Combinatorics Feb23

What is a group? One operation $*$ or $+$
Identity and inverses
Associative: $(ab)c = a(bc)$

$\mathbb{Z}_2$:

+	0	1
0	0	1
1	1	0

mod 2

$\approx$

+	even	odd
even	even	odd
odd	odd	even

$\approx$

*	1	-1
1	1	-1
-1	-1	1

$\approx$

*	1	2
1	1	2
2	2	1

mod 3

$\mathbb{Z}_3$:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

mod 3

$\mathbb{Z}_4$:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

mod 4

$\approx$

*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

mod 5

+ $*$
$0 \leftrightarrow 1$
$1 \leftrightarrow 2$
$2 \leftrightarrow 4$
$3 \leftrightarrow 3$

$\mathbb{Z}_2 \times \mathbb{Z}_2$:

+	0,0	0,1	1,0	1,1
0,0	0,0	0,1	1,0	1,1
0,1	0,1	0,0	1,1	1,0
1,0	1,0	1,1	0,0	0,1
1,1	1,1	1,0	0,1	0,0

mod 2,2

$\mathbb{Z}_5$:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\mathbb{Z}_6$:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

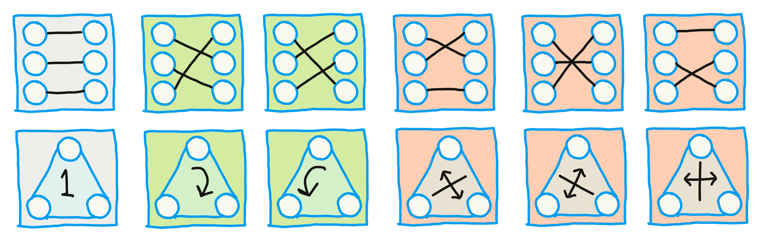
$\mathbb{Z}_2 \times \mathbb{Z}_3$:

+	0,0	0,1	0,2	1,0	1,1	1,2
0,0	0,0	0,1	0,2	1,0	1,1	1,2
0,1	0,1	0,2	0,0	1,1	1,2	1,0
0,2	0,2	0,0	0,1	1,2	1,0	1,1
1,0	1,0	1,1	1,2	0,0	0,1	0,2
1,1	1,1	1,2	1,0	0,1	0,2	0,0
1,2	1,2	1,0	1,1	0,2	0,0	0,1

mod 2,3

Inverses $\Leftrightarrow$ Each row is a permutation of the first row
Each col is a permutation of the first col

The symmetric group S_3: Permutations of $\{1,2,3\}$
Symmetries of a triangle



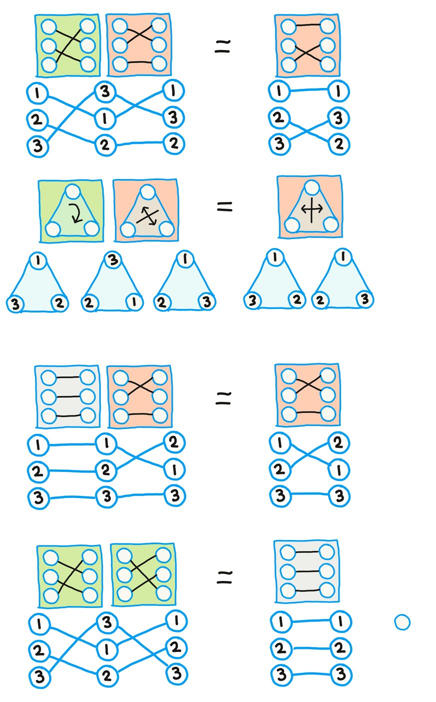
How to multiply?

Pull tight

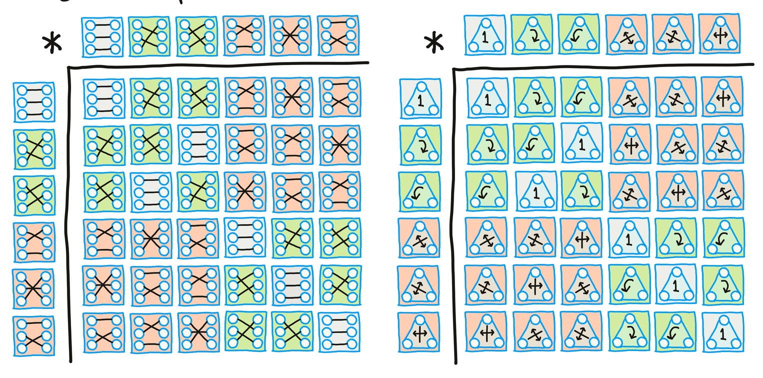
Watch test triangle

Identity

Inverses



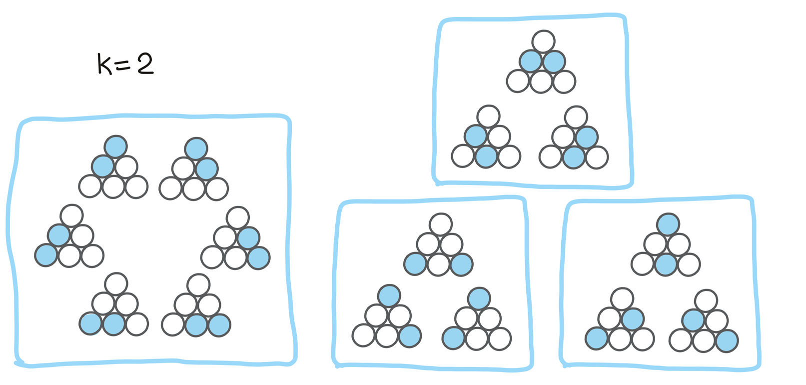
S_3 multiplication tables



Not commutative

Counting problem: Mark k cells in a triangular grid
How many patterns, up to S_3 symmetry?

k=2



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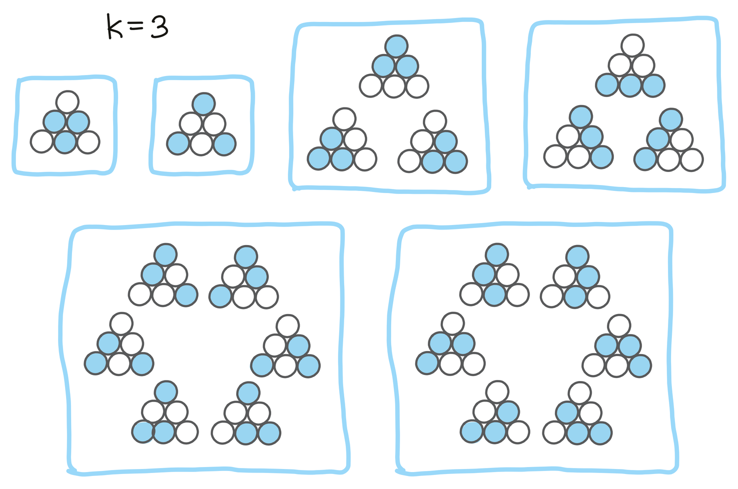
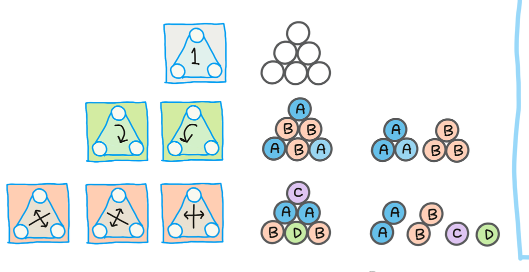
k=3



$$|P| = \frac{1}{|G|}\sum_{g\in G}|X_g|$$

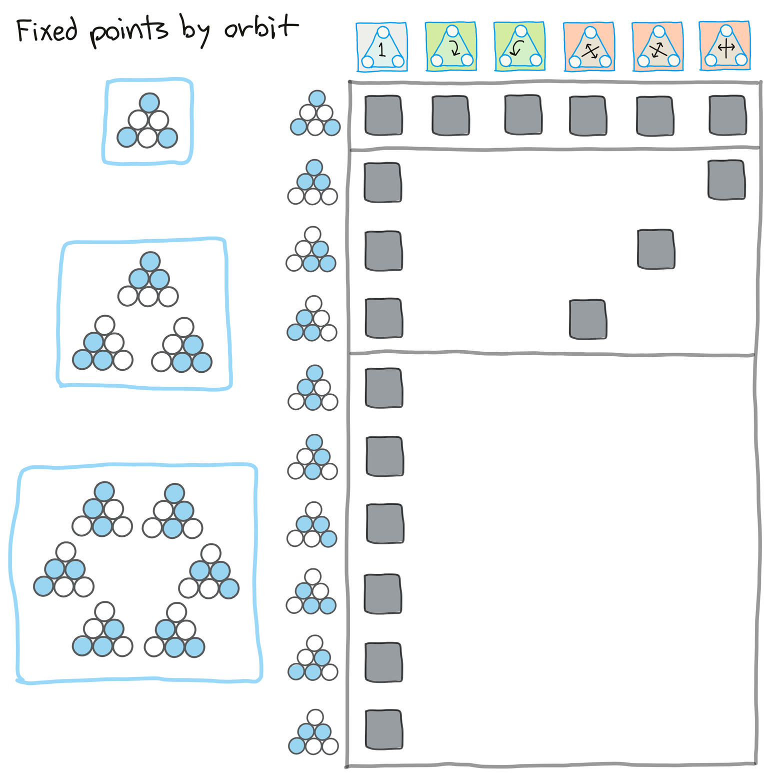


k=2	k=3
$\binom{6}{2}$	$\binom{6}{3}$
0	$\binom{2}{1}$
$\binom{2}{1}+\binom{2}{2}$	$\binom{2}{1}\binom{2}{1}$

k=2: $\frac{1}{6}\left[\binom{6}{2}+3\left(\binom{2}{1}+\binom{2}{2}\right)\right] = \frac{1}{6}(15+3\cdot 3) = 4$ ☑

k=3: $\frac{1}{6}\left[\binom{6}{3}+2\binom{2}{1}+3\binom{2}{1}\binom{2}{1}\right] = \frac{1}{6}(20+2\cdot 2+3\cdot 4) = 6$ ☑

Fixed points by orbit

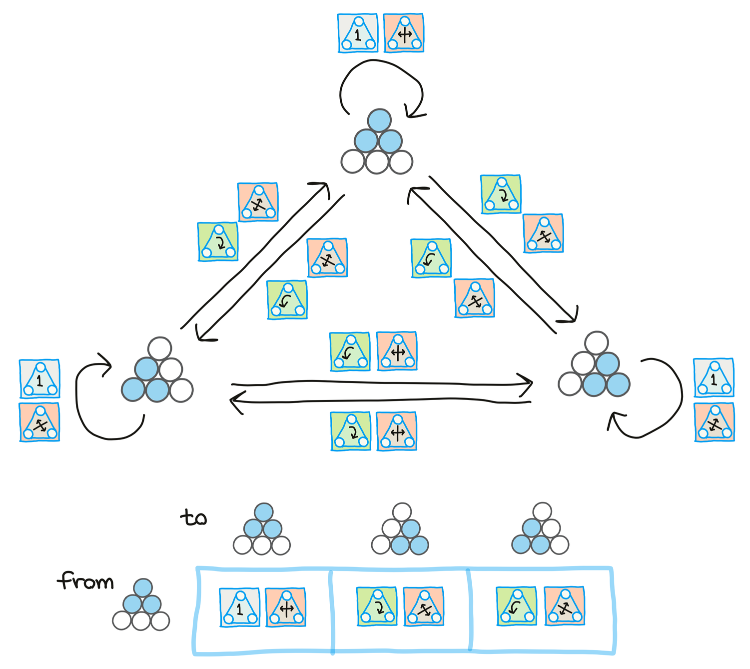


$\sum_{g\in G}|X_g|$ counts all fixed points (g,x) where $gx=x$

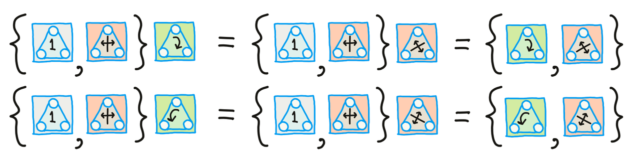
If we can understand why there are $|G|$ fixed points per orbit,

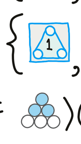
then we understand $|P| = \frac{1}{|G|}\sum_{g\in G}|X_g|$

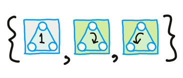
Look closely at how G acts on a particular orbit

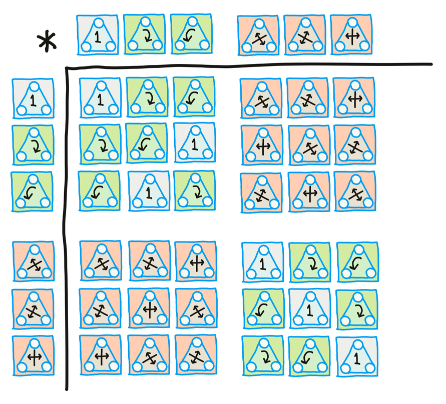


These subsets of G (cosets) are always in 1:1 correspondence with each other, so they divide G into equal sized subsets.



(# Fixed points of )(size of orbit) = $|G|$

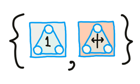
Quotients: mod out by "normal subgroup" 



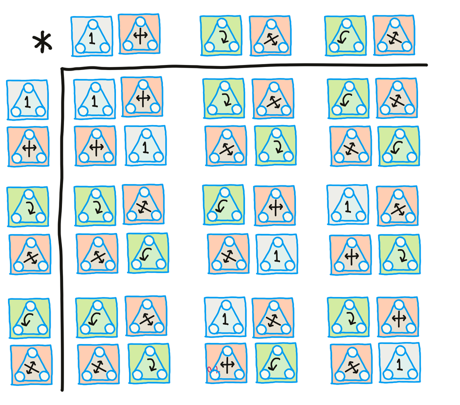
$\Longrightarrow$

+	0	1
0	0	1
1	1	0

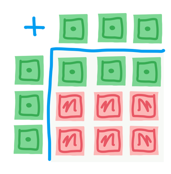
mod 2

 is not normal

and we don't get a coherent table when we try to mod out.



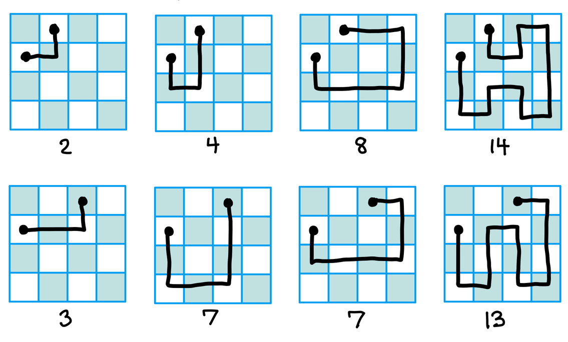
$\Longrightarrow$



Various entries are inconsistent

Expand on class questions:
Even-odd parity.

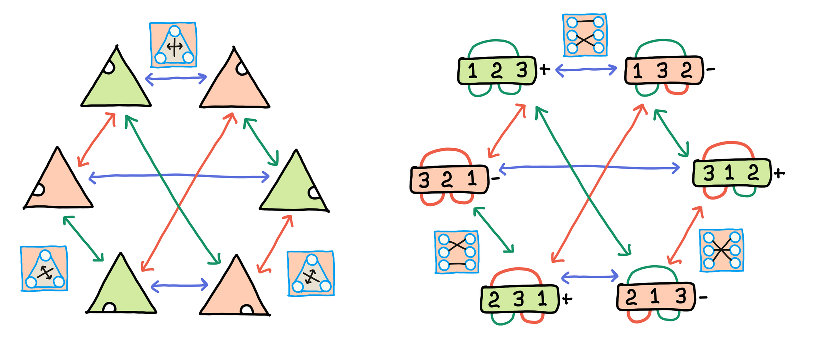
Walks alternate square colors



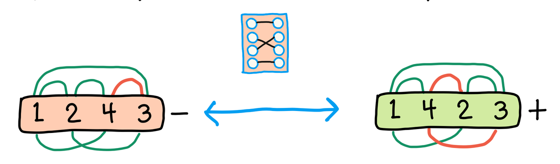
Walks between squares of the same color: even # steps

Walks between squares of the opposite color: odd # steps

We can checkerboard the graph of all triangle positions.
Flips all change checkerboard color



We can checkerboard the graph of all permutations of $\{1,\dots,n\}$
Even-odd: How many pairs are out of order?
Adjacent pair swaps change this count by 1



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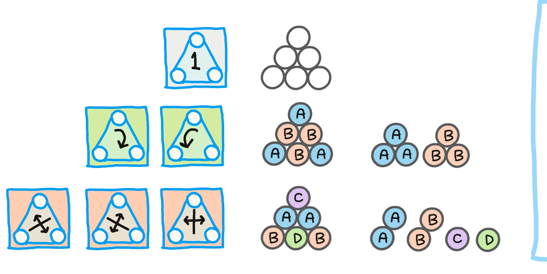
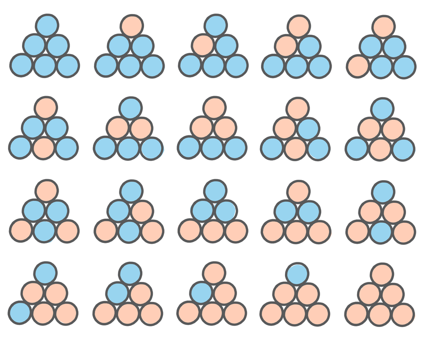
k colors $|P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$



k=2

$|P| = \frac{1}{6}(k^6+2k^2+3k^4)$
$= \frac{1}{6}(\underbrace{64+2\cdot 4}_{12}+\underbrace{3\cdot 16}_{8})$
$= 20$

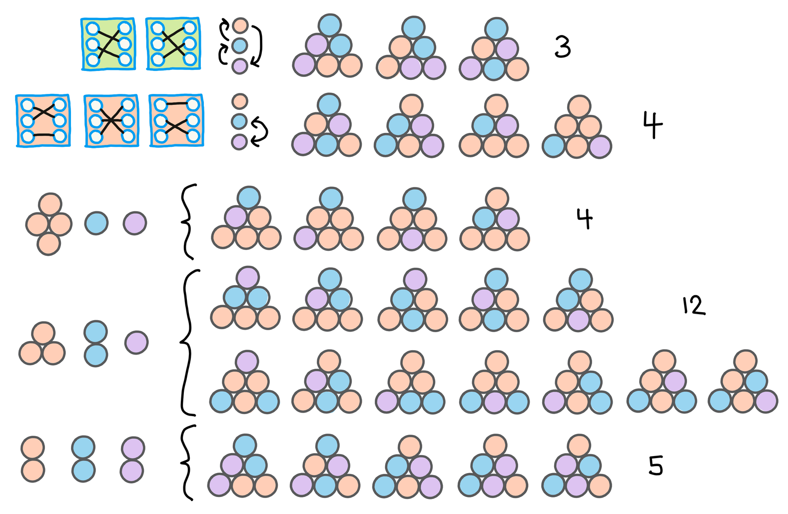


k=3 $|P| = \frac{1}{6}(k^6+2k^2+3k^4) = \frac{1}{6}(729+2\cdot 9+3\cdot 81) = 165$

use 1 color: 3
use 2 colors: $\binom{3}{2}18$ (From above)
$\Rightarrow$ use 3 colors: $165-3-\binom{3}{2}18 = 108$

Not easily checked
(This way lies madness)

Let S_3 act on the colors, for this $|X| = 108$



108 $\quad \frac{1}{6}(108+2\cdot 3+3\cdot 4) = 21$
18 1 2

3

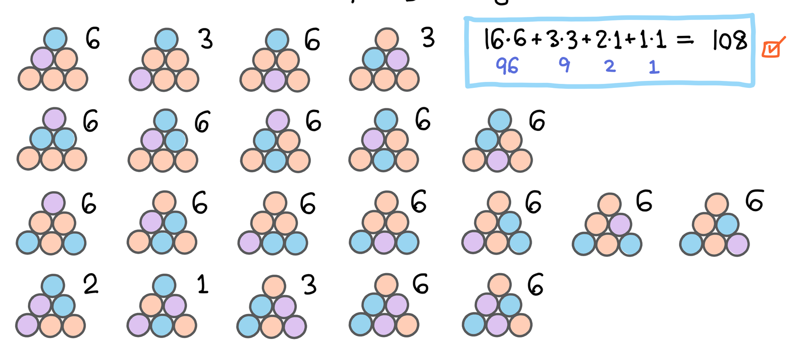
4

4

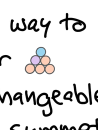
12

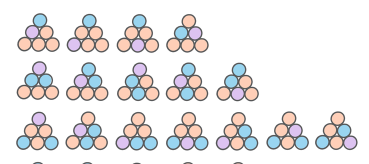
5

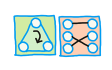
Now count orbit sizes by S_3 acting on colors



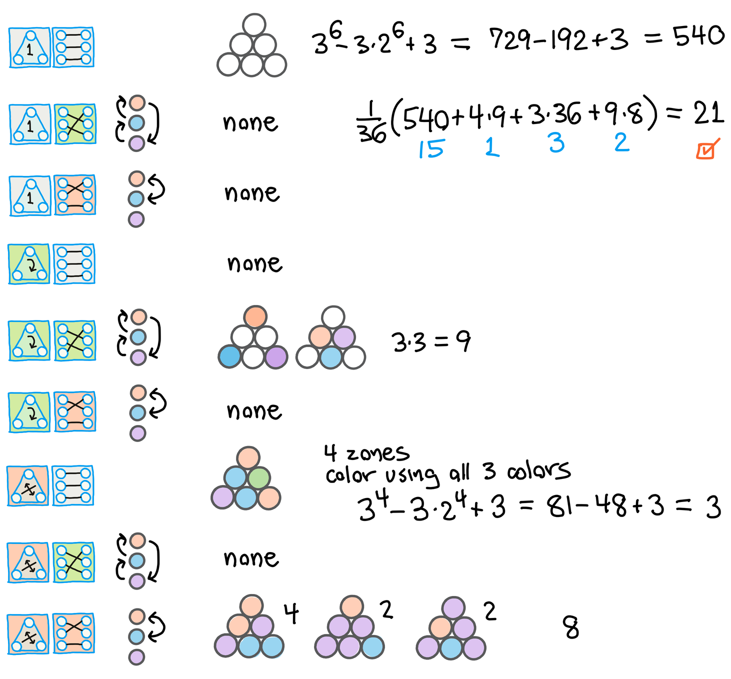
$16\cdot 6+3\cdot 3+2\cdot 1+1\cdot 1 = 108$ ☑
96 9 2 1

More systematic way to get 21 ways to color  using 3 interchangeable colors up to triangle symmetries:



Let $G = S_3 \times S_3$, group of pairs of actions of form  acting on triangle and then color choices

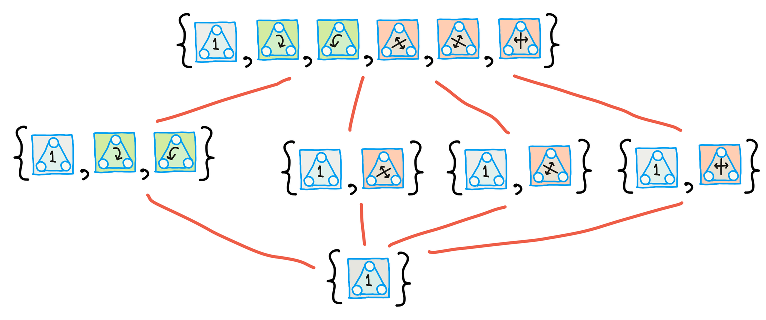
$|G| = |S_3||S_3| = 6\cdot 6 = 36$



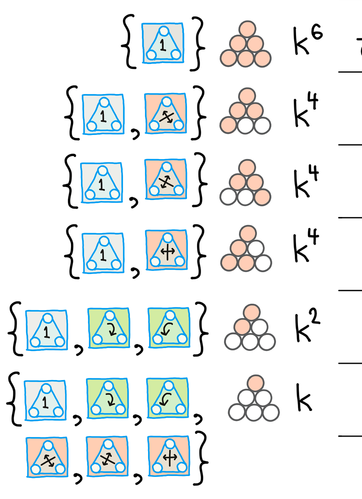
15	1	$3^6-3\cdot 2^6+3 = 729-192+3 = 540$
	2	none
	3	none
	2	none
1	4	$3\cdot 3 = 9$
	6	none
3	3	4 zones color using all 3 colors $3^4-3\cdot 2^4+3 = 81-48+3 = 3$
	6	none
2	9	4 2 2 — 8
21 ☑	36 ☑	

$\frac{1}{36}(540+4\cdot 9+3\cdot 36+9\cdot 8) = 21$ ☑
15 1 3 2

Can we use inclusion-exclusion instead of Burnside's lemma?
Need to consider poset of subgroups of S_3. Möbius inversion.



k colors $|P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$

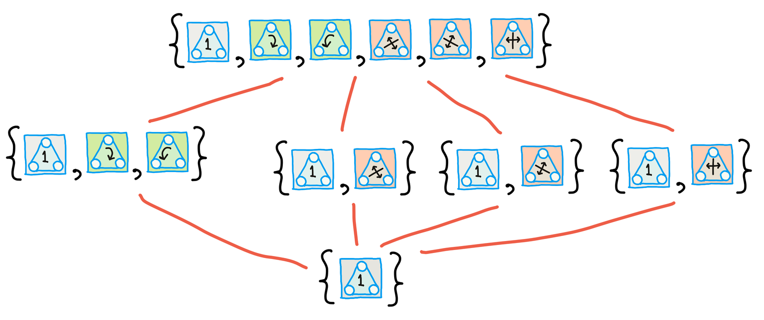


type of symmetry / at least	exactly, divided by symmetries	k^6	k^4	k^2	k
k^6	$\frac{1}{6}(k^6-3k^4-k^2+3k)$	1	-3	-1	3
k^4	$\frac{1}{3}(k^4-k)$		2		-2
k^4	$\frac{1}{3}(k^4-k)$		2		-2
k^4	$\frac{1}{3}(k^4-k)$		2		-2
k^2	$\frac{1}{2}(k^2-k)$			3	-3
k	k				6
		1	3	2	0

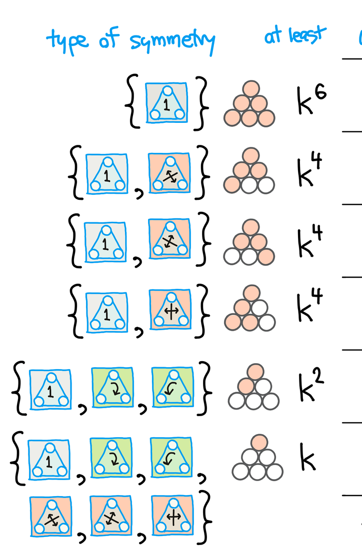
$(k^6 \; k^4 \; k^2 \; k)/6$

$\frac{1}{6}(k^6+2k^2+3k^4)$ ☑

Better approach: Skip Möbius inversion to compute "exactly".
Rather, when a pattern has d versions, we want to count each one with weight 1/d.
Work up the poset, adjusting weights based on count so far from below.



k colors $|P| = \frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{6}(k^6+2k^2+3k^4)$

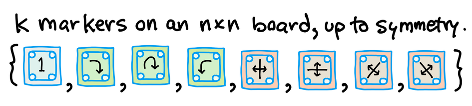


type of symmetry / at least	desired weight	subtract below	net contribution
k^6	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}k^6$
k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
k^4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}k^4$
k^2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}k^2$
k	1	0	

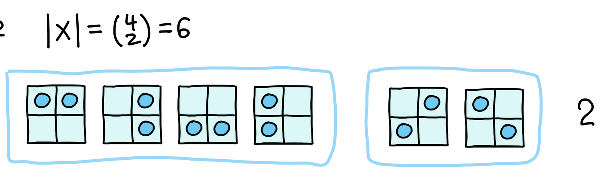
$\frac{1}{6}(k^6+2k^2+3k^4)$ ☑

This can be easier than Burnside's lemma.

Placing k markers on an n×n board, up to symmetry.

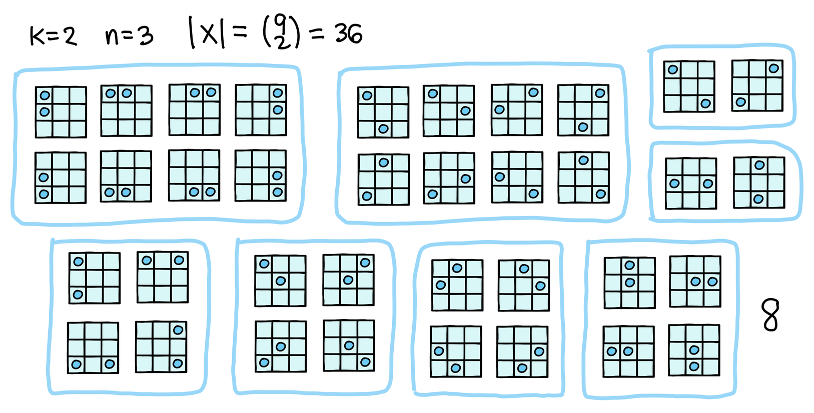
G =  $|G| = 8$

k=n=2 $|X| = \binom{4}{2} = 6$



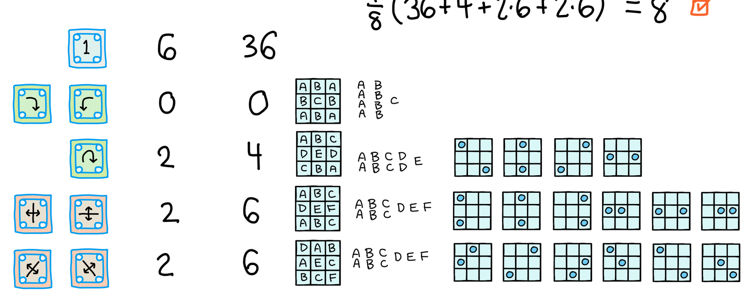
2

k=2 n=3 $|X| = \binom{9}{2} = 36$



8

$\frac{1}{8}(6+2+2\cdot 2+2\cdot 2) = 2$ ☑
$\frac{1}{8}(36+4+2\cdot 6+2\cdot 6) = 8$ ☑



6	36		
0	0	ABA / BCB / ABA	
2	4	ABC / DED / CBA	A B C D E / A B C D E
2	6	ABC / DEF / ABC	A B C D E F / A B C
2	6	DAB / AEC / BCF	A B C D E F / A B C

A B C D E / A B C D E Subproblem is picking up piles to form a count K.
A B C D E F / A B C Constrained subset problem.
Is there a good solution?

A B C D E $\binom{5}{2}$

coefficient of t^2 in $(1+At)(1+Bt)\cdots(1+Et)$ gives the sets themselves.

A B C D E F / A B C $\quad (1+A^2t^2)(1+B^2t^2)(1+C^2t^2)(1+Dt)(1+Et)(1+Ft)$
$(1+t^2)^3(1+t)^3$

	1	0	3	0	3	0	1	
	1		3		3		1	1
1	3		9		9		3	3
3	3		9		9		3	3
6	1		3		3		1	1
10	12							

	1		3		3		1	
	1		3		3		1	1
1	3		9		9		3	3
3	3		9		9		3	3
6	1		3		3		1	1
10	12	12	10	6	3	1		