Finite Time Analysis of Linear Two-timescale Stochastic Approximation with Markovian Noise

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Abstract

Linear two-timescale stochastic approximation (SA) scheme is an important class of algorithms which has become popular in reinforcement learning (RL), particularly for the policy evaluation problem. Recently, a number of works have been devoted to establishing the finite time analysis of the scheme, especially under the Markovian (non-i.i.d.) noise settings that are ubiquitous in practice. In this paper, we provide a finite-time analysis for linear two timescale SA. Our bounds show that there is no discrepancy in the convergence rate between Markovian and martingale noise, only the constants are affected by the mixing time of the Markov chain. With an appropriate step size schedule, the transient term in the expected error bound is $o(1/k^c)$ and the steady-state term is $\mathcal{O}(1/k)$, where $c > 1$ and k is the iteration number. Furthermore, we present an asymptotic expansion of the expected error with a matching lower bound of $\Omega(1/k)$. A simple numerical experiment is presented to support our theory.

Keywords: stochastic approximation, reinforcement learning, GTD learning, Markovian noise

1. Introduction

Since its introduction close to 70 years ago, the stochastic approximation (SA) scheme (Robbins and Monro, 1951) has been a powerful tool for root finding when only noisy samples are available. During the past two decades, considerable progresses in the practical and theoretical research of SA have been made, see (Benaïm, 1999; Kushner and Yin, 2003; Borkar, 2008) for an overview. Among others, linear SA schemes are popular in reinforcement learning (RL) as they lead to policy evaluation methods with linear function approximation, of particular importance is temporal difference (TD) learning (Sutton, 1988) for which finite time analysis has been reported in (Srikant and Ying, 2019; Lakshminarayanan and Szepesvari, 2018; Bhandari et al., 2018; Dalal et al., 2018a).

The TD learning scheme based on classical (linear) SA is known to be inadequate for the off-policy learning paradigms in RL, where data samples are drawn from a *behavior policy* different

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from the policy being evaluated (Baird, 1995; Tsitsiklis and Van Roy, 1997). To circumvent this problem, Sutton et al. (2009a,b) have suggested to replace TD learning with the gradient TD (GTD) method or the TD with gradient correction (TDC) method. These methods fall within the scope of linear two-timescale SA scheme introduced by Borkar (1997):

$$\theta_{k+1} = \theta_k + \beta_k\{\widetilde{b}_1(X_{k+1}) - \widetilde{A}_{11}(X_{k+1})\theta_k - \widetilde{A}_{12}(X_{k+1})w_k\}, \tag{1}$$

$$w_{k+1} = w_k + \gamma_k\{\widetilde{b}_2(X_{k+1}) - \widetilde{A}_{21}(X_{k+1})\theta_k - \widetilde{A}_{22}(X_{k+1})w_k\}. \tag{2}$$

The above recursion involves two iterates, $\theta_k \in \mathbb{R}^{d_\theta}$, $w_k \in \mathbb{R}^{d_w}$, whose updates are coupled with each other. In the above, $\widetilde{b}_i(x)$, $\widetilde{A}_{ij}(x)$ are measurable vector/matrix valued functions on X and the random sequence $(X_k)_{k\geq0}, X_k \in X$ forms an ergodic Markov chain. The scalars $\gamma_k, \beta_k > 0$ are step sizes. The above SA scheme is said to have two timescales as the step sizes satisfy $\lim_{k\to\infty} \beta_k/\gamma_k < 1$ such that w_k is updated at a faster timescale. In fact, w_k is a 'tracking' term which seeks solution to a linear system characterized by θ_k.

The goal of this paper is to characterize the finite time expected error bound with improved convergence rate for the two timescale SA (1),(2). The almost sure convergence of two timescale SA have been established in (Borkar, 1997; Tadic, 2004, 2006; Borkar, 2008), among others; the asymptotic convergence rates have been characterized in (Konda and Tsitsiklis, 2004; Mokkadem et al., 2006). However, finite-time risk bounds for two timescale SA have not been analyzed until recently. With martingale samples, Liu et al. (2015) provided the first finite time analysis of GTD method, Dalal et al. (2018b, 2019) provided improved finite time error bounds. Unlike our analysis, they analyzed modified two timescale SA with projection and their bounds hold with high probability. With Markovian noise, Gupta et al. (2019) studied the finite time expected error bound with constant step sizes; Xu et al. (2019) and Doan (2019) provided similar analysis for general step sizes. It is important to notice that with homogeneous martingale noise, the asymptotic rate of (1), (2) without a projection step, as shown in (Konda and Tsitsiklis, 2004, Theorem 2.6), is in the order $\mathbb{E}[\|\theta_k - \theta^\star\|^2] = \mathcal{O}(\beta_k), \mathbb{E}[\|w_k - A_{22}^{-1}(b_2 - A_{21}\theta_k)\|^2] = \mathcal{O}(\gamma_k)$, where $\theta^\star$ is a stationary point of the SA scheme. However, the latter rate is not achieved in the finite-time error bounds analyzed by the above works except for (Dalal et al., 2019). It remains an open problem whether this error bound holds for the Markovian noise setting and for linear two time-scale SA scheme without projection.

Contributions This paper has the following contributions:

- *Improved Convergence Rate* – We perform finite-time expected error bound analysis of the linear two timescale SA in both martingale and Markovian noise settings, in Theorems 1 & 2. Our analysis allow for general step sizes schedules [cf. A2, B4], including constant, piecewise constant, and diminishing step sizes explored in the prior works (Gupta et al., 2019; Dalal et al., 2019; Xu et al., 2019; Doan, 2019). We show that the error bound consists of a transient and a steady-state term, and the asymptotic rate is obtained from the latter. We show that this asymptotic rate matches those in (Konda and Tsitsiklis, 2004, Theorem 2.6), i.e., $\mathbb{E}[\|\theta_k - \theta^\star\|^2] = \mathcal{O}(\beta_k), \mathbb{E}[\|w_k - A_{22}^{-1}(b_2 - A_{21}\theta_k)\|^2] = \mathcal{O}(\gamma_k)$. In particular, the fastest achievable rate for $\mathbb{E}[\|\theta_k - \theta^\star\|^2]$ will be $\mathcal{O}(1/k)$ when we set $\beta_k = \mathcal{O}(1/k), \gamma_k = \mathcal{O}(1/k^\upsilon)$ with $\upsilon < 1$.

- *Novel Analysis without A-prori Stability Assumption* – Unlike the prior works (Liu et al., 2015; Dalal et al., 2019; Xu et al., 2019), our convergence results are obtained *without* requiring a projection step throughout the SA iterations. In fact, Dalal et al. (2019) have pointed out that the projection step is merely included to ensure *a-priori* stability of the algorithm, and is often

not used in practice. Our relaxation and the ability to achieve the optimal convergence rate are obtained through a tight analysis of the recursive inequalities of the (cross-)variances of θ_k, w_k, see Section 3.

- *Asymptotic Expansion* – With an additional assumption on the step size, we compute an exact asymptotic expansion of the expected error $\mathbb{E}[\|\theta_k - \theta^\star\|^2]$, see Theorem 10. With an appropriate diminishing step sizes schedule, we show that the expected error cannot be smaller than $\Omega(\beta_k)$, which matches our upper bound results in Theorem 1 & 2.

The rest of this paper is organized as follows. In Section 2, we present the detailed conditions for two timescale linear SA, and the main results on finite-time performance bounds. In Section 3, we provide an outline of the proof, illustrating the insights behind the main steps. In Section 4, we show that the finite-time error bounds are tight by quantifying an exact expansion of the covariance of iterates. In Section 5, we illustrate the theoretical findings using numerical experiments.

Notations   Let $n \in \mathbb{N}$ and Q be a symmetric definite $n \times n$ matrix. For $x \in \mathbb{R}^n$, we denote $\|x\|_Q = \{x^\top Q x\}^{1/2}$. For brevity, we set $\|x\| = \|x\|_I$. Let $m \in \mathbb{N}$, P be a symmetric definite $m \times m$ matrix, A be an $n \times m$ matrix. A matrix A is said to be Hurwitz if the real parts of its eigenvalues are strictly negative. We denote $\|A\|_{P,Q} = \max_{\|x\|_P=1} \|Ax\|_Q$. If A is a $n \times n$ matrix, we denote $\|A\|_Q = \|A\|_{Q,Q}$. Lastly, we give a number of auxiliary lemmas in Appendix D that are instrumental to our analysis.

2. Linear Two Time-scale Stochastic Approximation (SA) Scheme

We investigate the linear two timescale SA given by the following equivalent form of (1), (2):

$$\theta_{k+1} = \theta_k + \beta_k(b_1 - A_{11}\theta_k - A_{12}w_k + V_{k+1}), \tag{3}$$

$$w_{k+1} = w_k + \gamma_k(b_2 - A_{21}\theta_k - A_{22}w_k + W_{k+1}), \tag{4}$$

where the mean fields are defined as $b_i := \lim_{k\to\infty} \mathbb{E}[\widetilde{b}_i(X_k)]$, $A_{ij} := \lim_{k\to\infty} \mathbb{E}[\widetilde{A}_{ij}(X_k)]$ (these limits exist as we recall that $(X_k)_{k\geq 0}$ is an ergodic Markov chain). The noise terms V_{k+1}, W_{k+1} are given by:

$$\begin{aligned} V_{k+1} &:= \widetilde{b}_1(X_{k+1}) - b_1 - (\widetilde{A}_{11}(X_{k+1}) - A_{11})\theta_k - (\widetilde{A}_{12}(X_{k+1}) - A_{12})w_k, \\ W_{k+1} &:= \widetilde{b}_2(X_{k+1}) - b_2 - (\widetilde{A}_{21}(X_{k+1}) - A_{21})\theta_k - (\widetilde{A}_{22}(X_{k+1}) - A_{22})w_k. \end{aligned} \tag{5}$$

The goal of the recursion (3), (4) is to find a stationary solution pair $(\theta^\star, w^\star)$ that solves the system of linear equations:

$$A_{11}\theta + A_{12}w = b_1, \quad A_{21}\theta + A_{22}w = b_2. \tag{6}$$

We are interested in the scenario when the solution pair $(\theta^\star, w^\star)$ is unique and is given by

$$\theta^\star = \Delta^{-1}(b_1 - A_{12}A_{22}^{-1}b_2), \quad w^\star = A_{22}^{-1}(b_2 - A_{21}\theta^\star). \tag{7}$$

where $\Delta := A_{11} - A_{12}A_{22}^{-1}A_{21}$. To analyze the convergence of $(\theta_k, w_k)_{k\geq 0}$ in (3), (4) to $(\theta^\star, w^\star)$, we require the following assumptions:

A1 *Matrices $-A_{22}$ and $-\Delta = -\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)$ are Hurwitz.*

The above assumption is common for linear two time-scale SA, see (Konda and Tsitsiklis, 2004). As a consequence, using the Lyapunov lemma (stated in Lemma 16 in the appendix for completeness), there exist positive definite matrices $Q_{22}^\top = Q_{22} \succ 0, Q_\Delta^\top = Q_\Delta \succ 0$ satisfying

$$A_{22}^\top Q_{22} + Q_{22} A_{22} = \mathrm{I}, \quad Q_\Delta \Delta + \Delta^\top Q_\Delta = \mathrm{I}. \qquad (8)$$

This ensures the contraction (see Lemma 17 in the appendix):

$$\|\mathrm{I} - \gamma_k A_{22}\|_{Q_{22}} \le 1 - a_{22}\gamma_k, \quad \|\mathrm{I} - \beta_k \Delta\|_{Q_\Delta} \le 1 - a_\Delta \beta_k, \qquad (9)$$

provided that $\gamma_k \in [0, 1/(2\|A_{22}\|_{Q_{22}}^2 \|Q_{22}\|)], \beta_k \in [0, 1/(2\|A_\Delta\|_{Q_\Delta}^2 \|Q_\Delta\|)]$. Moreover, we have set $a_{22} := 1/(4\|Q_{22}\|), a_\Delta := 1/(4\|Q_\Delta\|)$. We consider the following conditions on the step sizes:

A2 $(\gamma_k)_{k\ge 0}$, $(\beta_k)_{k\ge 0}$ *are nonincreasing sequences of positive numbers that satisfy the following.*

1. There exist constants κ such that for all $k \in \mathbb{N}$, we have $\beta_k/\gamma_k \le \kappa$.

2. For all $k \in \mathbb{N}$, it holds

$$\gamma_k/\gamma_{k+1} \le 1 + (a_{22}/8)\gamma_{k+1}, \quad \beta_k/\beta_{k+1} \le 1 + (a_\Delta/16)\beta_{k+1}, \quad \gamma_k/\gamma_{k+1} \le 1 + (a_\Delta/16)\beta_{k+1}.$$

As a consequence, we can define $\varsigma := 1 + \{\gamma_0 a_{22}/8 \vee \beta_0 a_\Delta/16\}$ such that $\gamma_k/\gamma_{k+1} \le \varsigma$, $\beta_k/\beta_{k+1} \le \varsigma$. Our conditions on step sizes are similar to (Konda and Tsitsiklis, 2004, Assumption 2.3, 2.5). These conditions encompass diminishing, piecewise constant and constant step sizes schedules which are common in the literature. For instance, a popular choice of diminishing step sizes satisfying A2 is

$$\beta_k = c^\beta/(k + k_0^\beta), \quad \gamma_k = c^\gamma/(k + k_0^\gamma)^{2/3} \qquad (10)$$

with some constants c^β, c^γ, k_0^γ, k_0^β, e.g., as suggested in (Dalal et al., 2018b, Remark 9); or a constant step size of $\beta_k = \beta, \gamma_k = \gamma$; or a piecewise constant step size, e.g., (Gupta et al., 2019).

We present new results on the convergence rate of (3), (4) depending on the types of noise with V_{k+1}, W_{k+1}. To discuss these cases, let us define the σ-field generated by the two timescale SA scheme and the initial error made by the SA scheme, respectively as:

$$\mathcal{F}_k := \sigma\{\theta_0, w_0, X_1, X_2, ..., X_k\}, \quad \mathrm{V}_0 := \mathbb{E}[\|\theta^0 - \theta^\star\|^2 + \|w^0 - w^\star\|^2]. \qquad (11)$$

Our main results are presented as follows.

Martingale Noise We consider a simple setting where the random elements X_k are drawn i.i.d. from the stationary distribution such that b_i, A_{ij} are the expected values of $\widetilde{b}_i(X_k), \widetilde{A}_{ij}(X_k)$. Furthermore, the random variables $\widetilde{b}_i(X_k), \widetilde{A}_{ij}(X_k)$ have bounded second order moment. Note that this implies $\mathbb{E}^{\mathcal{F}_k}[V_{k+1}] = \mathbb{E}^{\mathcal{F}_k}[W_{k+1}] = 0$, i.e., the sequences $(V_{k+1})_{k\in\mathbb{N}}, (W_{k+1})_{k\in\mathbb{N}}$ are martingale difference sequences. Formally, we describe this setting as the following conditions on V_{k+1}, W_{k+1}:

A3 *The noise terms are zero-mean conditioned on $\mathcal{F}_k$, i.e., $\mathbb{E}^{\mathcal{F}_k}[V_{k+1}] = \mathbb{E}^{\mathcal{F}_k}[W_{k+1}] = 0$.*

A4 *There exist constants m_W, m_V such that*

$$\|\mathbb{E}[V_{k+1}V_{k+1}^\top]\| \le m_V(1 + \|\mathbb{E}[\theta_k\theta_k^\top]\| + \|\mathbb{E}[w_k w_k^\top]\|),$$
$$\|\mathbb{E}[W_{k+1}W_{k+1}^\top]\| \le m_W(1 + \|\mathbb{E}[\theta_k\theta_k^\top]\| + \|\mathbb{E}[w_k w_k^\top]\|).$$

Theorem 1 *Assume A1–4 and for all $k \in \mathbb{N}$, we have $\gamma_k \in [0, \gamma_\infty^{\mathtt{mtg}}]$, $\beta_k \in [0, \beta_\infty^{\mathtt{mtg}}]$ and $\kappa \in [0, \kappa_\infty]$, where $\gamma_\infty^{\mathtt{mtg}}, \beta_\infty^{\mathtt{mtg}}, \kappa_\infty$ are constants defined in (20), (16). Then*

$$\mathbb{E}[\|\theta_k - \theta^*\|^2] \le d_\theta \left\{ \mathrm{C}_0^{\tilde{\theta},\mathtt{mtg}} \prod_{\ell=0}^{k-1} \left(1 - \beta_\ell \frac{a_\Delta}{4}\right) \mathrm{V}_0 + \mathrm{C}_1^{\tilde{\theta},\mathtt{mtg}} \beta_k \right\} \tag{12}$$

$$\mathbb{E}[\|w_k - A_{22}^{-1}(b_2 - A_{21}\theta_k)\|^2] \le d_w \left\{ \mathrm{C}_0^{\widehat{w},\mathtt{mtg}} \prod_{\ell=0}^{k-1} \left(1 - \beta_\ell \frac{a_\Delta}{4}\right) \mathrm{V}_0 + \mathrm{C}_1^{\widehat{w},\mathtt{mtg}} \gamma_k \right\} \tag{13}$$

The exact constants are provided in the appendix, see (55), (58).

Markovian Noise   Consider the sequence $(X_k)_{k\ge 0}$ to be samples from an exogenous Markov chain on X with the transition kernel $\mathrm{P} : \mathsf{X} \times \mathsf{X} \to \mathbb{R}_+$. For any measurable function f, we have

$$\mathbb{E}^{\mathcal{F}_k}[f(X_{k+1})] = \mathrm{P}\, f(X_k) = \int_{\mathsf{X}} f(x)\, \mathrm{P}(X_k, \mathrm{d}x)$$

We state the following assumptions:

B1   *The Markov kernel P has a unique invariant distribution $\mu : \mathsf{X} \to \mathbb{R}_+$. Moreover, it is irreducible and aperiodic.*

Observe that

$$b_i = \int_{\mathsf{X}} \widetilde{b}_i(x)\,\mu(\mathrm{d}x), \quad A_{ij} = \int_{\mathsf{X}} \widetilde{A}_{ij}(x)\,\mu(\mathrm{d}x),\ i,j = 1,2.$$

We show that the linear two time-scale SA (1), (2) converges to a unique fixed point defined by the above mean field vectors/matrices, see (7). An important condition that enables our analysis is the existence of solutions to the following Poisson equations:

B2   *For any $i,j = 1,2$, consider $\widetilde{b}_i(x), \widetilde{A}_{ij}(x)$, there exists vector/matrix valued measurable functions $\widehat{b}_i(x), \widehat{A}_{ij}(x)$ which satisfy*

$$\widetilde{b}_i(x) - b_i = \widehat{b}_i(x) - \mathrm{P}\,\widehat{b}_i(x), \quad \widetilde{A}_{ij}(x) - A_{ij} = \widehat{A}_{ij}(x) - \mathrm{P}\,\widehat{A}_{ij}(x)$$

for any $x \in \mathsf{X}$ and b_i, A_{ij} are the mean fields of $\widetilde{b}_i(x), \widetilde{A}_{ij}(x)$ with the stationary distribution μ.

The above assumption can be guaranteed under B1 together with some regularity conditions, see (**?**, Section 21.2). Moreover,

B3   *Under B2, the vector/matrix valued functions $\widehat{b}_i(x), \widehat{A}_{ij}(x)$ are uniformly bounded: for any $i,j = 1,2$, $x \in \mathsf{X}$,*

$$\|\widehat{b}_i(x)\| \le \overline{\mathrm{b}},\ \|\widehat{A}_{ij}(x)\| \le \overline{\mathrm{A}}.$$

B4   *There exists constant ρ_0 such that for any $k \ge 1$, we have $\gamma_{k-1}^2 \le \rho_0 \beta_k$.*

To satisfy B3, we observe that the bounds $\overline{\mathrm{b}}, \overline{\mathrm{A}}$ depend on the mixing time of the chain $(X_k)_{k\ge 0}$ and a uniform bound on $\widetilde{b}_i(\cdot), \widetilde{A}_{ij}(\cdot)$. In the context of reinforcement learning, the latter can be satisfied when the feature vectors and reward are bounded. Note that B3 implies A4, see Section 3.2. Meanwhile, B4 imposes further restriction on the step size. The latter can also be satisfied by (10).

The challenges of analysis with Markovian noise lie in the biasedness of the noise term as $\mathbb{E}^{\mathcal{F}_k}[V_{k+1}] \ne 0$, $\mathbb{E}^{\mathcal{F}_k}[W_{k+1}] \ne 0$. With a careful analysis, we obtain:

Theorem 2 *Assume A1–2, B1–4 hold and for all $k \in \mathbb{N}$, we have $\beta_k \in (0, \beta_\infty^{\mathsf{mark}}], \gamma_k \in (0, \gamma_\infty^{\mathsf{mark}}], \kappa \leq \kappa_\infty$, where $\beta_\infty^{\mathsf{mark}}, \gamma_\infty^{\mathsf{mark}}, \kappa_\infty$ are defined in (27), (16). Then*

$$\mathbb{E}[\|\theta_k - \theta^\star\|^2] \leq d_\theta \left\{ \mathrm{C}_0^{\tilde{\theta},\mathsf{mark}} \prod_{\ell=0}^{k-1} \left(1 - \beta_\ell \frac{a_\Delta}{8}\right)(1 + \mathrm{V}_0) + \mathrm{C}_1^{\tilde{\theta},\mathsf{mark}} \beta_k \right\}, \tag{14}$$

$$\mathbb{E}[\|w_k - A_{22}^{-1}(b_2 - A_{21}\theta_k)\|^2] \leq d_w \left\{ \mathrm{C}_0^{\widehat{w},\mathsf{mark}} \prod_{\ell=0}^{k-1} \left(1 - \beta_\ell \frac{a_\Delta}{8}\right)(1 + \mathrm{V}_0) + \mathrm{C}_1^{\widehat{w},\mathsf{mark}} \gamma_k \right\}. \tag{15}$$

The exact constants are given in the appendix, see (78), (80).

While Theorem 2 relaxes the martingale difference assumption A4 in Theorem 1, we remark that the results here do not generalize that in Theorem 1 due to the additional B3, B4. Particularly, with martingale noise, the convergence of linear two timescale SA only requires the noise to have bounded *second order moment*, yet the Markovian noise needs to be uniformly bounded.

Convergence Rate of Linear Two Timescale SA   The upper bounds in Theorem 1 and 2 consist of two terms – the first term is a 'transient' error with product such as $\prod_{i=0}^{k-1}(1 - \beta_i a_\Delta/8)$ decays to zero at the rate $o(1/k^c)$ for some $c > 1$ under an appropriate choice of step sizes such as (10); the second term is a 'steady-state' error. We observe that the 'steady-state' error of the iterates θ_k, w_k exhibit different behaviors. Taking the step size choices in (10) as an example, the steady-state error of the slow-update iterates θ_k is $\mathcal{O}(1/k)$ while the error of fast-update iterates w_k is $\mathcal{O}(1/k^{\frac{2}{3}})$. Furthermore, similar bounds hold for *both* martingale and Markovian noise. In Section 4 we show that the obtained rates are also tight.

Comparison to Related Works   Our results improve the convergence rate analysis of linear two timescale SA in a number of recent works. In the martingale noise setting (Theorem 1), the closest work to ours is (Dalal et al., 2019) which analyzed the linear two timescale SA with martingale samples and diminishing step sizes. The authors improved on (Dalal et al., 2018b) and obtained the same convergence rate (in high probability) as our Theorem 1, furthermore it is demonstrated that the obtained rates are tight. Their bounds also exhibit a sublinear dependence on the dimensions d_θ, d_w. However, their algorithm involves a sparsely executed projection step and the error bound holds only for a sufficiently large k. These restrictions are lifted in our analysis.

In the Markovian noise setting (Theorem 2), the closest works to ours are (Doan, 2019; Gupta et al., 2019; Xu et al., 2019). In particular, Gupta et al. (2019) analyzed the linear two timescale SA with constant step sizes and showed that the steady-state error for both θ_k, w_k is $\mathcal{O}(\gamma^2/\beta)$. Xu et al. (2019) analyzed the TDC algorithm with a projection step and showed that the steady-state error for θ_k is $\mathcal{O}(1/k^{\frac{2}{3}})$ if the step sizes in (10) is used. Doan (2019) analyzed the linear two timescale SA with diminishing step size and showed that the steady state error for both θ_k, w_k is $\mathcal{O}(1/k^{\frac{2}{3}})$. Interestingly, the above works do not obtain the fast rate in Theorem 2, i.e., $\mathbb{E}[\|\theta_k - \theta^\star\|^2] = \mathcal{O}(1/k)$. One of the reasons for the sub-optimality in their rates is that their analysis are based on building a single Lyapunov function that controls both errors in θ_k and w_k. In contrast, our analysis relies on a set of coupled inequalities to obtain tight bounds for each of the iterates θ_k, w_k.

3. Convergence Analysis

While much of the technical details and the complete constants of non-asymptotic bounds will be postponed to the appendix, this section offers insights into our main theoretical results through

sketching the major steps involved in proving Theorem 1 & 2. Throughout, we shall consider the following bounds on the step sizes and step size ratio:

$$\beta_\infty^{(0)} := \frac{1}{2\|Q_\Delta\|\|\Delta\|_{Q_\Delta}^2} \wedge \frac{1}{2\|\Delta\|_{Q_\Delta} + a_\Delta}, \quad \gamma_\infty^{(0)} := \frac{1}{2\|Q_{22}\|\|A_{22}\|_{Q_{22}}^2},$$

$$\kappa_\infty := \left(\frac{a_{22}/2}{\|A_{12}\|_{Q_{22},Q_\Delta}\|A_{22}^{-1}A_{21}\|_{Q_\Delta,Q_{22}} + \frac{a_\Delta}{2}} \left\{ 1 \wedge \frac{a_\Delta/2}{\|\Delta\|_{Q_\Delta} + \frac{a_\Delta}{2}} \right\} \right) \wedge \frac{a_{22}}{4a_\Delta}. \quad (16)$$

To begin with, let us present the reformulation of the two time-scale SA scheme (3), (4) that is borrowed from (Konda and Tsitsiklis, 2004). Define:

$$L_{k+1} := \left(L_k - \gamma_k A_{22} L_k + \beta_k A_{22}^{-1} A_{21}(\Delta - A_{12}L_k) \right)\left(I - \beta_k(\Delta - A_{12}L_k) \right)^{-1}, \quad L_0 := 0,$$

and $L_\infty := a_\Delta/(2\|A_{12}\|_{Q_{22},Q_\Delta})$. As shown in Lemma 18 of the appendix, with the step sizes $\gamma_k \leq \gamma_\infty^{(0)}$, $\beta_k \leq \beta_\infty^{(0)}$, $\kappa \leq \kappa_\infty$, the above recursion on L_k is well defined where it holds that $\|L_k\|_{Q_\Delta,Q_{22}} \leq L_\infty$ for any $k \geq 0$. In addition, define the matrices:

$$B_{11}^k := \Delta - A_{12}L_k, \quad B_{22}^k := \frac{\beta_k}{\gamma_k}\left(L_{k+1} + A_{22}^{-1}A_{21} \right)A_{12} + A_{22}, \quad C_k := L_{k+1} + A_{22}^{-1}A_{21}.$$

In a similar vein as performing Gaussian elimination, we obtain a simplified two timescale SA recursions (proof in Appendix A):

Observation 1 *Consider the following change-of-variables:*

$$\tilde{\theta}_k := \theta_k - \theta^\star, \quad \tilde{w}_k = w_k - w^\star + C_{k-1}\tilde{\theta}_k. \quad (17)$$

The two time-scale SA (3), (4) is equivalent to the following iterations:

$$\tilde{\theta}_{k+1} = (I - \beta_k B_{11}^k)\tilde{\theta}_k - \beta_k A_{12}\tilde{w}_k - \beta_k V_{k+1}. \quad (18)$$

$$\tilde{w}_{k+1} = (I - \gamma_k B_{22}^k)\tilde{w}_k - \beta_k C_k V_{k+1} - \gamma_k W_{k+1}. \quad (19)$$

Observe that $\tilde{\theta}_k = 0, \tilde{w}_k = 0$ is equivalent to having $\theta_k = \theta^\star, w_k = w^\star$, i.e., the two timescale SA solves the linear system of equations (6). The simplified recursion (18), (19) *decouples* the update of $\tilde{w}_k$ from $\tilde{\theta}_k$. This allows one to treat the $\tilde{w}_k$ update as a one timescale linear SA, and therefore provides a shortcut to perform a tight analysis. We focus on estimating the following operator norms of covariances:

$$M_k^{\tilde{w}} := \|\mathbb{E}[\tilde{w}_k\tilde{w}_k^\top]\|, \quad M_k^{\tilde{\theta}} := \|\mathbb{E}[\tilde{\theta}_k\tilde{\theta}_k^\top]\|, \quad M_k^{\tilde{\theta},\tilde{w}} := \|\mathbb{E}[\tilde{\theta}_k\tilde{w}_k^\top]\|,$$

which are respectively the covariance for w_k, θ_k and the cross-variance between w_k, θ_k.

3.1. Proof Outline of Theorem 1

For this theorem, we assume the step sizes and their ratio are chosen such that

$$\gamma_k \leq \gamma_\infty^{\mathsf{mtg}} := \gamma_\infty^{(0)} \wedge \frac{1}{\frac{a_{22}}{2} + \frac{2}{a_{22}}p_{22}(\tilde{m}_V + \kappa^2\tilde{m}_W)} \wedge \frac{a_\Delta}{4\,C_2^{\tilde{\theta}}}, \quad \beta_k \leq \beta_\infty^{\mathsf{mtg}} := \beta_\infty^{(0)}, \quad (20)$$

where $p_{22} = \lambda_{\min}^{-1}(Q_{22})\lambda_{\max}(Q_{22})$ and $C_2^{\tilde{\theta}}$ is defined in (55) in the appendix.

While the property which the noise terms satisfy $\mathbb{E}^{\mathcal{F}_k}[V_{k+1}] = 0$, $\mathbb{E}^{\mathcal{F}_k}[W_{k+1}] = 0$ has greatly simplified the analysis, the challenge with our analysis lies in the coupling between slow and fast updating iterates whose convergence rates must be carefully characterized in order to obtain the desired rate in Theorem 1. To summarize, our proof consists of three steps in order: (i) we bound $M_k^{\tilde{w}}$ with an inequality that is coupled with $M_k^{\tilde{\theta}}$; then (ii) we bound the cross term $M_k^{\tilde{\theta},\tilde{w}}$ using an inequality coupled with $M_k^{\tilde{\theta}}$; lastly, (iii) these bounds are combined to bound $M_k^{\tilde{\theta}}$.

Step 1: Bounding $M_k^{\tilde{w}}$ Upon applying the variable transformation in Observation 1, (19) can be treated as a one-timescale SA which updates $\tilde{w}_k$ independently, and the contributions from $\tilde{\theta}_k$ are only found in the noise term, as seen from (34). This leads to:

Proposition 3 *Assume A1–4 and the step sizes satisfy* (20). *For any* $k \in \mathbb{N}$, *it holds*

$$M_{k+1}^{\tilde{w}} \leq \prod_{\ell=0}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{2}\right)\frac{\lambda_{\max}(Q_{22})}{\lambda_{\min}(Q_{22})}M_0^{\tilde{w}} + C_1^{\tilde{w}}\gamma_{k+1} + C_2^{\tilde{w}}\sum_{j=0}^{k}\gamma_j^2\prod_{\ell=j+1}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{2}\right)M_j^{\tilde{\theta}}, \quad (21)$$

where the constants $C_1^{\tilde{w}}, C_2^{\tilde{w}}$ *can be found in* (42) *in the appendix.*

The right hand side of (21) consists of three components: (i) a fast decaying term relying on the product $\prod_{\ell=0}^{k}(1 - \gamma_\ell a_{22}/2)$, (ii) an $\mathcal{O}(\gamma_k)$ term, and (iii) a convolutive term between $M_k^{\tilde{\theta}}$ and the fast decaying term depending on the step size sequence $(\gamma_k)_{k\geq 0}$. In the above, the second term can be viewed as a 'steady-state' term.

Step 2: Bounding $M_k^{\tilde{\theta},\tilde{w}}$ Observe that $M_k^{\tilde{\theta},\tilde{w}}$ refers to the cross variance between $\tilde{w}_k$ and $\tilde{\theta}_k$. We show that utilizing (18), (19), (21) allows us to derive:

Proposition 4 *Assume A1–4 and the step sizes satisfy* (20). *For any* $k \in \mathbb{N}$, *it holds*

$$M_{k+1}^{\tilde{\theta},\tilde{w}} \leq C_0^{\tilde{\theta},\tilde{w}}\prod_{\ell=0}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{2}\right) + C_1^{\tilde{\theta},\tilde{w}}\beta_{k+1} + C_2^{\tilde{\theta},\tilde{w}}\sum_{j=0}^{k}\gamma_j^2\prod_{\ell=j+1}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{2}\right)M_j^{\tilde{\theta}}, \quad (22)$$

where the constants $C_0^{\tilde{\theta},\tilde{w}}, C_1^{\tilde{\theta},\tilde{w}}, C_2^{\tilde{\theta},\tilde{w}}$ *can be found in* (49) *in the appendix.*

The above bound is a crucial step in obtaining the $\mathcal{O}(\beta_k)$ rate for $M_k^{\tilde{\theta}}$. To better appreciate it, note that as $M_k^{\tilde{\theta},\tilde{w}} \leq (\sqrt{d_\theta d_w}/2)\{M_k^{\tilde{\theta}} + M_k^{\tilde{w}}\}$ (see Lemma 23 in the appendix), one can derive a similar result to (22) by merely applying Proposition 3. However, doing so results in an overestimated 'steady-state' error of $\mathcal{O}(\gamma_k)$ which is worse than the $\mathcal{O}(\beta_k)$ error in (22). On the other hand, we take care of the two timescale nature of the algorithm to obtain (22) with the fast rate.

Step 3: Bounding $M_k^{\tilde{\theta}}$ Having equipped ourselves with Proposition 3 and 4, we can analyze $M_k^{\tilde{\theta}}$ using (18) and the derived bounds on $M_k^{\tilde{w}}, M_k^{\tilde{\theta},\tilde{w}}$, this leads to

Proposition 5 *Assume A1–4 and the step sizes satisfy* (20). *For any* $k \in \mathbb{N}$, *it holds*

$$M_{k+1}^{\tilde{\theta}} \leq C_0^{\tilde{\theta}}\prod_{\ell=0}^{k}\left(1 - \frac{\beta_\ell a_\Delta}{2}\right) + C_1^{\tilde{\theta}}\beta_{k+1} + C_2^{\tilde{\theta}}\sum_{j=0}^{k}\gamma_j\beta_j\prod_{\ell=j+1}^{k}\left(1 - \frac{\beta_\ell a_\Delta}{2}\right)M_j^{\tilde{\theta}}, \quad (23)$$

where the constants $C_0^{\tilde{\theta}}, C_1^{\tilde{\theta}}, C_2^{\tilde{\theta}}$ *are given in* (55) *in the appendix.*

Besides that the middle term is now $\mathcal{O}(\beta_k)$, we also observe that the convolution term with $(\mathrm{M}_j^{\tilde{\theta}})_{j\geq 0}$ depends on the *product of step sizes* $\beta_j\gamma_j$. This bound is obtained using Proposition 4 and the fact that the cross variance $\mathrm{M}_k^{\tilde{\theta},\tilde{w}}$ has a steady-state error of $\mathcal{O}(\beta_k)$.

Eq. (23) is a recursive inequality as $\mathrm{M}_k^{\tilde{\theta}}$ are found on both sides. In the appendix, we show that there exists a sequence $(\mathrm{U}_k)_{k\geq 0}$ satisfying $\mathrm{M}_k^{\tilde{\theta}} \leq \mathrm{U}_k$ and

$$\mathrm{U}_{k+1} \leq (1 - \beta_k a_\Delta/4)\, \mathrm{U}_k + \mathrm{C}_1^{\tilde{\theta}}(a_\Delta/2)\beta_k^2 \tag{24}$$

for some constant $\mathrm{C}_1^{\tilde{\theta}}$. This immediately leads to (12), followed by (13) similarly.

3.2. Proof Outline of Theorem 2

While our proof has largely followed the same strategy as in the martingale noise case, now that the main challenge in handling the Markovian noise case is that the noise terms V_{k+1}, W_{k+1} are no longer (conditionally) zero-mean. To circumvent this difficulty, we recall B2 and define the following using the solution of the Poisson equation: for any $i, j = 1, 2$,

$$\psi_k^{b_i} := \mathrm{P}\,\widehat{b}_i(X_k), \quad \Psi_k^{A_{ij}} := \mathrm{P}\,\widehat{A}_{ij}(X_k),$$
$$\xi_k^{b_i} := \widehat{b}_i(X_{k+1}) - \mathrm{P}\,\widehat{b}_i(X_k), \quad \Xi_k^{A_{ij}} := \widehat{A}_{ij}(X_{k+1}) - \mathrm{P}\,\widehat{A}_{ij}(X_k),$$

where $\xi_k^{b_i}, \Xi_k^{A_{ij}}$ are zero mean when conditioned on $\mathcal{F}_k$. The noise terms (5) can be rewritten as

$$V_{k+1} = \underbrace{\xi_k^{b_1} + \Xi_k^{A_{11}}\theta_k + \Xi_k^{A_{12}}w_k}_{=:V_{k+1}^{(0)}} + \underbrace{(\psi_k^{b_1} - \psi_{k+1}^{b_1}) + (\Psi_k^{A_{11}} - \Psi_{k+1}^{A_{11}})\theta_k + (\Psi_k^{A_{12}} - \Psi_{k+1}^{A_{12}})w_k}_{=:V_{k+1}^{(1)}}$$

$$W_{k+1} = \underbrace{\xi_k^{b_2} + \Xi_k^{A_{21}}\theta_k + \Xi_k^{A_{22}}w_k}_{=:W_{k+1}^{(0)}} + \underbrace{(\psi_k^{b_2} - \psi_{k+1}^{b_2}) + (\Psi_k^{A_{21}} - \Psi_{k+1}^{A_{21}})\theta_k + (\Psi_k^{A_{22}} - \Psi_{k+1}^{A_{22}})w_k}_{=:W_{k+1}^{(1)}}.$$

$$\tag{25}$$

We observe that $\mathbb{E}^{\mathcal{F}_k}\left[V_{k+1}^{(0)}\right] = 0, \mathbb{E}^{\mathcal{F}_k}\left[W_{k+1}^{(0)}\right] = 0$ and therefore (25) separates the noise terms into their martingale $(V_k^{(0)}, W_k^{(0)})$ and Markovian $(V_k^{(1)}, W_k^{(1)})$ components. Under B3, the second order moment of these noise components satisfy A4. Accordingly, we define $\tilde{\theta}_0^{(0)} = \tilde{\theta}_0, \tilde{\theta}_0^{(1)} = 0$, and $\tilde{w}_0^{(0)} = \tilde{w}_0, \tilde{w}_0^{(1)} = 0$ and the recursions:

$$\tilde{\theta}_{k+1}^{(i)} = (\mathrm{I} - \beta_k B_{11}^k)\tilde{\theta}_k^{(i)} - \beta_k A_{12}\tilde{w}_k^{(i)} - \beta_k V_{k+1}^{(i)}, \; i = 0, 1,$$
$$\tilde{w}_{k+1}^{(i)} = (\mathrm{I} - \gamma_k B_{22}^k)\tilde{w}_k^{(i)} - \beta_k C_k V_{k+1}^{(i)} - \gamma_k W_{k+1}^{(i)}, \; i = 0, 1, \tag{26}$$

where it holds that $\tilde{\theta}_k = \tilde{\theta}_k^{(0)} + \tilde{\theta}_k^{(1)}$, $\tilde{w}_k = \tilde{w}_k^{(0)} + \tilde{w}_k^{(1)}$ following from Observation 1. Clearly, $\tilde{\theta}_k^{(0)}, \tilde{w}_k^{(0)}$ (resp. $\tilde{\theta}_k^{(1)}, \tilde{w}_k^{(1)}$) are iterates of the two timescale SA driven by martingale (resp. Markovian) noise. The two sets of recursions are independent except the second order moments of noise are bounded by $\mathrm{M}_k^{\tilde{\theta}}, \mathrm{M}_k^{\tilde{w}}$, containing the contributions from $\tilde{\theta}_k^{(0)}, \tilde{w}_k^{(0)}$ and $\tilde{\theta}_k^{(1)}, \tilde{w}_k^{(1)}$.

In the sequel, we show the martingale noise driven terms $\| \mathbb{E}[\tilde{w}_k^{(0)}(\tilde{w}_k^{(0)})^\top]\|$, $\| \mathbb{E}[\tilde{w}_k^{(0)}(\tilde{\theta}_k^{(0)})^\top]\|$, $\| \mathbb{E}[\tilde{\theta}_k^{(0)}(\tilde{\theta}_k^{(0)})^\top]\|$ can be estimated using similar procedures as in Proposition 3–5 from the previous

subsection. Meanwhile the Markovian noise driven terms $\|\mathbb{E}[\tilde{w}_k^{(1)}(\tilde{w}_k^{(1)})^\top]\|$ vanish at a faster rate than the former. Throughout this subsection, we set the step sizes to satisfy:

$$\gamma_k \leq \gamma_\infty^{\mathsf{mark}} := \gamma_\infty^{(0)} \wedge \frac{1/\sqrt{d_\theta \vee d_w}}{6p_{22}\,\mathrm{E}_0^{WV}} \wedge \frac{a_{22}/4}{\widetilde{\mathrm{C}}_0 + \widetilde{\mathrm{C}}_3}, \quad \beta_k \leq \beta_\infty^{\mathsf{mark}} := \beta_\infty^{(0)} \wedge \frac{1}{\sqrt{6\widetilde{\mathrm{C}}_3^{(1,1)}}} \wedge \frac{a_\Delta}{8\widetilde{\mathrm{C}}_2^{\tilde{\theta}}}, \quad (27)$$

where $p_{22} = \lambda_{\mathsf{min}}^{-1}(Q_{22})\lambda_{\mathsf{max}}(Q_{22})$, $\widetilde{\mathrm{C}}_0$, $\widetilde{\mathrm{C}}_3$, E_0^{WV} are defined in (62), (67), (64), respectively, and $\widetilde{\mathrm{C}}_3^{(1,1)}$, $\widetilde{\mathrm{C}}_2^{\tilde{\theta}}$ are defined in (77), (78), respectively, in the appendix.

Step 1: Bounding $\mathrm{M}_k^{\tilde{w}}$ We first show that the martingale and Markov noise driven iterates converge with different rates as follows:

Lemma 6 *Assume A1–2, B1–4 and the step sizes satisfy* (27). *For any $k \in \mathbb{N}$, it holds*

$$\begin{aligned}
\|\mathbb{E}[\tilde{w}_{k+1}^{(0)}(\tilde{w}_{k+1}^{(0)})^\top]\| &\leq \prod_{\ell=0}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{2}\right)^2 \frac{\lambda_{\mathsf{max}}(Q_{22})}{\lambda_{\mathsf{min}}(Q_{22})} \mathrm{M}_0^{\tilde{w}} \\
&\quad + \widetilde{\mathrm{C}}_0 \sum_{j=0}^{k} \gamma_j^2 \prod_{\ell=j+1}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{2}\right)^2 (1 + \mathrm{M}_j^{\tilde{w}} + \mathrm{M}_j^{\tilde{\theta}}),
\end{aligned} \quad (28)$$

$$\begin{aligned}
\|\mathbb{E}[\tilde{w}_{k+1}^{(1)}(\tilde{w}_{k+1}^{(1)})^\top]\| &\leq \widetilde{\mathrm{C}}_1 \prod_{\ell=0}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{2}\right)^2 + \widetilde{\mathrm{C}}_2 \gamma_k^2 (\mathrm{M}_{k+1}^{\tilde{\theta}} + \mathrm{M}_{k+1}^{\tilde{w}}) + \widetilde{\mathrm{C}}_4 \gamma_k^2 \\
&\quad + \widetilde{\mathrm{C}}_3 \gamma_{k+1} \sum_{j=0}^{k} \gamma_j^2 \prod_{\ell=j+1}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{2}\right)^2 (\mathrm{M}_j^{\tilde{\theta}} + \mathrm{M}_j^{\tilde{w}}),
\end{aligned} \quad (29)$$

where $\widetilde{\mathrm{C}}_0, \widetilde{\mathrm{C}}_1, \widetilde{\mathrm{C}}_2, \widetilde{\mathrm{C}}_3, \widetilde{\mathrm{C}}_4$ are constants defined in (62), (67) *in the appendix.*

Let us compare the 'steady-state' error on the right hand side of both inequalities: second term of (28) and the second to fourth term of (29). We observe those in the Markovian noise driven iterates $\tilde{w}_k^{(1)}$ are $\mathcal{O}(\gamma_k)$ times smaller than the martingale noise driven counterparts, indicating a faster convergence. This is roughly due to the special structure of the Markovian noise in $V_k^{(1)}, W_k^{(1)}$, where each term can be written as successive differences of a bounded sequence, e.g., $V_k^{(1)} \approx \xi_k - \xi_{k+1}$. When the linear SA (26) is run over a long time horizon, the noise terms from consecutive iterations (roughly) cancels each other, leading to a significantly a smaller 'steady-state' error.

Using $\tilde{w}_k = \tilde{w}_k^{(0)} + \tilde{w}_k^{(1)}$ together with the above lemma give the following estimate for $\mathrm{M}_k^{\tilde{w}}$:

Proposition 7 *Assume A1–2, B1–4 and the step sizes satisfy* (27). *For any $k \in \mathbb{N}$, it holds*

$$\mathrm{M}_{k+1}^{\tilde{w}} \leq \prod_{\ell=0}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{4}\right)\widetilde{\mathrm{C}}_0^{\tilde{w}} + \widetilde{\mathrm{C}}_1^{\tilde{w}}\gamma_{k+1} + \widetilde{\mathrm{C}}_2^{\tilde{w}} \sum_{j=0}^{k} \gamma_j^2 \prod_{\ell=j+1}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{4}\right)\mathrm{M}_j^{\tilde{\theta}} + \widetilde{\mathrm{C}}_3^{\tilde{w}}\gamma_k^2 \mathrm{M}_{k+1}^{\tilde{\theta}},$$

where $\widetilde{\mathrm{C}}_0^{\tilde{w}}, \widetilde{\mathrm{C}}_1^{\tilde{w}}, \widetilde{\mathrm{C}}_2^{\tilde{w}}, \widetilde{\mathrm{C}}_3^{\tilde{w}}$ are defined in (67) *in the appendix.*

We note in passing that by considering a special case with $\mathrm{M}_k^{\tilde{\theta}} = 0$ for all k, the above proposition generalizes (Srikant and Ying, 2019, Theorem 7) for linear one timescale SA with Markovian noise.

In a similar vein to the proof of Theorem 1, we bound the cross term $\|\mathbb{E}[\tilde{\theta}_k^{(0)}(\tilde{w}_k^{(0)})^\top]\|$ as:

Lemma 8 *Assume A1–2, B1–4 and the step sizes satisfy* (27). *For any $k \in \mathbb{N}$, it holds*

$$\|\mathbb{E}[\tilde{\theta}_{k+1}^{(0)}(\tilde{w}_{k+1}^{(0)})^\top]\| \leq \widetilde{\mathrm{C}}_0^{\tilde{\theta},\tilde{w}} \prod_{\ell=0}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{4}\right) + \widetilde{\mathrm{C}}_1^{\tilde{\theta},\tilde{w}}\beta_{k+1} + \widetilde{\mathrm{C}}_2^{\tilde{\theta},\tilde{w}} \sum_{j=0}^{k} \gamma_j^2 \prod_{\ell=j+1}^{k}\left(1 - \frac{\gamma_\ell a_{22}}{4}\right)\mathrm{M}_j^{\tilde{\theta}},$$

where the constants $\widetilde{\mathrm{C}}_0^{\tilde{\theta},\tilde{w}}, \widetilde{\mathrm{C}}_1^{\tilde{\theta},\tilde{w}}, \widetilde{\mathrm{C}}_2^{\tilde{\theta},\tilde{w}}$ are defined in (73) *in the appendix.*

However, we observe that it is unnecessary to derive a similar (tight) bound for $\|\mathbb{E}[\tilde{\theta}_k^{(1)}(\tilde{w}_k^{(1)})^\top]\|$ as in the above lemma. The reason is that as observed in Lemma 6, the Markovian noise driven terms are anticipated to be sufficiently small compared to the martingale noise driven terms. In particular, a crude bound suffices to obtain the desirable convergence rate of $\mathrm{M}_k^{\tilde{\theta}}$, as we observe next.

Step 2: Bounding $\mathrm{M}_k^{\tilde{\theta}}$ Again we consider the bounds on $\|\mathbb{E}[\tilde{\theta}_{k+1}^{(0)}(\tilde{\theta}_{k+1}^{(0)})^\top]\|$ and $\mathbb{E}[\|\tilde{\theta}_{k+1}^{(1)}\|^2]$ separately. As we show in the appendix, both bounds are comparable as the Markovian noise term admits a successive difference structure. Using the decomposition $\tilde{\theta}_k = \tilde{\theta}_k^{(0)} + \tilde{\theta}_k^{(1)}$, we obtain:

Proposition 9 *Assume A1–2, B1–4 and the step sizes satisfy* (27). *For any $k \in \mathbb{N}$, it holds*

$$\mathrm{M}_{k+1}^{\tilde{\theta}} \leq \widetilde{\mathrm{C}}_0^{\tilde{\theta}} \prod_{\ell=0}^{k} \left(1 - \tfrac{\beta_\ell a_\Delta}{4}\right) + \widetilde{\mathrm{C}}_1^{\tilde{\theta}} \beta_{k+1} + \widetilde{\mathrm{C}}_2^{\tilde{\theta}} \sum_{i=0}^{k} \beta_i^2 \prod_{\ell=j+1}^{k} \left(1 - \tfrac{\beta_\ell a_\Delta}{4}\right) \mathrm{M}_i^{\tilde{\theta}}, \qquad (30)$$

where the constants $\widetilde{\mathrm{C}}_0^{\tilde{\theta}}, \widetilde{\mathrm{C}}_1^{\tilde{\theta}}, \widetilde{\mathrm{C}}_2^{\tilde{\theta}}$ are defined in (78) *in the appendix.*

Equipped with Proposition 9, we can repeat the same steps as in (24) to derive an upper bound for $\mathrm{M}_k^{\tilde{\theta}}$ through solving the recursive inequality (30). Similar steps also apply for yielding (15).

4. Tightness of the Finite-time Error Bounds

This section examines the tightness of our finite time error bounds in Theorem 1, 2 through characterizing the squared error $\mathbb{E}[\|\theta_k - \theta^\star\|^2]$ with expansion. We consider the assumption:

A5 *There exist matrices $\Sigma^{11}, \Sigma^{12}, \Sigma^{22}$, and a constant $m_{VW}^{\exp} \geq 0$ such that for all $j \in \mathbb{N}$, it holds*

$$\|\mathbb{E}[V_j V_j^\top] - \Sigma^{11}\| \vee \|\mathbb{E}[W_j W_j^\top] - \Sigma^{22}\| \vee \|\mathbb{E}[V_j W_j^\top] - \Sigma^{12}\| \leq m_{VW}^{\exp}(\|\mathbb{E}[\theta_k \theta_k^\top]\| + \|\mathbb{E}[w_k w_k^\top]\|).$$

Note that A5 implies A4 and therefore poses a stronger assumption. We have

Theorem 10 *Assume A1–3, A5 and for all $k \in \mathbb{N}$, we have $\gamma_k \in [0, \gamma_\infty^{\mathsf{mtg}}]$, $\beta_k \in [0, \beta_\infty^{\exp}]$ and $\kappa \in [0, \kappa_\infty^{\exp}]$, where $\gamma_\infty^{\mathsf{mtg}}, \beta_\infty^{\exp}, \kappa_\infty^{\exp}$ are constants defined in* (20), (83), (82) *in the appendix. Then for any $k \geq k_0^{\exp} := \min\{\ell : \sum_{j=0}^{\ell-1} \beta_j \geq \log(2)/(2\|\Delta\|)\}$, the following expansion holds*

$$\mathbb{E}\left[\|\theta_k - \theta^\star\|^2\right] = I_k + J_k. \qquad (31)$$

The leading term I_k is given by the following explicit formula

$$I_k := \sum_{j=0}^{k} \beta_j^2 \operatorname{Tr}\left(\prod_{\ell=j+1}^{k}(I - \beta_\ell \Delta)\,\Sigma\,\left\{\prod_{\ell=j+1}^{k}(I - \beta_\ell \Delta)\right\}^\top\right),$$

where $\Sigma := \Sigma^{11} + A_{12} A_{22}^{-1} \Sigma^{22} A_{22}^{-\top} A_{12}^\top + \Sigma^{12} A_{22}^{-\top} A_{12}^\top + A_{12} A_{22}^{-1} \Sigma^{21}$. Meanwhile, the following two-sided inequality holds

$$\mathrm{C}_3^{\exp} \operatorname{Tr}(\Sigma) \leq \frac{I_k}{\beta_k} \leq \mathrm{C}_4^{\exp} \operatorname{Tr}(\Sigma), \qquad (32)$$

and J_k is bounded by

$$|J_k| \leq \mathrm{C}_0^{\exp} \prod_{\ell=0}^{k-1} \left(1 - \frac{a_\Delta}{4} \beta_\ell\right) \mathrm{V}_0 + \mathrm{C}_1^{\exp} \beta_k \left(\gamma_k + \frac{\beta_k}{\gamma_k}\right), \qquad (33)$$

where V_0 was defined in (11). *All constants $\mathrm{C}_0^{\exp}, \mathrm{C}_1^{\exp}, \mathrm{C}_3^{\exp}, \mathrm{C}_4^{\exp}$ are given in* (109), (89) *and* (91) *in the appendix, respectively, and they are independent of β_k, γ_k.*

The proof is skipped in the interest of space, and it can be found in Appendix C. Observe that from (33), the dominant term for J_k is given by $\mathcal{O}(\beta_k \gamma_k + \frac{\beta_k^2}{\gamma_k})$. As such, using (32), we observe that

$$|J_k|/I_k = \mathcal{O}\left(\gamma_k + \beta_k/\gamma_k\right)$$

If $\lim_{k \to \infty} \beta_k/\gamma_k = 0$, we have $\lim_{k \to \infty} |J_k|/I_k = 0$. Combining (31), (32) shows that the expected error $\mathbb{E}[\|\theta_k - \theta^\star\|^2]$ is lower bounded by $\Omega(\beta_k)$.

We note that the assumptions A1–3, A5 imposed by the theorem imply A1–A4 required by Theorem 1. Hence, together with (12) in Theorem 1, the above observations constitute a *matching* lower bound on the convergence rate of linear two timescale SA with martingale noise. For the Markovian noise setting, we observe that if we impose the assumption that the random elements $(X_k)_{k \geq 0}$ are i.i.d., and $\widetilde{b}_i(x), \widetilde{A}_{ij}(x)$ are bounded above for any $i, j = 1, 2$ and $x \in \mathsf{X}$, then A5, B2–B3 can be satisfied. Therefore, the lower bound on the convergence rate also holds.

5. Numerical Experiments, Conclusions

We present numerical experiments to support our theoretical claims. We consider (a) a toy example with a randomly generated problem parameters b_i, A_{ij} and i.i.d. samples $(X_k)_{k \in \mathbb{N}}$ such that $\mathbb{E}[\widetilde{b}_i(X_k)] = b_i, \mathbb{E}[\widetilde{A}_{ij}(X_k)] = A_{ij}$, (b) the Garnet problem (Geist and Scherrer, 2014) with the GTD algorithm (Sutton et al., 2009a) using X_k from a simulated Markov chain. For example (a), we compute the stationary point $\theta^\star, w^\star$ exactly using (7); for example (b), while it is known that $w^\star = 0$, the solution $\theta^\star$ is computed using Monte Carlo simulation of the matrices $\widetilde{b}_i(X_k), \widetilde{A}_{ij}(X_k)$ with $2 \cdot 10^9$ iterations. The step sizes are chosen as $\beta_k = c^\beta/(k_0^\beta + k), \gamma_k = c^\gamma/(k_0^\beta + k)^\sigma$ with $\sigma \in \{0.5, 0.67, 0.75\}$. In the toy example (a), we have $d_\theta = d_w = 10, k_0^\beta = 10^4, k_0^\gamma = 10^7, c^\beta = 140, c^\gamma = 300$; while for the Garnet problem (b), we have $k_0^\beta = 8 \cdot 10^5, k_0^\gamma = 2 \cdot 10^5, c^\beta = 2300, c^\gamma = 120$. Garnet problem is generated from family $n_S = 30, n_A = 2, b = 2, p = 8$, see (Geist and Scherrer, 2014). Further details about both experiments are described in Appendix E.
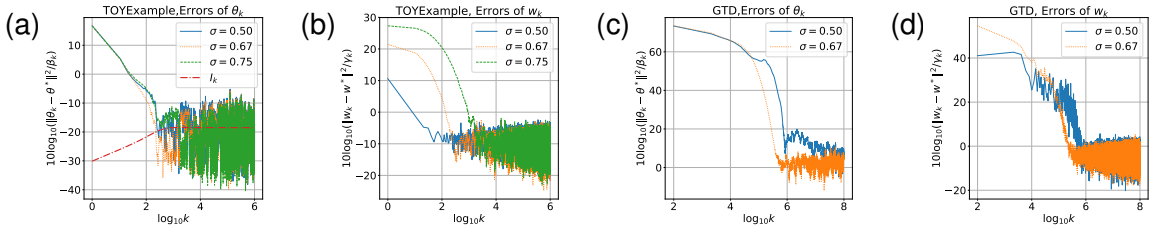

Figure 1: Deviations from stationary point $(\theta^\star, w^\star)$ *normalized* by step sizes β_k, γ_k: (a,b) the toy example, note we also show I_k using the exact formula in Theorem 10 (unnormalized plot also available in the Appendix); (c,d) the Garnet problem.

We illustrate the convergence rates of the linear two timescale SA on the two problems in Figure 1. Note that the plots show the (normalized) steady state errors are $\mathbb{E}[\|\theta_k - \theta^\star\|^2] = \mathcal{O}(\beta_k)$, $\mathbb{E}[\|w_k - w^\star\|^2] = \mathcal{O}(\gamma_k)$, which hold for both examples on martingale and Markovian noise. In addition, they are independent of the choice of σ. These observations agree with our main results.

Conclusions We have provided an improved finite time convergence analysis of the linear two timescale SA on both martingale and Markovian noises with relaxed conditions. Our analysis show that a tight analysis is possible through deriving and solving a sequence of recursive error bounds. Future works include the finite time analysis of nonlinear two timescale SA.

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Appendix A. Proof of Observation 1

The following derivation is largely borrowed from (Konda and Tsitsiklis, 2004) and is repeated here for completeness. We begin by substituting $\tilde{\theta}_k$ into (3) to obtain

$$\begin{aligned}\tilde{\theta}_{k+1} &= (I - \beta_k A_{11})\theta_k - \beta_k A_{12} w_k - \theta^\star + \beta_k b_1 - \beta_k V_{k+1} \\ &= (I - \beta_k A_{11})\tilde{\theta}_k - \beta_k A_{11}\theta^\star - \beta_k A_{12}(\tilde{w}_k + w^\star - C_{k-1}\tilde{\theta}_k) + \beta_k b_1 - \beta_k V_{k+1} \\ &= (I - \beta_k(A_{11} - A_{12}A_{22}^{-1}A_{21} - A_{12}L_k))\tilde{\theta}_k - \beta_k A_{12}\tilde{w}_k - \beta_k(A_{12}w^\star + A_{11}\theta^\star - b_1) - \beta_k V_{k+1}.\end{aligned}$$

Notice that

$$A_{12}w^\star + A_{11}\theta^\star - b_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\theta^\star + A_{12}A_{22}^{-1}b_2 - b_1 = 0.$$

The above yields

$$\tilde{\theta}_{k+1} = (I - \beta_k B_{11}^k)\tilde{\theta}_k - \beta_k A_{12}\tilde{w}_k - \beta_k V_{k+1}.$$

Next, we observe that

$$\begin{aligned}w_{k+1} - w^\star &= (I - \gamma_k A_{22})w_k - \gamma_k A_{21}\theta_k - w^\star + \gamma_k b_2 - \gamma_k W_{k+1} \\ &= (I - \gamma_k A_{22})(w_k - w^\star) - \gamma_k A_{22}w^\star - \gamma_k A_{21}\theta_k + \gamma_k b_2 - \gamma_k W_{k+1} \\ &= (I - \gamma_k A_{22})(w_k - w^\star) - \gamma_k A_{21}(\theta_k - \theta^\star) - \gamma_k W_{k+1}\end{aligned}$$

Substitute $\tilde{w}_k$ into (4) and using (18) yield:

$$\begin{aligned}\tilde{w}_{k+1} &= (I - \gamma_k A_{22})(w_k - w^\star) - \gamma_k A_{21}\tilde{\theta}_k + C_k\tilde{\theta}_{k+1} - \gamma_k W_{k+1} \\ &= (I - \gamma_k A_{22})\tilde{w}_k - \big((I - \gamma_k A_{22})C_{k-1} + \gamma_k A_{21}\big)\tilde{\theta}_k + C_k\big((I - \beta_k B_{11}^k)\tilde{\theta}_k - \beta_k A_{12}\tilde{w}_k\big) \\ &\quad - \beta_k C_k V_{k+1} - \gamma_k W_{k+1} \\ &= (I - \gamma_k B_{22}^k)\tilde{w}_k - \Big(C_{k-1} - \gamma_k(A_{22}C_{k-1} - A_{21}) - C_k(I - \beta_k B_{11}^k)\Big)\tilde{\theta}_k - \beta_k C_k V_{k+1} - \gamma_k W_{k+1}\end{aligned}$$

We observe that

$$\begin{aligned}&C_{k-1} - \gamma_k(A_{22}C_{k-1} - A_{21}) - C_k(I - \beta_k B_{11}^k) \\ &= L_k + A_{22}^{-1}A_{21} - (L_{k+1} + A_{22}^{-1}A_{21})(I - \beta_k B_{11}^k) - \gamma_k(A_{22}C_{k-1} - A_{21}) \\ &= L_k - (L_k - \gamma_k A_{22}L_k + \beta_k A_{22}^{-1}A_{21}B_{11}^k) - \beta_k A_{22}^{-1}A_{21}B_{11}^k - \gamma_k(A_{22}C_{k-1} - A_{21}) \\ &= \gamma_k A_{22}L_k - \gamma_k(A_{22}(L_k + A_{22}^{-1}A_{21}) - A_{21}) = 0.\end{aligned}$$

The above yields

$$\tilde{w}_{k+1} = (I - \gamma_k B_{22}^k)\tilde{w}_k - \beta_k C_k V_{k+1} - \gamma_k W_{k+1}.$$

Appendix B. Detailed Proofs for Section 3

Before we proceed to proving the main results of Section 3, we first study a few properties of the two timescale linear SA scheme.

To facilitate our discussions next, we define the constant:

$$\mathrm{C}_\infty := \sqrt{\lambda_{\mathsf{min}}(Q_\Delta)^{-1}\lambda_{\mathsf{max}}(Q_{22})}\, \mathrm{L}_\infty + \|A_{22}^{-1}A_{21}\|,$$

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where $\|C_k\| \leq \mathrm{C}_\infty$ for any $k \geq 0$. Then, as we have $\theta_k \theta_k^\top \preceq 2\tilde{\theta}_k \tilde{\theta}_k^\top + 2\theta^\star (\theta^\star)^\top$, it holds

$$\|\mathbb{E}[\theta_k \theta_k^\top]\| \leq 2\big\{ \mathrm{M}_k^{\tilde\theta} + \|\theta^\star (\theta^\star)^\top\| \big\}, \quad \|\mathbb{E}[w_k w_k^\top]\| \leq 3\big\{ \mathrm{M}_k^{\tilde w} + \mathrm{M}_k^{\tilde\theta}\, \mathrm{C}_\infty^2 + \|w^\star (w^\star)^\top\| \big\}$$

The noise terms V_k, W_k can then be estimated in terms of the transformed variables $\tilde{\theta}_k, \tilde{w}_k$ and their variances $\mathrm{M}_k^{\tilde\theta}, \mathrm{M}_k^{\tilde w}$. In particular, combining with A4 yields

$$\|\mathbb{E}[V_{k+1} V_{k+1}^\top]\| \leq \tilde{m}_V (1 + \mathrm{M}_k^{\tilde\theta} + \mathrm{M}_k^{\tilde w}), \quad \|\mathbb{E}[W_{k+1} W_{k+1}^\top]\| \leq \tilde{m}_W (1 + \mathrm{M}_k^{\tilde\theta} + \mathrm{M}_k^{\tilde w}) \quad (34)$$

$$\|\mathbb{E}[V_{k+1} W_{k+1}^\top]\| \leq \tilde{m}_{VW} (1 + \mathrm{M}_k^{\tilde\theta} + \mathrm{M}_k^{\tilde w}) \quad (35)$$

where

$$\begin{aligned}
\frac{\tilde{m}_V}{m_V} = \frac{\tilde{m}_W}{m_W} &= (1 + 2\|\theta^\star (\theta^\star)^\top\| + 3\|w^\star (w^\star)^\top\|) \vee (2 + 3\,\mathrm{C}_\infty^2) \vee 3 \\
\tilde{m}_{VW} &= \frac{\sqrt{d_\theta d_w}}{2}\,(\tilde{m}_W + \tilde{m}_V)
\end{aligned} \quad (36)$$

We also define a few constants related to the matrices Q_Δ, Q_{22} associated with the Hurwitz matrices Δ, A_{22} in (8). Set $p_\Delta := \lambda_{\min}^{-1}(Q_\Delta)\lambda_{\max}(Q_\Delta)$, $p_{22} := \lambda_{\min}^{-1}(Q_{22})\lambda_{\max}(Q_{22})$, $p_{22,\Delta} := \sqrt{p_{22} p_\Delta}$. Moreover, for any $a > 0$, we set

$$\varrho^a := \frac{2}{a}\varsigma \max\{1, a_{22}/(4a_\Delta)\} \vee \frac{4}{a}\,(\varsigma)^3.$$

Next, we study the contraction properties of $\mathrm{I} - \beta_k B_{11}^k$ and $\mathrm{I} - \gamma_k B_{22}^k$ that appear in the transformed two timescale SA (18),(19). Using (9), we observe that

$$\begin{aligned}
\|\mathrm{I} - \beta_k B_{11}^k\|_{Q_\Delta} = \|\mathrm{I} - \beta_k \Delta + \beta_k A_{12} L_k\|_{Q_\Delta} &\leq \|\mathrm{I} - \beta_k \Delta\|_{Q_\Delta} + \beta_k \|A_{12}\|_{Q_{22},Q_\Delta} \|L_k\|_{Q_\Delta,Q_{22}} \\
&\leq (1 - \beta_k a_\Delta) + \beta_k \|A_{12}\|_{Q_{22},Q_\Delta} \|L_k\|_{Q_\Delta,Q_{22}}.
\end{aligned}$$

Recalling that $\|L_k\|_{Q_\Delta,Q_{22}} \leq \mathrm{L}_\infty$, the above inequality yields

$$\|\mathrm{I} - \beta_k B_{11}^k\|_{Q_\Delta} \leq 1 - (1/2)\beta_k a_\Delta. \quad (37)$$

Since $\|\mathrm{I} - \gamma_k B_{22}^k\|_{Q_{22}} \leq \|\mathrm{I} - \gamma_k A_{22}\|_{Q_{22}} + \beta_k \|C_k A_{12}\|_{Q_{22}}$, we obtain the contraction:

$$\begin{aligned}
\|\mathrm{I} - \gamma_k B_{22}^k\|_{Q_{22}} &\leq 1 - \gamma_k a_{22} + \beta_k (\mathrm{L}_\infty + \|A_{22}^{-1} A_{21}\|_{Q_\Delta,Q_{22}})\|A_{12}\|_{Q_{22},Q_\Delta} \\
&\leq 1 - (1/2)\gamma_k a_{22}.
\end{aligned} \quad (38)$$

The last inequality is due to $\kappa \leq (a_{22}/2)\{(\mathrm{L}_\infty + \|A_{22}^{-1} A_{21}\|_{Q_\Delta,Q_{22}})\|A_{12}\|_{Q_{22},Q_\Delta}\}^{-1}$. Lastly, the following quantities will be used throughout the analysis:

$$\begin{aligned}
\Gamma_{m:n}^{(1)} &:= \prod_{i=m}^{n} (\mathrm{I} - \beta_i B_{11}^i), \quad \Gamma_{m:n}^{(2)} := \prod_{i=m}^{n} (\mathrm{I} - \gamma_i B_{22}^i), \\
G_{m:n}^{(1)} &:= \prod_{i=m}^{n} \Big(1 - (1/2)\beta_i a_\Delta \Big), \quad G_{m:n}^{(2)} := \prod_{i=m}^{n} \Big(1 - (1/2)\gamma_i a_{22} \Big).
\end{aligned}$$

As a convention, we define $\Gamma^{(1)}_{m:n} = \Gamma^{(2)}_{m:n} = I$ if $m > n$. In particular, for any $n, m \geq 0$, we observe the following bound on the operator norm of $\Gamma^{(1)}_{m:n}$,

$$\|\Gamma^{(1)}_{m:n}\| = \sqrt{p_\Delta}\|\Gamma^{(1)}_{m:n}\|_{Q_\Delta} \leq \sqrt{p_\Delta} \prod_{i=m}^{n} \| I - \beta_i B^i_{11} \|_{Q_\Delta} \leq \sqrt{p_\Delta} G^{(1)}_{m:n}$$

Similarly, we have $\|\Gamma^{(1)}_{m:n}\| \leq \sqrt{p_{22}} G^{(2)}_{m:n}$. Lastly, we define

$$\Sigma_k := \mathbb{E}\left[\tilde{w}_k \tilde{w}_k^\top\right], \quad \Omega_k := \mathbb{E}\left[\tilde{\theta}_k \tilde{w}_k^\top\right], \quad \Theta_k := \mathbb{E}\left[\tilde{\theta}_k \tilde{\theta}_k^\top\right],$$

whose operator norms correspond to $\mathrm{M}^{\tilde{w}}_k, \mathrm{M}^{\tilde{\theta},\tilde{w}}_k, \mathrm{M}^{\tilde{\theta}}_k$, respectively.

B.1. Detailed Proof of Theorem 1

This subsection provides proofs to the propositions stated in Section 3.1, as well as providing detailed steps in establishing Theorem 1.

Bounding $\mathrm{M}^{\tilde{w}}_k$ (Proof of Proposition 3)   Using (19), as the noise terms are martingale, we get

$$\mathbb{E}^{\mathcal{F}_k}\left[\tilde{w}_{k+1}\tilde{w}_{k+1}^\top\right] = (I - \gamma_k B^k_{22})\tilde{w}_k \tilde{w}_k^\top (I - \gamma_k B^k_{22})^\top + \gamma_k^2 \mathbb{E}^{\mathcal{F}_k}\left[W_{k+1}W_{k+1}^\top\right]$$
$$+ \beta_k^2 C_k \mathbb{E}^{\mathcal{F}_k}\left[V_{k+1}V_{k+1}^\top\right] C_k^\top + \beta_k \gamma_k \left(\mathbb{E}^{\mathcal{F}_k}\left[W_{k+1}V_{k+1}^\top\right] C_k^\top + C_k \mathbb{E}^{\mathcal{F}_k}\left[V_{k+1}W_{k+1}^\top\right]\right). \quad (39)$$

Repeatedly applying (39) and taking the total expectation on both sides show

$$\Sigma_{k+1} = \Gamma^{(2)}_{0:k}\Sigma_0(\Gamma^{(2)}_{0:k})^\top + \sum_{j=0}^{k} \Gamma^{(2)}_{j+1:k}D_{j+1}(\Gamma^{(2)}_{j+1:k})^\top, \quad (40)$$

where

$$D_{k+1} = \gamma_k^2\, \mathbb{E}[W_{k+1}W_{k+1}^\top] + \beta_k^2 C_k\, \mathbb{E}[V_{k+1}V_{k+1}^\top]C_k^\top + \beta_k\gamma_k\left(\mathbb{E}[W_{k+1}V_{k+1}^\top]C_k^\top + C_k\, \mathbb{E}[V_{k+1}W_{k+1}^\top]\right).$$

Using Lemma 23, we observe that

$$\gamma_k\beta_k\| \mathbb{E}[W_{k+1}V_{k+1}^\top]C_k^\top\| \leq \frac{\sqrt{d_\theta d_w}}{2} C_\infty\left(\gamma_k^2\| \mathbb{E}[W_{k+1}W_{k+1}^\top]\| + \beta_k^2\| \mathbb{E}[V_{k+1}V_{k+1}^\top]\|\right),$$

Let $K_C := \max\{C_\infty^2, 1\} + \sqrt{d_\theta d_w}C_\infty$, we have

$$\|D_{k+1}\| \leq \gamma_k^2\left(1 + C_\infty\sqrt{d_\theta d_w}\right)\| \mathbb{E}[W_{k+1}W_{k+1}^\top]\| + \beta_k^2 C_\infty\left(C_\infty + \sqrt{d_\theta d_w}\right)\| \mathbb{E}[V_{k+1}V_{k+1}^\top]\|$$
$$\leq K_C\left(\gamma_k^2\{\tilde{m}_V + \tilde{m}_V\, \mathrm{M}^{\tilde{\theta}}_k + \tilde{m}_V\, \mathrm{M}^{\tilde{w}}_k\} + \beta_k^2\{\tilde{m}_W + \tilde{m}_W\, \mathrm{M}^{\tilde{\theta}}_k + \tilde{m}_W\, \mathrm{M}^{\tilde{w}}_k\}\right)$$

where the last inequality is due to (34). Taking the operator norm on both sides of (40) yields

$$\mathrm{M}^{\tilde{w}}_{k+1} \leq p_{22}\left\{(G^{(2)}_{0:k})^2\, \mathrm{M}^{\tilde{w}}_0 + K_C \sum_{j=0}^{k}(G^{(2)}_{j+1:k})^2(\gamma_j^2\tilde{m}_V + \beta_j^2\tilde{m}_W)\{1 + \mathrm{M}^{\tilde{\theta}}_j + \mathrm{M}^{\tilde{w}}_j\}\right\}.$$

Using that $\beta_k \leq \kappa\gamma_k$ one writes

$$M_{k+1}^{\tilde{w}} \leq C_0^{\tilde{w}'}\big(G_{0:k}^{(2)}\big)^2 + c_0 C_1^{\tilde{w}'}\sum_{j=0}^{k}\gamma_j^2\big(G_{j+1:k}^{(2)}\big)^2(1+M_j^{\tilde{\theta}}) + C_2^{\tilde{w}'}\sum_{j=0}^{k}\gamma_j^2\big(G_{j+1:k}^{(2)}\big)^2 M_j^{\tilde{w}} \qquad (40)$$

where $c_0 = \tilde{m}_V + \kappa^2\tilde{m}_W$, $C_0^{\tilde{w}'} = p_{22}\,M_0^{\tilde{w}}$, $C_1^{\tilde{w}'} = p_{22}K_C$, and $C_2^{\tilde{w}'} = p_{22}c_0$. Define:

$$\tilde{U}_k = C_0^{\tilde{w}'}\big(G_{0:k-1}^{(2)}\big)^2 + c_0 C_1^{\tilde{w}'}\sum_{j=0}^{k-1}\gamma_j^2\big(G_{j+1:k-1}^{(2)}\big)^2(1+M_j^{\tilde{\theta}}) + C_2^{\tilde{w}'}\sum_{j=0}^{k-1}\gamma_j^2\big(G_{j+1:k-1}^{(2)}\big)^2\tilde{U}_j,$$

It is easily seen that the sequence $(\tilde{U}_k)_{k\geq 0}$ is given by the following recursion

$$\tilde{U}_{k+1} = (1 - a_{22}\gamma_k/2)^2\tilde{U}_k + c_0 C_1^{\tilde{w}'}\gamma_k^2(1+M_k^{\tilde{\theta}}) + C_2^{\tilde{w}'}\gamma_k^2\tilde{U}_k, \quad \tilde{U}_0 = C_0^{\tilde{w}'}.$$

Since the step size was chosen such that $\gamma_k(C_2^{\tilde{w}'} + (a_{22}^2/4)) \leq \frac{a_{22}}{2}$ [cf. (20)], we have

$$\tilde{U}_{k+1} \leq (1 - a_{22}\gamma_k/2)\tilde{U}_k + C_1^{\tilde{w}'}\gamma_k^2(c_0 + c_1\,M_k^{\tilde{\theta}})$$

which implies

$$\tilde{U}_{k+1} \leq C_0\,G_{0:k}^{(2)} + c_0 C_1^{\tilde{w}'}\sum_{j=0}^{k}\gamma_j^2(1+M_j^{\tilde{\theta}})G_{j+1:k}^{(2)}.$$

Observe that $M_k^{\tilde{w}} \leq \tilde{U}_k$. Applying Corollary 14 shows that $\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)} \leq \varrho^{a_{22}/2}\gamma_{k+1}$, we get

$$\boxed{M_{k+1}^{\tilde{w}} \leq C_0^{\tilde{w}}\,G_{0:k}^{(2)} + C_1^{\tilde{w}}\,\gamma_{k+1} + C_2^{\tilde{w}}\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}\,M_j^{\tilde{\theta}},} \qquad (41)$$

where we recall $K_C := \max\{C_\infty^2, 1\} + \sqrt{d_\theta d_w}C_\infty$, and

$$C_0^{\tilde{w}} := p_{22}\,M_0^{\tilde{w}}, \quad C_1^{\tilde{w}} := p_{22}(\tilde{m}_V + \kappa^2\tilde{m}_W)K_C\varrho^{a_{22}/2}, \quad C_2^{\tilde{w}} := p_{22}K_C(\tilde{m}_V + \kappa^2\tilde{m}_W). \qquad (42)$$

This concludes the proof for Proposition 3.

Bounding $M_k^{\tilde{\theta},\tilde{w}}$ (Proof of Proposition 4) We proceed by observing the following recursion of Ω_k:

$$\Omega_{k+1} = (I - \beta_k B_{11}^k)\Omega_k(I - \gamma_k B_{22}^k)^\top - \beta_k A_{12}\Sigma_k(I - \gamma_k B_{22}^k)^\top$$
$$+ \beta_k\gamma_k\,\mathbb{E}[V_{k+1}W_{k+1}^\top] + \beta_k^2\,\mathbb{E}[V_{k+1}V_{k+1}^\top]C_k^\top.$$

Repeatedly applying the recursion gives

$$\Omega_{k+1} = \Gamma_{0:k}^{(1)}\Omega_0\left(\Gamma_{0:k}^{(2)}\right)^\top - \sum_{j=0}^{k}\beta_j\Gamma_{j+1:k}^{(1)}A_{12}\Sigma_j\left(\Gamma_{j:k}^{(2)}\right)^\top$$

$$+ \sum_{j=0}^{k}\beta_j\gamma_j\Gamma_{j+1:k}^{(1)}\,\mathbb{E}\left[V_{j+1}W_{j+1}^\top\right]\left(\Gamma_{j+1:k}^{(2)}\right)^\top + \sum_{j=0}^{k}\beta_j^2\Gamma_{j+1:k}^{(1)}\,\mathbb{E}\left[V_{j+1}V_{j+1}^\top\right]C_j^\top\left(\Gamma_{j+1:k}^{(2)}\right)^\top.$$

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The contraction properties (38), (37) result in

$$\mathrm{M}_{k+1}^{\tilde{\theta},\tilde{w}} \le p_{22,\Delta}\Big\{ G_{0:k}^{(1)} G_{0:k}^{(2)} \mathrm{M}_0^{\tilde{\theta},\tilde{w}} + \|A_{12}\| \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(2)} \mathrm{M}_j^{\tilde{w}} \Big\} \tag{43}$$

$$+ p_{22,\Delta}\Big\{ \sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \|\mathbb{E}[V_{j+1} W_{j+1}^{\top}]\| + \mathrm{C}_\infty \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \|\mathbb{E}[V_{j+1} V_{j+1}^{\top}]\| \Big\}$$

Applying (41), we bound the third last term of (43) as

$$\sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(2)} \mathrm{M}_j^{\tilde{w}} \le \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(2)} \big(\mathrm{C}_0^{\tilde{w}} G_{0:j-1}^{(2)} + \mathrm{C}_1^{\tilde{w}} \gamma_j + \mathrm{C}_2^{\tilde{w}} \sum_{i=0}^{j-1} \gamma_i^2 G_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} \big)$$

$$\le \frac{2\,\mathrm{C}_0^{\tilde{w}}\,G_{0:k}^{(2)}}{a_\Delta} + \mathrm{C}_1^{\tilde{w}} \sum_{j=0}^{k} \beta_j G_{j:k}^{(2)} \gamma_j + \mathrm{C}_2^{\tilde{w}} \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(2)} \sum_{i=0}^{j-1} \gamma_i^2 G_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}}$$

where we have used Lemma 12 and $G_{j+1:k}^{(1)} \le 1$ in the last inequality. Applying Corollary 14 and Lemma 15, A2 to the second and the last term on the right hand side, respectively, we obtain the following upper bound:

$$\sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(2)} \mathrm{M}_j^{\tilde{w}} \le \frac{2\,\mathrm{C}_0^{\tilde{w}}\,G_{0:k}^{(2)}}{a_\Delta} + \mathrm{C}_1^{\tilde{w}}\, \varrho^{a_{22}/2} \beta_{k+1} + \frac{2\,\mathrm{C}_2^{\tilde{w}}}{a_\Delta} \sum_{i=0}^{k} \gamma_i^2 G_{i+1:k}^{(2)} \mathrm{M}_i^{\tilde{\theta}}. \tag{44}$$

Applying (35), we bound the second last term of (43) as

$$\sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \|\mathbb{E}[V_{j+1} W_{j+1}^{\top}]\| \le \tilde{m}_{VW} \sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \big(1 + \mathrm{M}_j^{\tilde{\theta}} + \mathrm{M}_j^{\tilde{w}} \big)$$

$$\le \tilde{m}_{VW}\Big\{ \sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(2)} + \sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{\theta}} + \sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{w}} \Big\} \tag{45}$$

$$\le \tilde{m}_{VW}\Big\{ \varrho^{a_{22}/2} \beta_{k+1} + \kappa \sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{\theta}} + \sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{w}} \Big\}$$

where the last inequality applied Corollary 14 again. We observe

$$\sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{w}} \le \frac{\gamma_0}{1 - \gamma_0 a_{22}/2} \sum_{j=0}^{k} \beta_j G_{j+1:k-1}^{(1)} G_{j:k-1}^{(2)} \mathrm{M}_j^{\tilde{w}} \tag{46}$$

Thirdly, we repeat the calculations above and exploit $\beta_k \le \kappa \gamma_k$ to bound

$$\sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \|\mathbb{E}[V_{j+1} V_{j+1}^{\top}]\| \le \tilde{m}_V \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \big(1 + \mathrm{M}_j^{\tilde{\theta}} + \mathrm{M}_j^{\tilde{w}} \big)$$

$$\le \tilde{m}_V\Big\{ \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} + \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{\theta}} + \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{w}} \Big\} \tag{47}$$

$$\le \tilde{m}_V\Big\{ \kappa \varrho^{a_{22}/2} \beta_{k+1} + \kappa^2 \sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{\theta}} + \kappa \sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{w}} \Big\}.$$

Combining (44), (45), (46), (47), we conclude that

$$\boxed{\mathrm{M}^{\tilde{\theta},\tilde{w}}_{k+1} \leq \mathrm{C}^{\tilde{\theta},\tilde{w}}_0 \, G^{(2)}_{0:k} + \mathrm{C}^{\tilde{\theta},\tilde{w}}_1 \, \beta_{k+1} + \mathrm{C}^{\tilde{\theta},\tilde{w}}_2 \sum_{j=0}^{k} \gamma_j^2 G^{(2)}_{j+1:k} \, \mathrm{M}^{\tilde{\theta}}_j} \tag{48}$$

where

$$
\begin{aligned}
\mathrm{C}^{\tilde{\theta},\tilde{w}}_0 &:= p_{22,\Delta}\Big(\mathrm{M}^{\tilde{\theta},\tilde{w}}_0 + \|A_{12}\|\frac{2\,\mathrm{C}^{\tilde{w}}_0}{a_\Delta} + (\tilde{m}_{VW} + \kappa C_\infty \tilde{m}_V)\frac{2\gamma_0\,\mathrm{C}^{\tilde{w}}_0}{a_\Delta(1-\gamma_0 a_{22}/2)} \Big), \\
\mathrm{C}^{\tilde{\theta},\tilde{w}}_1 &:= p_{22,\Delta}\varrho^{a_{22}/2}\Big(\mathrm{C}^{\tilde{w}}_1\big(\|A_{12}\| + \frac{\gamma_0}{1-\gamma_0 a_{22}/2}(\tilde{m}_{VW} + C_\infty \kappa \tilde{m}_V)\big) + \tilde{m}_{VW} + C_\infty \kappa \tilde{m}_V \Big), \\
\mathrm{C}^{\tilde{\theta},\tilde{w}}_2 &:= p_{22,\Delta}\Big(\frac{2\,\mathrm{C}^{\tilde{w}}_2}{a_\Delta}\big(\|A_{12}\| + \frac{\gamma_0}{1-\gamma_0 a_{22}/2}(\tilde{m}_{VW} + C_\infty \kappa \tilde{m}_V)\big) + \kappa\big(\tilde{m}_{VW} + C_\infty \kappa \tilde{m}_V\big) \Big).
\end{aligned}
\tag{49}
$$

This concludes the proof of Proposition 4.

Bounding $\mathrm{M}^{\tilde{\theta}}_k$ (Proof of Proposition 5) We observe the following recursion:

$$
\begin{aligned}
\mathbb{E}^{\mathcal{F}_k}\left[\tilde{\theta}_{k+1}\tilde{\theta}_{k+1}^\top\right] &= (\mathrm{I}-\beta_k B_{11}^k)\mathbb{E}^{\mathcal{F}_k}\left[\tilde{\theta}_k\tilde{\theta}_k^\top\right](\mathrm{I}-\beta_k B_{11}^k)^\top + \beta_k^2 A_{12}\mathbb{E}^{\mathcal{F}_k}\left[\tilde{w}_k\tilde{w}_k^\top\right]A_{12}^\top \\
&\quad + \beta_k^2\mathbb{E}^{\mathcal{F}_k}\left[V_{k+1}V_{k+1}^\top\right] - \beta_k\Big((\mathrm{I}-\beta_k B_{11}^k)\mathbb{E}^{\mathcal{F}_k}\left[\tilde{\theta}_k\tilde{w}_k^\top\right]A_{12}^\top + A_{12}\mathbb{E}^{\mathcal{F}_k}\left[\tilde{w}_k\tilde{\theta}_k^\top\right](\mathrm{I}-\beta_k B_{11}^k)^\top\Big)
\end{aligned}
$$

Taking total expectations and evaluating the recursion gives

$$
\begin{aligned}
\Theta_{k+1} &= \Gamma^{(1)}_{0:k}\Theta_0(\Gamma^{(1)}_{0:k})^\top + \sum_{j=0}^{k}\beta_j^2\Gamma^{(1)}_{j+1:k}(A_{12}\Sigma_j A_{12}^\top + \mathbb{E}[V_{j+1}V_{j+1}^\top])\big(\Gamma^{(1)}_{j+1:k}\big)^\top \\
&\quad - \sum_{j=0}^{k}\beta_j\Gamma^{(1)}_{j+1:k}((\mathrm{I}-\beta_j B_{11}^j)\Omega_j A_{12}^\top + A_{12}\Omega_j^\top(\mathrm{I}-\beta_j B_{11}^j)^\top)\big(\Gamma^{(1)}_{j+1:k}\big)^\top
\end{aligned}
$$

The above implies

$$
\begin{aligned}
\mathrm{M}^{\tilde{\theta}}_{k+1} &\leq p_\Delta\Bigg\{ (G^{(1)}_{0:k})^2\,\mathrm{M}^{\tilde{\theta}}_0 + 2\|A_{12}\|\sum_{j=0}^{k}\beta_j(G^{(1)}_{j+1:k})^2(1-\beta_j a_\Delta/2)\,\mathrm{M}^{\tilde{\theta},\tilde{w}}_j \Bigg\} \\
&\quad + p_\Delta\sum_{j=0}^{k}\beta_j^2(G^{(1)}_{j+1:k})^2\big(\|A_{12}\|^2\,\mathrm{M}^{\tilde{w}}_j + \|\mathbb{E}[V_{j+1}V_{j+1}^\top]\|\big)
\end{aligned}
\tag{50}
$$

Applying (34) and Corollary 14 yield

$$
\begin{aligned}
\mathrm{M}^{\tilde{\theta}}_{k+1} &\leq p_\Delta\Bigg\{ (G^{(1)}_{0:k})^2\,\mathrm{M}^{\tilde{\theta}}_0 + \tilde{m}_V\varrho^{a_\Delta/2}\beta_{k+1} + \tilde{m}_V\sum_{j=0}^{k}\beta_j^2(G^{(1)}_{j+1:k})^2\,\mathrm{M}^{\tilde{\theta}}_j \Bigg\} \\
&\quad + 2p_\Delta\|A_{12}\|\sum_{j=0}^{k}\beta_j G^{(1)}_{j:k}G^{(1)}_{j+1:k}\,\mathrm{M}^{\tilde{\theta},\tilde{w}}_j + p_\Delta\big(\|A_{12}\|^2 + \tilde{m}_V\big)\sum_{j=0}^{k}\beta_j^2(G^{(1)}_{j+1:k})^2\,\mathrm{M}^{\tilde{w}}_j,
\end{aligned}
\tag{51}
$$

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Applying (48), we can bound the second last term in (51) as

$$
\sum_{j=0}^{k} \beta_j G_{j:k}^{(1)} G_{j+1:k}^{(1)} \mathrm{M}_j^{\tilde{\theta},\tilde{w}} \leq \sum_{j=0}^{k} \beta_j G_{j:k}^{(1)} G_{j+1:k}^{(1)} \Big(\mathrm{C}_0^{\tilde{\theta},\tilde{w}} \, G_{0:j-1}^{(2)} + \mathrm{C}_1^{\tilde{\theta},\tilde{w}} \, \beta_j + \mathrm{C}_2^{\tilde{\theta},\tilde{w}} \sum_{i=0}^{j-1} \gamma_i^2 G_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} \Big)
$$

$$
\leq \frac{2\, \mathrm{C}_0^{\tilde{\theta},\tilde{w}}}{a_\Delta} G_{0:k}^{(1)} + \mathrm{C}_1^{\tilde{\theta},\tilde{w}} \, \varrho^{a_{22}/2} \beta_{k+1} + \mathrm{C}_2^{\tilde{\theta},\tilde{w}} \sum_{j=0}^{k} \beta_j G_{j:k}^{(1)} G_{j+1:k}^{(1)} \sum_{i=0}^{j-1} \gamma_i^2 G_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}}
$$

where the second inequality is derived using Corollary 14. To bound the last term in the above we start from the following observation. Indeed, taking into account definition of β_∞ in (20), we get $(1 - \beta_\ell a_\Delta/2)^{-1} \leq 1 + \beta_\ell a_\Delta$. This inequality and assumption A2-2 yield that

$$
\frac{\gamma_{\ell-1}}{\gamma_\ell} \frac{1 - \gamma_\ell a_{22}/2}{1 - \beta_\ell a_\Delta/2} \leq (1 + \epsilon_\gamma \gamma_\ell)(1 - \gamma_\ell a_{22}/2)(1 + \beta_\ell a_\Delta)
$$

$$
\leq 1 - \gamma_\ell \left\{ a_{22}/2 - a_\Delta \kappa - \epsilon \right\} + \epsilon_\gamma \gamma_\ell^2 \left\{ \kappa a_\Delta - a_{22}/2 \right\} \leq 1 - (1/8) a_{22} \gamma_\ell, \tag{52}
$$

since $\kappa_\infty \leq (1/4) a_{22}/a_\Delta$, see (16). We observe the following chain

$$
\sum_{j=0}^{k} \beta_j G_{j:k}^{(1)} G_{j+1:k}^{(1)} \sum_{i=0}^{j-1} \gamma_i^2 G_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} = \sum_{i=0}^{k-1} \gamma_i^2 \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \beta_j G_{j:k}^{(1)} G_{j+1:k}^{(1)} G_{i+1:j-1}^{(2)}
$$

$$
= \sum_{i=0}^{k-1} \gamma_i^2 G_{i+1:k}^{(1)} \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \beta_j G_{j+1:k}^{(1)} \frac{G_{i+1:j-1}^{(2)}}{G_{i+1:j-1}^{(1)}} \overset{(a)}{\leq} \sum_{i=0}^{k-1} \beta_i \gamma_i G_{i+1:k}^{(1)} \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \gamma_{j-1} \prod_{\ell=i+1}^{j-1} \frac{\gamma_{\ell-1}}{\gamma_\ell} \frac{G_{i+1:j-1}^{(2)}}{G_{i+1:j-1}^{(1)}}
$$

$$
\overset{(b)}{\leq} \sum_{i=0}^{k-1} \beta_i \gamma_i G_{i+1:k}^{(1)} \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \gamma_{j-1} \prod_{\ell=i+1}^{j-1} (1 - (1/8) \gamma_\ell a_{22}) \overset{(c)}{\leq} \frac{8\varsigma}{a_{22}} \sum_{i=0}^{k-1} \beta_i \gamma_i G_{i+1:k}^{(1)} \mathrm{M}_i^{\tilde{\theta}}. \tag{53}
$$

where (a) is due to $\beta_j \leq \beta_i$ and $G_{j+1:k}^{(1)} \leq 1$, (b) is due to (52), (c) is due to A2-1 and $\sum_{j=i+1}^{k} \gamma_j \prod_{\ell=i+1}^{j-1}(1 - \gamma_\ell \tilde{a}) \leq (8/a_{22})$ for any i, k.

Moreover, applying (41), we can bound the last term of (51) as:

$$
\sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \mathrm{M}_j^{\tilde{w}} \leq \sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \Big(\mathrm{C}_0^{\tilde{w}} \, G_{0:j-1}^{(2)} + \mathrm{C}_1^{\tilde{w}} \, \gamma_j + \mathrm{C}_2^{\tilde{w}} \sum_{i=0}^{j-1} \gamma_i^2 G_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} \Big)
$$

$$
\leq \frac{\mathrm{C}_0^{\tilde{w}} \, G_{0:k}^{(1)}}{1 - \beta_0 a_\Delta/2} \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} + \gamma_0 \, \mathrm{C}_1^{\tilde{w}} \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} + \mathrm{C}_2^{\tilde{w}} \sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \sum_{i=0}^{j-1} \gamma_i^2 G_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}}
$$

$$
\leq \Big(\frac{\mathrm{C}_0^{\tilde{w}} \, G_{0:k}^{(1)}}{1 - \beta_0 a_\Delta/2} + \gamma_0 \, \mathrm{C}_1^{\tilde{w}} \Big) \varrho^{a_{22}/2} \beta_{k+1} + \mathrm{C}_2^{\tilde{w}} \sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \sum_{i=0}^{j-1} \gamma_i^2 G_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}},
$$

where the last inequality is due to Corollary 14. In addition, similar to (53), we can derive the bound

$$
\sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \sum_{i=0}^{j-1} \gamma_i^2 G_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} \leq \frac{(8\varsigma)/a_{22}}{1 - \beta_0 a_\Delta/2} \sum_{i=0}^{k} \beta_i^2 \gamma_i G_{i+1:k}^{(1)} \mathrm{M}_i^{\tilde{\theta}}
$$

Substituting the above inequalities into (51) leads to

$$\boxed{M_{k+1}^{\tilde{\theta}} \leq C_0^{\tilde{\theta}} \, G_{0:k}^{(1)} + C_1^{\tilde{\theta}} \, \beta_{k+1} + C_2^{\tilde{\theta}} \sum_{j=0}^{k} \gamma_j \beta_j G_{j+1:k}^{(1)} \, M_j^{\tilde{\theta}}} \tag{54}$$

where

$$
\begin{aligned}
C_0^{\tilde{\theta}} &:= p_\Delta \Big(M_0^{\tilde{\theta}} + \frac{4\|A_{12}\| \, C_0^{\tilde{\theta},\tilde{w}}}{a_\Delta} \Big), \\
C_1^{\tilde{\theta}} &:= p_\Delta \Big\{ \tilde{m}_V \varrho^{a_\Delta/2} + 2\|A_{12}\| \, C_1^{\tilde{\theta},\tilde{w}} \, \varrho^{a_{22}/2} + (\|A_{12}\|^2 + \tilde{m}_V)\Big(\gamma_0 \, C_1^{\tilde{w}} + \frac{C_0^{\tilde{w}}}{1 - \beta_0 a_\Delta/2} \Big) \varrho^{a_{22}/2} \Big\}, \\
C_2^{\tilde{\theta}} &:= p_\Delta \Big\{ \frac{16\varsigma \|A_{12}\| \, C_2^{\tilde{\theta},\tilde{w}}}{a_{22}} + \tilde{m}_V + (\|A_{12}\|^2 + \tilde{m}_V)\frac{8 \, C_2^{\tilde{w}} \, \varsigma / a_{22}}{1 - \beta_0 a_\Delta/2} \Big\}.
\end{aligned} \tag{55}
$$

This completes the proof for Proposition 5.

Completing the Proof of Theorem 1 We complete the proof by analyzing the convergence rate of $M_k^{\tilde{\theta}}$ using (54). Consider the following recursion which upper bounds $M_k^{\tilde{\theta}}$:

$$U_{k+1} = C_0^{\tilde{\theta}} \, G_{0:k}^{(1)} + C_1^{\tilde{\theta}} \, \beta_{k+1} + C_2^{\tilde{\theta}} \sum_{j=0}^{k} \gamma_j \beta_j G_{j+1:k}^{(1)} \, U_j,$$

where we have set $U_0 = C_0^{\tilde{\theta}}$. Observe that

$$U_{k+1} - (1 - \beta_k a_\Delta/2) \, U_k = C_1^{\tilde{\theta}}(\beta_{k+1} - (1 - \beta_k a_\Delta/2)\beta_k) + C_2^{\tilde{\theta}} \gamma_k \beta_k \, U_k$$
$$\Longleftrightarrow U_{k+1} = (1 - \beta_k(a_\Delta/2 - C_2^{\tilde{\theta}} \gamma_k)) \, U_k + C_1^{\tilde{\theta}}(\beta_{k+1} - \beta_k + \beta_k^2 a_\Delta/2)$$

Since $\gamma_k \leq \gamma_0 \leq \frac{a_\Delta}{4 \, C_2^{\tilde{\theta}}}$, we have

$$U_{k+1} \leq (1 - \beta_k a_\Delta/4) \, U_k + C_1^{\tilde{\theta}} \beta_k^2 a_\Delta/2$$

Evaluating the recursion gives

$$U_{k+1} \leq \prod_{\ell=0}^{k}(1 - \beta_\ell a_\Delta/4) \, U_0 + C_1^{\tilde{\theta}}(a_\Delta/2) \sum_{j=0}^{k} \beta_j^2 \prod_{\ell=j+1}^{k}(1 - \beta_\ell a_\Delta/4)$$

Applying Corollary 14 shows $\sum_{j=0}^{k} \beta_j^2 \prod_{\ell=j+1}^{k}(1 - \beta_\ell a_\Delta/4) \leq \varrho^{a_\Delta/4}\beta_{k+1}$. Lastly, observing that $M_k^{\tilde{\theta}} \leq U_k$ gives

$$\boxed{M_{k+1}^{\tilde{\theta}} \leq C_0^{\tilde{\theta}} \prod_{\ell=0}^{k} \Big(1 - \beta_\ell \frac{a_\Delta}{4} \Big) + C_1^{\tilde{\theta}} \varrho^{a_\Delta/4}\frac{a_\Delta}{2}\beta_{k+1}.} \tag{56}$$

To finish the proof of (12), we observe (i) the constant $C_0^{\tilde{\theta}} \leq C_0^{\tilde{\theta},\mathsf{mtg}} V_0$ for some constant $C_0^{\tilde{\theta},\mathsf{mtg}}$, (ii) the inequality that $\mathbb{E}[\|\theta_k - \theta^\star\|^2] \leq d_\theta \, M_k^{\tilde{\theta}}$, and (iii) setting the constant $C_1^{\tilde{\theta},\mathsf{mtg}} := C_1^{\tilde{\theta}} \varrho^{a_\Delta/4}(a_\Delta/2)$.

Our last endeavor is to prove (13). Observe that the tracking error $\widehat{w}_k := w_k - A_{22}^{-1}(b_2 - A_{21}\theta_k)$ may be represented as

$$
\begin{aligned}
\widehat{w}_k &= w_k - w^\star + w^\star - A_{22}^{-1}(b_2 - A_{21}\theta_k) \\
&= \tilde{w}_k - C_{k-1}\tilde{\theta}_k + A_{22}^{-1}\big((b_2 - A_{21}\theta^\star) - (b_2 - A_{21}\theta_k)\big) = \tilde{w}_k - L_k\tilde{\theta}_k
\end{aligned}
$$

using the definitions in (17). This leads to the following estimate of $M_k^{\widehat{w}} := \|\mathbb{E}[\widehat{w}_k \widehat{w}_k^\top]\|$:

$$
M_{k+1}^{\widehat{w}} \leq 2\, M_{k+1}^{\tilde{w}} + 2\|L_{k+1}\|^2 \, M_{k+1}^{\tilde{\theta}} \leq 2\, M_{k+1}^{\tilde{w}} + 2\, L_\infty^2 \frac{\lambda_{\mathsf{max}}(Q_{22})}{\lambda_{\mathsf{min}}(Q_\Delta)} \, M_{k+1}^{\tilde{\theta}} \tag{57}
$$

In particular, substituting (56) into (41), we obtain:

$$
\begin{aligned}
M_{k+1}^{\tilde{w}} &\leq C_0^{\tilde{w}} G_{0:k}^{(2)} + C_1^{\tilde{w}} \gamma_{k+1} + C_2^{\tilde{w}} \sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \left\{ C_0^{\tilde{\theta}} \prod_{\ell=0}^{j-1} \left(1 - \beta_\ell \frac{a_\Delta}{4}\right) + C_1^{\tilde{\theta}} \varrho^{a_\Delta/4} \frac{a_\Delta}{2} \beta_j \right\} \\
&\leq C_0^{\tilde{w}} G_{0:k}^{(2)} + \gamma_{k+1} \left\{ C_1^{\tilde{w}} + C_1^{\tilde{\theta}} \varrho^{a_\Delta/4} \frac{a_\Delta}{2} C_2^{\tilde{w}} \varrho^{a_{22}/2} + \frac{C_2^{\tilde{w}} C_0^{\tilde{\theta}} \varrho^{a_{22}/2}}{1 - \beta_0 a_\Delta/4} \prod_{\ell=0}^{k} \left(1 - \beta_\ell \frac{a_\Delta}{4}\right) \right\}
\end{aligned}
$$

where the last inequality is due to the observation $G_{j+1:k}^{(2)} \leq \prod_{i=j+1}^{k}(1 - \gamma_i a_{22}/4)^2$ and the application of Corollary 14. Furthermore using $G_{0:k}^{(2)} \leq \prod_{\ell=0}^{k}(1 - \beta_\ell a_\Delta/4)$ and applying (57) gives

$$
\boxed{M_{k+1}^{\widehat{w}} \leq C_0^w \prod_{\ell=0}^{k}(1 - \beta_\ell a_\Delta/4) + C_1^{\widehat{w},\mathsf{mtg}} \gamma_{k+1},}
$$

where

$$
\begin{aligned}
C_0^w &:= 2\left\{ L_\infty^2 \frac{\lambda_{\mathsf{max}}(Q_{22})}{\lambda_{\mathsf{min}}(Q_\Delta)} C_0^{\tilde{\theta}} + \frac{\varrho^{a_{22}/2} C_2^{\tilde{w}} C_0^{\tilde{\theta}}}{1 - \beta_0 a_\Delta/4} + C_0^{\tilde{w}} \right\} \\
C_1^{\widehat{w},\mathsf{mtg}} &:= 2\left\{ \kappa\, L_\infty^2 \frac{\lambda_{\mathsf{max}}(Q_{22})}{\lambda_{\mathsf{min}}(Q_\Delta)} \varrho^{a_\Delta/4} \frac{a_\Delta}{2} C_1^{\tilde{\theta}} + C_1^{\tilde{w}} + \varrho^{a_\Delta/4} \frac{a_\Delta}{2} C_2^{\tilde{w}} \varrho^{a_{22}/2} C_1^{\tilde{\theta}} \right\}
\end{aligned} \tag{58}
$$

We conclude the proof for Theorem 1 by observing that $C_0^w \leq C_0^{\widehat{w},\mathsf{mtg}} V_0$ for some constant $C_0^{\widehat{w},\mathsf{mtg}}$.

B.2. Detailed Proofs of Theorem 2

To facilitate our discussions next, define a few additional constants as:

$$
\widetilde{G}_{m:n}^{(1)} := \prod_{i=m}^{n}(1 - \beta_i a_\Delta/4), \quad \widetilde{G}_{m:n}^{(2)} := \prod_{i=m}^{n}(1 - \gamma_i a_{22}/4)
$$

$$
B_{11,\infty} := \|\Delta\| + \sqrt{\lambda_{\mathsf{min}}(Q_\Delta)^{-1}\lambda_{\mathsf{max}}(Q_{22})}\, L_\infty \|A_{12}\|, \quad B_{22,\infty} := \kappa\, C_\infty \|A_{12}\| + \|A_{22}\|.
$$

Before we begin the proof, notice by observing the form of (25) that that A4 is satisfied by the Markovian noise through setting

$$m_V = \overline{b} \vee (3\overline{A}), \ m_W = \overline{b} \vee (3\overline{A}),$$

and furthermore (34) is satisfied with $\tilde{m}_V, \tilde{m}_W, \tilde{m}_{VW}$ defined in (36) and the above m_V, m_W. Moreover, for $i = 0, 1$, the second order moments of the decomposed noise satisfy:

$$\|\mathbb{E}[V_{k+1}^{(i)}(V_{k+1}^{(i)})^\top]\| \le \tilde{m}_V^{(i)}(1 + M_k^{\tilde{\theta}} + M_k^{\tilde{w}}), \ \|\mathbb{E}[W_{k+1}^{(i)}(W_{k+1}^{(i)})^\top]\| \le \tilde{m}_W^{(i)}(1 + M_k^{\tilde{\theta}} + M_k^{\tilde{w}}) \quad (59)$$

$$\|\mathbb{E}[V_{k+1}^{(i)}(W_{k+1}^{(i)})^\top]\| \le \tilde{m}_{VW}^{(i)}(1 + M_k^{\tilde{\theta}} + M_k^{\tilde{w}}), \quad (60)$$

for some constants $\tilde{m}_V^{(i)}, \tilde{m}_W^{(i)}, \tilde{m}_{VW}^{(i)}, i = 1, 2$. We proceed with the proof for Theorem 2 as follows.

Bounding $M_k^{\tilde{w}}$ (Proof of Lemma 6 and Proposition 7) Repeating the analysis that leaded to (40) and using the martingale property of $V_{k+1}^{(0)}, W_{k+1}^{(0)}$ shows that

$$\| \mathbb{E}[\tilde{w}_{k+1}^{(0)}(\tilde{w}_{k+1}^{(0)})^\top]\| \le (G_{0:k}^{(2)})^2 p_{22} M_0^{\tilde{w}} + \widetilde{C}_0 \sum_{j=0}^{k} \gamma_j^2 (G_{j+1:k}^{(2)})^2 (1 + M_j^{\tilde{w}} + M_j^{\tilde{\theta}}), \quad (61)$$

where

$$\widetilde{C}_0 = p_{22} \Big[\{ C_\infty^2 \vee 1 + \sqrt{d_w d_\theta} \, C_\infty \} \{ \tilde{m}_V + \kappa^2 \tilde{m}_W^{(0)} \vee \tilde{m}_V^\theta + \kappa^2 \tilde{m}_W^{(0)} \} \Big] \vee \big[\tilde{m}_V^{(0)} + \kappa^2 \tilde{m}_W^{(0)} \big]. \quad (62)$$

Our next endeavor is to bound $\mathbb{E}[\|\tilde{w}_{k+1}^{(1)}\|^2]$. Evaluating the recursion in (26) gives

$$\tilde{w}_{k+1}^{(1)} = \Gamma_{0:k}^{(2)} \tilde{w}_0^{(1)} + \sum_{j=0}^{k} \gamma_j \Gamma_{j+1:k}^{(2)} (W_{j+1}^{(1)} + C_j V_{j+1}^{(1)}) \quad (63)$$

Set $\widetilde{\psi}_j^{b_i} := \psi_j^{b_i} + \Psi_j^{A_{i1}} \theta^\star + \Psi_j^{A_{i2}} w^\star$ for $i = 1, 2$. Using the definitions, the combined noise has the following expression

$$W_{j+1}^{(1)} + C_j V_{j+1}^{(1)} = \big(\widetilde{\psi}_j^{b_2} - \widetilde{\psi}_{j+1}^{b_2} \big) + C_j \big(\widetilde{\psi}_j^{b_1} - \widetilde{\psi}_{j+1}^{b_1} \big) + \Big\{ \Psi_j^{A_{22}} - \Psi_{j+1}^{A_{22}} + C_j \big(\Psi_j^{A_{12}} - \Psi_{j+1}^{A_{12}} \big) \Big\} \tilde{w}_j$$

$$+ \Big\{ \Psi_j^{A_{21}} - \Psi_{j+1}^{A_{21}} - (\Psi_j^{A_{22}} - \Psi_{j+1}^{A_{22}}) C_{j-1} + C_j (\Psi_j^{A_{11}} - \Psi_{j+1}^{A_{11}}) - C_j (\Psi_j^{A_{12}} - \Psi_{j+1}^{A_{12}}) C_{j-1} \Big\} \tilde{\theta}_j$$

Upon some algebra manipulations that are detailed in Appendix B.2.1, we deduce that the combined noise may be decomposed as:

$$W_{j+1}^{(1)} + C_j V_{j+1}^{(1)} \equiv \psi_j^{WV} - \psi_{j+1}^{WV} + \widetilde{\Psi}_j^{WV,\tilde{\theta}} \tilde{\theta}_j + \widetilde{\Psi}_j^{WV,\tilde{w}} \tilde{w}_j$$

$$+ \big(\Upsilon_j^{WV,\tilde{\theta}} \tilde{\theta}_j - \Upsilon_{j+1}^{WV,\tilde{\theta}} \tilde{\theta}_{j+1} \big) + \big(\Upsilon_j^{WV,\tilde{w}} \tilde{w}_j - \Upsilon_{j+1}^{WV,\tilde{w}} \tilde{w}_{j+1} \big) \quad (63)$$

$$+ \Phi^{WV,\tilde{\theta}} \big(\tilde{\theta}_{j+1} - \tilde{\theta}_j \big) + \Phi^{WV,\tilde{w}} \big(\tilde{w}_{j+1} - \tilde{w}_j \big),$$

where it holds that

$$\|\psi_j^{WV}\| \vee \|\Upsilon_j^{WV,\tilde{\theta}}\| \vee \|\Upsilon_j^{WV,\tilde{w}}\| \vee \|\Phi^{WV,\tilde{w}}\| \vee \|\Phi^{WV,\tilde{\theta}}\| \le E_0^{WV}, \ \ \|\widetilde{\Psi}_j^{WV,\tilde{\theta}}\| \vee \|\widetilde{\Psi}_j^{WV,\tilde{w}}\| \le E_0^{WV} \gamma_j,$$

with

$$
\mathrm{E}_0^{WV} := \max\{\overline{\mathrm{b}}(1 + \mathrm{C}_\infty), \overline{\mathrm{A}}(1 + 2\,\mathrm{C}_\infty + \mathrm{C}_\infty^2), \overline{\mathrm{A}}C_2^U \varrho^{a_{22}/2}(1 + \mathrm{C}_\infty)(1 + \varsigma)\}. \quad (64)
$$

Let us bound the second term in (63) one by one as follows. Using Lemma 11, we obtain

$$
\sum_{j=0}^{k} \gamma_j \Gamma_{j+1:k}^{(2)}\Big(\psi_j^{WV} - \psi_{j+1}^{WV} + \big(\Upsilon_j^{WV,\tilde\theta}\tilde\theta_j - \Upsilon_{j+1}^{WV,\tilde\theta}\tilde\theta_{j+1}\big) + \big(\Upsilon_j^{WV,\tilde{w}}\tilde{w}_j - \Upsilon_{j+1}^{WV,\tilde{w}}\tilde{w}_{j+1}\big)\Big)
$$
$$
= \gamma_0 \Gamma_{1:k}^{(2)}\big(\psi_0^{WV} + \Upsilon_0^{WV,\tilde\theta}\tilde\theta_0 + \Upsilon_0^{WV,\tilde{w}}\tilde{w}_0\big) - \gamma_k\big(\psi_{k+1}^{WV} + \Upsilon_{k+1}^{WV,\tilde\theta}\tilde\theta_{k+1} + \Upsilon_{k+1}^{WV,\tilde{w}}\tilde{w}_{k+1}\big)
$$
$$
+ \sum_{j=1}^{k}\big(\gamma_j^2 B_{22}^j \Gamma_{j+1:k}^{(2)} + (\gamma_j - \gamma_{j-1})\Gamma_{j:k}^{(2)}\big)\big(\psi_j^{WV} + \Upsilon_j^{WV,\tilde\theta}\tilde\theta_j + \Upsilon_j^{WV,\tilde{w}}\tilde{w}_j\big),
$$

Secondly,

$$
\sum_{j=0}^{k} \gamma_j \Gamma_{j+1:k}^{(2)} \Phi^{WV,\tilde\theta}\big(\tilde\theta_{j+1} - \tilde\theta_j\big) = -\sum_{j=0}^{k} \gamma_j \beta_j \Gamma_{j+1:k}^{(2)} \Phi^{WV,\tilde\theta}\big(A_{12}\tilde{w}_{j+1} + V_{j+1}\big)
$$

$$
\sum_{j=0}^{k} \gamma_j \Gamma_{j+1:k}^{(2)} \Phi^{WV,\tilde{w}}\big(\tilde{w}_{j+1} - \tilde{w}_j\big) = -\sum_{j=0}^{k} \gamma_j^2 \Gamma_{j+1:k}^{(2)} \Phi^{WV,\tilde{w}}\big(W_{j+1} + C_j V_{j+1}\big)
$$

As a consequence of (59)–(60), we have

$$
\mathbb{E}[\|A_{12}\tilde{w}_{j+1} + V_{j+1}\|^2] \le \tilde{m}_{\Delta\tilde\theta}\big(1 + \mathrm{M}_j^{\tilde{w}} + \mathrm{M}_j^{\tilde\theta}\big), \quad \mathbb{E}[\|W_{j+1} + C_j V_{j+1}\|^2] \le \tilde{m}_{\Delta\tilde{w}}\big(1 + \mathrm{M}_j^{\tilde{w}} + \mathrm{M}_j^{\tilde\theta}\big)
$$

where

$$
\tilde{m}_{\Delta\tilde\theta} := 2\big\{\|A_{12}\|^2 + \tilde{m}_V\big\}, \quad \tilde{m}_{\Delta\tilde{w}} := 2(\tilde{m}_W + \mathrm{C}_\infty \tilde{m}_V).
$$

Noting that $\tilde{w}_0^{(1)} = 0$, taking Euclidean norm on both sides of (63) yields

$$
\|\tilde{w}_{k+1}^{(1)}\| \le \sqrt{p_{22}}\Big\{ \mathrm{E}_0^{WV}\big[G_{1:k}^{(2)}\gamma_0(1 + \|\tilde\theta_0\| + \|\tilde{w}_0\|) + \gamma_k(1 + \|\tilde\theta_{k+1}\| + \|\tilde{w}_{k+1}\|)\big]\Big\}
$$
$$
+ \sqrt{p_{22}}\,\mathrm{E}_0^{WV}\Big\{ \sum_{j=1}^{k} G_{j+1:k}^{(2)}(\gamma_j^2 + \|\gamma_j^2 B_{22}^j + (\gamma_j - \gamma_{j-1})(I - B_{22}^j)\|)(1 + \|\tilde{w}_j\| + \|\tilde\theta_j\|)\Big\} \quad (65)
$$
$$
+ \sqrt{p_{22}}\,\mathrm{E}_0^{WV}\sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)}(\kappa\|A_{12}\tilde{w}_{j+1} + V_{j+1}\| + \|W_{j+1} + C_j V_{j+1}\|)
$$

Note that for any sequence $(b_j)_{j\ge 0}$, the following inequality holds:

$$
\Big(\sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} b_j\Big)^2 \le \Big(\sum_{i=0}^{k} \gamma_i^2 G_{i+1:k}^{(2)}\Big)\sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} b_j^2 \le \gamma_{k+1}\varrho^{a_{22}/2}\sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} b_j^2, \quad (66)
$$

where the first inequality is due to Jensen's inequality and the second inequality is due to Corollary 14. Using $\|B_{22}^j\| \le \mathrm{B}_{22,\infty}$, $|\gamma_j - \gamma_{j-1}| \le \frac{a_{22}}{8}\gamma_j^2$ [cf. it is a direct consequence of A2-2 and the

fact $\gamma_j \leq \gamma_{j-1}$] and applying the above inequality to (65) yields

$$
\|\tilde{w}_{k+1}^{(1)}\|^2 \leq 9p_{22}(\mathrm{E}_0^{WV})^2\Big\{(G_{1:k}^{(2)})^2\gamma_0^2(1+\|\tilde{w}_0\|+\|\tilde{\theta}_0\|)^2 + \gamma_k^2(1+\|\tilde{\theta}_{k+1}\|^2+\|\tilde{w}_{k+1}\|^2)\Big\}
$$
$$
+ 9p_{22}(\mathrm{E}_0^{WV})^2(\mathrm{B}_{22,\infty}+\frac{a_{22}}{8}+1)^2\varrho^{a_{22}/2}\gamma_{k+1}\Big\{\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}(1+\|\tilde{w}_j\|^2+\|\tilde{\theta}_j\|^2)\Big\}
$$
$$
+ 9p_{22}(\mathrm{E}_0^{WV})^2\varrho^{a_{22}/2}\gamma_{k+1}\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}\big(\kappa\|A_{12}\tilde{w}_{j+1}+V_{j+1}\|^2+\|W_{j+1}+C_jV_{j+1}\|^2\big)
$$

Using the fact $\mathbb{E}[\|\tilde{w}_k\|^2] \leq d_w\|\mathbb{E}[\tilde{w}_k\tilde{w}_k^\top]\|$, $\mathbb{E}[\|\tilde{\theta}_k\|^2] \leq d_\theta\|\mathbb{E}[\tilde{\theta}_k\tilde{\theta}_k^\top]\|$ (cf. Corollary 22), taking the expectation on both sides yields

$$
\mathbb{E}[\|\tilde{w}_{k+1}^{(1)}\|^2] \leq 9p_{22}(\mathrm{E}_0^{WV})^2\Big\{(G_{1:k}^{(2)})^2\gamma_0^2(1+\|\tilde{w}_0\|+\|\tilde{\theta}_0\|)^2 + \gamma_k^2(1+d_\theta\,\mathrm{M}_{k+1}^{\tilde{\theta}}+d_w\,\mathrm{M}_{k+1}^{\tilde{w}})\Big\}
$$
$$
+ 9p_{22}(\mathrm{E}_0^{WV})^2(\mathrm{B}_{22,\infty}+\frac{a_{22}}{8}+1)^2\varrho^{a_{22}/2}\gamma_{k+1}\Big\{\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}(1+d_\theta\,\mathrm{M}_j^{\tilde{\theta}}+d_w\,\mathrm{M}_j^{\tilde{w}})\Big\}
$$
$$
+ 9p_{22}(\mathrm{E}_0^{WV})^2(\kappa\tilde{m}_{\Delta\tilde{\theta}}+\tilde{m}_{\Delta\tilde{w}})\varrho^{a_{22}/2}\gamma_{k+1}\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}\big(1+\mathrm{M}_j^{\tilde{\theta}}+\mathrm{M}_j^{\tilde{w}}\big)
$$

The above simplifies to

$$
\mathbb{E}[\|\tilde{w}_{k+1}^{(1)}\|^2] \leq \widetilde{\mathrm{C}}_1(G_{0:k}^{(2)})^2 + \widetilde{\mathrm{C}}_2\gamma_k^2(\mathrm{M}_{k+1}^{\tilde{\theta}}+\mathrm{M}_{k+1}^{\tilde{w}}) + \widetilde{\mathrm{C}}_3\gamma_{k+1}\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}(\mathrm{M}_j^{\tilde{\theta}}+\mathrm{M}_j^{\tilde{w}}) + \widetilde{\mathrm{C}}_4\gamma_k^2
$$
$$(66)$$

where we have used $\gamma_{k+1} \leq \gamma_k$ and defined

$$
\begin{aligned}
\widetilde{\mathrm{C}}_1 &= 9p_{22}(\mathrm{E}_0^{WV})^2(1+\|\tilde{w}_0\|+\|\tilde{\theta}_0\|)^2(\gamma_0/(1-\gamma_0 a_{22}/2))^2,\\
\widetilde{\mathrm{C}}_2 &= 9p_{22}(\mathrm{E}_0^{WV})^2(d_\theta \vee d_w),\\
\widetilde{\mathrm{C}}_3 &= 9p_{22}(\mathrm{E}_0^{WV})^2\varrho^{a_{22}/2}\big[(d_\theta \vee d_w)(\mathrm{B}_{22,\infty}+\frac{a_{22}}{8}+1)^2 + \big(\kappa\tilde{m}_{\Delta\tilde{\theta}}+\tilde{m}_{\Delta\tilde{w}}\big)\big],\\
\widetilde{\mathrm{C}}_4 &= \widetilde{\mathrm{C}}_2 + \varrho^{a_{22}/2}\widetilde{\mathrm{C}}_3.
\end{aligned}
$$
$$(67)$$

Notice that the intermediate results (61), (66) lead to Lemma 6.

Compared to (61), an important feature of the bound (66) is that the latter contains an extra γ_k factor. This indicates that the iterate $\tilde{w}_{k+1}^{(1)}$ driven by Markovian noise decays at a faster rate. As we will demonstrate below, the effect of the additional Markov noise is thus negligible compared to the martingale noise driven terms.

As the operator norm $\|\cdot\|$ is convex, applying Jensen's inequality yields

$$
\mathrm{M}_{k+1}^{\tilde{w}} \leq 2\|\mathbb{E}[\tilde{w}_{k+1}^{(1)}(\tilde{w}_{k+1}^{(1)})^\top]\| + 2\mathbb{E}[\tilde{w}_{k+1}^{(0)}(\tilde{w}_{k+1}^{(0)})^\top]\| \leq 2\mathbb{E}[\|\tilde{w}_{k+1}^{(1)}\|^2] + 2\|\mathbb{E}[\tilde{w}_{k+1}^{(0)}(\tilde{w}_{k+1}^{(0)})^\top]\|
$$

Substituting (61) and (66) gives

$$\mathrm{M}_{k+1}^{\tilde{w}} \leq 2\Big\{ \widetilde{\mathrm{C}}_1(G_{0:k}^{(2)})^2 + \widetilde{\mathrm{C}}_2\gamma_k^2(\mathrm{M}_{k+1}^{\tilde{w}} + \mathrm{M}_{k+1}^{\tilde{\theta}}) + \widetilde{\mathrm{C}}_3\gamma_{k+1}\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}(\mathrm{M}_j^{\tilde{w}} + \mathrm{M}_j^{\tilde{\theta}}) + \widetilde{\mathrm{C}}_4\gamma_k^2 \Big\}$$

$$+ 2\Big\{ p_{22}(G_{0:k}^{(2)})^2\, \mathrm{M}_0^{\tilde{w}} + \widetilde{\mathrm{C}}_0\varrho^{a_{22}/2}\gamma_{k+1} + \widetilde{\mathrm{C}}_0\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}(\mathrm{M}_j^{\tilde{w}} + \mathrm{M}_j^{\tilde{\theta}}) \Big\}$$

The assumption on step size in (27) guarantees $2\widetilde{\mathrm{C}}_2\gamma_k^2 \leq (1/2)$, which further implies

$$\mathrm{M}_{k+1}^{\tilde{w}} \leq 4\Big\{ \widetilde{\mathrm{C}}_1(G_{0:k}^{(2)})^2 + \widetilde{\mathrm{C}}_2\gamma_k^2\, \mathrm{M}_{k+1}^{\tilde{\theta}} + \widetilde{\mathrm{C}}_3\gamma_{k+1}\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}(\mathrm{M}_j^{\tilde{w}} + \mathrm{M}_j^{\tilde{\theta}}) + \widetilde{\mathrm{C}}_4\gamma_k^2 \Big\}$$

$$+ 4\Big\{ p_{22}(G_{0:k}^{(2)})^2\, \mathrm{M}_0^{\tilde{w}} + \widetilde{\mathrm{C}}_0\varrho^{a_{22}/2}\gamma_{k+1} + \widetilde{\mathrm{C}}_0\sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}(\mathrm{M}_j^{\tilde{w}} + \mathrm{M}_j^{\tilde{\theta}}) \Big\}$$

Like in the proof of Theorem 1, we set

$$\mathrm{U}_{k+1} = G_{0:k}^{(2)}(\widetilde{\mathrm{C}}_1 + p_{22}\, \mathrm{M}_0^{\tilde{w}}) + \widetilde{\mathrm{C}}_0\varrho^{a_{22}/2}\gamma_{k+1} + \sum_{j=0}^{k}\gamma_j^2 G_{j+1:k}^{(2)}(\widetilde{\mathrm{C}}_3\gamma_j + \widetilde{\mathrm{C}}_0)(\mathrm{U}_j + \mathrm{M}_j^{\tilde{\theta}})$$

with $\mathrm{U}_0 = \widetilde{\mathrm{C}}_1 + p_{22}\, \mathrm{M}_0^{\tilde{w}}$. Through evaluating the recursion, we observe that for any $k \geq 0$, it holds

$$\mathrm{M}_{k+1}^{\tilde{w}} \leq 4\Big\{ \mathrm{U}_{k+1} + \sum_{j=1}^{k+1}\gamma_{j-1}^2 G_{j:k}^{(2)}(\widetilde{\mathrm{C}}_2\, \mathrm{M}_j^{\tilde{\theta}} + \widetilde{\mathrm{C}}_4) \Big\}$$

$$\tag{66}$$

$$\leq 4\Big\{ \mathrm{U}_{k+1} + \gamma_k^2(\widetilde{\mathrm{C}}_2\, \mathrm{M}_{k+1}^{\tilde{\theta}} + \widetilde{\mathrm{C}}_4) + \sum_{j=1}^{k}\gamma_j^2 G_{j+1:k}^{(2)}(\widetilde{\mathrm{C}}_2\, \mathrm{M}_j^{\tilde{\theta}} + \widetilde{\mathrm{C}}_4) \Big\}$$

where the last inequality is due to A2-2 which guarantees that $\gamma_{j-1}^2 G_{j:k}^{(2)} \leq \gamma_j^2 G_{j+1:k}$. Moreover, the sequence U_{k+1} can be expressed as follows:

$$\mathrm{U}_{k+1} - (1 - \gamma_k a_{22}/2)\, \mathrm{U}_k = \widetilde{\mathrm{C}}_0\varrho^{a_{22}/2}(\gamma_{k+1} - \gamma_k(1 - \gamma_k a_{22}/2)) + \gamma_k^2(\widetilde{\mathrm{C}}_3\gamma_k + \widetilde{\mathrm{C}}_0)(\mathrm{U}_k + \mathrm{M}_k^{\tilde{\theta}})$$

$$\leq \widetilde{\mathrm{C}}_0\varrho^{a_{22}/2}(a_{22}/2)\gamma_k^2 + \gamma_k^2(\widetilde{\mathrm{C}}_3\gamma_k + \widetilde{\mathrm{C}}_0)(\mathrm{U}_k + \mathrm{M}_k^{\tilde{\theta}})$$

As the step size satisfies $\gamma_k(\widetilde{\mathrm{C}}_3\gamma_0 + \widetilde{\mathrm{C}}_0) \leq \frac{a_{22}}{4}$, we get

$$\mathrm{U}_{k+1} \leq (1 - \gamma_k a_{22}/4)\, \mathrm{U}_k + \gamma_k^2(\widetilde{\mathrm{C}}_3\gamma_0 + \widetilde{\mathrm{C}}_0)\, \mathrm{M}_k^{\tilde{\theta}} + \widetilde{\mathrm{C}}_0\varrho^{a_{22}/2}(a_{22}/2)\gamma_k^2$$

$$\implies \mathrm{U}_{k+1} \leq \widetilde{G}_{0:k}^{(2)}\, \mathrm{U}_0 + \sum_{j=0}^{k}\gamma_j^2\widetilde{G}_{j+1:k}^{(2)}\Big\{ (\widetilde{\mathrm{C}}_3\gamma_0 + \widetilde{\mathrm{C}}_0)\, \mathrm{M}_j^{\tilde{\theta}} + (\widetilde{\mathrm{C}}_0\varrho^{a_{22}/2}(a_{22}/2)) \Big\}.$$

Substituting the above into (66) yields

$$\mathrm{M}_{k+1}^{\tilde{w}} \leq 4\Big\{ \widetilde{G}_{0:k}^{(2)}\, \mathrm{U}_0 + \sum_{j=0}^{k}\gamma_j^2\widetilde{G}_{j+1:k}^{(2)}\big\{ (\widetilde{\mathrm{C}}_3\gamma_0 + \widetilde{\mathrm{C}}_0)\, \mathrm{M}_j^{\tilde{\theta}} + (\widetilde{\mathrm{C}}_0\varrho^{a_{22}/2}(a_{22}/2)) \big\} \Big\}$$

$$+ 4\Big\{ \gamma_k^2(\widetilde{\mathrm{C}}_2\, \mathrm{M}_{k+1}^{\tilde{\theta}} + \widetilde{\mathrm{C}}_4) + \sum_{j=1}^{k}\gamma_j^2 G_{j+1:k}^{(2)}(\widetilde{\mathrm{C}}_2\, \mathrm{M}_j^{\tilde{\theta}} + \widetilde{\mathrm{C}}_4) \Big\}$$

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Finally, using the fact that $\gamma_k^2 \leq \varsigma \gamma_{k+1}$ yields

$$\mathrm{M}_{k+1}^{\tilde{w}} \leq \widetilde{G}_{0:k}^{(2)} \widetilde{\mathrm{C}}_0^{\tilde{w}} + \widetilde{\mathrm{C}}_1^{\tilde{w}} \gamma_{k+1} + \widetilde{\mathrm{C}}_2^{\tilde{w}} \sum_{j=0}^{k} \gamma_j^2 \widetilde{G}_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{\theta}} + \widetilde{\mathrm{C}}_3^{\tilde{w}} \gamma_k^2 \mathrm{M}_{k+1}^{\tilde{\theta}}. \tag{66}$$

where

$$\widetilde{\mathrm{C}}_0^{\tilde{w}} := 4(\widetilde{\mathrm{C}}_1 + p_{22} \mathrm{M}_0^{\tilde{w}}), \quad \widetilde{\mathrm{C}}_1^{\tilde{w}} := 4(\widetilde{\mathrm{C}}_4(\varsigma + \varrho^{a_{22}/2}) + \widetilde{\mathrm{C}}_0(\varrho^{a_{22}/2})^2 (a_{22}/2)) \tag{67}$$

$$\widetilde{\mathrm{C}}_2^{\tilde{w}} := 4(\widetilde{\mathrm{C}}_3 \gamma_0 + \widetilde{\mathrm{C}}_2 + \widetilde{\mathrm{C}}_0), \quad \widetilde{\mathrm{C}}_3^{\tilde{w}} := 4\widetilde{\mathrm{C}}_2.$$

This concludes the proof for Proposition 7.

Before we proceed, we need to bound $\| \mathbb{E}[\tilde{w}_{k+1}^{(0)}(\tilde{w}_{k+1}^{(0)})^\top] \|$ and $\mathbb{E}[\|\tilde{w}_{k+1}^{(1)}\|^2]$ as well. Substituting (66) into (61) yields

$$\| \mathbb{E}[\tilde{w}_{k+1}^{(0)}(\tilde{w}_{k+1}^{(0)})^\top] \| \leq (G_{0:k}^{(2)})^2 p_{22} \mathrm{M}_0^{\tilde{w}} + \widetilde{\mathrm{C}}_0 \sum_{j=0}^{k} \gamma_j^2 (G_{j+1:k}^{(2)})^2 (1 + \mathrm{M}_j^{\tilde{\theta}})$$

$$+ \widetilde{\mathrm{C}}_0 \sum_{j=0}^{k} \gamma_j^2 (G_{j+1:k}^{(2)})^2 \Big(\widetilde{\mathrm{C}}_0^{\tilde{w}} \widetilde{G}_{0:j-1}^{(2)} + \widetilde{\mathrm{C}}_1^{\tilde{w}} \gamma_j + \widetilde{\mathrm{C}}_2^{\tilde{w}} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} + \widetilde{\mathrm{C}}_3^{\tilde{w}} \gamma_{j-1}^2 \mathrm{M}_j^{\tilde{\theta}} \Big) \tag{68}$$

We observe

$$\sum_{j=0}^{k} \gamma_j^2 (G_{j+1:k}^{(2)})^2 \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} = \sum_{i=0}^{k-1} \gamma_i^2 \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \gamma_j^2 (G_{j+1:k}^{(2)})^2 \widetilde{G}_{i+1:j-1}^{(2)}$$

$$\overset{(a)}{\leq} \frac{1}{1 - \gamma_0 a_{22}/4} \sum_{i=0}^{k-1} \gamma_i^2 \mathrm{M}_i^{\tilde{\theta}} \widetilde{G}_{i+1:k}^{(2)} \sum_{j=i+1}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \overset{(b)}{\leq} \frac{\varrho^{a_{22}/2} \gamma_{k+1}}{1 - \gamma_0 a_{22}/4} \sum_{i=0}^{k-1} \gamma_i^2 \mathrm{M}_i^{\tilde{\theta}} \widetilde{G}_{i+1:k}^{(2)}$$

where (a) is due to $G_{j+1:k}^{(2)} \leq \widetilde{G}_{j+1:k}^{(2)}$ and (b) is due to Corollary 14. As such, combining terms in (68) yields:

$$\| \mathbb{E}[\tilde{w}_{k+1}^{(0)}(\tilde{w}_{k+1}^{(0)})^\top] \| \leq \widetilde{\mathrm{C}}_0^{\tilde{w}'} \widetilde{G}_{0:k}^{(2)} + \widetilde{\mathrm{C}}_1^{\tilde{w}'} \gamma_{k+1} + \widetilde{\mathrm{C}}_2^{\tilde{w}'} \sum_{j=0}^{k} \gamma_j^2 \widetilde{G}_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{\theta}}, \tag{69}$$

where

$$\widetilde{\mathrm{C}}_0^{\tilde{w}'} := p_{22} \mathrm{M}_0^{\tilde{\theta}}, \quad \widetilde{\mathrm{C}}_1^{\tilde{w}'} := \widetilde{\mathrm{C}}_0 \varrho^{a_{22}/2} \big(1 + \widetilde{\mathrm{C}}_0^{\tilde{w}} + \widetilde{\mathrm{C}}_1^{\tilde{w}}\big)$$

$$\widetilde{\mathrm{C}}_2^{\tilde{w}'} := \widetilde{\mathrm{C}}_0 \Big(1 + \widetilde{\mathrm{C}}_3^{\tilde{w}} + \widetilde{\mathrm{C}}_2^{\tilde{w}} \varrho^{a_{22}/2} \frac{\gamma_0}{1 - \gamma_0 a_{22}/4}\Big)$$

Similarly, we can compute the bound for $\mathbb{E}[\|\tilde{w}_{k+1}^{(1)}\|^2]$ as follows. Using (66):

$$\mathbb{E}[\|\tilde{w}_{k+1}^{(1)}\|^2] \leq \widetilde{\mathrm{C}}_1 (G_{0:k}^{(2)})^2 + \widetilde{\mathrm{C}}_2 \gamma_k^2 \mathrm{M}_{k+1}^{\tilde{\theta}} + \widetilde{\mathrm{C}}_3 \gamma_{k+1} \sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{\theta}}$$

$$+ \widetilde{\mathrm{C}}_2 \gamma_k^2 \mathrm{M}_{k+1}^{\tilde{w}} + \widetilde{\mathrm{C}}_3 \gamma_{k+1} \sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \mathrm{M}_j^{\tilde{w}} \tag{70}$$

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Notice that

$$\sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \, \mathrm{M}_j^{\tilde{w}} \leq \sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \Big(\widetilde{\mathrm{C}}_0^{\tilde{w}} \widetilde{G}_{0:j-1}^{(2)} + \widetilde{\mathrm{C}}_1^{\tilde{w}} \gamma_j + \widetilde{\mathrm{C}}_2^{\tilde{w}} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} \, \mathrm{M}_i^{\tilde{\theta}} + \widetilde{\mathrm{C}}_3^{\tilde{w}} \gamma_{j-1}^2 \, \mathrm{M}_j^{\tilde{\theta}} \Big)$$

$$\leq \varrho^{a_{22}/2} (\widetilde{\mathrm{C}}_0^{\tilde{w}} + \widetilde{\mathrm{C}}_1^{\tilde{w}} \gamma_0) \gamma_{k+1} + \widetilde{\mathrm{C}}_3^{\tilde{w}} \sum_{j=0}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \, \mathrm{M}_j^{\tilde{\theta}} + \widetilde{\mathrm{C}}_2^{\tilde{w}} \sum_{i=0}^{k-1} \gamma_i^2 \, \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \widetilde{G}_{i+1:j-1}^{(2)}$$

Since $(1 - \gamma a_{22}/2) \leq (1 - \gamma a_{22}/4)^2$ for any $\gamma > 0$, we have $G_{j+1:k}^{(2)} \leq (\widetilde{G}_{j+1:k}^{(2)})^2$, therefore together with Corollary 14 it yields

$$\sum_{i=0}^{k-1} \gamma_i^2 \, \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \gamma_j^2 G_{j+1:k}^{(2)} \widetilde{G}_{i+1:j-1}^{(2)} \leq \frac{\varrho^{a_{22}/4} \gamma_{k+1}}{1 - \gamma_0 a_{22}/4} \sum_{i=0}^{k-1} \gamma_i^2 \, \mathrm{M}_i^{\tilde{\theta}} \widetilde{G}_{i+1:k}^{(2)}.$$

Collecting terms and substituting them in (70) yield

$$\boxed{\mathbb{E}[\|\tilde{w}_{k+1}^{(1)}\|^2] \leq \widetilde{\mathrm{C}}_0^{\tilde{w}''} \widetilde{G}_{0:k}^{(2)} + \widetilde{\mathrm{C}}_1^{\tilde{w}''} \gamma_{k+1}^2 + \widetilde{\mathrm{C}}_2^{\tilde{w}''} \gamma_{k+1} \sum_{j=0}^{k} \gamma_j^2 \widetilde{G}_{j+1:k}^{(2)} \, \mathrm{M}_j^{\tilde{\theta}} + \widetilde{\mathrm{C}}_3^{\tilde{w}''} \gamma_k^2 \, \mathrm{M}_{k+1}^{\tilde{\theta}},} \quad (70)$$

where we use again the fact that $\gamma_k^2 \leq \varsigma \gamma_{k+1}$ and

$$\widetilde{\mathrm{C}}_0^{\tilde{w}''} := \widetilde{\mathrm{C}}_1 + \gamma_0^2 \widetilde{\mathrm{C}}_2 \widetilde{\mathrm{C}}_0^{\tilde{w}}, \quad \widetilde{\mathrm{C}}_1^{\tilde{w}''} := \widetilde{\mathrm{C}}_3 \varrho^{a_{22}/2} (\widetilde{\mathrm{C}}_0^{\tilde{w}} + \widetilde{\mathrm{C}}_1^{\tilde{w}} \gamma_0) + \varsigma \widetilde{\mathrm{C}}_2 \widetilde{\mathrm{C}}_1^{\tilde{w}}$$

$$\widetilde{\mathrm{C}}_2^{\tilde{w}''} := \widetilde{\mathrm{C}}_3 \Big(1 + \widetilde{\mathrm{C}}_3^{\tilde{w}} + \widetilde{\mathrm{C}}_2^{\tilde{w}} \frac{\varrho^{a_{22}/4} \gamma_0}{1 - \gamma_0 a_{22}/4} \Big) + \varsigma \widetilde{\mathrm{C}}_2 \widetilde{\mathrm{C}}_2^{\tilde{w}}, \quad \widetilde{\mathrm{C}}_3^{\tilde{w}''} := \widetilde{\mathrm{C}}_2 (1 + \gamma_0^2 \widetilde{\mathrm{C}}_3^{\tilde{w}}).$$

Bounding the Cross Term (Proof of Lemma 8) Our next endeavor is to bound the cross variance between the *martingale noise* driven terms $\tilde{w}_{k+1}^{(0)}$ and $\tilde{\theta}_{k+1}^{(0)}$. Here, the steps involved are similar to those in bounding $\mathrm{M}_k^{\tilde{\theta},\tilde{w}}$ in the proof of Theorem 1. Particularly, in a similar vein as the derivation of (43), we obtain

$$\| \mathbb{E}[\tilde{\theta}_{k+1}^{(0)} (\tilde{w}_{k+1}^{(0)})^\top] \| \leq p_{22,\Delta} \Big\{ G_{0:k}^{(2)} \, \mathrm{M}_0^{\tilde{\theta},\tilde{w}} + \|A_{12}\| \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(2)} \| \mathbb{E}[\tilde{w}_j^{(0)} (\tilde{w}_j^{(0)})^\top] \| \Big\}$$

$$+ p_{22,\Delta} \Big\{ \sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \| \mathbb{E}[V_{j+1}^{(0)} (W_{j+1}^{(0)})^\top] \| + \mathrm{C}_\infty \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \| \mathbb{E}[V_{j+1}^{(0)} (V_{j+1}^{(0)})^\top] \| \Big\}$$

By observing that $G_{j:k}^{(2)} \leq \widetilde{G}_{j:k}^{(2)}$, we have

$$\| \mathbb{E}[\tilde{\theta}_{k+1}^{(0)} (\tilde{w}_{k+1}^{(0)})^\top] \| \leq p_{22,\Delta} \widetilde{G}_{0:k}^{(2)} \, \mathrm{M}_0^{\tilde{\theta},\tilde{w}} + p_{22,\Delta} \|A_{12}\| \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(2)} \| \mathbb{E}[\tilde{w}_j^{(0)} (\tilde{w}_j^{(0)})^\top] \|$$

$$+ p_{22,\Delta} \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j+1:k}^{(2)} \big(\tilde{m}_{VW}^{(0)} \gamma_j + \tilde{m}_V^{(0)} \mathrm{C}_\infty \beta_j \big) \big(1 + \mathrm{M}_j^{\tilde{\theta}} + \mathrm{M}_j^{\tilde{w}} \big) \quad (71)$$

When combined with (66), (69), it can be verified using similar steps as in deriving (44) that:

$$\sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(2)} \| \mathbb{E}[\tilde{w}_j^{(0)} (\tilde{w}_j^{(0)})^\top] \| \le \frac{2\widetilde{C}_0^{\tilde{w}'} \widetilde{G}_{0:k}^{(2)}}{a_\Delta} + \widetilde{C}_1^{\tilde{w}'} \varrho^{a_{22}/4} \beta_{k+1} + \frac{2\widetilde{C}_2^{\tilde{w}'}}{a_\Delta} \sum_{i=0}^{k} \gamma_i^2 \widetilde{G}_{i+1:k}^{(2)} M_i^{\tilde{\theta}},$$

$$\sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(2)} M_j^{\tilde{w}} \le \frac{2\widetilde{C}_0^{\tilde{w}} \widetilde{G}_{0:k}^{(2)}}{a_\Delta} + \widetilde{C}_1^{\tilde{w}} \varrho^{a_{22}/4} \beta_{k+1} + \Big(\frac{2\widetilde{C}_2^{\tilde{w}}}{a_\Delta} + \widetilde{C}_3^{\tilde{w}} \Big) \sum_{i=0}^{k} \gamma_i^2 \widetilde{G}_{i+1:k}^{(2)} M_i^{\tilde{\theta}},$$

Substituting the above into (71) gives:

$$\boxed{\| \mathbb{E}[\tilde{\theta}_{k+1}^{(0)} (\tilde{w}_{k+1}^{(0)})^\top] \| \le \widetilde{C}_0^{\tilde{\theta},\tilde{w}} \widetilde{G}_{0:k}^{(2)} + \widetilde{C}_1^{\tilde{\theta},\tilde{w}} \beta_{k+1} + \widetilde{C}_2^{\tilde{\theta},\tilde{w}} \sum_{j=0}^{k} \gamma_j^2 \widetilde{G}_{j+1:k}^{(2)} M_j^{\tilde{\theta}},} \qquad (72)$$

where

$$\begin{aligned}
\widetilde{C}_0^{\tilde{\theta},\tilde{w}} &:= p_{22,\Delta} \Big(M_0^{\tilde{\theta},\tilde{w}} + \frac{\widetilde{C}_0^{\tilde{w}'} \|A_{12}\| + \widetilde{C}_0^{\tilde{w}} \big(\tilde{m}_{VW}^{(0)} \gamma_0 + \tilde{m}_V^{(0)} C_\infty \beta_0\big)}{a_\Delta/2} \Big) \\
\widetilde{C}_1^{\tilde{\theta},\tilde{w}} &:= p_{22,\Delta} \varrho^{a_{22}/4} \Big(\widetilde{C}_1^{\tilde{w}'} \|A_{12}\| + \widetilde{C}_1^{\tilde{w}} \big(\tilde{m}_{VW}^{(0)} \gamma_0 + \tilde{m}_V^{(0)} C_\infty \beta_0\big) \Big) \qquad (73) \\
\widetilde{C}_2^{\tilde{\theta},\tilde{w}} &:= p_{22,\Delta} \Big\{ \frac{2\widetilde{C}_2^{\tilde{w}'}}{a_\Delta} \|A_{12}\| + \Big(\frac{2\widetilde{C}_2^{\tilde{w}}}{a_\Delta} + \widetilde{C}_3^{\tilde{w}} \Big) \big(\tilde{m}_{VW}^{(0)} \gamma_0 + \tilde{m}_V^{(0)} C_\infty \beta_0\big) \Big\}
\end{aligned}$$

Notice that this concludes the proof of Lemma 8.

Bounding $M_k^{\tilde{\theta}}$ (Proof of Proposition 9) Like in the proof of Theorem 1, we begin by bounding $\| \mathbb{E}[\tilde{w}_k^{(0)} (\tilde{w}_k^{(0)})^\top] \|$ as follows. Evaluating the recursion in (26) and following the derivations that lead to (50), we obtain

$$\| \mathbb{E} \tilde{\theta}_{k+1}^{(0)} (\tilde{\theta}_{k+1}^{(0)})^\top] \| \le p_\Delta \Big\{ (G_{0:k}^{(1)})^2 M_0^{\tilde{\theta}} + 2\|A_{12}\| \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(1)} \| \mathbb{E}[\tilde{\theta}_j^{(0)} (\tilde{w}_j^{(0)})^\top] \| \Big\}$$

$$+ p_\Delta \Big\{ \sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \big(\|A_{12}\|^2 \| \mathbb{E}[\tilde{w}_j^{(0)} (\tilde{w}_j^{(0)})^\top] \| + \| \mathbb{E}[V_{j+1}^{(0)} (V_{j+1}^{(0)})^\top] \| \big) \Big\}$$

We apply (72) and note that

$$\sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(1)} \| \mathbb{E}[\tilde{\theta}_j^{(0)} (\tilde{w}_j^{(0)})^\top] \|$$

$$\le \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(1)} \Big(\widetilde{C}_0^{\tilde{\theta},\tilde{w}} \widetilde{G}_{0:j-1}^{(2)} + \widetilde{C}_1^{\tilde{\theta},\tilde{w}} \beta_j + \widetilde{C}_2^{\tilde{\theta},\tilde{w}} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} M_i^{\tilde{\theta}} \Big)$$

$$\overset{(a)}{\le} \widetilde{C}_0^{\tilde{\theta},\tilde{w}} \frac{G_{0:k}^{(1)}}{a_\Delta/2} + \widetilde{C}_1^{\tilde{\theta},\tilde{w}} \varrho^{a_\Delta/2} \beta_{k+1} + \widetilde{C}_2^{\tilde{\theta},\tilde{w}} \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(1)} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} M_i^{\tilde{\theta}}$$

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where (a) is due to the observation that $1 - \gamma_j a_{22}/4 \le 1 - \beta_j a_\Delta/2$ and the application of Lemma 12. Moreover, by a slight modification of (53), we have

$$\sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(1)} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} \le \frac{16\varsigma}{a_{22}} \sum_{i=0}^{k-1} \beta_i \gamma_i G_{i+1:k}^{(1)} \mathrm{M}_i^{\tilde{\theta}} \tag{73}$$

Therefore,

$$\sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} G_{j:k}^{(1)} \| \mathbb{E}[\tilde{\theta}_j^{(0)} (\tilde{w}_j^{(0)})^\top] \|$$

$$\le \widetilde{\mathrm{C}}_0^{\tilde{\theta},\tilde{w}} \frac{G_{0:k}^{(1)}}{a_\Delta/2} + \widetilde{\mathrm{C}}_1^{\tilde{\theta},\tilde{w}} \varrho^{a_\Delta/2} \beta_{k+1} + \widetilde{\mathrm{C}}_2^{\tilde{\theta},\tilde{w}} \frac{16\varsigma}{a_{22}} \sum_{i=0}^{k-1} \beta_i \gamma_i G_{i+1:k}^{(1)} \mathrm{M}_i^{\tilde{\theta}}$$

Similarly, we apply (69), (73) and note that

$$\sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \| \mathbb{E}[\tilde{w}_j^{(0)} (\tilde{w}_j^{(0)})^\top] \|$$

$$\le \sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \left\{ \widetilde{\mathrm{C}}_0^{\tilde{w}'} \widetilde{G}_{0:j-1}^{(2)} + \widetilde{\mathrm{C}}_1^{\tilde{w}'} \gamma_j + \widetilde{\mathrm{C}}_2^{\tilde{w}'} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} \right\} \tag{74}$$

$$\le (\widetilde{\mathrm{C}}_0^{\tilde{w}'} + \widetilde{\mathrm{C}}_1^{\tilde{w}'} \gamma_0) \varrho^{a_\Delta/2} \beta_{k+1} + \widetilde{\mathrm{C}}_2^{\tilde{w}'} \frac{\beta_0 (16\varsigma/a_{22})}{1 - \beta_0 a_\Delta/2} \sum_{i=0}^{k-1} \beta_i \gamma_i G_{i+1:k}^{(1)} \mathrm{M}_i^{\tilde{\theta}}$$

Finally, we obtain that

$$\sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \| \mathbb{E}[V_{j+1}^{(0)} (V_{j+1}^{(0)})^\top] \| \le \tilde{m}_V^{(0)} \sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 (1 + \mathrm{M}_j^{\tilde{\theta}} + \mathrm{M}_j^{\tilde{w}})$$

$$\le \tilde{m}_V^{(0)} \left\{ \varrho^{a_\Delta/2} \beta_{k+1} + \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} \mathrm{M}_j^{\tilde{\theta}} + \sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \mathrm{M}_j^{\tilde{w}} \right\}$$

Using the bound in (66) and the derivations in (74), we have

$$\sum_{j=0}^{k} \beta_j^2 (G_{j+1:k}^{(1)})^2 \mathrm{M}_j^{\tilde{w}} \le (\widetilde{\mathrm{C}}_0^{\tilde{w}} + \widetilde{\mathrm{C}}_1^{\tilde{w}} \gamma_0) \varrho^{a_\Delta/2} \beta_{k+1} + \widetilde{\mathrm{C}}_2^{\tilde{w}} \frac{\beta_0 (16\varsigma/a_{22})}{1 - \beta_0 a_\Delta/2} \sum_{i=0}^{k-1} \beta_i \gamma_i G_{i+1:k}^{(1)} \mathrm{M}_i^{\tilde{\theta}}$$

$$+ \widetilde{\mathrm{C}}_3^{\tilde{w}} \gamma_0 \beta_0 \sum_{i=0}^{k} \beta_i \gamma_i G_{i+1:k}^{(1)} \mathrm{M}_i^{\tilde{\theta}}$$

Combining the above results, we obtain that

$$\| \mathbb{E}[\tilde{\theta}_{k+1}^{(0)} (\tilde{\theta}_{k+1}^{(0)})^\top] \| \le \widetilde{\mathrm{C}}_0^{(0)} G_{0:k}^{(1)} + \widetilde{\mathrm{C}}_1^{(0)} \beta_{k+1} + \widetilde{\mathrm{C}}_2^{(0)} \sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} \mathrm{M}_j^{\tilde{\theta}}, \tag{75}$$

where

$$\widetilde{C}_0^{(0)} = p_\Delta \Big(M_0^{\tilde\theta} + \widetilde{C}_0^{\tilde\theta,\tilde w} \frac{4}{a_\Delta/4} \|A_{12}\| \Big),$$

$$\widetilde{C}_1^{(0)} = p_\Delta \varrho^{a_\Delta/2} \Big(2\|A_{12}\|\widetilde{C}_1^{\tilde\theta,\tilde w} + \|A_{12}\|^2 (\widetilde{C}_0^{\tilde w'} + \widetilde{C}_1^{\tilde w'} \gamma_0) + \tilde m_V^{(0)} (\widetilde{C}_0^{\tilde w} + \widetilde{C}_1^{\tilde w} \gamma_0) \Big),$$

$$\widetilde{C}_2^{(0)} = p_\Delta \Big\{ 2\|A_{12}\| \frac{\widetilde{C}_2^{\tilde\theta,\tilde w}}{a_{22}} + \|A_{12}\|^2 \frac{\beta_0(16\varsigma/a_{22})}{1-\beta_0 a_\Delta/2} + \tilde m_V^{(0)} \Big(\widetilde{C}_2^{\tilde w} \frac{\beta_0(16\varsigma/a_{22})}{1-\beta_0 a_\Delta/2} + \widetilde{C}_3^{\tilde w} \gamma_0\beta_0 \Big) \Big\}$$

To bound the term $\mathbb{E}[\|\tilde\theta_{k+1}^{(1)}\|^2]$, we proceed by considering the following decomposition:

$$\tilde\theta_{k+1}^{(1)} = \underbrace{\Gamma_{0:k}^{(1)}\tilde\theta_0^{(1)} + \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} A_{12}\tilde w_j^{(1)}}_{=\tilde\theta_{k+1}^{(1,0)}} + \underbrace{\sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} V_{j+1}^{(1)}}_{=\tilde\theta_{k+1}^{(1,1)}}$$

As $\tilde\theta_0^{(1)} = 0$, we observe that

$$\|\tilde\theta_{k+1}^{(1,0)}\| \leq \sqrt{p_\Delta} \|A_{12}\| \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} \|\tilde w_j^{(1)}\|$$

Taking square on both sides and applying the Jensen's inequality (66) yields

$$\mathbb{E}[\|\tilde\theta_{k+1}^{(1,0)}\|^2] \leq \frac{p_\Delta \|A_{12}\|^2}{a_\Delta/2} \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} \mathbb{E}[\|\tilde w_j^{(1)}\|^2]$$

Applying (70) gives

$$\begin{aligned}
\mathbb{E}[\|\tilde\theta_{k+1}^{(1,0)}\|^2] \leq{}& \frac{p_\Delta \|A_{12}\|^2}{a_\Delta/2} \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} \Big(\widetilde{C}_0^{\tilde w''} \widetilde{G}_{0:j-1}^{(2)} + \widetilde{C}_3^{\tilde w''} \gamma_{j-1}^2 M_j^{\tilde\theta} + \widetilde{C}_1^{\tilde w''} \gamma_j^2 \Big) \\
&+ \frac{p_\Delta \|A_{12}\|^2}{a_\Delta/2} \widetilde{C}_2^{\tilde w''} \sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} M_i^{\tilde\theta}
\end{aligned} \tag{76}$$

Let us bound the right hand side one by one, we observe

$$\sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} \widetilde{G}_{0:j-1}^{(2)} \leq \sum_{j=0}^{k} \beta_j (\widetilde{G}_{j+1:k}^{(1)})^2 \widetilde{G}_{0:j-1}^{(2)} \leq \frac{G_{0:k}^{(1)}}{1-\beta_0 a_\Delta/2} \sum_{j=0}^{k} \beta_j \widetilde{G}_{j+1:k}^{(1)} \leq \frac{(4/a_\Delta)G_{0:k}^{(1)}}{1-\beta_0 a_\Delta/2}$$

$$\sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} \gamma_{j-1}^2 M_j^{\tilde\theta} \leq \rho_0 \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} M_j^{\tilde\theta}, \quad \sum_{j=0}^{k} \beta_j G_{j+1:k}^{(1)} \gamma_j^2 \leq \rho_0 \varrho^{a_\Delta/2} \beta_{k+1}$$

where the last two inequalities are due to $\gamma_{j-1}^2 \leq \rho_0 \beta_j$, see B4. In addition, using the fact $G_{m:n}^{(1)} \leq (\widetilde{G}_{m:n}^{(1)})^2$, we have

$$\sum_{j=0}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} \, \mathrm{M}_i^{\tilde{\theta}} = \sum_{i=0}^{k-1} \gamma_i^2 \, \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \beta_j \gamma_j G_{j+1:k}^{(1)} \widetilde{G}_{i+1:j-1}^{(2)}$$

$$\leq \rho_0 \sum_{i=0}^{k-1} \beta_i \, \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \beta_j \gamma_j (\widetilde{G}_{j+1:k}^{(1)})^2 \widetilde{G}_{i+1:j-1}^{(2)} \leq \frac{\rho_0}{1 - \beta_0 a_\Delta/4} \sum_{i=0}^{k-1} \beta_i \, \mathrm{M}_i^{\tilde{\theta}} \, \widetilde{G}_{i+1:k}^{(1)} \sum_{j=i+1}^{k} \beta_j \gamma_j \widetilde{G}_{j+1:k}^{(1)}$$

$$\leq \frac{\varrho^{a_\Delta/4} \rho_0}{1 - \beta_0 a_\Delta/4} \sum_{i=0}^{k-1} \beta_i^2 \, \mathrm{M}_i^{\tilde{\theta}} \, \widetilde{G}_{i+1:k}^{(1)}$$

Substituting these back into (76) yields

$$\mathbb{E}[\|\tilde{\theta}_{k+1}^{(1,0)}\|^2] \leq \widetilde{\mathrm{C}}_0^{(1,0)} \widetilde{G}_{0:k}^{(1)} + \widetilde{\mathrm{C}}_1^{(1,0)} \beta_{k+1} + \widetilde{\mathrm{C}}_2^{(1,0)} \sum_{i=0}^{k} \beta_i^2 \, \mathrm{M}_i^{\tilde{\theta}} \, \widetilde{G}_{i+1:k}^{(1)}, \tag{77}$$

where

$$\widetilde{\mathrm{C}}_0^{(1,0)} = \widetilde{\mathrm{C}}_0^{\tilde{w}''} \frac{8 p_\Delta \|A_{12}\|^2}{a_\Delta^2 (1 - \beta_0 a_\Delta/2)}, \quad \widetilde{\mathrm{C}}_1^{(1,0)} = \widetilde{\mathrm{C}}_1^{\tilde{w}''} \frac{p_\Delta \|A_{12}\|^2}{a_\Delta/2} \rho_0 \varrho^{a_\Delta/2},$$

$$\widetilde{\mathrm{C}}_2^{(1,0)} = \frac{p_\Delta \|A_{12}\|^2}{a_\Delta/2} \rho_0 \Big(\widetilde{\mathrm{C}}_3^{\tilde{w}''} + \widetilde{\mathrm{C}}_2^{\tilde{w}''} \frac{\varrho^{a_\Delta/4}}{1 - \beta_0 a_\Delta/4} \Big).$$

Next, we bound $\mathbb{E}[\|\tilde{\theta}_k^{(1,1)}\|^2]$. Set $\widetilde{\psi}_j^{b_1} := \psi_j^{b_1} + \Psi_j^{A_{11}} \theta^\star + \Psi_j^{A_{12}} w^\star$, upon some algebraic manipulations (details in Appendix B.2.1) we observe the following decomposition

$$
\begin{aligned}
V_{j+1}^{(1)} \equiv \widetilde{\psi}_j^{b_1} - \widetilde{\psi}_{j+1}^{b_1} + \big(\Psi_j^{A_{11}} \tilde{\theta}_j - \Psi_{j+1}^{A_{11}} \tilde{\theta}_{j+1} \big) + \big(\Psi_j^{A_{12}} \tilde{w}_j - \Psi_{j+1}^{A_{12}} \tilde{w}_{j+1} \big) \\
+ \Psi_j^{A_{11}} \big(\tilde{\theta}_{j+1} - \tilde{\theta}_j \big) + \Psi_j^{A_{12}} \big(\tilde{w}_{j+1} - \tilde{w}_j \big),
\end{aligned} \tag{78}
$$

and from B3 we have

$$\|\widetilde{\psi}_j^{b_1}\| \vee \|\Psi_j^{A_{11}}\| \vee \|\Psi_j^{A_{12}}\| \leq \mathrm{E}_0^V := \overline{\mathrm{A}} \vee (\overline{\mathrm{b}} + \overline{\mathrm{A}}(\|\theta^\star\| + \|w^\star\|)).$$

Applying Lemma 11, we can show

$$\sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} \Big(\widetilde{\psi}_j^{b_1} - \widetilde{\psi}_{j+1}^{b_1} + \big(\Psi_j^{A_{11}} \tilde{\theta}_j - \Psi_{j+1}^{A_{11}} \tilde{\theta}_{j+1} \big) + \big(\Psi_j^{A_{12}} \tilde{w}_j - \Psi_{j+1}^{A_{12}} \tilde{w}_{j+1} \big) \Big)$$

$$= \beta_0 \Gamma_{1:k}^{(1)} \big(\widetilde{\psi}_0^{b_1} + \Psi_0^{A_{11}} \tilde{\theta}_0 + \Psi_0^{A_{12}} \tilde{w}_0 \big) - \beta_k \big(\widetilde{\psi}_{k+1}^{b_1} + \Psi_{k+1}^{A_{11}} \tilde{\theta}_{k+1} + \Psi_{k+1}^{A_{12}} \tilde{w}_{k+1} \big)$$

$$+ \sum_{j=1}^{k} \big(\beta_j^2 B_{11}^k \Gamma_{j+1:k}^{(1)} + (\beta_j - \beta_{j-1}) \Gamma_{j:k}^{(1)} \big) \big(\widetilde{\psi}_j^{b_1} + \Psi_j^{A_{11}} \tilde{\theta}_j + \Psi_j^{A_{12}} \tilde{w}_j \big).$$

Moreover,

$$\sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} \Psi_j^{A_{11}} \big(\tilde{\theta}_{j+1} - \tilde{\theta}_j \big) = - \sum_{j=0}^{k} \beta_j^2 \Gamma_{j+1:k}^{(1)} \Psi_j^{A_{11}} (A_{12} \tilde{w}_j + W_{j+1})$$

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$$\sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} \Psi_j^{A_{12}} \big(\tilde{w}_{j+1} - \tilde{w}_j \big) = -\sum_{j=0}^{k} \beta_j \gamma_j \Gamma_{j+1:k}^{(1)} \Psi_j^{A_{12}} (W_{j+1} + C_j V_{j+1})$$

The above inequalities allow us to upper bound $\|\tilde{\theta}_{k+1}^{(1,1)}\|$. Note that as $|\beta_j - \beta_{j-1}| \leq \frac{a_\Delta}{16} \beta_j^2$ [cf. A2], we have

$$\|\tilde{\theta}_{k+1}^{(1,1)}\| \leq \sqrt{p_\Delta} \, \mathrm{E}_0^V \left\{ \frac{G_{0:k}^{(1)}}{1 - \beta_0 a_\Delta/2} (1 + \|\tilde{w}_0\| + \tilde{\theta}_0\|) + \beta_k (1 + \|\tilde{\theta}_{k+1}\| + \|\tilde{w}_{k+1}\|) \right\}$$

$$+ \sqrt{p_\Delta} \, \mathrm{E}_0^V (\mathrm{B}_{11,\infty} + a_\Delta/16) \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} \big(1 + \|\tilde{\theta}_j\| + \|\tilde{w}_j\| \big)$$

$$+ \sqrt{p_\Delta} \sum_{j=0}^{k} G_{j+1:k}^{(1)} \big(\beta_j^2 \|A_{12} \tilde{w}_j + W_{j+1}\| + \beta_j \gamma_j \|W_{j+1} + C_j V_{j+1}\| \big),$$

Applying the Jensen's inequality (66) and taking square on both sides give

$$\|\tilde{\theta}_{k+1}^{(1,1)}\|^2 \leq 7 p_\Delta (\mathrm{E}_0^V)^2 \left\{ (G_{0:k}^{(1)})^2 \left(\frac{1 + \|\tilde{w}_0\| + \|\tilde{\theta}_0\|}{1 - \beta_0 a_\Delta/2} \right)^2 + \beta_k^2 (1 + \|\tilde{\theta}_{k+1}\|^2 + \|\tilde{w}_{k+1}\|^2) \right\}$$

$$+ 7 p_\Delta (\mathrm{E}_0^V)^2 (\mathrm{B}_{11,\infty} + a_\Delta/16)^2 \varrho^{a_\Delta/2} \beta_{k+1} \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} (1 + \|\tilde{\theta}_j\|^2 + \|\tilde{w}_j\|^2) \right\}$$

$$+ 7 p_\Delta \varrho^{a_\Delta/2} \left\{ \beta_{k+1} \sum_{j=0}^{k} G_{j+1:k}^{(1)} \beta_j^2 \|A_{12} \tilde{w}_j + W_{j+1}\|^2 + \gamma_{k+1} \sum_{j=0}^{k} G_{j+1:k}^{(1)} \beta_j \gamma_j \|W_{j+1} + C_j V_{j+1}\|^2 \right\},$$

Note the subtle difference that the last term takes γ_{k+1}. Taking expectation on both sides leads to

$$\mathbb{E}[\|\tilde{\theta}_{k+1}^{(1,1)}\|^2] \leq 7 p_\Delta \left\{ (G_{0:k}^{(1)})^2 \left(\frac{1 + \|\tilde{w}_0\| + \|\tilde{\theta}_0\|}{1 - \beta_0 a_\Delta/2} \right)^2 + \beta_k^2 (1 + d_\theta \, \mathrm{M}_{k+1}^{\tilde{\theta}} + d_w \, \mathrm{M}_{k+1}^{\tilde{w}}) \right\}$$

$$+ 7 p_\Delta (\mathrm{E}_0^V)^2 (\mathrm{B}_{11,\infty} + a_\Delta/16)^2 \varrho^{a_\Delta/2} \beta_{k+1} \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} (1 + d_\theta \, \mathrm{M}_j^{\tilde{\theta}} + d_w \, \mathrm{M}_j^{\tilde{w}}) \right\}$$

$$+ 7 p_\Delta \varrho^{a_\Delta/2} \sum_{j=0}^{k} G_{j+1:k}^{(1)} (\beta_0 \beta_j^2 \tilde{m}_{\Delta\tilde{\theta}} + \beta_j \gamma_j^2 \tilde{m}_{\Delta\tilde{w}})(1 + \mathrm{M}_j^{\tilde{\theta}} + \mathrm{M}_j^{\tilde{w}}),$$

where we have used $\beta_{k+1} \leq \beta_0$ and $\gamma_{k+1} \leq \gamma_j$. Again, using the bound $\gamma_j^2 \leq \rho_0 \beta_j$ from B4, we can simplify the above inequality into

$$
\mathbb{E}[\|\tilde{\theta}_{k+1}^{(1,1)}\|^2] \leq 7p_\Delta \Big\{ (G_{0:k}^{(1)})^2 \Big(\frac{1 + \|\tilde{w}_0\| + \|\tilde{\theta}_0\|}{1 - \beta_0 a_\Delta/2} \Big)^2 + \beta_k^2 (1 + d_\theta \, \mathrm{M}_{k+1}^{\tilde{\theta}} + d_w \, \mathrm{M}_{k+1}^{\tilde{w}}) \Big\}
$$

$$
+ 7p_\Delta (\mathrm{E}_0^V)^2 (\mathrm{B}_{11,\infty} + a_\Delta/16)^2 \varrho^{a_\Delta/2} \beta_{k+1} \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} (1 + d_\theta \, \mathrm{M}_j^{\tilde{\theta}} + d_w \, \mathrm{M}_j^{\tilde{w}}) \Big\}
$$

$$
+ 7p_\Delta \varrho^{a_\Delta/2} (\beta_0 \tilde{m}_{\Delta\tilde{\theta}} + \rho_0 \tilde{m}_{\Delta\tilde{w}}) \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} (1 + \mathrm{M}_j^{\tilde{\theta}} + \mathrm{M}_j^{\tilde{w}})
$$

$$
\leq \widehat{\mathrm{C}}_0^{(1,1)} (G_{0:k}^{(1)})^2 + \widehat{\mathrm{C}}_1^{(1,1)} \beta_k^2 (1 + \mathrm{M}_{k+1}^{\tilde{\theta}} + \mathrm{M}_{k+1}^{\tilde{w}}) + \widehat{\mathrm{C}}_2^{(1,1)} \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} (\mathrm{M}_j^{\tilde{\theta}} + \mathrm{M}_j^{\tilde{w}}) + \widehat{\mathrm{C}}_3^{(1,1)} \beta_{k+1},
$$

$$
\tag{76}
$$

where

$$
\widehat{\mathrm{C}}_0^{(1,1)} = 7p_\Delta \Big(\frac{1 + \|\tilde{w}_0\| + \|\tilde{\theta}_0\|}{1 - \beta_0 a_\Delta/2} \Big)^2, \quad \widehat{\mathrm{C}}_1^{(1,1)} = 7p_\Delta (d_\theta \vee d_w)
$$

$$
\widehat{\mathrm{C}}_2^{(1,1)} = 7p_\Delta \varrho^{a_\Delta/2} \big\{ (d_\theta \vee d_w)(\mathrm{E}_0^V)^2 (\mathrm{B}_{11,\infty} + a_\Delta/16)^2 \beta_0 + (\beta_0 \tilde{m}_{\Delta\tilde{\theta}} + \rho_0 \tilde{m}_{\Delta\tilde{w}}) \big\}
$$

$$
\widehat{\mathrm{C}}_3^{(1,1)} = 7p_\Delta (\varrho^{a_\Delta/2})^2 \big((\mathrm{E}_0^V)^2 (\mathrm{B}_{11,\infty} + a_\Delta/16)^2 + \beta_0 \tilde{m}_{\Delta\tilde{\theta}} + \rho_0 \tilde{m}_{\Delta\tilde{w}} \big)
$$

Observe that

$$
\sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} \mathrm{M}_j^{\tilde{w}} \leq \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} \Big\{ \widetilde{\mathrm{C}}_0^{\tilde{w}} \widetilde{G}_{0:j-1}^{(2)} + \widetilde{\mathrm{C}}_1^{\tilde{w}} \gamma_j + \widetilde{\mathrm{C}}_2^{\tilde{w}} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} + \widetilde{\mathrm{C}}_3^{\tilde{w}} \gamma_{j-1}^2 \mathrm{M}_j^{\tilde{\theta}} \Big\}
$$

$$
\leq \big(\widetilde{\mathrm{C}}_0^{\tilde{w}} + \widetilde{\mathrm{C}}_1^{\tilde{w}} \gamma_0 \big) \varrho^{a_\Delta/2} \beta_{k+1} + \widetilde{\mathrm{C}}_3^{\tilde{w}} \gamma_0^2 \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} \mathrm{M}_j^{\tilde{\theta}} + \widetilde{\mathrm{C}}_2^{\tilde{w}} \sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}}.
$$

Furthermore, using $G_{j+1:k}^{(1)} \leq (\widetilde{G}_{j+1:k}^{(1)})^2$ and $\widetilde{G}_{i+1:j-1}^{(2)} \leq \widetilde{G}_{i+1:j-1}^{(1)}$, we have

$$
\sum_{j=0}^{k} \beta_j^2 G_{j+1:k}^{(1)} \sum_{i=0}^{j-1} \gamma_i^2 \widetilde{G}_{i+1:j-1}^{(2)} \mathrm{M}_i^{\tilde{\theta}} = \sum_{i=0}^{k-1} \gamma_i^2 \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \beta_j^2 G_{j+1:k}^{(1)} \widetilde{G}_{i+1:j-1}^{(2)}
$$

$$
\leq \sum_{i=0}^{k-1} \gamma_i^2 \mathrm{M}_i^{\tilde{\theta}} \sum_{j=i+1}^{k} \beta_j^2 (\widetilde{G}_{j+1:k}^{(1)})^2 \widetilde{G}_{i+1:j-1}^{(1)} \leq \frac{\varrho^{a_\Delta/4} \beta_{k+1}}{1 - \beta_0 a_\Delta/4} \sum_{i=0}^{k-1} \gamma_i^2 \mathrm{M}_i^{\tilde{\theta}} \widetilde{G}_{i+1:k}^{(1)}
$$

Moreover, through applying $\widetilde{G}_{i+1:j-1}^{(2)} \leq \widetilde{G}_{i+1:j-1}^{(1)}$ and $\beta_k \leq \beta_j$ for any $j \leq k$, we have

$$
\beta_k^2 \mathrm{M}_{k+1}^{\tilde{w}} \leq \beta_0^2 \widetilde{\mathrm{C}}_0^{\tilde{w}} \widetilde{G}_{0:k}^{(1)} + \varsigma \gamma_0 \widetilde{\mathrm{C}}_1^{\tilde{w}} \beta_{k+1} + \widetilde{\mathrm{C}}_2^{\tilde{w}} \gamma_0^2 \sum_{j=0}^{k} \beta_j^2 \widetilde{G}_{j+1:k}^{(1)} \mathrm{M}_j^{\tilde{\theta}} + \widetilde{\mathrm{C}}_3^{\tilde{w}} \gamma_0^2 \mathrm{M}_{k+1}^{\tilde{\theta}},
$$

where we have used $\beta_k^2 \leq \varsigma\beta_{k+1}$. The above results simplify (76) into

$$\mathbb{E}[\|\tilde{\theta}_{k+1}^{(1,1)}\|^2] \leq \widetilde{C}_0^{(1,1)}\widetilde{G}_{0:k}^{(1)} + \widetilde{C}_1^{(1,1)}\beta_{k+1} + \widetilde{C}_2^{(1,1)}\sum_{j=0}^{k}\beta_j^2\widetilde{G}_{j+1:k}^{(1)}\,\mathrm{M}_j^{\tilde{\theta}} + \widetilde{C}_3^{(1,1)}\beta_k^2\,\mathrm{M}_{k+1}^{\tilde{\theta}}, \qquad (76)$$

where

$$\begin{aligned}
\widetilde{C}_0^{(1,1)} &= \widehat{C}_0^{(1,1)} + \beta_0^2\widehat{C}_1^{(1,1)}\widetilde{C}_0^{\tilde{w}}, \\
\widetilde{C}_1^{(1,1)} &= \widehat{C}_1^{(1,1)}(1 + \varsigma\gamma_0\widetilde{C}_1^{\tilde{w}}) + \widehat{C}_3^{(1,1)} + \widehat{C}_2^{(1,1)}(\widetilde{C}_0^{\tilde{w}} + \widetilde{C}_1^{\tilde{w}}\gamma_0)\varrho^{a_\Delta/2} \\
\widetilde{C}_2^{(1,1)} &= \widehat{C}_2^{(1,1)}\Big(1 + \widetilde{C}_3^{\tilde{w}}\gamma_0^2 + \frac{\widetilde{C}_2^{\tilde{w}}\varrho^{a_\Delta/4}\beta_0}{1 - \beta_0 a_\Delta/4}\Big) + \widehat{C}_1^{(1,1)}\widetilde{C}_2^{\tilde{w}}\gamma_0^2, \\
\widetilde{C}_3^{(1,1)} &= \widehat{C}_1^{(1,1)}(1 + \widetilde{C}_3^{\tilde{w}}\gamma_0^2).
\end{aligned} \qquad (77)$$

Finally, combining (75), (77), (76) gives

$$\begin{aligned}
\mathrm{M}_{k+1}^{\tilde{\theta}} &\leq 3\big(\|\mathbb{E}[\tilde{\theta}_{k+1}^{(0)}(\tilde{\theta}_{k+1}^{(0)})^\top]\| + \mathbb{E}[\|\tilde{\theta}_{k+1}^{(1,0)}\|^2] + \mathbb{E}[\|\tilde{\theta}_{k+1}^{(1,1)}\|^2]\big) \\
&\leq 3\Big\{\big(\widetilde{C}_0^{(0)} + \widetilde{C}_0^{(1,0)} + \widetilde{C}_0^{(1,1)}\big)\widetilde{G}_{0:k}^{(1)} + \big(\widetilde{C}_1^{(0)} + \widetilde{C}_1^{(1,0)} + \widetilde{C}_1^{(1,1)}\big)\beta_{k+1}\Big\} \\
&\quad + 3\Big\{\big(\widetilde{C}_2^{(0)} + \widetilde{C}_2^{(1,0)} + \widetilde{C}_2^{(1,1)}\big)\sum_{i=0}^{k}\beta_i^2\widetilde{G}_{i+1:k}^{(1)}\,\mathrm{M}_i^{\tilde{\theta}} + \widetilde{C}_3^{(1,1)}\beta_k^2\,\mathrm{M}_{k+1}^{\tilde{\theta}}\Big\}
\end{aligned}$$

As we have $3\widetilde{C}_3^{(1,1)}\beta_k^2 \leq 1/2$, we have

$$\boxed{\mathrm{M}_{k+1}^{\tilde{\theta}} \leq \widetilde{C}_0^{\tilde{\theta}}\widetilde{G}_{0:k}^{(1)} + \widetilde{C}_1^{\tilde{\theta}}\beta_{k+1} + \widetilde{C}_2^{\tilde{\theta}}\sum_{i=0}^{k}\beta_i^2\widetilde{G}_{i+1:k}^{(1)}\,\mathrm{M}_i^{\tilde{\theta}},}$$

where

$$\begin{aligned}
\widetilde{C}_0^{\tilde{\theta}} &:= 6\big(\widetilde{C}_0^{(0)} + \widetilde{C}_0^{(1,0)} + \widetilde{C}_0^{(1,1)}\big), \quad \widetilde{C}_1^{\tilde{\theta}} := 6\big(\widetilde{C}_1^{(0)} + \widetilde{C}_1^{(1,0)} + \widetilde{C}_1^{(1,1)}\big), \\
\widetilde{C}_2^{\tilde{\theta}} &:= 6\big(\widetilde{C}_2^{(0)} + \widetilde{C}_2^{(1,0)} + \widetilde{C}_2^{(1,1)}\big).
\end{aligned} \qquad (78)$$

This concludes the proof of Proposition 9.

Completing the Proof of Theorem 2 From (30) we can derive a bound for $\mathrm{M}_k^{\tilde{\theta}}$ as follows. Let $\tilde{U}_0 = \widetilde{C}_0^{\tilde{\theta}}$, observe the following equivalent forms of the recursion

$$\tilde{U}_{k+1} = \widetilde{C}_0^{\tilde{w}}\widetilde{G}_{0:k}^{(1)} + \widetilde{C}_1^{\tilde{\theta}}\beta_{k+1} + \widetilde{C}_2^{\tilde{\theta}}\sum_{i=0}^{k}\beta_i^2\widetilde{G}_{i+1:k}^{(1)}\tilde{U}_i$$

$$\Longleftrightarrow \tilde{U}_{k+1} = (1 - \beta_k a_\Delta/4)\tilde{U}_k + \widetilde{C}_1^{\tilde{\theta}}(\beta_{k+1} - \beta_k + \beta_k^2 a_\Delta/4) + \widetilde{C}_2^{\tilde{\theta}}\beta_k^2\tilde{U}_k$$

$$\leq (1 - \beta_k a_\Delta/8)\tilde{U}_k + \widetilde{C}_1^{\tilde{\theta}}\beta_k^2 a_\Delta/4$$

where the last inequality is due to the fact $\beta_k \widetilde{C}_2^{\tilde{\theta}} \leq a_\Delta/8$. Subsequently, we have

$$\tilde{U}_{k+1} \leq \prod_{i=0}^k (1 - \beta_i a_\Delta/8)\tilde{U}_0 + \frac{\widetilde{C}_1^{\tilde{\theta}} a_\Delta}{4} \sum_{j=0}^k \gamma_j^2 \prod_{i=j+1}^k (1 - \beta_i a_\Delta/8)$$

$$\leq \prod_{i=0}^k (1 - \beta_i a_\Delta/8)\tilde{U}_0 + \frac{\widetilde{C}_1^{\tilde{\theta}} a_\Delta}{4} \varrho^{a_\Delta/8} \beta_{k+1},$$

Observing that $M_{k+1}^{\tilde{\theta}} \leq \tilde{U}_{k+1}$, we obtain

$$\boxed{M_{k+1}^{\tilde{\theta}} \leq \widetilde{C}_0^{\tilde{\theta}} \prod_{i=0}^k (1 - \beta_i a_\Delta/8) + \widetilde{C}_1^{\tilde{\theta}} \frac{a_\Delta}{4} \varrho^{a_\Delta/8} \beta_{k+1},} \tag{79}$$

We obtain (14) by setting $C_1^{\tilde{\theta},\mathsf{mark}} = \frac{a_\Delta}{4}\widetilde{C}_1^{\tilde{\theta}}\varrho^{a_\Delta/8}$ and observing $\widetilde{C}_0^{\tilde{\theta}} \leq C_0^{\tilde{\theta},\mathsf{mark}}(1 + V_0)$ for some constant $C_0^{\tilde{\theta},\mathsf{mark}}$.

Finally, we bound the tracking error $\widehat{w}_k := w_k - A_{22}^{-1}(b_2 - A_{21}\theta_k)$ as follows. Similarly to the martingale noise case, we set $M_k^{\widehat{w}} := \|\mathbb{E}[\widehat{w}_k \widehat{w}_k^\top]\|$ and observe:

$$M_{k+1}^{\widehat{w}} \leq 2\,M_{k+1}^{\tilde{w}} + 2\|L_{k+1}\|^2\,M_{k+1}^{\tilde{\theta}} \leq 2\,M_{k+1}^{\tilde{w}} + 2\,L_\infty^2 \frac{\lambda_{\mathsf{max}}(Q_{22})}{\lambda_{\mathsf{min}}(Q_\Delta)}\,M_{k+1}^{\tilde{\theta}}$$

Substituting (79) into (66) gives

$$M_{k+1}^{\tilde{w}} \leq \widetilde{G}_{0:k}^{(2)}\widetilde{C}_0^{\tilde{w}} + \widetilde{C}_1^{\tilde{w}} \gamma_{k+1} + \widetilde{C}_3^{\tilde{w}} \gamma_k^2 M_{k+1}^{\tilde{\theta}} + \widetilde{C}_2^{\tilde{w}} \sum_{j=0}^k \gamma_j^2 \widetilde{G}_{j+1:k}^{(2)} \left\{\widetilde{C}_0^{\tilde{\theta}} \prod_{i=0}^{j-1}\left(1 - \beta_i\frac{a_\Delta}{8}\right) + \widetilde{C}_1^{\tilde{\theta}}\frac{a_\Delta}{4}\varrho^{a_\Delta/8}\beta_j\right\}$$

$$\overset{(a)}{\leq} \widetilde{G}_{0:k}^{(2)}\widetilde{C}_0^{\tilde{w}} + \widetilde{C}_1^{\tilde{w}} \gamma_{k+1} + \widetilde{C}_3^{\tilde{w}} \gamma_k^2 M_{k+1}^{\tilde{\theta}} + \widetilde{C}_2^{\tilde{w}}\gamma_{k+1}\left\{\frac{C_0^{\tilde{\theta}}\varrho^{a_{22}/8}}{1 - \beta_0 a_\Delta/8}\prod_{i=0}^k\left(1 - \beta_i\frac{a_\Delta}{8}\right) + \widetilde{C}_1^{\tilde{\theta}}\frac{a_\Delta}{4}\varrho^{a_\Delta/8}\varrho^{a_{22}/4}\right\}$$

$$\leq \left\{\widetilde{C}_1^{\tilde{w}} + \widetilde{C}_2^{\tilde{w}}\widetilde{C}_1^{\tilde{\theta}}\frac{a_\Delta}{4}\varrho^{a_\Delta/8}\varrho^{a_{22}/4}\right\}\gamma_{k+1} + \left\{\widetilde{C}_0^{\tilde{w}} + \frac{\widetilde{C}_2^{\tilde{w}} C_0^{\tilde{\theta}}\varrho^{a_{22}/8}}{1 - \beta_0 a_\Delta/8}\right\}\prod_{i=0}^k\left(1 - \beta_i\frac{a_\Delta}{8}\right) + \widetilde{C}_3^{\tilde{w}}\gamma_k^2 M_{k+1}^{\tilde{\theta}}$$

where we have used $\widetilde{G}_{j+1:k}^{(2)} \leq \left(\prod_{i=j+1}^k(1 - \gamma_i a_{22}/8)\right)^2$ in (a). As such, together with (79) this gives

$$\boxed{M_{k+1}^{\widehat{w}} \leq \widetilde{C}_0^{\widehat{w}} \prod_{\ell=0}^k\left(1 - \beta_\ell\frac{a_\Delta}{8}\right) + C_1^{\widehat{w},\mathsf{mark}}\gamma_{k+1},}$$

where

$$\begin{aligned}\widetilde{C}_0^{\widehat{w}} &:= 2\left\{\widetilde{C}_0^{\tilde{w}} + \frac{\widetilde{C}_2^{\tilde{w}} C_0^{\tilde{\theta}}\varrho^{a_{22}/8}}{1 - \beta_0 a_\Delta/8} + (1 + \widetilde{C}_3^{\tilde{w}})L_\infty^2\frac{\lambda_{\mathsf{max}}(Q_{22})}{\lambda_{\mathsf{min}}(Q_\Delta)}\widetilde{C}_0^{\tilde{\theta}}\right\} \\ C_1^{\widehat{w},\mathsf{mark}} &:= 2\left\{\widetilde{C}_1^{\tilde{w}} + \widetilde{C}_2^{\tilde{w}}\widetilde{C}_1^{\tilde{\theta}}\frac{a_\Delta}{4}\varrho^{a_\Delta/8}\varrho^{a_{22}/4} + \kappa(1 + \widetilde{C}_3^{\tilde{w}})L_\infty^2\frac{\lambda_{\mathsf{max}}(Q_{22})}{\lambda_{\mathsf{min}}(Q_\Delta)}\widetilde{C}_1^{\tilde{\theta}}\frac{a_\Delta}{4}\varrho^{a_\Delta/8}\right\}\end{aligned} \tag{80}$$

Similarly, as $\widetilde{C}_0^{\widehat{w}} \leq C_0^{\widehat{w},\mathsf{mark}}(1 + V_0)$ for some constant $C_0^{\widehat{w},\mathsf{mark}}$, the above yields (15). We conclude the proof of Theorem 2.

B.2.1. AUXILIARY RESULTS FOR THE MARKOVIAN NOISE CASE

Lemma 11 *Let $(a_j)_{j\geq 0}$ be a sequence of d_θ-dimensional vectors. The following equality holds:*

$$\sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)}(a_j - a_{j+1})$$
$$= \beta_0 \Gamma_{1:k}^{(1)} a_0 - \beta_k a_{k+1} + \sum_{j=1}^{k} \left(\beta_j^2 B_{11}^j \Gamma_{j+1:k}^{(1)} + (\beta_j - \beta_{j-1})\Gamma_{j:k}^{(1)} \right) a_j \qquad (81)$$

Similarly, for $(b_j)_{j\geq 0}$ being a sequence of d_w-dimensional vectors, it holds:

$$\sum_{j=0}^{k} \gamma_j \Gamma_{j+1:k}^{(2)}(b_j - b_{j+1})$$
$$= \gamma_0 \Gamma_{1:k}^{(2)} b_0 - \gamma_k b_{k+1} + \sum_{j=1}^{k} \left(\gamma_j^2 B_{22}^j \Gamma_{j+1:k}^{(2)} + (\gamma_j - \gamma_{j-1})\Gamma_{j:k}^{(1)} \right) b_j.$$

Proof We only prove (81). Observe the following chain

$$\sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)}(a_j - a_{j+1}) = \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} a_j - \sum_{j=0}^{k} \beta_j \Gamma_{j+1:k}^{(1)} a_{j+1}$$

$$= \beta_0 \Gamma_{1:k}^{(1)} a_0 - \beta_k a_{k+1} + \sum_{j=1}^{k} \left(\beta_j \Gamma_{j+1:k}^{(1)} - \beta_{j-1} \Gamma_{j:k}^{(1)} \right) a_j$$

Using $\beta_j \Gamma_{j+1:k}^{(1)} - \beta_{j-1} \Gamma_{j:k}^{(1)} = \beta_j^2 B_{11}^j \Gamma_{j+1:k}^{(1)} + (\beta_j - \beta_{j-1})\Gamma_{j:k}^{(1)}$ concludes the proof. ∎

Derivation of Eq. (63)   The decomposition is obtained through repeatedly adding/subtracting terms. Particularly, we observe that the individual terms can be expressed as:

$$
\begin{aligned}
C_j\big(\widetilde{\psi}_j^{b_1} - \widetilde{\psi}_{j+1}^{b_1}\big) &= C_j \widetilde{\psi}_j^{b_1} - C_{j-1}\widetilde{\psi}_j^{b_1} + C_{j-1}\widetilde{\psi}_j^{b_1} - C_j \widetilde{\psi}_{j+1}^{b_1}\\
\big(\Psi_j^{A_{22}} - \Psi_{j+1}^{A_{22}}\big)\tilde{w}_j &= \Psi_j^{A_{22}}\tilde{w}_j - \Psi_{j+1}^{A_{22}}\tilde{w}_{j+1} + \Psi_{j+1}^{A_{22}}(\tilde{w}_{j+1} - \tilde{w}_j)\\
C_j\big(\Psi_j^{A_{12}} - \Psi_{j+1}^{A_{12}}\big)\tilde{w}_j &= (C_j - C_{j-1})\Psi_j^{A_{12}}\tilde{w}_j + C_{j-1}\Psi_j^{A_{12}}(\tilde{w}_j - \tilde{w}_{j-1})\\
&\quad + C_{j-1}\Psi_j^{A_{12}}\tilde{w}_{j-1} - C_j\Psi_{j+1}^{A_{12}}\tilde{w}_j\\
\big(\Psi_j^{A_{21}} - \Psi_{j+1}^{A_{21}}\big)\tilde{\theta}_j &= \Psi_j^{A_{21}}\tilde{\theta}_j - \Psi_{j+1}^{A_{21}}\tilde{\theta}_{j+1} + \Psi_{j+1}^{A_{21}}(\tilde{\theta}_{j+1} - \tilde{\theta}_j)\\
C_j\big(\Psi_j^{A_{11}} - \Psi_{j+1}^{A_{11}}\big)\tilde{\theta}_j &= (C_j - C_{j-1})\Psi_j^{A_{11}}\tilde{\theta}_j + C_{j-1}\Psi_j^{A_{11}}(\tilde{\theta}_j - \tilde{\theta}_{j-1})\\
&\quad + C_{j-1}\Psi_j^{A_{11}}\tilde{\theta}_{j-1} - C_j\Psi_{j+1}^{A_{11}}\tilde{\theta}_j\\
\big(\Psi_j^{A_{22}} - \Psi_{j+1}^{A_{22}}\big)C_{j-1}\tilde{\theta}_j &= \Psi_j^{A_{22}}(C_{j-1} - C_{j-2})\tilde{\theta}_j + \Psi_j^{A_{22}}C_{j-2}(\tilde{\theta}_j - \tilde{\theta}_{j-1})\\
&\quad + \Psi_j^{A_{22}}C_{j-2}\tilde{\theta}_{j-1} - \Psi_{j+1}^{A_{22}}C_{j-1}\tilde{\theta}_j\\
C_j(\Psi_j^{A_{12}} - \Psi_{j+1}^{A_{12}})C_{j-1}\tilde{\theta}_j &= (C_j - C_{j-1})\Psi_j^{A_{12}}C_{j-1}\tilde{\theta}_j + C_{j-1}\Psi_j^{A_{12}}(C_{j-1} - C_{j-2})\tilde{\theta}_j\\
&\quad + C_{j-1}\Psi_j^{A_{12}}C_{j-2}(\tilde{\theta}_j - \tilde{\theta}_{j-1})\\
&\quad + C_{j-1}\Psi_j^{A_{12}}C_{j-2}\tilde{\theta}_{j-1} - C_j\Psi_{j+1}^{A_{12}}C_{j-1}\tilde{\theta}_j.
\end{aligned}
$$

Collecting terms on the right hand side of the above equations yields (63). Moreover, we the vectors/matrices that appear in (63) can be bounded as

$$\|\psi_j^{WV}\| \leq \overline{b}(1 + C_\infty), \quad \|\Upsilon_j^{WV,\tilde{\theta}}\| \leq \overline{A}(1 + 2\,C_\infty + C_\infty^2), \quad \|\Upsilon_j^{WV,\tilde{w}}\| \leq \overline{A}(1 + C_\infty)$$

$$\|\Phi^{WV,\tilde{\theta}}\| \leq \overline{A}(1 + 2\,C_\infty + C_\infty^2), \quad \|\Phi^{WV,\tilde{w}}\| \leq \overline{A}(1 + C_\infty)$$

$$\|\widetilde{\Psi}_j^{WV,\tilde{\theta}}\| \leq \overline{A}C_2^U \varrho^{a_{22}/2}(1 + C_\infty)(1 + \varsigma)\gamma_j, \quad \|\widetilde{\Psi}_j^{WV,\tilde{w}}\| \leq \overline{A}C_2^U \varrho^{a_{22}/2}\gamma_j.$$

where the last inequality is due to Lemma 20 and we have used $\gamma_{j-1} \leq \varsigma\gamma_j$ [cf. A2-1]. Consequently, we can establish the bounds on the matrix/vector norms by setting

$$E_0^{WV} := \max\{\overline{b}(1 + C_\infty), \overline{A}(1 + 2\,C_\infty + C_\infty^2), \overline{A}C_2^U \varrho^{a_{22}/2}(1 + C_\infty)(1 + \varsigma)\}.$$

Derivation of Eq. (78)   Setting $\widetilde{\psi}_j^{b_1} := \psi_j^{b_1} + \Psi_j^{A_{11}}\theta^\star + \Psi_j^{A_{12}}w^\star$, we observe

$$\begin{aligned} V_{j+1}^{(1)} &= (\psi_j^{b_1} - \psi_{j+1}^{b_1}) + (\Psi_j^{A_{11}} - \Psi_{j+1}^{A_{11}})\theta_j + (\Psi_j^{A_{12}} - \Psi_{j+1}^{A_{12}})w_j \\ &= \widetilde{\psi}_j^{b_1} - \widetilde{\psi}_{j+1}^{b_1} + (\Psi_j^{A_{11}} - \Psi_{j+1}^{A_{11}})\tilde{\theta}_j + (\Psi_j^{A_{12}} - \Psi_{j+1}^{A_{12}})\tilde{w}_j \end{aligned}$$

Similar to the previous paragraph, the decomposition is obtained through repeatedly adding/subtracting terms. We observe

$$\begin{aligned} (\Psi_j^{A_{11}} - \Psi_{j+1}^{A_{11}})\tilde{\theta}_j &= \Psi_j^{A_{11}}\tilde{\theta}_j - \Psi_{j+1}^{A_{11}}\tilde{\theta}_{j+1} + \Psi_{j+1}^{A_{11}}(\tilde{\theta}_{j+1} - \tilde{\theta}_j) \\ (\Psi_j^{A_{12}} - \Psi_{j+1}^{A_{12}})\tilde{w}_j &= \Psi_j^{A_{12}}\tilde{w}_j - \Psi_{j+1}^{A_{12}}\tilde{w}_{j+1} + \Psi_{j+1}^{A_{12}}(\tilde{w}_{j+1} - \tilde{w}_j) \end{aligned}$$

Appendix C. Detailed Proof of Theorem 10

Throughout this section we will use additional notations. We denote

$$\kappa_\ell := \frac{\beta_\ell}{\gamma_\ell}.$$

Let

$$\kappa_\infty^{\mathsf{exp}} := \kappa_\infty \wedge (1/2)(\|A_{12}\|\,C_\infty\,\varrho^{a_{22}})^{-1} \tag{82}$$

and

$$\beta_\infty^{\mathsf{exp}} := \beta_\infty^{\mathsf{mtg}} \wedge 1/(4\|\Delta\|) \tag{83}$$

We assume $\beta_k \leq \beta_\infty^{\mathsf{exp}}, \gamma_k \leq \gamma_\infty^{\mathsf{mtg}}, \kappa_k \leq \kappa \leq \kappa_\infty^{\mathsf{exp}}$. Furthermore, let us define

$$\widetilde{\Gamma}_{m,n}^{(1)} := \prod_{j=m}^{n}(I - \beta_j\Delta), \quad \widetilde{\Gamma}_{m,n}^{(2)} := \prod_{j=m}^{n}(I - \gamma_j A_{22}).$$

Using standard arguments we may bound operator norm of these matrices

$$\|\widetilde{\Gamma}_{m:n}^{(1)}\| \leq \sqrt{p_\Delta}\prod_{j=m}^{n}(1 - a_\Delta\beta_j), \quad \|\widetilde{\Gamma}_{m:n}^{(2)}\| \leq \sqrt{p_{22}}\prod_{j=m}^{n}(1 - a_{22}\gamma_j) \tag{84}$$

We set quantities m_W, m_V from the assumption 4 to be equal to

$$m_W := m_V := \max(m_{VW}^{\mathsf{exp}}, \|\Sigma^{11}\|, \|\Sigma^{12}\|, \|\Sigma^{22}\|)$$

All conditions of Theorem 1 are satisfied. We will use this theorem in the following form

$$\mathrm{M}_k^{\tilde{\theta}} \leq \mathrm{C}_{0,\theta}^{\mathsf{exp}} \prod_{\ell=0}^{k-1}(1 - (a_\Delta/4)\beta_\ell)\mathrm{V}_0 + \mathrm{C}_{1,\theta}^{\mathsf{exp}}\beta_k, \tag{85}$$

$$\mathrm{M}_k^{\tilde{w}} \leq \mathrm{C}_{0,w}^{\mathsf{exp}} \prod_{\ell=0}^{k-1}(1 - (a_\Delta/4)\beta_\ell)\mathrm{V}_0 + \mathrm{C}_{1,w}^{\mathsf{exp}}\gamma_k, \tag{86}$$

where $\mathrm{C}_{0,\theta}^{\mathsf{exp}}$, $\mathrm{C}_{1,\theta}^{\mathsf{exp}}$, $\mathrm{C}_{0,w}^{\mathsf{exp}}$ and $\mathrm{C}_{1,w}^{\mathsf{exp}}$ denote corresponding constants from Theorem 1. Similarly to (34) and (35) we can define $\widetilde{m}_{VW}^{\mathsf{exp}}$. Hence, the following inequality holds

$$\|\mathbb{E}[V_j V_j^T]\| \vee \|\mathbb{E}[W_j W_j^T]\| \vee \|\mathbb{E}[V_j W_j^T]\| \leq \widetilde{m}_{VW}^{\mathsf{exp}}(1 + \mathrm{M}_k^{\tilde{\theta}} + \mathrm{M}_k^{\tilde{w}})$$

Applying (18) and (19) (compare with Konda and Tsitsiklis (2004)[Formula 4.4]) we may write down the following expansion for $\tilde{\theta}_{k+1}$:

$$\tilde{\theta}_{k+1} = S_{k+1}^{(0)} + \ldots + S_{k+1}^{(6)},$$

where

$$S_{k+1}^{(0)} := \widetilde{\Gamma}_{0:k}^{(1)}\tilde{\theta}_0;$$

$$S_{k+1}^{(1)} := \sum_{j=0}^{k} \beta_j \widetilde{\Gamma}_{j+1:k}^{(1)} A_{12}\widetilde{\Gamma}_{0:j}^{(2)}\tilde{w}_0;$$

$$S_{k+1}^{(2)} := \sum_{j=0}^{k} \beta_j \widetilde{\Gamma}_{j+1:k}^{(1)} \delta_j^{(1)};$$

$$S_{k+1}^{(3)} := \sum_{j=0}^{k} \beta_j \widetilde{\Gamma}_{j+1:k}^{(1)} (V_{j+1} + A_{12}A_{22}^{-1}W_{j+1});$$

$$S_{k+1}^{(4)} := \sum_{j=0}^{k} \beta_j \widetilde{\Gamma}_{j+1:k}^{(1)} A_{12}\bigg(\sum_{\ell=0}^{j-1} \beta_\ell \widetilde{\Gamma}_{\ell+1:j-1}^{(2)}\delta_\ell^{(2)}\bigg);$$

$$S_{k+1}^{(5)} := \sum_{j=0}^{k} \beta_j \widetilde{\Gamma}_{j+1:k}^{(1)} A_{12}\bigg(\sum_{\ell=0}^{j-1} \beta_\ell \widetilde{\Gamma}_{\ell+1:j}^{(2)}C_\ell V_{\ell+1}\bigg);$$

$$S_{k+1}^{(6)} := \sum_{j=0}^{k} \beta_j \widetilde{\Gamma}_{j+1:k}^{(1)} A_{12}\sum_{\ell=0}^{j-1} \gamma_\ell \widetilde{\Gamma}_{\ell+1:j-1}^{(2)}W_{\ell+1} - \sum_{\ell=0}^{k} \beta_\ell \widetilde{\Gamma}_{j+1:k}^{(1)} A_{12}A_{22}^{-1}W_{\ell+1},$$

where

$$\delta_\ell^{(1)} := A_{12}L_\ell\tilde{\theta}_\ell, \; \delta_\ell^{(2)} := -C_k A_{12}\tilde{w}_\ell.$$

We will group all terms in the expansion into 5 blocks, $S_{k+1}^{(0)} + S_{k+1}^{(1)}$, $S_{k+1}^{(2)} + S_{k+1}^{(5)}$, $S_{k+1}^{(3)}$, $S_{k+1}^{(4)}$ and $S_{k+1}^{(6)}$. It is easy to see that $S_{k+1}^{(0)} + S_{k+1}^{(1)}$ is uncorrelated with $S_{k+1}^{(3)}$, $S_{k+1}^{(6)}$ (moreover it is uncorrelated

with $S_{k+1}^{(5)}$, but we ignore this fact). Since $\mathbb{E}[\|\tilde{\theta}_{k+1}\|^2] = \mathbb{E}[\mathrm{Tr}(\tilde{\theta}_{k+1}\tilde{\theta}_{k+1}^\top)]$ and by the linearity of trace using expansion we show

$$\mathbb{E}[\|\tilde{\theta}_{k+1}\|^2] = \mathbb{E}\left[\,\mathrm{Tr}(S_{k+1}^{(3)}(S_{k+1}^{(3)})^\top)\right] + J_{k+1}',$$

where for J_{k+1}' we will use the following crude estimate

$$\begin{aligned}
|J_{k+1}'| \leq{}& 3\,\mathbb{E}\left[\,\mathrm{Tr}(\{S_{k+1}^{(0)} + S_{k+1}^{(1)}\}\{S_{k+1}^{(0)} + S_{k+1}^{(1)}\}^\top)\right]\\
&+ 5\,\mathbb{E}\left[\,\mathrm{Tr}(\{S_{k+1}^{(2)} + S_{k+1}^{(5)}\}\{S_{k+1}^{(2)} + S_{k+1}^{(5)}\}^\top) + 5\,\mathbb{E}\left[\,\mathrm{Tr}(S_{k+1}^{(6)}(S_{k+1}^{(6)})^\top)\right]\right]\\
&+ 2\,\mathbb{E}\left[\,\mathrm{Tr}(S_{k+1}^{(3)}(S_{k+1}^{(6)})^\top)\right] + 2\,\mathbb{E}\left[\,\mathrm{Tr}(S_{k+1}^{(3)}\{S_{k+1}^{(2)} + S_{k+1}^{(5)}\}^\top)\right]\\
&+ 5\,\mathbb{E}\left[\,\mathrm{Tr}(S_{k+1}^{(4)}(S_{k+1}^{(4)})^\top)\right] + 2\,\mathbb{E}\left[\,\mathrm{Tr}(S_{k+1}^{(3)}(S_{k+1}^{(4)})^\top)\right]
\end{aligned}$$

Using martingale property and definition of Σ we rewrite the term $\mathbb{E}[\mathrm{Tr}(S_{k+1}^{(3)}(S_{k+1}^{(3)})^\top)]$ as follows

$$\begin{aligned}
\mathrm{Tr}(\mathbb{E}[S_{k+1}^{(3)}(S_{k+1}^{(3)})^\top]) ={}& \sum_{j=0}^{k} \beta_j^2\,\mathrm{Tr}(\widetilde{\Gamma}_{j+1:k}^{(1)}\Sigma[\widetilde{\Gamma}_{j+1:k}^{(1)}]^\top)\\
&+ \sum_{j=0}^{k} \beta_j^2\,\mathrm{Tr}(\widetilde{\Gamma}_{j+1:k}^{(1)}(\Sigma_j - \Sigma)[\widetilde{\Gamma}_{j+1:k}^{(1)}]^\top)
\end{aligned} \tag{87}$$

where

$$\Sigma_j := \mathbb{E}[V_j V_j^\top] + A_{12}A_{22}^{-1}\,\mathbb{E}[W_j W_j^\top]A_{22}^{-\top}A_{12}^\top + \mathbb{E}[V_j W_j^\top]A_{22}^{-\top}A_{12}^\top + A_{12}A_{22}^{-1}\,\mathbb{E}[W_j V_j^\top]$$

Leading term in (87)   For lower bound of the first term in (87) we will use the following fact. Since for any $s \in [j+1, k]$

$$(\mathrm{I} - \beta_s\Delta)^\top(\mathrm{I} - \beta_s\Delta) = \mathrm{I} - \beta_s(\Delta + \Delta^\top) + \beta_s^2\Delta^\top\Delta \succeq (1 - 2\beta_s\|\Delta\|)\,\mathrm{I},$$

we obtain using Lemma 12 (and remark after this lemma)

$$\sum_{j=0}^{k} \beta_j^2\,\mathrm{Tr}(\widetilde{\Gamma}_{j+1:k}^{(1)}\Sigma[\widetilde{\Gamma}_{j+1:k}^{(1)}]^\top) \geq \beta_{k+1}\,\mathrm{Tr}\,\Sigma\sum_{j=0}^{k}\beta_j\prod_{\ell=j+1}^{k}(1 - 2\beta_\ell\|\Delta\|) \geq \mathrm{C}_3^{\mathsf{exp}}\beta_{k+1}\,\mathrm{Tr}\,\Sigma \tag{88}$$

where

$$\mathrm{C}_3^{\mathsf{exp}} := 1/(8\|\Delta\|) \tag{89}$$

and we used $\beta_\infty^{\mathsf{exp}} \leq 1/(4\|\Delta\|)$ and $k \geq k_0^{\mathsf{exp}}$. To obtain upper bound we apply von Neumann trace inequality (i.e. $\mathrm{Tr}(AB) \leq \sum_{j=1}^{n} a_j b_j$, where $\{a_j\}$ and $\{b_j\}$ are non-increasing sequences of eigenvalues of Hermitian matrices A and B resp.) and Lemma 14

$$\sum_{j=0}^{k} \beta_j^2\,\mathrm{Tr}(\widetilde{\Gamma}_{j+1:k}^{(1)}\Sigma[\widetilde{\Gamma}_{j+1:k}^{(1)}]^\top) \leq p_\Delta\,\mathrm{Tr}(\Sigma)\sum_{j=0}^{k}\beta_j^2\prod_{\ell=j+1}^{k}(1 - a_\Delta\beta_\ell)^2 \leq \mathrm{C}_4^{\mathsf{exp}}\,\mathrm{Tr}(\Sigma)\beta_{k+1} \tag{90}$$

where

$$\mathrm{C}_4^{\mathsf{exp}} := p_\Delta\varrho^{a_\Delta} \tag{91}$$

Inequalities (88) and (90) together imply (32).

Remainder term in (87)   The second term in (87), which we denote by R_{k+1} may be estimated as follows

$$|R_{k+1}| \leq p_\Delta d_\theta \widetilde{m}_{VW}^{\text{exp}} (1 + \|A_{12}A_{22}^{-1}\|)^2 \sum_{j=0}^{k} \beta_j^2 \prod_{\ell=j+1}^{k} (1 - a_\Delta\beta_\ell)^2 (M_j^{\tilde{\theta}} + M_j^{\tilde{w}})$$

Applying (85) and (86) and Lemma 14

$$\boxed{|R_{k+1}| \leq C_{3,0}^{\text{exp}} \prod_{\ell=0}^{k} (1 - (a_\Delta/4)\beta_\ell) V_0 \beta_{k+1} + C_{3,1}^{\text{exp}} \beta_{k+1} \gamma_{k+1},} \tag{92}$$

where

$$C_{3,0}^{\text{exp}} := p_\Delta d_\theta \widetilde{m}_{VW}^{\text{exp}} (1 + \|A_{12}A_{22}^{-1}\|)^2 \varrho^{a_\Delta} (C_{0,\theta}^{\text{exp}} + C_{0,w}^{\text{exp}})/(1 - a_\Delta\beta_\infty^{\text{exp}}),$$
$$C_{3,1}^{\text{exp}} := p_\Delta d_\theta \widetilde{m}_{VW}^{\text{exp}} (1 + \|A_{12}A_{22}^{-1}\|)^2 \varrho^{a_\Delta} (\kappa_\infty^{\text{exp}} C_{1,\theta}^{\text{exp}} + C_{1,w}^{\text{exp}})$$

Estimation of J'_{k+1} To finish the proof of the theorem it remains to estimate J'_{k+1}. Applying (84) it is easy to check that

$$\text{Tr}(\mathbb{E}[S_{k+1}^{(0)}(S_{k+1}^{(0)})^\top]) = \text{Tr}(\widetilde{\Gamma}_{0:k}^{(1)} \mathbb{E}[\tilde{\theta}_0 \tilde{\theta}_0^\top][\widetilde{\Gamma}_{0:k}^{(1)}]^\top) \leq p_\Delta \prod_{\ell=0}^{k} (1 - a_\Delta\beta_\ell)^2 \mathbb{E}[\|\tilde{\theta}_0\|^2]$$

Similarly, recalling that $\kappa_\infty^{\text{exp}} \leq (1/4)a_{22}/a_\Delta$ and using $\prod_{s=0}^{j}(1 - a_{22}\gamma_s)(1 - a_\Delta\beta_s)^{-1} \leq \prod_{s=0}^{j}(1 - (a_{22}/2)\gamma_s)$ we obtain

$$\text{Tr}(\mathbb{E}[S_{k+1}^{(1)}(S_{k+1}^{(1)})^\top]) = \sum_{j=0}^{k} \sum_{\ell=0}^{k} \beta_j \beta_\ell \text{Tr}\left(\widetilde{\Gamma}_{j+1:k}^{(1)} A_{12} \widetilde{\Gamma}_{0:j}^{(2)} \mathbb{E}[\tilde{w}_0 \tilde{w}_0^\top][\widetilde{\Gamma}_{0:\ell}^{(2)}]^\top A_{12}^\top [\widetilde{\Gamma}_{\ell+1:k}^{(1)}]^\top\right)$$

$$\leq C_1^{\text{exp}} \prod_{\ell=0}^{k} (1 - a_\Delta\beta_\ell)^2 \mathbb{E}[\|\tilde{w}_0\|^2],$$

where

$$C_1^{\text{exp}} := (4/a_{22}^2) p_{22} p_\Delta \|A_{12}\|^2 (\kappa_\infty^{\text{exp}})^2$$

Hence, we may conclude from the previous two inequalities that

$$\boxed{\mathbb{E}[\text{Tr}(\{S_{k+1}^{(0)} + S_{k+1}^{(1)}\}\{S_{k+1}^{(0)} + S_{k+1}^{(1)}\}^\top)] \leq C_{0+1}^{\text{exp}} \prod_{\ell=0}^{k} (1 - a_\Delta\beta_\ell)^2 V_0,} \tag{93}$$

where

$$C_{0+1}^{\text{exp}} := 2p_\Delta + 4C_1^{\text{exp}} (1 + C_\infty^2)$$

For the next term in the expansion we have

$$\text{Tr}(\mathbb{E}[S_{k+1}^{(2)}(S_{k+1}^{(2)})^\top]) \leq \sum_{j=0}^{k} \sum_{\ell=0}^{k} \beta_j \beta_\ell \mathbb{E}\left[| \text{Tr}\left(\widetilde{\Gamma}_{j+1:k}^{(1)} A_{12} L_j \tilde{\theta}_j \tilde{\theta}_l^\top L_l^\top A_{12}^T [\Gamma_{\ell+1:k}^{(1)}]^\top\right)|\right]$$

We apply Cauchy-Schwartz inequality twice, first $|\operatorname{Tr}(AB^\top)| \leq \operatorname{Tr}^{1/2}(AA^\top)\operatorname{Tr}^{1/2}(BB^\top)$ and then for expectation. We obtain

$$\operatorname{Tr}(\mathbb{E}[S_{k+1}^{(2)}(S_{k+1}^{(2)})^\top]) \leq \left(\sum_{j=0}^{k} \beta_j \big(\operatorname{Tr}\big(\widetilde{\Gamma}_{j+1:k}^{(1)}A_{12}L_j\,\mathbb{E}[\tilde{\theta}_j\tilde{\theta}_j^\top]L_j^\top A_{12}^\top[\widetilde{\Gamma}_{j+1:k}^{(1)}]^\top\big)\big)^{1/2} \right)^2$$

From Lemma 19 we conclude that $\|L_j\|_{Q_\Delta,Q_{22}} \leq \mathrm{C}_L^{\mathsf{exp}}\beta_j\gamma_j^{-1}$, where $\mathrm{C}_L^{\mathsf{exp}} := C_D(L_\infty)\varrho^{a_{22}}$. This inequality and Jensen's inequality imply

$$\operatorname{Tr}(\mathbb{E}[S_{k+1}^{(2)}(S_{k+1}^{(2)})^\top]) \leq (\mathrm{C}_L^{\mathsf{exp}})^2 p_\Delta\|A_{12}\|_{Q_{22},Q_\Delta}^2 \left(\sum_{j=0}^{k} \beta_j\kappa_j \prod_{\ell=j+1}^{k}(1-a_\Delta\beta_\ell)\{\mathbb{E}[\|\tilde{\theta}_j\|^2]\}^{1/2} \right)^2$$

$$\leq d_\theta(\mathrm{C}_L^{\mathsf{exp}})^2 p_\Delta a_\Delta^{-1}\|A_{12}\|_{Q_{22},Q_\Delta}^2 \sum_{j=0}^{k} \beta_j\kappa_j^2 \prod_{\ell=j+1}^{k}(1-a_\Delta\beta_\ell)\,\mathrm{M}_j^{\tilde{\theta}}$$

Applying (85) and Lemma 14 we get

$$\operatorname{Tr}(\mathbb{E}[S_{k+1}^{(2)}(S_{k+1}^{(2)})^\top]) \leq \mathrm{C}_{2,0}^{\mathsf{exp}} \prod_{\ell=0}^{k}(1-(a_\Delta/4)\beta_\ell)\mathrm{V}_0\kappa_{k+1}^2 + \mathrm{C}_{2,1}^{\mathsf{exp}}\beta_{k+1}\kappa_{k+1}^2\}$$

where

$$\mathrm{C}_{2,0}^{\mathsf{exp}} := d_\theta a_\Delta^{-1}(\mathrm{C}_L^{\mathsf{exp}})^2 p_\Delta\|A_{12}\|_{Q_{22},Q_\Delta}^2 \mathrm{C}_{0,\theta}^{\mathsf{exp}}\varrho^{a_\Delta/2}/(1-a_\Delta\beta_\infty^{\mathsf{exp}}),$$
$$\mathrm{C}_{2,1}^{\mathsf{exp}} := d_\theta a_\Delta^{-1}(\mathrm{C}_L^{\mathsf{exp}})^2 p_\Delta\|A_{12}\|_{Q_{22},Q_\Delta}^2 \mathrm{C}_{1,\theta}^{\mathsf{exp}}\varrho^{a_\Delta}$$

To estimate the next term we rewrite it as follows

$$S_{k+1}^{(4)} = \sum_{\ell=0}^{k} \beta_\ell N_{\ell+1,k}\delta_\ell^{(2)},$$

where

$$N_{\ell+1,k} := \sum_{j=\ell+1}^{k} \beta_j\widetilde{\Gamma}_{j+1:k}^{(1)}A_{12}\widetilde{\Gamma}_{\ell+1:j-1}^{(2)}.$$

It is straightforward to check

$$\|N_{\ell+1,k}\| \leq \sqrt{p_{22}p_\Delta}\|A_{12}\|\kappa_\ell \prod_{s=\ell+1}^{k}(1-a_\Delta\beta_s) \sum_{j=\ell+1}^{k} \gamma_j \prod_{s=\ell+1}^{j-1}(1-(a_{22}/2)\gamma_s)$$

$$\leq \mathrm{C}_N^{\mathsf{exp}}\kappa_\ell(1-a_\Delta\beta_\infty^{\mathsf{exp}})^{-1} \prod_{s=\ell+1}^{k-1}(1-a_\Delta\beta_s),$$

$$\tag{94}$$

where $\mathrm{C}_N^{\mathsf{exp}} := (2/a_{22})\sqrt{p_{22}p_\Delta}\|A_{12}\|(1-a_\Delta\beta_\infty^{\mathsf{exp}})^{-1}$ and we used (compare with Lemma 12)

$$\sum_{j=\ell+1}^{k} \gamma_j \prod_{s=\ell+1}^{j-1}(1-(a_{22}/2)\gamma_s) = \frac{1}{a_{22}}\left\{1 - \prod_{s=\ell+1}^{k}(1-(a_{22}/2)\gamma_s)\right\} \leq 2/a_{22}. \tag{95}$$

Applying (96), Jensen's inequality and observation

$$\mathbb{E}[\|\delta_\ell^{(2)}\|^2] \leq \mathrm{C}_{22}^{\mathsf{exp}} \, \mathbb{E}[\|\tilde{w}_\ell\|^2], \tag{96}$$

where $\mathrm{C}_{22}^{\mathsf{exp}} := \mathrm{C}_\infty^2 \, \|A_{12}\|^2$, we obtain

$$\mathrm{Tr}(\mathbb{E}[S_{k+1}^{(4)}(S_{k+1}^{(4)})^\top]) \leq d_w a_\Delta^{-1} \mathrm{C}_{22}^{\mathsf{exp}}(\mathrm{C}_N^{\mathsf{exp}})^2 \sum_{\ell=0}^{k} \beta_\ell \kappa_\ell^2 \prod_{s=\ell+1}^{k} (1 - a_\Delta \beta_s) \, \mathrm{M}_\ell^{\tilde{w}}.$$

Applying (86) and Lemma 14 we get

$$\boxed{\mathrm{Tr}(\mathbb{E}[S_{k+1}^{(4)}(S_{k+1}^{(4)})^\top]) \leq \mathrm{C}_{4,0}^{\mathsf{exp}} \prod_{s=0}^{k}(1 - (a_\Delta/4)\beta_s)\mathrm{V}_0 + \mathrm{C}_{4,0}^{\mathsf{exp}} \beta_{k+1} \kappa_{k+1},} \tag{97}$$

where

$$\mathrm{C}_{4,0}^{\mathsf{exp}} := d_w \mathrm{C}_{0,w}^{\mathsf{exp}} \mathrm{C}_{22}^{\mathsf{exp}}(\mathrm{C}_N^{\mathsf{exp}})^2(a_\Delta)^{-1}\varrho^{a_\Delta/2}(1 - a_\Delta \beta_\infty^{\mathsf{exp}})^{-1}\kappa_\infty^2,$$
$$\mathrm{C}_{4,1}^{\mathsf{exp}} := d_w \mathrm{C}_{1,w}^{\mathsf{exp}} \mathrm{C}_{22}^{\mathsf{exp}}(\mathrm{C}_N^{\mathsf{exp}})^2 a_\Delta^{-1}\varrho^{a_\Delta}.$$

To estimate the next term we proceed similarly. Using martingale property we obtain

$$\mathrm{Tr}(\mathbb{E}[S_{k+1}^{(5)}(S_{k+1}^{(5)})^\top]) \leq d_\theta \widetilde{m}_{VW}^{\mathsf{exp}} \, \mathrm{C}_\infty^2 \, \|A_{12}\|^2 (\mathrm{C}_N^{\mathsf{exp}})^2 \sum_{\ell=0}^{k} \beta_\ell^2 \kappa_\ell^2 \prod_{s=\ell+1}^{k} (1 - a_\Delta \beta_s)^2 (\beta_\ell \gamma_\ell^{-1})^2 (1 + \mathrm{M}_\ell^{\tilde{\theta}} + \mathrm{M}_\ell^{\tilde{w}})$$

Hence, due to Corollary

$$\mathrm{Tr}(\mathbb{E}[S_{k+1}^{(5)}(S_{k+1}^{(5)})^\top]) \leq \mathrm{C}_{5,0}^{\mathsf{exp}} \prod_{\ell=0}^{k}(1 - (a_\Delta/4)\beta_\ell)\mathrm{V}_0 \beta_{k+1} \kappa_{k+1}^2 + \mathrm{C}_{5,1}^{\mathsf{exp}} \beta_{k+1} \kappa_{k+1}^2$$

where

$$\mathrm{C}_{5,0}^{\mathsf{exp}} := d_\theta \widetilde{m}_{VW}^{\mathsf{exp}} \, \mathrm{C}_\infty^2 \, \|A_{12}\|^2 (\mathrm{C}_N^{\mathsf{exp}})^2 \varrho^{a_\Delta}(\mathrm{C}_{0,w}^{\mathsf{exp}} + \mathrm{C}_{0,\theta}^{\mathsf{exp}})/(1 - a_\Delta \beta_\infty^{\mathsf{exp}}),$$
$$\mathrm{C}_{5,1}^{\mathsf{exp}} := d_\theta \widetilde{m}_{VW}^{\mathsf{exp}} \, \mathrm{C}_\infty^2 \, \|A_{12}\|^2 (\mathrm{C}_N^{\mathsf{exp}})^2(1 + \mathrm{C}_{1,w}^{\mathsf{exp}} \gamma_\infty^{\mathsf{mtg}} + \mathrm{C}_{1,\theta}^{\mathsf{exp}} \beta_\infty^{\mathsf{exp}})$$

It follows from the previous inequalities that

$$\boxed{\mathbb{E}\left[\mathrm{Tr}(\{S_{k+1}^{(2)} + S_{k+1}^{(5)}\}\{S_{k+1}^{(2)} + S_{k+1}^{(5)}\}^\top)\right] \leq \mathrm{C}_{2+5,0}^{\mathsf{exp}} \prod_{\ell=0}^{k}(1 - (a_\Delta/4)\beta_\ell)\mathrm{V}_0 \kappa_{k+1}^2 + \mathrm{C}_{2+5,1}^{\mathsf{exp}} \beta_{k+1} \kappa_{k+1}^2}$$

$$\tag{98}$$

where

$$\mathrm{C}_{2+5,0}^{\mathsf{exp}} := 2\mathrm{C}_{5,0}^{\mathsf{exp}} \beta_\infty^{\mathsf{exp}} + 2\mathrm{C}_{2,0}^{\mathsf{exp}},$$
$$\mathrm{C}_{2+5,1}^{\mathsf{exp}} := 2\mathrm{C}_{5,1}^{\mathsf{exp}} + 2\mathrm{C}_{2,1}^{\mathsf{exp}}$$

For the term $\mathbb{E}\left[\operatorname{Tr}(S_{k+1}^{(3)}\{S_{k+1}^{(2)}+S_{k+1}^{(5)}\}^{\top})\right]$ we write

$$\mathbb{E}\left[\operatorname{Tr}(S_{k+1}^{(3)}\{S_{k+1}^{(2)}+S_{k+1}^{(5)}\}^{\top})\right] \leq \mathbb{E}^{1/2}\left[\operatorname{Tr}(S_{k+1}^{(3)}(S_{k+1}^{(3)})^{\top})\right]\mathbb{E}^{1/2}\left[\operatorname{Tr}(\{S_{k+1}^{(2)}+S_{k+1}^{(5)}\}\{S_{k+1}^{(2)}+S_{k+1}^{(5)}\}^{\top})\right]$$

$$\leq \Big\{ C_{3,0}^{\mathsf{exp}}\prod_{\ell=0}^{k}(1-(a_{\Delta}/4)\beta_{\ell})V_0\beta_{k+1}+C_{3,1}^{\mathsf{exp}}\beta_{k+1}\gamma_{k+1}+p_{\Delta}\varrho^{a_{\Delta}}\operatorname{Tr}(\Sigma)\beta_{k+1}\Big\}^{1/2}$$

$$\times \Big\{ C_{2+5,0}^{\mathsf{exp}}\prod_{\ell=0}^{k}(1-(a_{\Delta}/4)\beta_{\ell})V_0\kappa_{k+1}^2+C_{2+5,1}^{\mathsf{exp}}\beta_{k+1}\kappa_{k+1}^2\Big\}^{1/2}$$

We obtain

$$\boxed{\mathbb{E}\left[\operatorname{Tr}(S_{k+1}^{(3)}\{S_{k+1}^{(2)}+S_{k+1}^{(5)}\}^{\top})\right] \leq C_{3/2+5,0}^{\mathsf{exp}}\prod_{\ell=0}^{k}(1-(a_{\Delta}/4)\beta_{\ell})V_0+C_{3/2+5,1}^{\mathsf{exp}}\beta_{k+1}\kappa_{k+1},} \quad (99)$$

where

$$C_{3/2+5,0}^{\mathsf{exp}}:=(C_{2+5,0}^{\mathsf{exp}}+C_{3,0}^{\mathsf{exp}}+C_{3,0}^{\mathsf{exp}}C_{2+5,0}^{\mathsf{exp}})/2,$$

$$C_{3/2+5,1}^{\mathsf{exp}}:=\kappa_{\infty}^{\mathsf{exp}}/2+C_{2+5,1}^{\mathsf{exp}}\beta_{\infty}^{\mathsf{exp}}\kappa_{\infty}^{\mathsf{exp}}/2+(C_{3,1}^{\mathsf{exp}}\gamma_{\infty}^{\mathsf{mtg}}+p_{\Delta}\varrho^{a_{\Delta}}\operatorname{Tr}(\Sigma))\kappa_{\infty}^{\mathsf{exp}}/2$$

$$+(C_{2+5,1}^{\mathsf{exp}}(C_{3,1}^{\mathsf{exp}}\gamma_{\infty}^{\mathsf{mtg}}+p_{\Delta}\varrho^{a_{\Delta}}\operatorname{Tr}(\Sigma)))^{1/2}$$

Let us consider the term

$$\mathbb{E}[\operatorname{Tr}(S_{k+1}^{(3)}S_{k+1}^{(4)})^{\top}] = \sum_{j=0}^{k}\beta_j\sum_{\ell=j+1}^{k}\beta_{\ell}\operatorname{Tr}(\widetilde{\Gamma}_{j+1:k}^{(1)}Z_{j+1}\tilde{w}_{\ell}^{\top}A_{12}^{\top}C_{\ell}^{\top}N_{\ell+1,k}^{\top}), \quad (100)$$

where $Z_{j+1}:=V_{j+1}+A_{12}A_{22}^{-1}W_{j+1}$. For $\tilde{w}_{\ell}$ we can use the following expansion

$$\tilde{w}_{\ell}=\widetilde{\Gamma}_{0:\ell-1}^{(2)}\tilde{w}_0+\sum_{i=0}^{\ell-1}\gamma_i\widetilde{\Gamma}_{i+1:\ell-1}^{(2)}\widetilde{Z}_{i+1}+\sum_{i=0}^{\ell-1}\beta_i\widetilde{\Gamma}_{i+1:\ell-1}^{(2)}C_iA_{12}\tilde{w}_i,$$

where $\widetilde{Z}_{i+1}:=W_{i+1}+\kappa_iC_iV_{i+1}$ Substituting this expansion into r.h.s of (100) and repeating this procedure until $\mathbb{E}[Z_{j+1}\tilde{w}_l^{\top}]=\gamma_j\,\mathbb{E}[Z_{j+1}\widetilde{Z}_{j+1}^{\top}(\widetilde{\Gamma}_{j+1:\ell_1-1}^{(2)})^{\top}]$ we come to the following expansion of (100)

$$\sum_{\ell=j+1}^{k}\beta_{\ell}\operatorname{Tr}(\widetilde{\Gamma}_{j+1:k}^{(1)}Z_{j+1}\tilde{w}_{\ell}^{\top}A_{12}^{\top}C_{\ell}^{\top}N_{\ell+1,k}^{\top})$$

$$=\sum_{\ell_1=j+1}^{k}\beta_{\ell_1}\sum_{\ell_2=j+1}^{\ell_1-1}\beta_{\ell_2}\operatorname{Tr}(\widetilde{\Gamma}_{j+1:k}^{(1)}Z_{j+1}\tilde{w}_{\ell_2}^{\top}A_{12}^{\top}C_{\ell_2}^{\top}(\widetilde{\Gamma}_{\ell_2+1:\ell_1-1}^{(2)})^{\top}A_{12}^{\top}C_{\ell_1}^{\top}N_{\ell_1+1,k}^{\top})$$

$$+\gamma_j\sum_{\ell_1=j+1}^{k}\beta_{\ell_1}\operatorname{Tr}(\widetilde{\Gamma}_{j+1:k}^{(1)}Z_{j+1}\widetilde{Z}_{j+1}^{\top}(\widetilde{\Gamma}_{j+1:\ell_1-1}^{(2)})^{\top}A_{12}^{\top}C_{\ell_1}^{\top}N_{\ell_1+1,k}^{\top})$$

$$=\gamma_j\sum_{s=1}^{k-j}\sum_{\ell_1=j+1}^{k}\beta_{\ell_1}\sum_{\ell_2=1}^{\ell_1-1}\beta_{\ell_2}\ldots\sum_{\ell_s=j+1}^{l_{s-1}-1}\beta_{\ell_s}$$

$$\times\operatorname{Tr}(\widetilde{\Gamma}_{j+1:k}^{(1)}Z_{j+1}\widetilde{Z}_{j+1}^{\top}(\widetilde{\Gamma}_{j+1:\ell_s-1}^{(2)})^{\top}A_{12}^{\top}C_{\ell_s}^{\top}(\widetilde{\Gamma}_{\ell_s+1:\ell_{s-1}-1}^{(2)})^{\top}A_{12}^{\top}C_{\ell_s}^{\top}\ldots(\widetilde{\Gamma}_{\ell_2+1:\ell_1-1}^{(2)})^{\top}A_{12}^{\top}C_{\ell_1}^{\top}N_{\ell_1+1,k}^{\top})$$

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where $\ell_0 := k+1$. Using iteratively Corollary and estimate (94) for $N_{\ell_1+1,k}$ we obtain the following bound

$$\left\| \sum_{\ell=j+1}^{k} \beta_\ell \operatorname{Tr}(\Gamma_{j:k-1}^{(1)} Z_{j+1} \tilde{w}_\ell^\top A_{12}^\top C_\ell^\top N_{\ell_1+1,k}^\top) \right\|$$

$$\leq \mathrm{C}_\infty\, \varrho^{a_{22}} \mathrm{C}_N^{\mathsf{exp}} \beta_j \kappa_j \prod_{\ell=j+1}^{k} (1 - a_\Delta \beta_\ell)\, \mathbb{E}^{1/2}[\|Z_{j+1}\|^2]\, \mathbb{E}^{1/2}[\|\widetilde{Z}_{j+1}\|^2] \sum_{s=1}^{k-j} (\kappa_j \|A_{12}\|\, \mathrm{C}_\infty\, \varrho^{a_{22}})^{s-1}$$

Since $\kappa_\infty^{\mathsf{exp}} \leq (1/2)(\|A_{12}\|\, \mathrm{C}_\infty\, \varrho^{a_{22}})^{-1}$ and

$$\mathbb{E}^{1/2}[\|Z_{j+1}\|^2]\, \mathbb{E}^{1/2}[\|\widetilde{Z}_{j+1}\|^2] \leq 2\widetilde{m}_{VW}^{\mathsf{exp}}(1 + \|A_{12}A_{22}^{-1}\|)(1 + \sqrt{\kappa_\infty^{\mathsf{exp}}\, \mathrm{C}_\infty})(1 + \mathrm{M}_j^{\tilde\theta} + \mathrm{M}_j^{\tilde{w}})$$

we obtain that

$$\left\| \sum_{\ell=j+1}^{k} \beta_\ell \operatorname{Tr}(\widetilde{\Gamma}_{j+1:k}^{(1)} Z_{j+1} \tilde{w}_\ell^\top A_{12}^\top C_\ell^\top N_{\ell_1+1,k-1}^\top) \right\|$$

$$\leq \mathrm{C}_\infty\, \mathrm{C}_N^{\mathsf{exp}} 2\widetilde{m}_{VW}^{\mathsf{exp}}(1 + \|A_{12}A_{22}^{-1}\|)(1 + \sqrt{\kappa_\infty^{\mathsf{exp}}\, \mathrm{C}_\infty}) \beta_j \kappa_j \prod_{s=j+1}^{k} (1 - a_\Delta \beta_s)(1 + \mathrm{M}_j^{\tilde\theta} + \mathrm{M}_j^{\tilde{w}})$$

This inequality and (100) together imply

$$\big| \mathbb{E}[\operatorname{Tr}(S_{k+1}^{(3)} S_{k+1}^{(4)})^\top]\big|$$

$$\leq \mathrm{C}_\infty\, \mathrm{C}_N^{\mathsf{exp}} 2\widetilde{m}_{VW}^{\mathsf{exp}}(1 + \|A_{12}A_{22}^{-1}\|)(1 + \sqrt{\kappa_\infty^{\mathsf{exp}}\, \mathrm{C}_\infty}) \sum_{j=0}^{k} \beta_j^2 \kappa_j \prod_{s=j+1}^{k} (1 - a_\Delta \beta_s)(1 + \mathrm{M}_j^{\tilde\theta} + \mathrm{M}_j^{\tilde{w}})$$

Finally, the standard arguments will lead to

$$\boxed{\big| \mathbb{E}[\operatorname{Tr}(S_{k+1}^{(3)} S_{k+1}^{(4)})^\top]\big| \leq \mathrm{C}_{3/4,0}^{\mathsf{exp}} \prod_{\ell=0}^{k} (1 - (a_\Delta/4)\beta_\ell) V_0 + \mathrm{C}_{3/4,0}^{\mathsf{exp}} \beta_{k+1} \kappa_{k+1},} \qquad (101)$$

where

$$\mathrm{C}_{3/4,0}^{\mathsf{exp}} := \mathrm{C}_\infty\, \mathrm{C}_N^{\mathsf{exp}} 2\widetilde{m}_{VW}^{\mathsf{exp}}(1 + \|A_{12}A_{22}^{-1}\|)(1 + \sqrt{\kappa_\infty^{\mathsf{exp}}\, \mathrm{C}_\infty}) \varrho^{a_\Delta/2}(\mathrm{C}_{0,w}^{\mathsf{exp}} + \mathrm{C}_{0,\theta}^{\mathsf{exp}})/(1 - a_\Delta \beta_\infty^{\mathsf{exp}}) \beta_\infty^{\mathsf{exp}} \kappa_\infty^{\mathsf{exp}},$$

$$\mathrm{C}_{3/4,1}^{\mathsf{exp}} := \mathrm{C}_\infty\, \mathrm{C}_N^{\mathsf{exp}} 2\widetilde{m}_{VW}^{\mathsf{exp}}(1 + \|A_{12}A_{22}^{-1}\|)(1 + \sqrt{\kappa_\infty^{\mathsf{exp}}\, \mathrm{C}_\infty}) \varrho^{a_\Delta}(1 + \mathrm{C}_{1,w}^{\mathsf{exp}} \gamma_\infty^{\mathsf{mtg}} + \mathrm{C}_{1,\theta}^{\mathsf{exp}} \beta_\infty^{\mathsf{exp}})$$

Finally, we estimate all terms involving $S_{k+1}^{(6)}$. We rewrite $S_{k+1}^{(6)}$ as follows

$$S_{k+1}^{(6)} = \sum_{\ell=0}^{k} \beta_l M_{\ell,k} W_{\ell+1}$$

where we defined

$$M_{\ell,k} := \gamma_\ell \beta_\ell^{-1} \sum_{j=\ell+1}^{k} \beta_j \widetilde{\Gamma}_{j+1:k}^{(1)} A_{12} \widetilde{\Gamma}_{\ell+1:j-1}^{(2)} - \widetilde{\Gamma}_{\ell+1:k}^{(1)} A_{12} A_{22}^{-1}.$$

Using martingale property we obtain

$$\text{Tr}(\mathbb{E}[S_{k+1}^{(6)}(S_{k+1}^{(6)})^\top]) \leq \sum_{\ell=0}^{k} \beta_\ell^2 \|M_{\ell,k}\|^2 \, \mathbb{E}[\|W_{\ell+1}\|^2]$$

We rewrite $M_{\ell,k}$ as follows

$$M_{\ell,k} = \sum_{j=\ell+1}^{k} \gamma_j \left[\frac{\beta_j \gamma_l}{\beta_l \gamma_j} \, \mathrm{I} - \widetilde{\Gamma}_{\ell+1:j}^{(1)} \right] \widetilde{\Gamma}_{j+1:k}^{(1)} A_{12} \widetilde{\Gamma}_{\ell+1:j-1}^{(2)} + \widetilde{\Gamma}_{\ell+1:k}^{(1)} A_{12} \left(\sum_{j=l+1}^{k} \gamma_j \widetilde{\Gamma}_{\ell+1:j-1}^{(2)} - A_{22}^{-1} \right)$$

Since,

$$\sum_{j=l+1}^{k} \gamma_j \Gamma_{\ell+1:j-1}^{(2)} = A_{22}^{-1} \left(\mathrm{I} - \widetilde{\Gamma}_{\ell+1:k}^{(2)} \right)$$

this equation leads to

$$M_{\ell,k} = \sum_{j=\ell+1}^{k} \gamma_j \left[\frac{\beta_j \gamma_l}{\beta_l \gamma_j} \, \mathrm{I} - \widetilde{\Gamma}_{\ell+1:j}^{(1)} \right] \widetilde{\Gamma}_{j+1:k}^{(1)} A_{12} \widetilde{\Gamma}_{\ell+1:j-1}^{(2)} - \widetilde{\Gamma}_{\ell+1:k}^{(1)} A_{12} A_{22}^{-1} \widetilde{\Gamma}_{\ell+1:k}^{(2)}$$

We rewrite the term in the square brackets as follows

$$\frac{\beta_j \gamma_l}{\beta_l \gamma_j} \, \mathrm{I} - \widetilde{\Gamma}_{\ell+1:j}^{(1)} = \prod_{s=\ell+1}^{j} \frac{\kappa_s}{\kappa_{s-1}} \, \mathrm{I} - \prod_{s=\ell+1}^{j} (\mathrm{I} - \beta_s \Delta) = \sum_{t=\ell+1}^{j} \frac{\kappa_{t-1}}{\kappa_l} \, \mathrm{I} \left\{ \beta_t \Delta + (\kappa_t/\kappa_{t-1} - 1) \, \mathrm{I} \right\} \widetilde{\Gamma}_{t+1:j}^{(1)}$$

Using assumption 2 we may show that

$$|\kappa_t/\kappa_{t-1} - 1| \leq (a_\Delta/16)\beta_t$$

Taking norm of the both sides of the previous equation we obtain

$$\left\| \frac{\beta_j \gamma_l}{\beta_l \gamma_j} \, \mathrm{I} - \widetilde{\Gamma}_{\ell+1:j}^{(1)} \right\| \leq \sqrt{p_\Delta}(\|\Delta\| + (a_\Delta/16))\kappa_l^{-1} \sum_{t=\ell+1}^{j} \beta_t \kappa_{t-1} \prod_{s=t+1}^{j} (1 - a_\Delta \beta_s)$$

Finally we arrive at the following bound for $M_{\ell,k}$

$$\begin{aligned}
\|M_{\ell,k}\| &\leq \sqrt{p_{22} p_\Delta} \|A_{12} A_{22}^{-1}\| \prod_{s=\ell+1}^{k} (1 - a_\Delta \beta_s) \prod_{s=\ell+1}^{k} (1 - a_{22} \gamma_s) \\
&+ \sqrt{p_\Delta p_{22}} \left(\|\Delta\| + \frac{a_\Delta}{16} \right) \kappa_l^{-1} \sum_{j=\ell+1}^{k} \gamma_j \sum_{t=\ell+1}^{j} \beta_t \kappa_{t-1} \prod_{s=t+1}^{k} (1 - a_\Delta \beta_s) \prod_{s=\ell+1}^{j-1} (1 - a_{22} \gamma_s)
\end{aligned} \tag{102}$$

This bound will yield

$$
\begin{aligned}
\mathrm{Tr}(\mathbb{E}[S_{k+1}^{(6)}(S_{k+1}^{(6)})^\top]) \leq\ & 2 p_{22} p_\Delta \|A_{12} A_{22}^{-1}\|^2 \sum_{\ell=0}^{k} \beta_\ell^2 \prod_{s=\ell+1}^{k} (1 - a_\Delta \beta_s) \prod_{s=\ell+1}^{k} (1 - a_{22} \gamma_s) \, \mathbb{E}[\|W_{\ell+1}\|^2] \\
& + 2 p_\Delta p_{22} (\|\Delta\| + (a_\Delta/16))^2 \sum_{\ell=0}^{k} \gamma_\ell^2 \, \mathbb{E}[\|W_{\ell+1}\|^2] \prod_{s=\ell+1}^{k} (1 - a_\Delta \beta_s) \\
& \times \left\{ \sum_{j=\ell+1}^{k} \gamma_j \prod_{s=\ell+1}^{j-1} (1 - (a_{22}/2)\gamma_s) \sum_{t=\ell+1}^{j} \beta_t \kappa_{t-1} \prod_{s=t+1}^{j} (1 - a_\Delta \beta_s) \right\}^2 \\
=:\ & A_1 + A_2
\end{aligned}
$$

The estimate of A_1 follows from Lemma 14

$$
A_1 \leq \mathrm{C}_{6,0,2}^{\mathsf{exp}} \prod_{\ell=0}^{k} (1 - (a_\Delta/4)\beta_\ell) \mathrm{V}_0 + \mathrm{C}_{6,1,1}^{\mathsf{exp}} \beta_{k+1} \kappa_{k+1}. \tag{103}
$$

where

$$
\begin{aligned}
\mathrm{C}_{6,0,1}^{\mathsf{exp}} &:= 2 d_w \widetilde{m}_{VW}^{\mathsf{exp}} p_{22} p_\Delta \|A_{12} A_{22}^{-1}\|^2 \varrho^{a_{22}} (\mathrm{C}_{0,w}^{\mathsf{exp}} + \mathrm{C}_{0,\theta}^{\mathsf{exp}}) \beta_\infty^{\mathsf{exp}} \kappa_\infty^{\mathsf{exp}} / (1 - a_\Delta \beta_\infty^{\mathsf{exp}}), \\
\mathrm{C}_{6,1,1}^{\mathsf{exp}} &:= 2 d_w \widetilde{m}_{VW}^{\mathsf{exp}} p_{22} p_\Delta \|A_{12} A_{22}^{-1}\|^2 (1 + \mathrm{C}_{1,w}^{\mathsf{exp}} \gamma_\infty^{\mathsf{mtg}} + \mathrm{C}_{1,\theta}^{\mathsf{exp}} \beta_\infty^{\mathsf{exp}}) \varrho^{a_{22}}
\end{aligned}
$$

Inequality (95) and Jensen's inequality together imply

$$
\begin{aligned}
A_2 \leq\ & 2(2/a_{22})^2 p_\Delta p_{22} (\|\Delta\| + (a_\Delta/16))^2 \sum_{\ell=0}^{k} \gamma_\ell^2 \, \mathbb{E}[\|W_{\ell+1}\|^2] \prod_{s=\ell+1}^{k} (1 - a_\Delta \beta_s) \\
& \times \sum_{j=\ell+1}^{k} \gamma_j \left\{ \sum_{t=\ell+1}^{j} \beta_t \kappa_{t-1} \prod_{s=t+1}^{j} (1 - a_\Delta \beta_s) \right\}^2 \prod_{s=\ell+1}^{j-1} (1 - (a_{22}/2)\gamma_s)
\end{aligned}
$$

Similarly, applying Jensen's inequality for the second time we come to the following inequality

$$
\begin{aligned}
A_2 \leq\ & 2(2/a_{22})^2 (1/a_\Delta)^2 p_\Delta p_{22} (\|\Delta\| + (a_\Delta/16))^2 \sum_{\ell=0}^{k} \gamma_\ell^2 \, \mathbb{E}[\|W_{\ell+1}\|^2] \prod_{s=\ell+1}^{k} (1 - a_\Delta \beta_s) \\
& \times \sum_{j=\ell+1}^{k} \gamma_j \sum_{t=\ell+1}^{j} \beta_t \kappa_{t-1}^2 \prod_{s=t+1}^{j} (1 - a_\Delta \beta_s) \prod_{s=\ell+1}^{j-1} (1 - (a_{22}/2)\gamma_s)
\end{aligned}
$$

Changing the order of summation we obtain

$$
\begin{aligned}
A_2 \leq\ & 2(2/a_{22})^2 (1/a_\Delta)^2 p_\Delta p_{22} (\|\Delta\| + (a_\Delta/16))^2 \sum_{j=0}^{k} \gamma_j \prod_{s=j+1}^{k} (1 - a_\Delta \beta_s) \sum_{t=0}^{j} \beta_t \kappa_{t-1}^2 \prod_{s=t+1}^{j} (1 - (a_{22}/2)\gamma_s) \\
& \times \sum_{\ell=0}^{t-1} \gamma_\ell^2 \prod_{s=\ell+1}^{t-1} (1 - (a_{22}/2)\gamma_s) \, \mathbb{E}[\|W_{\ell+1}\|^2]
\end{aligned}
$$

Finally, estimating $\mathbb{E}[\|W_{\ell+1}\|^2]$ by (85) and (86) and applying Lemma 14 we obtain

$$A_2 \le C_{6,0,2}^{\mathsf{exp}} \prod_{\ell=0}^{k}(1 - (a_\Delta/4)\beta_\ell)V_0 + C_{6,1,2}^{\mathsf{exp}}\beta_{k+1}\kappa_{k+1}, \tag{104}$$

where

$$C_{6,0,2}^{\mathsf{exp}} := d_w \widetilde{m}_{VW}^{\mathsf{exp}}(\varrho^{a_{22}/4})^2 \varrho^{a_\Delta/2}2(2/a_{22})^2(1/a_\Delta)^2 p_\Delta p_{22}(\|\Delta\| + (a_\Delta/16))^2$$
$$\times (C_{0,w}^{\mathsf{exp}} + C_{0,\theta}^{\mathsf{exp}})(\kappa_\infty^{\mathsf{exp}})^2 \gamma_\infty^{\mathsf{mtg}}/(1 - a_\Delta \beta_\infty^{\mathsf{exp}})^2,$$

$$C_{6,1,2}^{\mathsf{exp}} := d_w \widetilde{m}_{VW}^{\mathsf{exp}}(\varrho^{a_{22}/2})^2 \varrho^{a_\Delta}2(2/a_{22})^2(1/a_\Delta)^2 p_\Delta p_{22}(\|\Delta\| + (a_\Delta/16))^2(1 + C_{1,w}^{\mathsf{exp}}\gamma_\infty^{\mathsf{mtg}} + C_{1,\theta}^{\mathsf{exp}}\beta_\infty^{\mathsf{exp}})$$

We conclude from (103) and (104) that

$$\boxed{\operatorname{Tr}(\mathbb{E}[S_{k+1}^{(6)}(S_{k+1}^{(6)})^\top]) \le C_{6,0}^{\mathsf{exp}} \prod_{\ell=0}^{k}(1 - (a_\Delta/4)\beta_\ell)V_0 + C_{6,1}^{\mathsf{exp}}\beta_{k+1}\kappa_{k+1},} \tag{105}$$

where

$$C_{6,0}^{\mathsf{exp}} := C_{6,0,1}^{\mathsf{exp}} + C_{6,0,2}^{\mathsf{exp}},$$
$$C_{6,1}^{\mathsf{exp}} := C_{6,1,1}^{\mathsf{exp}} + C_{6,1,2}^{\mathsf{exp}}$$

It remains to consider $\mathbb{E}\left[\operatorname{Tr}(S_{k+1}^{(3)}(S_{k+1}^{(6)})^\top)\right]$. We proceed similarly and use (102) to get

$$\left|\mathbb{E}\left[\operatorname{Tr}(S_{k+1}^{(3)}(S_{k+1}^{(6)})^\top)\right]\right| \le \sum_{\ell=0}^{k}\beta_\ell^2\|M_{\ell,k}\| \prod_{s=\ell+1}^{k}(1 - a_\Delta\beta_s)\,\mathbb{E}^{1/2}[\|Z_{\ell+1}\|^2]\,\mathbb{E}^{1/2}[\|W_{\ell+1}\|^2]$$
$$= A_1' + A_2'$$

The following estimate holds for A_1'

$$|A_1'| \le \sqrt{p_{22}p_\Delta}\|A_{12}A_{22}^{-1}\|\max(d_\theta, d_w)(2 + 2\|A_{12}A_{22}^{-1}\|)^{1/2} \sum_{\ell=0}^{k}\beta_\ell^2 \prod_{s=\ell+1}^{k}(1 - a_\Delta\beta_s)$$
$$\times \prod_{s=\ell+1}^{k}(1 - a_{22}\gamma_s)(1 + \mathrm{M}_\ell^{\tilde{\theta}} + \mathrm{M}_\ell^{\tilde{w}})$$

Applying standard arguments we get

$$A_1' \le C_{3/6,0,1}^{\mathsf{exp}} \prod_{\ell=0}^{k}(1 - (a_\Delta/4)\beta_\ell)V_0 + C_{3/6,1,1}^{\mathsf{exp}}\beta_{k+1}\kappa_{k+1}, \tag{106}$$

where

$$C_{3/6,0,1}^{\mathsf{exp}} := \widetilde{m}_{VW}^{\mathsf{exp}}\sqrt{p_{22}p_\Delta}\|A_{12}A_{22}^{-1}\|\max(d_\theta, d_w)(2 + 2\|A_{12}A_{22}^{-1}\|)^{1/2}$$
$$\times (C_{0,w}^{\mathsf{exp}} + C_{0,\theta}^{\mathsf{exp}})\varrho^{a_{22}}\beta_\infty^{\mathsf{exp}}\kappa_\infty^{\mathsf{exp}}/(1 - a_\Delta\beta_\infty^{\mathsf{exp}}),$$

$$C_{3/6,1,1}^{\mathsf{exp}} := 2\widetilde{m}_{VW}^{\mathsf{exp}}\sqrt{p_{22}p_\Delta}\|A_{12}A_{22}^{-1}\|\max(d_\theta, d_w)(2 + 2\|A_{12}A_{22}^{-1}\|)^{1/2}(1 + C_{1,w}^{\mathsf{exp}}\gamma_\infty^{\mathsf{mtg}} + C_{1,\theta}^{\mathsf{exp}}\beta_\infty^{\mathsf{exp}})\varrho^{a_{22}}$$

For A_2' we write the following bound

$$A_2' \leq \sqrt{p_\Delta p_{22}}(\|\Delta\| + (a_\Delta/16)) \max(d_\theta, d_w)(2 + 2\|A_{12}A_{22}^{-1}\|)^{1/2} \sum_{\ell=0}^{k} \gamma_\ell^2 (1 + M_\ell^{\tilde{\theta}} + M_\ell^{\tilde{w}}) \prod_{s=\ell+1}^{k} (1 - a_\Delta \beta_s)$$

$$\times \left\{ \sum_{j=\ell+1}^{k} \gamma_j \prod_{s=\ell+1}^{j-1} (1 - a_{22}\gamma_s) \sum_{t=\ell+1}^{j} \beta_t \kappa_{t-1} \prod_{s=t+1}^{j} (1 - a_\Delta \beta_s) \right\}$$

Changing the order of summation and applying arguments from the estimation of A_2 we come to the following bound

$$A_2' \leq C_{3/6,0,2}^{\text{exp}} \prod_{\ell=0}^{k} (1 - (a_\Delta/4)\beta_\ell) V_0 + C_{3/6,1,2}^{\text{exp}} \beta_{k+1} \kappa_{k+1}, \tag{107}$$

where

$$C_{3/6,0,2}^{\text{exp}} := \sqrt{p_\Delta p_{22}}(\|\Delta\| + (a_\Delta/16)) \max(d_\theta, d_w)(2 + 2\|A_{12}A_{22}^{-1}\|)^{1/2}$$

$$\times (C_{0,w}^{\text{exp}} + C_{0,\theta}^{\text{exp}})(\varrho^{a_{22}/4})^2 \varrho^{a_\Delta/2} \gamma_\infty^{\text{mtg}} (\kappa_\infty^{\text{exp}})^2 / (1 - a_\Delta \beta_\infty^{\text{exp}})^2,$$

$$C_{3/6,1,2}^{\text{exp}} := \sqrt{p_\Delta p_{22}}(\|\Delta\| + (a_\Delta/16)) \max(d_\theta, d_w)(2 + 2\|A_{12}A_{22}^{-1}\|)^{1/2} (1 + C_{1,w}^{\text{exp}}\gamma_\infty^{\text{mtg}} + C_{1,\theta}^{\text{exp}}\beta_\infty^{\text{exp}})(\varrho^{a_{22}/2})^2 \varrho^{a_\Delta}$$

We conclude from (106) and (107)

$$\boxed{\left| \text{Tr}(\mathbb{E}[S_{k+1}^{(3)}(S_{k+1}^{(6)})^\top]) \right| \leq C_{3/6,0}^{\text{exp}} \prod_{\ell=0}^{k} (1 - (a_\Delta/4)\beta_\ell) V_0 + C_{3/6,1}^{\text{exp}} \beta_{k+1} \kappa_{k+1},} \tag{108}$$

where

$$C_{3/6,0}^{\text{exp}} := C_{3/6,0,1}^{\text{exp}} + C_{3/6,0,2}^{\text{exp}},$$
$$C_{3/6,1}^{\text{exp}} := C_{3/6,1,1}^{\text{exp}} + C_{3/6,1,2}^{\text{exp}}$$

Final estimate of the remainder term J_{k+1} Collecting (92), (93),(97), (98), (99), (101), (105), (108) we obtain the estimate of $J_{k+1} := R_{k+1} + J_{k+1}'$

$$\boxed{|J_{k+1}| \leq C_0^{\text{exp}} \prod_{\ell=0}^{k} (1 - (a_\Delta/4)\beta_\ell) V_0 + C_0^{\text{exp}} \beta_{k+1}(\gamma_{k+1} + \kappa_{k+1}),}$$

where

$$C_0^{\text{exp}} := C_{3,0}^{\text{exp}} \beta_\infty^{\text{exp}} + 3C_{0/1}^{\text{exp}} + 5C_{4,0}^{\text{exp}} + 5C_{2+5,0}^{\text{exp}}(\kappa_\infty^{\text{exp}})^2 + 2C_{3/2+5,0}^{\text{exp}} + 2C_{3/4,0}^{\text{exp}}$$

$$+ 5C_{6,0}^{\text{exp}} + 2C_{3/6,0}^{\text{exp}}, \tag{109}$$

$$C_1^{\text{exp}} := C_{3,1}^{\text{exp}} + 5C_{4,1}^{\text{exp}} + 5C_{2+5,1}^{\text{exp}} + 2C_{3/2+5,1}^{\text{exp}} + 2C_{3/4,1}^{\text{exp}} + 5C_{6,1}^{\text{exp}} + 2C_{3/6,1}^{\text{exp}}$$

Hence, we obtained (33).

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Appendix D. Auxiliary Lemmas

Lemma 12 *Let $a > 0$ and $(\gamma_k)_{k \geq 0}$ be a nonincreasing sequence such that $\gamma_0 < 1/a$. Then, for any integer $k \geq 1$,*

$$\sum_{j=0}^{k-1} \gamma_j \prod_{l=j+1}^{k-1} (1 - \gamma_l a) = \frac{1}{a} \left\{ 1 - \prod_{l=0}^{k-1} (1 - \gamma_l a) \right\}$$

Remark 13 *If k_0 is such that $\sum_{l=0}^{k_0-1} \gamma_l \geq \log(2)/a$ then the r.h.s. of the previous equation is lower bounded by $1/(2a)$ for any $k \geq k_0$.*

Proof Let us denote $u_{j:k-1} = \prod_{l=j}^{k-1} (1 - \gamma_l a)$. Then, for $j \in \{0, \ldots, k-1\}$, $u_{j+1:k-1} - u_{j:k-1} = a\gamma_j u_{j+1}$. Hence,

$$\sum_{j=0}^{k-1} \gamma_j \prod_{l=j+1}^{k-1} (1 - \gamma_l a) = \frac{1}{a} \sum_{j=0}^{k-1} (u_{j+1:k-1} - u_{j:k-1}) = a^{-1}(1 - u_{0:k-1}).$$

∎

Lemma 14 *Assume A2 and set*

$$\varrho^a = \frac{2}{a} \varsigma \max\{1, a_{22}/(4a_\Delta)\} \vee \frac{4}{a} \varsigma^3.$$

The following holds

(i) For any $a \in [a_{22}/4, \gamma_0^{-1}]$ and $k \in \mathbb{N}$, if in addition, we have $\kappa \leq a_{22}/(4a_\Delta)$, then

$$\sum_{j=0}^{k-1} \gamma_j^2 \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varrho^a \gamma_k, \quad \sum_{j=0}^{k-1} \beta_j \gamma_j \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varrho^a \beta_k, \quad \sum_{j=0}^{k-1} \beta_j^2 \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varrho^a \beta_k$$

(ii) For any $a \in [a_\Delta/8, \beta_0^{-1}]$ and $k \in \mathbb{N}$,

$$\sum_{j=0}^{k-1} \beta_j \gamma_j \prod_{\ell=j+1}^{k-1} (1 - a\beta_\ell) \leq \varrho^a \gamma_k, \quad \sum_{j=0}^{k-1} \beta_j^2 \prod_{\ell=j+1}^{k-1} (1 - a\beta_\ell) \leq \varrho^a \beta_k$$

(iii) For any $a \in [a_\Delta/4, \beta_0^{-1}]$ and $k \in \mathbb{N}$,

$$\sum_{j=0}^{k-1} \beta_j^3/\gamma_j \prod_{\ell=j+1}^{k-1} (1 - a\beta_\ell) \leq \varrho^a \beta_k^2/\gamma_k, \quad \sum_{j=0}^{k-1} \beta_j^4/\gamma_j^2 \prod_{\ell=j+1}^{k-1} (1 - a\beta_\ell) \leq \varrho^a \beta_k^3/\gamma_k^2,$$

$$\sum_{j=0}^{k-1} \beta_j^2 \gamma_j \prod_{\ell=j+1}^{k-1} (1 - a\beta_\ell) \leq \varrho^a \beta_k \gamma_k$$

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(iv) For any $a \in [a_{22}/4, \gamma_0^{-1}]$ and $k \in \mathbb{N}$,

$$\sum_{j=0}^{k-1} \beta_j \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varrho^a \beta_k / \gamma_k, \qquad \sum_{j=0}^{k-1} \beta_j^2 \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varrho^a \beta_k^2 / \gamma_k, \qquad \sum_{j=0}^{k-1} \beta_j^3 / \gamma_j \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varrho^a \beta_k^3 / \gamma_k^2$$

Proof Part i) of the corollary, consider the first inequality and observe that

$$\sum_{j=0}^{k-1} \gamma_j^2 \prod_{l=j+1}^{k-1} (1 - \gamma_l a) = \gamma_k \sum_{j=0}^{k-1} \frac{\gamma_{k-1}}{\gamma_k} \frac{\gamma_j}{\gamma_{k-1}} \gamma_j \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varsigma \gamma_k \sum_{j=0}^{k-1} \gamma_j \prod_{l=j+1}^{k-1} \frac{\gamma_{l-1}}{\gamma_l} (1 - \gamma_l a)$$

Note that as $a \geq a_{22}/4$, we have

$$\frac{\gamma_{l-1}}{\gamma_l} (1 - \gamma_l a) \leq (1 + \frac{a_{22}}{8} \gamma_l)(1 - \gamma_l a) \leq 1 - a\gamma_l/2$$

Substituting into the above inequality yields

$$\sum_{j=0}^{k-1} \gamma_j^2 \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varsigma \gamma_k \sum_{j=0}^{k-1} \gamma_j \prod_{l=j+1}^{k-1} (1 - \gamma_l a/2) \leq \varsigma \frac{2}{a} \gamma_k$$

where we have applied Lemma 12 in the last inequality. Next and applying similar steps as before, we observe that

$$\sum_{j=0}^{k-1} \beta_j \gamma_j \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varsigma \beta_k \sum_{j=0}^{k-1} \gamma_j \prod_{l=j+1}^{k-1} \frac{\beta_{l-1}}{\beta_l} (1 - \gamma_l a)$$

As we have

$$\frac{\beta_{l-1}}{\beta_l} (1 - \gamma_l a) \leq 1 - \gamma_l (a - \kappa a_\Delta / 16) \leq 1 - \gamma_l (a - a_{22}/64) \leq 1 - \gamma_l a/2$$

we obtain

$$\sum_{j=0}^{k-1} \beta_j \gamma_j \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varsigma \frac{2}{a} \beta_k$$

Similarly, using $\beta_j \leq \kappa \gamma_j \leq a_{22}/(4 a_\Delta) \gamma_j$, we obtain

$$\sum_{j=0}^{k-1} \beta_j^2 \prod_{l=j+1}^{k-1} (1 - \gamma_l a) \leq \varsigma \frac{a_{22}}{2 a \, a_\Delta} \beta_k$$

For part ii) of the corollary, we observe that the first inequality can be proven by:

$$\sum_{j=0}^{k-1} \beta_j \gamma_j \prod_{\ell=j+1}^{k-1} (1 - a\beta_\ell) = \gamma_k \sum_{j=0}^{k-1} \beta_j \frac{\gamma_{k-1}}{\gamma_k} \frac{\gamma_j}{\gamma_{k-1}} \prod_{\ell=j+1}^{k-1} (1 - a\beta_\ell)$$

$$\leq \varsigma \gamma_k \sum_{j=0}^{k-1} \beta_j \frac{\gamma_j}{\gamma_{k-1}} \prod_{\ell=j+1}^{k-1} (1 - a\beta_\ell) \leq \varsigma \gamma_k \sum_{j=0}^{k-1} \beta_j \prod_{\ell=j+1}^{k-1} \frac{\gamma_{\ell-1}}{\gamma_\ell} (1 - a\beta_\ell)$$

Note that as $a \geq a_\Delta/8$, we have

$$\frac{\gamma_{\ell-1}}{\gamma_\ell}(1 - a\beta_\ell) \leq (1 + \epsilon_\beta\beta_\ell)(1 - a\beta_\ell) \leq 1 - a\beta_\ell/2$$

Using Lemma 12, this yields

$$\sum_{j=0}^{k-1}\beta_j\gamma_j \prod_{\ell=j+1}^{k-1}(1 - a\beta_\ell) \leq \varsigma\gamma_k \sum_{j=0}^{k-1}\beta_j \prod_{\ell=j+1}^{k-1}(1 - a\beta_\ell/2) \leq \frac{2}{a}\varsigma\gamma_k$$

Similarly, the second inequality is:

$$\sum_{j=0}^{k-1}\beta_j^2 \prod_{\ell=j+1}^{k-1}(1 - a\beta_\ell) \leq \varsigma\beta_k \sum_{j=0}^{k-1}\beta_j\frac{\beta_j}{\beta_{k-1}} \prod_{\ell=j+1}^{k-1}(1 - a\beta_\ell)$$

$$\leq \varsigma\beta_k \sum_{j=0}^{k-1}\beta_j \prod_{\ell=j+1}^{k-1}\frac{\beta_{\ell-1}}{\beta_\ell}(1 - a\beta_\ell) \overset{(a)}{\leq} \varsigma\beta_k \sum_{j=0}^{k-1}\beta_j \prod_{\ell=j+1}^{k-1}(1 - a\beta_\ell/2) \overset{(b)}{\leq} \frac{2}{a}\varsigma\beta_k$$

where (a) is due to the fact that we have $\frac{\beta_{\ell-1}}{\beta_\ell}(1 - a\beta_\ell) \leq 1 - a\beta_\ell/2$, and (b) is obtained by applying Lemma 12.

For the proof of part iii) we proceed similarly. We prove the second inequality only. The proof of the remaining follows the same lines. Denote $\kappa_\ell := \beta_\ell/\gamma_\ell$. Clearly, $\kappa_{\ell-1}/\kappa_\ell \leq \beta_{\ell-1}/\beta_\ell \leq \varsigma$. Then

$$\sum_{j=0}^{k-1}\beta_j^2\kappa_j^2 \prod_{\ell=j+1}^{k-1}(1 - a\beta_\ell) \leq \varsigma^3\beta_k\kappa_k^2 \sum_{j=0}^{k-1}\beta_j\frac{\beta_j}{\beta_{k-1}}\left(\frac{\kappa_j}{\kappa_{k-1}}\right)^2 \prod_{\ell=j+1}^{k-1}(1 - a\beta_\ell)$$

$$\leq \varsigma^3\beta_k\kappa_k^2 \sum_{j=0}^{k-1}\beta_j \prod_{\ell=j+1}^{k-1}\left(\frac{\beta_{\ell-1}}{\beta_\ell}\right)^3(1 - a\beta_\ell) \overset{(a)}{\leq} \varsigma^3\beta_k\kappa_k^2 \sum_{j=0}^{k-1}\beta_j \prod_{\ell=j+1}^{k-1}(1 - a\beta_\ell/2) \overset{(b)}{\leq} \frac{4}{a}\varsigma^3\beta_k\kappa_k^2$$

where (a) is due to the fact that we have $(\frac{\beta_{\ell-1}}{\beta_\ell})^3(1 - a\beta_\ell) \leq 1 - a\beta_\ell/4$, and (b) is obtained by applying Lemma 12. Part iv) may be proved in the similar way. ∎

Lemma 15 *Let $a > 0$, $p \geq 0$, $(\gamma_j)_{j \geq 0}$, $(\kappa_j)_{j \geq 0}$ and $(u_j)_{j \geq 0}$ be nonnegative sequences. Then, for all integers k,*

$$\sum_{j=0}^{k-1}\kappa_j \prod_{\ell=j}^{k-1}(1 - a\gamma_\ell)\sum_{i=0}^{j-1}\gamma_i^p \prod_{n=i+1}^{j-1}(1 - a\gamma_n)u_i = \sum_{i=0}^{k-1}\gamma_i^p u_i\left(\sum_{j=i+1}^{k-1}\kappa_j\right)\prod_{\ell=i+1}^{k-1}(1 - a\gamma_\ell)$$

Lemma 16 (Lyapunov Lemma) *A matrix A is Hurwitz if and only if for any positive symmetric matrix $P = P^\top \succ 0$ there is $Q = Q^\top \succ 0$ that satisfies the Lyapunov equation*

$$A^\top Q + QA = -P\,.$$

In addition, Q is unique.

Proof See (Poznyak, 2008, Lemma 9.1, p. 140). ∎

Lemma 17 *Assume that $-A$ is a Hurwitz matrix. Let Q be the unique solution of the Lyapunov equation*

$$A^\top Q + QA = \mathrm{I} .$$

Then, for any $\zeta \in [0, \zeta_A]$, where

$$\zeta_A := (1/2)\|A\|_Q^{-2}\|Q\|^{-1},$$

we get

$$\|\mathrm{I} - \zeta A\|_Q^2 \le (1 - a\zeta) \quad with \quad a = (1/2)\|Q\|^{-1} .$$

If in addition $\zeta \le \|Q\|$ then

$$1 - a\zeta \ge 1/2 .$$

Proof For any $x \in \mathbb{R}^d$, we get

$$\frac{x^\top(\mathrm{I} - \gamma A)^\top Q(\mathrm{I} - \gamma A)x}{x^\top Qx} = 1 - \gamma\frac{\|x\|^2}{x^\top Qx} + \gamma^2\frac{x^\top A^\top QAx}{x^\top Qx}$$

Hence, we get that for all $\gamma \in [0, (1/2)\|A\|_Q^{-2}\|Q\|^{-1}]$,

$$1 - \gamma\frac{\|x\|^2}{x^\top Qx} + \gamma^2\frac{x^\top A^\top QAx}{x^\top Qx} \le 1 - \gamma\|Q\|^{-1} + \gamma^2\|A\|_Q^2$$
$$\le 1 - (1/2)\|Q\|^{-1}\gamma .$$

The proof follows. ∎

Lemma 18 *Assume that $\|L\|_{Q_\Delta, Q_{22}} \le \varepsilon$ for some $\varepsilon > 0$ and*

$$0 \le \beta \le (1/2)\{\|\Delta\|_{Q_\Delta} + \varepsilon\|A_{12}\|_{Q_{22}, Q_\Delta}\}^{-1} \tag{110}$$
$$0 \le \gamma \le (1/2)\|Q_{22}\|^{-1}\|A\|_{Q_{22}}^{-2} . \tag{111}$$

Set $B_{11}(L) = \Delta - A_{12}L$. Then, the equation

$$L'\{\mathrm{I} - \beta B_{11}(L)\} = (\mathrm{I} - \gamma A_{22})L + \beta A_{22}^{-1}A_{21}B_{11}(L) \tag{112}$$

has a unique solution satisfying

$$\|L'\|_{Q_\Delta, Q_{22}} \le (1 - \gamma a_{22})\|L\|_{Q_\Delta, Q_{22}} + \beta\,\mathrm{C}_D(\varepsilon)$$

where

$$\mathrm{C}_D(\varepsilon) = 2\{\|A_{22}^{-1}A_{21}\|_{Q_\Delta, Q_{22}} + \varepsilon\}\{\|\Delta\|_{Q_\Delta} + \varepsilon\|A_{12}\|_{Q_{22}, Q_\Delta}\} .$$

If $\beta/\gamma \le \varepsilon a_{22}/\mathrm{C}_D(\varepsilon)$, then $\|L'\|_{Q_\Delta, Q_{22}} \le \varepsilon$.

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Proof Since $\|L\|_{Q_\Delta,Q_{22}} \leq \varepsilon$, we get that $\|B_{11}(L)\|_{Q_\Delta} \leq \|\Delta\|_{Q_\Delta} + \varepsilon\|A_{12}\|_{Q_{22},Q_\Delta}$. Hence, using (110) and the triangular inequality, we get that $\beta\|B_{11}(L)\|_{Q_\Delta} \leq 1/2$ and thus

$$\|I - \beta B_{11}(L)\|_{Q_\Delta} \geq 1/2. \tag{113}$$

Hence, $I - \beta B_{11}(L)$ is invertible and (112) has a unique solution given by

$$L' = \left\{(I - \gamma A_{22})L + \beta A_{22}^{-1} A_{21} B_{11}(L)\right\}\{I - \beta B_{11}(L)\}^{-1}$$
$$= (I - \gamma A_{22})L + \beta D(L)$$

where

$$D(L) = \{A_{22}^{-1} A_{21} + (I - \gamma A_{22})L\}B_{11}(L)\{I - \beta B_{11}(L)\}^{-1}.$$

Using (113) and $\|L\|_{Q_\Delta,Q_{22}} \leq \varepsilon$, we get that $\|D(L)\|_{Q_\Delta,Q_{22}} \leq C_D(\varepsilon)$. Hence, for γ satisfying (111), we get that

$$\|L'\|_{Q_\Delta,Q_{22}} \leq (1 - \gamma a_{22})\|L\|_{Q_\Delta,Q_{22}} + \beta\, C_D(\varepsilon) \leq \varepsilon + \gamma\Big(\frac{\beta}{\gamma} C_D(\varepsilon) - a_{22}\varepsilon\Big) \leq \varepsilon,$$

where the last inequality is due to $\frac{\beta}{\gamma} \leq \varepsilon a_{22}/C_D(\varepsilon)$. ∎

Lemma 19 *Let $L_0 = 0$. Assume that $\|L_k\|_{Q_\Delta,Q_{22}} \leq L_\infty$ and*

$$0 \leq \beta_0 \leq (1/2)\{\|\Delta\|_{Q_\Delta} + L_\infty\|A_{12}\|_{Q_{22},Q_\Delta}\}^{-1}$$
$$0 \leq \gamma_0 \leq (1/2)\|Q_{22}\|^{-1}\|A\|_{Q_{22}}^{-2}$$

Then for any $k \in \mathbb{N}$

$$\|L_k\|_{Q_\Delta,Q_{22}} \leq C_D(L_\infty)\varrho^{a_{22}}\beta_k/\gamma_k,$$

where

$$C_D(L_\infty) := 2\{\|A_{22}^{-1}A_{21}\|_{Q_\Delta,Q_{22}} + L_\infty\}\{\|\Delta\|_{Q_\Delta} + L_\infty\|A_{12}\|_{Q_{22},Q_\Delta}\}$$

Proof Similarly to Lemma 18 we may show that

$$L_{k+1} = (I - \gamma A_{22})L_k + \beta_k D(L_k)$$

where $\|D(L_k)\|_{Q_\Delta,Q_{22}} \leq C_D(L_\infty)$. Hence,

$$\|L_k\|_{Q_\Delta,Q_{22}} \leq C_D(L_\infty)\sum_{j=0}^{k}\beta_j \prod_{s=j+1}^{k}(1 - a_{22}\gamma_s)$$

Application of Lemma 14 to the right hand side of the above completes the proof. ∎

Lemma 20 *Let $L_1 := L_0 := 0$. Assume that $\|L_k\|_{Q_\Delta,Q_{22}} \le L_\infty$ and*

$$0 \le \beta_0 \le (1/2)\{\|\Delta\|_{Q_\Delta} + L_\infty\|A_{12}\|_{Q_{22},Q_\Delta}\}^{-1}$$
$$0 \le \gamma_0 \le (1/2)\|Q_{22}\|^{-1}\|A\|_{Q_{22}}^{-2}$$
$$\beta_{k-1} - \beta_k \le \rho_\beta \beta_k^2, \gamma_{k-1} - \gamma_k \le \rho_\gamma \gamma_k^2$$
$$\beta_k/\gamma_k \le (1/(2C_1^U))a_{22}$$

with

$$C_1^U := 2(\|\Delta\|_{Q_\Delta} + \|A_{22}^{-1}A_{21}\|_{Q_\Delta,Q_{22}}\|A_{12}\|_{Q_{22},Q_\Delta} + 2L_\infty\|A_{12}\|_{Q_{22},Q_\Delta}).$$

Then

$$\|L_{k+1} - L_k\|_{Q_\Delta,Q_{22}} \le C_2^U \varrho^{a_{22}/2}\gamma_{k+1},$$

where

$$C_2^U := 2\rho_\gamma L_\infty\|A_{22}\|_{Q_{22}} + 2\rho_\beta(L_\infty + \|A_{22}^{-1}A_{21}\|_{Q_\Delta,Q_{22}})(\|\Delta\|_{Q_\Delta} + L_\infty\|A_{12}\|_{Q_{22},Q_\Delta})$$

Proof Recall that $B_{11}(L) = \Delta - A_{12}L$. It follows from Lemma 18 that $\mathrm{I} - \beta_k B_{11}(L_k)$ is invertible matrix with bounded norm. Equation

$$L_k(\mathrm{I} - \beta_{k-1}B_{11}(L_{k-1})) = \left\{ (\mathrm{I} - \gamma_{k-1}A_{22})L_{k-1} + \beta_{k-1}A_{22}^{-1}A_{21}B_{11}(L_{k-1}) \right\}$$

may be rewritten as follows

$$L_k(\mathrm{I} - \beta_k B_{11}(L_k)) = (\mathrm{I} - \gamma_k A_{22})L_{k-1} + \beta_k B_{11}(L_k) + E_k,$$

where $E_k := (\gamma_k - \gamma_{k-1})A_{22}L_{k-1} + (L_k + A_{22}^{-1}A_{21})D_k$, $D_k := -\beta_k A_{12}(L_k - L_{k-1}) + (\beta_k - \beta_{k-1})B_{11}(L_{k-1})$. Let $U_k = L_k - L_{k-1}$. Then

$$U_{k+1}(\mathrm{I} - \beta_k B_{11}(L_k)) = (\mathrm{I} - \gamma_k A_{22})U_k - E_k.$$

Then

$$U_{k+1} = (\mathrm{I} - \gamma_k A_{22})U_k + \beta_k(\mathrm{I} - \gamma_k A_{22})U_k B_{11}(L_k)(\mathrm{I} - \beta_k B_{11}(L_k))^{-1} - E_k(\mathrm{I} - \beta_k B_{11}(L_k))^{-1}$$

It is easy to check that

$$\|(\mathrm{I} - \gamma_k A_{22})U_k B_{11}(L_k)(\mathrm{I} - \beta_k B_{11}(L_k))^{-1}\|_{Q_\Delta,Q_{22}} \le 2\|U_k\|_{Q_\Delta,Q_{22}}\{\|\Delta\|_{Q_\Delta} + L_\infty\|A_{12}\|_{Q_{22},Q_\Delta}\}$$

Moreover,

$$\|E_k(\mathrm{I} - \beta_k B_{11}(L_k))^{-1}\|_{Q_\Delta,Q_{22}} \le 2\rho_\gamma \gamma_k^2 L_\infty\|A_{22}\|_{Q_{22}}$$
$$+ 2(L_\infty + \|A_{22}^{-1}A_{21}\|_{Q_\Delta,Q_{22}})\{\rho_\beta \beta_k^2(\|\Delta\|_{Q_\Delta} + L_\infty\|A_{12}\|_{Q_{22},Q_\Delta}) + \beta_k\|A_{12}\|_{Q_{22},Q_\Delta}\|U_k\|_{Q_\Delta,Q_{22}}\}$$

Applying previous inequalities we obtain

$$\|U_{k+1}\|_{Q_\Delta,Q_{22}} \le (1 - \gamma_k a_{22} + C_1^U \beta_k)\|U_k\|_{Q_\Delta,Q_{22}} + C_2^U \gamma_k^2$$

Since $\beta_k/\gamma_k \le (1/(2C_1^U))a_{22}$ we obtain

$$\|U_{k+1}\|_{Q_\Delta,Q_{22}} \le C_2^U \varrho^{a_{22}/2}\gamma_{k+1}$$

∎

Lemma 21 *Let Q be a symmetric definite positive $n \times n$ matrix and Σ be a $n \times n$ matrix. Then*

$$\mathrm{Tr}(Q\Sigma) \leq \|\Sigma\|_Q \, \mathrm{Tr}(Q) \, .$$

Proof Denote by $(e_i)_{i=1}^n$ an orthonomal basis of eigenvectors of Q, $Qe_i = \lambda_i(Q)e_i$, $i = 1, \dots, n$, $\langle e_i, e_j \rangle = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol. We get that

$$\mathrm{Tr}(Q\Sigma) = \sum_{i=1}^n \langle e_i, Q\Sigma e_i \rangle = \sum_{i=1}^n \langle e_i, \Sigma e_i \rangle_Q$$

$$\leq \|\Sigma\|_Q \sum_{i=1}^n \|e_i\|_Q^2 = \|\Sigma\|_Q \, \mathrm{Tr}\, Q$$

where we have used $\|e_i\|_Q = \lambda_i$ and $\mathrm{Tr}\, Q = \sum_{i=1}^n \lambda_i(Q)$. ∎

Corollary 22 *If X is a $n \times 1$ random vector such that $\mathbb{E}[\|X\|_2] < \infty$. Then,*

$$\mathbb{E}[\|X\|_Q^2] \leq \mathrm{Tr}(Q)\|\mathbb{E}[XX^\top]\|_Q \, .$$

Proof Note that $\mathbb{E}[\|X\|_Q^2] = \mathrm{Tr}(Q\,\mathbb{E}[XX^\top]) \leq \|\mathbb{E}[XX^\top]\|_Q \, \mathrm{Tr}\, Q$ ∎

Lemma 23 *Let m and n be two integers, P and Q be $m \times m$ and $n \times n$ symmetric positive definite matrices. Let X and Y be $m \times 1$ and $n \times 1$ random vectors such that $\mathbb{E}[\|X\|^2] < \infty$ and $\mathbb{E}[\|Y\|^2] < \infty$. Then,*

$$\|\mathbb{E}[XY^\top]\|_{Q,P} \leq \lambda_{\min}(Q)^{-1}\{\mathrm{Tr}(Q)\}^{1/2}\{\mathrm{Tr}(P)\}^{1/2}\|\mathbb{E}[XX^\top]\|_P^{1/2}\|\mathbb{E}[YY^\top]\|_Q^{1/2}$$

Proof Note that $\|\mathbb{E}[XY^\top]\|_{Q,P} \leq \mathbb{E}[\|XY^\top\|_{Q,P}]$ and

$$\|XY^\top\|_{Q,P} = \sup_{\|y\|_Q=1} \|X\langle Y, y\rangle_Q\| = \|X\|_P \sup_{\|y\|_Q=1} \langle Q^{-1}Y, y\rangle_Q$$

$$= \|X\|_P\|Q^{-1}Y\|_Q = \|X\|_P\|Y\|_{Q^{-1}} \leq \lambda_{\min}^{-1}(Q)\|X\|_P\|Y\|_Q \, .$$

By applying the Cauchy-Schwarz inequality, we obtain

$$\|\mathbb{E}[XY^\top]\|_{Q,P} \leq \lambda_{\min}^{-1}(Q) \left\{\mathbb{E}[\|X\|_P^2]\right\}^{1/2} \left\{\mathbb{E}[\|Y\|_Q^2]\right\}^{1/2} \, .$$

The proof follows from Corollary 22. ∎

Appendix E. Details on Numerical Experiments

This section provides details about the numerical experiments and verification that the convergence conditions are satisfied.

E.1. Toy Example

In this toy example, we consider randomly generated instances of linear two timescale SA in the form (1), (2) with i.i.d. samples (and thus the martingale noise setting). In particular, we let the iterates $\theta_k, w_k \in \mathbb{R}^d$ be d-dimensional and construct a problem instance as follows:

1. Sample a random matrix T whose entries are drawn i.i.d. from the uniform distribution $U[-1, 1]$; Compute the QR-decomposition as $T = QR$.

2. Set $A_{12} = Q$ and $A_{22} = Q^\top \Lambda_0 Q$, where Λ_0 is a diagonal matrix with i.i.d. entries from $U[-1, 1]$.

3. Sample a random matrix R whose entries are drawn i.i.d. from the uniform distribution $U[-1, 1]$.

4. Set $A_{11} = RR^\top + I$ and $A_{21} = Q^\top \Lambda_1$, where Λ_1 is a diagonal matrix with i.i.d. entries from $U[-1, 1]$.

5. Sample a stationary solution pair θ^*, w^* with i.i.d. entries from $U[-1, 1]$.

6. Compute b_1, b_2 using the generated matrices and stationary points, i.e.,

$$b_1 = A_{11}\theta^\star + A_{12}w^\star, \quad b_2 = A_{21}\theta^\star + A_{22}w^\star.$$

During the linear two timescale SA iteration, the noise terms are generated as

$$V_{k+1} = F_V^k + A_{V,\theta}^k \theta_k + A_{V,w}^k w_k, \quad W_{k+1} = F_W^k + A_{W,\theta}^k \theta_k + A_{W,w}^k w_k$$

where $F_V^k, A_{V,\theta}^k, A_{V,w}^k$ are vectors/matrices with entries drawn i.i.d. from the standard normal distribution $\mathcal{N}(0, 0.1)$, and $F_W^k, A_{W,\theta}^k, A_{W,w}^k$ are vectors/matrices with entries drawn i.i.d. from the standard normal distribution $\mathcal{N}(0, 0.5)$. With the above constructions, it can be verified that the required assumptions A1, A3, A4 of the martingale noise setting hold. It remains to verify that the step sizes chosen satisfy A2.

Below, we show the plots of deviations in θ_k and w_k without normalization by the step sizes (see Fig. 2).
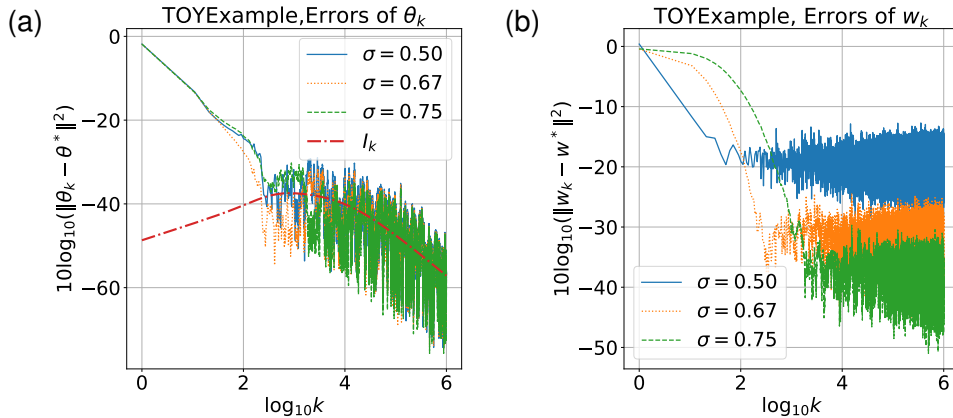

Figure 2: Unnormalized deviations from stationary point $(\theta^\star, w^\star)$ and term I_k : the toy example.

E.2. Garnet Problems

GTD Algorithm and Policy Evaluation Problem The specific form of linear two timescale SA used in this example follows from that of the classical GTD algorithm (Sutton et al., 2009a,b), which is described below for completeness. Let $\mathcal{S}, \mathcal{A}$ be some discrete state and action spaces (for clarity we bound ourselves by discrete setting, but one could formulate it in more general way), $\rho \in (0,1)$ and $\pi : \mathcal{S} \rightarrow \mathcal{P}(\mathcal{A})$ be a *stochastic policy*, i.e. mapping from states to probability measures over actions. When in state s the agent performs action a (distributed according to its policy π), it transitions randomly to state s' with probability $p(s'|s,a)$ and obtains reward $r(s,a)$. This induces a Markov chain with transition probabilities $p_\pi(s'|s) := \sum_{a \in \mathcal{A}} \pi(a|s)p(s'|s,a)$.

The goal of policy evaluation is to estimate the average discounted cumulative reward obtained with the policy π. In detail, we evaluate the value function $V_\pi(s) := \mathbb{E}\left[r(s,a) + \sum_{k=1} \rho^k r(s_k, a_k)\right]$ with ρ being the *discounting factor*. As the state space $|\mathcal{S}|$ is often large, we use the linear approximation $V_\pi(s) \approx V_\theta(s) := \langle \theta, \phi(s) \rangle$, where $\phi : \mathcal{S} \rightarrow \mathbb{R}^d$ is a pre-defined feature map. Define also *temporal difference* at iteration $k \in \mathbb{Z}_+$ for transition $s_k \rightarrow s_{k+1}$ as $\delta_k := r(s_k, a_k) + \rho V_{\theta_k}(s_{k+1}) - V_{\theta_k}(s_k)$. For brevity, denote the observation at iteration $k \in \mathbb{Z}_+$, namely, $\phi(s_k), \phi(s_{k+1}), r(s_k, a_k)$ as ϕ_k, ϕ_{k+1}, r_k respectively. The GTD algorithm iterations are described as:

$$\theta_{k+1} = \theta_k + \beta_k \left[\phi_k - \rho \phi_{k+1}\right] \langle \phi_k, w_k \rangle, \quad w_{k+1} = w_k + \gamma_k \left[\phi_k \delta_k - w_k\right].$$

The above is a special case of our linear two timescale SA in (3), (4) with the notations:

$$b_1 = 0, \ A_{11} = 0, \ A_{12} = -\mathbb{E}[(\phi_k - \rho\phi_{k+1})\phi_k^\top],$$
$$b_2 = \mathbb{E}[\phi_k r_k], \ A_{21} = -\mathbb{E}[\phi_k(\rho\phi_{k+1} - \phi_k)^\top], \ A_{22} = \mathrm{I}_d,$$
$$V_{k+1} = \left((\phi_k - \rho\phi_{k+1})\phi_k^\top - \mathbb{E}[(\phi_k - \rho\phi'_k)\phi_k^\top]\right) w_k,$$
$$W_{k+1} = \phi_k r_k - \mathbb{E}[\phi_k r_k] + \left((\phi_k - \rho\phi_{k+1})\phi_k^\top - \mathbb{E}[(\phi_k - \rho\phi_{k+1})\phi_k^\top]\right)\theta_k,$$

where the expectations above are taken with respect to the stationary distribution of the MDP under policy π. Particularly, the noise terms V_{k+1}, W_{k+1} follow the Markovian noise setting.

Garnet Problem The Garnet problem refers to a set of policy evaluation problems with randomly generated problem instances, originally proposed in (Archibald et al., 1995). Here, we consider a simpler version of Garnet problems described in (Geist and Scherrer, 2014). Particularly, we consider a finite-state MDP with the parameters n_S as the number of states, n_A as the number of possible actions in each state, b as the branching factor, i.e., the number of transitions from each state-action pair to a new state, p as the number of features in the linear function approximation applied. For any pair $(s, a) \in \mathcal{S} \times \mathcal{A}$ we choose b states $\mathcal{S}' \subset \mathcal{S}$ out of $|\mathcal{S}|$ at random and then draw the probabilities of going from (s,a) to $s' \in \mathcal{S}'$. For the features, for each state $s \in \mathcal{S}$ the corresponding feature vector $\phi(s)$ is generated from $(U[0,1])^p$. In our numerical example, we consider a particular problem from the family $n_S = 30, n_A = 8, b = 2, p = 8$.

By the above constructions, we observe that the assumptions A1, B1–B3 are all satisfied. It remains to verify that the step sizes chosen satisfy A2, B4.

E.3. Step Size Parameters

We consider the family of step size schedules:

$$\beta_k = c^\beta/(k_0^\beta + k), \ \ \gamma_k = c^\gamma/(k_0^\gamma + k)^\sigma,$$

with $\sigma \in [0.5, 1]$ and the parameters $c^\beta, c^\gamma, k_0^\beta, k_0^\gamma$. Note that

$$\frac{\beta_k}{\gamma_k} = \frac{c^\beta(k_0^\gamma + k)^\sigma}{c^\gamma(k_0^\beta + k)} \leq \frac{c^\beta}{c^\gamma} \left(\frac{k_0^\gamma}{k_0^\beta}\right)^\sigma =: \kappa$$

since we have $\sigma \leq 1$. This ensures A2-1. Furthermore, we observe that

$$\frac{\gamma_{k-1}}{\gamma_k} = \left(1 + \frac{1}{k_0^\gamma + k - 1}\right)^\sigma \leq 1 + \frac{\sigma}{k_0^\gamma + k - 1} \leq 1 + \frac{\sigma k_0^\gamma}{c^\gamma(k_0^\gamma - 1)}\frac{c^\gamma}{(k_0^\gamma + k)^\sigma} = 1 + \frac{\sigma k_0^\gamma}{c^\gamma(k_0^\gamma - 1)}\gamma_k,$$

On the other hand, we also have

$$\frac{\gamma_{k-1}}{\gamma_k} \leq 1 + \frac{\sigma k_0^\beta}{c^\beta(k_0^\gamma - 1)}\frac{c^\beta}{k_0^\beta + k} = 1 + \frac{\sigma k_0^\beta}{c^\beta(k_0^\gamma - 1)}\beta_k$$

Similar upper bound can be derived for β_{k-1}/β_k. Setting c^γ, c^β large enough ensures A2-2. Lastly, B4 can be guaranteed by observing that $\sigma \geq 0.5$.

The above discussions illustrate that the satisfaction of A2 hinge on setting a large c^γ, c^β. However, this requirement can be hard to satisfy since we also have requirements such as $\gamma_k \leq \gamma_\infty^{\mathsf{mark}}$, $\beta_k \leq \beta_\infty^{\mathsf{mark}}$. To this end, we have to set a large k_0^β, k_0^γ. As a result, there are four inter-related hyper parameters to be tuned in order to ensure the desired convergence of linear two timescale SA. We remark that tuning the step size parameters for SA scheme is generally difficult.