Question 1

Please prove properties (i) to (v) of the covariance derivatives assuming it arises from a tangential projection $D_X Y$, note that $[X, Y]f = X(Yf) - Y(Xf)$.

(i) $\nabla_{X+fY} Z = \nabla_X Z + f\nabla_Y Z$

(ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$

(iii) $\nabla_X(fY) = (\nabla_X f)Y + f\nabla_X Y = X(f)Y + f\nabla_X Y$

(iv) $\nabla_X \langle Y, Z\rangle = X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

(v) $[X, Y] = \nabla_X Y - \nabla_Y X$

Proof.

(i) This follows from property of directional derivative and bilinearity of II.

$$\begin{aligned}
\nabla_{X+fY} Z &= D_{X+fY} Z - \mathrm{II}(X + fY, Z)N \\
&= D_X Z + fD_Y Z - \mathrm{II}(X, Z)N - f\mathrm{II}(Y, Z)N \\
&= \nabla_X Z + f\nabla_Y Z
\end{aligned}$$

(ii) This holds for similar reasons.

$$\begin{aligned}
\nabla_X(Y + Z) &= D_X(Y + Z) - \mathrm{II}(X, Y + Z)N \\
&= D_X Y + D_X Z - \mathrm{II}(X, Y)N - \mathrm{II}(X, Z)N \\
&= \nabla_X Y + \nabla_X Z
\end{aligned}$$

(iii) Apply product rule of directional derivative and bilinearity of II, note that $D_X f = \nabla_X f$ as f is real-valued.

$$\begin{aligned}
\nabla_X(fY) &= D_X(fY) - \mathrm{II}(X, fY) \\
&= (D_X f)Y + f(D_X Y) - f\mathrm{II}(X, Y) \\
&= (\nabla_X f)Y + f\nabla_X Y = X(f)Y + f\nabla_X Y
\end{aligned}$$

(iv) For reasons similar to above, it suffices to show $D_X \langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$. We illustrate in the case that X, Y, Z are vector fields in $\mathbb{R}^3$.

$$\begin{aligned}
D_X \langle Y, Z\rangle &= \frac{d}{dt} \langle Y(\alpha(t)), Z(\alpha(t))\rangle\bigg|_{t=0} \\
&= \sum_{i=1}^{3} Y^i Z^i\bigg|_{t=0} \\
&= \sum_{i=1}^{3} \frac{dY^i}{dt}\bigg|_{t=0} Z^i + \sum_{i=1}^{3} Y^i \frac{dZ^i}{dt}\bigg|_{t=0} \\
&= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle
\end{aligned}$$

where we appeal to product rule in the second-last line.

(v) For smooth real-valued f,

$$\begin{aligned}
[X,Y]f &= X(Yf) - Y(Xf) \\
&= \sum_i X^i \frac{\partial}{\partial x^i}\left(\sum_j Y^j \frac{\partial f}{\partial x^j}\right) - \sum_j Y^j \frac{\partial}{\partial x^j}\left(\sum_i X^i \frac{\partial f}{\partial x^i}\right) \\
&= \dots \\
&= \sum_{i,j}\left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j} \\
&= \left(D_X Y - D_Y X\right) f
\end{aligned}$$

Question 2

We have the parameterization

$$\begin{aligned}
x &= a\cos(\theta) \\
y &= a\sin(\theta) \\
z &= h
\end{aligned}$$

Then the metric tensor is $g = \mathrm{d}a \otimes \mathrm{d}a + a^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + \mathrm{d}h \otimes \mathrm{d}h$ with non-vanishing Christoffel symbols

$$\begin{array}{rcl}
\Gamma^{a}{}_{\theta\,\theta} &=& -a \\
\Gamma^{\theta}{}_{a\,\theta} &=& \frac{1}{a} \\
\Gamma^{\theta}{}_{\theta\,a} &=& \frac{1}{a}
\end{array} \tag{1}$$

Similarly, for the parameterization

$$\begin{aligned}
x &= r\sin(\phi)\cos(\theta) \\
y &= r\sin(\phi)\sin(\theta) \\
z &= \cos(\phi)
\end{aligned}$$

Then the metric tensor is $g = \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \sin\left(\theta\right)^2 \mathrm{d}\phi \otimes \mathrm{d}\phi$ with non-vanishing Christoffel symbols

$$\begin{array}{rcl}
\Gamma^{r}{}_{\theta\,\theta} &=& -r \\
\Gamma^{r}{}_{\phi\,\phi} &=& -r\sin\left(\theta\right)^2 \\
\Gamma^{\theta}{}_{r\,\theta} &=& \frac{1}{r} \\
\Gamma^{\theta}{}_{\theta\,r} &=& \frac{1}{r} \\
\Gamma^{\theta}{}_{\phi\,\phi} &=& -\cos\left(\theta\right)\sin\left(\theta\right) \\
\Gamma^{\phi}{}_{r\,\phi} &=& \frac{1}{r} \\
\Gamma^{\phi}{}_{\theta\,\phi} &=& \frac{\cos(\theta)}{\sin(\theta)} \\
\Gamma^{\phi}{}_{\phi\,r} &=& \frac{1}{r} \\
\Gamma^{\phi}{}_{\phi\,\theta} &=& \frac{\cos(\theta)}{\sin(\theta)}
\end{array} \tag{2}$$

Question 3

Let e_1, e_2 be a basis of $T_p(S)$ and $X = X^1 e_1 + X^2 e_2, Y = Y^1 e_1 + Y^2 e_2$. express $[X,Y]$ in terms of X^i, Y^i and e_i.

Solution. We assume e_1 and e_2 can be smoothly extended into tangent fields on S. Then we have

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^1 e_1 + X^2 e_2}(Y^1 e_1 + Y^2 e_2) \\
&= X^1 \nabla_{e_1} Y^1 e_1 + X^1 \nabla_{e_1} Y^2 e_2 + X^2 \nabla_{e_2} Y^1 e_1 + X^2 \nabla_{e_2} Y^2 e_2 \\
&= X^1 Y^1 \nabla_{e_1} e_1 + X^1 Y^2 \nabla_{e_1} e_2 + X^2 Y^1 \nabla_{e_2} e_1 + X^2 Y^2 \nabla_{e_2} e_2 \\
&\quad + X^1 e_1(Y^1) e_1 + X^1 e_1(Y^2) e_2 + X^2 e_2(Y^1) e_1 + X^2 e_2(Y^2) e_2; \\
\nabla_Y X &= \nabla_{Y^1 e_1 + Y^2 e_2}(X^1 e_1 + X^2 e_2) \\
&= Y^1 \nabla_{e_1} X^1 e_1 + Y^1 \nabla_{e_1} X^2 e_2 + Y^2 \nabla_{e_2} X^1 e_1 + Y^2 \nabla_{e_2} X^2 e_2 \\
&= X^1 Y^1 \nabla_{e_1} e_1 + X^2 Y^1 \nabla_{e_1} e_2 + X^1 Y^2 \nabla_{e_2} e_1 + X^2 Y^2 \nabla_{e_2} e_2 \\
&\quad + Y^1 e_1(X^1) e_1 + Y^1 e_1(X^2) e_2 + Y^2 e_2(X^1) e_1 + Y^2 e_2(X^2) e_2;
\end{aligned}$$

Therefore, $X^i, Y^j \neq 0$, we have

$$\begin{aligned}
[X, Y] &= \nabla_X Y - \nabla_Y X \\
&= (X^1 Y^2 \nabla_{e_1} e_2 + X^2 Y^1 \nabla_{e_2} e_1) - (Y^1 X^2 \nabla_{e_1} e_2 + Y^2 X^1 \nabla_{e_2} e_1) \\
&\quad + [X^1 e_1(Y^1) e_1 + X^1 e_1(Y^2) e_2 + X^2 e_2(Y^1) e_1 + X^2 e_2(Y^2) e_2] \\
&\quad - [Y^1 e_1(X^1) e_1 + Y^1 e_1(X^2) e_2 + Y^2 e_2(X^1) e_1 + Y^2 e_2(X^2) e_2] \\
&= \det \begin{pmatrix} X^1 & Y^1 \\ X^2 & Y^2 \end{pmatrix} [e_1, e_2] \\
&\quad + [X^1 e_1(Y^1) + X^2 e_2(Y^1) - Y^1 e_1(X^1) - Y^2 e_2(X^1)] e_1 \\
&\quad + [X^1 e_1(Y^2) + X^2 e_2(Y^2) - Y^1 e_1(X^2) - Y^2 e_2(X^2)] e_2
\end{aligned}$$

If $e_1 = \underset{\sim}{X}_u$ and $e_2 = \underset{\sim}{X}_v$, then we have $[e_1, e_2] = 0$ and hence

$$\begin{aligned}
[X, Y] &= [X^1 e_1(Y^1) + X^2 e_2(Y^1) - Y^1 e_1(X^1) - Y^2 e_2(X^1)] e_1 \\
&\quad + [X^1 e_1(Y^2) + X^2 e_2(Y^2) - Y^1 e_1(X^2) - Y^2 e_2(X^2)] e_2
\end{aligned}$$