18.100B PROBLEM SET 7

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Problem 1. Suppose f is a real function defined on $\mathbb{R}$. We call $x \in \mathbb{R}$ a fixed point of f if $f(x) = x$.

(a) If f is differentiable and $f'(x) \neq 1$ for every real x, prove that f has at most one fixed point.

(b) Show that the function f defined by

$$f(x) = x + \frac{1}{1+e^x}$$

has no fixed point, although $0 < f'(x) < 1$ for all real x.

(c) However, if there is a constant $0 < \lambda < 1$ such that $|f'(x)| \leq \lambda$ for all real x, prove that a fixed point x_0 of f exists, and that $x_n \to x_0$, where x_1 is an arbitrary real number and

$$\forall n \in \mathbb{N},\ x_{n+1} := f(x_n).$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be any function.

(a) Suppose that f is differentiable on $\mathbb{R}$ and

$$\forall x \in \mathbb{R},\ f'(x) :\neq 1.$$

Assume, for the purpose of contradiction, that f has more than one fixed point. Then f is continuous on $\mathbb{R}$, and $\exists x_0, y_0 \in \mathbb{R}$ where

$$x_0 \neq y_0 \wedge f(x_0) = x_0 \wedge f(y_0) = y_0.$$

By the Mean Value Theorem, $\exists t \in \mathbb{R}$ such that

$$f'(t)(x_0 - y_0) = f(x_0) - f(y_0) = (x_0 - y_0) \implies f'(t) = 1$$

(a contradiction); hence, f can only have at most one fixed point.

(b) Suppose that

$$\forall x \in \mathbb{R},\ f(x) := x + \frac{1}{1+e^x}.$$

Then

$$\forall x \in \mathbb{R},\ f(x) > x$$

(f has no fixed point); however, f is differentiable on $\mathbb{R}$ and

$$\forall x \in \mathbb{R},\ f'(x) = 1 - \frac{e^x}{(1+e^x)^2} \implies 0 < f'(x) < 1$$

(Chain Rule).

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(c) Suppose that $\exists \lambda \in \mathbb{R}$ where $0 < \lambda < 1$ and

$$\forall x \in \mathbb{R},\ |f'(x)| \leq \lambda.$$

Pick $x_1 \in \mathbb{R}$. Given that $x_1, ..., x_n$ are fixed, define

$$x_{n+1} := f(x_n).$$

Inductively, assume that $|x_n - x_{n+1}| \leq \lambda^{n-1}|x_1 - x_2|$. By the Mean Value Theorem, $\exists t \in \mathbb{R}$ such that

$$\begin{aligned} |x_{n+1} - x_{n+2}| \leq |f(x_n) - f(x_{n+1})| &= |f'(t)(x_n - x_{n+1})| \\ &= |f'(t)||x_n - x_{n+1}| \\ &\leq \lambda^n |x_1 - x_2|. \end{aligned}$$

Choose $\epsilon > 0$. Then

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m+1}| + \cdots + |x_{n-1} - x_n| \\ &\leq (\lambda^m + \cdots + \lambda^{n-1})|x_1 - x_2| \\ &= \lambda^m(1 + \cdots + \lambda^{n-m-1})|x_1 - x_2| \\ &= \lambda^m \frac{1 - \lambda^{n-m}}{1 - \lambda}|x_1 - x_2| \\ &< \frac{\lambda^m}{1 - \lambda}|x_1 - x_2|; \end{aligned}$$

however, $\exists M \in \mathbb{N}$ where

$$\lambda^M |x_1 - x_2| < (1 - \lambda)\epsilon$$

and $\forall m, n \geq M$,

$$|x_m - x_n| < \frac{\lambda^m}{1 - \lambda}|x_1 - x_2| \leq \frac{\lambda^M}{1 - \lambda}|x_1 - x_2| < \epsilon,$$

i.e. (x_n) is convergent. Indeed, $\exists x_0 \in \mathbb{R}$ such that $x_n \to x_0$ and

$$\begin{aligned} |x_0 - f(x_0)| &\leq |x_0 - x_{N+1}| + |x_{N+1} - f(x_0)| \\ &< \frac{\epsilon}{1 + \lambda} + |f(x_N) - f(x_0)| \\ &\leq \frac{\epsilon}{1 + \lambda} + \lambda |x_N - x_0| \\ &< \frac{\epsilon}{1 + \lambda} + \frac{\lambda \epsilon}{1 + \lambda} = \epsilon \end{aligned}$$

for some $N \in \mathbb{N}$; hence, $f(x_0) = x_0$.

□

Problem 2. Let f be a continuous real function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 7$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Suppose that f is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$ where

$$\lim_{x \to 0} f'(x) = 7.$$

By L'Hospital's Rule,

$$f'(0) := \lim_{x\to 0}\frac{f(x)-f(0)}{x}\ \left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{(f(x)-f(0))'}{(x)'} = \lim_{x\to 0} f'(x) = 7.$$

□

Problem 3. Let f be a real function on $[a,b]$ and suppose $n \geq 2$ is an integer, $f^{(n-1)}$ is continuous on $[a,b]$, and $f^{(n)}(x)$ exists for all $x \in (a,b)$. Moreover, assume there exists $x_0 \in (a,b)$ such that

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0, f^{(n)}(x_0) \neq 0.$$

Prove the following criteria: If n is even, then f has a local minimum at x_0 when $f^{(n)}(x_0) > 0$, and f has a local maximum at x_0 when $f^{(n)}(x_0) < 0$. If n is odd, then f does not have a local minimum or maximum at x_0. *Hint:* Use Taylor's Theorem.

Proof. Let $f : [a,b] \to \mathbb{R}$ be any function. Suppose that $n \in \mathbb{N}$, $f^{(n)}$ is continuous on $[a,b]$, and $f^{(n+1)}$ exists on $[a,b]$. Assume $\exists x_0 \in (a,b)$ where

$$f'(x_0) = f''(x_0) = \cdots = f^{(n)}(x_0) = 0, f^{(n+1)}(x_0) \neq 0.$$

By Taylor's Theorem,

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \\ &= f(x_0) + 0 + \cdots + 0 + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \\ &= f(x_0) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \end{aligned}$$

for some $c \in (\inf\{x_0, x), \sup\{x_0, x\})$; however,

$$\lim_{x\to x_0} f^{(n+1)}(x) = \lim_{h\to 0}\frac{f^{(n)}(x_0+h) - f^{(n)}(x_0)}{h} = f^{(n+1)}(x_0) \neq 0.$$

If n is odd, then $\exists k \in \mathbb{N}$ such that $n = 2k-1$ and

$$f(x) = f(x_0) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} = f(x_0) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{2k};$$

hence,

$$f^{(n+1)}(x_0) > 0 \implies \lim_{x\to x_0} f^{(n+1)}(x) > 0 \implies f^{(n+1)}(c) > 0 \implies f(x) \geq f(x_0)$$

and

$$f^{(n+1)}(x_0) < 0 \implies \lim_{x\to x_0} f^{(n+1)}(x) < 0 \implies f^{(n+1)}(c) < 0 \implies f(x) \leq f(x_0)$$

for all x in some δ-neighborhood of x_0, so f has a local extrema at x_0. If n is even, then $\exists k \in \mathbb{N}$ such that $n = 2k$ and

$$f(x) = f(x_0) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} = f(x_0) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{2k+1} \neq f(x_0)$$

for all x in every δ-neighborhood of x_0; hence, f cannot have a local extrema at x_0. □

Problem 4. Let $I \subset \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is called Hölder continuous of order $\alpha > 0$ if there is constant $C > 0$ such that

$$|f(x) - f(y)| < C|x - y|^\alpha$$

holds for all $x, y \in I$.

(a) Show that any Hölder continuous function is uniformly continuous.
(b) Prove that $f(x) = \sqrt{|x|}$ is Hölder continuous of order $\alpha = 1/2$.
(c) Prove that Hölder continuity of order α implies Hölder continuity of order $0 < \beta \le \alpha$, provided that I is bounded. What happens if I is unbounded?
(d) Show that if f is Hölder continuous of order $\alpha > 1$, then f has to be constant.

Proof. Let $f : I \to \mathbb{R}$ be any function.

(a) Suppose that f is Hölder continuous of order $\alpha > 0$. Then $\exists C > 0$,

$$x, y \in I \implies |f(x) - f(y)| < C|x - y|^\alpha.$$

Fix $\epsilon > 0$. Make $\delta := \sqrt[\alpha]{\epsilon/C} > 0$. Then $\forall x, y \in I$,

$$|x - y| < \delta \implies |f(x) - f(y)| < C|x - y|^\alpha < C\delta^\alpha = \epsilon.$$

Ergo, f is uniformly continuous.

(b) Suppose that $I = (-\infty, \infty)$ and $f(x) = \sqrt{|x|}$. By the Triangle Inequality,

$$x, y \in I \implies \sqrt{|x + y|} \le \sqrt{|x|} + \sqrt{|y|};$$

hence,

$$x, y \in I \implies \sqrt{|x|} - \sqrt{|y|} \le \sqrt{|x - y|}.$$

Then $\forall x, y \in I$,

$$|f(x) - f(y)| = |\sqrt{|x|} - \sqrt{|y|}| \le |\sqrt{|x - y|}| = \sqrt{|x - y|}.$$

Ergo, f is Hölder continuous of order $\alpha = 1/2$.

(c) Suppose that f is Hölder continuous of order $\alpha > 0$. Assume that I is bounded. Fix $0 < \beta < \alpha$. Then $\exists C > 0$ where

$$x, y \in I \implies |f(x) - f(y)| < C|x - y|^\alpha,$$

and $\exists \gamma > 0$ where

$$x, y \in I \implies |x - y| \le \gamma;$$

hence,

$$\begin{aligned} x, y \in I \implies |f(x) - f(y)| &< C|x - y|^\alpha \\ &= C|x - y|^{\alpha - \beta}|x - y|^\beta \\ &\le C\gamma^{\alpha - \beta}|x - y|^\beta. \end{aligned}$$

Ergo, f is Hölder continuous of order $0 < \beta < \alpha$. Assume that I is unbounded and $f(x) = \sqrt{|x|}$. Given that f is Hölder continuous, $\alpha = 1/2$ and $\beta = 1/3$. Then $\forall C > 0$, $\exists x, y \in I$ where

$$|f(x) - f(y)| = |\sqrt{|x|} - \sqrt{|y|}| = \sqrt{|x|} \ge C\sqrt[3]{|x|} = C\sqrt[3]{|x - y|}.$$

Ergo, f is not Hölder continuous of order $0 < \beta < \alpha$.

(d) Suppose that f is Hölder continuous of order $\alpha > 1$. Then $\exists C > 0$,

$$x \in I \implies |f(x+h) - f(x)| < C|h|^\alpha$$

for all sufficiently small $h \neq 0$. By the Squeeze Theorem,

$$\begin{aligned} x \in I \implies 0 &\leq \left| \frac{f(x+h) - f(x)}{h} \right| < C|h|^{\alpha - 1} \\ \implies 0 &\leq \lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \leq C \lim_{h \to 0} |h|^{\alpha-1} = 0 \\ \implies f'(x) &= 0. \end{aligned}$$

Then f is differentiable and continuous on I. By the Mean Value Theorem,

$$x, y \in I \implies f(x) = f(y) + f'(c)(x-y) = f(y) + 0 = f(y)$$

for some $c \in I$. Ergo, f is constant.

□

Problem 5. Let $a \in \mathbb{R}$, and suppose $f : (a, \infty) \to \mathbb{R}$ is twice-differentiable. Define

$$M_0 := \sup_{a<x<\infty} |f(x)|, \; M_1 := \sup_{a<x<\infty} |f'(x)|, \; M_2 := \sup_{a<x<\infty} |f''(x)|,$$

which we assume to be finite numbers. Prove the inequality

$$M_1^2 \leq 4M_0M_2.$$

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1 & -1 < x < 0 \\ \frac{x^2-1}{x^2+1} & 0 \leq x \leq \infty \end{cases}$$

and show that $M_0 = 1$, $M_1 = 4$, and $M_2 = 4$. *Hint:* If $h > 0$, Taylor's Theorem shows that

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - f''(\xi)h$$

for some $\xi \in (x, x+2h)$. Hence

$$|f'(x)| \leq \frac{M_0}{h} + M_2h.$$

Proof. Let $f : (a, \infty) \to \mathbb{R}$ be any mapping. Suppose that f is twice-differentiable. Define

$$M_0 := \sup_{a<x<\infty} |f(x)|, \; M_1 := \sup_{a<x<\infty} |f'(x)|, \; M_2 := \sup_{a<x<\infty} |f''(x)| \in \mathbb{R}.$$

Make $x \in (a, \infty)$. Choose $h := \sqrt{M_0/M_2} > 0$. By Taylor's Theorem, $\exists \xi \in (x, x+h)$ such that

$$f'(x) = \frac{f(x+2h) - f(x)}{2h} - f''(\xi)h.$$

By the Triangle Inequality, we have that

$$\begin{aligned} |f'(x)| = \left| \frac{f(x+2h) - f(x)}{2h} - f''(\xi)h \right| &\leq \frac{|f(x+2h) - f(x)|}{2h} + |f''(\xi)|h \\ &\leq \frac{|f(x+2h)|}{2h} + \frac{|f(x)|}{2h} + |f''(\xi)|h \\ &\leq \frac{M_0}{2h} + \frac{M_0}{2h} + M_2h = \frac{M_0}{h} + M_2h. \end{aligned}$$

Indeed, we obtain that

$$\begin{aligned} M_1^2 \leq \left(\frac{M_0}{h} + M_2 h\right)^2 &= \left(\frac{M_0}{h}\right)^2 + 2M_0M_2 + (M_2h)^2 \\ &= M_0M_2 + 2M_0M_2 + M_0M_2 = 4M_0M_2. \end{aligned}$$

Define $a := -1$ and

$$f(x) := \begin{cases} 2x^2 - 1 & -1 < x < 0 \\ \frac{x^2-1}{x^2+1} & 0 \leq x \leq \infty \end{cases} \implies M_0 = 1.$$

Indeed,

$$f'(x) := \begin{cases} 4x & -1 < x < 0 \\ \frac{4x}{(x^2+1)^2} & 0 \leq x \leq \infty \end{cases} \implies M_1 = 4$$

and

$$f'(x) := \begin{cases} 4 & -1 < x < 0 \\ \frac{20x^2+4}{(x^2+1)^3} & 0 \leq x \leq \infty \end{cases} \implies M_2 = 4;$$

hence, $M_1^2 = 16 \leq 16 = 4M_0M_2$. $\square$