# 1 Section 1.3

## Question 2

A circular disk of radius 1 in the plane $xy$ rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is calIed a cycloid.

- Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.
- Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Solution:
Let us first parameterize the location of the centre of the circle. When it has rotated by $\theta$, it will also have moved $\theta$ to the right. Hence, the position of the centre with respect to amount of rotation is $(\theta, 1)$.
Now consider the positional vector from the centre to the marked point on the circumference. At $\theta = 0$, this is at $(0, -1)$. Then, for general $\theta$, this is at $(-\sin\theta, -\cos\theta)$. Summing this, it shows that the parameterization $\alpha(\theta) = (\theta - \sin\theta, 1 - \cos\theta)$.
We shall use the arc-length formula to yield

$$\begin{aligned}\int_0^{2\pi} |\nabla \cdot \alpha(x)| \mathrm{d}x &= \int_0^{2\pi} \sqrt{(1-\cos x)^2 + (\sin x)^2} \mathrm{d}x \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos x} \mathrm{d}x \\ &= \left[-4\cos\frac{x}{2}\right]_{x=0}^{x=2\pi} \\ &= 8.\end{aligned}$$

## Question 3

## Question 4

## Question 5

Let $\alpha : (-1, \infty) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right)$$

Solution:
(a) For $t = 0$, $\alpha$ is tangent to the $x$-axis. Computing the derivative, $\frac{\partial\alpha(t)}{\partial t} = \left(\frac{a(3-6t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2}\right)$. Thus, $\alpha(0) = (0, 0)$ and $\frac{\partial\alpha(0)}{\partial t} = (3a, 0)$. Thus, it is tangent to the $x$-axis.
(b) As $t \to \infty$, $\alpha(t) = \frac{\partial\alpha(t)}{\partial t} = (0, 0)$. We take limits, $\lim_{t\to\infty} \alpha(t) = \left(\lim_{t\to\infty} \frac{3at}{1+t^3}, \lim_{t\to\infty} \frac{3at^2}{1+t^3}\right) = (0, 0)$. Similarly, $\lim_{t\to\infty} \frac{\partial\alpha(t)}{\partial t} = \left(\lim_{t\to\infty} \frac{a(3-6t^3)}{(1+t^3)^2}, \lim_{t\to\infty} \frac{3at(2-t^3)}{(1+t^3)^2}\right) = (0, 0)$.

(c)Take the curve with the opposite orientation. Now, as $t \to -1$, the curve and its tangent approach the line $x + y + a = 0$.
Let us compute $\lim_{t\to -1} \frac{\alpha(t)_y}{\alpha(t)_x} = \lim_{t\to -1} \frac{1}{t} = -1$. Now, consider

$$\lim_{t\to -1} \alpha(t)_y - (-1)\alpha(t)_x = \lim_{t\to -1} \frac{3a(t+t^2)}{1+t^3} = -a.$$

Thus, the tangent approaches the line $y = (-1)x + (-a)$, or the line $x + y + a = 0$.

### Question 10

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve. Let $[a, b] \subseteq I$ and set $\alpha(a) = p, \alpha(b) = q$.
(a) Show that, for any constant vector $v$, $|v| = 1$.

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v\mathrm{d}t \leq \int_a^b |\alpha'(t)|\mathrm{d}t.$$

*Proof.* By Fundamental Theorem of Calculus in 1-Dimension,

$$\int_a^b \alpha'(t) \cdot v\mathrm{d}t = (\alpha(b) - \alpha(a)) \cdot v = (q - p) \cdot v.$$

Then, by Hölder's inequality, we have $|\alpha'(t) \cdot v| \leq |\alpha'(t)||v| = |\alpha'(t)|$. Thus,

$$\left|\int_a^b \alpha'(t) \cdot v\mathrm{d}t\right| \leq \int_a^b |\alpha'(t) \cdot v|\mathrm{d}t \leq \int_a^b |\alpha'(t)|\mathrm{d}t.$$

$\square$

(b) Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|\mathrm{d}t.$$

That is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

**Corollary 1.1.** *Use* $(q - p) \cdot v = \frac{|q-p|^2}{|q-p|} = |q - p| = |\alpha(b) - \alpha(a)|$ *with 10(a).*

## 2 Section 1.4

### Question 10

The natural orientation of $R^2$ makes it possible to associate a sign to the area $A$ of a parallelogram generated by two linearly independent vectors $u, v \in R^2$. To do this, let $\{e_i\}, i = 1, 2$, be the natural ordered basis of $R^2$, and write $u = u_1e_1 + u_2e_2, v = v_1e_1 + v_2e_2$. Observe the matrix relation

$$\begin{pmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

and conclude that

$$A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.$$

Since the last determinant has the same sign as the basis $\{u, v\}$, we can say that $A$ is positive or negative according to whether the orientation of $\{u, v\}$ is positive or negative. This is called the *oriented area* in $R^2$.

Solution:

$$\begin{aligned}
A^2 &= (u \wedge v) \cdot (u \wedge v) \\
&= u \cdot (v \wedge (u \wedge v)) \\
&= u \cdot [(v \cdot v)u - (v \cdot u)v] \\
&= (u \cdot u)(v \cdot v) - (v \cdot u)(v \cdot u) \\
&= \begin{vmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.
\end{aligned}$$

## Question 11

a. Show that the volume $V$ of a parallelepiped generated by three linearly independent vectors $u, v, w \in R^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an oriented volume in $R^3$.
b. Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}$$

Solution. (a) Let $n = \frac{u \wedge v}{||u \wedge v||}$ be the normal vector of the plane generated by $u$ and $v$. Then

$$\begin{aligned}
V &= (||u|| \times ||v|| \times |\sin(u, v)|) \times ||w|| \times |\cos(n, w)| \\
&= ||u \wedge v|| \times ||w|| \times |\cos(n, w)| \\
&= ||u \wedge v|| \times ||w|| \times |\cos(u \times v, w)| \\
&= |(u \wedge v) \cdot w|
\end{aligned}$$

(b) We know that

$$\begin{aligned}
|(u \wedge v) \cdot w| &= | \begin{pmatrix} u_2 v_3 - v_2 u_3 \\ u_3 v_1 - v_3 u_1 \\ u_1 v_2 - v_1 u_2 \end{pmatrix} \cdot w| \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
&= \det(u, v, w).
\end{aligned}$$

Hence,

$$
\begin{aligned}
V^2 &= \det(u, v, w)^2 \\
&= | \begin{pmatrix} u^T \\ v^T \\ w^T \end{pmatrix} \begin{pmatrix} u & v & w \end{pmatrix} | \\
&= \begin{vmatrix} u^T u & u^T v & u^T w \\ v^T u & v^T v & v^T w \\ w^T u & w^T v & w^T w \end{vmatrix} \\
&= \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w. \end{vmatrix}
\end{aligned}
$$

## Question 12

Given the vectors $v \neq 0$ and $w$, show that there exists a vector $u$ such that $u \wedge v = w$ if and only if $v$ is perpendicular to $w$. Is this vector $u$ uniquely determined? If not, what is the most general solution?

Solution: ($\Rightarrow$) By the properties of cross product, $u \wedge v = w$ implies that $v \cdot w = 0$.
($\Leftarrow$) If $v \cdot w = 0$, we have

$$(v \wedge w) \wedge v = (v \cdot v)w - (v \cdot w)v = ||v||^2 w.$$

Then $v \neq 0$ implies that $w = \frac{v \wedge w}{||v||^2} \wedge v$. Let $u = \frac{v \wedge w}{||v||^2}$ and we have $u \wedge v = w$.
Suppose there exist $u'$ other than $u$ such that $u' \times v = w$. Then

$$
\begin{aligned}
& u \wedge v = u' \wedge v \\
\Rightarrow & (u' - u) \wedge v = 0 \\
\Rightarrow & u' - u = kv, \quad k \in R \\
\Rightarrow & u' = u + kv, \quad k \in R.
\end{aligned}
$$

Therefore, the most general solution of $u \wedge v = w$ is

$$u = \frac{v \wedge w}{||v||^2} + kv, \quad k \in R.$$