18.100B PROBLEM SET 2

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Problem 1. In vector spaces, metrics are usually defined in terms of norms which measure the length of a vector. If V is a vector space over $\mathbb{R}$, then a norm is a function from vectors to real numbers, denoted by $\|\cdot\|$ satisfying:

(i) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$;
(ii) $\forall \lambda \in \mathbb{R},\ \|\lambda x\| = |\lambda|\|x\|$;
(iii) $\|x+y\| \leq \|x\| + \|y\|$.

Prove that every norm defines a metric.

Proof. Let V be any vector space over $\mathbb{R}$. Make a function $d : V \times V \to \mathbb{R}$ such that

$$d(x,y) = \|x-y\|.$$

We want to show that d is a metric on V.

(a) (Positive Definite) If $x, y \in V$, then

$$d(x,y) = \|x-y\| \geq 0;$$

however,

$$d(x,y) = 0 \iff \|x-y\| = 0 \iff x-y = 0 \iff x = y.$$

(b) (Symmetry) If $x, y \in V$, then

$$\begin{aligned} d(x,y) = \|x-y\| &= \|(-1)(y-x)\| \\ &= |(-1)|\|y-x\| = \|y-x\| = d(y,x). \end{aligned}$$

(c) (Triangle Inequality) If $x, y, z \in V$, then

$$\begin{aligned} d(x,z) &= \|x-z\| \\ &= \|(x-y)+(y-z)\| \\ &\leq \|x-y\| + \|y-z\| = d(x,y) + d(y,z). \end{aligned}$$

Thus, V is a metric space, as desired. $\square$

Problem 2. Let (X,d) be a metric space. Show that $d'(x,y) = \sqrt{d(x,y)}$ is also a metric on X, and that the open sets for d' are the same as the open sets for d.

Proof. Let (X,d) be any metric space. Note that

$$0 \leq a \leq b \implies b - a \geq 0 \implies (\sqrt{b}+\sqrt{a})(\sqrt{b}-\sqrt{a}) \geq 0 \implies \sqrt{a} \leq \sqrt{b}.$$

Define a function $d' : X \times X \to \mathbb{R}$ by

$$d'(x,y) = \sqrt{d(x,y)}.$$

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Note that

$$\begin{aligned} a, b \geq 0 \implies \sqrt{ab} \geq 0 &\implies a + b \leq a + 2\sqrt{ab} + b \\ &\implies (\sqrt{a+b})^2 \leq (\sqrt{a} + \sqrt{b})^2 \implies \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}. \end{aligned}$$

We want to show that d' is a metric on X.

(a) (Positive Definite) If $x, y \in X$, then

$$d(x, y) \geq 0 \implies d'(x, y) = \sqrt{d(x, y)} \geq 0;$$

however,

$$x = y \iff d(x, y) = 0 \iff \sqrt{d(x, y)} = 0 \iff d'(x, y) = 0.$$

(b) (Symmetry) If $x, y \in X$, then $d(x, y) = d(y, x)$; hence,

$$d'(x, y) = \sqrt{d(x, y)} = \sqrt{d(y, x)} = d'(y, x).$$

(c) (Triangle Inequality) If $x, y, z \in X$, then $d(x, z) \leq d(x, y) + d(y, z)$; hence,

$$\begin{aligned} d'(x, z) = \sqrt{d(x, z)} &\leq \sqrt{d(x, y) + d(y, z)} \\ &\leq \sqrt{d(x, y)} + \sqrt{d(y, z)} = d'(x, y) + d'(y, z). \end{aligned}$$

Thus, (X, d') is a metric space, as desired. Note that

$$0 \leq a \leq b \implies b \pm a \geq 0 \implies b^2 - a^2 = (b + a)(b - a) \geq 0 \implies a^2 \leq b^2.$$

Choose $E \subset X$ to be open for d. Given $x \in E$, $\exists r > 0$ such that

$$N_r(x) \subset E.$$

Thus,

$$\begin{aligned} z \in N'_{\sqrt{r}}(x) &\implies d'(x, z) < \sqrt{r} \\ &\implies d(x, z) = d'(x, z)^2 < r \implies z \in N_r(x); \end{aligned}$$

hence,

$$N'_{\sqrt{r}}(x) \subset N_r(x) \subset E.$$

Indeed, E is open for d'. Choose $E' \subset X$ to be open for d'. Given $x \in E'$, $\exists r > 0$ such that

$$N'_r(x) \subset E'.$$

Thus,

$$\begin{aligned} z \in N_{r^2}(x) &\implies d(x, z) < r^2 \\ &\implies d'(x, z) = \sqrt{d(x, z)} < r \implies z \in N'_r(x); \end{aligned}$$

hence,

$$N_{r^2}(x) \subset N'_r(x) \subset E'.$$

Indeed, E' is open for d. Thus, the open subsets for d and d' are the same, as desired. □

Problem 3. Let E be a subset of a metric space X. The interior E° is defined by

$$E^{\circ} := \{x \in E : x \text{ is an interior point}\}.$$

(a) Prove that E° is always open.
(b) Prove that E is open if and only if $E = E^{\circ}$.
(c) If $G \subset E$ and G is open, prove that $G \subset E^{\circ}$.

(d) Prove that $X \setminus E^{\circ} = \overline{X \setminus E}$.
(e) Do E and $\overline{E}$ always have the same interiors?
(f) Do E and E° always have the same closures?

Proof. Let X be any metric space. Assume $E \subset X$.

(a) Suppose $x \in E^{\circ}$. Then x is an interior point of E; hence, $\exists r > 0$ such that $N_r(x) \subset E$. Choose $y \in N_r(x)$. By Theorem 2.19,
$$N_{r-d(x,y)}(y) \subset N_r(x) \subset E.$$
Indeed, $N_r(x) \subset E^{\circ}$.

(b) By definition,
$$E \text{ is open} \iff \forall x \in E,\ x \in E^{\circ} \iff E \subset E^{\circ};$$
however, $E \supset E^{\circ}$, so
$$E \text{ is open} \iff E = E^{\circ}.$$

(c) Suppose $G \subset E$ is open. If $x \in G$, then $x \in G^{\circ}$; hence, $\exists r > 0$ such that
$$N_r(x) \subset G \subset E;$$
hence, $x \in E^{\circ}$. Thus, $G \subset E^{\circ}$.

(d) By definition,
$$\begin{aligned} x \in X \setminus E^{\circ} &\iff x \notin E^{\circ} \\ &\iff \forall r > 0,\ N_r(x) \not\subset E \\ &\iff \forall r > 0,\ N_r(x) \cap (X \setminus E) \neq \emptyset \iff x \in \overline{X \setminus E}; \end{aligned}$$
hence, $X \setminus E^{\circ} = \overline{X \setminus E}$.

(e) No, E and $\overline{E}$ can have different interiors. Define $X = \mathbb{R}$ and $E = \mathbb{Q}$. If $x \in X$, then
$$\forall r > 0, \exists q \in E \text{ such that } q \in (N_r(x) \cap E) \setminus \{x\};$$
hence, $x \in \overline{E}$. Therefore, $\overline{E} = X$. Similarly, $\overline{X \setminus E} = X$; hence,
$$X \setminus E^{\circ} = \overline{X \setminus E} = X \implies E^{\circ} = \emptyset.$$
Thus, $E^{\circ} = \emptyset \neq X = (\overline{E})^{\circ}$; hence, $E^{\circ} \neq (\overline{E})^{\circ}$.

(f) No, E and E° can have different closures. If $X = \mathbb{R}$ and $E = \mathbb{Q}$, then
$$\overline{E} = X \neq \emptyset = \overline{E^{\circ}};$$
hence, $\overline{E} \neq \overline{E^{\circ}}$.

□

Problem 4. Consider $\mathbb{R}$ with the standard metric. Let $E \subset \mathbb{R}$ be a subset that has no limit points. Show that E is countable.

Proof. Let E be any subset of $\mathbb{R}$. Suppose E has no limit points. Then $\forall x \in E$, x is an isolated point; hence, $\exists r_x > 0$ such that $B_{r_x}(x) \cap E = \{x\}$. By the Density of $\mathbb{Q}$, there are infinitely many rational numbers in each neighborhood of real numbers. Make a function $f : E \to \mathbb{Q}$ such that $f(x) \in \mathbb{Q}$ and
$$|f(x) - x| < \frac{r_x}{2}.$$

Then $\forall x \in E$, $f(x) \notin E$. We want to show that f is $1-1$. Assume $f(x) = f(y)$. Then $\exists q \in \mathbb{Q}$, $|x-q| < r_x$ and $|y-q| < r_y$ (namely, $q := f(x) = f(y)$). Without loss of generality, fix $r_x \geq r_y$. By the Triangle Inequality,

$$|x-y| \leq |x-q| + |q-y| < \frac{r_x}{2} + \frac{r_y}{2} = \frac{r_x + r_y}{2} \leq r_x \implies x = y;$$

otherwise, x is not an isolated point of E. Then $x = y$. Indeed, f is $1-1$. Thus, $|E| \leq |\mathbb{Q}| = |\mathbb{N}|$; hence, E is countable. $\square$

Problem 5. Let E be a subset of a metric space X. Recall that $\overline{E}$, the closure of E, is the union of E and its limit points. Recall that a point $x \in X$ belongs to the boundary of E, ∂E, if every open ball centered at $x \in X$ contains points of E and points of E^c, the complement of E. Prove that:

(a) $\partial E = \overline{E} \cap \overline{E^c}$,
(b) $x \in \partial E \iff x \in \overline{E} \setminus E^{\mathrm{o}}$,
(c) ∂E is a closed set,
(d) E is closed $\iff \partial E \subset E$.

Proof. Let X be any metric space. Suppose $E \subset X$.

(a) Then

$$\begin{aligned} x \in \partial E &\iff \forall r > 0, N_r(x) \cap E \neq \emptyset \wedge N_r(x) \cap E^c \neq \emptyset \\ &\iff x \in \overline{E} \wedge x \in \overline{E^c} \iff x \in \overline{E} \cap \overline{E^c}; \end{aligned}$$

hence, $\partial E = \overline{E} \cap \overline{E^c}$.

(b) Then

$$\begin{aligned} x \in \partial E &\iff \forall r > 0, N_r(x) \cap E \neq \emptyset \wedge N_r(x) \not\subset E \\ &\iff x \in \overline{E} \wedge x \notin E^{\mathrm{o}} \iff x \in \overline{E} \setminus E^{\mathrm{o}}; \end{aligned}$$

hence, $x \in \partial E \iff x \in \overline{E} \setminus \overline{E^c}$.

(c) By Theorems 2.24 and 2.27, $\overline{E}$ and $\overline{E^c}$ are closed subsets (Theorem 2.27); hence, $\partial E = \overline{E} \cap \overline{E^c}$ is a closed subset (Theorem 2.24).

(d) By Theorem 2.27,

$$E \text{ is closed} \implies E = \overline{E} \implies \partial E \subset \overline{E} = E \implies \partial E \subset E;$$

however,

$$\partial E \subset E \implies E \supset E^{\mathrm{o}} \cup \partial E = \overline{E} \implies E = \overline{E} \implies E \text{ is closed},$$

so

$$E \text{ is closed} \iff \partial E \subset E.$$

$\square$

Problem 6. Prove that every open set in $\mathbb{R}$ is the union of a countable collection of disjoint open intervals.

Proof. Let E be any open set in $\mathbb{R}$. We want to show that E is a union of disjoint open intervals. Assume $x \in E$. Define the sets

$$F_x := \{y \in \mathbb{R} : y \leq x \wedge [y, x] \subset E\}$$

and

$$U_x := \{y \in \mathbb{R} : y \geq x \wedge [x, y] \subset E\}.$$

Note that $F_x, U_x \subset E$. Given E is open, $\exists \epsilon > 0$ such that

$$[x-\epsilon, x+\epsilon] \subset E \implies x-\epsilon \in F_x \wedge x+\epsilon \in U_x;$$

hence, F_x and U_x both contain elements other than x. Make $c_x := \inf F_x \geq -\infty$ and $d_x := \sup U_x \leq \infty$.

Claim 1: $(c_x, x] \subset E$ and $[x, d_x) \subset E$.

If $y \in (c_x, x)$, then $\exists y' \in F_x$ such that $c_x < y' < y$ (else, $c_x \neq \inf F_x$); hence, $[y', x] \subset E$ and $y \in E$ (moreover, $y \in F_x$). Similarly, $[x, d_x) \subset E$.

Claim 2: $c_x \notin E$ and $d_x \notin E$.

Suppose, for obtaining a contradiction, $d_x \in E$. If $d_x < \infty$, then $\exists \epsilon > 0$ such that

$$[x, d_x+\epsilon] = [x, d_x] \cup [d_x, d_x+\epsilon] \subset E \implies d_x+\epsilon \in U_x;$$

hence, d_x cannot be an upper bound of U_x, so $d_x \notin E$. If $d_x = \infty$, then $d_x \notin E$. Similarly, $c_x \notin E$.

Claim 3: $(c_x, x] = F_x$ and $[x, d_x) = U_x$.

By Claim 1, $(c_x, x] \subset F_x$. If $y \in F_x$, then $c_x \leq y \leq x$; however, $c_x \notin F_x$, so

$$c_x < y \leq x \implies y \in (c_x, x]$$

and $F_x \subset (c_x, x]$. Indeed, $(c_x, x] = F_x$. Similarly, $[x, d_x) = U_x$.

Define the set $E_x := (c_x, d_x) = F_x \cup U_x$. Thus,

$$x \in E \iff x \in E_x \iff x \in \bigcup_{x \in E} E_x;$$

hence,

$$E = \bigcup_{x \in E} E_x.$$

Define the collection

$$\mathcal{U} := \{E_x \subset \mathbb{R} : x \in E\}.$$

Claim 4: If $x, y \in E$, then either $E_x = E_y$ or $E_x \cap E_y = \emptyset$.

If $x, y \in E$, then either $E_x = E_y$ or $E_x \neq E_y$. Suppose $E_x \neq E_y$. Make $(c, d) = E_x$ and $(e, f) = E_y$. Without loss of generality, fix $c \leq e$. If $c < e$, then

$$e < d \implies e \in E_x \subset E;$$

however, $e \notin E$ (Claim 2), so

$$e \geq d \implies E_x \cap E_y = \emptyset.$$

If $c = e$, then $d \neq f$; hence, either $d < f$ or $d > f$. Without loss of generality, take $d < f$. Then $d \in E_y \subset E$; however, $d \notin E$ (Claim 2), a contradiction. Indeed,

$$E_x \neq E_y \implies E_x \cap E_y = \emptyset.$$

Thus, $\mathcal{U}$ is a collection of disjoint open intervals; hence, E can be written as a union of disjoint open intervals. We want to show that $\mathcal{U}$ is countable. By the Axiom of Choice, we can construct a function $f : \mathcal{U} \to \mathbb{Q}$ such that

$$f(S) \in S \cap \mathbb{Q}.$$

Claim 5: f is $1-1$.

If $S, T \in \mathcal{U}$, then

$$\begin{aligned} f(S) = f(T) &\implies \exists q \in S \cap T \text{ (namely, } q := f(S) = f(T)) \\ &\implies S \cap T \neq \emptyset \\ &\implies S = T \text{ (Claim 4);} \end{aligned}$$

hence, f is $1 - 1$.

Thus, $|\mathcal{U}| < |\mathbb{Q}| = |\mathbb{N}|$; hence, $\mathcal{U}$ is countable. □