# 1 Section 2.3

## Question 1

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $A : S^2 \to S^2$ be the (antipodal) map $A(x, y, z) = (-x, -y, -z)$. Prove that $A$ is a diffeomorphism.

*Solution.* Let $U = \{(x, y, \sqrt{1 - x^2 - y^2}) : x^2 + y^2 < 1\}$. Then we have

$$\begin{aligned} A(U) &= \{-(x, y, \sqrt{1 - x^2 - y^2}) : x^2 + y^2 < 1\} \\ &= \{(x, y, -\sqrt{1 - x^2 - y^2}) : x^2 + y^2 < 1\}. \end{aligned}$$

Let $\phi(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$ be a parametrization for $U$ and $\psi(u, v) = (u, v, -\sqrt{1 - u^2 - v^2})$ a parametrization for $A(U)$. Then

$$\begin{aligned} \psi^{-1} \circ A \circ \phi(u, v) &= \psi^{-1}(A(\phi(u, v))) \\ &= \psi^{-1}(A((u, v, \sqrt{1 - u^2 - v^2}))) \\ &= \psi^{-1}((-u, -v, -\sqrt{1 - u^2 - v^2})) \\ &= -(u, v). \end{aligned}$$

Therefore, $\psi^{-1} \circ A \circ \phi$ is a differentiable function from $U \to A(U)$. This shows that $A$ is a diffeomorphism.

## Question 3

Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to the plane.
*Solution.* Luckily, we can pick one local chart each to cover the paraboloid $U$ and the plane $V$. For the paraboloid, we can have $(U, f^{-1})$ where $f : (x, y) \to (x, y, x^2 + y^2)$ and for the plane, we can choose our local chart to be $(V, g^{-1})$ where to be $g : (x, y) \to (x, y)$.
Let us now define the diffeomorphism $\pi : U \to V$. Let $\pi(a, b, c) = (a, b)$. Then, $g^{-1} \circ \pi \circ f = \mathrm{id}_{\mathbb{R}^2}$. Thus, $\pi$ is smooth. Then, $\pi^{-1}$ is well-defined, as we the fiber over the point $(a, b)$ must be of the form $(a, b, a^2 + b^2)$. $\pi^{-1}$ is smooth as $f^{-1} \circ \pi^{-1} \circ g = \mathrm{Id}_{\mathbb{R}^2}$. Thus, $\pi$ is a diffeomorphism.

## Question 5

Let $S \subset R^3$ be a regular surface, and let $d : S \to R$ be given by $d(p) = |p - p_0|$, where $p \in S, p \in R^3, p_0 \notin S$; that is, $d$ is the distance from $p$ to a fixed point $p_0$, not in $S$. Prove that $d$ is differentiable.

*Solution.* We select $p \in S$, a open neighborhood $U$ of $p$ and a parametrization $\underset{\sim}{x} : V \to U$, where $V \subset R^2$ and $\underset{\sim}{x}(u_0, v_0) = p$. Let $p_0 = (p_1, p_2, p_3)$. Then

$$d^2 \circ \underset{\sim}{x}(u, v) = (x_1(u, v) - p_1)^2 + (x_2(u, v) - p_2)^2 + (x_3(u, v) - p_3)^2$$

Since $x_i(u, v)$ are all smooth functions, $d^2 \circ \underset{\sim}{x}$ should also be smooth. Since $p_0 \notin S$, we have $d^2 \circ \underset{\sim}{x}(S) \in (0, \infty)$. Note that $f(x) = \sqrt{x}$ is smooth on $(0, \infty)$. Therefore, $\sqrt{d^2 \circ \underset{\sim}{x}} = d \circ \underset{\sim}{x}$ is also smooth. This shows that $d$ is differentiable.

## Question 7

**Theorem 1.1.** *Diffeomorphism is an isomorphism in the category of Differentiable Manifolds.*

*Proof.* We need to show reflexivity, symmetric and transititive. Let $(M, \mathscr{A})$ and $(N, \mathscr{B})$ be two manifolds where $\mathscr{A}, \mathscr{B}$ are maximal atlas. Let $f : M \to N$ be a diffeomorphism.

- (Reflexive) The diffeomorphism has the property that $x \in \mathscr{B}$ if and only if $x \circ f \in \mathscr{A}$, and $f$ is bijective (as sets). Thus, $y \in \mathscr{A}$ implies, $y \circ f^{-1} \circ f \in \mathscr{A}$, implying $y \circ f^{-1} \in \mathscr{B}$. As the implication are reversible, this shows that $f^{-1}$ is also a diffeomorphism.
- (Symmetric) Consider the identity map on $M$. This has the property that $x \in \mathscr{A}$ if and only if $x \circ \mathrm{Id}_M \in A$.
- (Transitivity) Let $(K, \mathscr{C})$ be a manifold where $\mathscr{C}$ is a maximal atlas, and $g$ be a diffeomorphism from $N$ to $K$. Then, $x \in \mathscr{C}$ if and only if $x \circ g \in \mathscr{B}$ if and only if $x \circ g \circ f \in \mathscr{A}$. As the composition of bijective functions is bijective, $(g \circ f)$ is a diffeomorphism.

□

## Question 9

**(a).** Define the notion of differentiable function on a regular curve. What does one need to prove for the definition to make sense?

*Solution.* Let $f : C \to \mathbb{R}$ be defined on a regular curve $S$. Then $f$ is differentiable at $p \in C$ if, for some parametrization $\alpha : I \to C$ with $p \in \alpha(I) \subset C$ where $I$ is an interval, the composition $f \circ \alpha : I \to \mathbb{R}$ is differentiable at $\alpha^{-1}(p)$. $f$ is differentiable in $C$ if it is differentiable at all points $p \in C$.

For the definition to make sense, one has to prove that the definition given does not depend on the choice of parametrization. One needs to prove a statement analogous to proposition 1, that is given two parametrizations of $C$ both containing a neighbourhood of $p$, the change of parameters between those is a diffeomorphism.

# 2 Section 2.4

## Question 15

Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

*Solution.* Let $p_0$ be such a fixed point. Then for any $p = \underset{\sim}{x}(u, v) \in S$, $p - p_0$ is normal to the tangent plane in $p$. That is

$$\underset{\sim}{x}_u \cdot (p - p_0) = \underset{\sim}{x}_v \cdot (p - p_0) = 0.$$

Then for the function $h(u, v) = (\underset{\sim}{x}(u, v) - p_0) \cdot (\underset{\sim}{x}(u, v) - p_0)$, we have

$$h_u = 2\underset{\sim}{x}_u \cdot (\underset{\sim}{x} - p_0) = 0$$
$$h_v = 2\underset{\sim}{x}_v \cdot (\underset{\sim}{x} - p_0) = 0.$$

This shows that $h$ is constant for any connected component of $S$. However, since $S$ is connected, $h$ should be constant on $S$, which implies that $S \subset p \in R^3 : ||p - p_0|| = K$ for some $K > 0$.

## Question 17

Two regular surfaces $S_1$ and $S_2$ intersect transversally if whenever $p \in S_1 \cap S_2$, then $T_p(S_1) \neq T_p(S_2)$. Prove that if $S_1$ intersects $S_2$ transversally, then a connected component of $S_1 \cap S_2$ is a regular curve.

*Proof.* Observe in $\mathbb{R}^3$, if $T_p(S_1) \neq T_p(S_2)$, then $\dim_{\mathbb{R}}(T_p(S_1) \cap T_p(S_2)) = 1$.
It suffices to show that $S_1 \cap S_2$ is a locally a curve. This can be seen as we pick $v \in T_p(S_1) \cap T_p(S_2)$. Then we have a curve $c : [0,1] \to S_1$ with initial conditions $c(0) = p$,$c'(0) = v$. Similarly, we have a curve $d : [0,1] \to S_2$ mutatis mutandis. As they share initial conditions, they are locally equal at $p$, and thus the intersection is locally a curve. □

## Question 19

Let $S \in \mathbb{R}^3$ be a regular surface and $P \in \mathbb{R}^3$ be a plane. If all points of $S$ are on the same side of $P$, prove that $P$ is tangent to $S$ at all points of $P \cap S$.
*Proof.* Let $p \in P$ and $n_P$ be a unit normal of the plane $P$, then consider

$$h(r) = (r - p) \cdot n_P$$

one can compute that $dh_r = n_P$ for all $r \in \mathbb{R}^3$.
Without loss of generality assume that for all $s \in S$, $h(s) \geq 0$, so the surface lies on the "positive" side of the plane. If $\gamma(u)$ is a regular path in $S$, then

$$(h \circ \gamma)'(u) = (\nabla h) \cdot \gamma'(u) = n_P \cdot \gamma'(u).$$

Now if $s \in P \cap S$, then $h(s) = 0$, and if $\gamma$ is a path that goes through $s$ such that $\gamma(u_0) = s$, then we know that $(h \circ \gamma)(u) \geq (h \circ \gamma)(u_0)$ and by extreme value theorem we have

$$(h \circ \gamma)'(u_0) = 0 = \gamma'(u_0) \cdot n_P.$$

Since the path $\gamma$ chosen was arbitrary, it means that the entire tangent plane of $s$ is normal to $n_P$ at $s$, which means that its tangent plane is $P$.

## Question 21

Let $f : S \to R$ be a differentiable function on a connected regular surface $S$. Assume that $df_p = 0$ for all $p \in S$. Prove that $f$ is constant on $S$.
*Proof.* Suppose for a contradiction that $f$ is non-constant, let $f(s_1) = y_1$ and $f(s_2) = y_2$ where $s_1 \neq s_2$ and $y_1 \neq y_2$. Since $S$ is connected, consider a regular path $\gamma$ from $s_1$ to $s_2$, let $\gamma(a) = s_1$ and $\gamma(b) = s_2$. By chain rule

$$(f \circ \gamma)'(x) = df_{\gamma(x)} \cdot \gamma'(x) = 0$$

from single variable calculus we have $f \circ \gamma$ is a constant function, but $(f \circ \gamma)(s_1) = y_1 \neq y_2 = (f \circ \gamma)(s_2)$.

## Question 23

Let $U = \{(x,y,z) \in \mathbb{R}^3 \mid z = -1\}$ be identified with the complex plane $\mathbb{C}$, by sending $(x,y,z) \in U$ to $x + iy$. Let $P(x) = \sum_{k=0}^{n} a_k x^{n-k} \in \mathbb{C}[x]$ be of degree $n$. Denote $\pi$ to be the stereographic projection of $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ from the north pole $(0,0,1)$ onto $U$.

Prove that the map $F : S^2 \to S^2$ given by

$$F(p) = \pi^{-1} \circ P \circ \pi(p) \text{ for } p \notin U \setminus \{(0,0,1)\}$$
$$F((0,0,1)) = (0,0,1)$$

has finitely many critical points.

*Proof.* Recall that $\pi(x,y,z) = \left(\frac{2x}{1-z}, \frac{2y}{1-z}, -1\right) \subseteq U$. Thus $\pi$ is a diffeomorphism between $U \setminus \{(0,0,1)\}$ and $\mathbb{R}^2$. Thus, it suffices to show that $\hat{P} : \mathbb{R}^2 \to \mathbb{R}^2$ sending $(x,y)$ to $(\mathrm{Re}P(x+iy), \mathrm{Im}P(x+iy))$ has finitely many critical points. (We can ignore the north pole, as it doesn't contain infinitely many distinct points). To see this, we require $\frac{\partial \mathrm{Re}P(x+iy)}{\partial x} = \frac{\partial \mathrm{Re}P(x+iy)}{\partial y} = \frac{\partial \mathrm{Im}P(x+iy)}{\partial x} = \frac{\partial \mathrm{Im}P(x+iy)}{\partial y} = 0$. By Complex Analysis, this implies that $\frac{\mathrm{d}P(z)}{dz} = 0$, but that can only occurs finitely many times as $P$ is a polynomial. $\square$

## Question 25

Prove that if two regular curves $C_1$ and $C_2$ of a regular surface $S$ are tangent at a point $p \in S$, and if $\varphi : S \to S$ is a diffeomorphism, then $\varphi(C_1)$ and $\varphi(C_2)$ are regular curves which are tangent at $\varphi(p)$.

*Solution.* Let $U$ be a neighborhood of $p$ with parametrization $\underset{\sim}{x}(u,v)$. Let $\alpha_1(t)$ and $\alpha_2(t)$ be such that $\underset{\sim}{x} \circ \alpha_1$ and $\underset{\sim}{x} \circ \alpha_2$ are regular parametrizations of $C_1$ and $C_2$ with $\underset{\sim}{x} \circ \alpha_1(0) = \underset{\sim}{x} \circ \alpha_2(0) = p$. Then that $C_1$ and $C_2$ are tangent at $p$ implies that

$$\alpha_1'(0) = \alpha_2'(0).$$

Now let $V$ be a neighborhood of $\varphi(p)$ with parametrization $\underset{\sim}{y}(w,z)$ and $\psi = (\psi_1, \psi_2) = \underset{\sim}{y}^{-1} \circ \varphi \circ \underset{\sim}{x}$. Let $\beta_1(t)$ and $\beta_2(t)$ be such that $\underset{\sim}{y} \circ \beta_1$ and $\underset{\sim}{y} \circ \beta_2$ are regular parametrizations of $\varphi(C_1)$ and $\varphi(C_2)$ with $\underset{\sim}{y} \circ \beta_1(0) = \underset{\sim}{y} \circ \beta_2(0) = \varphi(p)$. Than

$$\begin{aligned}
\beta_1'(0) &= \left(\frac{\partial \psi_1}{\partial u}(\alpha_1(0))\alpha_{1u}'(0) + \frac{\partial \psi_1}{\partial v}(\alpha_1(0))\alpha_{1v}'(0), \frac{\partial \psi_2}{\partial u}(\alpha_1(0))\alpha_{1u}'(0) + \frac{\partial \psi_2}{\partial v}(\alpha_1(0))\alpha_{1v}'(0)\right) \\
&= \begin{pmatrix} \frac{\partial \psi_1}{\partial u}(\alpha_1(0)) & \frac{\partial \psi_1}{\partial v}(\alpha_1(0)) \\ \frac{\partial \psi_2}{\partial u}(\alpha_1(0)) & \frac{\partial \psi_2}{\partial v}(\alpha_1(0)) \end{pmatrix} \alpha_1'(0) \\
&= J_\psi(\alpha_1(0))\alpha_1'(0).
\end{aligned}$$

Likewise, we have

$$\beta_2'(0) = J_\psi(\alpha_2(0))\alpha_2'(0).$$

However, since $\alpha_1(0) = \alpha_2(0) = \underset{\sim}{x}^{-1}(p)$ and $\alpha_1'(0) = \alpha_2'(0)$, we have $\beta_1'(0) = \beta_2'(0)$. This implies that $\varphi(C_1)$ and $\varphi(C_2)$ is tangent at $\varphi(p)$.