Section 4.3

Question 2

Show that if $\mathbf{x}$ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then

$$K = -\frac{1}{2\lambda}\Delta(\log\lambda),$$

where $\Delta\varphi$ denotes the Laplacian $\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2}$ of the function φ. Conclude that when $E = G = (u^2 + v^2 + c)^{-2}$ and $F = 0$, then $K = \text{const.} = 4c$.
Proof. As $F = 0$, $\mathbf{x}$ is orthogonal, so we can apply the formula from the previous exercise, which is

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{E_u}{\sqrt{EG}}\right)_u\right\}.$$

Note that $\sqrt{EG} = \lambda$, also

$$\begin{aligned}\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{E_u}{\sqrt{EG}}\right)_u &= \left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u \\ &= \left(\frac{\partial\log\lambda}{\partial v}\right)_v + \left(\frac{\partial\log\lambda}{\partial u}\right)_u \\ &= \Delta(\log\lambda)\end{aligned}$$

which completes the proof.
When $E = G = \lambda = (u^2 + v^2 + c)^{-2}$ and $F = 0$,

$$\begin{aligned}\log\lambda &= -2\log(u^2 + v^2 + c) \\ \frac{\partial\log\lambda}{\partial u} &= -4\frac{u}{u^2 + v^2 + c} \\ \frac{\partial^2\log\lambda}{\partial u^2} &= -4\frac{u^2 + v^2 + c - 2u^2}{(u^2 + v^2 + c)^2} \\ &= -4\lambda(-u^2 + v^2 + c) \\ \frac{\partial^2\log\lambda}{\partial v^2} &= -4\lambda(u^2 - v^2 + c) \\ \Delta(\log\lambda) &= -4\lambda(2c) = -8c\lambda \\ K &= -\frac{1}{2\lambda}\Delta(\log\lambda) = 4c.\end{aligned}$$

Question 3

Verify that the surfaces

$$\begin{aligned}\mathbf{x}(u, v) &= (u\cos v, u\sin v, \log u), \\ \overline{\mathbf{x}}(u, v) &= (u\cos v, u\sin v, v),\end{aligned}$$

have equal Gaussian curvature at the points $\mathbf{x}(u,v)$ and $\overline{\mathbf{x}}(u,v)$ but that the mapping $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry. This shows that the "converse" of the Gauss theorem is not true.
Solution. We compute

$$\begin{aligned}
\mathbf{x}_u &= \left(\cos v, \sin v, \frac{1}{u}\right) \\
\mathbf{x}_v &= (-u\sin v, u\cos v, 0) \\
E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle \\
&= 1 + \frac{1}{u^2} \\
F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\
&= 0 \\
G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle \\
&= u^2 \\
\overline{\mathbf{x}}_u &= (\cos v, \sin v, 0) \\
\overline{\mathbf{x}}_v &= (-u\sin v, u\cos v, 1) \\
\overline{E} &= \langle \overline{\mathbf{x}}_u, \overline{\mathbf{x}}_u \rangle \\
&= 1 \\
\overline{F} &= \langle \overline{\mathbf{x}}_u, \overline{\mathbf{x}}_v \rangle \\
&= 0 \\
\overline{G} &= \langle \overline{\mathbf{x}}_v, \overline{\mathbf{x}}_v \rangle \\
&= u^2 + 1
\end{aligned}$$

Now as $F = \overline{F} = 0$, we can simplify the checking of $K = \overline{K}$ by applying the formula in exercise 1. We compute

$$EG = u^2 + 1 = \overline{EG}$$

and

$$E_v = 0 = \overline{E}_v$$

and

$$G_u = 2u = \overline{G}_u$$

which shows the two surfaces have the same Gaussian curvature.
To see that $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry, consider this curve

$$\alpha(t) = \mathbf{x}(t,\pi) = (-t, 0, \log t), \quad 1 < t < 2$$

note that

$$\overline{\mathbf{x}} \circ \mathbf{x}^{-1} \circ \alpha(t) = \overline{\mathbf{x}}(t,\pi) = (-t, 0, \pi).$$

We can compute and check that $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ fails to preserve the arc length of this curve.

Question 7

Does there exists a surface with $E = 1$, $F = 0$ and $G = \cos(u)^2$, and $e = \cos(u)^2$, $f = 0$ and $g = 1$.
Solution. No. Observe that the Codazzi-Gauss-Mainardi equation states

$$M_v - N_u = \Gamma^1_{22}L + \Gamma^2_{22}M - \Gamma^1_{12}M - \Gamma^2_{12}N.$$

Substituting, this says, $-\sin(u)(\cos(u)^3 + 1) = 0$, which is a contradiction.

Section 4.4

Question 1

(a) Let the curve be given an arc-length parameterization as $c(s) : [0, 1] \to C \subseteq S$. Our goal is to show that c''' is a linear combination of c' and c''. This would imply that the torsion vanishes, and hence a plane curve. As c is a geodesic, c'' is parallel to the unit normal $\tilde{N}$. Moreover as c is a line of curvature, $\frac{\mathrm{d}\tilde{N}}{\mathrm{d}s} = -\kappa c$. Combining this gives our result.

(b) Conversely, we have that c''' is a linear combination, and we wish to show that $\frac{\mathrm{d}c''}{\mathrm{d}s} = -\kappa c$. This can be seen by reversing the argument above.

(c) Pick our surface as a plane, then all such lines of curvatures are necessarily planar, but not necessarily a geodesic (straight line).

Question 10

Show that the geodesic curvature of an oriented curve $C \subset S$ at a point $p \in C$ is equal to the curvature of the plane curve obtained by projecting C onto the tangent plane $T_p(S)$ along the normal to the surface at p.

Solution. We parametrize C by $\alpha(t)$ with $\alpha(0) = p$ and $|\alpha'(t)| = 1$. Then the projection of

$$\beta(t) = \alpha(t) - \alpha(0) - \langle \alpha(t) - \alpha(0), N \rangle N,$$

where N is the unit normal of $T_p(S)$. Then we have

$$\beta'(t) = \alpha'(t) - \langle \alpha'(t), N \rangle N, \beta''(t) = \alpha''(t) - \langle \alpha''(t), N \rangle N.$$

This implies

$$\begin{aligned}
&\beta'(0) = \alpha'(0) - \langle \alpha'(0), N \rangle N = \alpha'(0), \\
&\beta''(t) = \alpha''(0) - \langle \alpha''(0), N \rangle N; \\
&\Rightarrow \beta'(0) \cdot \beta''(0) = 0, \qquad |\beta'(0)| = 1; \\
&\Rightarrow \kappa_\beta(p) = \frac{|\beta'(0) \times \beta''(0)|}{|\beta'(0)|^3} = |\beta''(0)| \\
&= \sqrt{|a''(0)|^2 + \langle a''(0), N \rangle^2 - 2\langle a''(0), N \rangle^2} \\
&= \sqrt{\kappa_\alpha(p)^2 - K^2}.
\end{aligned}$$

Question 20

Let T be a torus of revolution which we shall assume to be parametrized by

$$X(u, v) = ((r \cos u + a) \cos v, (r \cos u + a) \sin v, r \sin u).$$

Prove that

a. If a geodesic is tangent to the parallel $u = \pi/2$, then it is entirely contained in the region of T given by

$$-\frac{\pi}{2} < u < \frac{\pi}{2}$$

b. A geodesic that intersects the parallel $u = 0$ under an angle $\theta (0 < \theta < \pi/2)$ also intersects the parallel $u = \pi$ if

$$\cos\theta < \frac{a-r}{a+r}.$$

Solution. a. According to Clairaut's relation, given a geodesic $\alpha(t) \subset T$, we have

$$\frac{d}{dt} R(t) \cos\theta(t) = 0,$$

where $R(t)$ is the distance from $\alpha(t)$ to the z-axis and $\theta(t)$ is the angel made by $\alpha'(t)$ and $\S_u$. Suppose $\alpha(0)$ is the tangential point on $u = \pi/2$. Then

$$R(t)\theta(t) = R(0)\cos\theta(0) = a.$$

In particular, we have $R(t) \geq a$. Hence, $\alpha(t)$ has to lie on the outer side of T, which is the region with $-\pi/2 < u < \pi/2$.

b. Suppose $\alpha(0)$ is the intersection on $u = 0$. We have

$$R(t)\theta(t) = R(0)\cos\theta(0) < (a+r) \cdot \frac{a-r}{a+r} = a - r.$$

Suppose $\alpha(t)$ does not intersect $u = \pi$. Let $k = \inf\{u_0 \in (-\pi, \pi) : \{u = u_0\} \text{ intersects } \alpha(t)\}$. By the continuity, $\alpha(t)$ will touch $\{u = k\}$ and therefore tangent to the parallel. Moreover, we can also see that $k \neq \pi$. Let the $\alpha(t_0)$ be a tangential point. Then we have

$$R(t_0)\cos\theta(t_0) = R(t_0) > a - r.$$

This contradicts the fact that $R(t)\cos\theta(t) < a - r$. Hence, $\alpha(t)$ has to intersect $\{u = \pi\}$.