18.100B PROBLEM SET 6

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Problem 1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $\forall x \in \mathbb{R}$,

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0.$$

Does this imply that f is continuous?

Proof. Start by defining $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) := \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}.$$

Then $\forall x \in \mathbb{R}$,

$$\lim_{h \to 0} [f(x+h) - f(x-h)] := \lim_{h \to 0} (1-1) = \lim_{h \to 0} 0 = 0;$$

however, f is discontinuous (namely, at $x = 0$). $\square$

Problem 2. Let X and Y be metric spaces and $f : X \to Y$ a function. Prove that f is continuous if and only if $f(\overline{E}) \subset \overline{f(E)}$ for any subset $E \subset X$.

Proof. Let $f : X \to Y$ be any function. Suppose that f is continuous on X. Assume that $E \subset X$. Then $\overline{f(E)} \subset Y$ is closed; hence, $f^{-1}(\overline{f(E)}) \subset X$ is closed and

$$E \subset f^{-1}(\overline{f(E)}) \implies \overline{E} \subset f^{-1}(\overline{f(E)}) \implies f(\overline{E}) \subset \overline{f(E)}$$

as desired. Suppose that $\forall E \subset X$, $f(\overline{E}) \subset \overline{f(E)}$. Assume that $F \subset Y$ is closed. Then $f^{-1}(F) \subset X$; hence,

$$f(\overline{f^{-1}(F)}) \subset \overline{f(f^{-1}(F))} \subset \overline{F} = F \implies f^{-1}(F) = \overline{f^{-1}(F)},$$

i.e. f is continuous on X. $\square$

Problem 3. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $x \in X$ at which $f(x) = 0$. Prove that $Z(f)$ is closed.

Proof. Let $f : (X, d) \to \mathbb{R}$ be any function. Assume that f is continuous on X. Suppose that $x \in X$ is any limit point of $Z(f)$. Then $\forall \epsilon > 0$, $\exists \delta > 0$ where

$$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon;$$

hence, $\exists z \in Z(f)$ where

$$d(x, z) < \delta \implies |f(x)| = |f(x) - f(z)| < \epsilon \implies f(x) = 0 \implies x \in Z(f).$$

$\square$

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Problem 4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $f(x) = g(x)$ for all $x \in E$, prove that $f(x) = g(x)$ for all $x \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

Proof. Let $f : X \to Y$ and $g : X \to Y$ be any mappings. Suppose that f and g are continuous on X. Choose $E \subset X$ to be any dense subset. Then

$$f(E) \subset f(X) \implies \overline{f(E)} \subset \overline{f(X)} \subset f(\overline{X}) = f(X)$$

(else, $\exists (x_n) \in X$ where $x_n \to x$ and $f(x_n) \not\to f(x)$); however,

$$f(X) = f(\overline{E}) \subset \overline{f(E)}$$

(Problem 2) and $\overline{f(E)} = f(X)$. Assume that $\forall x \in E,\ f(x) = g(x)$. Then

$$f(E) = g(E) \implies f(X) = \overline{f(E)} = \overline{g(E)} = g(X).$$

Indeed

$$\forall x \in E,\ f(x) = g(x) \implies \forall x \in X,\ f(x) = g(x).$$

□

Problem 5. Suppose that $f : X \to Y$ is a uniformly continuous mapping between metric spaces.

(a) Prove that if (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is a Cauchy sequence in Y.
(b) Use the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ to show that it is possible for a continuous function to send Cauchy sequences to Cauchy sequences without being uniformly continuous.

Proof. Let $f : X \to Y$ be any mapping.

(a) Suppose that f is uniformly continuous on X. Make any Cauchy sequence (x_n) in X. Then $\forall \epsilon > 0$, $\exists \delta > 0$ where

$$d(x_m, x_n) < \delta \implies d(f(x_m), f(x_n)) < \epsilon;$$

however, $\exists M \in \mathbb{N}$ where

$$m, n \geq M \implies d(x_m, x_n) < \delta$$

and

$$m, n \geq M \implies d(f(x_m), f(x_n)) < \epsilon,$$

i.e. $(f(x_n))$ is a Cauchy sequence in Y.

(b) Suppose that $X = Y = \mathbb{R}$ and $f(x) = x^2$. Then $\forall \epsilon > 0$, $\exists \delta > 0$ (namely, $\delta := \epsilon/2\alpha$ where α is an upper bound of (x_n)) where

$$\begin{aligned} |x_m - x_n| < \delta \implies |f(x_m) - f(x_n)| &= |x_m^2 - x_n^2| \\ &= |x_m + x_n||x_m - x_n| \\ &\leq (|x_m| + |x_n|)(|x_m - x_n|) \\ &\leq 2\alpha\delta = \epsilon, \end{aligned}$$

(f is continuous on X); however, $\exists M \in \mathbb{N}$ where

$$m, n \geq M \implies |x_m - x_n| < \delta$$

and

$$m, n \geq M \implies |f(x_m) - f(x_n)| < \epsilon$$

($(f(x_n))$ is a Cauchy sequence in Y).

□

Problem 6. In class, we showed that a continuous function $f : [a,b] \to \mathbb{R}$ is uniformly continuous. Prove this by either:

(a) Assume it is false, so for some $\epsilon > 0$ no choice of $\delta > 0$ works everywhere. Find, for each $n \in \mathbb{N}$ a point x_n where $\delta = \frac{1}{n}$ does not work. Extract a convergent subsequence, (x_{n_k}) and derive a contradiction from the convergence of $(f(x_{n_k}))$.

(b) Fix $\epsilon > 0$, and for each $x \in [a,b]$ let $\delta(x)$ be the length of the largest open interval centered at x such that $|f(y) - f(z)| < \epsilon$ (really $\delta(x)$ is defined as a supremum of course). Show that $\delta(x) > 0$ and $\delta(x)$ is continuous. Because $[a,b]$ is compact, $\delta(x)$ must achieve a minimum, say δ_0. Show that δ_0 works in the definition of uniform continuity.

Proof. Let $f : [a,b] \to \mathbb{R}$ be any continuous function.

(a) Assume, for the purpose of contradiction, that f is not uniformly continuous on $[a,b]$. Then $\exists \epsilon > 0$, $\forall \delta > 0$ where

$$|x - y| < \delta \implies |f(x) - f(y)| \geq \epsilon.$$

Make a sequence (x_n) of $[a,b]$ in the following construction: Pick $x_1 \in [a,b]$. Given that $x_1, ..., x_n \in [a,b]$ are fixed, define $x_{n+1} \in [a,b]$ such that

$$|x_{n+1} - x_n| < \frac{1}{n+1} \implies |f(x_{n+1}) - f(x_n)| \geq \epsilon.$$

Extract any convergent subsequence (x_{n_k}) of $[a,b]$ (Bolzano-Weierstrass Theorem) with limit x. Since f is continuous on $[a,b]$, $\exists \delta > 0$ where

$$|x_{n_k} - x| < \delta \implies |f(x_{n_k}) - f(x)| < \epsilon;$$

however, $\exists M \in \mathbb{N}$ where

$$k \geq M \implies |x_{n_k} - x| < \delta$$

and

$$k \geq M \implies |f(x_{n_k}) - f(x)| < \epsilon,$$

i.e. $(f(x_{n_k}))$ is a convergent subsequence of $[a,b]$ with limit $f(x)$. Then $(f(x_{n_k}))$ is a Cauchy subsequence of $[a,b]$; hence, $\exists M \in \mathbb{N}$ where

$$i, j \geq M \implies |f(x_{n_i}) - f(x_{n_j})| < \epsilon$$

(a contradiction).

(b) Fix $\epsilon > 0$. Make $\delta : [a,b] \to \mathbb{R}$ such that

$$\delta(x) := \sup\{r \in \mathbb{R} : \forall y, z \in B_r(x),\ |f(y) - f(z)| < \epsilon\}.$$

Given that f is continuous on $[a,b]$, $\exists r > 0$ where

$$y, z \in B_r(x) \implies |f(y) - f(x)| < \frac{\epsilon}{2} \wedge |f(z) - f(x)| < \frac{\epsilon}{2}$$

and

$$\begin{aligned} y, z \in B_r(x) \implies |f(y) - f(z)| &= |(f(y) - f(x)) + (f(x) - f(z))| \\ &\leq |f(y) - f(x)| + |f(z) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon; \end{aligned}$$

hence, $\delta(x) > 0$. Similarly,

$$|x - y| < \delta(x) \implies |f(x) - f(y)| < \epsilon;$$

however,

$$|x - y| < \delta(y) \implies |f(x) - f(y)| < \epsilon$$

and

$$\delta(x) = \delta(y) \implies |\delta(x) - \delta(y)| < \epsilon,$$

so δ is continuous on $[a, b]$. By the Extreme Value Theorem, δ achieves a minimum $\delta_0 \in \mathbb{R}$ on $[a, b]$; hence,

$$|x - y| < \delta_0 \implies |f(x) - f(y)| < \epsilon.$$

$\square$