

1 Section 2.5

Question 1

The general recipe is to $(E, F, G) = (\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial u} \rangle, \langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \rangle, \langle \frac{\partial x}{\partial v}, \frac{\partial x}{\partial v} \rangle)$.

Direct Computation shows,

1. We have that

- (a) $\frac{\partial x}{\partial u} = (a \cos(u) \cos(v), b \cos(u) \sin(v), -c \sin(u))$
- (b) $\frac{\partial x}{\partial v} = (-a \sin(u) \sin(v), b \cos(v) \sin(u), 0)$
- (c) $E = a^2 \cos(u)^2 \cos(v)^2 + b^2 \cos(u)^2 \sin(v)^2 + c^2 \sin(u)^2$
- (d) $F = -(a^2 - b^2) \cos(u) \cos(v) \sin(u) \sin(v)$
- (e) $G = b^2 \cos(v)^2 \sin(u)^2 + a^2 \sin(u)^2 \sin(v)^2$

2. We have that

- (a) $\frac{\partial x}{\partial u} = (a \cos(v), b \sin(v), 2u)$
- (b) $\frac{\partial x}{\partial v} = (-au \sin(v), bu \cos(v), 0)$
- (c) $E = a^2 \cos(v)^2 + b^2 \sin(v)^2 + 4u^2$
- (d) $F = -(a^2 - b^2)u \cos(v) \sin(v)$
- (e) $G = b^2 u^2 \cos(v)^2 + a^2 u^2 \sin(v)^2$

3. We have that

- (a) $\frac{\partial x}{\partial u} = (a \cosh(v), b \sinh(v), 2u)$
- (b) $\frac{\partial x}{\partial v} = (au \sinh(v), bu \cosh(v), 0)$
- (c) $E = (a^2 + b^2) \sinh(v)^2 + a^2 + 4u^2$
- (d) $F = (a^2 + b^2)u \cosh(v) \sinh(v)$
- (e) $G = b^2 u^2 \cosh(v)^2 + a^2 u^2 \sinh(v)^2$

4. We have that

- (a) $\frac{\partial x}{\partial u} = (a \cos(v) \cosh(u), b \cosh(u) \sin(v), c \sinh(u))$
- (b) $\frac{\partial x}{\partial v} = (-a \sin(v) \sinh(u), b \cos(v) \sinh(u), 0)$
- (c) $E = a^2 \cos(v)^2 \cosh(u)^2 + b^2 \cosh(u)^2 \sin(v)^2 + c^2 \sinh(u)^2$
- (d) $F = -(a^2 - b^2) \cos(v) \cosh(u) \sin(v) \sinh(u)$
- (e) $G = b^2 \cos(v)^2 \sinh(u)^2 + a^2 \sin(v)^2 \sinh(u)^2$

Question 3

We have the parameterization

$$x(u, v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right).$$

Hence,

1. $\frac{\partial x}{\partial u} = \left(-\frac{8u^2}{(u^2+v^2+4)^2} + \frac{4}{u^2+v^2+4}, -\frac{8uv}{(u^2+v^2+4)^2}, \frac{4u}{u^2+v^2+4} - \frac{4(u^2+v^2)u}{(u^2+v^2+4)^2} \right)$
2. $\frac{\partial x}{\partial v} = \left(-\frac{8uv}{(u^2+v^2+4)^2}, -\frac{8v^2}{(u^2+v^2+4)^2} + \frac{4}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4} - \frac{4(u^2+v^2)v}{(u^2+v^2+4)^2} \right)$
3. $E = \frac{16}{u^4+v^4+2(u^2+4)v^2+8u^2+16}$
4. $F = 0$
5. $G = \frac{16}{u^4+v^4+2(u^2+4)v^2+8u^2+16}$

Question 5

We have the parameterization

$$x(u, v) = (u, v, f(u, v)).$$

Hence,

1. $\frac{\partial x}{\partial u} = (1, 0, \frac{\partial f}{\partial u})$
2. $\frac{\partial x}{\partial v} = (0, 1, \frac{\partial f}{\partial v})$

And thus, $\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} = (-\frac{\partial x}{\partial u}, -\frac{\partial x}{\partial v}, 1)$ and so

$$A = \iint_Q |(-\frac{\partial x}{\partial u}, -\frac{\partial x}{\partial v}, 1)| dx dy = \iint_Q \sqrt{\frac{\partial x^2}{\partial u} + \frac{\partial x^2}{\partial v} + 1^2} dx dy$$

Question 7

The conditions translate to for all $a, b \in \mathbb{R}$ such that $a < b$.

$$\begin{aligned} \hat{E}(v) &= \int_a^b \sqrt{E(u, v)} du \\ \hat{G}(u) &= \int_a^b \sqrt{G(u, v)} dv \end{aligned} \tag{1}$$

are constant functions. Differentiation the first with respect to v , and the second with respect to u yields the answer.

Question 9

We consider the parameterization

$$x(u, v) = (f(u) \cos(v), f(u) \sin(v), g(u))$$

Hence,

1. $\frac{\partial x}{\partial u} = (f'(u) \cos(v), f'(u) \sin(v), g'(u))$
2. $\frac{\partial x}{\partial v} = (-f(u) \sin(v), f(u) \cos(v), 0)$
3. $E = f'(u)^2 + g'(u)^2$
4. $F = 0$
5. $G = 1$

Swapping the variable names, $u \leftrightarrow v$ suffices.

Question 11

Let S be a surface of revolution and C be its generating curve. Let s be the arc length of C and denote by $\rho = \rho(s)$ the distance to the rotation axis of the point of C corresponding to s .

- a. (*Pappus' Theorem.*) Show that the area of S is

$$2\pi \int_0^l \rho(s) \, ds,$$

where l is the length of C .

Proof. Similarly to Example 4 of Sec. 2-3, we assume that the curve C lies in the xz -plane and the rotation axis is the z -axis. Let

$$\alpha(s) = (\rho(s), 0, h(s)), \quad 0 < s < l$$

be parametrization of C , denote by θ the rotation angle about the z -axis. Thus we obtain the map

$$\mathbf{x}(\theta, s) = (\rho(s) \cos \theta, \rho(s) \sin \theta, h(s))$$

which is a parametrisation $U \rightarrow S$ where $U = \{(\theta, s) : 0 < \theta < 2\pi, 0 < s < l\}$ is open. Now we can compute

$$\begin{aligned} |\mathbf{x}_\theta \wedge \mathbf{x}_s| &= \sqrt{\rho^2(s)(\rho'(s))^2 + \rho^2(s)(h'(s))^2} = \rho(s) \\ \text{Area of } S &= \iint_U |\mathbf{x}_\theta \wedge \mathbf{x}_s| \, d\theta \, ds \\ &= \int_0^l \int_0^{2\pi} \pi \rho(s) \, d\theta \, ds \\ &= 2\pi \int_0^l \rho(s) \, ds \end{aligned}$$

as desired.

b. Apply part a to compute the area of a torus of revolution.

Solution. Let R be the distance from the center of the tube to the center of the torus, and r be the radius of the tube. The torus of revolution can be generated by the curve C parametrised by

$$\alpha(s) = \left(R + r \cos \frac{s}{r}, 0, r \sin \frac{s}{r} \right), \quad 0 < s < 2\pi r.$$

In this case $\rho(s) = R + r \cos \frac{s}{r}$, so we have

$$\begin{aligned} \text{Area of torus} &= 2\pi \int_0^{2\pi r} R + r \cos \frac{s}{r} ds \\ &= 2\pi \left[Rs + r^2 \sin \frac{s}{r} \right]_0^{2\pi r} \\ &= 4\pi^2 Rr. \end{aligned}$$

2 Section 3.2

Question 1

Show that at a hyperbolic point, the principal directions bisect the asymptotic directions.

Proof. Let p be a hyperbolic point and let v be an asymptotic direction with angle θ from e_1 , where we write v as

$$v = e_1 \cos \theta + e_2 \sin \theta.$$

As v is asymptotic by Euler's formula we have

$$0 = \Pi_p(v) = k_1 \cos^2 \theta + k_2 \sin^2 \theta.$$

Since we have the identities $\cos(\pi - \theta) = -\cos \theta$ and $\sin(\pi - \theta) = \sin(\theta)$ we have

$$v' = e_1 \cos(\pi - \theta) + e_2 \sin(\pi - \theta)$$

is also asymptotic. Now we see that e_2 bisects v and v' .

Question 3

Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature k of C at p satisfies

$$k \geq \min(|k_1|, |k_2|),$$

where k_1 and k_2 are the principal curvatures of S at p .

Proof. WLOG suppose the S is oriented such that both k_1, k_2 are positive, furthermore for simplicity suppose $0 < k_1 < k_2$. Note that $k = k_n + k \sin^2 \theta$ so $k \geq k_n$, it suffices to prove that $k_n \geq \min(k_1, k_2)$. Let v be the tangent of the curve C at p , then

$$v = e_1 \cos \theta + e_2 \sin \theta$$

for some angle θ . By Euler's formula we have

$$k_n = k_1 \cos^2(\theta) + k_2 \sin^2(\theta).$$

Putting everything together we have

$$k \geq k_n = k_1 \cos^2(\theta) + k_2 \sin^2(\theta) \geq k_1 \cos^2(\theta) + k_1 \sin^2(\theta) = k_1 = \min(k_1, k_2).$$

3 Section 3.3

Question 1

Show that at the origin $(0, 0, 0)$ of the hyperboloid $z = axy$ we have $K = -a^2$ and $H = 0$.

Solution. We parametrize the hyperboloid

$$\mathbf{x}(u, v) = (u, v, auv).$$

We compute the following at origin $(0, 0, 0)$

$$\mathbf{x}_u = (1, 0, av)$$

$$\mathbf{x}_v = (0, 1, au)$$

$$\mathbf{x}_{uu} = \mathbf{0}$$

$$\mathbf{x}_{uv} = (0, 0, a)$$

$$\mathbf{x}_{vv} = \mathbf{0}$$

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = (-av, -au, 1)$$

$$e = \langle N, \mathbf{x}_{uu} \rangle = 0$$

$$f = \langle N, \mathbf{x}_{uv} \rangle = a$$

$$g = \langle N, \mathbf{x}_{vv} \rangle = 0$$

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1 + a^2v^2 = 1$$

$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = a^2uv = 0$$

$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1 + a^2u^2 = 1$$

$$K = \frac{eg - f^2}{EG - F^2} \\ = -a^2$$

$$H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} = 0$$

Question 3

Determine the asymptotic curves of the catenoid

$$\mathbf{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v).$$

Solution. We have

$$\mathbf{x}_u = (-\cosh v \sin u, \cosh v \cos u, 0),$$

$$\mathbf{x}_v = (\sinh v \cos u, \sinh v \sin u, 1),$$

$$\mathbf{x}_{uu} = (-\cosh v \cos u, -\cosh v \sin u, 0),$$

$$\mathbf{x}_{uv} = (-\sinh v \sin u, \sinh v \cos u, 0),$$

$$\mathbf{x}_{vv} = (\cosh v \cos u, \cosh v \sin u, 0).$$

Hence,

$$\begin{aligned}
N &= \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} \\
&= \frac{(\cosh v \cos u, \cosh v \sin u, -\cosh v \sinh v)}{(\cosh v)^2} \\
&= \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v).
\end{aligned}$$

Then

$$\begin{aligned}
e &= \langle N, \mathbf{x}_{uu} \rangle = -1, \\
f &= \langle N, \mathbf{x}_{uv} \rangle = 0, \\
g &= \langle N, \mathbf{x}_{vv} \rangle = 1.
\end{aligned}$$

This implies that the asymptotic directions correspond to $\langle u', v' \rangle$ satisfies

$$\begin{aligned}
e(u')^2 + 2fu'v' + g(v')^2 &= 0 \\
\Rightarrow -(u')^2 + (v')^2 &= 0 \\
\Rightarrow u' = v' \quad \text{or} \quad u' = -v'.
\end{aligned}$$

This shows that the asymptotic curves are the traces of $v = u + a$ or $v = -u + b$ for some $a, b \in \mathbb{R}$, which are

$$\begin{aligned}
\alpha_a(u) &= \mathbf{x}(u, u + a) = (\cosh(u + a) \cos u, \cosh(u + a) \sin u, u + a), \\
\beta_b(u) &= \mathbf{x}(u, -u + b) = (\cosh(-u + b) \cos u, \cosh(-u + b) \sin u, -u + b).
\end{aligned}$$

The collection of all α_a, β_b are all the asymptotic curves.

Question 9

(Contact of Curves.) Define contact of order $\geq n$ (n integer ≥ 1) for regular curves in \mathbb{R}^3 with a common point p and prove that

- The notion of contact of order $\geq n$ is invariant by diffeomorphisms.
- Two curves have contact of order ≥ 1 at p if and only if they are tangent at p .

Solution. We say two surfaces S and \bar{S} with a common point p to have contact of order $\geq n$ at p if there exist parametrizations $\mathbf{x}(u, v)$ and $\tilde{\mathbf{x}}(u, v)$ in p of S and \bar{S} such that the partial derivatives of \mathbf{x} and $\tilde{\mathbf{x}}(u, v)$ agree up to order n .

- Let $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a diffeomorphism. Then for any partial derivative operator ∂_I of order less than n on $u - v$ space. Then

$$\partial_I \psi(\mathbf{x}) = J_\psi(\mathbf{x}) \partial_I \mathbf{x} = J_\psi(\tilde{\mathbf{x}}) \partial_I \tilde{\mathbf{x}} = \partial_I \psi(\tilde{\mathbf{x}}),$$

where $J\psi(\cdot)$ is the Jacobian of ψ . This shows that the notion of contact is invariant by diffeomorphisms.

- It is easy to see that the contact of order ≥ 1 implies that the two surfaces are tangent. For

the converse, we suppose S and \bar{S} are tangent at p with parametrizations $\mathbf{x}(\mathbf{u}, \mathbf{v})$ and $\tilde{\mathbf{x}}(u, v)$ respectively. Then at the point p , $\tilde{\mathbf{x}}_u, \tilde{\mathbf{x}}_v \in T_{\mathbf{x}}(p)$ we can write

$$\begin{aligned}\tilde{\mathbf{x}}_u &= a_1 \mathbf{x}_u + a_2 \mathbf{x}_v, \\ \tilde{\mathbf{x}}_v &= b_1 \mathbf{x}_u + b_2 \mathbf{x}_v.\end{aligned}$$

Note that since $\tilde{\mathbf{x}}_u, \tilde{\mathbf{x}}_v$ are linearly independent, we have $a_1 b_2 - a_2 b_1 \neq 0$. Now let $w = \frac{b_2 u - a_2 v}{a_1 b_2 - a_2 b_1}$, $l = \frac{b_1 u - a_1 v}{a_2 b_1 - a_1 b_2}$ and $\mathbf{y}(w, l) = \tilde{\mathbf{x}}(u, v)$. Then

$$\begin{aligned}\mathbf{y}_w &= \frac{b_2}{a_1 b_2 - a_2 b_1} \tilde{\mathbf{x}}_u - \frac{a_2}{a_1 b_2 - a_2 b_1} \tilde{\mathbf{x}}_v = \mathbf{x}_u, \\ \mathbf{y}_l &= \frac{b_1}{a_2 b_1 - a_1 b_2} \tilde{\mathbf{x}}_u - \frac{a_1}{a_2 b_1 - a_1 b_2} \tilde{\mathbf{x}}_v = \mathbf{x}_v.\end{aligned}$$

This shows that S and \bar{S} have contact of order ≥ 1 at p .

Question 15

Give an example of a surface which has an isolated parabolic point p (that is, no other parabolic point is contained in some neighborhood of p).

Solution. Consider the graph $(x, y, x^4 + x^2 y^2 + y^2)$. Let $h(x, y) = x^4 + x^2 y^2 + y^2$. Then we have

$$\begin{aligned}K &= \frac{h_{xx} h_{yy} - (h_{xy})^2}{(1 + h_x^2 + h_y^2)^2} = \frac{24x^4 - 12x^2 y^2 + 24x^2 + 4y^2}{(1 + h_x^2 + h_y^2)^2}, \\ e &= \frac{h_{xx}}{(1 + h_x^2 + h_y^2)^{1/2}} = \frac{12x^2 + 2y^2}{(1 + h_x^2 + h_y^2)^{1/2}}, \\ f &= \frac{h_{xy}}{(1 + h_x^2 + h_y^2)^{1/2}} = \frac{2x^2 + 2}{(1 + h_x^2 + h_y^2)^{1/2}}, \\ g &= \frac{h_{yy}}{(1 + h_x^2 + h_y^2)^{1/2}} = \frac{4xy}{(1 + h_x^2 + h_y^2)^{1/2}}.\end{aligned}$$

Then $K = 0$ only at $(0, 0, 0)$, at which f is nonzero. This shows that the graph has an isolated parabolic point. $K = 0$ only at $(0, 0, 0)$, at which f and g are nonzero. This shows that the graph has an isolated parabolic point.

Question 19

Obtain the asymptotic curves of the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution. Note that the hyperboloid is a surface of revolution parametrized by

$$\mathbf{x}(u, v) = (\phi(v) \cos u, \phi(v) \sin u, \psi(v)),$$

where $\phi(v) = \cosh v$, $\psi(v) = \sinh v$ and $u \in (0, 2\pi)$. Then

$$\begin{aligned}e &= -\phi\psi' = -\cosh^2(v), \\ f &= 0, \\ g &= \psi'\phi'' - \psi''\phi' = \cosh^2(v) - \sinh^2(v) = 1.\end{aligned}$$

Then solving $e(u')^2 + 2fu'v' + g(v')^2 = 0$, we have

$$v' = u' \cosh(v) \quad \text{or} \quad v' = -u' \cosh(v).$$

Solving the ODE, we have

$$u(t) = \pm \tan^{-1}(\sinh v(t)) + C, \quad C \in \mathbb{R}$$

Hence, the asymptotic curves will be the trace of $\gamma_C(v) = (\pm \tan^{-1}(\sinh v) + C, v)$, $v \in \mathbb{R}$. They are

$$\alpha_C(v) = \mathbf{x}(\tan^{-1}(\sinh v) + C, v)$$

or

$$\beta_C(v) = \mathbf{x}(-\tan^{-1}(\sinh v) + C, v)$$

Question 21

Let S be a surface with orientation N . Let $V \subset S$ be an open set in S and let $f : V \subset S \rightarrow \mathbb{R}$ be any nowhere-zero differentiable function in V . Let v_1 and v_2 be two differentiable (tangent) vector fields in V such that at each point of V , v_1 and v_2 are orthonormal and $v_1 \wedge v_2 = N$.

a. Prove that the Gaussian curvature K of V is given by

$$K = \frac{\langle dfN(v_1) \wedge dfN(v_2), fN \rangle}{f^3}.$$

b. Apply the above result to show that iff is the restriction of

$$\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

then the Gaussian curvature of the ellipsoid is

$$K = \frac{1}{a^2 b^2 c^2} \frac{1}{f^4}.$$

Solution. Since f is a smooth function on $V = \mathbf{x}(u, v)$, if $\alpha'(0) = v_i = \frac{d}{dt}\mathbf{x}(\beta(t))|_{t=0}$, we have

$$\begin{aligned} dfN(v_i) &= \frac{d}{dt}f(\alpha(t))N(\alpha(t)) \\ &= \left(\frac{d}{dt}f(\alpha(t))\right)N(\alpha(t))|_{t=0} + f(\alpha(t))\frac{d}{dt}N(\alpha(t))|_{t=0} \\ &= (\nabla(f \circ \mathbf{x}) \cdot \beta'(0))N + fdN(v_i). \end{aligned}$$

Hence,

$$\begin{aligned} dfN(v_1) \wedge dfN(v_2) &= (C_1N + fdN(v_1)) \wedge (C_2N + fdN(v_2)) \\ &= C_1N \wedge fdN(v_2) - C_2N \wedge fdN(v_1) + f^2(dN(v_1) \wedge dN(v_2)) \\ &= C_1N \wedge fdN(v_2) - C_2N \wedge fdN(v_1) + f^2 \det(dN)(v_1 \wedge v_2) \\ &= C_1N \wedge fdN(v_2) - C_2N \wedge fdN(v_1) + f^2 \det(dN)N. \end{aligned}$$

Therefore,

$$\begin{aligned} dfN(v_1) \wedge dfN(v_2) \cdot fN &= C_1 N \wedge fdN(v_2) \cdot fN - C_2 N \wedge fdN(v_1) \cdot fN + f^2 \det(dN) N \cdot fN \\ &= f^3 KN \cdot N = f^3 K. \end{aligned}$$

Thus

$$\frac{dfN(v_1) \wedge dfN(v_2) \cdot fN}{f^3} = K.$$

b. We know that

$$\begin{aligned} N(x, y, z) &= \frac{\left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)}{\left|\left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)\right|} \\ &= \frac{\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)}{\left|\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)\right|} \\ &= \frac{\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)}{f(x, y, z)}. \end{aligned}$$

Therefore, $fN = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$. Then

$$\begin{aligned} \frac{d}{dt}fN(\alpha(t)) &= \left(\frac{x'(t)}{a^2}, \frac{y'(t)}{b^2}, \frac{z'(t)}{c^2}\right) \\ &= \begin{pmatrix} a^{-2} & & \\ & b^{-2} & \\ & & c^{-2} \end{pmatrix} \alpha'(t). \end{aligned}$$

Hence, $dfN(v_i) = \begin{pmatrix} a^{-2} & & \\ & b^{-2} & \\ & & c^{-2} \end{pmatrix} v_i$ and thus

$$\begin{aligned} K &= \frac{dfN(v_1) \wedge dfN(v_2) \cdot fN}{f^3} \\ &= \det(dfN) \frac{(dfN^{-1})^T(v_1 \wedge v_2) \cdot fN}{f^3} \\ &= (abc)^{-2} \frac{(dfN^{-1})N \cdot fN}{f^3} \\ &= (abc)^{-2} \frac{1}{f^3} \begin{pmatrix} a^2 & & \\ & b^2 & \\ & & c^2 \end{pmatrix} \frac{\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)}{f(x, y, z)} \cdot \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right) \\ &= (abc)^{-2} \frac{1}{f^3} \frac{(x, y, z)}{f} \cdot \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right) \\ &= \frac{1}{a^2 b^2 c^2} \frac{1}{f^4} \end{aligned}$$