Universal Enveloping Algebras

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1 Background and Motivation

$$\left\{\begin{array}{c}\text{Representations}\\ \text{of}\\ \text{a Group } G/\mathbb{C}\end{array}\right\} \sim \left\{\begin{array}{c}\text{Modules}\\ \text{of algebra}\\ \mathbb{C}G\end{array}\right\}$$

Naturally,

$$\left\{\begin{array}{c}\text{Representations}\\ \text{of a}\\ \text{Lie algebre } \mathfrak{g}/\mathbb{C}\end{array}\right\} \sim \left\{\begin{array}{c}\text{Modules}\\ \text{of algebra}\\ ??\end{array}\right\}$$

! For today all representations are taken over a field alg. closed and characteristic 0 denoted $\mathbb{C}$

Why Algebras?

Key problem of representation theory

Given an algebra A find

$$\mathrm{Irr}\, A \overset{\text{Def}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Equivalence classes of} \\ \text{irreducible representations} \\ \text{of } A \end{array} \right\}$$

But we have the A-(left) module A. For any $S \in \mathrm{Irr}\, A$, we have a homomorphism

$$f : A \longrightarrow S$$
$$\alpha \longmapsto \alpha s$$

S irreducible so f is epi

so $S \cong A/I$ where I is a maximal ideal of A

☆ Apply techniques of algebra

2 Definition.

Recall we have a functor

$$\bullet_{Lie} : \mathbb{C}\text{-Alg} \mapsto \mathbb{C}\text{-Lie}$$

Take V a $\mathbb{C}$-Alg,

$$[u, v] = uv - vu \quad , \quad u, v \in V$$

multiplication as an algebra

Restated goal

Find a functor $\mathcal{U} : \mathbb{C}\text{-Lie} \mapsto \mathbb{C}\text{-Alg}$

left adjoint to $\bullet_{Lie}$

Explicitly for any Lie algebra $\mathfrak{g}$

get $\mathcal{U}\mathfrak{g}$ satisfying the universal

property: for any algebra A

$$\mathrm{Hom}(\mathcal{U}\mathfrak{g}, A) \cong \mathrm{Hom}(\mathfrak{g}, A_{Lie})$$

as algebras $\quad \forall \mathfrak{g} \quad$ as Lie algebras functorial in both $\mathfrak{g}$ and A

$\mathcal{U}_{\mathrm{Lie}} : \mathfrak{g} \longmapsto \mathfrak{h}$

> Aside A

A representation of a Lie algebra $\mathfrak{g}$ is a Lie homomorphism

$$\mathfrak{g} \longmapsto \mathfrak{gl}(V)$$

But $\mathfrak{gl}(V) = \mathrm{End}(V)_{\mathrm{Lie}}$ so

$$\mathrm{Hom}(\mathfrak{g}, \mathfrak{gl}(V)) \cong \mathrm{Hom}(\mathcal{U}\mathfrak{g}, \mathrm{End}\, V)$$

$\mathcal{U}\mathfrak{g} \circlearrowright V.$

3 Construction

We shall construct $\mathcal{U}\mathfrak{g}$ with a canonical[§1] Lie homomorphism

$$\iota_{\mathfrak{g}} : \mathfrak{g} \longmapsto (\mathcal{U}\mathfrak{g})_{\mathrm{Lie}}$$

~~$\in \mathrm{Hom}(\mathfrak{g}, (\mathcal{U}\mathfrak{g})_{\mathrm{Lie}})$~~

For any $f : \mathfrak{g} \xmapsto{\mathrm{Lie}} A_{\mathrm{Lie}}$

where A is an algebra

Exists unique $\hat{f} : U\mathfrak{g} \xrightarrow{Alg} A$
such that

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, "f"] \arrow[d, "\iota_{\mathfrak{g}}"'] & A \\
U\mathfrak{g} \arrow[ur, dashed, "\exists!\, \hat{f}"'] &
\end{tikzcd}
$$

commutes.

§1
($\iota_{\mathfrak{g}}$ must exist by our universal property as the unit of adjunction)

① Let $T\mathfrak{g}$ be the tensor algebra

$$\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$$

Example

$$\mathfrak{g} = sl_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$$

$$\mathfrak{g}^{\otimes 0} = \mathrm{span}(1) = \mathbb{C}$$

$$\mathfrak{g}^{\otimes 1} = \mathrm{span}(sl_2)$$

$$= \mathrm{span}(e, f, h) = sl_2.$$

$(e+f) \otimes (h) = e \otimes h + f \otimes h$

$$\mathfrak{g}^{\otimes 2} = sl_2 \otimes sl_2 \quad \mathrm{span}\{\alpha \otimes \beta \mid \alpha, \beta \in sl_2\}.$$

$$= \mathrm{span}\begin{pmatrix} e\otimes e & e\otimes f & e\otimes h \\ f\otimes e & f\otimes f & f\otimes h \\ h\otimes e & h\otimes f & h\otimes h \end{pmatrix}$$

ee fe he fh

$T\mathfrak{g}$ = 'non-commutative polynomials of e, f & h'

② $\boxed{\text{Define}}$ $U\mathfrak{g} = T\mathfrak{g}/L$ where

$$L = (x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g})$$

Examples

$[x,y] = 0.$

I) If $\mathfrak{g}$ is commutative then

$L = (x \otimes y - y \otimes x)$

$$U\mathfrak{g} = \mathrm{Sym}\,\mathfrak{g}.$$

III) Let $\mathfrak{g} = sl_2 = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$

$[e,f]=h, \quad [h,e]=2e, \quad [h,f]=-2f$

$-(e+f)2h \quad -2h(e+f)$

$-[e+f,h]$

Claim

$$U\mathfrak{g} = T\mathfrak{g} \Big/ \left\langle \begin{array}{l} ef-fe=h \\ he-eh=2e \\ hf-fh=-2f \end{array} \right\rangle$$

$2eh$

$[e,c]=kc$

associative algebra generated by e,f,h

'missing' relations are generated by these

Recall the casimir element c

$$c \in Z(U\mathfrak{g}) = \mathbb{C}[c]$$

III) Recall the Heisenberg Lie Algebra $\mathcal{H}$ with basis

$$e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This has Lie bracket

$$[f,e]=h, \quad [h,e]=[h,f]=0$$

So

$$UH = TH \Big/ \left\langle \begin{array}{l} fe-ef=h \\ he-eh=0 \\ hf-fh=0 \end{array} \right\rangle$$

Consider the algebra of operators on $\Bbbk[x]$ generated by x and $\frac{d}{dx}$. This is called the

T_p T' Σ $\mathbb{C}$ $p(x)$

Weyl Algebra W_1. But

$$\left(\frac{d}{dx}x - x\frac{d}{dx}\right)g = \frac{d}{dx}(xg) - xg' = g$$

$xg'+g$

e f

shows $W_1 \cong UH/\langle h-1\rangle$

This turns out to be an infinite dimensional irreducible representation of H

! Cannot grade by degree

$$\underbrace{x\otimes y - y\otimes x}_{\text{degree } 2} - \underbrace{[x,y]}_{\text{degree } 1}$$

$(\mathfrak{g}\ \mathcal{U}\mathfrak{g} = (\mathcal{U}\mathfrak{g})_{\mathfrak{s}}$

$\mathcal{U}\mathfrak{g}$ is non-zero. Why? $L \subseteq T^{+}\mathfrak{g}$

$$= \bigoplus_{n>0} \mathfrak{g}^{\otimes n}$$

What is $\iota_{\mathfrak{g}}$?

$\boxed{\text{Define}}$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota_{\mathfrak{g}}"] \arrow[d, dashed] & \mathfrak{g}^{\otimes 1} \arrow[d, hook] \\ \mathcal{U}\mathfrak{g} = T\mathfrak{g}/L & T\mathfrak{g} \arrow[l, two heads] \end{tikzcd}$$

$$\iota_{\mathfrak{g}}(x) = x + L, \quad x \in \mathfrak{g}$$

Check $\iota_{\mathfrak{g}}$ is a Lie homomorphism

$x, y \in \mathfrak{g}$

$$[\iota x, \iota y] = \iota x \iota y - \iota y \iota x$$
$$= \iota[x,y]$$

$x\otimes y - y\otimes x - [x,y]$

"L"

Bijection

$$\underset{\text{Alg}}{\mathrm{Hom}(\mathcal{U}\mathfrak{g}, A)} = \left\{ f \in \underset{\text{Alg}}{\mathrm{Hom}(T\mathfrak{g}, A)} \;\middle|\; \begin{array}{c} (fx)(fy) - (fy)(fx) \\ = f[x,y] \\ \forall\, x,y \in \mathfrak{g} \end{array} \right\}$$

$(\star)$ $[fx, fy]$

$$\underset{\text{Lie}}{\mathrm{Hom}(\mathfrak{g}, A_{Lie})} = \left\{ f \in \mathrm{Hom}(\mathfrak{g}, A_{vect}) \;\middle|\; \begin{array}{c} (fx)(fy) - (fy)(fx) \\ = f[x,y] \\ \forall\, x,y \in \mathfrak{g} \end{array} \right\}$$

(!) Linear map viewing A as a Vector Space

$(\star)$ is the universal property of tensor algebras.

(Tensor functor is the left adjoint of the forgetful functor)

Functoriality? Is $\mathcal{U}$ a functor

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "f"] \arrow[d, "\iota_{\mathfrak{g}}"'] \arrow[dr, "s"] & \mathfrak{h} \arrow[d, "\iota_{\mathfrak{h}}"] \\
\mathcal{U}\mathfrak{g} \arrow[r, dashed, "\exists!\,\hat{s}"'] & \mathcal{U}\mathfrak{h}
\end{tikzcd}$$

$s = \iota_{\mathfrak{h}} \circ f$

$: \mathfrak{g} \longrightarrow (\mathcal{U}\mathfrak{h})_{Lie}$

Left check $\mathcal{U}$ respects identity
and composition

AND

$$\mathrm{Hom}(\mathcal{U}\mathfrak{g}, A) \cong \mathrm{Hom}(\mathfrak{g}, A_{Lie})$$

as algebras   as Lie algebras

functorial in both $\mathfrak{g}$ and A

Omitted.

3 Poincaré-Birkhoff-Witt

Qn: Is $\iota_{\mathfrak{g}}$ injective?

Qn: What is a basis of $\mathcal{U}\mathfrak{g}$?

Fix a basis $X = (x_i)_{i \in I}$ of $\mathfrak{g}$

Then $T\mathfrak{g} = \mathbb{C}\langle X \rangle$ ← free algebra.

and

$$U\mathfrak{g} \cong \mathbb{C}\langle X \rangle \Big/ \left\langle \begin{array}{c} x_i x_j - x_j x_i - [x_i, x_j] \\ \text{for } i, j \in I \end{array} \right\rangle$$

Aside B

Old problem on mathoverflow.

Square matrices satisfying certain relations must have dimension divisible by 3

Asked 7 years, 8 months ago Active 7 years, 7 months ago Viewed 1k times

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I saw this tucked away in a MathOverflow comment and am asking this question to preserve (and advertise?) it. It's a nice problem!

Problem: Suppose A and B are real $n \times n$ matrices with $A^2 + B^2 = AB$. If $AB - BA$ is invertible, prove n is a multiple of 3.

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linear-algebra matrices contest-math

Hot Meta Posts

8 Should [tag:hypergeomet… generalized to admit high…

7 Why do very few question… stochastic PDE actually h… PDE…

15 Problem with Question Sr…

How to approach such problems?

Step 1

Consider the algebra

$$\Gamma = \mathbb{C}\langle a, b\rangle \Big/ \langle a^2 + b^2 = ab \rangle$$

free algebra

ideal generated

Then we have a representation of Γ sending $\underline{a} \longrightarrow \underline{A}$

$\underline{b} \longrightarrow \underline{B}$

Step 2

What is the basis?

Give a lexicographical order $a < b$, then

$abbaba$

try and reduce words $a(ab-a)aba$

using $b^2 = ab - a^2$

Formalizing, we ask if the set of basis elements without b^2 is linearly independent

Technique (Bergman's Diamond Lemma)

Consider a reduction system of an algebra with presentation

$(b^2 \mapsto ab - a^2)$

$$A = \mathbb{C}\langle X\rangle / \langle w_k - f_k \mid k \in S\rangle$$

where

monomials of X

from $\mathbb{C}\langle X\rangle$

The reduction system is the set of (w_k, f_k).

Examples

$$W_1 = \mathbb{C}\langle x, y\rangle \big/ \langle yx - xy - 1\rangle$$

has reduction system

$$\{(yx,\ xy+1)\}$$

$$U\mathfrak{sl}_2 = \mathbb{C}\langle e, f, h\rangle \Big/ \left\langle \begin{array}{l} ef - fe = h \\ he - eh = 2e \\ hf - fh = -2f \end{array} \right\rangle$$

has reduction system

$$\{(ef,\ fe+h),\ (eh,\ he-2e),\ (hf,\ fh-2f)\}$$

Good Reduction System

- ☆ af_ib must be better aw_ib
- ☆ after finite steps, must have no possible reductions
- ☆ unambiguous, does not depend on what order we choose $f_i \to w_i$

Formally, let W be the set of monomials and $\hat{W}$ be the set of irreducible monomials. Define a partial order $\leq$ on W. For $t \in A$ we denote $t \leq s$ if $t_r \leq S$ for all the monomials t_r of t.

☆ $a f_i b$ must be better $a w_i b$

For all (w_k, f_k) of S,

$$f_k > w_k$$

For all $\alpha < \beta$, $a\alpha b < a\beta b$ for all $a, b \in W$

* after finite steps, must have no possible reductions

$\leq$ satisfies the descending chain condition (i.e. $w_1 \geq w_2 \geq \dots$ stabilizes after finitely many steps)

* unambiguous, does not depend on what order we choose $f_i \to w_i$

Two possible issues

1) Overlap ambiguities

$$ab \longmapsto \alpha$$

$$bc \longmapsto \beta$$

substring

$$\begin{tikzcd}
 & abc \arrow[dl] \arrow[dr] & \\
\alpha c \arrow[dr, dashed] & & a\beta \arrow[dl, dashed] \\
 & T &
\end{tikzcd}$$

2) Inclusion ambiguities

$$abc \longmapsto \alpha$$

$$b \longmapsto \beta$$

$$\begin{tikzcd}
 & abc \arrow[dl] \arrow[dr, mapsto] & \\
\alpha \arrow[dr, dashed] & & a\beta c \arrow[dl, dashed] \\
 & T &
\end{tikzcd}$$

Theorem (Diamond Lemma) For any Good Reduction System, $\hat{W}$ (set of irreducible monomials) form a basis.

Examples

$$W_1 = \mathbb{C}\langle x, y\rangle / \langle yx - xy - 1\rangle$$

has reduction system $\{(yx, xy+1)\}$

Using $x < y$ and degree lexicographical order shows

$$\{x^\alpha y^\beta\}$$ as a basis

Similarly for

$$U\mathfrak{sl}_2 = \mathbb{C}\langle e,f,h\rangle \Big/ \left\langle \begin{matrix} ef-fe=h \\ he-eh=2e \\ hf-fh=-2f \end{matrix} \right\rangle$$

has reduction system

$$\{(\underline{ef}, fe+h), (\underline{eh}, he-2e), (hf, fh-2f)\}$$

$f < h < e$ and degree lexicographic

$$\hat{W} = \{f^\alpha h^\beta e^\gamma\}$$

But overlap ambiguity

$$ehf$$

$$e(fh-2f) = efh - 2ef \qquad (he-eh)f = hef - 2ef$$

$$(fe+h)h - 2(fe+h) \qquad h(fe+h) - 2(fe+h)$$

$$= feh + h^2 - 2fe - 2h \qquad = hfe + h^2 - 2fe - 2h$$

$$f(he-2e) = fhe - 2fe = (fh-2f)e$$

$$+h^2 - 2fe - 2h = +h^2 - 2fe - 2h = +h^2 - 2fe - 2h$$

Theorem (Poincaré-Birkhoff-Witt)

Let $(X_i)_{i\in I}$ be a basis of $\mathfrak{g}$ and $\leq$ be a total order on I, then $U\mathfrak{g}$ has basis $\underline{X_{\alpha_1} X_{\alpha_2} \cdots X_{\alpha_k}}$ images where $\alpha_1 \leq \alpha_2 \cdots \leq \alpha_k$.

Caveat with Proof.

Use $\leq$ the misordering index.

If degree is the same tie break by score $x_1 x_2 \cdots x_k$ has

score $|\{(i,j) \mid \substack{x_i > x_j \\ i<j}\}|$

Corollary

$\iota_{\mathfrak{g}}$ is injective

Corollary

Let $\mathfrak{h} \hookrightarrow \mathfrak{g}$ be an injective Lie homomorphism.

Then $U\mathfrak{h} \hookrightarrow U\mathfrak{g}$ is injective.

Qn: Is $U\mathfrak{g}$ noetherian? If $\mathfrak{g}$ f.d. Yes.

Qn: How does the structure of $\mathfrak{g}$ affect $U\mathfrak{g}$?

4 Structure of $\mathcal{U}\mathfrak{g}$

Theorem

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebra

1) $\mathcal{U}(\mathfrak{g}_1 \times \mathfrak{g}_2) \cong \mathcal{U}\mathfrak{g}_1 \otimes \mathcal{U}\mathfrak{g}_2$

2) $\mathcal{U}(\mathfrak{g}_1^{op}) \cong (\mathcal{U}\mathfrak{g}_1)^{op}$

Proof

1)

$$\begin{tikzcd}
\mathfrak{g}_1 \arrow[r, "x \mapsto (x,0)"] \arrow[d, hook] & \mathfrak{g}_1 \times \mathfrak{g}_2 \arrow[d, hook] & \mathfrak{g}_2 \arrow[l, "(0,y) \mapsfrom y"'] \arrow[d, hook] \\
\mathcal{U}\mathfrak{g}_1 \arrow[r] \arrow[dr] & \mathcal{U}(\mathfrak{g}_1 \times \mathfrak{g}_2) & \mathcal{U}\mathfrak{g}_2 \arrow[l] \arrow[dl] \\
& \mathcal{U}\mathfrak{g}_1 \otimes \mathcal{U}\mathfrak{g}_2 \arrow[u, dashed, "\star"] &
\end{tikzcd}$$

($\star$) exists as $[(x,0),(0,y)] = 0$

Surjective follows from PBW

2) Use $(\mathcal{U}\mathfrak{g})^{op} \cong \mathcal{U}\mathfrak{g}$;

$$\begin{tikzcd} \mathfrak{g}^{op} \arrow[r, mapsto, "f"'] \arrow[d, hook] & (\mathcal{U}\mathfrak{g})^{op}_{Lie} \\ \mathcal{U}(\mathfrak{g}^{op}) \arrow[ur, dashed, mapsto] & \end{tikzcd}$$

f is Lie Hom as

$$[x^{op}, y^{op}] = [y, x]^{op}$$

Counit

$\mathfrak{g} \mapsto 0$ is a Lie homomorphism

$$\varepsilon : \mathcal{U}\mathfrak{g} \longmapsto \mathcal{U}0 = \mathbb{C}$$

$$x \longmapsto 0 \text{ if } x \in \mathfrak{g}$$

We call ε the counit.

Comultiplication

$\mathfrak{g} \longmapsto \mathfrak{g} \times \mathfrak{g}$ is a Lie homomorphism
$x \longmapsto (x, x)$

$$\Delta : U\mathfrak{g} \longmapsto U(\mathfrak{g} \times \mathfrak{g}) = U\mathfrak{g} \otimes U\mathfrak{g}$$

$$\forall x \in \mathfrak{g} : \quad x \longmapsto x \otimes 1 + 1 \otimes x$$

We call Δ the <u>comultiplication</u>.

Moreover one can check that

$\mathfrak{g} \longmapsto \mathfrak{g}^{op}$ is a Lie Homomorphism
$x \longmapsto -x^{op}$

So
$$S : U\mathfrak{g} \xrightarrow{\sim} (U\mathfrak{g})^{op}$$

$$\forall x \in \mathfrak{g} : \quad x \longmapsto -x^{op}$$

S is the <u>antipode</u> and $S^2 = Id$.

This gives Ug the structure of a coalgebra. So Ug is a

<u>Hopf</u> <u>Algebra</u>.

$$\begin{tikzcd}
Ug \arrow[r, mapsto, "\Delta"] \arrow[d, "\Delta"'] & Ug \otimes Ug \arrow[d, "\Delta \otimes Id"] \\
Ug \otimes Ug \arrow[r, mapsto, "Id \otimes \Delta"'] & Ug \otimes Ug \otimes Ug
\end{tikzcd}$$

Coassociativity

Coidentity

$$\begin{tikzcd}
 & Ug \arrow[dl, "\sim"'] \arrow[d, "\Delta"] \arrow[dr, "\sim"] & \\
\mathbb{C} \otimes Ug & & Ug \otimes \mathbb{C} \\
 & Ug \otimes Ug \arrow[ul, "\varepsilon \otimes Id"] \arrow[ur, "Id \otimes \varepsilon"'] &
\end{tikzcd}$$

Algebra-Coalgebra Compatibility.

$$\begin{tikzcd}
 & Ug \otimes Ug \arrow[r, "S \otimes Id"] & Ug \otimes Ug \arrow[dr, "mult."] & \\
Ug \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & \mathbb{C} \arrow[rr] & & Ug \\
 & Ug \otimes Ug \arrow[r, "Id \otimes S"'] & Ug \otimes Ug \arrow[ur, "mult."'] &
\end{tikzcd}$$

Thank You.

Any Questions?