1 Section 4.2

Question 1

We have that

1. $\frac{\partial x}{\partial u} = (\cos(v)\sin(a), \sin(a)\sin(v), \cos(a))$
2. $\frac{\partial x}{\partial v} = (-u\sin(a)\sin(v), u\cos(v)\sin(a), 0)$
3. $E = 1$
4. $F = 0$
5. $G = u^2\sin(a)^2$

As $E \neq G$, it is not conformal and hence not a local isometry.
To see that it is a local diffeomorphism onto a cone with angle 2α, observe that for any point $(u,v) \in \mathbb{R}^2$, construct the set $S = \{(x,y)|x > 0, v - \frac{\pi}{2} \leq y < \frac{\pi}{2}\}$. Then $(u,v) \in S \subseteq \mathbb{R}^2$, and we had that $F|_S$ is a diffeomorphism from S to the cone needed (we had seen this noting that (u,v) is in polar coordinates where u is the radius and v the angle).

Question 4

Let us consider the stereographic projection $\pi : \mathbb{R}^2 \to S^2 \setminus \mathbf{N} \subseteq \mathbb{R}^3$ where $\mathbf{N}$ denotes the north pole.

$$\pi(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2}\right).$$

Note that this is sufficient to show local conformality as we can rotate the sphere. We had from previous homework that the local chart is a diffeomorphism and

1. $\frac{\partial x}{\partial u} = (-\frac{4u^2}{(u^2+v^2+1)^2} + \frac{2}{u^2+v^2+1}, -\frac{4uv}{(u^2+v^2+1)^2}, \frac{2u}{u^2+v^2+1} - \frac{2\left(u^2+v^2-1\right)u}{(u^2+v^2+1)^2})$
2. $\frac{\partial x}{\partial v} = (-\frac{4uv}{(u^2+v^2+1)^2}, -\frac{4v^2}{(u^2+v^2+1)^2} + \frac{2}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1} - \frac{2\left(u^2+v^2-1\right)v}{(u^2+v^2+1)^2})$
3. $E = \frac{4}{u^4+v^4+2(u^2+1)v^2+2u^2+1}$
4. $F = 0$
5. $G = \frac{4}{u^4+v^4+2(u^2+1)v^2+2u^2+1}$

As $E = G$ and $F = 0$, it is a conformal chart.

Question 7

Let V and W be (finite-dimensional) vector spaces with inner product denoted by $\langle , \rangle$ and let $F : V \to W$ be a linear map. Prove that the following conditions are equivalent:

a. $\langle F(v_1), F(v_2) \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in V$.

b. $|F(v)| = |v|$ for all $v \in V$.

c. If $\{v_1, \ldots, v_n\}$ is an orthonormal basis in V, then $\{F(v_1), \ldots, F(v_n)\}$ is an orthonormal basis in W.

d. There exists an orthonormal basis $\{v_1, \ldots, v_n\}$ in V such that $\{F(v_1), \ldots, F(v_n)\}$ is an orthonormal basis in W.

(a)⇒(b). is obvious.
(a)⇒(c). Assume (a), if $\{v_1, \ldots, v_n\}$ is an orthonormal basis in V then whenever $i \neq j$, $0 = \langle v_i, v_j \rangle = \langle F(v_i), F(v_j) \rangle$ and furthermore for each i, by (b) $|F(v_i)| = |v_i|$. This shows $\{F(v_1), \ldots, F(v_n)\}$ is orthogonal set, for it to be a basis we need to assume in addition that V and W have the same dimension (so F is surjective).
(c)⇒(d). is trivial as orthonormal bases can always be produced by Gram-Schmidt.
(d)⇒(a). Assume (d), let $v, v' \in V$. We can express them in our orthonormal basis as follows

$$\begin{aligned} v &= \sum_{i=1}^{n} c_i v_i, \\ v' &= \sum_{j=1}^{n} d_j v_j. \end{aligned}$$

as $\{v_1, \ldots, v_n\}$ is an orthonormal basis, $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$ and $\langle v_i, v_i \rangle = 1$, so

$$\begin{aligned} \langle v, v' \rangle &= \left\langle \sum_{i=1}^{n} c_i v_i, \sum_{j=1}^{n} d_j v_j \right\rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i d_j \langle v_i, v_j \rangle \\ &= \sum_{i=1}^{n} c_i d_i \end{aligned}$$

and by linearity

$$\begin{aligned} F(v) &= \sum_{i=1}^{n} c_i F(v_i), \\ F(v') &= \sum_{j=1}^{n} d_j F(v_j). \end{aligned}$$

and as $\{F(v_1), \ldots, F(v_n)\}$ is also an orthonormal basis we can perform a similar computation to get

$$\langle F(v), F(v') \rangle = \sum_{i=1}^{n} c_i d_i = \langle v, v' \rangle$$

which shows (a).

Question 10

Let S be a surface of revolution. Prove that the rotations about its axis are isometries of S.

Proof. Suppose S is a surface formed by rotating around z-axis, let S be parametrised by

$$\mathbf{x}(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v))$$

for some φ, ψ. Then the rotation about z-axis by θ can be given by

$$T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

§2-3 exercise 11 shows that T restricted to S is a diffeomorphism onto S, we just need to show that it is a local isometry at every point.

Now let $\overline{\mathbf{x}} = T \circ \mathbf{x}$ and we can compute that

$$\overline{\mathbf{x}}(u,v) = (\varphi(v)\cos(u+\theta), \varphi(v)\sin(u+\theta), \psi(v))$$

Note that §2-3 Example 4 computes the coefficients of the first fundamental form of S as

$$E = \varphi^2, \quad F = 0, \quad G = (\varphi')^2 + (\psi')^2.$$

We can compute $\overline{E}, \overline{F}, \overline{G}$ by hand as

$$\begin{aligned}
\overline{\mathbf{x}}_u &= (-\varphi(v)\sin(u+\theta), \varphi(v)\cos(u+\theta), 0) \\
\overline{\mathbf{x}}_v &= (\varphi'(v)\cos(u+\theta), \varphi'(v)\sin(u+\theta), \psi'(v)) \\
\overline{E} &= \langle \overline{\mathbf{x}}_u, \overline{\mathbf{x}}_u \rangle \\
&= \varphi^2 = E \\
\overline{F} &= \langle \overline{\mathbf{x}}_u, \overline{\mathbf{x}}_v \rangle \\
&= 0 = F \\
\overline{G} &= \langle \overline{\mathbf{x}}_v, \overline{\mathbf{x}}_v \rangle \\
&= (\varphi')^2 + (\psi')^2 = G
\end{aligned}$$

Applying proposition 1 we have $\overline{\mathbf{x}} \circ \mathbf{x} = T$ is a local isometry at some arbitrary point, which suffices.

Question 13

Let V and W be (finite-dimensional) vector spaces with inner products $\langle , \rangle$. Let $G : V \to W$ be a linear map. Prove that the following conditions are equivalent:

1. There exists a real constant $\lambda \neq 0$ such that

$$\langle G(v_1), G(v_2) \rangle = \lambda^2 \langle v_1, v_2 \rangle \quad \text{for all} v_1, v_2 \in V.$$

2. There exists a real constant $\lambda > 0$ such that

$$|G(v)| = \lambda |v| \quad \text{for all} \quad v \in V.$$

3. There exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of V such that $\{G(v_1), ..., G(v_n)\}$ is an orthogonal basis of W and, also, the vectors $G(v_i), i = 1, ..., n$, have the same (nonzero) length.

If any of these conditions is satisfied, G is called a linear conformal map (or a similitude).

Solution. $(1 \Rightarrow 2)$ We have

$$\begin{aligned}|G(v)| &= \sqrt{\langle G(v), G(v)\rangle} \\ &= \sqrt{\lambda^2\langle v, v\rangle} \\ &= |\lambda|\sqrt{\langle v, v\rangle} \\ &= |\lambda||v|,\end{aligned}$$

where $|\lambda| > 0$ is the positive constant desired.

$(2 \Rightarrow 1)$ We have

$$\begin{aligned}\langle G(v_1), G(v_2)\rangle &= \frac{1}{2}(|G(v_1) + G(v_2)|^2 - |G(v_1)|^2 - |G(v_2)|^2), \\ &= \frac{\lambda^2}{2}(|v_1 + v_2|^2 - |v_1|^2 - |v_2|^2) \\ &= \lambda^2\langle v_1, v_2\rangle\end{aligned}$$

$(1\&2 \Rightarrow 3)$ For $\{v_i, v_j\}$ orthonormal we have

$$\begin{aligned}|G(v_i)| = \lambda|v_i| = \lambda|v_j| = |G(v_j)| \\ \langle G(v_i), G(v_j)\rangle = \lambda^2\langle v_i, v_j\rangle\end{aligned}$$

This shows that $\{G(v_1), ..., G(v_n)\}$ is an orthogonal basis of W and $G(v_i), i = 1, ..., n$, have the same (nonzero) length.

$(3 \Rightarrow 2)$ For any $v \in V$, let $v = \sum_{i=1}^n a_i v_i$. Then

$$\begin{aligned}|G(v)|^2 = \langle G(v), G(v)\rangle &= \langle \sum_{i=1}^n a_i G(v_i), \sum_{i=1}^n a_i G(v_i)\rangle \\ &= \sum_{i=1}^n \sum_{i=1}^n a_i a_j \langle G(v_i), G(v_j)\rangle \\ &= \sum_{i=1}^n a_i^2 \langle G(v_i), G(v_i)\rangle + \sum_{i\neq j; i,j\in 1,\ldots,n} a_i a_j \langle G(v_i), G(v_j)\rangle \\ &= \sum_{i=1}^n \lambda^2 a_i^2 \langle v_i, v_j\rangle + 0 \\ &= \lambda^2\langle v, v\rangle = (|\lambda||v|)^2\end{aligned}$$

Hence, $|G(v)| = |\lambda||v|$.

Question 16

Let $\mathbf{x} : U \subset R^2 \to R^3$, where

$$U = \{(\theta, \varphi) \in R^2 : 0 < \theta < \pi, 0 < \varphi < 2\pi\},$$
$$x(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

be a parametrization of the unit sphere S^2. Let

$$\log\tan\frac{1}{2}\theta = u, \qquad \varphi = v$$

and show that a new parametrization of the coordinate neighborhood $\mathbf{x}(U) = V$ can be given by

$$\mathbf{y}(u, v) = (\text{sech } u\cos v, \text{sech } u\sin v, \tanh u).$$

Prove that in the parametrization y the coefficients of the first fundamental form are

$$E = G = \text{sech}^2\ u, \qquad F = 0.$$

Thus, $\mathbf{y}^{-1} : V \subset S^2 \to R^2$ is a conformal map which takes the meridians and parallels of S^2 into straight lines of the plane. This is called *Mercator's projection*.

Solution. We have

$$\theta = 2\arctan e^u, \qquad v = \varphi.$$

Hence,

$$\begin{aligned}
\mathbf{y}(u, v) &= \mathbf{x}(2\arctan e^u, v) \\
&= (\sin 2\arctan e^u \cos v, \sin 2\arctan e^u \sin v, \cos 2\arctan e^u) \\
&= (\text{sech } u\cos v, \text{sech } u\sin v, \tanh u).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbf{y}_u &= \langle -\tanh u \text{ sech } u\cos v, -\tanh u \text{ sech } u\sin v, 1 - \tanh^2 u\rangle \\
\mathbf{y}_v &= (-\text{sech } u\sin v, \text{sech } u\cos v, 0) \\
E &= |\mathbf{y}_u|^2 = \tanh^2 u \text{ sech}^2\ u + (1 - \tanh^2 u)^2 \\
&= \tanh^2 u(1 - \tanh^2 u) + (1 - \tanh^2 u)^2 \\
&= 1 - \tanh^2 u = \text{sech}^2\ u \\
G &= |\mathbf{y}_v|^2 = \text{sech}^2\ u(\sin^2 u + \cos^2 u) = \text{sech}^2\ u \\
F &= \tanh u\text{sech}^2\ u\sin v\cos v - \tanh u\text{sech}^2\ u\sin v\cos v = 0
\end{aligned}$$