MAST 680 Assignment 2: Learning Dynamics from Video

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March 07, 2023

Abstract

Many phenomena in the world are complex and some may have dynamics which have yet to be discovered. A method of identifying models called SINDy may be used to obtain a simple yet accurate model of a system. In this paper, SINDy is used to attempt to learn the dynamics of a mass-spring system. Videos of different angles were given and the mass positions were obtained. PCA was also needed to extract additional an additional component. The model produced by SINDy was very close to the known system, however, there are still some discrepancies.

1 Introduction and Overview

Traditional approaches to modelling physical systems have been to find the governing equations analytically through fundamental principles. The models could then be verified through experimentation. However, many physical phenomenons are complex and finding their governing equations that accurately model their behaviour may be challenging[3]. It is not until the 1980s that people started to characterize dynamic systems through the use of time series data[3]. There have been many more advances in this field. One of them is the Sparse Identification of Non-Linear Dynamics or SINDy for short. The SINDy method allows us to uncover the underlying dynamics of a system with few terms through sparse regression[1].

In this current paper, the SINDy method was implemented to learn the dynamics of a mass-spring system. Two sets of videos were given: one with relatively low noise and the other with the noise from the camera shaking. Each set contains three videos representing the different camera angles of the system.

2 Theoretical Background

Before working with SINDy, the steps leading to the method will be defined. A model of a dynamical system in its spatial coordinates (x,y,z) is usually described as equation 1.

$$\dot{x} = f(t, x, y, z); \quad \dot{y} = g(t, x, y, z); \quad \dot{z} = h(t, x, y, z) \tag{1}$$

where $\dot{x}$, $\dot{y}$ and $\dot{z}$ are the time derivatives of the spatial coordinates; f, g and h are functions in terms of the spatial coordinates and elapse time t. equation 1 can be reformulated into matrices where similar operations to DMD seen in assignment 1 can be performed. The left-hand side will be represented by the matrix Y containing the time derivatives of the collected data matrix X. The data matrix X contains N columns representing the coordinates collected at each time step and the rows representing each variable. Since the data are collected discretely, the time derivatives may be estimated using the finite difference method $\frac{dx_n}{dt} \approx \frac{x_{n+1} - x_n}{\Delta t}$ where $n \in \{0, 1, 2, ..., N-1\}$ and Δt is the time difference between each measurement. The functions on the right-hand side can be approximated with the sum of different basic functions with coefficients. These different functions called θ form a dictionary $\mathcal{D}$ defined by the user.

$$\mathcal{D} = \{\theta_1, \theta_2, ..., \theta_K\} \tag{2}$$

The subscript K represents the number of user-defined functions. These functions can be linear, polynomials, sine functions, exponential, etc. of x, y or z. Equation 3 illustrates the matrix reformulation for a single

time frame to save on space, but it can be extended to a matrix of multiple time frames.

$$
\begin{aligned}
\dot{x} &\approx a_1\theta_1 + a_2\theta_2 + a_3\theta_3 + ... + a_K\theta_K \\
\dot{y} &\approx b_1\theta_1 + b_2\theta_2 + b_3\theta_3 + ... + b_K\theta_K \rightarrow \\
\dot{z} &\approx c_1\theta_1 + c_2\theta_2 + b_3\theta_3 + ... + c_K\theta_K
\end{aligned}
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \approx
\begin{bmatrix} a_1 & a_2 & a_3 & ... & a_K \\ b_1 & b_2 & b_3 & ... & b_K \\ c_1 & c_2 & c_3 & ... & b_K \end{bmatrix}
\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ ... \\ \theta_K \end{bmatrix} \rightarrow Y \approx \Xi\Theta(X) \tag{3}
$$

Since Y is estimated from the data and $\Theta(X)$ is user-defined, only Ξ is unknown. As with DMD, the matrices are not square thus the coefficient matrix Ξ should be obtained such that it minimizes the difference between both sides of the approximate equality. Just as with DMD, the pseudo-inverse, denoted with the † symbol, will be used to find the coefficient matrix Ξ.

$$
\Xi = Y\Theta(X)^\dagger \tag{4}
$$

Performing equation 4, a model can be fitted to describe the system approximately. However, it is advantageous to find a system in which the model is easier to work with yet remains accurate.

The user-defined dictionary, in the beginning, may contain some insignificant terms. This will be shown by coefficient values of smaller magnitudes. The SINDy method helps iron out the terms that are more significant. The method imposes a sparsity threshold λ to collect the coefficients higher than this value. It is an iterative method imposing this constraint and re-evaluating equation 4 to obtain a new set of coefficients. The iteration stops when the new coefficients does not change from the previous one. This is different than just finding Ξ once and picking the coefficients higher than the threshold. In doing so, the model is less accurate due to some loss of information. In contrast, the SINDy method allows most of the information to be kept while best fitting the model to the smallest amount of terms.

Up to now, it is assumed the data matrix X is noise free. However, in reality, the data is messy. Thus finding the matrix Y with the finite derivative method can lead to inaccuracies. Any noise present will be amplified as shown in section 3 of [4]. Instead of using finite derivatives, an integration formulation can be used to average out the data. Starting from any of the formulations in equation 3, both sides are integrated to obtain equation 5.

$$
X(t) - X(0) \approx \Xi \int_0^t \Theta(X(s))\, ds \tag{5}
$$

The variable s is only a change of variable for integration. $X(t)$ is the position at time t and $X(0)$ is the initial position or simply the first data. The integration of discrete data sets can be estimated by performing a simple Riemann Sum. This means at each moment in time, we multiply the function by Δt and add the value to the total approximating the area under the curve.

$$
\mathcal{I} = \int_0^{t_n} \Theta(X(s))\, ds = \sum_{i=1}^n \Delta t\Theta(X(t_n)) \tag{6}
$$

Rewriting equation 5 by replacing $X(t) - X(0)$ by $\tilde{X}$ and using I from equation 6, the same form of equation 3 can be obtained. Thus the pseudo-inverse can be used to find Ξ.

$$
\Xi = \tilde{X}\mathcal{I}^\dagger \tag{7}
$$

The SINDy method can be performed the same way as stated earlier to find a sparse representation of the system.

Regarding the problem of a one-dimensional damped mass-spring system travelling in the x direction, the dynamics of the system, in state-space representation, can be shown to be equation 8 using Newton's second law.

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -\frac{K}{M}z_1 - \frac{c}{M}z_2
\end{aligned} \tag{8}
$$

z_1 represents the x-position, z_2 represents the x-velocity, K is the spring constant in $[N/m]$, M is the mass of the system in $[kg]$ and c is the damping coefficient in $[Ns/m]$. For a single-dimension problem with the

involvement of acceleration, such as a mass-spring system, the use of different camera angles to collect data along with a Principal Component Analysis (PCA) may be used to uncover the velocity component of the system.

PCA, in essence, redefines the data in a way that variables become uncorrelated. To do this, the variance and covariance of the mean subtracted data are used. Variance and covariance, in statistics, are measures of the spread and correlation of between measurements respectively. These variances and covariances can be calculated in a single matrix (C_x) by multiplying X with its transpose.

$$C_x = \frac{1}{N-1} X X^T \tag{9}$$

The factor $\frac{1}{N-1}$ comes from the calculation of variances in the sampled population where N is the number of data points. C_x is a symmetric matrix thus its eigenvectors can be found. These vectors are called the principal components. Since the eigenvectors are orthogonal to each other, they can be projected to the data using equation 10 to create new variables which are uncorrelated. Furthermore, the eigenvalues provide the magnitude of correlation associated with the principal component direction.

$$X_{new} = U^T X \tag{10}$$

X_{new} is the new uncorrelated data and U is a matrix containing the eigenvectors. For the mass-spring system, applying PCA will uncover two dominant principal components which provide the position as well as the velocity directions of the system. The SINDy procedure described earlier could then be used on the resulting PCA data to find the model.

3   Algorithm Implementation and Development

To solve the problem, the algorithms were written in Octave. The scripts can be found in Appendix B. As mentioned in assignment 1, Octave scripts are interchangeable with Matlab with few exceptions. Any pre-made functions used will be described in Appendix A. The problem was first solved by extracting the position of the mass-spring system in each of the video files in a set, performing PCA on the data obtained and then finally applying SINDy to find the underlying dynamics. The algorithms described in the coming sections were performed on the first video set containing non-noisy videos and will not work on the second set containing noisy videos. A separate approach will be suggested to tackle the noisy video set in the future.

3.1   Locating the Mass

The first step was to load the video files into Octave. The video was stored as a matrix with 4 dimensions. Each of the dimensions represents, in order, the vertical pixels, the horizontal pixels, the RGB values and the different frames. The video can be viewed by taking the values of the first three dimensions and viewing the frames in succession. The video was turned into a grayscale for ease of computation. The data was then converted into a single matrix of dimensions M and N, with M representing the number of pixels in a frame and N representing the number of frames.

To locate the mass in each of the frames, a simple background-subtraction method was used. There are many different ways of recreating the background in a simple manner [2]. Here, we used, at each pixel, the median or mode value of the pixel intensity from all the frames to construct the background depending on which method gave the better result. This will give a static image of the background since it is capturing the pixels appearing the most and groups all N pixels in a row into a single frame. The background was then subtracted from the original gray video, leaving only the objects in motion visible. A threshold intensity of 50 was set to remove some of the noise. Higher intensities were set to 255 and the lower intensities were set to 0 to produce a pure black-and-white image. This method worked well for the first and third videos but had some issues with the second due to a larger level of camera shake. In this case, the median of the pixel intensity was calculated in sections of 4 frames to provide a background that is adapting to the video.

The background subtracting method gave a bright area on each frame which has a larger surface area than the noise elements that appeared. Thus the x and y centroid positions of the mass in each frame can

3

be calculated by taking the mean pixel intensities of each column (x-centroid) and row (y-centroid) and then using equation 11.

$$x_c = \frac{\sum I_{col} x}{\sum I}; \quad y_c = \frac{\sum I_{row} y}{\sum I} \tag{11}$$

x_c and y_c are the centroid positions and I are the mean pixel intensity values at each x and y pixel position. This equation is analogous to the mass centroid calculation of a physical system. If the noise is large, the centroid could be shifted. To counter this, the pixel range containing the mass was manually selected in the first frame, and the pixels outside this boundary were turned black. The centroid in this boundary could be found using equation 11. Using previous positions, a new slightly larger boundary can be set in the next frame. Repeating the steps leads to a fairly accurate tracking algorithm.

Algorithm 1: Locating the Mass

Import data from `cami_j.mat`, $i \in \{1, 2, 3\}, j \in \{1, 2\}$
for $k = 1 : Number of frames$ **do**
   Convert to grayscale
   Data(:,k) = Reshape grayscale k^{th} frame
end for
for $k = 1 : M$ **do**
   Background(k,:) = median(I,:)
end for
Subtract background and apply threshold
for $k = 1 : N$ **do**
   Reshape k^{th} frame of Data into frame width × height
   if $k == 1$, create boundary around mass
   Else, take the previous frame location and create a new bounded region
   Calculate the mean intensities of each row and col of frame
   Calculate the x and y centroid positions and collect value into array
   Reshape back into MxN Data
end for
Save collected positions into `pos.mat` file

3.2   PCA Implementation

Once the x and y positions have been obtained for the three videos, their mean values were subtracted and the values were brought together in a single matrix stacking one on top of the other. This matrix is of size $6 \times N$. PCA was then performed on this matrix. Instead of finding C_x and computing its eigenvectors, SVD will be used. To do this, a matrix A was created by taking X and dividing by $\sqrt{N-1}$. Since multiplying A by its transpose will find C_x, Performing SVD on A will give the principal components U but now the vectors are in order of importance. Thus we can use equation 10 to readily find the new coordinates using just the first two principal components. Lastly, a filtering function was used to reduce the noise in the new data. Furthermore, only a section (125 frames) in the middle of the data was kept.

Algorithm 2: Implementation of PCA

Import data from `pos.mat`
Create matrix X by stacking the 6 positions
Create A by dividing elements of X by $\sqrt{N-1}$
[U,S,V] = svd(A,'econ')
X_{new} = U'X

3.3 SINDy Implementation

The X_{new} obtained from PCA contains two rows of information. The first (z_1) indicates the position and the second (z_2) indicates the velocity directions. Here, we use the variable z instead of x to be consistent with equation 8. Since the second principal component does not actually give the exact magnitude of the velocity, this principal component was multiplied by a scale factor. This scale factor was found using the finite derivative method on the position and taking the ratio of the maximum values between the finite derivative and the second principal component. To further reduce the effect of noise, the integral method was implemented instead of the time derivatives. The dictionary was defined as the set of polynomials up to degree 2. The higher degree terms would allow us to see how SINDy performs.

$$D = \{1, z_1, z_2, z_1^2, z_1 z_2, z_2^2\} \tag{12}$$

Θ was created by performing element-wise multiplication of the data from X_{new}. Each row of Θ would be a different function. By applying equation 5, 6 and 7 the first set of coefficients was found. A time step Δt of 1 was used as it curiously provided a better result. A loop was then created to continuously select indices which contain coefficients larger than the sparsity threshold value and re-apply equation 7 with those indices until the coefficients stay the same as those from the last iteration. The sparsity threshold was taken as 0.1 to obtain the closest result to the actual system. The system of equations was displayed with very small values in the coefficients omitted. Finally, the SINDy algorithm was validated against a dataset of a known solution of the damped mass-spring system which it was able to obtain the exact system of 8 even with small artificially induced noise.

Algorithm 3: Implementation of SINDy

Create $\tilde{X}$ by subtracting $X_{new}(0)$ from $X_{new}(:, 2 : end)$
Perform $\Theta(i, :) = D_i$ using element-wise operation of X_{new}
Calculate $\mathcal{I}$
$coeff = \tilde{X} pinv(\mathcal{I})$
while $|coeff - prevCoeff| \neq 0$ or first iteration **do**
    Find indices of coeff smaller than threshold λ and make them 0 in the new coeff
    Calculate new coeff using the indices with coeff bigger than λ using pinv
end while
Display system on screen

---

4   Computational Results

The data obtained from the mass locating algorithm showed almost perfect sinusoidal waves as seen in figure 1. This indicates that the first algorithm performs well in capturing the movement of the mass. In one of the videos, the mass seemed to be out of phase with the other two. This phase shift allowed PCA to discover both the position and the velocity components. These were the 2 dominant principal components. We can see from the bottom portion of figure 1 that the blue curve (second principal component) resembles the velocity of the red curve. If the second principal component data was scaled to where the amplitude matches the amplitude of the actual derivative of the first principal component, then the exact velocity is uncovered (see Appendix C for complementary figures). This confirms that it is the velocity component.

Performing SINDy method on the filtered data with a time step (dt) of 1 and a sparsity threshold (λ) of 0.1, we arrive at a system close to the theoretical system of equation 8.

$$\dot{z}_1 = 0.986784 z_2$$
$$\dot{z}_2 = 0.150838 - 0.130678 z_1 \tag{13}$$

However, there are still quite a few differences. The first is that the coefficient in front of z_2 is not exactly 1. This difference could be due to a combination of the remaining noise in the data, the pseudoinverse which can only best fit the data as well as the sparsity threshold. The second is that there is a constant term
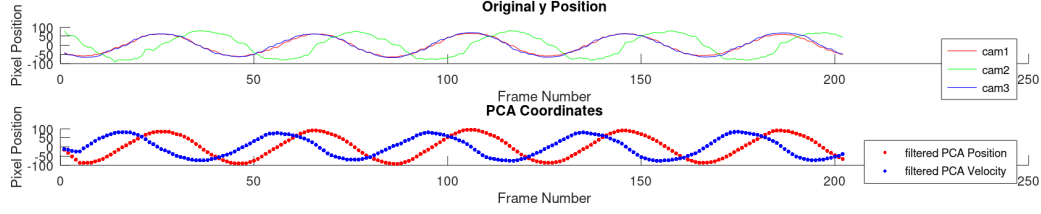
Figure 1: Comparison between the mean adjusted y-positions obtained with the mass locating algorithm (TOP) and the PCA variables (BOTTOM)
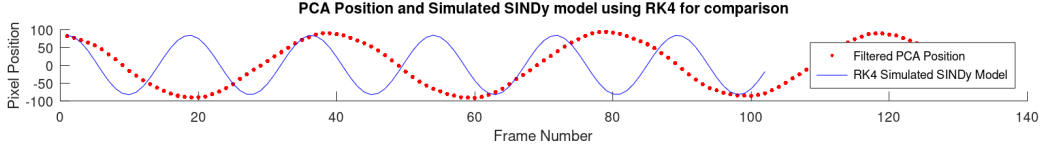

Figure 2: PCA position compared with the simulated SINDy model

which appears in the $\dot{z}_2$ equation. This may be due to noise in the data as well. Particularly, the blue velocity curve from figure 1 appears less smooth than the red position curve. Theoretically, the constant means there is a constant external force (other than gravity and the spring) acting on the mass. The noise could be producing this effect. The third discrepancy is that there is no z_2 term in $\dot{z}_2$. However, this may not be surprising since the decay in the sine wave seems to be negligible. Lastly, and perhaps less obvious if the theoretical system was unknown, the coefficient in front of z_1 shows that the system is oscillating at a higher frequency than what the data shows. In fact, the coefficient K/M represents the square of the natural frequency $\omega_n^2 = \frac{K}{M}$. If we simulate the result with an initial condition matching the data and plotting it with the first principal component, then we see a faster oscillation than the original (figure 2).

This difference might be due to the sparsity threshold being higher than the actual natural frequency. However, this should mean that the z_1 would be completely gone. Lowering the threshold resulted in some undesirable coefficients appearing which further increases the discrepancies. The time step could also be an area of concern since 1 second between each frame seems slow and unrealistic. However, decreasing this value only increases the differences.

Note that the described algorithm to locate the mass would not be able to extract the positions of the second set of videos which contain a larger amount of noise. Due to time constraints, a new tracking algorithm was not able to be developed. However, one could use a cross-correlation technique by comparing each frame to an image of the mass to locate the position based on the highest correlation index [5]. Based on current results with the low-noise videos, it is suspected that SINDy would have a hard time extracting the model of equation 8 without excessive filtering of data.

5   Summary and Conclusions

In summary, the SINDy method is used to obtain a simple yet accurate model of a dynamical system. This technique was examined on a mass-spring system. PCA was first applied to data collected from three perspectives of this one-dimension system to uncover the velocity component. The results from SINDy with a sparsity parameter of 0.1 and a time difference of 1 second resulted in a better model than other combinations. However, there were still some discrepancies possibly due to noise or outliers in the data. The reasons for the differences concerning the frequencies are not yet understood.

References

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[2] First Principles of Computer Vision. *Change Detection — Object Tracking.* May 2021. URL: `https://www.youtube.com/watch?v=0ANl4bZAxjI`.

[3] Amin Ghadami and Bogdan I. Epureanu. "Data-driven prediction in dynamical systems: recent developments". In: (2022). DOI: `10.1098/rsta.2021.0213`. URL: `https://royalsocietypublishing.org/doi/10.1098/rsta.2021.0213`.

[4] Jason J. Bramburger. *Data-Driven Methods for Dynamic Systems.* Concordia University, 2023.

[5] MathWorks. *xcorr2 2-D cross-correlation.* URL: `https://www.mathworks.com/help/signal/ref/xcorr2.html`. (accessed: 07.03.2023).

Appendix A   Premade Functions Used

The following premade functions are found in Octave which are the same as in Matlab. The last function was created during a course on Numerical Simulations. It has the same function as ODE45.

- `grayImage = rgb2gray(readFrame(vid))` converts each pixel containing of RGB values to a single value by a weighted sum. These values represent the grayscale of each pixel.

- `reshape(X, M, N)` converts the matrix X into a matrix of size M × N.

- `movie(h, mov, 1, frameRate)` displays the frames inside `mov` structure in the figure box `h`. The number 1 indicates the number of loops and `frameRate` indicates the video fps.

- `[U,S,V] = svd(X,'econ')` returns the unitary matrices `U` and `V`, and the singular values of matrix X. The argument `'econ'` removes all zeros rows or columns to make the matrices compact.

- `round(x)` rounds the value `x` down to the nearest integer.

- `imshow(mov(k).cdata,mov(k).colormap)` displays the kth frame in `mov` structure with its color map onto a figure window.

- `pinv(X)` returns the pseudoinverse $(V\Sigma^{-1}U^*)$ of `X` matrix.

- `abs(x)` returns the norm of `x`.

- `sort(X)` returns a vector sorted from smallest to largest of vector `X`.

- `linspace(x,y,n)` produces an array from x to y in n equal spaces.

- `filter(x,y,A)` filters array A according to a transfer function x/y. For a moving average filter, x is an array having a length of the window size and its values are 1 over the window size. y is simply 1.

- `RK4_Sys_Inputs(f, to, yo, n, tf, inputs)` solves numerically system function f with initial time to, initial conditions yo, number of divisions, final time tf and function inputs.

Appendix B   Octave Code

B.1   Example Mass Location Code

```
%Mass position extracting on Cam 1
%Will Sze
%40096561

```

```octave
 5  clear all; close all; clc
 6  pause(0.1);
 7
 8  nfigure = 1;
 9  FrameInterest = 1;
10
11  fprintf('\nImporting Video...\n');
12  tic
13      %Load data into Octave
14      load cam1_1.mat;
15
16      %Collects video dimensions
17      [vidHeight1, vidWidth1, nRGB1, vidNFrames1] = size(vidFrames1_1);
18      FrameRate = 60;
19
20      %Movie Struct to collect all original frames to run movie using movie
             function
21      mov1 = struct('cdata', zeros(vidHeight1,vidWidth1,3,'uint8'), 'colormap'
             ,[]);
22      mov1_Gray = struct('cdata', zeros(vidHeight1,vidWidth1,1,'uint8'), '
             colormap',gray(255));
23      mov1_BW = struct('cdata', zeros(vidHeight1,vidWidth1,1,'uint8'), '
             colormap',gray(255));
24      mov1_int = struct('cdata', zeros(vidHeight1*vidWidth1,1,'uint8'), '
             colormap',gray(255));
25
26      for k = 1:vidNFrames1
27        mov1(k).cdata = vidFrames1_1(:,:,:,k);
28      end
29  toc
30
31  %View original video1
32  fprintf('\nDisplaying Original Video1...\n');
33  tic
34      hf = figure(nfigure);
35      set(hf, 'position', [150 150 vidWidth1 vidHeight1]);
36      movie(hf, mov1, 1, FrameRate);
37      nfigure = nfigure + 1;
38  toc
39
40  fprintf('\nDisplaying Video1 image at frame %d in the movie...\n',
             FrameInterest);
41  tic
42    hf = figure(nfigure);
43    set(hf, 'position', [150 150 vidWidth1 vidHeight1]);
44    imshow(mov1(FrameInterest).cdata,mov1(FrameInterest).colormap);
45    nfigure = nfigure + 1;
46  toc
47
48
49  %Converting to grayscale1
50  tic
51      for k = 1:vidNFrames1
52        mov1_Gray(k).cdata = rgb2gray(mov1(k).cdata);
```

```matlab
         mov1_Gray(k).colormap = gray(255);
     end
toc


%View GrayScale video1
fprintf('\nDisplaying Gray Scale Video1...\n');
tic
    hf = figure(nfigure);
    set(hf, 'position', [150 150 vidWidth1 vidHeight1]);
    movie(hf, mov1_Gray, 1, FrameRate);
    nfigure = nfigure + 1;
toc

fprintf('\nDisplaying Video1 image at frame %d in the movie...\n',
    FrameInterest);
tic
   hf = figure(nfigure);
   set(hf, 'position', [150 150 vidWidth1 vidHeight1]);
   imshow(mov1_Gray(FrameInterest).cdata,mov1_Gray(FrameInterest).colormap)
       ;
   nfigure = nfigure + 1;
toc



Data1 = zeros(vidWidth1*vidHeight1, vidNFrames1);

%---------- Tracking algorithm -------------%
fprintf('\nCreating Data1...\n');
tic
   for k = 1:vidNFrames1
      mov1_int(k).cdata = reshape(mov1_Gray(k).cdata,[],1);
      mov1_int(k).colormap = gray(255);
      Data1(:,k) = [mov1_int(k).cdata];
   end
toc
[M N] = size(Data1);


IntensityThreshold = 50;
%Mode Background Method
fprintf('\nCreating background on Data1...\n');
tic
   Background = zeros(M,N);
   for i = 1:M
      Background(i,:) = mode(Data1(i,:));
   end
   Data_Test2 = Data1 - Background;

   Data_Test2(Data_Test2>IntensityThreshold) = 255;
   Data_Test2(Data_Test2<=IntensityThreshold) = 0;
toc
```

```matlab
%Horizontal and Vertical Intensities
fprintf('\nLocating the mass...\n');
cc_F_Interest = 1;
x_pos = [];
y_pos = [];
x_Intensity = zeros(vidNFrames1,vidWidth1);
y_Intensity = zeros(vidNFrames1,vidHeight1);
for i = 1:vidNFrames1

  tic
  Frame_Analysis = reshape(Data_Test2(:,i),vidHeight1, vidWidth1);
  Gray_Frame = reshape(Data1(:,i),vidHeight1, vidWidth1);

  %Set Boundary on first frame
  if(i == 1)
    Frame_Analysis(:,1:300) = 0;
    Frame_Analysis(:,390:vidWidth1) = 0;
    Frame_Analysis(1:210,:) = 0;
    Frame_Analysis(320:vidHeight1,:) = 0;
  end
  %Use previous position to locate next boundary
  if(i ~= 1)
    Y_UpperBound = round(vidHeight1-y_pos(i-1))+60;
    Y_LowerBound = round(vidHeight1-y_pos(i-1))-60;
    if(Y_UpperBound > vidHeight1)
      Y_UpperBound = vidHeight1;
    end
    if(Y_LowerBound < 1)
      Y_LowerBound = 1;
    end
    Frame_Analysis(1:Y_LowerBound,:) = 0;
    Frame_Analysis(Y_UpperBound:vidHeight1,:) = 0;

    X_UpperBound = round(x_pos(i-1))+50;
    X_LowerBound = round(x_pos(i-1))-50;
    if(X_UpperBound > vidWidth1)
      X_UpperBound = vidWidth1;
    end
    if(X_LowerBound < 1)
      X_LowerBound = 1;
    end
    Frame_Analysis(:,1:X_LowerBound) = 0;
    Frame_Analysis(:,X_UpperBound:vidWidth1) = 0;
  end

  %Calculating Centroid
  for j = 1:vidWidth1
      x_Intensity(i,j) = mean(Frame_Analysis(:,j));
  end
  for j = 1:vidHeight1
    y_Intensity(i,vidHeight1+1-j) = mean(Frame_Analysis(j,:));
  end

  %X-centroid
```

```matlab
159      SumWeightPos = 0;
160      SumWeight = 0;
161      for j = 1:vidWidth1
162        SumWeightPos = SumWeightPos + j*x_Intensity(i,j);
163        SumWeight = SumWeight + x_Intensity(i,j);
164      end
165      x_pos = [x_pos,SumWeightPos/SumWeight];
166
167      %Y-centroid
168      SumWeightPos = 0;
169      SumWeight = 0;
170      for j = 1:vidHeight1
171        SumWeightPos = SumWeightPos + j*y_Intensity(i,j);
172        SumWeight = SumWeight + y_Intensity(i,j);
173      end
174      y_pos = [y_pos,SumWeightPos/SumWeight];
175
176      %Visualize Point in video
177      Frame_Analysis(:,round(x_pos(i))) = 180;
178      Gray_Frame(:,round(x_pos(i))) = 0;
179      Frame_Analysis(vidHeight1 - round(y_pos(i)),:) = 180;
180      Gray_Frame(vidHeight1 - round(y_pos(i)),:) = 0;
181
182      Data_Test2(:,i) = reshape(Frame_Analysis,vidHeight1*vidWidth1,1);
183      Data1(:,i) = reshape(Gray_Frame,vidHeight1*vidWidth1,1);
184      toc
185  end
186
187
188  fprintf('\nPlotting intensity curves at frame %d in the movie...\n',
         cc_F_Interest);
189  tic
190      figure(nfigure);
191      plot(x_Intensity(cc_F_Interest,:),'-o');
192      nfigure = nfigure + 1;
193
194      y = linspace(1,vidHeight1,vidHeight1);
195      figure(nfigure);
196      plot(y_Intensity(cc_F_Interest,:),y,'-o');
197      nfigure = nfigure + 1;
198  toc
199
200
201  %View Results
202  tic
203      for k = 1:vidNFrames1
204        mov1_int(k).cdata = [cast(Data_Test2(:,k),'uint8')];
205        mov1_BW(k).cdata = reshape(mov1_int(k).cdata,vidHeight1, vidWidth1);
206        mov1_BW(k).colormap = gray(255);
207
208      end
209  toc
210
211  fprintf('\nDisplaying Black and White Video1...\n');
```

```
212  tic
213     hf = figure(nfigure);
214     set(hf, 'position', [150 150 vidWidth1 vidHeight1]);
215     movie(hf, mov1_BW, 1, FrameRate);
216     nfigure = nfigure + 1;
217  toc
218
219  fprintf('\nDisplaying Video1 image at frame %d in the movie...\n',
         FrameInterest);
220  tic
221     FrameInterest = cc_F_Interest;
222     hf = figure(nfigure);
223     set(hf, 'position', [150 150 vidWidth1 vidHeight1]);
224     imshow(mov1_BW(FrameInterest).cdata,mov1_BW(FrameInterest).colormap);
225     nfigure = nfigure + 1;
226  toc
227
228
229
230  tic
231     for k = 1:vidNFrames1
232        mov1_int(k).cdata = [cast(Data1(:,k),'uint8')];
233        mov1_Gray(k).cdata = reshape(mov1_int(k).cdata,vidHeight1, vidWidth1);
234        mov1_Gray(k).colormap = gray(255);
235
236     end
237  toc
238
239  fprintf('\nDisplaying Gray with centroid of Video1...\n');
240  tic
241     hf = figure(nfigure);
242     set(hf, 'position', [150 150 vidWidth1 vidHeight1]);
243     movie(hf, mov1_Gray, 1, FrameRate);
244     nfigure = nfigure + 1;
245  toc
```

B.2 PCA and SINDy Implementation Code

```
1  %PCA and SINDy Implementation
2  %Will Sze
3  %40096561
4  clear all; close all; clc;
5
6  %Loading Data from extracted location of each camera
7  load cam1_1_xPos2.mat
8  load cam1_1_yPos2.mat
9  cam1_xPos = x_pos;
10 cam1_yPos = y_pos;
11
12 load cam2_1_xPos.mat
13 load cam2_1_yPos.mat
14 cam2_xPos = x_pos(1:length(cam1_xPos));
15 cam2_yPos = y_pos(1:length(cam1_yPos));
```

```matlab
load cam3_1_xPos.mat
load cam3_1_yPos.mat
cam3_xPos = x_pos(1:length(cam1_xPos));
cam3_yPos = y_pos(1:length(cam1_yPos));

clear 'x_pos';
clear 'y_pos';

nfigure = 1;

%Plots all three camera y positions
figure(nfigure);
hold on
  plot(cam1_yPos);
  plot(cam2_yPos);
  plot(cam3_yPos);
hold off
nfigure++;

%Removes first few data
Start = 25;
X_og(1,:) = cam1_xPos(Start:end);
X_og(2,:) = cam1_yPos(Start:end);
X_og(3,:) = cam2_xPos(Start:end);
X_og(4,:) = cam2_yPos(Start:end);
X_og(5,:) = cam3_xPos(Start:end);
X_og(6,:) = cam3_yPos(Start:end);

for i = 1:6
  X_og_mean(i) = mean(X_og(i,:));
  X_PCA(i,:) = X_og(i,:) - X_og_mean(i);
end

%Plotting the mean subtracted data
figure(nfigure);
hold on
  plot(X_PCA(2,:),'r.','MarkerSize', 10);
  plot(X_PCA(4,:),'g.','MarkerSize', 10);
  plot(X_PCA(6,:),'b.','MarkerSize', 10);
hold off
nfigure++;


%--------- Applying PCA -----------%
A = 1/sqrt(length(X_og)-1)*X_PCA;

%A_AT = A*A'; %Alternative
%[eig_Vect, eig_Val] = eig(A_AT); %Alternative

[U,S,V] = svd(A,'econ');
Rank_S = 2; %First 2 principal components
Y_PCA = U(:,1:Rank_S)'*X_PCA;
```

13

```matlab
70  %Ploting Original mean subtracted data and the data obtained from PCA
71  figure(nfigure);
72
73    subplot (2, 1, 1)
74    hold on
75    plot(X_PCA(2,:),'r','MarkerSize', 10);
76    plot(X_PCA(4,:),'g','MarkerSize', 10);
77    plot(X_PCA(6,:),'b','MarkerSize', 10);
78    hold off
79
80    title("Original y Position");
81    h = legend ("cam1","cam2","cam3");
82    xlabel ("Frame Number");
83    ylabel ("Pixel Position");
84
85    subplot (2, 1, 2)
86    hold on
87    plot(Y_PCA(1,:),'r.','MarkerSize', 10);
88    plot(Y_PCA(2,:),'b.','MarkerSize', 10);
89    hold off
90
91    title("PCA Coordinates");
92    h = legend ("Unfiltered PCA Position","Unfiltered PCA Velocity");
93    xlabel ("Frame Number");
94    ylabel ("Pixel Position");
95
96  nfigure++;
97
98  %Comparing with a true sine curve
99  t = linspace(0,202,203);
100 y_sin = 90*sin(2*pi/40*t-pi/2-0.5);
101
102 figure(nfigure)
103
104   hold on
105   plot(Y_PCA(1,:),'r.','MarkerSize', 10);
106   plot(y_sin,'b','MarkerSize', 10);
107
108   %Filtering
109   windowSize = 5;
110   b = (1/windowSize)*ones(1,windowSize);
111   a = 1;
112
113   Y_PCA(1,:) = filter(b,a,Y_PCA(1,:));
114   Y_PCA(2,:) = filter(b,a,Y_PCA(2,:));
115
116   plot(Y_PCA(1,:),'g','MarkerSize', 10);
117   hold off
118
119   title("Comparison Between PCA Position, true sine curve and filtered PCA
            Position");
120   h = legend ("PCA Position","True sine curve", "Filtered PCA Position");
121   xlabel ("Frame Number");
122   ylabel ("Pixel Position");
```

```matlab
nfigure++;

%Plotting the filtered PCA Data
figure(nfigure);

  subplot (2, 1, 1)
  hold on
  plot(X_PCA(2,:),'r','MarkerSize', 10);
  plot(X_PCA(4,:),'g','MarkerSize', 10);
  plot(X_PCA(6,:),'b','MarkerSize', 10);
  hold off

  title("Original y Position");
  h = legend ("cam1","cam2","cam3");
  xlabel ("Frame Number");
  ylabel ("Pixel Position");

  subplot (2, 1, 2)
  hold on
  plot(Y_PCA(1,:),'r.','MarkerSize', 10);
  plot(Y_PCA(2,:),'b.','MarkerSize', 10);
  hold off

  title("PCA Coordinates");
  h = legend ("filtered PCA Position","filtered PCA Velocity");
  xlabel ("Frame Number");
  ylabel ("Pixel Position");


nfigure++;

save("Y_PCA.mat",'Y_PCA');


%--------- Applying SINDy -----------%
Framerate = 1;
dt = 1/Framerate;
N_Start = 28;
N_End = 153;
X_SINDy(1,:) = Y_PCA(1,N_Start:(N_End-1));
X_SINDy(2,:) = Y_PCA(2,N_Start:(N_End-1));

Y_SINDy_dxdt = (Y_PCA(:,N_Start+1:(N_End))-Y_PCA(:,N_Start:(N_End-1)))/dt;
    %derivative
Y_SINDy = Y_PCA(:,N_Start+1:(N_End)) - Y_PCA(:,N_Start).*ones([size(Y_PCA
    (:,N_Start+1:(N_End)))]);

dxdt_Max = max(Y_SINDy_dxdt(1,:));
scaling = dxdt_Max/max(X_SINDy(2,:))
X_SINDy(2,:) = Y_PCA(2,N_Start:(N_End-1))*scaling;

%Plotting the principal components and their derivatives
figure(nfigure);
```

```matlab
175
176     hold on
177     plot(X_SINDy(1,:),'r.','MarkerSize', 10);
178     plot(X_SINDy(2,:),'b.','MarkerSize', 10);
179     plot(Y_SINDy_dxdt(1,:),'g','MarkerSize', 10);
180     hold off
181
182     title("PCA Coordinates and derivative comparison");
183     h = legend ("Filtered PCA Position","Filtered & Scaled PCA Velocity","
           Derivative PCA Position");
184     xlabel ("Frame Number");
185     ylabel ("Pixel Position");
186
187  nfigure++;
188
189  figure(nfigure);
190
191     hold on
192     plot(X_SINDy(1,:),'r.','MarkerSize', 10);
193     plot(Y_SINDy_dxdt(1,:),'g.','MarkerSize', 10);
194     hold off
195
196     title("PCA Position and its derivative");
197     h = legend ("PCA Position","Derivative PCA Position");
198     xlabel ("Frame Number");
199     ylabel ("Pixel Position");
200
201  nfigure++;
202
203  figure(nfigure);
204
205     hold on
206     plot(X_SINDy(2,:),'b.','MarkerSize', 10);
207     plot(Y_SINDy_dxdt(2,:),'o.','MarkerSize', 10);
208     hold off
209
210     title("PCA Velocity and its derivative");
211     h = legend ("PCA Velocity","Derivative PCA Velocity");
212     xlabel ("Frame Number");
213     ylabel ("Pixel Position");
214
215  nfigure++;
216
217  % Defining Dictionary (1, x1, x2, x1^2, x1*x2, x2^2);
218  Theta = ones([1,length(X_SINDy)]);
219  Theta(2:3,:) = X_SINDy;
220  Theta(4,:) = X_SINDy(1,:).^2;
221  Theta(5,:) = X_SINDy(1,:).*X_SINDy(2,:);
222  Theta(6,:) = X_SINDy(2,:).^2;
223
224  %Applying integration method
225  ThetaInt = zeros([size(Theta)]);
226  ThetaInt(:,1) = dt*Theta(:,1);
227
```

```matlab
228   %reman sum
229   for i = 2:length(Theta(1,:))
230      ThetaInt(:,i) = ThetaInt(:,i-1)+dt*Theta(:,i);
231   end
232
233   coeff = Y_SINDy*pinv(ThetaInt);
234
235   %SINDy method begins %adapted from Jason Bramburger's SINDy example code:
         https://github.com/jbramburger/DataDrivenDynSyst/tree/main/Identifying
         %20Nonlinear%20Dynamics
236   lam = 0.1;
237   k = 1;
238
239   Coeff_New = coeff;
240   Err = sum(sum((abs(coeff - Coeff_New)))); %Coefficient checking
241   while k == 1 || Err > 0;
242
243      coeff = Coeff_New; %Save new coeff
244      smallinds = (abs(coeff)<lam);
245      Coeff_New(smallinds) = 0; %Set new coeff small indices to 0;
246
247      for i = 1:2
248         biginds = ~smallinds(i,:);
249         Coeff_New(i,biginds) = Y_SINDy(i,:)*pinv(ThetaInt(biginds,:)); %Find
                new coeff for big indices;
250      end
251
252      k = k + 1;
253      Err = sum(sum((abs(coeff - Coeff_New))));
254   end
255
256   %Display results
257   fprintf('Expected Output: \n')
258   fprintf('z1_dot = z2 \n')
259   fprintf('z2_dot = -Az2 - Bz1 \n')
260   fprintf('B is K/m (2pi/T)^2 should be around 0.025\n')
261
262   mons2 = {''; 'z1'; 'z2'; 'z1^2'; 'z1z2'; 'z2^2'};
263
264   fprintf('\nSINDy Output with lam = %d', lam)
265   fprintf(' and dt = %d:\n', dt)
266   fprintf('z1_dot = ')
267   for i = 1:length(Coeff_New)
268
269      if Coeff_New(1,i) < -1e-5;
270         mons2_get = mons2{i};
271         fprintf(" - %d %s", abs(Coeff_New(1,i)),mons2_get);
272      elseif Coeff_New(1,i) > 1e-5
273         mons2_get = mons2{i};
274         fprintf(" + %d %s", abs(Coeff_New(1,i)),mons2_get);
275      end
276
277   end
278   fprintf('\n')
```

```matlab
279
280
281  fprintf('z2_dot = ')
282  for i = 1:length(Coeff_New)
283
284    if Coeff_New(2,i) < -1e-5
285      mons2_get = mons2{i};
286      fprintf(" - %d %s", abs(Coeff_New(2,i)),mons2_get);
287    elseif Coeff_New(2,i) > 1e-5
288      mons2_get = mons2{i};
289      fprintf(" + %d %s", abs(Coeff_New(2,i)),mons2_get);
290    end
291
292  end
293  fprintf('\n')
294
295  %Simulate results and plot
296  fprintf('\nSimulating SINDy discovered model...\n')
297  [t_sim,Y_PCA_Sim] = RK4_Sys_Inputs(@f, 0, [85,0], 126, 125, 0);
298  figure(nfigure);
299
300  hold on
301  plot(Y_PCA(1,N_Start:(N_End-1)),'r.','MarkerSize', 10);
302  plot(Y_PCA_Sim(:,1),'b','MarkerSize', 10);
303  hold off
304
305  title("PCA Position and Simulated SINDy model using RK4 for comparison");
306  h = legend ("Filtered PCA Position","RK4 Simulated SINDy Model");
307  xlabel ("Frame Number");
308  ylabel ("Pixel Position");
309
310  nfigure++;
```

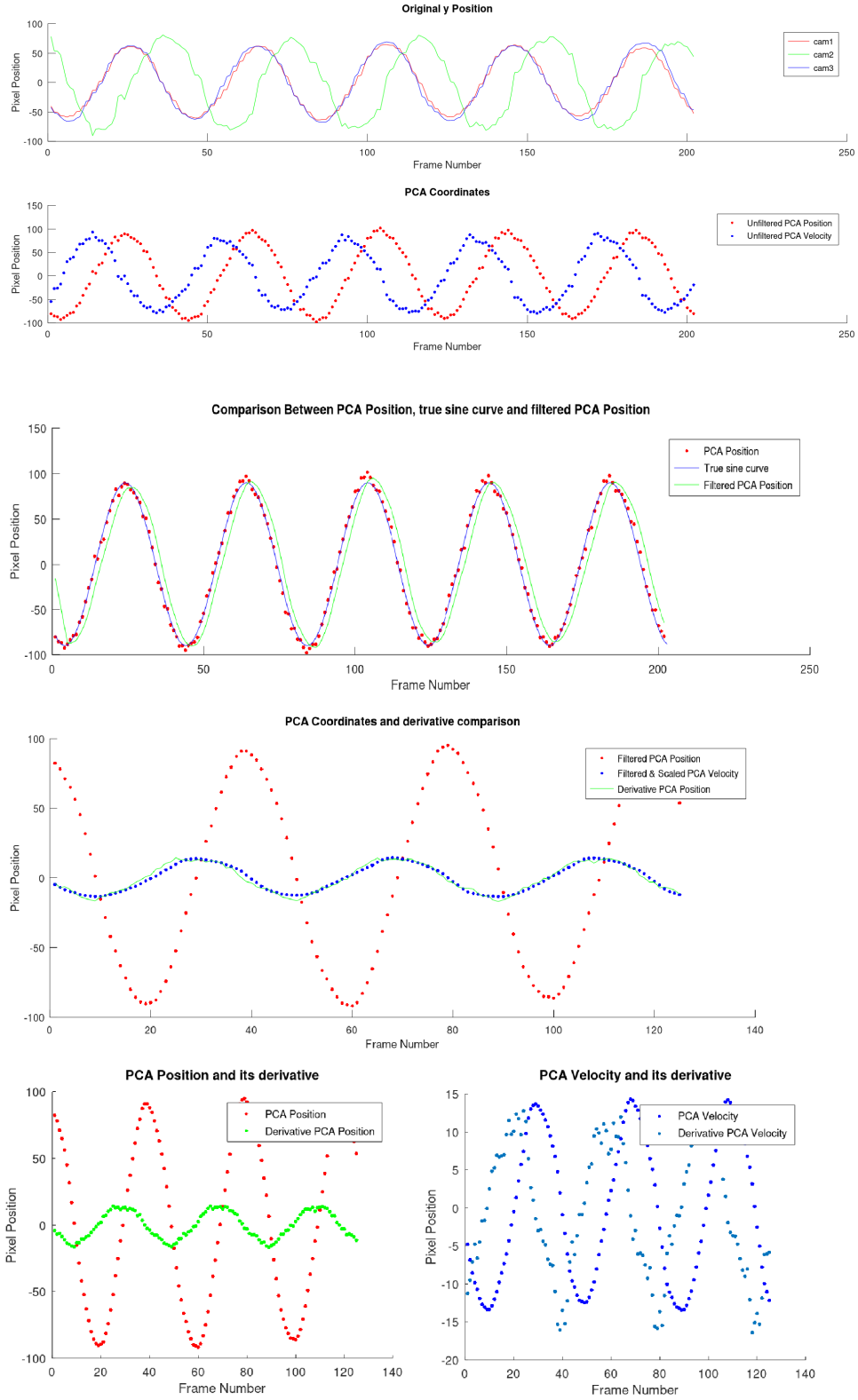## Appendix C   Complementary Figures

Figure 3: From top to bottom: plot of the original mean subtracted y positions of three different camera angles, plot of the resulting PCA component variables; plot comparing the filtered PCA positions with the true sine curves and the raw PCA positions; plot of the scaled velocity principal component with the true velocity superimposed, plot of the time derivatives of the PCA component variables.

19