18.100B PROBLEM SET 3

SHUO ZHENG

Problem 1. Prove directly from the definition that the set

$$K = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\} \subset \mathbb{R}$$

is compact.

Proof. Let $\{\mathcal{U}_\alpha\}$ be any open cover of K in $\mathbb{R}$. Then $\exists \alpha_0$ where $0 \in \mathcal{U}_{\alpha_0}$; however, $\mathcal{U}_{\alpha_0}$ is open in $\mathbb{R}$, so $\exists \epsilon > 0$ where $N_\epsilon(0) \subset \mathcal{U}_{\alpha_0}$. By the Archimedean Property of $\mathbb{R}$, $\exists M \in \mathbb{N}$ such that $M\epsilon > 1$; hence,

$$n > M \implies \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{M} < \epsilon,$$

i.e. $N_\epsilon(0)$ contains all except finitely many elements of K, namely

$$1, ..., \frac{1}{M}.$$

Given $\{\mathcal{U}_\alpha\}$ covers K, $\exists \alpha_1, ..., \alpha_M$ such that

$$1 \in \mathcal{U}_{\alpha_1} \wedge \cdots \wedge \frac{1}{M} \in \mathcal{U}_{\alpha_M}.$$

Thus,

$$K \subset \bigcup_{i=0}^{M} \mathcal{U}_{\alpha_i};$$

hence, $\{\mathcal{U}_\alpha\}$ has a finite subcover in $\mathbb{R}$. □

Problem 2. Let K be a compact subset of a metric space X, and let $\{\mathcal{U}_\alpha\}$ be an open cover of K. Show that there is a positive real number δ with the property that for every $x \in K$ there is some α with

$$B_\delta(x) \subset \mathcal{U}_\alpha.$$

Proof. Let X be any metric space. Assume K is a compact subset of X. Suppose $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is an open cover of K in X. Then $\forall x \in K$, $x \in \mathcal{U}_\alpha$ for some $\alpha \in I$; however, $\mathcal{U}_\alpha$ is open, so

$$B_\epsilon(x) \subset \mathcal{U}_\alpha$$

for some $\epsilon > 0$. Define the open subsets

$$\mathcal{V}_{\alpha,n} := \{x \in \mathcal{U}_\alpha : B_{\frac{1}{n}}(x) \subset \mathcal{U}_\alpha\}^\circ$$

of X for all $\alpha \in I$ and $n \in \mathbb{N}$. Fix $\alpha \in I$. By the Archimedean Property of $\mathbb{R}$, $\forall \epsilon > 0$ $\exists M \in \mathbb{N}$ such that $M\epsilon > 1$. Then $\forall x \in \mathcal{U}_\alpha$, $\exists M \in \mathbb{N}$ where

$$B_{\frac{1}{M}}(x) \subset B_\epsilon(x) \subset \mathcal{U}_\alpha$$

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for some $\epsilon > 0$ and

$$\mathcal{U}_\alpha = \bigcup_{n\in\mathbb{N}} \mathcal{V}_{\alpha,n}.$$

Note α is arbitrary. Then

$$K \subset \bigcup_{\alpha\in I} \mathcal{U}_\alpha \subset \bigcup_{\alpha\in I} \bigcup_{n\in\mathbb{N}} \mathcal{V}_{\alpha,n};$$

hence, $\{\mathcal{V}_{\alpha,n}\}_{\alpha\in I, n\in\mathbb{N}}$ is an open cover of K in X. Given K is compact, there exists a finite subcover $\{\mathcal{V}_{\alpha_i,n_i}\}_{i=1}^N$. Define the number $\delta := \min\{n_1^{-1}, ..., n_N^{-1}\} > 0$. Then $\forall x \in K$, $\exists i \in \{1, ..., N\}$ such that $x \in \mathcal{V}_{\alpha_i,n_i}$ and

$$B_{\frac{1}{n_i}}(x) \subset \mathcal{U}_{\alpha_i};$$

however, since $\delta \le n_i^{-1}$,

$$B_\delta(x) \subset B_{\frac{1}{n_i}}(x) \subset \mathcal{U}_{\alpha_i}.$$

Note δ is universal. Thus, $\forall x \in K$ $\exists \alpha \in I$ where

$$B_\delta(x) \subset \mathcal{U}_\alpha,$$

as desired. □

Problem 3. Prove that the set

$$E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

is not compact. (As usual, $\mathbb{R}^2$ is equipped with the standard Euclidean metric.)

Proof. Start with defining

$$\mathcal{U}_n := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 - \frac{1}{n}\right\}$$

for all $n \in \mathbb{N}$. By the Archimedean Property of $\mathbb{R}$, $\forall\epsilon > 0$ $\exists M \in \mathbb{N}$ such that $M\epsilon > 1$. Then $\forall(x, y) \in E$, $\exists\epsilon > 0$ such that

$$x^2 + y^2 \le 1 - \epsilon;$$

however, $\exists N \in \mathbb{N}$ such that $N\epsilon > 1$ and

$$x^2 + y^2 \le 1 - \epsilon < 1 - \frac{1}{N} \implies (x, y) \in \mathcal{U}_N,$$

i.e. $\{\mathcal{U}_n\}$ is an open cover of E. Given $M, N \in \mathbb{N}$,

$$M \le N \implies 1 - \frac{1}{M} \le 1 - \frac{1}{N} \implies \mathcal{U}_M \subset \mathcal{U}_N;$$

hence,

$$E \Subset X \implies \exists N \in \mathbb{N},\ E \subset \mathcal{U}_N.$$

By the Density of $\mathbb{R}$, $\forall N \in \mathbb{N}$ $\exists(x, y) \in \mathbb{R}^2$ such that

$$1 - \frac{1}{N} < x^2 + y^2 < 1;$$

hence, $(x, y) \in E \setminus \mathcal{U}_N$ (E is not compact). □

Problem 4. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space with $d(x, y) = |x - y|$. Define the set $E = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

Proof. Start with defining

$$E := \{x \in \mathbb{Q} : 2 < x^2 < 3\}.$$

We want to prove that E is closed and bounded in $\mathbb{Q}$. Consider $x \in E'$. Then $\forall r > 0$, $\exists y \in E$ where

$$|x - y| < r \iff y - r < x < y + r.$$

By the Density of $\mathbb{R}$, $\exists \epsilon > 0$ such that

$$\sqrt{2} \leq z - \epsilon < x < z + \epsilon \leq \sqrt{3}$$

for some $z \in E$; hence,

$$2 \leq (z - \epsilon)^2 < x^2 < (z + \epsilon)^2 \leq 3.$$

Then $x \in E$ (E is closed in $\mathbb{Q}$). Consider $x \in E$. Then

$$|x| \geq 2 \implies x^2 \geq 4 \implies x \notin E;$$

hence,

$$x \in E \implies |x| < 2$$

(E is bounded in $\mathbb{Q}$). We want to prove that E is not compact in $\mathbb{Q}$. Consider the open subsets

$$\mathcal{U}_n = \left\{x \in \mathbb{Q} : 2 + \frac{1}{n} < x^2 < 3 - \frac{1}{n}\right\}$$

for all $n \in \mathbb{N}$. Then $\{\mathcal{U}_n\}$ covers E; however, no finite subcollection of $\{\mathcal{U}_n\}$ covers E (E is not compact). We want to prove that E is open in $\mathbb{Q}$. Given $x \in E$, there exists a neighborhood $N(x)$ of $\mathbb{R}$ such that

$$N(x) \subset \{x \in \mathbb{R} : 2 < x^2 < 3\};$$

hence,

$$N(x) \cap \mathbb{Q} \subset E$$

(E is open in $\mathbb{Q}$). □

Problem 5. Prove that the finite union of compact sets is always compact. Does this assertion also hold for their intersection?

Proof. Let X be any metric space. Consider any sequence $\{K_n\}$ of compact sets in X. We want to show that any finite union of $\{K_n\}$ is always compact in X. Given K_1 is compact in X, the base case is true (trivially). Inductively, assume

$$\bigcup_{i=1}^{n} K_i \Subset X.$$

Consider any open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of

$$\bigcup_{i=1}^{n+1} K_i \subset X$$

in X. Then $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ covers $K_1 \cup \cdots \cup K_n$ because

$$K_1 \cup \cdots \cup K_n \subset K_1 \cup \cdots \cup K_{n+1};$$

hence, $\exists \alpha_1, ..., \alpha_M \in I$ where

$$K_1 \cup \cdots \cup K_n \subset \mathcal{U}_{\alpha_1} \cup \cdots \cup \mathcal{U}_{\alpha_M}.$$

Similarly, $\{\mathcal{U}_\beta\}_{\beta\in I}$ covers K_{n+1}; hence, $\exists\beta_1, ..., \beta_N \in I$ where

$$K_{n+1} \subset \mathcal{U}_{\beta_1} \cup \cdots \cup \mathcal{U}_{\beta_N}.$$

Then

$$\bigcup_{i=1}^{n+1} K_i \subset \bigcup_{i=1}^{M} \mathcal{U}_{\alpha_i} \cup \bigcup_{i=1}^{N} \mathcal{U}_{\beta_i};$$

hence, $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ has a finite subcover of

$$\bigcup_{i=1}^{n+1} K_i \subset X.$$

Thus,

$$\bigcup_{i=1}^{n+1} K_i \Subset X,$$

as desired. We want to show that any finite intersection of $\{K_n\}$ is always compact. Given K_1 is compact, the base case is true (trivially). Inductively, assume

$$\bigcap_{i=1}^{n} K_i \Subset X.$$

By Theorem 2.34, $K_1 \cap \cdots \cap K_n$ and K_{n+1} are closed sets in X; hence,

$$K_1 \cap \cdots \cap K_{n+1}$$

is closed in X. Note $\forall i \in \{1, ..., n+1\}$,

$$K_1 \cap \cdots \cap K_{n+1} \subset K_i.$$

By Theorem 2.35,

$$K_1 \cap \cdots \cap K_{n+1}$$

is compact in X. Thus,

$$\bigcap_{i=1}^{n+1} K_i \Subset X,$$

as desired. □

Problem 6. Let (X, d) be a compact metric space, and $f : X \to X$ a map such that $d(f(x), f(y)) \leq d(x, y)$ for all $x \neq y$. Prove that there exists a point $x \in X$ such that $f(x) = x$. *Hint:* How small can $d(x, f(x))$ get?

Proof. Let (X, d) be any compact metric space. Make a function $f : X \to X$ such that

$$d(f(x), f(y)) \leq d(x, y).$$

We want to show that $d(x, f(x)) = 0$ for some $x \in X$. Consider the set

$$E := \{d(x, f(x)) \in \mathbb{R} : x \in X\}.$$

By the Greatest Upper Bound Property of $\mathbb{R}$,

$$\forall x \in X,\ d(x, f(x)) \geq 0 \implies \inf E \geq 0.$$

Define the number $\alpha := \inf E$. Since $\alpha \in \mathbb{R}$, either $\alpha \in E$ or $\alpha \notin E$ and either $\alpha = 0$ or $\alpha > 0$. Assume, for the purpose of contradiction, $\alpha \notin E$. Consider the subsets

$$\mathcal{U}_n := \{x \in X : d(x, f(x)) > \alpha + \frac{1}{n}\}$$

for all $n \in \mathbb{N}$. We want to show that $\{\mathcal{U}_n\}$ is an open cover of X.Then $\forall x \in X$, $\exists \epsilon > 0$ where

$$d(x, f(x)) = \alpha + \epsilon > 0;$$

hence, $\exists N \in \mathbb{N}$ where $N\epsilon > 1$ and

$$d(x, f(x)) > \alpha + \frac{1}{N},$$

i.e. $\{\mathcal{U}_n\}$ covers X. Consider $x \in \mathcal{U}_n$ for some fixed $n \in \mathbb{N}$. Define the number

$$r := \frac{1}{2}\left(d(x, f(x)) - \alpha - \frac{1}{n}\right) > 0.$$

By the Reverse Triangle Inequality,

$$\begin{aligned} y \in N_r(x) \implies d(y, f(y)) &\geq d(y, f(x)) - d(f(x), f(y)) \\ &\geq d(x, f(x)) - d(x, y) - d(f(x), f(y)) \\ &\geq d(x, f(x)) - 2d(x, y) \\ &> d(x, f(x)) - 2r \\ &= \alpha + \frac{1}{n} \implies y \in \mathcal{U}_n; \end{aligned}$$

hence, $N_r(x) \subset \mathcal{U}_n$. Thus, $\{\mathcal{U}_n\}$ is an open cover of X, as desired. Given that X is compact, $\exists M_1, ..., M_k \in \mathbb{N}$ such that

$$X \subset \bigcup_{i=1}^{k} \mathcal{U}_{M_i}.$$

Define the number $M := \max\{M_1, ..., M_k\} \in \mathbb{N}$. Then

$$x \in X \implies d(x, f(x)) > \alpha + \frac{1}{M_i} > \alpha + \frac{1}{M} \implies x \in \mathcal{U}_M$$

for some $i \in \{1, ..., k\}$; hence, $\mathcal{U}_M$ covers X and

$$\forall x \in X,\ d(x, f(x)) > \alpha + \frac{1}{M},$$

i.e. α cannot be the greatest lower bound of E, a contradiction. Thus, $\alpha \in E$, i.e. $\exists x \in X$ where $d(x, f(x)) = \alpha$. Consider $\alpha = 0$. Then $\exists x \in X$,

$$d(x, f(x)) = 0 \implies x = f(x).$$

Consider $\alpha > 0$. Then $\exists x \in X$,

$$d(x, f(x)) > 0 \implies x \neq f(x).$$

Define the point $y := f(x) \in X$. Then $d(y, f(y)) \in E$ and

$$d(y, f(y)) = d(f(x), f(y)) < d(x, y) = d(x, f(x)) = \alpha$$

(because $x \neq y$); hence, α is not a lower bound of E, a contradiction. Thus, $\exists x \in X$ where $f(x) = x$, as desired. $\square$