1 Section 3.5

Question 1

Show that the helicoid (cf. Example 3, Sec. 2-5) is a ruled surface, its line of striction is the z-axis, and its distribution parameter is constant.

Solution. The helicoid is a surface with parametrization $\mathbf{x}(u,v) = (a\sinh v\cos u, a\sinh v\sin u, au)$ with some $a \neq 0$. We let $\mathbf{y}(t,v) = \mathbf{x}(t, \sinh^{-1}(v/a))$ be a reparametrization of helicoid. Then

$$\begin{aligned}\mathbf{y}(t,v) &= (a\sinh(\sinh^{-1}(v/a))\cos t, a\sinh(\sinh^{-1}(v/a))\sin t, at)\\ &= (0,0,at) + v(\cos t, \sin t, 0).\end{aligned}$$

Let $\alpha(t) = (0,0,at)$ and $w(t) = (\cos t, \sin t, 0)$. We can see let the helicoid is a ruled surface and $|w(t)| \equiv 1$.

To find the line of restriction and the distribution parameter, we notice that

$$\alpha'(t) = (0,0,a) \qquad w'(t) = (-\sin t, \cos t, 0).$$

Hence,

$$u(t) = -\frac{\langle \alpha'(t), w'(t)\rangle}{\langle w'(t), w'(t)\rangle} = 0.$$

This shows that $\beta(t) = \alpha(t) + u(t)w(t) = \alpha(t)$, which is the z-axis. Besides,

$$\begin{aligned}\lambda(t) &= \frac{\langle \beta'(t)\wedge w(t), w'(t)\rangle}{|w'(t)|^2}\\ &= \frac{\langle \alpha'(t)\wedge w(t), w'(t)\rangle}{|w'(t)|^2}\\ &= \frac{\langle (0,0,a)\wedge(\cos t,\sin t,0), (-\sin t,\cos t,0)\rangle}{|(-\sin t,\cos t,0)|^2}\\ &= \langle(-a\sin t, a\cos t, 0), (-\sin t, \cos t, 0)\rangle\\ &= a,\end{aligned}$$

which is a constant.

Question 6

Let

$$\mathbf{x}(t,v) = \alpha(t) + vw(t)$$

be a developable surface. Prove that at a regular point we have

$$\langle N_v, \mathbf{x}_v\rangle = \langle N_v, \mathbf{x}_t\rangle = 0.$$

Conclude that the tangent plane of a developable surface is constant along (the regular points of) a fixed ruling.

Solution. We have

$$\begin{aligned}
\mathbf{x}_t &= a'(t) + vw'(t), \\
\mathbf{x}_v &= w(t), \\
\mathbf{x}_{vt} &= w'(t), \\
\mathbf{x}_{vv} &= 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\langle N_v, \mathbf{x}_v \rangle &= -\langle N, \mathbf{x}_{vv} \rangle \\
&= -\langle N, 0 \rangle \\
&= 0 \\
\langle N_v, \mathbf{x}_t \rangle &= -\langle N, \mathbf{x}_{vt} \rangle \\
&= -\langle \frac{(a'(t) + vw'(t)) \wedge w(t)}{|(a'(t) + vw'(t)) \wedge w(t)|}, w'(t) \rangle \\
&= C\langle (a'(t) + vw'(t)) \wedge w(t), w'(t) \rangle \\
&= C\langle a'(t) \wedge w(t), w'(t) \rangle + Cv\langle w'(t) \wedge w(t), w'(t) \rangle \\
&= C\langle a'(t) \wedge w(t), w'(t) \rangle \\
&= C\langle a'(t), w(t) \wedge w'(t) \rangle \\
&= 0,
\end{aligned}$$

where $C = -\frac{1}{|(a'(t)+vw'(t)) \wedge w(t)|}$. Since $\langle N_v, \mathbf{x}_v \rangle = \langle N_v, \mathbf{x}_t \rangle = 0$ and $N_v \in T_p(S) = span\{\mathbf{x}_v, \mathbf{x}_t\}$ for all the regular points, we have $N_v = 0$ for all the regular points. In particular, $N_v = 0$ for a fixed ruling and hence N is constant for a fixed ruling. This implied that the tangent plane of a developable surface is constant along a fixed ruling.