1 Section 2.3

Question 1

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $A : S^2 \to S^2$ be the (antipodal) map $A(x, y, z) = (-x, -y, -z)$. Prove that A is a diffeomorphism.

Solution. Let $U = \{(x, y, \sqrt{1 - x^2 - y^2}) : x^2 + y^2 < 1\}$. Then we have

$$\begin{aligned} A(U) &= \{-(x, y, \sqrt{1 - x^2 - y^2}) : x^2 + y^2 < 1\} \\ &= \{(x, y, -\sqrt{1 - x^2 - y^2}) : x^2 + y^2 < 1\}. \end{aligned}$$

Let $\phi(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$ be a parametrization for U and $\psi(u, v) = (u, v, -\sqrt{1 - u^2 - v^2})$ a parametrization for $A(U)$. Then

$$\begin{aligned} \psi^{-1} \circ A \circ \phi(u, v) &= \psi^{-1}(A(\phi(u, v))) \\ &= \psi^{-1}(A((u, v, \sqrt{1 - u^2 - v^2}))) \\ &= \psi^{-1}((-u, -v, -\sqrt{1 - u^2 - v^2})) \\ &= -(u, v). \end{aligned}$$

Therefore, $\psi^{-1} \circ A \circ \phi$ is a differentiable function from $U \to A(U)$. This shows that A is a diffeomorphism.

Question 3

Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to the plane.
Solution. Luckily, we can pick one local chart each to cover the paraboloid U and the plane V. For the paraboloid, we can have (U, f^{-1}) where $f : (x, y) \to (x, y, x^2 + y^2)$ and for the plane, we can choose our local chart to be (V, g^{-1}) where to be $g : (x, y) \to (x, y)$.
Let us now define the diffeomorphism $\pi : U \to V$. Let $\pi(a, b, c) = (a, b)$. Then, $g^{-1} \circ \pi \circ f = \mathrm{id}_{\mathbb{R}^2}$. Thus, π is smooth. Then, π^{-1} is well-defined, as we the fiber over the point (a, b) must be of the form $(a, b, a^2 + b^2)$. π^{-1} is smooth as $f^{-1} \circ \pi^{-1} \circ g = \mathrm{Id}_{\mathbb{R}^2}$. Thus, π is a diffeomorphism.

Question 5

Let $S \subset R^3$ be a regular surface, and let $d : S \to R$ be given by $d(p) = |p - p_0|$, where $p \in S, p \in R^3, p_0 \notin S$; that is, d is the distance from p to a fixed point p_0, not in S. Prove that d is differentiable.

Solution. We select $p \in S$, a open neighborhood U of p and a parametrization $\underset{\sim}{x} : V \to U$, where $V \subset R^2$ and $\underset{\sim}{x}(u_0, v_0) = p$. Let $p_0 = (p_1, p_2, p_3)$. Then

$$d^2 \circ \underset{\sim}{x}(u, v) = (x_1(u, v) - p_1)^2 + (x_2(u, v) - p_2)^2 + (x_3(u, v) - p_3)^2$$

Since $x_i(u, v)$ are all smooth functions, $d^2 \circ \underset{\sim}{x}$ should also be smooth. Since $p_0 \notin S$, we have $d^2 \circ \underset{\sim}{x}(S) \in (0, \infty)$. Note that $f(x) = \sqrt{x}$ is smooth on $(0, \infty)$. Therefore, $\sqrt{d^2 \circ \underset{\sim}{x}} = d \circ \underset{\sim}{x}$ is also smooth. This shows that d is differentiable.

Question 7

Theorem 1.1. *Diffeomorphism is an isomorphism in the category of Differentiable Manifolds.*

Proof. We need to show reflexivity, symmetric and transititive. Let $(M, \mathscr{A})$ and $(N, \mathscr{B})$ be two manifolds where $\mathscr{A}, \mathscr{B}$ are maximal atlas. Let $f : M \to N$ be a diffeomorphism.

- (Reflexive) The diffeomorphism has the property that $x \in \mathscr{B}$ if and only if $x \circ f \in \mathscr{A}$, and f is bijective (as sets). Thus, $y \in \mathscr{A}$ implies, $y \circ f^{-1} \circ f \in \mathscr{A}$, implying $y \circ f^{-1} \in \mathscr{B}$. As the implication are reversible, this shows that f^{-1} is also a diffeomorphism.
- (Symmetric) Consider the identity map on M. This has the property that $x \in \mathscr{A}$ if and only if $x \circ \mathrm{Id}_M \in A$.
- (Transitivity) Let $(K, \mathscr{C})$ be a manifold where $\mathscr{C}$ is a maximal atlas, and g be a diffeomorphism from N to K. Then, $x \in \mathscr{C}$ if and only if $x \circ g \in \mathscr{B}$ if and only if $x \circ g \circ f \in \mathscr{A}$. As the composition of bijective functions is bijective, $(g \circ f)$ is a diffeomorphism.

□

Question 9

2 Section 2.4

Question 15

Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

Solution. Let p_0 be such a fixed point. Then for any $p = \underset{\sim}{x}(u, v) \in S$, $p - p_0$ is normal to the tangent plane in p. That is

$$\underset{\sim}{x}_u \cdot (p - p_0) = \underset{\sim}{x}_v \cdot (p - p_0) = 0.$$

Then for the function $h(u, v) = (\underset{\sim}{x}(u, v) - p_0) \cdot (\underset{\sim}{x}(u, v) - p_0)$, we have

$$h_u = 2\underset{\sim}{x}_u \cdot (\underset{\sim}{x} - p_0) = 0$$
$$h_v = 2\underset{\sim}{x}_v \cdot (\underset{\sim}{x} - p_0) = 0.$$

This shows that h is constant for any connected component of S. However, since S is connected, h should be constant on S, which implies that $S \subset p \in R^3 : ||p - p_0|| = K$ for some $K > 0$.

Question 17

Question 19

Question 21

Question 23

Question 25

Prove that if two regular curves C_1 and C_2 of a regular surface S are tangent at a point $p \in S$, and if $\varphi : S \to S$ is a diffeomorphism, then $\varphi(C_1)$ and $\varphi(C_2)$ are regular curves which are tangent at $\varphi(p)$.

Solution. Let U be a neighborhood of p with parametrization $\underset{\sim}{x}(u, v)$. Let $\alpha_1(t)$ and $\alpha_2(t)$ be such that $\underset{\sim}{x} \circ \alpha_1$ and $\underset{\sim}{x} \circ \alpha_2$ are regular parametrizations of C_1 and C_2 with $\underset{\sim}{x} \circ \alpha_1(0) = \underset{\sim}{x} \circ \alpha_2(0) = p$. Then that C_1 and C_2 are tangent at p implies that

$$\alpha_1'(0) = \alpha_2'(0).$$

Now let V be a neighborhood of $\varphi(p)$ with parametrization $\underset{\sim}{y}(w, z)$ and $\psi = (\psi_1, \psi_2) = \underset{\sim}{y}^{-1} \circ \varphi \circ \underset{\sim}{x}$. Let $\beta_1(t)$ and $\beta_2(t)$ be such that $\underset{\sim}{y} \circ \beta_1$ and $\underset{\sim}{y} \circ \beta_2$ are regular parametrizations of $\varphi(C_1)$ and $\varphi(C_2)$ with $\underset{\sim}{y} \circ \beta_1(0) = \underset{\sim}{y} \circ \beta_2(0) = \varphi(p)$. Than

$$\begin{aligned}
\beta_1'(0) &= \left(\frac{\partial \psi_1}{\partial u}(\alpha_1(0))\alpha_{1u}'(0) + \frac{\partial \psi_1}{\partial v}(\alpha_1(0))\alpha_{1v}'(0), \frac{\partial \psi_2}{\partial u}(\alpha_1(0))\alpha_{1u}'(0) + \frac{\partial \psi_2}{\partial v}(\alpha_1(0))\alpha_{1v}'(0) \right) \\
&= \begin{pmatrix} \frac{\partial \psi_1}{\partial u}(\alpha_1(0)) & \frac{\partial \psi_1}{\partial v}(\alpha_1(0)) \\ \frac{\partial \psi_2}{\partial u}(\alpha_1(0)) & \frac{\partial \psi_2}{\partial v}(\alpha_1(0)) \end{pmatrix} \alpha_1'(0) \\
&= J_\psi(\alpha_1(0))\alpha_1'(0).
\end{aligned}$$

Likewise, we have

$$\beta_2'(0) = J_\psi(\alpha_2(0))\alpha_2'(0).$$

However, since $\alpha_1(0) = \alpha_2(0) = \underset{\sim}{x}^{-1}(p)$ and $\alpha_1'(0) = \alpha_2'(0)$, we have $\beta_1'(0) = \beta_2'(0)$. This implies that $\varphi(C_1)$ and $\varphi(C_2)$ is tangent at $\varphi(p)$.