1 Section 1.7

Question 6

Question 7

Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be a plane curve defined in the entire real line $\mathbb{R}$. Assume that α does not pass through the origin $O = (0,0)$ and that both limits

$$\lim_{t\to-\infty} |\alpha(t)| = \lim_{t\to\infty} |\alpha(t)| = \infty.$$

(a) Prove that there exists a point t_0 such that $|\alpha(t_0)| \leq |\alpha(t)|$ for all $t \in \mathbb{R}$.
(b) Show, by an example, that the assertion in part a is false if one does not assume that both $\lim_{t\to-\infty} |\alpha(t)| = \infty$ and $\lim_{t\to\infty} |\alpha(t)| = \infty$.
Solution.
(a) Consider $f : \mathbb{R} \to \mathbb{R}$ sending t to $|\alpha(t)|$. Pick an arbitrary $t_1 \in \mathbb{R}$. We can find a, b such that for all $t < a$, $f(t) > f(t_1)$ and for all $t > b$, we have $f(t) > f(t_1)$. Then, we can consider the restriction $f|_{[a,b]}$ to the compact interval $[a, b]$. By the Extreme Value Theorem, we have that this takes a minimum on $[a, b]$, say at t_0. Then this t_0 satisfy the required properties as for $t \in [a, b]$, $|\alpha(t_0)| \leq |\alpha(t)|$, and for $t \notin [a, b]$, $|\alpha(t)| > |\alpha(t_1)| \geq |\alpha(t_0)|$.
(b) Consider $f(t) = (e^t, 0)$. Then, $\inf_t |\alpha(t)| = 0$, but this is non-zero for all t.

Question 8

Question 9

2 Section 2.2

Question 4

Let $f(x, y, z) = z^2$. Prove that 0 is not a regular value off and yet that $f^{-1}(0)$ is a regular surface.

Solution. Note that

$$\mathrm{d}f = (f_x, f_y, f_z) = (0, 0, 2z),$$

which is not surjective only when $z = 0$. Hence, $(0, 0, 0)$ is a critical point and thus $f(0, 0, 0) = 0$ is not a regular value.

However,

$$\begin{aligned} f^{-1}(0) &= \{(x, y, z) \in \mathbb{R}^3 | z^2 = 0\} \\ &= \{(x, y, 0) | x, y \in \mathbb{R}\} \\ &= \mathbb{R}^2 \times \{0\}. \end{aligned}$$

Hence, $f^{-1}(0)$ is homeomorphic to $\mathbb{R}^2$ and therefore is regular.

Question 5

Let $P = \{(x,y,z) \in \mathbb{R}^3 | x = y\}$ (a plane) and let $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$x(u,v) = (u+v, u+v, uv),$$

where $U = \{(u,v) \in \mathbb{R}^2 | u > v\}$. Clearly, $x(U) \subset P$.Is x a parametrization of P?

Solution. Yes, x is a parametrization. Clearly, x is differentiable in U with

$$\mathrm{d}x(u,v) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ u & v \end{pmatrix}.$$

Note that for $(u,v) \in U$, we have $u > v$. Then

$$\begin{vmatrix} 1 & 1 \\ u & v \end{vmatrix} = v - u \neq 0.$$

This implies $\mathrm{d}x(u,v)$ is injective for all $(u,v) \in U$. Now let (a,a,b) be any point in $x(U)$. Then

$$\begin{aligned} &u+v = a, uv = b \\ \Rightarrow &u(a-u) = b \\ \Rightarrow &(u - \frac{a}{2})^2 = \frac{a^2}{4} - b. \end{aligned}$$

Notice that here one must have $\frac{a^2}{4} - b \geq 0$ as the equations $\begin{cases} u+v = a \\ uv = b \end{cases}$ should have real solutions for $(a,a,b) \in x(U)$. Then given $u > v$, we have

$$\begin{aligned} u &= \frac{a}{2} + \sqrt{\frac{a^2}{4} - b} \\ v &= \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}. \end{aligned}$$

These are the unique (u,v) solving $x(u,v) = (a,b)$, which shows x is injective. Hence, by Prop. 4, x^{-1} must be continuous and we can conclude that x is indeed a parametrization.

Question 6

Give another proof of Prop. 1 by applying Prop. 2 to $h(x,y,z) = f(x,y) - z$.

Solution. Since f is differentiable in U, for any point in $U \times \mathbb{R}$, we have

$$\mathrm{d}h = (f_x, f_y, -1),$$

which is always surjective regardless of the value of f_x, f_y. Hence, any $z_0 \in f(U)$ with $f(x_0, y_0) = z_0$, we have

$$h(x_0, y_0, z_0) = f(x_0, y_0) - z_0 = 0,$$

being a regular value. This implies that

$$\begin{aligned} h^{-1}(0) &= \{(x, y, z) \in U \times \mathbb{R} | h(x, y, z) = 0\} \\ &= \{(x, y, z) \in U \times \mathbb{R} | f(x, y) = z\} \\ &= \{(x, y, f(x, y)) | (x, y) \in U\} \end{aligned}$$

is a regular surface.