

## 1 Section 1.3

### Question 2

A circular disk of radius 1 in the plane  $xy$  rolls without slipping along the  $x$  axis. The figure described by a point of the circumference of the disk is called a cycloid.

- Obtain a parametrized curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$  the trace of which is the cycloid, and determine its singular points.
- Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Solution:

Let us first parameterize the location of the centre of the circle. When it has rotated by  $\theta$ , it will also have moved  $\theta$  to the right. Hence, the position of the centre with respect to amount of rotation is  $(\theta, 1)$ .

Now consider the positional vector from the centre to the marked point on the circumference. At  $\theta = 0$ , this is at  $(0, -1)$ . Then, for general  $\theta$ , this is at  $(-\sin \theta, -\cos \theta)$ . Summing this, it shows that the parameterization  $\alpha(\theta) = (\theta - \sin \theta, 1 - \cos \theta)$ .

We shall use the arc-length formula to yield

$$\begin{aligned} \int_0^{2\pi} |\nabla \alpha(x)| dx &= \int_0^{2\pi} \sqrt{(1 - \cos x)^2 + (\sin x)^2} dx \\ &= \int_0^{2\pi} \sqrt{2 - 2 \cos x} dx \\ &= \left[ -4 \cos \frac{x}{2} \right]_{x=0}^{x=2\pi} \\ &= 8. \end{aligned}$$

### Question 3

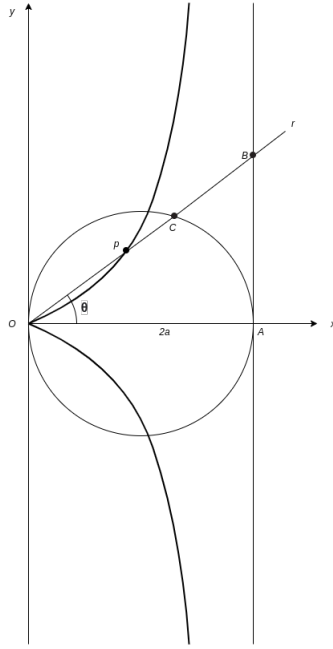
From errata: p. 8, Figure 1-8: The labelling is wrong: the points  $p$  and  $C$  should lie on the same half-line  $r$  through 0 as  $B$ .

Let  $0A = 2a$  be the diameter of a circle  $S^1$  and  $0Y$  and  $AV$  be the tangents to  $S^1$  at 0 and  $A$ , respectively. A half-line  $r$  is drawn from 0 which meets the circle  $S^1$  at  $C$  and the line  $AV$  at  $B$ . On  $0B$  mark off the segment  $0p = CB$  (means both segments are equally long). If we rotate  $r$  about 0, the point  $p$  will describe a curve called *cisoid of Diocles*. By taking  $0A$  as  $x$  axis and  $0Y$  as  $y$  axis, prove that

a. The trace of

$$\alpha(t) = \left( \frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2} \right), \quad t \in \mathbb{R}$$

is the cisoid of Diocles ( $t = \tan \theta$ , see Figure 1-8)



*Proof.*

Let  $t = \tan \theta$  for  $\theta \in (-\pi, \pi)$ , fix  $\theta$  and we show that  $p = \alpha(\tan \theta)$  satisfies the requirements when the half line  $r$  is of angle  $\theta$  from origin. Refer to Figure ??

First we check that  $p$  lies on the half-line, where

$$p = \left( \frac{2a \tan^2 \theta}{1 + \tan^2 \theta}, \frac{2a \tan^3 \theta}{1 + \tan^2 \theta} \right)$$

and we see that  $\tan \arg p = \tan \theta$ .

Finally we check that the segments  $Op = CB$ , we know the point  $C$  can be given by  $(a, 0) + (a \cos 2\theta, a \sin 2\theta)$  and  $B$  is given by  $(2a, 2a \tan \theta)$ . So

$$\begin{aligned} \|CB\| &= \|(a - a \cos 2\theta, 2a \tan \theta - a \sin 2\theta)\| \\ \|CB\|^2 &= (a - a \cos 2\theta)^2 + (2a \tan \theta - a \sin 2\theta)^2 \\ &= 4a^2 \sin^2 \theta \tan^2 \theta \\ \|p\|^2 &= \frac{4a^2(t^4 + t^6)}{(1 + t^2)^2} \\ &= \frac{4a^2 \tan^4 \theta}{1 + \tan^2 \theta} \\ &= 4a^2 \sin^2 \theta \tan^2 \theta \end{aligned}$$

b. The origin  $(0, 0)$  is a singular point of the cissoid.

*Proof.* At point  $(0, 0)$ ,  $t = 0$ , we just need to check that  $\alpha'(0) = 0$ . Now

$$\alpha'(t) = \left( \frac{4at}{(1 + t^2)^2}, \frac{2at^2(t^2 + 3)}{(1 + t^2)^2} \right)$$

which is 0 when  $t = 0$ .

- c. As  $t \rightarrow \infty$ ,  $\alpha(t)$  approaches the line  $x = 2a$ , and  $\alpha'(t) \rightarrow (0, 2a)$  (book typo'd this I think).  
Thus as  $t \rightarrow \infty$ , the curve and its tangent approach the line  $x = 2a$ .

*Proof.* To show that the curve approaches the line  $x = 2a$  we check that

$$\lim_{t \rightarrow \infty} \frac{2at^2}{1+t^2} = 2a.$$

To verify the other claim we compute

$$\lim_{t \rightarrow \infty} \alpha'(t) = (0, 2a).$$

## Question 4

Let  $\alpha : (0, \pi) \rightarrow \mathbb{R}^2$  be given by

$$\alpha(t) = \left( \sin t, \cos t + \log \tan \frac{t}{2} \right)$$

where  $t$  is the angle that the  $y$  axis makes with the vector  $\alpha(t)$ . The trace of  $\alpha$  is called the tractrix. Show that

(a)  $\alpha$  is a differentiable parameterized curve, regular except  $t = \frac{\pi}{2}$ . Consider the derivative,  $\alpha'(t) = (\cos t, -\sin t + (\sin t)^{-1})$ . This is differentiable except when  $\cos t = 0$  and  $-\sin t + (\sin t)^{-1} = 0$ , which occurs when  $t = \frac{\pi}{2}$ .

(b) The length of the segment of the tangent of the tractrix between the point of tangency and the  $y$  axis is constantly equal to 1.

Consider  $\frac{\alpha'(t)_y}{\alpha'(t)_x} = \frac{-\sin t + (\sin t)^{-1}}{\cos t} = \frac{\cos t}{\sin t}$ . This is as  $(\sin x)^2 + (\cos x)^2 = 1$ . Thus, the line of the tangent at  $\alpha(t)$  is  $y - \cos t - \log \tan \frac{t}{2} = \frac{\cos t}{\sin t}(x - \sin t)$ . This has  $y$ -intersect  $(0, \log \tan \frac{t}{2})$ . Then, the distance is  $(\sin t)^2 + (\cos t + \log \tan \frac{t}{2} - \log \tan \frac{t}{2})^2 = 1$ .

## Question 5

Let  $\alpha : (-1, \infty) \rightarrow \mathbb{R}^2$  be given by

$$\alpha(t) = \left( \frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right)$$

Solution:

(a) For  $t = 0$ ,  $\alpha$  is tangent to the  $x$ -axis. Computing the derivative,  $\frac{\partial \alpha(t)}{\partial t} = \left( \frac{a(3-6t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right)$ . Thus,  $\alpha(0) = (0, 0)$  and  $\frac{\partial \alpha(0)}{\partial t} = (3a, 0)$ . Thus, it is tangent to the  $x$ -axis.

(b) As  $t \rightarrow \infty$ ,  $\alpha(t) = \frac{\partial \alpha(t)}{\partial t} = (0, 0)$ . We take limits,  $\lim_{t \rightarrow \infty} \alpha(t) = \left( \lim_{t \rightarrow \infty} \frac{3at}{1+t^3}, \lim_{t \rightarrow \infty} \frac{3at^2}{1+t^3} \right) = (0, 0)$ . Similarly,  $\lim_{t \rightarrow \infty} \frac{\partial \alpha(t)}{\partial t} = \left( \lim_{t \rightarrow \infty} \frac{a(3-6t^3)}{(1+t^3)^2}, \lim_{t \rightarrow \infty} \frac{3at(2-t^3)}{(1+t^3)^2} \right) = (0, 0)$ .

(c) Take the curve with the opposite orientation. Now, as  $t \rightarrow -1$ , the curve and its tangent approach the line  $x + y + a = 0$ .

Let us compute  $\lim_{t \rightarrow -1} \frac{\alpha(t)_y}{\alpha(t)_x} = \lim_{t \rightarrow -1} \frac{1}{t} = -1$ . Now, consider

$$\lim_{t \rightarrow -1} \alpha(t)_y - (-1)\alpha(t)_x = \lim_{t \rightarrow -1} \frac{3a(t+t^2)}{1+t^3} = -a.$$

Also  $\lim_{t \rightarrow -1} \frac{\alpha'(t)_y}{\alpha'(t)_x} = \lim_{t \rightarrow -1} \frac{3t(2-t^3)}{3-6t^3} = \frac{-9}{9} = -1$ , which is a slope of  $x+y+a=0$ .

Thus, the curve and its tangent approach the line  $y = (-1)x + (-a)$ , or the line  $x + y + a = 0$ .

## Question 10

Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve. Let  $[a, b] \subseteq I$  and set  $\alpha(a) = p, \alpha(b) = q$ .

(a) Show that, for any constant vector  $v$ ,  $|v| = 1$ .

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

*Proof.* By Fundamental Theorem of Calculus in 1-Dimension,

$$\int_a^b \alpha'(t) \cdot v dt = (\alpha(b) - \alpha(a)) \cdot v = (q - p) \cdot v.$$

Then, by Hölder's inequality, we have  $|\alpha'(t) \cdot v| \leq |\alpha'(t)| |v| = |\alpha'(t)|$ . Thus,

$$\left| \int_a^b \alpha'(t) \cdot v dt \right| \leq \int_a^b |\alpha'(t) \cdot v| dt \leq \int_a^b |\alpha'(t)| dt.$$

□

(b) Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt.$$

That is, the curve of shortest length from  $\alpha(a)$  to  $\alpha(b)$  is the straight line joining these points.

**Corollary 1.1.** Use  $(q - p) \cdot v = \frac{|q-p|^2}{|q-p|} = |q - p| = |\alpha(b) - \alpha(a)|$  with 10(a).

## 2 Section 1.4

### Question 10

The natural orientation of  $\mathbb{R}^2$  makes it possible to associate a sign to the area  $A$  of a parallelogram generated by two linearly independent vectors  $u, v \in \mathbb{R}^2$ . To do this, let  $\{e_i\}, i = 1, 2$ , be the natural ordered basis of  $\mathbb{R}^2$ , and write  $u = u_1 e_1 + u_2 e_2, v = v_1 e_1 + v_2 e_2$ . Observe the matrix relation

$$\begin{pmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

and conclude that

$$A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.$$

Since the last determinant has the same sign as the basis  $\{u, v\}$ , we can say that  $A$  is positive or negative according to whether the orientation of  $\{u, v\}$  is positive or negative. This is called the *oriented area* in  $\mathbb{R}^2$ .

Solution:

We first observe that

$$\begin{pmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{pmatrix} = \begin{pmatrix} u_1^2 + u_2^2 & u_1 v_1 + u_2 v_2 \\ v_1 u_1 + v_2 u_2 & v_1^2 + v_2^2 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

The identity then holds as,

$$\begin{aligned} A^2 &= (u \wedge v) \cdot (u \wedge v) \\ &= u \cdot (v \wedge (u \wedge v)) \\ &= u \cdot [(v \cdot v)u - (v \cdot u)v] \\ &= (u \cdot u)(v \cdot v) - (v \cdot u)(v \cdot u) \\ &= \begin{vmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2. \end{aligned}$$

## Question 11

a. Show that the volume  $V$  of a parallelepiped generated by three linearly independent vectors  $u, v, w \in R^3$  is given by  $V = |(u \wedge v) \cdot w|$ , and introduce an oriented volume in  $R^3$ .

b. Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}$$

Solution. (a) Let  $n = \frac{u \wedge v}{||u \wedge v||}$  be the normal vector of the plane generated by  $u$  and  $v$ . Then

$$\begin{aligned} V &= (||u|| \times ||v|| \times |\sin(u, v)|) \times ||w|| \times |\cos(n, w)| \\ &= ||u \wedge v|| \times ||w|| \times |\cos(n, w)| \\ &= ||u \wedge v|| \times ||w|| \times |\cos(u \times v, w)| \\ &= |(u \wedge v) \cdot w| \end{aligned}$$

(b) We know that

$$\begin{aligned} |(u \wedge v) \cdot w| &= \left| \begin{pmatrix} u_2 v_3 - v_2 u_3 \\ u_3 v_1 - v_3 u_1 \\ u_1 v_2 - v_1 u_2 \end{pmatrix} \cdot w \right| \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \det(u, v, w). \end{aligned}$$

Hence,

$$\begin{aligned}
 V^2 &= \det(u, v, w)^2 \\
 &= \left| \begin{pmatrix} u^T \\ v^T \\ w^T \end{pmatrix} \begin{pmatrix} u & v & w \end{pmatrix} \right| \\
 &= \begin{vmatrix} u^T u & u^T v & u^T w \\ v^T u & v^T v & v^T w \\ w^T u & w^T v & w^T w \end{vmatrix} \\
 &= \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}
 \end{aligned}$$

## Question 12

Given the vectors  $v \neq 0$  and  $w$ , show that there exists a vector  $u$  such that  $u \wedge v = w$  if and only if  $v$  is perpendicular to  $w$ . Is this vector  $u$  uniquely determined? If not, what is the most general solution?

Solution: ( $\Rightarrow$ ) By the properties of cross product,  $u \wedge v = w$  implies that  $v \cdot w = 0$ .

( $\Leftarrow$ ) If  $v \cdot w = 0$ , we have

$$(v \wedge w) \wedge v = (v \cdot v)w - (v \cdot w)v = \|v\|^2 w.$$

Then  $v \neq 0$  implies that  $w = \frac{v \wedge w}{\|v\|^2} \wedge v$ . Let  $u = \frac{v \wedge w}{\|v\|^2}$  and we have  $u \wedge v = w$ .

Suppose there exist  $u'$  other than  $u$  such that  $u' \times v = w$ . Then

$$\begin{aligned}
 u \wedge v &= u' \wedge v \\
 \Rightarrow (u' - u) \wedge v &= 0 \\
 \Rightarrow u' - u &= kv, \quad k \in R \\
 \Rightarrow u' &= u + kv, \quad k \in R.
 \end{aligned}$$

Therefore, the most general solution of  $u \wedge v = w$  is

$$u = \frac{v \wedge w}{\|v\|^2} + kv, \quad k \in R.$$