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Question 1

Question 3

Determine the asymptotic curves of the catenoid

$$\mathbf{x}(u,v) = (\cosh v \cos u, \cosh v \sin u, v).$$

Solution. We have

$$\begin{aligned}
\mathbf{x}_u &= (-\cosh v \sin u, \cosh v \cos u, 0), \\
\mathbf{x}_v &= (\sinh v \cos u, \sinh v \sin u, 1), \\
\mathbf{x}_{uu} &= (-\cosh v \cos u, -\cosh v \sin u, 0), \\
\mathbf{x}_{uv} &= (-\sinh v \sin u, \sinh v \cos u, 0), \\
\mathbf{x}_{vv} &= (\cosh v \cos u, \cosh v \sin u, 0).
\end{aligned}$$

Hence,

$$\begin{aligned}
N &= \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} \\
&= \frac{(\cosh v \cos u, \cosh v \sin u, -\cosh v \sinh v)}{(\cosh v)^2} \\
&= \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v).
\end{aligned}$$

Then

$$
\begin{aligned}
e &= \langle N, \mathbf{x}_{uu} \rangle = -1, \\
f &= \langle N, \mathbf{x}_{uv} \rangle = 0, \\
g &= \langle N, \mathbf{x}_{vv} \rangle = 1.
\end{aligned}
$$

This implies that the asymptotic directions correspond to $\langle u', v' \rangle$ satisfies

$$
\begin{aligned}
& e(u')^2 + 2fu'v' + g(v')^2 = 0 \\
\Rightarrow & -(u')^2 + (v')^2 = 0 \\
\Rightarrow & u' = v' \quad \text{or} \quad u' = -v'.
\end{aligned}
$$

This shows that the asymptotic curves are the traces of $v = u + a$ or $v = -u + b$ for some $a, b \in \mathbb{R}$, which are

$$
\begin{aligned}
\alpha_a(u) &= \mathbf{x}(u, u + a) = (\cosh(u + a)\cos u, \cosh(u + a)\sin u, u + a), \\
\beta_b(u) &= \mathbf{x}(u, -u + b) = (\cosh(-u + b)\cos u, \cosh(-u + b)\sin u, -u + b).
\end{aligned}
$$

The collection of all α_a, β_b are all the asymptotic curves.

Question 9

(Contact of Curves.) Define contact of order $\geq n$ (n integer ≥ 1) for regular curves in R3 with a common point p and prove that
a. The notion of contact of order $\geq n$ is invariant by diffeomorphisms.
b. Two curves have contact of order ≥ 1 at p if and only if they are tangent at p.

Solution. We say two surfaces S and $\bar{S}$ with a common point p to have contact of order $\geq n$ at p if there exist parametrizations $\mathbf{x}(u, v)$ and $\tilde{\mathbf{x}}(u, v)$ in p of S and $\bar{S}$ such that the partial derivatives of $\mathbf{x}$ and $\tilde{\mathbf{x}}(u, v)$ agree up to order n.

a. Let $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism. Then for any partial derivative operator ∂_I of order less than n on $u - v$ space. Then

$$
\partial_I \psi(\mathbf{x}) = J_\psi(\mathbf{x})\partial_I \mathbf{x} = J_\psi(\tilde{\mathbf{x}})\partial_I \tilde{\mathbf{x}} = \partial_I \psi(\tilde{\mathbf{x}}),
$$

where $J\psi(\cdot)$ is the Jacobian of ψ. This shows that the notion of contact is invariant by diffeomorphisms.

b. It is easy to see that the contact of order ≥ 1 implies that the two surfaces are tangent. For the converse, we suppose S and $\bar{S}$ are tangent at p with parametrizations $\mathbf{x}(\mathbf{u}, \mathbf{v})$ and $\tilde{\mathbf{x}}(u, v)$ respectively. Then at the point p, $\tilde{\mathbf{x}}_u, \tilde{\mathbf{x}}_v \in T_{\mathbf{x}}(p)$ we can write

$$
\begin{aligned}
\tilde{\mathbf{x}}_u &= a_1 \mathbf{x}_u + a_2 \mathbf{x}_v, \\
\tilde{\mathbf{x}}_v &= b_1 \mathbf{x}_u + b_2 \mathbf{x}_v.
\end{aligned}
$$

Note that since $\tilde{\mathbf{x}}_u, \underset{\sim}{\mathbf{x}}_v$ are linearly independent, we have $a_1b_2 - a_2b_1 \neq 0$. Now let $w = \frac{b_2u - a_2v}{a_1b_2 - a_2b_1}$, $l = \frac{b_1u - a_1v}{a_2b_1 - a_1b_2}$ and $\mathbf{y}(w,l) = \tilde{\mathbf{x}}(u,v)$. Then

$$\begin{aligned} \mathbf{y}_w &= \frac{b_2}{a_1b_2 - a_2b_1}\tilde{\mathbf{x}}_u - \frac{a_2}{a_1b_2 - a_2b_1}\tilde{\mathbf{x}}_v = \mathbf{x}_u, \\ \mathbf{y}_l &= \frac{b_1}{a_2b_1 - a_1b_2}\tilde{\mathbf{x}}_u - \frac{a_1}{a_2b_1 - a_1b_2}\tilde{\mathbf{x}}_v = \mathbf{x}_v. \end{aligned}$$

This shows that S and $\bar{S}$ have contact of order ≥ 1 at p.

Question 15

Give an example of a surface which has an isolated parabolic point p (that is, no other parabolic point is contained in some neighborhood of p).
Solution. Consider the graph (x, y, xy^2). Let $h(x,y) = xy^2$. Then we have

$$\begin{aligned} K &= \frac{h_{xx}h_{yy} - (h_{xy})^2}{(1 + h_x^2 + h_y^2)^2} = \frac{-2y}{(1 + y^4 + 4x^2y^2)^2}, \\ e &= \frac{h_{xx}}{(1 + h_x^2 + h_y^2)^{1/2}} = 0, \\ f &= \frac{h_{xy}}{(1 + h_x^2 + h_y^2)^{1/2}} = \frac{2y}{(1 + h_x^2 + h_y^2)^{1/2}}, \\ g &= \frac{h_{yy}}{(1 + h_x^2 + h_y^2)^{1/2}} = \frac{2}{(1 + h_x^2 + h_y^2)^{1/2}}. \end{aligned}$$

Then $K = 0$ only at $(0,0,0)$, at which f and g are nonzero. This shows that the graph has an isolated parabolic point.

Question 19

Obtain the asymptotic curves of the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.
Solution. Note that the hyperboloid is a surface of revolution parametrized by

$$\mathbf{x}(u,v) = (\phi(v)\cos u, \phi(v)\sin u, \psi(v)),$$

where $\phi(v) = \cosh v, \psi(v) = \sinh v$ and $u \in (0, 2\pi)$. Then

$$\begin{aligned} e &= -\phi\psi' = -\cosh^2(v), \\ f &= 0, \\ g &= \psi'\phi'' - \psi''\phi' = \cosh^2(v) - \sinh^2(v) = 1. \end{aligned}$$

Then solving $e(u')^2 + 2fu'v' + g(v')^2 = 0$, we have

$$v' = u'\cosh(v) \qquad \text{or} \qquad v' = -u'\cosh(v).$$

Solving the ODE, we have

$$u(t) = \pm\tan^{-1}(\sinh v(t)) + C, \qquad C \in \mathbb{R}$$

Hence, the asymptotic curves will be the trace of $\gamma_C(v) = (\pm\tan^{-1}(\sinh v) + C, v), v \in R$. They are

$$\alpha_C(v) = \mathbf{x}(\tan^{-1}(\sinh v) + C, v)$$

or

$$\beta_C(v) = \mathbf{x}(-\tan^{-1}(\sinh v) + C, v)$$

Question 21

Let S be a surface with orientation N. Let $V \subset S$ be an open set in S and let $f : V \subset S \to R$ be any nowhere-zero differentiable function in V. Let v_1 and v_2 be two differentiable (tangent) vector fields in V such that at each point of V, v_1 and v_2 are orthonormal and $v_1 \wedge v_2 = N$.
a. Prove that the Gaussian curvature K of V is given by

$$K = \frac{\langle dfN(v_1) \wedge dfN(v_2), fN\rangle}{f^3}.$$

b. Apply the above result to show that iff is the restriction of

$$\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

then the Gaussian curvature of the ellipsoid is

$$K = \frac{1}{a^2b^2c^2}\frac{1}{f^4}.$$

Solution. Since f is a smooth function on $V = \mathbf{x}(u, v)$, if $\alpha'(0) = v_i = \frac{d}{dt}\mathbf{x}(\beta(t))|_{t=0}$, we have

$$\begin{aligned}
dfN(v_i) &= \frac{d}{dt} f(\alpha(t))N(\alpha(t)) \\
&= (\frac{d}{dt} f(\alpha(t)))N(\alpha(t))|_{t=0} + f(\alpha(t))\frac{d}{dt}N(\alpha(t))|_{t=0} \\
&= (\nabla(f \circ \mathbf{x}) \cdot \beta'(0))N + f dN(v_i).
\end{aligned}$$

Hence,

$$\begin{aligned}
&dfN(v_1) \wedge dfN(v_2) = (C_1N + f dN(v_1)) \wedge (C_2N + f dN(v_2)) \\
&= C_1N \wedge f dN(v_2) - C_2N \wedge f dN(v_1) + f^2(dN(v_1) \wedge dN(v_2)) \\
&= C_1N \wedge f dN(v_2) - C_2N \wedge f dN(v_1) + f^2 \det(dN)(v_1 \wedge v_2) \\
&= C_1N \wedge f dN(v_2) - C_2N \wedge f dN(v_1) + f^2 \det(dN)N.
\end{aligned}$$

Therefore,

$$\begin{aligned}
dfN(v_1) \wedge dfN(v_2) \cdot fN &= C_1N \wedge f dN(v_2) \cdot fN - C_2N \wedge f dN(v_1) \cdot fN + f^2 \det(dN)N \cdot fN \\
&= f^3 KN \cdot N = f^3 K.
\end{aligned}$$

Thus

$$\frac{dfN(v_1) \wedge dfN(v_2) \cdot fN}{f^3} = K.$$

b. We know that

$$\begin{aligned}
N(x, y, z) &= \frac{(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2})}{|(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2})|} \\
&= \frac{(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})}{|(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})|} \\
&= \frac{(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})}{f(x, y, z)}.
\end{aligned}$$

Therefore, $fN = (\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})$. Then

$$\begin{aligned}
\frac{d}{dt} fN(\alpha(t)) &= (\frac{x'(t)}{a^2}, \frac{y'(t)}{b^2}, \frac{z'(t)}{c^2}) \\
&= \begin{pmatrix} a^{-2} & & \\ & b^{-2} & \\ & & c^{-2} \end{pmatrix} \alpha'(t).
\end{aligned}$$

Hence, $dfN(v_i) = \begin{pmatrix} a^{-2} & & \\ & b^{-2} & \\ & & c^{-2} \end{pmatrix} v_i$ and thus

$$\begin{aligned}
K &= \frac{dfN(v_1) \wedge dfN(v_2) \cdot fN}{f^3} \\
&= \det(dfN) \frac{(dfN^{-1})^T (v_1 \wedge v_2) \cdot fN}{f^3} \\
&= (abc)^{-2} \frac{(dfN^{-1}) N \cdot fN}{f^3} \\
&= (abc)^{-2} \frac{1}{f^3} \begin{pmatrix} a^2 & & \\ & b^2 & \\ & & c^2 \end{pmatrix} \frac{(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})}{f(x,y,z)} \cdot (\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}) \\
&= (abc)^{-2} \frac{1}{f^3} \frac{(x,y,z)}{f} \cdot (\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}) \\
&= \frac{1}{a^2 b^2 c^2} \frac{1}{f^4}
\end{aligned}$$