1 Section 1.7

Question 6

Let $\alpha(s), s \in [0, l]$ be a closed convex plane curve positively oriented. The curve

$$\beta(s) = \alpha(s) - rn(s)$$

where r is a positive constant and n is a normal vector, is called a *parallel* curve to α (textbook figure is wrong, curve needs to be convex). Show that
(a) Length of $\beta =$ length of $\alpha + 2\pi r$.
Solution. WLOG assume α is parametrised by arc length, since α is a plane curve the torsion is zero, so $n' = -kt$.

$$\begin{aligned}
\beta'(s) &= \alpha'(s) - rn'(s) \\
&= \alpha'(s) - r(-k(s)t(s)) \\
&= (1 + rk(s))t(s)
\end{aligned}$$

now as α is a simple closed curve with positive orientation, the rotation index is 1 and $\int_0^l k(s)\, ds = 2\pi$, so

$$\begin{aligned}
\text{length of } \beta &= \int_0^l |\beta'(s)| \ ds \\
&= \int_0^l 1 + rk(s) \ ds \\
&= l + 2\pi r.
\end{aligned}$$

(b) $A(\beta) = A(\alpha) + rl + \pi r^2$.
Solution. For each $t \in [0, r]$ let

$$\beta_t(s) = \alpha(s) - tn(s)$$

which is a curve that is also parallel to α and lies between α and β. Then we have the expression

$$\begin{aligned}
A(\beta) - A(\alpha) &= \int_0^r \text{length of } \beta_t \ dt \\
&= \int_0^r l + 2\pi t \ dt \\
&= rl + \pi r^2
\end{aligned}$$

as required.
(c) $k_\beta(s) = k_\alpha(s)/(1 + rk_\alpha(s))$ (question typo'd).
Solution. We re-parametrise β by arc length. Let

$$t(s) = \int_0^s |\beta'(u)| \ du$$

and we have $\frac{ds}{dt} = \frac{1}{|\beta'(s)|}$. Define $\hat{\beta}$ such that $\hat{\beta}(t(s)) = \beta(s)$, then

$$\hat{\beta}''(l) = \frac{\alpha''(s)}{|\beta'(s)|}$$

taking lengths we have

$$|\hat{\beta}''(l)| = k_\beta(s) = \frac{k_\alpha(s)}{1 + rk_\alpha(s)}.$$

Question 7

Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be a plane curve defined in the entire real line $\mathbb{R}$. Assume that α does not pass through the origin $O = (0,0)$ and that both limits

$$\lim_{t\to-\infty} |\alpha(t)| = \lim_{t\to\infty} |\alpha(t)| = \infty.$$

(a) Prove that there exists a point t_0 such that $|\alpha(t_0)| \leq |\alpha(t)|$ for all $t \in \mathbb{R}$.
(b) Show, by an example, that the assertion in part a is false if one does not assume that both $\lim_{t\to-\infty} |\alpha(t)| = \infty$ and $\lim_{t\to\infty} |\alpha(t)| = \infty$.
Solution.
(a) Consider $f : \mathbb{R} \to \mathbb{R}$ sending t to $|\alpha(t)|$. Pick an arbitrary $t_1 \in \mathbb{R}$. We can find a, b such that for all $t < a$, $f(t) > f(t_1)$ and for all $t > b$, we have $f(t) > f(t_1)$. Then, we can consider the restriction $f|_{[a,b]}$ to the compact interval $[a, b]$. By the Extreme Value Theorem, we have that this takes a minimum on $[a, b]$, say at t_0. Then this t_0 satisfy the required properties as for $t \in [a, b]$, $|\alpha(t_0)| \leq |\alpha(t)|$, and for $t \notin [a, b]$, $|\alpha(t)| > |\alpha(t_1)| \geq |\alpha(t_0)|$.
(b) Consider $f(t) = (e^t, 0)$. Then, $\inf_t |\alpha(t)| = 0$, but this is non-zero for all t.

Question 8

(a) Let $\alpha(s)$, $s \in [0, l]$, be a plane simple closed curve. Assume that the curvature $k(s)$ satisfies $0 < k(s) \leq c$ where c is a constant (thus, α is less curved than a circle of radius $1/c$). Prove that

$$\text{length of } \alpha \geq \frac{2\pi}{c}.$$

Solution. We have a simple closed curve so we know that

$$\int_0^l k(s)\ ds = 2\pi$$

substituting the bounds $0 < k(s) \leq c$ we have

$$2\pi \leq cl \implies l \geq \frac{2\pi}{c}.$$

(b) In part (a) replace the assumption of being simple by "α has rotation index N." Prove that

$$\text{length of } \alpha \geq \frac{2\pi N}{c}.$$

Solution. Similarly, as rotation index is N,

$$\int_0^l k(s)\ ds = 2\pi N$$

similarly substitute the bounds and we have

$$l \geq \frac{2\pi N}{c}.$$

Question 9

We here define the notion of being "inside" a simple closed curve.

Definition 1.1. *A point p is inside a simple closed curve C if there exists a ray from p that intersects exactly one point in C, say q, such that $q \neq p$. Moreover, the tangent line at q is not parallel to the ray from p to q.*

Theorem 1.2. *A set $K \subseteq \mathbb{R}^2$ is convex if for any two points $p, q \in K$, the segment of straight line $\overline{pq}$ is contained in K.*
Let C be a simple closed curve and K be the points on or inside C. Then if C is convex so is K.

Proof. For each point $c \in C$, let U_c be the closed half-plane, with the tangent line through c being the boundary and $C \subseteq U_c$. By convexity, we have $K \subseteq U = \cap_{c \in C} U_c$. As the intersection of two convex sets is convex, and half-planes are convex, it remains to show that $U \subseteq K$.
Consider a point $p \in \mathbb{R}^2 \setminus K$. As C is compact, we can pick $q \in C$ such that $d(p, q)$ is minimized. Then, the tangent line at q must be perpendicular to the ray from p to q. Suppose then that $p \in U_q$. Then the ray can not intersect another point of C, as C cannot intersect the line segment $\overline{pq}$ as q is the closest point to p on C, and C cannot intersect the ray after q, as all of C lies in U_q. By our definition, this implies that p is inside the curve, which is a contradiction. □

2 Section 2.2

Question 4

Let $f(x, y, z) = z^2$. Prove that 0 is not a regular value off and yet that $f^{-1}(0)$ is a regular surface.

Solution. Note that

$$\mathrm{d}f = (f_x, f_y, f_z) = (0, 0, 2z),$$

which is not surjective only when $z = 0$. Hence, $(0, 0, 0)$ is a critical point and thus $f(0, 0, 0) = 0$ is not a regular value.

However,

$$\begin{aligned} f^{-1}(0) &= \{(x, y, z) \in \mathbb{R}^3 | z^2 = 0\} \\ &= \{(x, y, 0) | x, y \in \mathbb{R}\} \\ &= \mathbb{R}^2 \times \{0\}. \end{aligned}$$

Hence, $f^{-1}(0)$ is homeomorphic to $\mathbb{R}^2$ and therefore is regular.

Question 5

Let $P = \{(x, y, z) \in \mathbb{R}^3 | x = y\}$ (a plane) and let $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$x(u, v) = (u + v, u + v, uv),$$

where $U = \{(u, v) \in \mathbb{R}^2 | u > v\}$. Clearly, $x(U) \subset P$.Is x a parametrization of P?

Solution. Yes, x is a parametrization. Clearly, x is differentiable in U with

$$\mathrm{d}x(u,v) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ u & v \end{pmatrix}.$$

Note that for $(u,v) \in U$, we have $u > v$. Then

$$\begin{vmatrix} 1 & 1 \\ u & v \end{vmatrix} = v - u \neq 0.$$

This implies $\mathrm{d}x(u,v)$ is injective for all $(u,v) \in U$. Now let (a,a,b) be any point in $x(U)$. Then

$$\begin{aligned} &u+v = a, uv = b \\ \Rightarrow &u(a-u) = b \\ \Rightarrow &(u - \frac{a}{2})^2 = \frac{a^2}{4} - b. \end{aligned}$$

Notice that here one must have $\frac{a^2}{4} - b \geq 0$ as the equations $\begin{cases} u+v=a \\ uv=b \end{cases}$ should have real solutions for $(a,a,b) \in x(U)$. Then given $u > v$, we have

$$\begin{aligned} u &= \frac{a}{2} + \sqrt{\frac{a^2}{4} - b} \\ v &= \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}. \end{aligned}$$

These are the unique (u,v) solving $x(u,v) = (a,b)$, which shows x is injective. Hence, by Prop. 4, x^{-1} must be continuous and we can conclude that x is indeed a parametrization.

Question 6

Give another proof of Prop. 1 by applying Prop. 2 to $h(x,y,z) = f(x,y) - z$.

Solution. Since f is differentiable in U, for any point in $U \times \mathbb{R}$, we have

$$\mathrm{d}h = (f_x, f_y, -1),$$

which is always surjective regardless of the value of f_x, f_y. Hence, any $z_0 \in f(U)$ with $f(x_0,y_0) = z_0$, we have

$$h(x_0,y_0,z_0) = f(x_0,y_0) - z_0 = 0,$$

being a regular value. This implies that

$$\begin{aligned} h^{-1}(0) &= \{(x,y,z) \in U \times \mathbb{R} | h(x,y,z) = 0\} \\ &= \{(x,y,z) \in U \times \mathbb{R} | f(x,y) = z\} \\ &= \{(x,y,f(x,y)) | (x,y) \in U\} \end{aligned}$$

is a regular surface.