1 Section 1.3

Question 2

A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a cycloid.

- Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.

- Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Solution:

Let us first parameterize the location of the centre of the circle. When it has rotated by θ, it will also have moved θ to the right. Hence, the position of the centre with respect to amount of rotation is $(\theta, 1)$.

Now consider the positional vector from the centre to the marked point on the circumference. At $\theta = 0$, this is at $(0, -1)$. Then, for general θ, this is at $(-\sin\theta, -\cos\theta)$. Summing this, it shows that the parameterization $\alpha(\theta) = (\theta - \sin\theta, 1 - \cos\theta)$.

We shall use the arc-length formula to yield

$$
\begin{aligned}
\int_0^{2\pi} |\nabla \cdot \alpha(x)| \mathrm{d}x &= \int_0^{2\pi} \sqrt{(1 - \cos x)^2 + (\sin x)^2} \mathrm{d}x \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos x}\, \mathrm{d}x \\
&= \left[-4 \cos \frac{x}{2} \right]_{x=0}^{x=2\pi} \\
&= 8.
\end{aligned}
$$

Question 3

Question 4

Question 5

Question 10

2 Section 1.4

Question 10

The natural orientation of R^2 makes it possible to associate a sign to the area A of a parallelogram generated by two linearly independent vectors $u, v \in R^2$. To do this, let $\{e_i\}, i = 1, 2$, be the natural ordered basis of R^2, and write $u = u_1 e_1 + u_2 e_2, v = v_1 e_1 + v_2 e_2$. Observe the matrix relation

$$
\begin{pmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}
$$

and conclude that

$$A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.$$

Since the last determinant has the same sign as the basis $\{u, v\}$, we can say that A is positive or negative according to whether the orientation of $\{u, v\}$ is positive or negative. This is called the *oriented area* in R^2.

Solution:

$$
\begin{aligned}
A^2 &= (u \wedge v) \cdot (u \wedge v) \\
&= u \cdot (v \wedge (u \wedge v)) \\
&= u \cdot [(v \cdot v)u - (v \cdot u)v] \\
&= (u \cdot u)(v \cdot v) - (v \cdot u)(v \cdot u) \\
&= \begin{vmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.
\end{aligned}
$$

Question 11

a. Show that the volume V of a parallelepiped generated by three linearly independent vectors $u, v, w \in R^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an oriented volume in R^3.

b. Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}$$

Solution. (a) Let $n = \frac{u \wedge v}{||u \wedge v||}$ be the normal vector of the plane generated by u and v. Then

$$
\begin{aligned}
V &= (||u|| \times ||v|| \times |\sin(u, v)|) \times ||w|| \times |\cos(n, w)| \\
&= ||u \wedge v|| \times ||w|| \times |\cos(n, w)| \\
&= ||u \wedge v|| \times ||w|| \times |\cos(u \times v, w)| \\
&= |(u \wedge v) \cdot w|
\end{aligned}
$$

(b) We know that

$$
\begin{aligned}
|(u \wedge v) \cdot w| &= \left| \begin{pmatrix} u_2 v_3 - v_2 u_3 \\ u_3 v_1 - v_3 u_1 \\ u_1 v_2 - v_1 u_2 \end{pmatrix} \cdot w \right| \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
&= \det(u, v, w).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
V^2 &= \det(u, v, w)^2 \\
&= \left| \begin{pmatrix} u^T \\ v^T \\ w^T \end{pmatrix} \begin{pmatrix} u & v & w \end{pmatrix} \right| \\
&= \begin{vmatrix} u^T u & u^T v & u^T w \\ v^T u & v^T v & v^T w \\ w^T u & w^T v & w^T w \end{vmatrix} \\
&= \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w. \end{vmatrix}
\end{aligned}
$$

Question 12

Given the vectors $v \neq 0$ and w, show that there exists a vector u such that $u \wedge v = w$ if and only if v is perpendicular to w. Is this vector u uniquely determined? If not, what is the most general solution?

Solution: ($\Rightarrow$) By the properties of cross product, $u \wedge v = w$ implies that $v \cdot w = 0$.
($\Leftarrow$) If $v \cdot w = 0$, we have

$$
(v \wedge w) \wedge v = (v \cdot v)w - (v \cdot w)v = ||v||^2 w.
$$

Then $v \neq 0$ implies that $w = \frac{v \wedge w}{||v||^2} \wedge v$. Let $u = \frac{v \wedge w}{||v||^2}$ and we have $u \wedge v = w$.
Suppose there exist u' other than u such that $u' \times v = w$. Then

$$
\begin{aligned}
u \wedge v &= u' \wedge v \\
\Rightarrow (u' - u) \wedge v &= 0 \\
\Rightarrow u' - u &= kv, \quad k \in R \\
\Rightarrow u' &= u + kv, \quad k \in R.
\end{aligned}
$$

Therefore, the most general solution of $u \wedge v = w$ is

$$
u = \frac{v \wedge w}{||v||^2} + kv, \quad k \in R.
$$