{\theta}{2\pi})(n - \frac{1}{2} - \frac{\theta}{2\pi}) \qquad C > 0. \tag{4.27}$$

Smoothly change θ from 0 to 2π.

$$\theta = 0 : \text{ all factors positive} \tag{4.28}$$

$$\theta = 2\pi : \text{ one factor is negative, } (1 - \frac{1}{2} - \frac{\theta}{2\pi}) \tag{4.29}$$

Conclusion:

$$[D\psi]_{\theta=2\pi} = -[D\psi]_{\theta=0}. \tag{4.30}$$

But $g(\theta = 2\pi) = 1$ as an element of $\mathrm{SO}(N)$.
The path integral measure has a sign ambiguity: an anomaly.
$g(\theta) : S^1 \to \mathrm{SO}(N)$ gives an element of $\pi_1(\mathrm{SO}(N))$.
An anomaly associated to $\pi_1(\mathrm{SO}(N))$.

Interpretations:

1. We have the $\mathrm{SO}(N)$ symmetry which is anomalous.
2. We have the $\mathrm{Spin}(N)$ symmetry which is anomaly free.

The first is convenient for 't Hooft anomaly matching of global symmetries.
The second is necessary for gauge symmetries.

4.3 Time reversal or reflection symmetry and Pin groups

Time reversal T : Important in many systems.

- Topological insulators and superconductors
- String worldsheet
- M-theory, Type IIA
- ...

$$\begin{array}{ccc} & \text{Bosonic} & \text{Fermionic} \\ \text{Without } \mathsf{T} & \mathrm{SO}(d) & \mathrm{Spin}(d) \\ \text{With } \mathsf{T} & \mathrm{O}(d) & \mathrm{Pin}^{\bullet}(d) \end{array} \tag{4.31}$$

Recall

$$\mathrm{SO}(d) = \{\Lambda = (\Lambda^{\mu}{}_{\nu}), \quad \det\Lambda = +1\} \tag{4.32}$$

$$\mathrm{O}(d) = \{\Lambda = (\Lambda^{\mu}{}_{\nu}), \quad \det\Lambda = \pm 1\} \tag{4.33}$$

$$\mathrm{Spin}(d) = \left\{\exp(\frac{1}{4}\gamma^{\mu\nu}\omega_{\mu\nu}), \quad \omega_{\mu\nu} = -\omega_{\nu\mu}\right\} \tag{4.34}$$

Spin version of $\mathrm{O}(d)$: Pin groups.

n^{μ} : unit vector.

$$\mathsf{R}(n) : x^{\mu} \mapsto x^{\mu} - 2n^{\mu}(n\cdot x). \tag{4.35}$$

$$\mathsf{R}(n)_{\mu\nu} = \delta_{\mu\nu} - 2n_{\mu}n_{\nu}. \qquad \det\mathsf{R}(n) = -1. \tag{4.36}$$

Reflection in the direction n^{μ}.

Uplift to spin : $\widetilde{\mathsf{R}}(n)$.

$$\widetilde{\mathsf{R}}(n) : \Psi \mapsto \alpha n_{\mu}\gamma^{\mu}\Psi \tag{4.37}$$

α : phase factor.
The reason :

$$\widetilde{\mathsf{R}}(n)^{\dagger}\gamma^{\mu}\widetilde{\mathsf{R}}(n) = -(\gamma^{\mu} - 2n^{\mu}(n\cdot\gamma)). \tag{4.38}$$

Then the fermion action $-\overline{\Psi}\gamma^{\mu}D_{\mu}\Psi$ is invariant under

$$\Psi(x) \to \widetilde{\mathsf{R}}\Psi(\mathsf{R}x), \qquad \overline{\Psi}(x) \to -\overline{\Psi}(\mathsf{R}x)\widetilde{\mathsf{R}}^{\dagger}. \tag{4.39}$$

$\mathsf{R}^2 = 1 \in \mathrm{O}(d)$. Thus $\widetilde{\mathsf{R}}^2 = 1$ or $\widetilde{\mathsf{R}} = (-1)^F$.
$(-1)^F = 2\pi$-rotation.

$$\widetilde{\mathsf{R}}_{+}^2 = 1, \qquad \widetilde{\mathsf{R}}_{-}^2 = (-1)^F. \tag{4.40}$$