1 Section 2.5

Question 1

The general recipe is to $(E, F, G) = (\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial u} \rangle, \langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \rangle, \langle \frac{\partial x}{\partial v}, \frac{\partial x}{\partial v} \rangle)$.
Direct Computation shows,

1. We have that
   (a) $\frac{\partial x}{\partial u} = (a\cos(u)\cos(v), b\cos(u)\sin(v), -c\sin(u))$
   (b) $\frac{\partial x}{\partial v} = (-a\sin(u)\sin(v), b\cos(v)\sin(u), 0)$
   (c) $E = a^2\cos(u)^2\cos(v)^2 + b^2\cos(u)^2\sin(v)^2 + c^2\sin(u)^2$
   (d) $F = -(a^2 - 2b^2)\cos(u)\cos(v)\sin(u)\sin(v)$
   (e) $G = b^2\cos(v)^2\sin(u)^2 + a^2\sin(u)^2\sin(v)^2$
2. We have that
   (a) $\frac{\partial x}{\partial u} = (a\cos(v), b\sin(v), 2u)$
   (b) $\frac{\partial x}{\partial v} = (-au\sin(v), bu\cos(v), 0)$
   (c) $E = a^2\cos(v)^2 + b^2\sin(v)^2 + 4u^2$
   (d) $F = -(a^2 - 2b^2)u\cos(v)\sin(v)$
   (e) $G = b^2u^2\cos(v)^2 + a^2u^2\sin(v)^2$
3. We have that
   (a) $\frac{\partial x}{\partial u} = (a\cosh(v), b\sinh(v), 2u)$
   (b) $\frac{\partial x}{\partial v} = (au\sinh(v), bu\cosh(v), 0)$
   (c) $E = (a^2 + b^2)\sinh(v)^2 + a^2 + 4u^2$
   (d) $F = (a^2 + 2b^2)u\cosh(v)\sinh(v)$
   (e) $G = b^2u^2\cosh(v)^2 + a^2u^2\sinh(v)^2$
4. We have that
   (a) $\frac{\partial x}{\partial u} = (a\cos(v)\cosh(u), b\cosh(u)\sin(v), c\sinh(u))$
   (b) $\frac{\partial x}{\partial v} = (-a\sin(v)\sinh(u), b\cos(v)\sinh(u), 0)$
   (c) $E = a^2\cos(v)^2\cosh(u)^2 + b^2\cosh(u)^2\sin(v)^2 + c^2\sinh(u)^2$
   (d) $F = -(a^2 - 2b^2)\cos(v)\cosh(u)\sin(v)\sinh(u)$
   (e) $G = b^2\cos(v)^2\sinh(u)^2 + a^2\sin(v)^2\sinh(u)^2$

Question 3

We have the parameterization

$$x(u,v) = (\frac{4\,u}{u^2+v^2+4}, \frac{4\,v}{u^2+v^2+4}, \frac{2\,(u^2+v^2)}{u^2+v^2+4}).$$

Hence,

1. $\frac{\partial x}{\partial u} = (-\frac{8\,u^2}{(u^2+v^2+4)^2} + \frac{4}{u^2+v^2+4}, -\frac{8\,uv}{(u^2+v^2+4)^2}, \frac{4\,u}{u^2+v^2+4} - \frac{4\left(u^2+v^2\right)u}{(u^2+v^2+4)^2})$
2. $\frac{\partial x}{\partial v} = (-\frac{8\,uv}{(u^2+v^2+4)^2}, -\frac{8\,v^2}{(u^2+v^2+4)^2} + \frac{4}{u^2+v^2+4}, \frac{4\,v}{u^2+v^2+4} - \frac{4\left(u^2+v^2\right)v}{(u^2+v^2+4)^2})$
3. $E = \frac{16}{u^4+v^4+2\,(u^2+4)v^2+8\,u^2+16}$
4. $F = \frac{32\left(uv^3-\left(u^3+12\,u\right)v\right)}{u^8+v^8+4\,(u^2+4)v^6+16\,u^6+6\,(u^4+8\,u^2+16)v^4+96\,u^4+4\,(u^6+12\,u^4+48\,u^2+64)v^2+256\,u^2+256}$
5. $G = \frac{16}{u^4+v^4+2\,(u^2+4)v^2+8\,u^2+16}$

Question 5

We have the parameterization

$$x(u,v) = (u, v, f(u,v)).$$

Hence,

1. $\frac{\partial x}{\partial u} = (1, 0, \frac{\partial f}{\partial u})$
2. $\frac{\partial x}{\partial v} = (0, 1, \frac{\partial f}{\partial v})$

And thus, $\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} = (-\frac{\partial x}{\partial u}, -\frac{\partial x}{\partial v}, 1)$ and so

$$A = \iint_Q |(-\frac{\partial x}{\partial u}, -\frac{\partial x}{\partial v}, 1)|\mathrm{d}x\mathrm{d}y = \iint_Q \sqrt{\frac{\partial x}{\partial u}^2 + \frac{\partial x}{\partial v}^2 + 1^2}\mathrm{d}x\mathrm{d}y$$

Question 7

The conditions translate to for all $a, b \in \mathbb{R}$ such that $a < b$.

$$\begin{aligned} \hat{E}(v) &= \int_a^b \sqrt{E(u,v)}\mathrm{d}u \\ \hat{G}(u) &= \int_a^b \sqrt{G(u,v)}\mathrm{d}v \end{aligned} \tag{1}$$

are constant functions. Differentiation the first with respect to v, and the second with respect to u yields the answer.

Question 9

We consider the parameterization

$$x(u, v) = (f(u)\cos(v), f(u)\sin(v), g(u))$$

.

Hence,

1. $\frac{\partial x}{\partial u} = (f'(u)\cos(v), f'(u)\sin(v), g'(u))$
2. $\frac{\partial x}{\partial v} = (-f(u)\sin(v), f(u)\cos(v), 0)$
3. $E = f'(u)^2 + g'(u)^2$
4. $F = 0$
5. $G = 1$

Swapping the variable names, $u \leftrightarrow v$ suffices.

Question 11

Let S be a surface of revolution and C be its generating curve. Let s be the arc length of C and denote by $\rho = \rho(s)$ the distance to the rotation axis of the point of C corresponding to s.

a. *(Pappus' Theorem.)* Show that the area of S is

$$2\pi \int_0^1 \rho(s)\ ds,$$

where l is the length of C.

Proof. Similarly to Example 4 of Sec. 2-3, we assume that the curve C lies in the xz-plane and the rotation axis is the z-axis. Let

$$\alpha(s) = (\rho(s), 0, h(s)), \quad 0 < s < l$$

be parametrization of C, denote by θ the rotation angle about the z-axis. Thus we obtain the map

$$\mathbf{x}(\theta, s) = (\rho(s)\cos\theta, \rho(s)\sin\theta, h(s))$$

which is a parametrisation $U \to S$ where $U = \{(\theta, s) : 0 < \theta < 2\pi, 0 < s < l\}$ is open. Now we can compute

$$\begin{aligned}
|\mathbf{x}_\theta \wedge \mathbf{x}_s| &= \sqrt{\rho^2(s)(\rho'(s))^2 + \rho^2(s)(h'(s))^2} = \rho(s) \\
\text{Area of } S &= \iint_U |\mathbf{x}_\theta \wedge \mathbf{x}_s|\ d\theta\ ds \\
&= \int_0^l \int_0^2 \pi\rho(s)\ d\theta\ ds \\
&= 2\pi \int_0^l \rho(s)\ ds
\end{aligned}$$

as desired.

b. Apply part a to compute the area of a torus of revolution.

Solution. Let R be the distance from the center of the tube to the center of the torus, and r be the radius of the tube. The torus of revolution can be generated by the curve C parametrised by

$$\alpha(s) = \left(R + r\cos\frac{s}{r}, 0, r\sin\frac{s}{r}\right), \quad 0 < s < 2\pi r.$$

In this case $\rho(s) = R + r\cos\frac{s}{r}$, so we have

$$\begin{aligned}\text{Area of torus} &= 2\pi\int_0^{2\pi r} R + r\cos\frac{s}{r}\,ds\\ &= 2\pi\left[Rs + r^2\sin\frac{s}{r}\right]_0^{2\pi r}\\ &= 4\pi^2 Rr.\end{aligned}$$

2 Section 3.2

Question 1

Show that at a hyperbolic point, the principal directions bissect the asymptotic directions.
Proof. Let p be a hyperbolic point and let v be an asymptotic direction with angle θ from e_1, where we write v as

$$v = e_1\cos\theta + e_2\sin\theta.$$

As v is asymptotic by Euler's formula we have

$$0 = \mathrm{II}_p(v) = k_1\cos^2\theta + k_2\sin^2\theta.$$

Since we have the identities $\cos(\pi-\theta) = -\cos\theta$ and $\sin(\pi-\theta) = \sin(\theta)$ we have

$$v' = e_1\cos(\pi-\theta) + e_2\sin(\pi-\theta)$$

is also asymptotic. Now we see that e_2 bisects v and v'.

Question 3

Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature k of C at p satisfies

$$k \geq \min(|k_1|, |k_2|),$$

where k_1 and k_2 are the principal curvatures of S at p.
Proof. WLOG suppose the S is oriented such that both k_1, k_2 are positive, furthermore for simplicity suppose $0 < k_1 < k_2$. Note that $k = k_n + k\sin\theta$ so $k \geq k_n$, it suffices to prove that $k_n \geq \min(k_1, k_2)$. Let v be the tangent of the curve C at p, then

$$v = e_1\cos\theta + e_2\sin\theta$$

for some angle θ. By Euler's formula we have

$$k_n = k_1\cos^2(\theta) + k_2\sin^2(\theta).$$

Putting everything together we have

$$k \geq k_n = k_1\cos^2(\theta) + k_2\sin^2(\theta) \geq k_1\cos^2(\theta) + k_1\sin^2(\theta) = k_1 = \min(k_1, k_2).$$

3 Section 3.3

Question 1

Show that at the origin $(0,0,0)$ of the hyperboloid $z = axy$ we have $K = -a^2$ and $H = 0$.
Solution. We parametrize the hyperboloid

$$\mathbf{x}(u,v) = (u, v, auv).$$

We compute the following at origin $(0,0,0)$

$$\begin{aligned}
\mathbf{x}_u &= (1, 0, av) \\
\mathbf{x}_v &= (0, 1, au) \\
\mathbf{x}_{uu} &= \mathbf{0} \\
\mathbf{x}_{uv} &= (0, 0, a) \\
\mathbf{x}_{vv} &= \mathbf{0} \\
N &= \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = (-av, -au, 1) \\
e &= \langle N, \mathbf{x}_{uu} \rangle = 0 \\
f &= \langle N, \mathbf{x}_{uv} \rangle = a \\
g &= \langle N, \mathbf{x}_{vv} \rangle = 0 \\
E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1 + a^2 v^2 = 1 \\
F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = a^2 uv = 0 \\
G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1 + a^2 u^2 = 1 \\
K &= \frac{eg - f^2}{EG - F^2} \\
&= -a^2 \\
H &= \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} = 0
\end{aligned}$$

Question 3

Determine the asymptotic curves of the catenoid

$$\mathbf{x}(u,v) = (\cosh v \cos u, \cosh v \sin u, v).$$

Solution. We have

$$\begin{aligned}
\mathbf{x}_u &= (-\cosh v \sin u, \cosh v \cos u, 0), \\
\mathbf{x}_v &= (\sinh v \cos u, \sinh v \sin u, 1), \\
\mathbf{x}_{uu} &= (-\cosh v \cos u, -\cosh v \sin u, 0), \\
\mathbf{x}_{uv} &= (-\sinh v \sin u, \sinh v \cos u, 0), \\
\mathbf{x}_{vv} &= (\cosh v \cos u, \cosh v \sin u, 0).
\end{aligned}$$

Hence,

$$\begin{aligned}
N &= \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} \\
&= \frac{(\cosh v \cos u, \cosh v \sin u, -\cosh v \sinh v)}{(\cosh v)^2} \\
&= \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v).
\end{aligned}$$

Then

$$\begin{aligned}
e &= \langle N, \mathbf{x}_{uu} \rangle = -1, \\
f &= \langle N, \mathbf{x}_{uv} \rangle = 0, \\
g &= \langle N, \mathbf{x}_{vv} \rangle = 1.
\end{aligned}$$

This implies that the asymptotic directions correspond to $\langle u', v' \rangle$ satisfies

$$\begin{aligned}
&e(u')^2 + 2fu'v' + g(v')^2 = 0 \\
\Rightarrow &-(u')^2 + (v')^2 = 0 \\
\Rightarrow &u' = v' \quad \text{or} \quad u' = -v'.
\end{aligned}$$

This shows that the asymptotic curves are the traces of $v = u + a$ or $v = -u + b$ for some $a, b \in \mathbb{R}$, which are

$$\begin{aligned}
\alpha_a(u) &= \mathbf{x}(u, u + a) = (\cosh(u + a) \cos u, \cosh(u + a) \sin u, u + a), \\
\beta_b(u) &= \mathbf{x}(u, -u + b) = (\cosh(-u + b) \cos u, \cosh(-u + b) \sin u, -u + b).
\end{aligned}$$

The collection of all α_a, β_b are all the asymptotic curves.

Question 9

(Contact of Curves.) Define contact of order $\geq n$ (n integer ≥ 1) for regular curves in R3 with a common point p and prove that
a. The notion of contact of order $\geq n$ is invariant by diffeomorphisms.
b. Two curves have contact of order ≥ 1 at p if and only if they are tangent at p.

Solution. We say two surfaces S and $\bar{S}$ with a common point p to have contact of order $\geq n$ at p if there exist parametrizations $\mathbf{x}(u, v)$ and $\tilde{\mathbf{x}}(u, v)$ in p of S and $\bar{S}$ such that the partial derivatives of $\mathbf{x}$ and $\tilde{\mathbf{x}}(u, v)$ agree up to order n.

a. Let $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism. Then for any partial derivative operator ∂_I of order less than n on $u - v$ space. Then

$$\partial_I \psi(\mathbf{x}) = J_\psi(\mathbf{x}) \partial_I \mathbf{x} = J_\psi(\tilde{\mathbf{x}}) \partial_I \tilde{\mathbf{x}} = \partial_I \psi(\tilde{\mathbf{x}}),$$

where $J\psi(\cdot)$ is the Jacobian of ψ. This shows that the notion of contact is invariant by diffeomorphisms.

b. It is easy to see that the contact of order ≥ 1 implies that the two surfaces are tangent. For the converse, we suppose S and $\bar{S}$ are tangent at p with parametrizations $\mathbf{x}(\mathbf{u},\mathbf{v})$ and $\tilde{\mathbf{x}}(u,v)$ respectively. Then at the point p, $\tilde{\mathbf{x}}_u, \tilde{\mathbf{x}}_v \in T_{\mathbf{x}}(p)$ we can write

$$\begin{aligned}\tilde{\mathbf{x}}_u &= a_1\mathbf{x}_u + a_2\mathbf{x}_v,\\ \tilde{\mathbf{x}}_v &= b_1\mathbf{x}_u + b_2\mathbf{x}_v.\end{aligned}$$

Note that since $\tilde{\mathbf{x}}_u, \underset{\sim}{\mathbf{x}}_v$ are linearly independent, we have $a_1b_2 - a_2b_1 \neq 0$. Now let $w = \frac{b_2u-a_2v}{a_1b_2-a_2b_1}$, $l = \frac{b_1u-a_1v}{a_2b_1-a_1b_2}$ and $\mathbf{y}(w,l) = \tilde{\mathbf{x}}(u,v)$. Then

$$\begin{aligned}\mathbf{y}_w &= \frac{b_2}{a_1b_2-a_2b_1}\tilde{\mathbf{x}}_u - \frac{a_2}{a_1b_2-a_2b_1}\tilde{\mathbf{x}}_v = \mathbf{x}_u,\\ \mathbf{y}_l &= \frac{b_1}{a_2b_1-a_1b_2}\tilde{\mathbf{x}}_u - \frac{a_1}{a_2b_1-a_1b_2}\tilde{\mathbf{x}}_v = \mathbf{x}_v.\end{aligned}$$

This shows that S and $\bar{S}$ have contact of order ≥ 1 at p.

Question 15

Give an example of a surface which has an isolated parabolic point p (that is, no other parabolic point is contained in some neighborhood of p).
Solution. Consider the graph $(x,y,x^4+x^2y^2+y^2)$. Let $h(x,y) = x^4+x^2y^2+y^2$. Then we have

$$\begin{aligned}K &= \frac{h_{xx}h_{yy}-(h_{xy})^2}{(1+h_x^2+h_y^2)^2} = \frac{24x^4-12x^2y^2+24x^2+4y^2}{(1+h_x^2+h_y^2)^2},\\ e &= \frac{h_{xx}}{(1+h_x^2+h_y^2)^{1/2}} = \frac{12x^2+2y^2}{(1+h_x^2+h_y^2)^{1/2}},\\ f &= \frac{h_{xy}}{(1+h_x^2+h_y^2)^{1/2}} = \frac{2x^2+2}{(1+h_x^2+h_y^2)^{1/2}},\\ g &= \frac{h_{yy}}{(1+h_x^2+h_y^2)^{1/2}} = \frac{4xy}{(1+h_x^2+h_y^2)^{1/2}}.\end{aligned}$$

Then $K = 0$ only at $(0,0,0)$, at which f is nonzero. This shows that the graph has an isolated parabolic point.n $K = 0$ only at $(0,0,0)$, at which f and g are nonzero. This shows that the graph has an isolated parabolic point.

Question 19

Obtain the asymptotic curves of the one-sheeted hyperboloid $x^2+y^2-z^2=1$.
Solution. Note that the hyperboloid is a surface of revolution parametrized by

$$\mathbf{x}(u,v) = (\phi(v)\cos u, \phi(v)\sin u, \psi(v)),$$

where $\phi(v) = \cosh v, \psi(v) = \sinh v$ and $u \in (0,2\pi)$. Then

$$\begin{aligned}&e = -\phi\psi' = -\cosh^2(v),\\ &f = 0,\\ &g = \psi'\phi'' - \psi''\phi' = \cosh^2(v) - \sinh^2(v) = 1.\end{aligned}$$

Then solving $e(u')^2 + 2fu'v' + g(v')^2 = 0$, we have

$$v' = u' \cosh(v) \qquad \text{or} \qquad v' = -u' \cosh(v).$$

Solving the ODE, we have

$$u(t) = \pm \tan^{-1}(\sinh v(t)) + C, \qquad C \in \mathbb{R}$$

Hence, the asymptotic curves will be the trace of $\gamma_C(v) = (\pm \tan^{-1}(\sinh v) + C, v), v \in R$. They are

$$\alpha_C(v) = \mathbf{x}(\tan^{-1}(\sinh v) + C, v)$$

or

$$\beta_C(v) = \mathbf{x}(-\tan^{-1}(\sinh v) + C, v)$$

Question 21

Let S be a surface with orientation N. Let $V \subset S$ be an open set in S and let $f : V \subset S \to R$ be any nowhere-zero differentiable function in V. Let v_1 and v_2 be two differentiable (tangent) vector fields in V such that at each point of V, v_1 and v_2 are orthonormal and $v_1 \wedge v_2 = N$.
a. Prove that the Gaussian curvature K of V is given by

$$K = \frac{\langle dfN(v_1) \wedge dfN(v_2), fN \rangle}{f^3}.$$

b. Apply the above result to show that iff is the restriction of

$$\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

then the Gaussian curvature of the ellipsoid is

$$K = \frac{1}{a^2b^2c^2}\frac{1}{f^4}.$$

Solution. Since f is a smooth function on $V = \mathbf{x}(u, v)$, if $\alpha'(0) = v_i = \frac{d}{dt}\mathbf{x}(\beta(t))|_{t=0}$, we have

$$\begin{aligned} dfN(v_i) &= \frac{d}{dt} f(\alpha(t))N(\alpha(t)) \\ &= (\frac{d}{dt} f(\alpha(t)))N(\alpha(t))|_{t=0} + f(\alpha(t))\frac{d}{dt}N(\alpha(t))|_{t=0} \\ &= (\nabla(f \circ \mathbf{x}) \cdot \beta'(0))N + fdN(v_i). \end{aligned}$$

Hence,

$$\begin{aligned} & dfN(v_1) \wedge dfN(v_2) = (C_1N + fdN(v_1)) \wedge (C_2N + fdN(v_2)) \\ &= C_1N \wedge fdN(v_2) - C_2N \wedge fdN(v_1) + f^2(dN(v_1) \wedge dN(v_2)) \\ &= C_1N \wedge fdN(v_2) - C_2N \wedge fdN(v_1) + f^2 \det(dN)(v_1 \wedge v_2) \\ &= C_1N \wedge fdN(v_2) - C_2N \wedge fdN(v_1) + f^2 \det(dN)N. \end{aligned}$$

Therefore,

$$\begin{aligned} dfN(v_1) \wedge dfN(v_2) \cdot fN &= C_1 N \wedge f dN(v_2) \cdot fN - C_2 N \wedge f dN(v_1) \cdot fN + f^2 \det(dN) N \cdot fN \\ &= f^3 K N \cdot N = f^3 K. \end{aligned}$$

Thus

$$\frac{dfN(v_1) \wedge dfN(v_2) \cdot fN}{f^3} = K.$$

b. We know that

$$\begin{aligned} N(x,y,z) &= \frac{(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2})}{|(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2})|} \\ &= \frac{(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})}{|(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})|} \\ &= \frac{(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})}{f(x,y,z)}. \end{aligned}$$

Therefore, $fN = (\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})$. Then

$$\begin{aligned} \frac{d}{dt} fN(\alpha(t)) &= (\frac{x'(t)}{a^2}, \frac{y'(t)}{b^2}, \frac{z'(t)}{c^2}) \\ &= \begin{pmatrix} a^{-2} & & \\ & b^{-2} & \\ & & c^{-2} \end{pmatrix} \alpha'(t). \end{aligned}$$

Hence, $dfN(v_i) = \begin{pmatrix} a^{-2} & & \\ & b^{-2} & \\ & & c^{-2} \end{pmatrix} v_i$ and thus

$$\begin{aligned} K &= \frac{dfN(v_1) \wedge dfN(v_2) \cdot fN}{f^3} \\ &= \det(dfN) \frac{(dfN^{-1})^T (v_1 \wedge v_2) \cdot fN}{f^3} \\ &= (abc)^{-2} \frac{(dfN^{-1}) N \cdot fN}{f^3} \\ &= (abc)^{-2} \frac{1}{f^3} \begin{pmatrix} a^2 & & \\ & b^2 & \\ & & c^2 \end{pmatrix} \frac{(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})}{f(x,y,z)} \cdot (\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}) \\ &= (abc)^{-2} \frac{1}{f^3} \frac{(x,y,z)}{f} \cdot (\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}) \\ &= \frac{1}{a^2 b^2 c^2} \frac{1}{f^4} \end{aligned}$$