18.100B PROBLEM SET 4

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Problem 1. Let $\{x_n\}$ be a convergent sequence in a metric space (X, d). Now permute its terms, forming another sequence $x'_n = x_{f(n)}$, where $f : \mathbb{N} \to \mathbb{N}$ is $1-1$ and onto. Show that $\{x'_n\}$ is convergent, and has the same limit as the original $\{x_n\}$. Is this still true if we drop the assumption that f should be $1-1$?

Proof. Let (X, d) be any metric space. Make a convergent sequence $\{x_n\}$ of X, with limit x. Permute its entries, obtaining a new sequence $x'_n := x_{f(n)}$ of X, such that $f : \mathbb{N} \to \mathbb{N}$ is $1-1$ and onto. Consider $\epsilon > 0$. Then $\exists M \in \mathbb{N}\ \forall n > M$,

$$d(x_n, x) < \epsilon.$$

Given f is $1-1$ and onto, there are only finitely many $n \in \mathbb{N}$ such that $f(n) \leq M$; in particular, exactly M such indices, called $n_1, ..., n_M \in \mathbb{N}$. Define $N = \max\{n_1, ..., n_M\} > 0$. Then

$$n > N \implies f(n) > M \implies d(x'_n, x) = d(x_{f(n)}, x) < \epsilon.$$

Thus, $x'_n \to x$. Consider $X = \mathbb{R}$ with $d(x, y) = |x - y|$. Define

$$x_n = \frac{1}{n} \to 0$$

and $x = 0$. Make a function $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) := \begin{cases} 1, & n \in \{2k - 1 \in \mathbb{N} : k \in \mathbb{N}\} \\ \frac{n}{2}, & n \in \{2k \in \mathbb{N} : k \in \mathbb{N}\} \end{cases}.$$

Then f is not $1-1$ and f is onto. Define

$$\epsilon_0 = 1 > 0.$$

Given M is any natural number, there exists a natural number $n \geq M$ such that

$$d(x'_n, x) = |x'_n - 0| = |x'_n| \geq \epsilon_0;$$

namely,

$$M \in \{2k - 1 \in \mathbb{N} : k \in \mathbb{N}\} \implies n = M$$

and

$$M \in \{2k \in \mathbb{N} : k \in \mathbb{N}\} \implies n = M + 1.$$

Thus, $x_n \to x$; however, $x'_n \not\to x$. □

Problem 2. Find a sequence $\{x_n\}$ with values in $[0, 1]$ that has the following property. For every $x \in [0, 1]$, we can find a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$. *Hint:* Think about the rational numbers between 0 and 1.

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Proof. Let $\{x_n\}$ be any sequence of $\mathbb{Q}$ in $[0,1]$. Choose $x \in [0,1]$. Given the Density of $\mathbb{Q}$, there are infinitely many rational numbers between any pair of real numbers. Suppose that $x = 0$. Then $\exists r \in (0,1)$, $0 = x < x + r < 1$; hence,

$$\forall k \in \mathbb{N},\ 0 = x < x + \frac{r}{k+1} < x + \frac{r}{k} < 1.$$

Choose x_{n_1} to be any rational number such that

$$0 = x < x_{n_1} < x + r < 1.$$

Given that $x_{n_1}, ..., x_{n_k}$ are fixed, define $x_{n_{k+1}}$ to be any rational number such that $n_k < n_{k+1}$ and

$$0 = x < x_{n_{k+1}} < x + \frac{r}{k+1} < 1.$$

By induction, $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$ with limit x. Suppose that $x \in (0,1)$. Then $\exists r \in (0,1)$, $0 < x - r < x < x + r < 1$; hence,

$$\forall k \in \mathbb{N},\ 0 < x - \frac{r}{k} < x - \frac{r}{k+1} < x < x + \frac{r}{k+1} < x + \frac{r}{k} < 1.$$

Choose x_{n_1} to be any rational number such that

$$0 < x - r < x_{n_1} < x + r < 1.$$

Given that $x_{n_1}, ..., x_{n_k}$ are fixed, define $x_{n_{k+1}}$ to be any rational number such that $n_k < n_{k+1}$ and

$$0 < x - \frac{r}{k+1} < x_{n_{k+1}} < x + \frac{r}{k+1} < 1.$$

By induction, $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$ with limit x. Suppose that $x = 1$. Then $\exists r \in (0,1)$, $0 < x - r < x = 1$; hence,

$$\forall k \in \mathbb{N},\ 0 < x - \frac{r}{k} < x - \frac{r}{k+1} < x = 1.$$

Choose x_{n_1} to be any rational number such that

$$0 < x - r < x_{n_1} < x = 1.$$

Given that $x_{n_1}, ..., x_{n_k}$ are fixed, define $x_{n_{k+1}}$ to be any rational number such that $n_k < n_{k+1}$ and

$$0 < x - \frac{r}{k+1} < x_{n_{k+1}} < x = 1.$$

By induction, $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$ with limit x. Thus,

$$x \in [0,1] \implies \lim_{k\to\infty} x_{n_k} = x.$$

□

Problem 3. Fix some prime p, and let $X = \mathbb{Z}$ with the p-adic metric. Show that the sequence $x_1 = 1$, $x_2 = 1 + p$, $x_3 = 1 + p + p^2$, . . . is a Cauchy sequence. For $p = 2$, show that this sequence converges.

Proof. Let p be any prime number. Suppose that $X = \mathbb{Z}$ with the p-adic metric. Then $\forall x, y \in X$, $\exists k \in \mathbb{N}$ where $p^k \mid (x - y)$ and $p^{k+1} \nmid (x - y)$; hence,

$$d(x,y) = \frac{1}{p^k}.$$

Make the sequence

$$x_n := \sum_{i=0}^{n-1} p^i.$$

Without loss of generality, assume that $m < n$. Then

$$x_n - x_m = \sum_{i=0}^{n-1} p^i - \sum_{i=0}^{m-1} p^i = \sum_{i=m}^{n-1} p^i = p^m \sum_{i=0}^{n-m-1} p^i = p^m x_{n-m}$$

hence, $p^m \mid (x_n - x_m)$ and $p^{m+1} \nmid (x_n - x_m)$, i.e.

$$d(x_m, x_n) = \frac{1}{p^m}.$$

Choose $\epsilon > 0$. Then $\exists M \in \mathbb{N}$ where $M\epsilon > 1$; hence,

$$m, n \geq M \implies d(x_m, x_n) = \frac{1}{p^m} < \frac{1}{m} \leq \frac{1}{M} < \epsilon$$

and (x_n) is Cauchy. Fix $p = 2$. Then $\forall n \in \mathbb{N}$,

$$x_n := \sum_{i=0}^{n-1} 2^i = 2^n - 1 \implies x_n - (-1) = x_n + 1 = 2^n;$$

hence, $\exists N \in \mathbb{N}$ where $N\epsilon > 1$ and

$$n \geq N \implies d(x_n, -1) = \frac{1}{2^n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

i.e. (x_n) converges to -1. □

Problem 4. Let X be a complete metric space, and let $Y \subset X$. Show that Y is complete if and only if Y is closed.

Proof. Let (X, d) be any metric space. Suppose that X is complete. Choose $Y \subset X$. Thus, $d|_Y$ induces a metric on Y; hence, $(Y, d|_Y)$ is a metric subspace of (X, d). Assume that Y is complete. If $x \in X$ is a limit point of Y, then $\exists (x_n) \in Y$ such that $\forall \epsilon > 0$, $\exists M \in \mathbb{N}$ where

$$n \geq M \implies d(x_n, x) < \epsilon \implies d|_Y(x_n, x) < \epsilon$$

(because Y is complete); hence, $x \in Y$ and Y is closed. Assume that Y is closed. If $(x_n) \in Y$ is Cauchy, then $(x_n) \in X$ is Cauchy and $\exists x \in X$ such that $\forall \epsilon > 0$, $\exists M \in \mathbb{N}$ where

$$n \geq M \implies d(x_n, x) < \epsilon$$

(because X is complete); hence, x is a limit point of Y and Y is complete (because Y is closed). □

Problem 5. If (x_n) and (y_n) are two bounded sequences of real numbers, show that

(a) $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$,
(b) $\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n$,
(c) $\limsup(x_n + y_n) = \limsup x_n + \limsup y_n$ if (x_n) or (y_n) converges,
(d) $\liminf(x_n + y_n) = \liminf x_n + \liminf y_n$ if (x_n) or (y_n) converges.

(*Hint:* Pick a subsequence of $(x_n + y_n)$ that converges, then, from these x_{n_k}'s pick a subsequence that converges and do the same for the y_{n_k}'s)

Proof. Let (x_n) and (y_n) be any bounded sequences of $\mathbb{R}$. Thus, $\exists \alpha, \beta \in \mathbb{R}$ where $\forall n \in \mathbb{N}$, $|x_n| \leq \alpha$ and $|y_n| \leq \beta$; hence,

$$|x_n + y_n| \leq |x_n| + |y_n| \leq \alpha + \beta,$$

so

$$\sup\{x_n + y_n : n \in \mathbb{N}\} \leq \sup\{x_n : n \in \mathbb{N}\} + \sup\{y_n : n \in \mathbb{N}\}$$

and

$$\inf\{x_n + y_n : n \in \mathbb{N}\} \geq \inf\{x_n : n \in \mathbb{N}\} + \inf\{y_n : n \in \mathbb{N}\}.$$

If $x_n \to x$ and $y_n \to y$, then

$$\forall n \in \mathbb{N},\ x_n \leq y_n \implies x \leq y;$$

else, $\exists M \in \mathbb{N}$ such that

$$\forall n \geq M,\ x_n > x - \frac{x-y}{2} = \frac{x+y}{2} = y + \frac{x-y}{2} > y_n.$$

(a) Ergo,

$$\begin{aligned}\limsup_{n\to\infty}(x_n + y_n) &:= \lim_{n\to\infty} \sup\{x_k + y_n : k \geq n\} \\ &\leq \lim_{n\to\infty} (\sup\{x_k : k \geq n\} + \sup\{y_k : k \geq n\}) \\ &= \lim_{n\to\infty} \sup\{x_k : k \geq n\} + \lim_{n\to\infty} \sup\{y_k : k \geq n\} \\ &=: \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.\end{aligned}$$

(b) Ergo,

$$\begin{aligned}\liminf_{n\to\infty}(x_n + y_n) &:= \lim_{n\to\infty} \inf\{x_k + y_n : k \geq n\} \\ &\geq \lim_{n\to\infty} (\inf\{x_k : k \geq n\} + \inf\{y_k : k \geq n\}) \\ &= \lim_{n\to\infty} \inf\{x_k : k \geq n\} + \lim_{n\to\infty} \inf\{y_k : k \geq n\} \\ &=: \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n.\end{aligned}$$

(c) Without loss of generality, assume that (x_n) converges. It follows that $\forall (n_k) \in \mathbb{N}$,

$$x_{n_k} \to \limsup_{n\to\infty} x_n;$$

however, $\exists (n_k) \in \mathbb{N}$ such that

$$y_{n_k} \to \limsup_{n\to\infty} y_n$$

and

$$x_{n_k} + y_{n_k} \to \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \leq \limsup_{n\to\infty}(x_n + y_n),$$

so

$$\limsup_{n\to\infty}(x_n + y_n) = \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

(d) Without loss of generality, assume that (y_n) converges. It follows that $\exists (n_k) \in \mathbb{N}$,

$$x_{n_k} \to \liminf_{n\to\infty} x_n;$$

hence,

$$x_{n_k} + y_{n_k} \to \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \geq \liminf_{n\to\infty}(x_n + y_n),$$

so

$$\liminf_{n\to\infty}(x_n + y_n) = \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n.$$

□

Problem 6. A metric space is called complete if every Cauchy sequence converges. Let (X, d) be a complete metric space, and $f : X \to X$ a map with the following property. There is some $0 \le \lambda < 1$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \le \lambda d(x, y).$$

Prove that then, there is a point $x \in X$ such that $f(x) = x$. *Hint:*

$$1 + \lambda + \lambda^2 + \cdots + \lambda^n = \frac{1 - \lambda^{n+1}}{1 - \lambda}.$$

Proof. Let (X, d) be any metric space. Assume that X is complete. Construct a mapping $f : X \to X$ so that $\exists \lambda \in [0, 1)$,

$$d(f(x), f(y)) \le \lambda d(x, y).$$

Pick $x_1 \in X$. Given that $x_1, ..., x_k$ $(k = 1, 2, 3, ...)$ are fixed, define $x_{k+1} = f(x_k)$ $(k = 1, 2, 3, ...)$. By induction, (x_n) is a sequence of X. Suppose that $\forall k \in \mathbb{N}$,

$$d(x_k, x_{k+1}) \le \lambda^{k-1} d(x_1, x_2) \ (k \in \mathbb{N}).$$

Then $\forall k \in \mathbb{N}$,

$$d(x_{k+1}, x_{k+2}) = d(f(x_k), f(x_{k+1})) \le \lambda d(x_k, x_{k+1}) \le \lambda^k d(x_1, x_2).$$

By induction,

$$d(x_n, x_{n+1}) \le \lambda^{n-1} d(x_1, x_2).$$

Without loss of generality, fix $m < n$. By the Triangle Inequality,

$$\begin{aligned}
d(x_m, x_n) &\le d(x_m, x_{m+1}) + \cdots + d(x_{n-1}, x_n) \\
&\le \lambda^{m-1} d(x_1, x_2) + \cdots + \lambda^{n-2} d(x_1, x_2) \\
&= \frac{\lambda^{m-1}(1 - \lambda^{n-m})}{1 - \lambda} \\
&< \frac{\lambda^{m-1}}{1 - \lambda}.
\end{aligned}$$

Choose $\epsilon > 0$. By Theorem 3.20(e), $\exists M \in \mathbb{N}$ such that $\lambda^M < (1 - \lambda)\epsilon$; hence,

$$m, n \ge M \implies d(x_m, x_n) < \frac{\lambda^{m-1}}{1 - \lambda} \le \frac{\lambda^M}{1 - \lambda} < \epsilon.$$

Indeed, (x_n) must be Cauchy; hence, (x_n) must be convergent. Given that (x_n) is convergent, $\exists x_0 \in X$ such that

$$\lim_{n \to \infty} x_n = x_0;$$

hence, $\exists N \in \mathbb{N}$ where

$$n \ge N \implies d(x_n, x_0) < \frac{\epsilon}{1 + \lambda}$$

and

$$\begin{aligned}
d(x_0, f(x_0)) &\le d(x_0, x_{N+1}) + d(x_{N+1}, f(x_0)) \\
&= d(x_0, x_{N+1}) + d(f(x_N), f(x_0)) \\
&\le d(x_0, x_{N+1}) + \lambda d(x_N, x_0) \\
&= \frac{\epsilon}{1 + \lambda} + \frac{\lambda \epsilon}{1 + \lambda} = \epsilon.
\end{aligned}$$

Thus, $d(x_0, f(x_0)) = 0$, i.e. $f(x_0) = x_0$ as desired. Suppose that $\exists x, y \in X$ where $f(x_0) = x_0$ and $f(y_0) = y_0$. Then

$$d(x_0, y_0) = d(f(x_0), f(y_0)) \leq \lambda d(x_0, y_0) \implies (1 - \lambda)d(x_0, y_0) \leq 0;$$

hence, $d(x_0, y_0) = 0$ (else $d(x_0, y_0) < 0$) and $x_0 = y_0$. $\square$