

1 Section 1.7

Question 6

Let $\alpha(s)$, $s \in [0, l]$ be a closed convex plane curve positively oriented. The curve

$$\beta(s) = \alpha(s) - rn(s)$$

where r is a positive constant and n is a normal vector, is called a *parallel* curve to α (textbook figure is wrong, curve needs to be convex). Show that

(a) Length of β = length of α + $2\pi r$.

Solution. WLOG assume α is parametrised by arc length, since α is a plane curve the torsion is zero, so $n' = -kt$.

$$\begin{aligned}\beta'(s) &= \alpha'(s) - rn'(s) \\ &= \alpha'(s) - r(-k(s)t(s)) \\ &= (1 + rk(s))t(s)\end{aligned}$$

now as α is a simple closed curve with positive orientation, the rotation index is 1 and $\int_0^l k(s) ds = 2\pi$, so

$$\begin{aligned}\text{length of } \beta &= \int_0^l |\beta'(s)| ds \\ &= \int_0^l (1 + rk(s)) ds \\ &= l + 2\pi r.\end{aligned}$$

(b) $A(\beta) = A(\alpha) + rl + \pi r^2$.

Solution. For each $t \in [0, r]$ let

$$\beta_t(s) = \alpha(s) - tn(s)$$

which is a curve that is also parallel to α and lies between α and β . Then we have the expression

$$\begin{aligned}A(\beta) - A(\alpha) &= \int_0^r \text{length of } \beta_t dt \\ &= \int_0^r (l + 2\pi t) dt \\ &= rl + \pi r^2\end{aligned}$$

as required.

(c) $k_\beta(s) = k_\alpha(s)/(1 + rk_\alpha(s))$ (question typo'd).

Solution. We re-parametrise β by arc length. Let

$$t(s) = \int_0^s |\beta'(u)| du$$

and we have $\frac{ds}{dt} = \frac{1}{|\beta'(s)|}$. Define $\hat{\beta}$ such that $\hat{\beta}(t(s)) = \beta(s)$, then

$$\hat{\beta}''(t) = \frac{\alpha''(s)}{|\beta'(s)|}$$

taking lengths we have

$$|\hat{\beta}''(t)| = k_{\beta}(s) = \frac{k_{\alpha}(s)}{1 + rk_{\alpha}(s)}.$$

Question 7

Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ be a plane curve defined in the entire real line \mathbb{R} . Assume that α does not pass through the origin $O = (0, 0)$ and that both limits

$$\lim_{t \rightarrow -\infty} |\alpha(t)| = \lim_{t \rightarrow \infty} |\alpha(t)| = \infty.$$

(a) Prove that there exists a point t_0 such that $|\alpha(t_0)| \leq |\alpha(t)|$ for all $t \in \mathbb{R}$.

(b) Show, by an example, that the assertion in part a is false if one does not assume that both $\lim_{t \rightarrow -\infty} |\alpha(t)| = \infty$ and $\lim_{t \rightarrow \infty} |\alpha(t)| = \infty$.

Solution.

(a) Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ sending t to $|\alpha(t)|$. Pick an arbitrary $t_1 \in \mathbb{R}$. We can find a, b such that for all $t < a$, $f(t) > f(t_1)$ and for all $t > b$, we have $f(t) > f(t_1)$. Then, we can consider the restriction $f|_{[a, b]}$ to the compact interval $[a, b]$. By the Extreme Value Theorem, we have that this takes a minimum on $[a, b]$, say at t_0 . Then this t_0 satisfy the required properties as for $t \in [a, b]$, $|\alpha(t_0)| \leq |\alpha(t)|$, and for $t \notin [a, b]$, $|\alpha(t)| > |\alpha(t_1)| \geq |\alpha(t_0)|$.

(b) Consider $f(t) = (e^t, 0)$. Then, $\inf_t |\alpha(t)| = 0$, but this is non-zero for all t .

Question 8

(a) Let $\alpha(s)$, $s \in [0, l]$, be a plane simple closed curve. Assume that the curvature $k(s)$ satisfies $0 < k(s) \leq c$ where c is a constant (thus, α is less curved than a circle of radius $1/c$). Prove that

$$\text{length of } \alpha \geq \frac{2\pi}{c}.$$

Solution. We have a simple closed curve so we know that

$$\int_0^l k(s) ds = 2\pi$$

substituting the bounds $0 < k(s) \leq c$ we have

$$2\pi \leq cl \implies l \geq \frac{2\pi}{c}.$$

(b) In part (a) replace the assumption of being simple by “ α has rotation index N .” Prove that

$$\text{length of } \alpha \geq \frac{2\pi N}{c}.$$

Solution. Similarly, as rotation index is N ,

$$\int_0^l k(s) ds = 2\pi N$$

similarly substitute the bounds and we have

$$l \geq \frac{2\pi N}{c}.$$

Question 9

We here define the notion of being “inside” a simple closed curve.

Definition 1.1. A point p is inside a simple closed curve C if there exists a ray from p that intersects exactly one point in C , say q , such that $q \neq p$. Moreover, the tangent line at q is not parallel to the ray from p to q .

Theorem 1.2. A set $K \subseteq \mathbb{R}^2$ is convex if for any two points $p, q \in K$, the segment of straight line \overline{pq} is contained in K .

Let C be a simple closed curve and K be the points on or inside C . Then if C is convex so is K .

Proof. For each point $c \in C$, let U_c be the closed half-plane, with the tangent line through c being the boundary and $C \subseteq U_c$. By convexity, we have $K \subseteq U = \bigcap_{c \in C} U_c$. As the intersection of two convex sets is convex, and half-planes are convex, it remains to show that $U \subseteq K$.

Consider a point $p \in \mathbb{R}^2 \setminus K$. As C is compact, we can pick $q \in C$ such that $d(p, q)$ is minimized. Then, the tangent line at q must be perpendicular to the ray from p to q . Suppose then that $p \in U_q$. Then the ray can not intersect another point of C , as C cannot intersect the line segment \overline{pq} as q is the closest point to p on C , and C cannot intersect the ray after q , as all of C lies in U_q . By our definition, this implies that p is inside the curve, which is a contradiction. \square

2 Section 2.2

Question 4

Let $f(x, y, z) = z^2$. Prove that 0 is not a regular value off and yet that $f^{-1}(0)$ is a regular surface.

Solution. Note that

$$df = (f_x, f_y, f_z) = (0, 0, 2z),$$

which is not surjective only when $z = 0$. Hence, $(0, 0, 0)$ is a critical point and thus $f(0, 0, 0) = 0$ is not a regular value.

However,

$$\begin{aligned} f^{-1}(0) &= \{(x, y, z) \in \mathbb{R}^3 | z^2 = 0\} \\ &= \{(x, y, 0) | x, y \in \mathbb{R}\} \\ &= \mathbb{R}^2 \times \{0\}. \end{aligned}$$

Hence, $f^{-1}(0)$ is homeomorphic to \mathbb{R}^2 and therefore is regular.

Question 5

Let $P = \{(x, y, z) \in \mathbb{R}^3 | x = y\}$ (a plane) and let $x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$x(u, v) = (u + v, u + v, uv),$$

where $U = \{(u, v) \in \mathbb{R}^2 | u > v\}$. Clearly, $x(U) \subset P$. Is x a parametrization of P ?

Solution. Yes, x is a parametrization. Clearly, x is differentiable in U with

$$dx(u, v) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ u & v \end{pmatrix}.$$

Note that for $(u, v) \in U$, we have $u > v$. Then

$$\begin{vmatrix} 1 & 1 \\ u & v \end{vmatrix} = v - u \neq 0.$$

This implies $dx(u, v)$ is injective for all $(u, v) \in U$. Now let (a, a, b) be any point in $x(U)$. Then

$$\begin{aligned} u + v &= a, uv = b \\ \Rightarrow u(a - u) &= b \\ \Rightarrow \left(u - \frac{a}{2}\right)^2 &= \frac{a^2}{4} - b. \end{aligned}$$

Notice that here one must have $\frac{a^2}{4} - b \geq 0$ as the equations $\begin{cases} u + v = a \\ uv = b \end{cases}$ should have real solutions for $(a, a, b) \in x(U)$. Then given $u > v$, we have

$$\begin{aligned} u &= \frac{a}{2} + \sqrt{\frac{a^2}{4} - b} \\ v &= \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}. \end{aligned}$$

These are the unique (u, v) solving $x(u, v) = (a, b)$, which shows x is injective. Hence, by Prop. 4, x^{-1} must be continuous and we can conclude that x is indeed a parametrization.

Question 6

Give another proof of Prop. 1 by applying Prop. 2 to $h(x, y, z) = f(x, y) - z$.

Solution. Since f is differentiable in U , for any point in $U \times \mathbb{R}$, we have

$$dh = (f_x, f_y, -1),$$

which is always surjective regardless of the value of f_x, f_y . Hence, any $z_0 \in f(U)$ with $f(x_0, y_0) = z_0$, we have

$$h(x_0, y_0, z_0) = f(x_0, y_0) - z_0 = 0,$$

being a regular value. This implies that

$$\begin{aligned} h^{-1}(0) &= \{(x, y, z) \in U \times \mathbb{R} | h(x, y, z) = 0\} \\ &= \{(x, y, z) \in U \times \mathbb{R} | f(x, y) = z\} \\ &= \{(x, y, f(x, y)) | (x, y) \in U\} \end{aligned}$$

is a regular surface.