

Question 1

Please prove properties (i) to (v) of the covariance derivatives assuming it arises from a tangential projection $D_X Y$, note that $[X, Y]f = X(Yf) - Y(Xf)$.

- (i) $\nabla_{X+fY} Z = \nabla_X Z + f\nabla_Y Z$
- (ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
- (iii) $\nabla_X(fY) = (\nabla_X f)Y + f\nabla_X Y = X(f)Y + f\nabla_X Y$
- (iv) $\nabla_X \langle Y, Z \rangle = X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$
- (v) $[X, Y] = \nabla_X Y - \nabla_Y X$

Proof.

- (i) This follows from property of directional derivative and bilinearity of Π .

$$\begin{aligned}\nabla_{X+fY} Z &= D_{X+fY} Z - \Pi(X + fY, Z)N \\ &= D_X Z + fD_Y Z - \Pi(X, Z)N - f\Pi(Y, Z)N \\ &= \nabla_X Z + f\nabla_Y Z\end{aligned}$$

- (ii) This holds for similar reasons.

$$\begin{aligned}\nabla_X(Y + Z) &= D_X(Y + Z) - \Pi(X, Y + Z)N \\ &= D_X Y + D_X Z - \Pi(X, Y)N - \Pi(X, Z)N \\ &= \nabla_X Y + \nabla_X Z\end{aligned}$$

- (iii) Apply product rule of directional derivative and bilinearity of Π , note that $D_X f = \nabla_X f$ as f is real-valued.

$$\begin{aligned}\nabla_X(fY) &= D_X(fY) - \Pi(X, fY) \\ &= (D_X f)Y + f(D_X Y) - f\Pi(X, Y) \\ &= (\nabla_X f)Y + f\nabla_X Y = X(f)Y + f\nabla_X Y\end{aligned}$$

- (iv) For reasons similar to above, it suffices to show $D_X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle$. We illustrate in the case that X, Y, Z are vector fields in \mathbb{R}^3 .

$$\begin{aligned}D_X \langle Y, Z \rangle &= \left. \frac{d}{dt} \langle Y(\alpha(t)), Z(\alpha(t)) \rangle \right|_{t=0} \\ &= \left. \sum_{i=1}^3 Y^i Z^i \right|_{t=0} \\ &= \sum_{i=1}^3 \left. \frac{dY^i}{dt} \right|_{t=0} Z^i + \sum_{i=1}^3 Y^i \left. \frac{dZ^i}{dt} \right|_{t=0} \\ &= \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle\end{aligned}$$

where we appeal to product rule in the second-last line.

(v) For smooth real-valued f ,

$$\begin{aligned}
[X, Y]f &= X(Yf) - Y(Xf) \\
&= \sum_i X^i \frac{\partial}{\partial x^i} \left(\sum_j Y^j \frac{\partial f}{\partial x^j} \right) - \sum_j Y^j \frac{\partial}{\partial x^j} \left(\sum_i X^i \frac{\partial f}{\partial x^i} \right) \\
&= \dots \\
&= \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial f}{\partial x^j} \\
&= (D_X Y - D_Y X) f
\end{aligned}$$

Question 2

Question 3

Let e_1, e_2 be a basis of $T_p(S)$ and $X = X^1 e_1 + X^2 e_2, Y = Y^1 e_1 + Y^2 e_2$. express $[X, Y]$ in terms of X^i, Y^i and e_i .

Solution. We assume e_1 and e_2 can be smoothly extended into tangent fields on S . Then we have

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^1 e_1 + X^2 e_2} (Y^1 e_1 + Y^2 e_2) \\
&= X^1 \nabla_{e_1} Y^1 e_1 + X^1 \nabla_{e_1} Y^2 e_2 + X^2 \nabla_{e_2} Y^1 e_1 + X^2 \nabla_{e_2} Y^2 e_2 \\
&= X^1 Y^1 \nabla_{e_1} e_1 + X^1 Y^2 \nabla_{e_1} e_2 + X^2 Y^1 \nabla_{e_2} e_1 + X^2 Y^2 \nabla_{e_2} e_2 \\
&\quad + X^1 e_1 (Y^1) e_1 + X^1 e_1 (Y^2) e_2 + X^2 e_2 (Y^1) e_1 + X^2 e_2 (Y^2) e_2; \\
\nabla_Y X &= \nabla_{Y^1 e_1 + Y^2 e_2} (X^1 e_1 + X^2 e_2) \\
&= Y^1 \nabla_{e_1} X^1 e_1 + Y^1 \nabla_{e_1} X^2 e_2 + Y^2 \nabla_{e_2} X^1 e_1 + Y^2 \nabla_{e_2} X^2 e_2 \\
&= Y^1 X^1 \nabla_{e_1} e_1 + Y^1 X^2 \nabla_{e_1} e_2 + Y^2 X^1 \nabla_{e_2} e_1 + Y^2 X^2 \nabla_{e_2} e_2 \\
&\quad + Y^1 e_1 (X^1) e_1 + Y^1 e_1 (X^2) e_2 + Y^2 e_2 (X^1) e_1 + Y^2 e_2 (X^2) e_2;
\end{aligned}$$

Therefore, $X^i, Y^j \neq 0$, we have

$$\begin{aligned}
[X, Y] &= \nabla_X Y - \nabla_Y X \\
&= (X^1 Y^2 \nabla_{e_1} e_2 + X^2 Y^1 \nabla_{e_2} e_1) - (Y^1 X^2 \nabla_{e_1} e_2 + Y^2 X^1 \nabla_{e_2} e_1) \\
&\quad + [X^1 e_1 (Y^1) e_1 + X^1 e_1 (Y^2) e_2 + X^2 e_2 (Y^1) e_1 + X^2 e_2 (Y^2) e_2] \\
&\quad - [Y^1 e_1 (X^1) e_1 + Y^1 e_1 (X^2) e_2 + Y^2 e_2 (X^1) e_1 + Y^2 e_2 (X^2) e_2] \\
&= \det \begin{pmatrix} X^1 & Y^1 \\ X^2 & Y^2 \end{pmatrix} [e_1, e_2] \\
&\quad + [X^1 e_1 (Y^1) + X^2 e_2 (Y^1) - Y^1 e_1 (X^1) - Y^2 e_2 (X^1)] e_1 \\
&\quad + [X^1 e_1 (Y^2) + X^2 e_2 (Y^2) - Y^1 e_1 (X^2) - Y^2 e_2 (X^2)] e_2
\end{aligned}$$

If $e_1 = \tilde{X}_u$ and $e_2 = \tilde{X}_v$, then we have $[e_1, e_2] = 0$ and hence

$$\begin{aligned}[X, Y] &= [X^1 e_1(Y^1) + X^2 e_2(Y^1) - Y^1 e_1(X^1) - Y^2 e_2(X^1)]e_1 \\ &\quad + [X^1 e_1(Y^2) + X^2 e_2(Y^2) - Y^1 e_1(X^2) - Y^2 e_2(X^2)]e_2\end{aligned}$$