Lecture 3: September 12

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3.1 Variance bound for binary classification with constant noise and threshold functions

Let $\mathcal{X} = [0,1]$, $\mathcal{Y} = \{0,1\}$, $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ and for $w \in [0,1]$ let $\ell_w : \mathcal{Z} \to \{0,1\}$ be defined by

$$\ell_w(x,y) = \mathbb{I}\{\mathbb{I}\{x \ge w\} \ne y\}\,.$$

Let

$$o_w(x,y) = \mathbb{I}\{x \notin [w-\varepsilon, w]\}\,.$$

Let P be a distribution over $\mathcal{Z}$. As usual, Pf stands for $\int f dP$. For some $w^* \in [0,1]$, we let P_{w^*} be a distribution over $\mathcal{Z}$ such that if $(X,Y) \sim P_{w^*}$ then X is uniformly distributed over $[0,1]$ and $\mathbb{P}(Y=1|X=x) = 1-p$ if $x \ge w^*$ and $\mathbb{P}(Y=1|X=x) = p$ otherwise.

We consider functions of the form $g_{w,w^*} = \ell_w o_w - \ell_{w^*}(1-\varepsilon)$. Note that from this choice, it follows that

$$P_{w^*} g_{w,w^*} \ge 0\,.$$

Indeed, writing $P = P_{w^*}$ to minimize clutter, on the one hand we have

$$P\ell_{w^*} = p\,,$$

and also

$$\min_{w\in[0,1]} P\ell_w o_w \ge \min_{w'\in[0,1]} \min_{w\in[0,1]} P\ell_w o_{w'} = \min_{w'\in[0,1]} P\ell_{w^*} o_{w'} = P\ell_{w^*} o_\varepsilon = p - p\varepsilon = p(1-\varepsilon)\,.$$

Hence,

$$Pg_{w,w^*} = P\ell_w o - (1-\varepsilon)P\ell_{w^*} \ge p(1-\varepsilon) - p(1-\varepsilon) = 0\,.$$

Our next goal is to find the "smallest possible" $c_0, c_1 \ge 0$ such that for any w, w^*, ε, p,

$$P_{w^*} g^2_{w,w^*} \le c_0^2 + c_1 P_{w^*} g_{w,w^*} \tag{3.1}$$

holds. In fact, we want to minimize first c_0 and keep c_1 bounded.

To minimize clutter, we let $P = P_{w^*}$, $o = o_w$, but these dependencies should still be kept in mind. First, algebra gives

$$\begin{aligned} Pg^2_{w,w^*} &= P\ell_w o - 2(1-\varepsilon)P\ell_w \ell_{w^*} o + (1-\varepsilon)^2 P\ell_{w^*} \\ &= P\ell_w o - 2(1-\varepsilon)P\ell_w \ell_{w^*} o + (1-\varepsilon)^2 p\,. \end{aligned}$$

Furthermore,

$$Pg = P\ell_w o - (1-\varepsilon)P\ell_{w^*} = P\ell_w o - (1-\varepsilon)p\,.$$

Hence, Eq. (3.1) is equivalent to

$$P\ell_w o - 2(1-\varepsilon)P\ell_w\ell_{w^*}o + (1-\varepsilon)^2 p \le c_0^2 + c_1(P\ell_w o - (1-\varepsilon)p)\,,$$

which is equivalent to

$$c_1(1-\varepsilon)p + (1-\varepsilon)^2 p \le c_0^2 + (c_1-1)P\ell_w o + 2(1-\varepsilon)P\ell_w\ell_{w^*}o\,. \tag{3.2}$$

Now, we consider three cases based on the location of w^* relative to $w-\varepsilon$ and w and we will in each case lower bound the value of

$$A = (c_1-1)P\ell_w o + 2(1-\varepsilon)P\ell_w\ell_{w^*}o\,,$$

while enforcing that

$$c_1 \ge 1\,. \tag{3.3}$$

Case 1: $w^* \le w-\varepsilon$. In this case we have

$$\begin{aligned} P\ell_w o &= (w-w^*)(1-2p) + p - \varepsilon(1-p)\,, \\ P\ell_w\ell_{w^*}o &= w^*(1-p) - wp + p\,. \end{aligned}$$

Then,

$$A = (c_1-1)\left\{(w-w^*)(1-2p) + p - \varepsilon(1-p)\right\} + 2(1-\varepsilon)\left\{w^*(1-p) - wp + p\right\}\,.$$

The coefficient of w^* in the above expression is

$$\alpha = (c_1-1)(2p-1) + 2(1-\varepsilon)(1-p)\,.$$

If $\alpha \le 0$, the value of A is minimized in w^* by choosing $w^* = w-\varepsilon$ and the minimum value is equal to

$$\begin{aligned} A' &= (c_1-1)\left\{\varepsilon(1-2p) + p - \varepsilon(1-p)\right\} + 2(1-\varepsilon)\left\{(w-\varepsilon)(1-p) - wp + p\right\} \\ &= (c_1-1)p(1-\varepsilon) + 2(1-\varepsilon)\left\{(w(1-2p) - \varepsilon(1-p) + p\right\} \\ &\ge (c_1-1)p(1-\varepsilon) + 2(1-\varepsilon)\left\{p - \varepsilon(1-p)\right\} =: A''\,, \end{aligned}$$

where the last inequality follows since $2(1-\varepsilon)w(1-2p) \ge 0$. Now, Eq. (3.2) will follow if $c_1(1-\varepsilon)p + (1-\varepsilon)^2 p \le c_0^2 + A''$. Plugging in the definition of A'', we get that the latter inequality is equivalent to

$$\begin{aligned} \cancel{c_1(1-\varepsilon)p} + (1-\varepsilon)^2 p &\le c_0^2 + (\cancel{c_1} - 1)p(1-\varepsilon) + 2(1-\varepsilon)\left\{p - \varepsilon(1-p)\right\} \\ &= c_0^2 - \cancel{p(1-\varepsilon)} + \cancel{2}(1-\varepsilon)p - 2(1-\varepsilon)\varepsilon(1-p)\,. \end{aligned}$$

Reordering gives

$$2(1-\varepsilon)\varepsilon(1-p) - \varepsilon p(1-\varepsilon) \le c_0^2$$

which gives

$$c_0^2 \ge (1-\varepsilon)\varepsilon(2-3p)\,. \tag{3.4}$$

Since the $p < 1/2$ by assumption, this constraint can be satisfied.

Finally, simple algebra shows that $\alpha \le 0$ is equivalent to

$$c_1 \ge 1 + \frac{(1-\varepsilon)(1-p)}{\frac{1}{2} - p}\,, \tag{3.5}$$

where we used that $p < 1/2$.

Case 2: $w - \varepsilon < w^* \le w$. In this case we have

$$\begin{aligned} P\ell_w o &= p(1-\varepsilon)\,, \\ P\ell_w \ell_{w^*} o &= p(1-\varepsilon)\,. \end{aligned}$$

Then,

$$A = (c_1 - 1)p(1-\varepsilon) + 2p(1-\varepsilon)^2\,. \tag{3.6}$$

Now, Eq. (3.2) will follow if $c_1(1-\varepsilon)p + (1-\varepsilon)^2 p \le c_0^2 + A$. Plugging in the definition of A, we get that the latter inequality is equivalent to

$$\cancel{c_1(1-\varepsilon)p} + \cancel{(1-\varepsilon)^2 p} \le c_0^2 + (\cancel{c_1} - 1)p(1-\varepsilon) + \cancel{2}p(1-\varepsilon)^2\,,$$

which gives

$$c_0^2 \ge \varepsilon p(1-\varepsilon)\,. \tag{3.7}$$

Case 3: $w < w^*$. In this case we have

$$\begin{aligned} P\ell_w o &= (w - w^*)(2p-1) + p(1-\varepsilon)\,, \\ P\ell_w \ell_{w^*} o &= (w - w^*)p + p(1-\varepsilon)\,. \end{aligned}$$

Then,

$$A = (c_1 - 1)\left\{(w - w^*)(2p-1) + p(1-\varepsilon)\right\} + 2(1-\varepsilon)\left\{(w-w^*)p + p(1-\varepsilon)\right\}\,.$$

Note that here $w - w^* \le 0$. The coefficient of $w - w^*$ in this expression is

$$\alpha = (c_1 - 1)(2p-1) + 2(1-\varepsilon)p.$$

When $\alpha \le 0$, the value of A is minimized by choosing the largest possible value for $w - w^*$, which is zero. In this case the value of A is

$$A' = (c_1 - 1)p(1-\varepsilon) + 2(1-\varepsilon)^2 p\,,$$

which is the value we got in the previous case (cf. Eq. (3.6)). As such, as long as Eq. (3.7) holds, Eq. (3.1) will be satisfied.

To guarantee that $\alpha \le 0$, we need to have

$$(c_1 - 1)(2p-1) + 2(1-\varepsilon)p \le 0\,.$$

Thanks to $2p - 1 < 0$, this holds if and only if

$$c_1 \ge 1 + \frac{(1-\varepsilon)p}{\frac{1}{2} - p}\,. \tag{3.8}$$

Thus, we derived the following:

Proposition 3.1. *Let $p < 1/2$,*

$$c_1 = 1 + \frac{(1-\varepsilon)(1-p)}{\frac{1}{2} - p}\,, \qquad c_0^2 = \varepsilon(1-\varepsilon)\max(2-3p, 1)\,.$$

Then, for any w, w^* it holds that

$$P_{w^*} g_{w,w^*}^2 \le c_0^2 + c_1 P_{w^*} g_{w,w^*}\,.$$

Proof. Algebra, together with Eqs. (3.5) and (3.8), Eqs. (3.4) and (3.7) gives the result. $\square$