1 Section 1.3

Question 2

A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is calIed a cycloid.

- Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.
- Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Solution:
Let us first parameterize the location of the centre of the circle. When it has rotated by θ, it will also have moved θ to the right. Hence, the position of the centre with respect to amount of rotation is $(\theta, 1)$.
Now consider the positional vector from the centre to the marked point on the circumference. At $\theta = 0$, this is at $(0, -1)$. Then, for general θ, this is at $(-\sin\theta, -\cos\theta)$. Summing this, it shows that the parameterization $\alpha(\theta) = (\theta - \sin\theta, 1 - \cos\theta)$.
We shall use the arc-length formula to yield

$$\begin{aligned}
\int_0^{2\pi} |\nabla \cdot \alpha(x)| \mathrm{d}x &= \int_0^{2\pi} \sqrt{(1-\cos x)^2 + (\sin x)^2} \mathrm{d}x \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos x} \mathrm{d}x \\
&= \left[-4\cos\frac{x}{2}\right]_{x=0}^{x=2\pi} \\
&= 8.
\end{aligned}$$

Question 3

From errata: p. 8, Figure 1-8: The labelling is wrong: the points p and C should lie on the same half-line r through 0 as B.
Let $0A = 2a$ be the diameter of a circle S^1 and $0Y$ and AV be the tangents to S^1 at 0 and A, respectively. A half-line r is drawn from 0 which meets the circle S^1 at C and the line AV at B. On $0B$ mark off the segment $0p = CB$ (means both segments are equally long). If we rotate r about 0, the point p will describe a curve called *cissoid of Diocles*. By taking $0A$ as x axis and $0Y$ as y axis, prove that

a. The trace of

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right), \quad t \in \mathbb{R}$$

is the cissoid of Diocles ($t = \tan\theta$, see Figure 1-8)

Proof.
Let $t = \tan\theta$ for $\theta \in (-\pi, \pi)$, fix θ and we show that $p = \alpha(\tan\theta)$ satisfies the requirements when the half line r is of angle θ from origin.
First we check that p lies on the half-line, where

$$p = \left(\frac{2a\tan^2\theta}{1+\tan^2\theta}, \frac{2a\tan^3\theta}{1+\tan^2\theta}\right)$$

and we see that $\tan\arg p = \tan\theta$.
Finally we check that the segments $0p = CB$, we know the point C can be given by $(a, 0) + (a\cos 2\theta, a\sin 2\theta)$ and B is given by $(2a, 2a\tan\theta)$. So

$$\begin{aligned}
\|CB\| &= \|(a - a\cos 2\theta, 2a\tan\theta - a\sin 2\theta)\| \\
\|CB\|^2 &= (a - a\cos 2\theta)^2 + (2a\tan\theta - a\sin 2\theta)^2 \\
&= 4a^2\sin^2\theta\tan^2\theta \\
\|p\|^2 &= \frac{4a^2(t^4 + t^6)}{(1+t^2)^2} \\
&= \frac{4a^2\tan^4\theta}{1+\tan^2\theta} \\
&= 4a^2\sin^2\theta\tan^2\theta
\end{aligned}$$

b. The origin $(0, 0)$ is a singular point of the cissoid.

Proof. At point $(0, 0)$, $t = 0$, we just need to check that $\alpha'(0) = 0$. Now

$$\alpha'(t) = \left(\frac{4at}{(1+t^2)^2}, \frac{2at^2(t^2+3)}{(1+t^2)^2}\right)$$

which is 0 when $t = 0$.

c. As $t \to \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \to (0, 2a)$ (book typo'd this I think). Thus as $t \to \infty$, the curve and its tangent approach the line $x = 2a$.

Proof. To show that the curve approaches the line $x = 2a$ we check that

$$\lim_{t\to\infty} \frac{2at^2}{1+t^2} = 2a.$$

To verify the other claim we compute

$$\lim_{t\to\infty} \alpha'(t) = (0, 2a).$$

Question 4

Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log\tan\frac{t}{2}\right)$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the tractrix. Show that

(a) α is a differentiable parameterized curve, regular except $t = \frac{\pi}{2}$. Consider the derivative, $\alpha'(t) = (\cos t, -\sin t + (\sin t)^{-1})$ This is differentiable except when $\cos t = 0$ and $-\sin t + (\sin t)^{-1} = 0$, which occurs when $t = \frac{\pi}{2}$.
(b) The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.
Consider $\frac{\alpha'(t)_y}{\alpha'(t)_x} = \frac{-\sin t+(\sin t)^{-1}}{\cos t} = \frac{\cos t}{\sin t}$. This is as $(\sin x)^2 + (\cos x)^2 = 1$. Thus, the line of the tanget at $\alpha(t)$ is $y - \cos t - \log\tan\frac{t}{2} = \frac{\cos t}{\sin t}(x - \sin t)$. This has y-intersect $(0, \log\tan\frac{t}{2})$. Then, the distance is $(\sin t)^2 + (\cos t + \log\tan\frac{t}{2} - \log\tan\frac{t}{2})^2 = 1$.

Question 5

Let $\alpha : (-1, \infty) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right)$$

Solution:
(a) For $t = 0$, α is tangent to the x-axis. Computing the derivative, $\frac{\partial\alpha(t)}{\partial t} = \left(\frac{a(3-6t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2}\right)$. Thus, $\alpha(0) = (0,0)$ and $\frac{\partial\alpha(0)}{\partial t} = (3a, 0)$. Thus, it is tangent to the x-axis.
(b) As $t \to \infty$, $\alpha(t) = \frac{\partial\alpha(t)}{\partial t} = (0,0)$. We take limits, $\lim_{t\to\infty}\alpha(t) = \left(\lim_{t\to\infty}\frac{3at}{1+t^3}, \lim_{t\to\infty}\frac{3at^2}{1+t^3}\right) = (0,0)$. Similarly, $\lim_{t\to\infty}\frac{\partial\alpha(t)}{\partial t} = \left(\lim_{t\to\infty}\frac{a(3-6t^3)}{(1+t^3)^2}, \lim_{t\to\infty}\frac{3at(2-t^3)}{(1+t^3)^2}\right) = (0,0)$.
(c)Take the curve with the opposite orientation. Now, as $t \to -1$, the curve and its tangent approach the line $x + y + a = 0$.
Let us compute $\lim_{t\to-1}\frac{\alpha(t)_y}{\alpha(t)_x} = \lim_{t\to-1}\frac{1}{t} = -1$. Now, consider

$$\lim_{t\to-1}\alpha(t)_y - (-1)\alpha(t)_x = \lim_{t\to-1}\frac{3a(t+t^2)}{1+t^3} = -a.$$

Also $\lim_{t\to-1}\frac{\alpha'(t)_y}{\alpha'(t)_x} = \lim_{t\to-1}\frac{3t(2-t^3)}{3-6t^3} = \frac{-9}{9} = -1$, which is a slope of x+y+a=0.
Thus, the curve and its tangent approach the line $y = (-1)x + (-a)$, or the line $x + y + a = 0$.

Question 10

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve. Let $[a,b] \subseteq I$ and set $\alpha(a) = p, \alpha(b) = q$.
(a) Show that, for any constant vector v, $|v| = 1$.

$$(q-p)\cdot v = \int_a^b \alpha'(t)\cdot v\mathrm{d}t \leq \int_a^b |\alpha'(t)|\mathrm{d}t.$$

Proof. By Fundamental Theorem of Calculus in 1-Dimension,

$$\int_a^b \alpha'(t)\cdot v\mathrm{d}t = (\alpha(b) - \alpha(a))\cdot v = (q-p)\cdot v.$$

Then, by Hölder's inequality, we have $|\alpha'(t)\cdot v| \leq |\alpha'(t)||v| = |\alpha'(t)|$. Thus,

$$\left|\int_a^b \alpha'(t)\cdot v\mathrm{d}t\right| \leq \int_a^b |\alpha'(t)\cdot v|\mathrm{d}t \leq \int_a^b |\alpha'(t)|\mathrm{d}t.$$

□

(b) Set

$$v = \frac{q-p}{|q-p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|\mathrm{d}t.$$

That is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Corollary 1.1. *Use* $(q-p)\cdot v = \frac{|q-p|^2}{|q-p|} = |q-p| = |\alpha(b)-\alpha(a)|$ *with 10(a).*

2 Section 1.4

Question 10

The natural orientation of R^2 makes it possible to associate a sign to the area A of a parallelogram generated by two linearly independent vectors $u, v \in R^2$. To do this, let $\{e_i\}, i = 1, 2$, be the natural ordered basis of R^2, and write $u = u_1e_1 + u_2e_2, v = v_1e_1 + v_2e_2$. Observe the matrix relation

$$\begin{pmatrix} u\cdot u & u\cdot v \\ v\cdot u & v\cdot v \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

and conclude that

$$A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.$$

Since the last determinant has the same sign as the basis $\{u, v\}$, we can say that A is positive or negative according to whether the orientation of $\{u, v\}$ is positive or negative. This is called the *oriented area* in R^2.

Solution:

$$\begin{aligned}
A^2 &= (u\wedge v)\cdot(u\wedge v) \\
&= u\cdot(v\wedge(u\wedge v)) \\
&= u\cdot[(v\cdot v)u - (v\cdot u)v] \\
&= (u\cdot u)(v\cdot v) - (v\cdot u)(v\cdot u) \\
&= \begin{vmatrix} u\cdot u & u\cdot v \\ v\cdot u & v\cdot v \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.
\end{aligned}$$

Question 11

a. Show that the volume V of a parallelepiped generated by three linearly independent vectors $u, v, w \in R^3$ is given by $V = |(u\wedge v)\cdot w|$, and introduce an oriented volume in R^3.

b. Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}$$

Solution. (a) Let $n = \frac{u \wedge v}{||u \wedge v||}$ be the normal vector of the plane generated by u and v. Then

$$\begin{aligned} V &= (||u|| \times ||v|| \times |\sin(u,v)|) \times ||w|| \times |\cos(n,w)| \\ &= ||u \wedge v|| \times ||w|| \times |\cos(n,w)| \\ &= ||u \wedge v|| \times ||w|| \times |\cos(u \times v, w)| \\ &= |(u \wedge v) \cdot w| \end{aligned}$$

(b) We know that

$$\begin{aligned} |(u \wedge v) \cdot w| &= | \begin{pmatrix} u_2 v_3 - v_2 u_3 \\ u_3 v_1 - v_3 u_1 \\ u_1 v_2 - v_1 u_2 \end{pmatrix} \cdot w| \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \det(u, v, w). \end{aligned}$$

Hence,

$$\begin{aligned} V^2 &= \det(u,v,w)^2 \\ &= | \begin{pmatrix} u^T \\ v^T \\ w^T \end{pmatrix} \begin{pmatrix} u & v & w \end{pmatrix} | \\ &= \begin{vmatrix} u^T u & u^T v & u^T w \\ v^T u & v^T v & v^T w \\ w^T u & w^T v & w^T w \end{vmatrix} \\ &= \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w. \end{vmatrix} \end{aligned}$$

Question 12

Given the vectors $v \neq 0$ and w, show that there exists a vector u such that $u \wedge v = w$ if and only if v is perpendicular to w. Is this vector u uniquely determined? If not, what is the most general solution?

Solution: ($\Rightarrow$) By the properties of cross product, $u \wedge v = w$ implies that $v \cdot w = 0$.
($\Leftarrow$) If $v \cdot w = 0$, we have

$$(v \wedge w) \wedge v = (v \cdot v)w - (v \cdot w)v = ||v||^2 w.$$

Then $v \neq 0$ implies that $w = \frac{v \wedge w}{||v||^2} \wedge v$. Let $u = \frac{v \wedge w}{||v||^2}$ and we have $u \wedge v = w$.

Suppose there exist u' other than u such that $u' \times v = w$. Then

$$\begin{aligned} & u \wedge v = u' \wedge v \\ \Rightarrow & (u' - u) \wedge v = 0 \\ \Rightarrow & u' - u = kv, \quad k \in R \\ \Rightarrow & u' = u + kv, \quad k \in R. \end{aligned}$$

Therefore, the most general solution of $u \wedge v = w$ is

$$u = \frac{v \wedge w}{||v||^2} + kv, \quad k \in R.$$