

## 1 Section 4.2

### Question 1

We have that

1.  $\frac{\partial x}{\partial u} = (\cos(v) \sin(a), \sin(a) \sin(v), \cos(a))$
2.  $\frac{\partial x}{\partial v} = (-u \sin(a) \sin(v), u \cos(v) \sin(a), 0)$
3.  $E = 1$
4.  $F = 0$
5.  $G = u^2 \sin(a)^2$

As  $E \neq G$ , it is not conformal and hence not a local isometry.

To see that it is a local diffeomorphism onto a cone with angle  $2\alpha$ , observe that for any point  $(u, v) \in \mathbb{R}^2$ , construct the set  $S = \{(x, y) | x > 0, v - \frac{\pi}{2} \leq y < \frac{\pi}{2}\}$ . Then  $(u, v) \in S \subseteq \mathbb{R}^2$ , and we had that  $F|_S$  is a diffeomorphism from  $S$  to the cone needed (we had seen this noting that  $(u, v)$  is in polar coordinates where  $u$  is the radius and  $v$  the angle).

### Question 4

Let us consider the stereographic projection  $\pi : \mathbb{R}^2 \rightarrow S^2 \setminus \mathbf{N} \subseteq \mathbb{R}^3$  where  $\mathbf{N}$  denotes the north pole.

$$\pi(u, v) = \left( \frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right).$$

Note that this is sufficient to show local conformality as we can rotate the sphere. We had from previous homework that the local chart is a diffeomorphism and

1.  $\frac{\partial x}{\partial u} = \left( -\frac{4u^2}{(u^2+v^2+1)^2} + \frac{2}{u^2+v^2+1}, -\frac{4uv}{(u^2+v^2+1)^2}, \frac{2u}{u^2+v^2+1} - \frac{2(u^2+v^2-1)u}{(u^2+v^2+1)^2} \right)$
2.  $\frac{\partial x}{\partial v} = \left( -\frac{4uv}{(u^2+v^2+1)^2}, -\frac{4v^2}{(u^2+v^2+1)^2} + \frac{2}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1} - \frac{2(u^2+v^2-1)v}{(u^2+v^2+1)^2} \right)$
3.  $E = \frac{4}{u^4+v^4+2(u^2+1)v^2+2u^2+1}$
4.  $F = 0$
5.  $G = \frac{4}{u^4+v^4+2(u^2+1)v^2+2u^2+1}$

As  $E = G$  and  $F = 0$ , it is a conformal chart.

## Question 7

Let  $V$  and  $W$  be (finite-dimensional) vector spaces with inner product denoted by  $\langle, \rangle$  and let  $F : V \rightarrow W$  be a linear map. Prove that the following conditions are equivalent:

- a.  $\langle F(v_1), F(v_2) \rangle = \langle v_1, v_2 \rangle$  for all  $v_1, v_2 \in V$ .
- b.  $|F(v)| = |v|$  for all  $v \in V$ .
- c. If  $\{v_1, \dots, v_n\}$  is an orthonormal basis in  $V$ , then  $\{F(v_1), \dots, F(v_n)\}$  is an orthonormal basis in  $W$ .
- d. There exists an orthonormal basis  $\{v_1, \dots, v_n\}$  in  $V$  such that  $\{F(v_1), \dots, F(v_n)\}$  is an orthonormal basis in  $W$ .

**(a) $\Rightarrow$ (b).** is obvious.

**(a) $\Rightarrow$ (c).** Assume (a), if  $\{v_1, \dots, v_n\}$  is an orthonormal basis in  $V$  then whenever  $i \neq j$ ,  $0 = \langle v_i, v_j \rangle = \langle F(v_i), F(v_j) \rangle$  and furthermore for each  $i$ , by (b)  $|F(v_i)| = |v_i|$ . This shows  $\{F(v_1), \dots, F(v_n)\}$  is orthogonal set, for it to be a basis we need to assume in addition that  $V$  and  $W$  have the same dimension (so  $F$  is surjective).

**(c) $\Rightarrow$ (d).** is trivial as orthonormal bases can always be produced by Gram-Schmidt.

**(d) $\Rightarrow$ (a).** Assume (d), let  $v, v' \in V$ . We can express them in our orthonormal basis as follows

$$v = \sum_{i=1}^n c_i v_i,$$
$$v' = \sum_{j=1}^n d_j v_j.$$

as  $\{v_1, \dots, v_n\}$  is an orthonormal basis,  $\langle v_i, v_j \rangle = 0$  whenever  $i \neq j$  and  $\langle v_i, v_i \rangle = 1$ , so

$$\begin{aligned} \langle v, v' \rangle &= \left\langle \sum_{i=1}^n c_i v_i, \sum_{j=1}^n d_j v_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n c_i d_j \langle v_i, v_j \rangle \\ &= \sum_{i=1}^n c_i d_i \end{aligned}$$

and by linearity

$$F(v) = \sum_{i=1}^n c_i F(v_i),$$
$$F(v') = \sum_{j=1}^n d_j F(v_j).$$

and as  $\{F(v_1), \dots, F(v_n)\}$  is also an orthonormal basis we can perform a similar computation to get

$$\langle F(v), F(v') \rangle = \sum_{i=1}^n c_i d_i = \langle v, v' \rangle$$

which shows (a).

## Question 10

Let  $S$  be a surface of revolution. Prove that the rotations about its axis are isometries of  $S$ .

*Proof.* Suppose  $S$  is a surface formed by rotating around  $z$ -axis, let  $S$  be parametrised by

$$\mathbf{x}(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v))$$

for some  $\varphi, \psi$ . Then the rotation about  $z$ -axis by  $\theta$  can be given by

$$T = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

§2-3 exercise 11 shows that  $T$  restricted to  $S$  is a diffeomorphism onto  $S$ , we just need to show that it is a local isometry at every point.

Now let  $\bar{\mathbf{x}} = T \circ \mathbf{x}$  and we can compute that

$$\bar{\mathbf{x}}(u, v) = (\varphi(v) \cos(u + \theta), \varphi(v) \sin(u + \theta), \psi(v))$$

Note that §2-3 Example 4 computes the coefficients of the first fundamental form of  $S$  as

$$E = \varphi^2, \quad F = 0, \quad G = (\varphi')^2 + (\psi')^2.$$

We can compute  $\bar{E}, \bar{F}, \bar{G}$  by hand as

$$\begin{aligned} \bar{\mathbf{x}}_u &= (-\varphi(v) \sin(u + \theta), \varphi(v) \cos(u + \theta), 0) \\ \bar{\mathbf{x}}_v &= (\varphi'(v) \cos(u + \theta), \varphi'(v) \sin(u + \theta), \psi'(v)) \\ \bar{E} &= \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_u \rangle \\ &= \varphi^2 = E \\ \bar{F} &= \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v \rangle \\ &= 0 = F \\ \bar{G} &= \langle \bar{\mathbf{x}}_v, \bar{\mathbf{x}}_v \rangle \\ &= (\varphi')^2 + (\psi')^2 = G \end{aligned}$$

Applying proposition 1 we have  $\bar{\mathbf{x}} \circ \mathbf{x} = T$  is a local isometry at some arbitrary point, which suffices.

## Question 13

Let  $V$  and  $W$  be (finite-dimensional) vector spaces with inner products  $\langle \cdot, \cdot \rangle$ . Let  $G : V \rightarrow W$  be a linear map. Prove that the following conditions are equivalent:

1. There exists a real constant  $\lambda \neq 0$  such that

$$\langle G(v_1), G(v_2) \rangle = \lambda^2 \langle v_1, v_2 \rangle \quad \text{for all } v_1, v_2 \in V.$$

2. There exists a real constant  $\lambda > 0$  such that

$$|G(v)| = \lambda |v| \quad \text{for all } v \in V.$$

3. There exists an orthonormal basis  $\{v_1, \dots, v_n\}$  of  $V$  such that  $\{G(v_1), \dots, G(v_n)\}$  is an orthogonal basis of  $W$  and, also, the vectors  $G(v_i), i = 1, \dots, n$ , have the same (nonzero) length.

If any of these conditions is satisfied,  $G$  is called a linear conformal map (or a similitude).

*Solution.*  $(1 \Rightarrow 2)$  We have

$$\begin{aligned} |G(v)| &= \sqrt{\langle G(v), G(v) \rangle} \\ &= \sqrt{\lambda^2 \langle v, v \rangle} \\ &= |\lambda| \sqrt{\langle v, v \rangle} \\ &= |\lambda| |v|, \end{aligned}$$

where  $|\lambda| > 0$  is the positive constant desired.

$(2 \Rightarrow 1)$  We have

$$\begin{aligned} \langle G(v_1), G(v_2) \rangle &= \frac{1}{2}(|G(v_1) + G(v_2)|^2 - |G(v_1)|^2 - |G(v_2)|^2), \\ &= \frac{\lambda^2}{2}(|v_1 + v_2|^2 - |v_1|^2 - |v_2|^2) \\ &= \lambda^2 \langle v_1, v_2 \rangle \end{aligned}$$

$(1 \& 2 \Rightarrow 3)$  For  $\{v_i, v_j\}$  orthonormal we have

$$\begin{aligned} |G(v_i)| &= \lambda |v_i| = \lambda |v_j| = |G(v_j)| \\ \langle G(v_i), G(v_j) \rangle &= \lambda^2 \langle v_i, v_j \rangle \end{aligned}$$

This shows that  $\{G(v_1), \dots, G(v_n)\}$  is an orthogonal basis of  $W$  and  $G(v_i), i = 1, \dots, n$ , have the same (nonzero) length.

$(3 \Rightarrow 2)$  For any  $v \in V$ , let  $v = \sum_{i=1}^n a_i v_i$ . Then

$$\begin{aligned} |G(v)|^2 &= \langle G(v), G(v) \rangle = \left\langle \sum_{i=1}^n a_i G(v_i), \sum_{i=1}^n a_i G(v_i) \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \langle G(v_i), G(v_j) \rangle \\ &= \sum_{i=1}^n a_i^2 \langle G(v_i), G(v_i) \rangle + \sum_{i \neq j; i, j \in 1, \dots, n} a_i a_j \langle G(v_i), G(v_j) \rangle \\ &= \sum_{i=1}^n \lambda^2 a_i^2 \langle v_i, v_i \rangle + 0 \\ &= \lambda^2 \langle v, v \rangle = (|\lambda| |v|)^2 \end{aligned}$$

Hence,  $|G(v)| = |\lambda| |v|$ .

## Question 16

Let  $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ , where

$$U = \{(\theta, \varphi) \in \mathbb{R}^2 : 0 < \theta < \pi, 0 < \varphi < 2\pi\},$$
$$\mathbf{x}(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),$$

be a parametrization of the unit sphere  $S^2$ . Let

$$\log \tan \frac{1}{2}\theta = u, \quad \varphi = v$$

and show that a new parametrization of the coordinate neighborhood  $\mathbf{x}(U) = V$  can be given by

$$\mathbf{y}(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u).$$

Prove that in the parametrization  $y$  the coefficients of the first fundamental form are

$$E = G = \operatorname{sech}^2 u, \quad F = 0.$$

Thus,  $\mathbf{y}^{-1} : V \subset S^2 \rightarrow \mathbb{R}^2$  is a conformal map which takes the meridians and parallels of  $S^2$  into straight lines of the plane. This is called *Mercator's projection*.

*Solution.* We have

$$\theta = 2 \arctan e^u, \quad v = \varphi.$$

Hence,

$$\begin{aligned} \mathbf{y}(u, v) &= \mathbf{x}(2 \arctan e^u, v) \\ &= (\sin 2 \arctan e^u \cos v, \sin 2 \arctan e^u \sin v, \cos 2 \arctan e^u) \\ &= (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathbf{y}_u &= \langle -\tanh u \operatorname{sech} u \cos v, -\tanh u \operatorname{sech} u \sin v, 1 - \tanh^2 u \rangle \\ \mathbf{y}_v &= \langle -\operatorname{sech} u \sin v, \operatorname{sech} u \cos v, 0 \rangle \\ E = |\mathbf{y}_u|^2 &= \tanh^2 u \operatorname{sech}^2 u + (1 - \tanh^2 u)^2 \\ &= \tanh^2 u (1 - \tanh^2 u) + (1 - \tanh^2 u)^2 \\ &= 1 - \tanh^2 u = \operatorname{sech}^2 u \\ G = |\mathbf{y}_v|^2 &= \operatorname{sech}^2 u (\sin^2 v + \cos^2 v) = \operatorname{sech}^2 u \\ F = \tanh u \operatorname{sech}^2 u \sin v \cos v - \tanh u \operatorname{sech}^2 u \sin v \cos v &= 0 \end{aligned}$$