1 Section 2.3

Question 1

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $A : S^2 \to S^2$ be the (antipodal) map $A(x, y, z) = (-x, -y, -z)$. Prove that A is a diffeomorphism.

Solution. Let $U = \{(x, y, \sqrt{1 - x^2 - y^2}) : x^2 + y^2 < 1\}$. Then we have

$$\begin{aligned} A(U) &= \{-(x, y, \sqrt{1 - x^2 - y^2}) : x^2 + y^2 < 1\} \\ &= \{(x, y, -\sqrt{1 - x^2 - y^2}) : x^2 + y^2 < 1\}. \end{aligned}$$

Let $\phi(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$ be a parametrization for U and $\psi(u, v) = (u, v, -\sqrt{1 - u^2 - v^2})$ a parametrization for $A(U)$. Then

$$\begin{aligned} \psi^{-1} \circ A \circ \phi(u, v) &= \psi^{-1}(A(\phi(u, v))) \\ &= \psi^{-1}(A((u, v, \sqrt{1 - u^2 - v^2}))) \\ &= \psi^{-1}((-u, -v, -\sqrt{1 - u^2 - v^2})) \\ &= -(u, v). \end{aligned}$$

Therefore, $\psi^{-1} \circ A \circ \phi$ is a differentiable function from $U \to A(U)$. This shows that A is a diffeomorphism.

Question 3

Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to the plane.
Solution. Luckily, we can pick one local chart each to cover the paraboloid U and the plane V. For the paraboloid, we can have (U, f^{-1}) where $f : (x, y) \to (x, y, x^2+y^2)$ and for the plane, we can choose our local chart to be (V, g^{-1}) where to be $g : (x, y) \to (x, y)$.
Let us now define the diffeomorphism $\pi : U \to V$. Let $\pi(a, b, c) = (a, b)$. Then, $g^{-1} \circ \pi \circ f = \mathrm{id}_{\mathbb{R}^2}$. Thus, π is smooth. Then, π^{-1} is well-defined as it is bijective as a function of sets. π^{-1} is smooth as $f^{-1} \circ \pi^{-1} \circ g = \mathrm{Id}_{\mathbb{R}^2}$. Thus, π is a diffeomorphism.

Question 5

Let $S \subset R^3$ be a regular surface, and let $d : S \to R$ be given by $d(p) = |p - p_0|$, where $p \in S, p \in R^3, p_0 \notin S$; that is, d is the distance from p to a fixed point p_0, not in S. Prove that d is differentiable.

Solution. We select $p \in S$, a open neighborhood U of p and a parametrization $\underset{\sim}{x} : V \to U$, where $V \subset R^2$ and $\underset{\sim}{x}(u_0, v_0) = p$. Let $p_0 = (p_1, p_2, p_3)$. Let $f(p) = (d(p, p_0))^2$Then

$$f \circ \underset{\sim}{x}(u, v) = (x_1(u, v) - p_1)^2 + (x_2(u, v) - p_2)^2 + (x_3(u, v) - p_3)^2$$

Since $x_i(u, v)$ are all smooth functions, $f \circ \underset{\sim}{x}$ should also be smooth. Since $p_0 \notin S$, we have $f \circ \underset{\sim}{x}(S) \in (0, \infty)$. Note that $g(x) = \sqrt{x}$ is smooth on $(0, \infty)$. Therefore, $\sqrt{f \circ \underset{\sim}{x}} = d \circ \underset{\sim}{x}$ is also smooth. This shows that d is differentiable.

Question 7

Definition 1.1. *A* ***manifold*** *is a second-countable Hausdorff topological space M such that for any point $m \in M$, there exists an open set N containing m such that N is homeomorphic to a subspace of $\mathbb{R}^k$ for some fixed $k \in \mathbb{Z}_{\geq 0}$.*
A ***chart*** *is a pair (U, ϕ) such that U is an open subset of M and $\phi : U \to \mathbb{R}^k$ is a homeomorphism. Then, an* ***atlas*** *is a set of charts such that for every point, there exists a chart (A, α) such that A contains the point.*
A ***differentiable manifold*** *is a manifold with an atlas with the property that for any two charts (A, α) and (B, β) in the atlas, if $A \cap B$ is nonempty, $\beta \circ \alpha^{-1}$ is a differentiable map when restricted such that the image of α^{-1} is $A \cap B$.*

Theorem 1.2. *Diffeomorphism is an isomorphism in the category of Differentiable Manifolds.*

Remark 1.3. *A surface is a manifold, as it is by definition Hausdorff and second countable as a subspace of $\mathbb{R}^3$, and locally is isomorphic to $\mathbb{R}^2$ as per the definiton of surface. A regular surface implies that we have a atlas with compatible differentiable structure.*

Proof. We need to show reflexivity, symmetric and transititive. Let $(M, \mathscr{A})$ and $(N, \mathscr{B})$ be two manifolds where $\mathscr{A}, \mathscr{B}$ are maximal atlas. Let $f : M \to N$ be a diffeomorphism.

- (Reflexive) The diffeomorphism has the property that $x \in \mathscr{B}$ if and only if $x \circ f \in \mathscr{A}$, and f is bijective (as sets). Thus, $y \in \mathscr{A}$ implies, $y \circ f^{-1} \circ f \in \mathscr{A}$, implying $y \circ f^{-1} \in \mathscr{B}$. As the implication are reversible, this shows that f^{-1} is also a diffeomorphism.

- (Symmetric) Consider the identity map on M. This has the property that $x \in \mathscr{A}$ if and only if $x \circ \mathrm{Id}_M \in A$.

- (Transitivity) Let $(K, \mathscr{C})$ be a manifold where $\mathscr{C}$ is a maximal atlas, and g be a diffeomorphism from N to K. Then, $x \in \mathscr{C}$ if and only if $x \circ g \in \mathscr{B}$ if and only if $x \circ g \circ f \in \mathscr{A}$. As the composition of bijective functions is bijective, $(g \circ f)$ is a diffeomorphism.

$\square$

Question 9

(a). Define the notion of differentiable function on a regular curve. What does one need to prove for the definition to make sense?
Solution. Let $f : C \to \mathbb{R}$ be defined on a regular curve S. Then f is differentiable at $p \in C$ if, for some parametrization $\alpha : I \to C$ with $p \in \alpha(I) \subset C$ where I is an interval, the composition $f \circ \alpha : I \to \mathbb{R}$ is differentiable at $\alpha^{-1}(p)$. f is differentiable in C if it is differentiable at all points $p \in C$.
For the definition to make sense, one has to prove that the definition given does not depend on the choice of parametrization. One needs to prove a statement analogous to proposition 1, that is given two parametrizations of C both containing a neighbourhood of p, the change of parameters between those is a diffeomorphism.

2 Section 2.4

Question 15

Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

Solution. Let p_0 be such a fixed point. Then for any $p = \underset{\sim}{x}(u, v) \in S$, $p - p_0$ is normal to the tangent plane in p. That is

$$\underset{\sim}{x}_u \cdot (p - p_0) = \underset{\sim}{x}_v \cdot (p - p_0) = 0.$$

Then for the function $h(u, v) = (\underset{\sim}{x}(u, v) - p_0) \cdot (\underset{\sim}{x}(u, v) - p_0)$, we have

$$h_u = 2\underset{\sim}{x}_u \cdot (\underset{\sim}{x} - p_0) = 0$$
$$h_v = 2\underset{\sim}{x}_v \cdot (\underset{\sim}{x} - p_0) = 0.$$

This shows that h is constant for any connected component of S. However, since S is connected, h should be constant on S, which implies that $S \subset p \in R^3 : ||p - p_0|| = K$ for some $K > 0$.

Question 17

Two regular surfaces S_1 and S_2 intersect transversally if whenever $p \in S_1 \cap S_2$, then $T_p(S_1) \neq T_p(S_2)$. Prove that if S_1 intersects S_2 transversally, then a connected component of $S_1 \cap S_2$ is a regular curve.

Proof. Observe in $\mathbb{R}^3$, if $T_p(S_1) \neq T_p(S_2)$, then $\dim_{\mathbb{R}}(T_p(S_1) \cap T_p(S_2)) = 1$.
It suffices to show that $S_1 \cap S_2$ is a locally a curve. This can be seen as we pick $v \in T_p(S_1) \cap T_p(S_2)$. Then we have a curve $c : [0, 1] \to S_1$ with initial conditions $c(0) = p$,$c'(0) = v$. Similarly, we have a curve $d : [0, 1] \to S_2$ mutatis mutandis. As they share initial conditions, they are locally equal at p, and thus the intersection is locally a curve. □

Question 19

Let $S \in \mathbb{R}^3$ be a regular surface and $P \in \mathbb{R}^3$ be a plane. If all points of S are on the same side of P, prove that P is tangent to S at all points of $P \cap S$.
Proof. Let $p \in P$ and n_P be a unit normal of the plane P, then consider

$$h(r) = (r - p) \cdot n_P$$

one can compute that $dh_r = n_P$ for all $r \in \mathbb{R}^3$.
Without loss of generality assume that for all $s \in S$, $h(s) \geq 0$, so the surface lies on the "positive" side of the plane. If $\gamma(u)$ is a regular path in S, then

$$(h \circ \gamma)'(u) = (\nabla h) \cdot \gamma'(u) = n_P \cdot \gamma'(u).$$

Now if $s \in P \cap S$, then $h(s) = 0$, and if γ is a path that goes through s such that $\gamma(u_0) = s$, then we know that $(h \circ \gamma)(u) \geq (h \circ \gamma)(u_0)$ and by extreme value theorem we have

$$(h \circ \gamma)'(u_0) = 0 = \gamma'(u_0) \cdot n_P.$$

Since the path γ chosen was arbitrary, it means that the entire tangent plane of s is normal to n_P at s, which means that its tangent plane is P.

Question 21

Let $f : S \to R$ be a differentiable function on a connected regular surface S. Assume that $df_p = 0$ for all $p \in S$. Prove that f is constant on S.

Proof. Suppose for a contradiction that f is non-constant, let $f(s_1) = y_1$ and $f(s_2) = y_2$ where $s_1 \neq s_2$ and $y_1 \neq y_2$. Since S is connected, consider a regular path γ from s_1 to s_2, let $\gamma(a) = s_1$ and $\gamma(b) = s_2$. By chain rule

$$(f \circ \gamma)'(x) = df_{\gamma(x)} \cdot \gamma'(x) = 0$$

from single variable calculus we have $f \circ \gamma$ is a constant function, but $(f \circ \gamma)(s_1) = y_1 \neq y_2 = (f \circ \gamma)(s_2)$.

Question 23

Let $U = \{(x, y, z) \in \mathbb{R}^3 \mid z = -1\}$ be identified with the complex plane $\mathbb{C}$, by sending $(x, y, z) \in U$ to $x + iy$. Let $P(x) = \sum_{k=0}^{n} a_k x^{n-k} \in \mathbb{C}[x]$ be of degree n. Denote π to be the stereographic projection of $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ from the north pole $(0, 0, 1)$ onto U.

Prove that the map $F : S^2 \to S^2$ given by

$$\begin{aligned} F(p) &= \pi^{-1} \circ P \circ \pi(p) \text{ for } p \notin U \setminus \{(0,0,1)\} \\ F((0,0,1)) &= (0,0,1) \end{aligned}$$

has finitely many critical points.

Proof. Recall that $\pi(x, y, z) = \left(\frac{2x}{1-z}, \frac{2y}{1-z}, -1\right) \subseteq U$. Thus π is a diffeomorphism between $U \setminus \{(0, 0, 1)\}$ and $\mathbb{R}^2$. Thus, it suffices to show that $\hat{P} : \mathbb{R}^2 \to \mathbb{R}^2$ sending (x, y) to $(\mathrm{Re}P(x + iy), \mathrm{Im}P(x + iy))$ has finitely many critical points. (We can ignore the north pole, as it doesn't contain infinitely many distinct points). To see this, we require $\frac{\partial \mathrm{Re}P(x+iy)}{\partial x} = \frac{\partial \mathrm{Re}P(x+iy)}{\partial y} = \frac{\partial \mathrm{Im}P(x+iy)}{\partial x} = \frac{\partial \mathrm{Im}P(x+iy)}{\partial y} = 0$. By Complex Analysis, this implies that $\frac{\mathrm{d}P(z)}{dz} = 0$, but that can only occurs finitely many times as P is a polynomial. ∎

Question 25

Prove that if two regular curves C_1 and C_2 of a regular surface S are tangent at a point $p \in S$, and if $\varphi : S \to S$ is a diffeomorphism, then $\varphi(C_1)$ and $\varphi(C_2)$ are regular curves which are tangent at $\varphi(p)$.

Solution. Let U be a neighborhood of p with parametrization $\underset{\sim}{x}(u, v)$. Let $\alpha_1(t)$ and $\alpha_2(t)$ be such that $\underset{\sim}{x} \circ \alpha_1$ and $\underset{\sim}{x} \circ \alpha_2$ are regular parametrizations of C_1 and C_2 with $\underset{\sim}{x} \circ \alpha_1(0) = \underset{\sim}{x} \circ \alpha_2(0) = p$. Then that C_1 and C_2 are tangent at p implies that

$$\alpha_1'(0) = \alpha_2'(0).$$

Now let V be a neighborhood of $\varphi(p)$ with parametrization $\underset{\sim}{y}(w, z)$ and $\psi = (\psi_1, \psi_2) = \underset{\sim}{y}^{-1} \circ \varphi \circ \underset{\sim}{x}$. Let $\beta_1(t)$ and $\beta_2(t)$ be such that $\underset{\sim}{y} \circ \beta_1$ and $\underset{\sim}{y} \circ \beta_2$ are regular parametrizations of $\varphi(C_1)$ and $\varphi(C_2)$ with $\underset{\sim}{y} \circ \beta_1(0) = \underset{\sim}{y} \circ \beta_2(0) = \varphi(p)$. Than

$$\begin{aligned} \beta_1'(0) &= \left(\frac{\partial \psi_1}{\partial u}(\alpha_1(0))\alpha_{1u}'(0) + \frac{\partial \psi_1}{\partial v}(\alpha_1(0))\alpha_{1v}'(0), \frac{\partial \psi_2}{\partial u}(\alpha_1(0))\alpha_{1u}'(0) + \frac{\partial \psi_2}{\partial v}(\alpha_1(0))\alpha_{1v}'(0) \right) \\ &= \begin{pmatrix} \frac{\partial \psi_1}{\partial u}(\alpha_1(0)) & \frac{\partial \psi_1}{\partial v}(\alpha_1(0)) \\ \frac{\partial \psi_2}{\partial u}(\alpha_1(0)) & \frac{\partial \psi_2}{\partial v}(\alpha_1(0)) \end{pmatrix} \alpha_1'(0) \\ &= J_\psi(\alpha_1(0))\alpha_1'(0). \end{aligned}$$

Likewise, we have

$$\beta_2'(0) = J_\psi(\alpha_2(0))\alpha_2'(0).$$

However, since $\alpha_1(0) = \alpha_2(0) = \underset{\sim}{x}^{-1}(p)$ and $\alpha_1'(0) = \alpha_2'(0)$, we have $\beta_1'(0) = \beta_2'(0)$. This implies that $\varphi(C_1)$ and $\varphi(C_2)$ is tangent at $\varphi(p)$.