18.100B PROBLEM SET 5

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Problem 1. Prove that if $\sum |x_n|$ converges, then $\sum |x_n|^k$ $(k = 1, 2, 3, ...)$ converges.

Proof. Let $\sum |x_n|$ be any convergent series. Inductively, assume that $\sum |x_n|^k$ $(k = 1, 2, 3, ...)$ converges. By Theorem 3.23,

$$\lim_{n\to\infty} |x_n|^k = 0;$$

hence, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies |x_n|^k < 1 \implies |x_n|^{k+1} < |x_n|.$$

By the Comparison Test,

$$\sum |x_n| \text{ converges} \implies \sum |x_n|^{k+1} \ (k = 1, 2, 3, ...) \text{ converges.}$$

Then $\sum |x_n|^k$ $(k = 1, 2, 3, ...)$ converges. □

Problem 2. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}.$$

Hint: Use a "telescope trick", i.e. an argument of the form $\sum x_n = \sum (y_n - y_{n+1}) = y_1$.

Proof. Start by defining

$$x_n := \frac{1}{n(n+1)(n+2)} \wedge y_n := \frac{1}{2n} - \frac{1}{2(n+1)} \ (n = 1, 2, 3, ...).$$

Then

$$\begin{aligned} x_n &= \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2n+2)} \\ &= \left(\frac{1}{2} - \frac{1}{2(n+1)}\right) - \left(\frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right) = y_n - y_{n+1} \ (n = 1, 2, 3, ...); \end{aligned}$$

hence,

$$\begin{aligned} \sum_{n=1}^{\infty} x_n := \lim_{k\to\infty} \sum_{n=1}^{k} x_n &= \lim_{k\to\infty} \sum_{n=1}^{k} (y_n - y_{n+1}) \\ &= \lim_{k\to\infty} (y_1 - y_{k+1}) \\ &= \lim_{k\to\infty} \left(\frac{1}{4} - \frac{1}{2(k+1)} + \frac{1}{2(k+2)}\right) \\ &= \frac{1}{4} - \frac{1}{2} \lim_{k\to\infty} \frac{1}{k+1} + \frac{1}{2} \lim_{k\to\infty} \frac{1}{k+2} = \frac{1}{4}. \end{aligned}$$

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□

Problem 3. Assume $x_0 \geq x_1 \geq x_2 \geq \cdots$ and suppose that $\sum x_n$ converges. Prove that

$$\lim_{n\to\infty} nx_n = 0.$$

Hint: Show and use the inequality $nx_{2n} \leq \sum_{k=n+1}^{2n} x_k$.

Proof. Let $\sum x_n$ be any convergent series, with $x_0 \geq x_1 \geq x_2 \geq \cdots$, of $\mathbb{R}$. Then $x_0 \geq x_1 \geq x_2 \geq \cdots \geq 0$; else, $x_n \not\to 0$ as required. By the Cauchy Criterion,

$$\forall \epsilon > 0,\ \exists M \in \mathbb{N} \text{ such that } n \geq M \implies nx_{2n} \leq \sum_{k=n+1}^{2n} x_k < \epsilon \implies nx_{2n} \to 0$$

and

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies nx_{2n+1} \leq \sum_{k=n+2}^{2n+1} x_k < \epsilon \implies nx_{2n+1} \to 0.$$

Thus,

$$\lim_{n\to\infty} 2nx_{2n} = 2\lim_{n\to\infty} nx_{2n} = 0$$

and

$$\lim_{n\to\infty} (2n+1)x_{2n+1} = 2\lim_{n\to\infty} nx_{2n+1} + \lim_{n\to\infty} x_{2n+1} = 0;$$

hence,

$$\lim_{n\to\infty} nx_n = 0$$

as desired. □

Problem 4. Prove Theorem 3.43, in the book, directly relying on the definition of convergence, and on the fact that the partial sums $s_n = x_1 + \cdots + x_n$ of an alternating series satisfy $s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_5 \leq s_3 \leq s_1$.

Proof. Let (x_n) be any monotone sequence of $\mathbb{R}$. Suppose that

(a) $|x_1| \geq |x_2| \geq |x_3| \geq \cdots$;
(b) $x_{2m-1} \geq 0$ and $x_{2m} \leq 0$ $(m = 1, 2, 3, ...)$;
(c) $x_n \to 0$.

Write that

$$\forall n \in \mathbb{N},\ s_n := x_1 + \cdots + x_n.$$

Then $\forall n \in \mathbb{N}$,

$$s_{2n} \leq s_{2n+2} \leq s_{2n+1};$$

hence, $s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_5 \leq s_3 \leq s_1$. Define the sets

$$E := \{s_{2n} \in \mathbb{R} : n \in \mathbb{N}\} \wedge F := \{s_{2n+1} \in \mathbb{R} : n \in \mathbb{N}\}.$$

Note that E is nonempty and bounded above (with s_1). By the Least Upper Bound Property of $\mathbb{R}$, $\exists s \in \mathbb{R}$ where $s = \sup E$. Choose $\epsilon > 0$. Then s is a lower bound of F; else, $\exists M \in \mathbb{N}$ where

$$s_{2M} \leq s_{2M+1} < s$$

and $s \neq \sup E$. Given that $x_n \to 0$, $\exists N_0 \in \mathbb{N}$ such that

$$n \geq N_0 \implies |x_{2n+1}| < \epsilon.$$

In fact, $\exists N \in \mathbb{N}$ where $N \geq N_0$ and

$$s - \epsilon < s_{2N} < s < s_{2N+1} < s + \epsilon$$

(else $s \neq \sup E$); hence, $s = \inf F$. Then $\forall n \in \mathbb{N}, \exists M \in \mathbb{N}$ (namely, $M := 2N$) where

$$n \geq M \implies s - \epsilon < s_{2N} \leq s_n \leq s_{2N+1} < s + \epsilon;$$

hence, $s_n \to s$. Note that s is hard to compute typically. Thus, $\sum x_n$ converges. $\square$

Problem 5. Take the convergent series from Example 3.53. It is shown there how to rearrange it so that it converges to a different number. Find another explicit rearrangement so that the rearranged series does not converge at all. Note that following our usual terminology, going to ∞ also counts as "does not converge" or "diverges".

Proof. Start by defining

$$\forall n \in \mathbb{N},\ x_n := \frac{(-1)^{n+1}}{n}.$$

By Theorem 3.43,

$$\sum_{n=1}^{\infty} x_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges. In fact,

$$2\sum_{m=1}^{n} x_{2m-1} > \sum_{m=1}^{n} x_{2m-1} - \sum_{m=1}^{n} x_{2m} = \sum_{m=1}^{2n} \frac{1}{m} \to \infty \implies \sum_{m=1}^{n} x_{2m-1} \to \infty;$$

hence,

$$\sum_{n=1}^{\infty} x_{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$$

does not converge. Make an increasing sequence (N_k) of $\mathbb{N}$ in the following way: Pick $N_1 = 1$. Given that $N_1, ..., N_k$ $(k = 1, 2, 3, ...)$ are fixed, define N_{k+1} such that

$$\sum_{m=1}^{N_{k+1}} x_{2m-1} - \sum_{m=1}^{N_k} x_{2m-1} > \frac{1}{2}$$

because

$$\sum_{m=1}^{n} x_{2m-1} \to \infty.$$

End by defining

$$\sum_{m=1}^{\infty} x'_m = 1 - \frac{1}{2} + \sum_{m=N_1+1}^{N_2} \frac{1}{2m-1} - \frac{1}{4} + \cdots + \sum_{m=N_{k-1}+1}^{N_k} \frac{1}{2m-1} - \frac{1}{2k} + \cdots$$

and

$$\forall k \in \mathbb{N},\ M_k := N_k + k.$$

Lemma 1:

$$\forall k \in \mathbb{N},\ M_k < n < M_{k+1} \implies \sum_{m=1}^{n} x'_m - \sum_{m=1}^{M_k} x'_m = \sum_{m=N_k+1}^{n-k} x_{2m-1} > 0$$

$$\implies \sum_{m=1}^{n} x'_m > \sum_{m=1}^{M_k} x'_m.$$

Lemma 2:

$$\forall k \in \mathbb{N},\ \sum_{m=1}^{M_{k+1}} x'_m - \sum_{m=1}^{M_k} x'_m = \sum_{m=N_k+1}^{N_{k+1}} x_{2m-1} + x'_{M_{k+1}} > \frac{1}{2} - \frac{1}{2k+2} > \frac{1}{4}.$$

Lemma 3:

$$\forall k \in \mathbb{N},\ \sum_{m=1}^{M_k} x'_m > \frac{k}{4} \implies \sum_{m=1}^{M_{k+1}} x'_m > \sum_{m=1}^{M_k} x'_m + \frac{1}{4} > \frac{k+1}{4}.$$

Lemma 4:

$$\sum_{m=1}^{n} x'_m \to \infty.$$

Thus,

$$\sum_{m=1}^{\infty} x'_m = 1 - \frac{1}{2} + \sum_{m=N_1+1}^{N_2} \frac{1}{2m-1} - \frac{1}{4} + \cdots + \sum_{m=N_{k-1}+1}^{N_k} \frac{1}{2m-1} - \frac{1}{2k} + \cdots$$

diverges as desired. □

Problem 6. Let (p_k) be the sequence of prime numbers, and let J_N denote the set of natural numbers whose factorization into primes only involves the primes $\{p_k \in \mathbb{N} : 1 \leq k \leq N\}$. Prove the following identity

$$\sum_{n \in J_N} \frac{1}{n^r} = \prod_{k=1}^{N} \frac{1}{1 - p_k^{-r}}$$

for any $N \in \mathbb{N}$ and $r \in \mathbb{Q} \cap (1, \infty)$. From this deduce *Euler's Formula*

$$\sum_{n=1}^{\infty} \frac{1}{n^r} = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-r}}.$$

Proof. Let (p_k) be the sequence of prime numbers. Suppose that

$$J_N := \left\{ \prod p_k \in \mathbb{N} : 1 \leq k \leq N \right\} \ (N = 1, 2, 3, \cdots).$$

Inductively, assume that $p_N \geq N$, $J_N \subset J_{N+1}$ and $\{1, ..., N\} \subset J_N$ $(N = 1, 2, 3, \cdots)$. Then

$$p_{N+1} \geq p_N + 1 \geq N + 1$$

(else, $p_{N+1} < p_N + 1 \implies p_{N+1} - p_N < 1 \implies p_N = p_{N+1}$) and

$$J_{N+1} \subset J_{N+2}$$

(else, $p_{N+1} > p_{N+2} \implies J_N \not\subset J_{N+1}$); hence,

$$\{1, ..., N+1\} = \{1, ..., N\} \cup \{N+1\} \subset J_N \cup \{N+1\} \subset J_{N+1}$$

(else, $p_{N+1} < N + 1$). Fix $r \in \mathbb{Q} \cap (1, \infty)$. Inductively, assume that

$$\sum_{n \in J_N} \frac{1}{n^r} = \prod_{k=1}^{N} \frac{1}{1 - p_k^{-r}} \ (N = 1, 2, 3, \cdots).$$

Then

$$\begin{aligned}\sum_{n\in J_{N+1}} \frac{1}{n^r} &= \sum_{n\in J_N} \frac{1}{n^r} + \frac{1}{p_{N+1}^r}\sum_{n\in J_N} \frac{1}{n^r} + \cdots + \frac{1}{p_{N+1}^{Nr}}\sum_{n\in J_N} \frac{1}{n^r} + \cdots \\ &= \sum_{n\in J_N} \frac{1}{n^r}\left(1 + \frac{1}{p_{N+1}^r} + \frac{1}{p_{N+1}^{2r}} + \cdots\right) \\ &= \prod_{k=1}^{N} \frac{1}{1-p_k^{-r}} \cdot \frac{1}{1-p_{N+1}^{-r}} = \prod_{k=1}^{N+1} \frac{1}{1-p_k^{-r}}.\end{aligned}$$

Thus,

$$\sum_{n=1}^{N} \frac{1}{n^r} < \sum_{n\in J_N} \frac{1}{n^r}\ (N = 1,2,3,\cdots) \implies \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^r} \le \lim_{N\to\infty} \sum_{n\in J_N} \frac{1}{n^r};$$

however,

$$J_N \subset \mathbb{N}\ (N = 1,2,3,\cdots) \implies \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^r} \ge \lim_{N\to\infty} \sum_{n\in J_N} \frac{1}{n^r}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^r} := \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^r} = \lim_{N\to\infty} \sum_{n\in J_N} \frac{1}{n^r} = \lim_{N\to\infty} \prod_{k=1}^{N} \frac{1}{1-p_k^{-1}} =: \prod_{k=1}^{\infty} \frac{1}{1-p_k^{-r}}.$$

□