1 Section 1.7

Question 6

Question 7

Question 8

Question 9

2 Section 2.2

Question 4

Let $f(x, y, z) = z^2$. Prove that 0 is not a regular value off and yet that $f^{-1}(0)$ is a regular surface.

Solution. Note that

$$\mathrm{d}f = (f_x, f_y, f_z) = (0, 0, 2z),$$

which is not surjective only when $z = 0$. Hence, $(0, 0, 0)$ is a critical point and thus $f(0, 0, 0) = 0$ is not a regular value.

However,

$$\begin{aligned} f^{-1}(0) &= \{(x, y, z) \in \mathbb{R}^3 | z^2 = 0\} \\ &= \{(x, y, 0) | x, y \in \mathbb{R}\} \\ &= \mathbb{R}^2 \times \{0\}. \end{aligned}$$

Hence, $f^{-1}(0)$ is homeomorphic to $\mathbb{R}^2$ and therefore is regular.

Question 5

Let $P = \{(x, y, z) \in \mathbb{R}^3 | x = y\}$ (a plane) and let $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$x(u, v) = (u + v, u + v, uv),$$

where $U = \{(u, v) \in \mathbb{R}^2 | u > v\}$. Clearly, $x(U) \subset P$.Is x a parametrization of P?

Solution. Yes, x is a parametrization. Clearly, x is differentiable in U with

$$\mathrm{d}x(u, v) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ u & v \end{pmatrix}.$$

Note that for $(u, v) \in U$, we have $u > v$. Then

$$\begin{vmatrix} 1 & 1 \\ u & v \end{vmatrix} = v - u \neq 0.$$

This implies $\mathrm{d}x(u,v)$ is injective for all $(u,v) \in U$. Now let (a,a,b) be any point in $x(U)$. Then

$$\begin{aligned} &u+v=a, uv=b \\ \Rightarrow &u(a-u)=b \\ \Rightarrow &(u-\frac{a}{2})^2=\frac{a^2}{4}-b. \end{aligned}$$

Notice that here one must have $\frac{a^2}{4}-b \geq 0$ as the equations $\begin{cases} u+v=a \\ uv=b \end{cases}$ should have real solutions for $(a,a,b) \in x(U)$. Then given $u>v$, we have

$$\begin{aligned} u &= \frac{a}{2}+\sqrt{\frac{a^2}{4}-b} \\ v &= \frac{a}{2}-\sqrt{\frac{a^2}{4}-b}. \end{aligned}$$

These are the unique (u,v) solving $x(u,v)=(a,b)$, which shows x is injective. Hence, by Prop. 4, x^{-1} must be continuous and we can conclude that x is indeed a parametrization.

Question 6

Give another proof of Prop. 1 by applying Prop. 2 to $h(x,y,z)=f(x,y)-z$.

Solution. Since f is differentiable in U, for any point in $U \times \mathbb{R}$, we have

$$\mathrm{d}h=(f_x,f_y,-1),$$

which is always surjective regardless of the value of f_x, f_y. Hence, any $z_0 \in f(U)$ with $f(x_0,y_0)=z_0$, we have

$$h(x_0,y_0,z_0)=f(x_0,y_0)-z_0=0,$$

being a regular value. This implies that

$$\begin{aligned} h^{-1}(0) &= \{(x,y,z) \in U \times \mathbb{R} | h(x,y,z)=0\} \\ &= \{(x,y,z) \in U \times \mathbb{R} | f(x,y)=z\} \\ &= \{(x,y,f(x,y)) | (x,y) \in U\} \end{aligned}$$

is a regular surface.