The Hamburger Moment Problem

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Abstract

In this project we examine the Hamburger Moment Problem, which asks when can we find a probability measure on the real line with a given sequence of moments. We approach this question through the lens of linear operators and spectral theory, providing an exposition along the way of the theory of Hilbert space adjoints, variants of the spectral theorem, as well as the von Neumann theory of self-adjoint extensions for unbounded operators.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Bence Szilágyi*)

To my family, much love and thank you all for supporting me unconditionally.

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Chapter 1

Preliminaries and introduction to the Hamburger Moment Problem

When dealing with a Borel measure μ on the real line $\mathbb{R}$, its sequence of moments $(\gamma_n)_{n=0}^{\infty}$ defined as

$$\gamma_n := \int_{\mathbb{R}} x^n \, d\mu(x)$$

is an object of frequent inquiry. Its moments associate to any Borel measure a sequence of real numbers. The Hamburger Moment Problem then asks the following natural question: Suppose we are given a sequence $(\gamma_n)_{n=0}^{\infty}$ of real numbers, does there exist a Borel measure μ on $\mathbb{R}$, such that for all $n \in \mathbb{N}$, γ_n is the nth moment of μ, and if such a measure exists, is it unique?

It turns out that in order to answer these questions, it is helpful to define a sesquilinear form $\mathcal{H} : \mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}$ on the complex polynomials $\mathbb{C}[X]$ that acts on $P(X) := \sum_{k=0}^{d_P} a_k X^k$ and $Q(X) := \sum_{k=0}^{d_Q} b_k X^k$ by

$$\mathcal{H} : (P, Q) \mapsto \mathcal{H}(P, Q) := \sum_{i=0}^{d_P} \sum_{j=0}^{d_Q} a_i \bar{b}_i \gamma_{i+j}.$$

We will later see that under certain conditions this defines a semi inner product on the polynomials, which we will use to construct a Hilbert space. Next, we define a linear operator A on the complex polynomials as $A : \mathbb{C}[X] \to \mathbb{C}[X] : P \mapsto XP$. These constructions will yield the following answers to the questions posed by the Hamburger Moment Problem: If A is bounded, there exists a unique Borel measure μ on $\mathbb{R}$ with moments γ_n if and only if $\mathcal{H}$ is a semi inner product on $\mathbb{C}[X]$. If A is unbounded the question of uniqueness becomes much more difficult to answer, and is beyond the scope of this project, however the answer to the question of existence remains the same, although more theory is required to show this than in the bounded case.

We will use several forms of the spectral theorem to arrive at these solutions, as well as the von Neumann theory of self-adjoint extensions for unbounded operators. Moment problems provide a good motivation for the exposition of this theory, as they provide concrete examples for many concepts that are introduced. They were also an important historical motivator for the development of functional analysis. Beside spectral theorems and self-adjoint extension theory, we make use of the Riesz-Markov-Kakutani Representation Theorem, which we quote almost verbatim from [1] without proof.

In chapters 2, 3, and 5, including in most places the method of proof, we follow the treatment of the theory by [2] quite closely, supplemented by [3], [4], and [5]. Additionally, our approach to the question of existence in the Hamburger Moment Problem is largely similar to those outlined in [4] and [5], with the notable difference that we do not develop the general unbounded functional calculus. Instead we use the functional calculus for bounded normal operators to show the relevant results directly. Wherever we borrow ideas or arguments in a proof, they will be referenced at the beginning of said proof. We will use basic results from the Edinburgh undergraduate curriculum without comment, deeper results will be referenced as usual.

We use the notation $\mathscr{B}(X)$ to denote the set of bounded linear operators on a normed vector space X. Additionally, our inner products are linear in the first argument and antilinear in the second argument. We will use the notation $C(X)$ to refer to continuous functions $X \to \mathbb{C}$. Unless otherwise stated, we use the convention $0 \in \mathbb{N}$. The following results are stated for reference:

Theorem 1.1 (BLT). *Suppose $T : X \to Y$ is a bounded linear operator, X a normed vector space, Y a complete normed vector space. Let $\tilde{X}$ be the completion of X. T can be uniquely extended to a bounded linear operator $\tilde{T} : \tilde{X} \to Y$.*

Theorem 1.2 (Heine-Borel). *If $S \subseteq \mathbb{R}$ is closed and bounded, then S is compact.*

Theorem 1.3 (Weierstrass Approximation). *For any continuous function $f : [a, b] \to \mathbb{R}$ on a closed (possibly unbounded) interval $[a, b]$ and $\epsilon > 0$ there exists a polynomial $P \in \mathbb{R}[X]$ such that $\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon$.*

Theorem 1.4 (Riesz-Markov-Kakutani [1]). *Let X be a locally compact Hausdorff space, and ℓ a positive linear functional on $C_c(X)$ the space of compactly supported, continuous, complex-valued functions on X. Then there exists a sigma-algebra $\mathbb{B}$ on X which contains all the Borel sets of X, and there exists a unique positive measure μ on $\mathbb{B}$ representing ℓ in the sense that $\ell(f) = \int_X f \, d\mu$ for all $f \in C_c(X)$. Also, μ is Borel-regular and takes finite values on all compact subsets of X.*

Theorem 1.5 (Closed Graph [2]). *A linear map $T : X \to Y$ between Banach spaces X, Y is bounded if and only if the graph of T is closed.*

Theorem 1.6 (Riesz Representation). *For any bounded linear functional $\phi : \mathscr{H} \to \mathbb{C}$ on a Hilbert space $\mathscr{H}$ there exists a unique element $y \in \mathscr{H}$ such that $\forall x \in \mathscr{H} : \phi(x) = \langle y, x\rangle$.*

Theorem 1.7 (Stone-Weierstrass (variant)). *The polynomials over a compact topological space X are dense in the of continuous functions $X \to \mathbb{C}$.*

Chapter 2

Adjoints and spectra of bounded linear operators

As mentioned before, the theory of linear operators is intimately related to the Moment Problem. In this chapter we go over important properties of bounded operators as well as the concept of resolvent sets and spectra. We will use the facts developed here extensively in later chapters. Of particular interest are self-adjoint operators, as we will see that the diagonalisation of finite dimensional Hermitian matrices can be generalised to self-adjoint linear operators on infinite dimensional spaces. The spectrum, the analogue of the set of eigenvalues in the case of bounded linear operators, remains a subset of the real line. Also of interest are positive operators, which are always self-adjoint in a Hilbert space over $\mathbb{C}$, and will be used to define positive linear functionals of the form $\langle Tv, w\rangle$. The Riesz-Markov-Kakutani Theorem, which we only quote in this project, tells us that positive linear functionals correspond to measures on the real line. In relation to the Moment Problem, this is in fact how we will acquire solutions, which will turn out to be so-called spectral measures, supported on the spectrum of the aforementioned operator A.

Definition 2.1. *Let X, Y be normed vector spaces, a linear map $T : X \to Y$ is bounded if $\sup_{x\in X} \frac{\|Tx\|}{\|x\|} < \infty$, and then $\|T\| := \sup_{x\in X} \frac{\|Tx\|}{\|x\|}$, such T is continuous.*

Definition 2.2. *Let $T \in \mathscr{B}(X)$ on a Banach space X. Then $\lambda \in \mathbb{C}$ is in the resolvent set $\rho(T)$ of T if $\lambda I - T$ is a bijection with a bounded inverse, and the operator $R_\lambda(T) = (\lambda I - T)^{-1}$ called the resolvent of T at λ. If $\lambda \notin \rho(T)$ then we say λ is in the spectrum $\sigma(T)$ of T. We further split the spectrum into three disjoint sets.*

The point spectrum of T consists of those $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is not one-to-one. The continuous spectrum consists of $\lambda_c \in \mathbb{C}$ such that $T - \lambda_c I$ is one-to-one but not onto, and $\mathrm{Ran}(T - \lambda_c I)$ is dense in X. And the residual spectrum consists of $\lambda_r \in \mathbb{C}$ such that $T - \lambda_r I$ is one-to-one but not onto, and $\mathrm{Ran}(T - \lambda_r I)$ is not dense in X.

Definition 2.3. *The adjoint of $A \in \mathscr{B}(\mathscr{H})$ where $\mathscr{H}$ is a Hilbert space is $A^* \in \mathscr{B}(\mathscr{H})$ such that $\forall x, y \in \mathscr{H} : \langle Ax, y\rangle = \langle x, A^*y\rangle$. A is self-adjoint if $A^* = A$.*

The following fact justifies the use of the definite article in the definition of the adjoint.

Theorem 2.4. *Every $T \in \mathscr{B}(\mathscr{H})$ has a unique adjoint $T^* \in \mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space.*

We recall that adjoints satisfy the following basic properties, which we will frequently use:

Theorem 2.5. *Consider $T, S \in \mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space. Then the map $\mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H}) : T \mapsto T^*$, which is well-defined by the previous theorem, is a conjugate linear isometric isomorphism. The following equalities hold: $(TS)^* = S^*T^*$, $(T^*)^* = T$, and $\|T^*T\| = \|T\|^2$. And if T has a bounded inverse T^{-1}, then T^* has a bounded inverse and $(T^*)^{-1} = (T^{-1})^*$.*

Now we need to introduce analytic operator-valued functions, the theory of which we will use extensively in the remainder of this chapter. Let $\mathscr{H}$ be a Hilbert space, then a map $F : D \to \mathscr{B}(\mathscr{H})$ defined on an open subset $D \subseteq \mathbb{C}$ of the complex plane is analytic at $z_0 \in D$ if there exists a sequence $(T_n)_{n=1}^{\infty} \subset \mathscr{B}(\mathscr{H})$ of operators such that the series $F(z) = \sum_{n=0}^{\infty}(z - z_0)^n T_n$ converges in the space $\mathscr{B}(\mathscr{H})$ for all $z \in \{z \in \mathbb{C} \mid |z - z_0| < \delta\}$ for some $\delta > 0$. We say that F is analytic on D if F is analytic at every $z_0 \in D$. This leads to a generalisation of complex analysis to operator-valued analytic functions, however the proof is beyond the scope of the present project. See [6] for more details. From here onwards we will assume that classic theorems from complex analysis can be invoked on operator-valued analytic functions.

Theorem 2.6. *Let $\mathscr{H}$ be a Hilbert space, and $T \in \mathscr{B}(\mathscr{H})$. Then $\rho(T) \supset \{\lambda \in \mathbb{C} \,|\, \lambda > \|A\|\}$ and $\rho(T)$ is an open subset of $\mathbb{C}$. Additionally, the mapping of $\lambda \in \rho(T)$ to the resolvent, $R : \lambda \mapsto R_\lambda(T) := (\lambda I - T)^{-1}$ is an analytic operator-valued function.*

Proof [2] [3]. Let $\lambda_0 \in \rho(T)$, and rewrite

$$\lambda I - T = (\lambda_0 I - T)(I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}).$$

Now suppose $\|(\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}\| < 1$, or equivalently

$$|\lambda_0 - \lambda| < \frac{1}{\|(\lambda_0 I - T)^{-1}\|} =: C.$$

Then the Taylor series $I + \sum_{n=1}^{\infty}\left(\frac{\lambda_0 - \lambda}{\lambda_0 - T}\right)^n$ converges in the space $\mathscr{B}(\mathscr{H})$ by the usual geometric series trick in complex analysis, and

$$(\lambda I - T)^{-1} = (I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1})^{-1} = I + \sum_{n=1}^{\infty}\left(\frac{\lambda_0 - \lambda}{\lambda_0 - T}\right)^n.$$

Hence we can write

$$R_\lambda(T) := (\lambda I - T)^{-1} = \left(I + \sum_{n=1}^{\infty}\left(\frac{\lambda_0 - \lambda}{\lambda_0 - T}\right)^n\right)^{-1}(\lambda_0 I - T)^{-1}.$$

So any λ within the open disc of radius C around λ_0 is contained in $\rho(T)$, and R is an analytic operator-valued function on $\rho(T)$, since $R_\lambda(T)$ has a power series expansion there. □

Corollary 2.7. *Let $\mathscr{H}, T$ as above, and $\lambda, \mu \in \rho(T)$. Then $R_\lambda(T), R_\mu(T)$ commute: $R_\lambda(T)R_\mu(T) = R_\mu(T)R_\lambda(T)$, and the First Resolvent Formula holds:*

$$R_\lambda(T) - R_\mu(T) = (\mu - \lambda)R_\mu(T)R_\lambda(T).$$

Proof [2]. Since $R_\lambda(T) = (\lambda I - T)^{-1}$, we have

$$\begin{aligned} R_\lambda(T) - R_\mu(T) &= R_\lambda(T)I - IR_\mu(T) \\ &= R_\lambda(T)(\mu I - T)R_\mu(T) - R_\lambda(T)(\lambda I - T)R_\mu(T) \\ &= (\mu - \lambda)R_\lambda(T)R_\mu(T), \end{aligned}$$

and

$$-(R_\lambda(T) - R_\mu(T)) = R_\mu(T) - R_\lambda(T) = (\lambda - \mu)R_\mu(T)R_\lambda(T),$$

so

$$(\mu - \lambda)(R_\lambda(T)R_\mu(T) - R_\mu(T)R_\lambda(T)) = 0,$$

hence $R_\lambda(T)R_\mu(T) = R_\mu(T)R_\lambda(T)$. □

Corollary 2.8. *Let $T \in \mathscr{B}(\mathscr{H})$ where $\mathscr{H}$ is a Hilbert space. Then $\sigma(T) \neq \emptyset$.*

Proof. Consider $\lambda \in \mathbb{C}$ such that $|\lambda| > \|T\|$. Then

$$R_\lambda(T) = (\lambda I - T)^{-1} = \frac{1}{\lambda}\left(I - \frac{1}{\lambda}T\right)^{-1} = \frac{1}{\lambda}\left(I + \sum_{n=0}^{\infty} \lambda^{-n}T^n\right),$$

where the second equality is again by the geometric series expansion. Hence $\lim_{|\lambda|\to\infty} \|R_\lambda(T)\| = 0$. Then by Liouville's Theorem if R were analytic on $\mathbb{C}$, it would have to be the case that $R_\lambda(T) = 0$ everywhere, which is false. Thus $\rho(T) \neq \mathbb{C}$, i.e. $\sigma(T) = \mathbb{C} \setminus \rho(T) \neq \emptyset$. □

Definition 2.9. *The spectral radius $r(T)$ of $T \in \mathscr{B}(X)$ on a Banach space X is $r(T) := \sup_{\lambda\in\sigma(T)} |\lambda|$.*

Theorem 2.10. *Let $A \in \mathscr{B}(X)$ where X is a Banach space. Then $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$. If X is a Hilbert space and A is self-adjoint then $r(A) = \|A\|$.*

Proof [2] [3]. First we show that $\lim_{n\to\infty} \|A^n\|^{1/n}$ exists. Consider the series $(a_n)_{n=1}^{\infty}$ defined by $a_n := \log \|A^n\|$, and recall that $\|A^{m+n}\| \leq \|A^m\|\|A^n\|$, so

$$a_{m+n} \leq a_m + a_n.$$

Let $n, m, p, q \in \mathbb{N}$ be such that $n = pm + q, 0 \leq q < m$. Then $a_n \leq a_{pm} + a_q$ and $a_{pm} \leq a_{(p-1)m} + a_m \leq \cdots \leq pa_m$, so

$$a_n \leq pa_m + a_q.$$

Now consider $\frac{a_n}{n} \leq \frac{p}{n} a_m + \frac{1}{n} a_q$ as $n \to \infty$ and m, q are fixed. Since $n = pm+q$ and $0 \leq q < m$, we have $\frac{p}{n} = \frac{1}{n}\frac{n-q}{m} = \frac{1}{m} - \frac{q}{mn} \to \frac{1}{m}$ as $n \to \infty$, so $\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m}$. Now let $m \to \infty$ and we can see that $\limsup_{n\to\infty} \frac{a_n}{n} \leq \liminf_{m\to\infty} \frac{a_m}{m}$. Thus $\left(\frac{a_n}{n}\right)_{n=1}^{\infty}$ has a finite limit, and by the monotonicity of the logarithm function $\lim_{n\to\infty} \|A^n\|^{1/n} < \infty$ exists.

Recall that for $|\lambda| > \|A\|$ we have

$$R_\lambda(A) = (\lambda I - A)^{-1} = \frac{1}{\lambda}\left(I - \frac{1}{\lambda}A\right)^{-1} = \frac{1}{\lambda}\left(I + \sum_{n=0}^{\infty} \lambda^{-n} A^n\right)$$

from the proof of 2.8, and note that the final expression is the Laurent series of $R_\lambda(A)$, as a function of λ, around ∞. Let C denote the radius of convergence of this Laurent series. Now note that $S := \mathbb{C} \setminus \{\lambda \in \mathbb{C} \mid |\lambda| \leq r(A)\} \subseteq \rho(A)$, and we have shown that $R : \lambda \mapsto R_\lambda(A)$ is analytic at all $\lambda \in \rho(A)$, so its series expansion converges outside the disc $\{\lambda \in \mathbb{C} \mid |\lambda| \leq r(A)\}$, hence $\frac{1}{r(A)} \leq C$ through a change of variable to $\frac{1}{\lambda}$ and considering the series around 0.

Additionally, we know that any Laurent series must converge absolutely on its disc of convergence, i.e. $R_\lambda(A)$ must exist there. This eliminates all points of the spectrum from being contained in the disc of convergence, so $C \geq \frac{1}{r(A)}$ as none of $\left\{\frac{1}{\lambda} \mid \lambda \in \mathbb{C}, \lambda \leq r(A)\right\}$ can lie on the disc of convergence. Thus $r(A) = \frac{1}{C}$.

Now recall the Cauchy-Hadamard Theorem, which states that the radius of convergence C of a power series $f(z) = \sum_{n=0}^{\infty} b_n(z-a)^n$ with $b_n, a \in \mathbb{C}$ is given by $\frac{1}{C} = \limsup_{n\to\infty} |b_n|^{1/n}$. In the case of our power series $\left(I + \sum_{n=0}^{\infty} \left(\frac{1}{\lambda}\right)^n A^n\right)$ in the variable $\frac{1}{\lambda}$, the coefficients b_n are replaced by A^n, and the operator-valued Cauchy-Hadamard Theorem gives $\frac{1}{C} = \limsup_{n\to\infty} \|A^n\|^{1/n}$. Finally, have already seen that $\lim_{n\to\infty} \|A^n\|^{1/n}$ exists, in which case it must be equal to $\frac{1}{C}$ by our previous argument.

Now let X be a Hilbert space and A be self-adjoint. Then $\|A\|^2 = \|A^*A\| = \|A^2\|$ where the first equality is a standard result for adjoints. By induction $\|A^{2^n}\| = \|A\|^{2^n}$ for $n \in \mathbb{N}$. Thus $r(A) = \lim_{k\to\infty} \|A^k\|^{1/k} = \lim_{n\to\infty} \|A^{2^n}\|^{2^{-n}} = \|A\|$. $\square$

Corollary 2.11. *The spectrum of a bounded self-adjoint operator on a Hilbert space is compact.*

Proof [2]. Let $\mathscr{H}$ be a Hilbert space and $A \in \mathscr{B}(\mathscr{H})$ self-adjoint. We have seen that $\sigma(A)$ is closed, and $r(A) = \|A\|$ so $\sigma(A)$ is bounded. Then the Heine-Borel theorem implies that $\sigma(A)$ is compact as a subset of $\mathbb{R}$. $\square$

Lemma 2.12. *For a (not necessarily bounded) linear operator T on a Hilbert space $\mathscr{H}$, we have* $\ker(T^*) = \operatorname{Ran}(T)^\perp$.

Proof [7]. In order to show this, we need to see that $\forall \phi \in \ker(T^*) : \forall \psi \in \operatorname{Ran}(T) : \langle \phi, \psi \rangle = 0$. Note that $\forall \psi \in \operatorname{Ran}(T) : \exists \xi \in \mathscr{H} : \psi = T\xi$. And then for $\phi \in \ker T^*$, $\psi \in \operatorname{Ran}(T)$ we have $\langle \phi, \psi \rangle = \langle T\xi, \psi \rangle = \langle \xi, T^*\psi \rangle = \langle \xi, 0 \rangle = 0$. $\square$

Lemma 2.13. *If λ is in the residual spectrum of a bounded operator A on a Hilbert space $\mathscr{H}$, then $\overline{\lambda}$ is in the point spectrum of A^*.*

Proof [2]. By definition $\overline{\operatorname{Ran}(A-\lambda I)} \neq \mathscr{H}$. Then $\overline{\operatorname{Ran}(A-\lambda I)}$ is a closed proper subspace, so there is some $x \in \operatorname{Ran}(A-\lambda I)^{\perp}$. So by the previous lemma $x \in \ker(A^* - \overline{\lambda} I)$. $\square$

Theorem 2.14. *If $T \in \mathscr{B}(\mathscr{H})$ is a self-adjoint operator on a Hilbert space $\mathscr{H}$, then:*

a) T has no residual spectrum.

b) $\sigma(T) \subseteq \mathbb{R}$.

Proof [2]. Let $\lambda, \mu \in \mathbb{R}$ so

$$\begin{aligned}
\|(T-(\lambda+i\mu))x\|^2 &= \|(T-\lambda I)x\|^2 + \|-i\mu x\|^2 + \langle Tx, -i\mu x\rangle + \langle -i\mu x, Tx\rangle \\
&= \|(T-\lambda I)x\|^2 + \mu^2\|x\|^2 \\
&\geq \mu^2\|x\|^2.
\end{aligned}$$

Suppose $\mu \neq 0$. Then the kernel of $T-(\lambda+i\mu)I$ is empty and thus $\lambda+i\mu$ is not an eigenvalue. So $T-(\lambda+i\mu)I$ has a bounded inverse on its range. Additionally, in this case $\operatorname{Ran}(T-(\lambda+i\mu)I)$ is closed, so $\lambda+i\mu$ cannot be in the continuous spectrum. The last possibility is that $\lambda+i\mu$ is in the residual spectrum of T, but T is self-adjoint, so by the previous lemma $\lambda+i\mu$ is also in the point spectrum of $T^* = T$, which is a contradiction. Hence $\mu = 0$, and by the previous statement the residual spectrum of T must be empty. $\square$

Definition 2.15. *An operator $B \in \mathscr{B}(\mathscr{H})$ on a Hilbert space $\mathscr{H}$ is positive if $b := \langle Bx, x\rangle \geq 0$ (so $b \in \mathbb{R}$) for all $x \in \mathscr{H}$, and in this case we write $B \geq 0$. Also we write $A \geq B$ if $A - B \geq 0$.*

Theorem 2.16. *Every positive linear operator B on a complex Hilbert space $\mathscr{H}$ is self-adjoint. Additionally, for any linear operator A on a real or complex Hilbert space $\mathscr{H}$ we have $A^*A \geq 0$.*

Proof [2]. Since $\langle Bx, x\rangle \in \mathbb{R}$, we have $\langle x, Bx\rangle = \overline{\langle Bx, x\rangle} = \langle Bx, x\rangle$ for all $x \in \mathscr{H}$. We extend this results using the Polarization Identity for complex Hilbert spaces, which gives $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathscr{H}$, showing that B is self-adjoint. To show that $A^*A \geq 0$, simply note $\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0$. $\square$

Remark 2.17. *Theorem 2.16 holds for general, so possibly unbounded, linear operators A, B. We will introduce unbounded operators in chapter 5.*

Lemma 2.18. *If $T \in \mathscr{B}(\mathscr{H})$ for a Hilbert space $\mathscr{H}$ and there is a sequence $(\psi_n)_{n=1}^{\infty} \subset \mathscr{H}$ such that $\|\psi_n\| = 1$ for all n and $(T-\lambda)\psi_n \to 0$ as $n \to 0$, then $\lambda \in \sigma(T)$.*

Proof. If $\lambda \in \rho(T)$ then $(T-\lambda)^{-1} \in \mathscr{B}(\mathscr{H})$ but $\|(T-\lambda)^{-1}\| \geq \frac{\|(T-\lambda)^{-1}((T-\lambda)\psi_n)\|}{\|(T-\lambda)\psi_n\|} = \frac{\|\psi_n\|}{\|(T-\lambda)\psi_n\|}$ for all $n \in \mathbb{N}$, but $\|\psi_n\| = 1$ and $\|(T-\lambda)\psi_n\| \to 0$ as $n \to \infty$, thus $(T-\lambda)^{-1}$ cannot be a bounded operator. Therefore $\lambda \in \sigma(T)$. $\square$

Chapter 3

The continuous functional calculus for bounded self-adjoint operators

In order to proceed with the moment problem, we first demonstrate that continuous functions on the spectrum of a bounded self-adjoint operator can, in a sense, be evaluated at the operator, giving rise to a functional calculus for bounded self-adjoint operators. In chapter 6 we will extend this functional calculus first to a pair of commuting bounded self-adjoint operators, and later to normal bounded operators. Such a functional calculus will allow us to define $f \mapsto \langle f(A)\psi, \psi \rangle$, and through the Riesz-Markov-Kakutani Representation Theorem will give rise to the so-called spectral measures of A, supported on the spectrum. We will first state the main theorem of this chapter, then discuss the main ideas behind the proof before proving it.

Theorem 3.1. *For a commuting self-adjoint operator A on a Hilbert space $\mathscr{H}$, and $C(\sigma(A))$ the space of continuous functions $\sigma(A) \to \mathbb{C}$ under the supremum norm, there is a unique map $\phi : C(\sigma(A)) \to \mathscr{B}(\mathscr{H})$ such that:*

- *ϕ is a $*$-homomorphism, which means the following:*
  - $\phi(f(A)g(A)) = \phi(f(A))\phi(g(A))$,
  - $\phi(\lambda f) = \lambda\phi(f)$,
  - $\phi(1) = I$, *and*
  - $\phi(\overline{f}) = \phi(f)^*$.
- *ϕ is an isometry, and hence continuous.*
- *ϕ respects the identity, i.e. if $f(x) = x$ then $\phi(f(A)) = A$.*
- *ϕ has the spectral mapping property: $\sigma(\phi(f)) = \{f(\lambda) \mid \lambda \in \sigma(A)\}$.*
- *If $f \geq 0$ then $\phi(f) \geq 0$, i.e. $\phi(f)$ is positive and thus self-adjoint.*
- *If $A\psi = \lambda\psi$, then $\phi(f)\psi = f(\lambda)\psi$ for $\lambda \in \sigma(A)$, $\psi \in \mathscr{H}$.*

We will use both $f(A)$ and $\phi(f)$ interchangeably to mean the function $f \in C(\sigma(A))$ evaluated at the operator A through the functional calculus.

We start our construction of this functional calculus by defining polynomials of bounded self-adjoint operators. The mapping of polynomials on the spectrum of a bounded self-adjoint operator to bounded operators ends up having the spectral mapping property, as well as being an isometry. This will enable us to extend our mapping to the space of continuous functions on the spectrum through the BLT Theorem.

Suppose we had a self-adjoint bounded operator $A : \mathscr{H} \to \mathscr{H}$ and a $*$-homomorphism $\phi'_A : \mathbb{C}[X] \to \mathscr{B}(\mathscr{H})$ such that $\phi'_A(X) = A$. The properties of $*$-homomorphisms then fully determine what ϕ'_A must be, $\phi'_A : P = \sum_{i=0}^d a_i X^i \mapsto P(A) := \sum_{i=0}^d a_i A^i$ where $A^i := \underbrace{A \circ \cdots \circ A}_{i \text{ times}}$.

Lemma 3.2 (Spectral Mapping)**.** *For $P \in \mathbb{C}[X]$ and $P(A)$ as before, we have $\sigma(P(A)) = \{P(\lambda) \,|\, \lambda \in \sigma(A)\}$.*

Proof [2]. First we show $P(\sigma(A)) \subseteq \sigma(P(A))$. Suppose $\lambda \in \sigma(A)$. Then $x = \lambda$ is a root of the polynomial $P(x) - P(\lambda)$. Therefore we can factor $P(x) - P(\lambda) = (x - \lambda)Q(x)$. Evaluating this polynomial at $x = A$, we can see it has $A - \lambda$ as a factor $P(A) - P(\lambda) = (A - \lambda)Q(A)$. And $\lambda \in \sigma(A)$, therefore $A - \lambda$ is not invertible, so $P(A) - P(\lambda)$ is not invertible either. Hence $P(\lambda) \in \sigma(P(A))$.

Next we show $\sigma(P(A)) \subseteq P(\sigma(A))$. Suppose $\rho \in \sigma(P(A))$ and $\lambda_1, \ldots, \lambda_n$ are the roots of $P(x)-\rho$. Then we have the factoring $P(x)-\rho = \gamma(x-\lambda_1)\ldots(x-\lambda_n)$. So if none of the roots λ_i are in $\sigma(A)$ then the inverse $(P(x) - \rho)^{-1} = \gamma^{-1}(x - \lambda_1)^{-1} \ldots (x-\lambda_n)^{-1}$ exists, contradicting our assumption that $\rho \in \sigma(P(A))$. Hence $\lambda_i \in \sigma(A)$ for one or more $i \in \{1, \ldots, n\}$, and then $\rho = P(\lambda_i)$ for some such i. $\square$

Lemma 3.3. *For a bounded self-adjoint operator A we have the equality*

$$\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|.$$

Proof [2]. We calculate:

$$\begin{aligned}
\|P(A)\|^2 &= \|P(A)^* P(A)\| \\
&= \|(\overline{P}P)(A)\| \\
&= \sup_{\lambda \in \sigma(\overline{P}P(A))} |\lambda| && \text{by Thm. 2.10} \\
&= \sup_{\lambda \in \sigma(A)} |\overline{P}P(\lambda)| && \text{by Lemma 3.2} \\
&= \left(\sup_{\lambda \in \sigma(A)} |P(\lambda)| \right)^2 .
\end{aligned}$$

Since $((\overline{P}P)(A))^* = (\overline{\overline{P}P})(A) = (P\overline{P})(A) = (\overline{P}P)(A)$, and thus $\overline{P}P$ is self-adjoint. $\square$

Next we need to see that $\mathbb{C}[X]$ on $\sigma(A)$ is dense in $C(\sigma(A))$, as we want to extend ϕ to all of $C(\sigma(A))$. We can extend the Weierstrass Approximation theorem to complex-valued continuous functions and polynomials with complex coefficients as follows:

Corollary 3.4. *Let $f : [a, b] \to \mathbb{C}$ be continuous with $[a, b]$ a closed (possibly unbounded) interval and $\epsilon > 0$. There exists a polynomial $P \in \mathbb{C}[X]$ such that $\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon$.*

Proof. It is a standard fact from complex analysis that f can be written as $f = g + ih$ where $g, h : [a, b] \to \mathbb{R}$ are continuous. Then fix some $\epsilon > 0$, and find $Q, R \in \mathbb{R}[X]$ such that $\sup_{x\in[a,b]} |g(x) - Q(x)| < \frac{\epsilon}{2}$ and $\sup_{x\in[a,b]} |h(x) - R(x)| < \frac{\epsilon}{2}$. Let $P = Q + iR$, then $|f(x) - P(x)| \leq |g(x) - Q(x)| + |h(x) - R(x)|$, so $\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon$. □

Lemma 3.5. *The polynomials $\mathbb{C}[X]$ on $\sigma(A)$ are dense in $C(\sigma(A))$.*

Proof. We know that $\sigma(A)$ is bounded, so there exists some closed and bounded interval $[a, b]$ such that $\sigma(A) \subseteq [a, b]$. Then by the complex-valued generalisation of the Weierstrass Approximation Theorem, for any continuous $f : [a, b] \to \mathbb{C}$ there is some $P \in \mathbb{C}[X]$ such that $\sup_{x\in\sigma(A)} |f(x) - P(x)| \leq \sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon$. So the polynomials $\mathbb{C}[X]$ on $\sigma(A)$ are dense in $C(\sigma(A))$ under the supremum norm. □

Lemma 3.6. *$\phi : C(\sigma(A)) \to \mathscr{B}(\mathscr{H})$ is an isometric $*$-homomorphism, has the property that $A\psi = \lambda\psi$ for $\psi \in \mathscr{H}$ implies $\phi(f)\psi = f(\lambda)\psi$, and its restriction to real-valued functions is positive.*

Proof [2]. The isometry property of ϕ'_A implies that ϕ'_A is continuous. Then the BLT Theorem implies that ϕ'_A has a unique extension $\tilde{\phi} : C(\sigma(A)) \to \mathscr{B}(\mathscr{H})$ and that $\tilde{\phi}$ is also continuous. By continuity, $\tilde{\phi}$ is an isometric $*$-homomorphism. Since $\phi(P)\psi = P(\lambda)\psi$, we can again apply continuity to see that $A\psi = \lambda\psi$ implies $\phi(f)\psi = f(\lambda)\psi$ for $\lambda \in \sigma(A)$, $\psi \in \mathscr{H}$. Now if $f \geq 0$, i.e. f is real-valued and non-negative, then there exists some real-valued $g \in \mathbb{C}(\sigma(A))$ with $f = g^2$. Since g is real-valued, $\phi(g)$ is self-adjoint by the same argument as in the proof of Thm. 2.16, hence $\phi(f) = \phi(g)^2 \geq 0$ by Thm. 2.16. □

Theorem 3.7 (Spectral Mapping)**.** $\sigma(\phi(f)) = \{f(\lambda) \,|\, \lambda \in \sigma(A)\}$

Proof [2], [8]. Suppose $f \in C(\sigma(A))$ and $\lambda \notin \mathrm{Ran}(f) = f(\sigma(A))$ and let $g = (f - \lambda)^{-1}$. Then $g \in C(\sigma(A))$ since $f(x) - \lambda \neq 0$ for $x \in \sigma(A)$. We note that $\phi\left(\frac{1}{h}\right) = \phi(h)^{-1}$ since $I = \phi(1) = \phi\left(\frac{1}{h}h\right) = \phi\left(h\frac{1}{h}\right) = \phi\left(\frac{1}{h}\right)\phi(h) = \phi(h)\phi\left(\frac{1}{h}\right)$ for any $h \in C(\sigma(A))$ such that $0 \notin \mathrm{Ran}(h)$, therefore $\phi(g) = (\phi(f) - \lambda)^{-1}$. Thus $f(\sigma(A))^C \subseteq \sigma(f(A))^C$, i.e. $\sigma(f(A)) \subseteq f(\sigma(A))$.

If $\lambda \in \mathrm{Ran}(f) = f(\sigma(A))$ then we want to show that for any $\epsilon > 0$ there exists some $\psi \in \mathscr{H}$ with $\|\psi\| = 1$ and $\|(\phi(f) - \lambda)\psi\| < \epsilon$, after which we are done by Lemma 2.18. First, fix $\epsilon > 0$. Then note that since $\lambda \in f(\sigma(A))$, there exists some $x_0 \in \sigma(A)$ such that $f(x_0) - \lambda = 0$, and, by continuity of f, also $|f(x) - \lambda| < \frac{\epsilon}{2}$ in some neighbourhood $N(x_0) \subseteq \sigma(A)$ of x_0. Now let $g \in C(\sigma(A))$ such that g vanishes outside of the neighbourhood $N(x_0)$, $g(x) = 1$ and $|g(y)| \leq 1$ for $y \in \sigma(A)$. Then $\|(f(x) - \lambda)g(x)\| < \frac{\epsilon}{2}$, and since ϕ is an isometric $*$-homomorphism,

$\|(\phi(f)-\lambda)\phi(g)\| = \|\phi((f-\lambda)g)\| < \frac{\epsilon}{2}$ and $\|\phi(g)\| = 1$. That $\|\phi(g)\| = 1$ implies that there exists some $\xi \in \mathscr{H}$ with $\|\xi\| = 1$ and $|1 - \|\phi(g)\xi\|| < \frac{\epsilon}{2}$. So

$$\|(\phi(f)-\lambda)\phi(g)\xi\| < \frac{\epsilon}{2}\|\phi(g)\xi\| \leq \frac{\epsilon}{2}\left(1+\frac{\epsilon}{2}\right) = \frac{\epsilon}{2} + \left(\frac{\epsilon}{2}\right)^2 < \epsilon$$

(for $\epsilon < 2$, but in the case of $\epsilon \geq 2$ we can simply substitute 1 in the place of ϵ in $\frac{\epsilon}{2}\|\phi(g)\xi\|$). Now set $\psi = \frac{1}{\|\phi(g)\xi\|}\phi(g)\xi$ and we have what we wanted to show.

Finally let ψ_n be the ψ corresponding to $\epsilon = \frac{1}{n}$ in the previous part. This yields a sequence of unit vectors such that $\|(\phi(f)-\lambda)\psi_n\| \to 0$ as $n \to \infty$. Thus $\lambda \in \sigma(f(A))$, which shows that $f(\sigma(A)) \subseteq \sigma(f(A))$. □

And with this we have shown all the claimed properties of the continuous functional calculus for bounded self-adjoint operators. To finish this chapter, we show a special case of the spectral theorem for bounded self-adjoint operators. Morally speaking this provides us, through a vector $\psi \in \mathscr{H}$ that is cyclic in A (we define below what this means precisely), with a kind of inverse to the map ϕ corresponding to the functional calculus. However, first we recall the following definition.

Definition 3.8. *[1] For a compact space X, a linear functional $\ell : C(X) \to \mathbb{C} : f \mapsto \ell(f)$ is positive if and only if $f(X) \subseteq [0,\infty)$ implies $\ell(f) \in [0,\infty)$.*

Let A be a bounded self-adjoint operator on a Hilbert space $\mathscr{H}$, and $\psi \in \mathscr{H}$. Consider the bounded linear functional $\ell : C(\sigma(A)) \to \mathbb{C} : f \mapsto \langle f(A)\psi, \psi\rangle$. Additionally, we recall that ϕ is a positive operator on the real-valued continuous functions, meaning that $\ell : C(\sigma(A)) \to \mathbb{C}$ is a positive linear functional. The Riesz-Markov-Kakutani Representation Theorem then gives that there exists a unique, finite, positive, Borel-regular measure μ_ψ on $\sigma(A)$ satisfying $\langle f(A)\psi, \psi\rangle = \int_{\sigma(A)} f(\lambda)\, d\mu_\psi(\lambda)$.

Definition 3.9. *We call such μ_ψ the spectral measure of A corresponding to ψ.*

Remark 3.10. *Through the Polarization Identity we can define $f \mapsto \langle f(A)\phi, \psi\rangle$ for any $\phi, \psi \in \mathscr{H}$. We can extend the class of spectral measures to include the measures corresponding to these linear functionals. Such a spectral measure is denoted $\mu_{\phi,\psi}$.*

Definition 3.11. *As spectral measures are Borel measures, they allow us to extend the functional calculus to bounded Borel-measurable functions. Let g be a bounded Borel-measurable function on $\sigma(A)$ for some self-adjoint operator A on a Hilbert space $\mathscr{H}$. Now we can define $g(A)\phi$ for any $\phi \in \mathscr{H}$ by the Riesz Representation Theorem as the unique element $g(A)\phi$ representing the bounded linear functional $\ell_\phi : \psi \mapsto \int_{\sigma(A)} g(A)\, d\mu_{\phi,\psi} =: \langle g(A)\phi, \psi\rangle$. The properties of this extended functional calculus are omitted in the interest of brevity.*

Definition 3.12. *An element $\psi \in \mathscr{H}$ of a Hilbert space $\mathscr{H}$ is called cyclic with regard to an operator $A \in \mathscr{B}(\mathscr{H})$ if $\operatorname{span}\{\psi, A\psi, A^2\psi, \dots\}$ is dense in $\mathscr{H}$.*

Theorem 3.13 (Spectral Theorem, with cyclic vector). *Let A be a bounded self-adjoint operator with a cyclic vector $\psi \in \mathscr{H}$. Then there is a unitary operator $U : \mathscr{H} \to L^2(\sigma(A), \mu_\psi)$, where μ_ψ is the spectral measure corresponding to ψ, such that*

$$(UAU^{-1}f)(\lambda) = \lambda f(\lambda),$$

where equality is meant in the space $L^2(\sigma(A), \mu_\psi)$. In other words, $\mathscr{H}$ is isomorphic to $L^2(\sigma(A), \mu_\psi)$ through U.

Proof [2]. Define U by $U\phi(f)\psi := f$ for $f \in C(\sigma(A))$. This results in a well-defined map, since

$$\|\phi(f)\psi\|^2 = \langle \phi^*(f)\phi(f)\psi, \psi\rangle = \langle \phi(\overline{f}f)\psi, \psi\rangle = \int_{\sigma(A)} |f(\lambda)|^2 \, d\mu_\psi(\lambda).$$

Therefore, if $f = g$, μ_ψ-almost everywhere, then $\phi(f)\psi = \phi(g)\psi$. Thus U is well-defined on $D := \{\phi(f)\psi \,|\, f \in C(\sigma(A))\}$, and U is also norm-preserving. Recall that ψ is cyclic in A, so $\overline{D} = \mathscr{H}$ and then the BLT Theorem yields an isometric extension $U : \mathscr{H} \to L^2(\sigma(A), \mu_\psi)$, and because $C(\sigma(A))$ is dense in $L^2(\sigma(A), \mu_\psi)$, this extension must be onto, as $U : \mathscr{H} \to C(\sigma(A))$ is norm-preserving. Finally, if $f \in C(\sigma(A))$, then

$$\begin{aligned}(UAU^{-1}f)(\lambda) &= (UA\phi(f)\psi)(\lambda) \\ &= (U\phi(xf)\psi)(\lambda) \\ &= \lambda f(\lambda).\end{aligned}$$

But by continuity of U and ϕ this also holds for $f \in L^2(\sigma(A), \mu_\psi)$. □

Chapter 4

Uniqueness in the bounded case

Now we have everything in place to show necessary and sufficient conditions on the uniqueness of solutions to the Hamburger Moment Problem in the case that $\mathcal{H}$ is a semi inner product on $\mathbb{C}[X]$, and $A : \mathbb{C}[X] \to \mathbb{C}[X] : P \mapsto XP$ is bounded under the seminorm induced by $\mathcal{H}$. Necessary and sufficient conditions for when $\mathcal{H}$ is a semi inner product on $\mathbb{C}[X]$ will be given in Chapter 7.

Our main theorem for this chapter is:

Theorem 4.1. *Whenever $\mathcal{H}$ is a semi inner product on $\mathbb{C}[X]$, and $A : \mathbb{C}[X] \to \mathbb{C}[X] : P \mapsto XP$ is bounded under the seminorm induced by $\mathcal{H}$, and there exists a measure μ on $\mathbb{R}$ satisfying the conditions of the Hamburger Moment Problem, then μ is unique as such.*

The sketch of the proof goes as follows: We first show that given there exists a measure μ satisfying the Hamburger Moment Problem, A is bounded if and only if μ is compactly supported. Then we consider two measures μ and ν satisfying the conditions of the moment problem. We will find, through the special case of the Spectral Theorem 3.13, that both μ and ν must be equal to the spectral measure of A corresponding to the constant function 1, guaranteeing uniqueness.

Lemma 4.2. *A is bounded if and only if* $\sup_{n\in\mathbb{N}\cup\{0\}} \frac{m_{2(n+1)}}{m_{2n}} < \infty$.

Proof. We calculate:

$$\begin{aligned}
\|A\| &= \sup_{P\in\mathbb{C}[X]} \frac{\|AP\|}{\|P\|} \\
&= \left| \frac{\sum_{n=0}^{d}\sum_{k=0}^{d} a_n\overline{a_k}m_{n+k+2}}{\sum_{n=0}^{d}\sum_{k=0}^{d} a_n\overline{a_k}m_{n+k}} \right| \\
&\leq \sup_{n\in\mathbb{N}\cup\{0\}} \left| \frac{m_{2(n+1)}}{m_{2n}} \right| && \text{if this supremum is finite} \\
&= \sup_{n\in\mathbb{N}\cup\{0\}} \frac{\|AX^n\|}{\|X^n\|} \\
&\leq \left\| A\restriction_{\{X^n|n\in\mathbb{N}\cup\{0\}\}} \right\| \\
&\leq \|A\|.
\end{aligned}$$

Therefore $\|A\| = \sup_{n\in\mathbb{N}\cup\{0\}} \left|\frac{m_{2(n+1)}}{m_{2n}}\right|$ if $\sup_{n\in\mathbb{N}\cup\{0\}} \left|\frac{m_{2(n+1)}}{m_{2n}}\right| \in \mathbb{R}$ exists, and if it does not exist then A is unbounded. $\square$

Lemma 4.3. *Our linear operator $A : \mathbb{C}[X] \to \mathbb{C}[X] : P \mapsto XP$ is a bounded linear operator if and only if μ is compactly supported.*

Proof. First suppose that μ is compactly supported, and m_n is its nth moment. Then

$$\begin{aligned} m_{n+2} &= \int_{\mathbb{R}} x^{n+2}\, d\mu(x) = \int_{\mathbb{R}} x^2 x^n\, d\mu(x) \\ &\leq m_2|m_n| && \text{by Cauchy-Schwartz.} \end{aligned}$$

Hence $\frac{m_{n+2}}{m_n} \leq m_2$, and $0 \leq m_2 \in \mathbb{R}$ since μ is compactly supported.

Next, suppose A is a bounded linear operator. Then there is some $C \in \mathbb{R}$ such that $\|A\| \leq C$. For a contradiction, suppose that for all $C > 0$ the measure μ is supported outside $[-C, C]$, i.e. there exists some subset $E \subseteq \mathbb{R}$ such that $E \cap [-C, C] = \emptyset$ and $\mu(E) > 0$. Therefore $m_n \geq \int_E x^n\, d\mu(x) \geq \int_E (C+\varepsilon)^n\, d\mu(x) \geq \mu(E)(C+\varepsilon)^n > C^n$, where the last inequality holds for sufficiently large n. Now note that $m_0 = 1$, so $m_{2n} = \prod_{k=1}^{n} \frac{m_{2k}}{m_{2(k-1)}} \leq \sup_{n\in\mathbb{N}} \left|\frac{m_{2(n+1)}}{m_{2n}}\right|^n$. But this means $C \leq \sup_{n\in\mathbb{N}} \left|\frac{m_{2(n+1)}}{m_{2n}}\right|$ for all $C > 0$, therefore by Lemma 4.2 the operator A must be unbounded, contradicting our assumptions. $\square$

Theorem 4.4. *$C^{\mathbb{R}}([a, b])$ is dense in $L^p([a, b], \mu)$ where $1 \leq p < \infty$ and μ is a Borel measure on $\mathbb{R}$, and $C^{\mathbb{R}}([a, b])$ is the space of real-valued continuous functions on $[a, b]$.*

Proof. All functions in $L^p([a, b], \mu)$ arise as limits (under the corresponding L^p norm) of simple functions, thus all we need to show is that for any simple function $h(x) = \sum_{i=0}^{N} c_i \chi_{[a_i,b_i]}$ with $a_i \leq b_i$ for $i \in \{0, 1, \ldots, N\}$ on $[a, b]$, and $\epsilon > 0$, there exists some $f \in C^{\mathbb{R}}([a, b])$ such that $\|h - f\|_{L^p_\mu}^p = \int_a^b |h(x) - f(x)|^p\, d\mu(x) < \epsilon$. Let $f(x) = c_0\chi_{[a_0,b_0]} \sum_{i=1}^{N} \Big((c_{i-1} + \frac{x-a_i}{\delta_i}(c_i - c_{i-1})\chi_{(a_i,a_i+\delta_i)} + c_i\chi_{[a_i+\delta_i,b_i]}\Big)$, on $[a, b]$, where $\delta_i = \min\left(\frac{b_i-a_i}{2}, \frac{\epsilon}{N}\right)$ for $i \in \{1, 2, \ldots, N\}$. Then $\|h - f\|_{L^p_\mu}^p < \epsilon$. $\square$

Lemma 4.5. *The operator $A : \mathbb{C}[X] \to \mathbb{C}[X]$ is self-adjoint.*

Proof. Note that $\gamma_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Then

$$\langle P, AQ\rangle = \mathcal{H}(P, A(Q)) = \sum_{i=0}^{d}\sum_{j=0}^{d'} a_i\overline{b_j}\gamma_{i+j+1} = \mathcal{H}(A(P), Q) = \langle AP, Q\rangle.$$

Where $P = \sum_{i=0}^{d} a_i X^i, Q = \sum_{j=0}^{d'} b_j X^j \in \mathbb{C}[X]$. $\square$

We extend A uniquely to the space $L^2([a, b], \mu)$ by the BLT Theorem by density. Call this extension $\tilde{A}_\mu$. Since A is self-adjoint, by continuity so is $\tilde{A}_\mu$.

Lemma 4.6. *The constant function $\mathbb{1}(x) = 1$ is cyclic with regard to $\tilde{A}_\mu$.*

Proof. Clearly $S := \operatorname{span}\{\mathbb{1}, \tilde{A}_\mu \mathbb{1} = x, \tilde{A}_\mu^{\,2}\mathbb{1} = x^2, \dots\} = \mathbb{C}[x]$ for $x \in [a,b]$. Then by the Weierstrass Approximation Theorem S is dense in $C([a,b])$, so by Theorem 4.4 S is dense in $L^2([a,b],\mu)$. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. For clarity, in the following argument we will denote variables taking values in $[a,b]$ as x, and variables taking values in $\sigma(\tilde{A}_\mu)$ as λ. Consider a polynomial $P = \sum_{i=0}^{d} a_i X^i \in \mathbb{C}[X]$, and for such P let $p(\lambda) := P(\lambda) \in L^2(\sigma(\tilde{A}_\mu), \mu_{\mathbb{1}})$ as well as $p_\mu(x) = P(x) \in L^2([a,b],\mu)$. Then let $\phi : C(\sigma(\tilde{A}_\mu)) \to \mathscr{B}(\mathscr{H})$ be the map defining the functional calculus for $\tilde{A}_\mu$. Recall ϕ is a $*$-homomorphism and $\phi(\lambda) = A$, so

$$\phi(p)\mathbb{1} = P(A)\mathbb{1} = p_\mu(x) \in L^2([a,b],\mu).$$

By Lemma 4.6, the cyclic case of the Spectral Theorem, 3.13, applies to the space $\mathscr{H} := L^2([a,b],\mu)$, the self-adjoint operator $\tilde{A}_\mu$, and $\psi := \mathbb{1}$, yielding a unitary

$$U : L^2([a,b],\mu) \to L^2(\sigma(\tilde{A}_\mu), \mu_{\mathbb{1}}).$$

Then by definition of U as in the proof of Theorem 3.13, we have

$$Up_\mu = U\phi(p)\mathbb{1} = p.$$

So U maps any polynomial $P \in \mathbb{C}[x]$ on $[a,b]$ to the same polynomial $P \in \mathbb{C}[\lambda]$ on $\sigma(\tilde{A}_\mu)$, i.e. we have found the restriction

$$U_P := U \restriction_{\mathbb{C}[x]} : \mathbb{C}[x] \to \mathbb{C}[\lambda] : P(x) \mapsto P(\lambda).$$

Now consider $f, g \in C([a,b])$, such that $f \restriction_{\sigma(\tilde{A}_\mu)} = g \restriction_{\sigma(\tilde{A}_\mu)}$, but $f \neq g$. By density of the polynomials on $C([a,b])$, there exist sequences $(P_n)_{n=0}^{\infty}, (Q_n)_{n=0}^{\infty}$ with $P_n, Q_n \in \mathbb{C}[X]$, such that we have $\sup_{x\in[a,b]} |P_n(x) - f(x)| \to 0$, as well as $\sup_{x\in[a,b]} |Q_n(x) - g(x)| \to 0$, but $\sup_{x\in[a,b]} |P_n(x) - Q_n(x)| \nrightarrow 0$ as $n \to \infty$. However, $f \restriction_{\sigma(\tilde{A}_\mu)} = g \restriction_{\sigma(\tilde{A}_\mu)}$ so $\sup_{\lambda\in\sigma(\tilde{A}_\mu)} |P_n(\lambda) - Q_n(\lambda)| \to 0$ as $n \to \infty$. By continuity of U we have

$$Uf = \lim_{n\to\infty} UP_n = f \restriction_{\sigma(\tilde{A}_\mu)} = g \restriction_{\sigma(\tilde{A}_\mu)} = \lim_{n\to\infty} UQ_n = Ug.$$

So $0 = \|Uf - Ug\|^2_{L^2(\sigma(\tilde{A}_\mu),\mu_{\mathbb{1}})} = \|U(f-g)\|^2_{L^2(\sigma(\tilde{A}_\mu),\mu_{\mathbb{1}})} = \int_{\sigma(\tilde{A}_\mu)} |f-g|^2 \, d\mu_{\mathbb{1}}$. But U is unitary, therefore $0 = \|f-g\|^2_{L^2([a,b],\mu)} = \int_a^b |f-g|^2 \, d\mu$. This immediately implies $\operatorname{supp}(\mu) = \sigma(\tilde{A}_\mu)$. Hence $f = f \restriction_{\sigma(\tilde{A}_\mu)}$ a.e. with regard to μ for any $f \in C([a,b])$.

Then $\langle \phi(f)\mathbb{1}, \mathbb{1}\rangle = \int_a^b f \, d\mu = \int_{\sigma(\tilde{A}_\mu)} f \, d\mu$ for all $f \in C([a,b])$, but we recall that $\langle \phi(f)\mathbb{1}, \mathbb{1}\rangle = \int_{\sigma(\tilde{A}_\mu)} f \, d\mu_{\mathbb{1}}$ by definition of $\mu_{\mathbb{1}}$ and by the Riesz-Markov-Kakutani Representation Theorem $\mu_{\mathbb{1}}$ is the unique Borel measure on $\sigma(\tilde{A}_\mu)$ with this property. Therefore $\mu = \mu_{\mathbb{1}}$. By an identical argument $\nu = \mu_{\mathbb{1}}$, and then $\mu = \nu$, so the solution of the Hamburger Moment Problem is unique in the case that the corresponding operator A is bounded. □

Chapter 5

Unbounded operators and the von Neumann theory of self-adjoint extensions

The operator A in the Moment Problem may well turn out to be unbounded. In this chapter we develop the theory of unbounded operators that is required for our approach to the Hamburger Moment Problem.

Definition 5.1. *The graph of a linear transformation T defined on a linear subspace of a Hilbert space $\mathscr{H}$ is the set $\Gamma(T) := \{(x, Tx) \,|\, x \in D(T)\}$ where $D(T) \subseteq \mathscr{H}$ is the domain of T. We say that T is densely defined if $D(T)$ is dense in $\mathscr{H}$. We say that an operator T is closed if $\Gamma(T)$ is a closed set in $\mathscr{H} \times \mathscr{H}$, which is a Hilbert space under the inner product $\langle (x, y), (x', y') \rangle = \langle x, x' \rangle + \langle y, y' \rangle$.*

Definition 5.2. *If we have two operators T and T' such that $\Gamma(T) \subseteq \Gamma(T')$, then we say T' is an extension of T and we write $T \subseteq T'$.*

Definition 5.3. *An operator T is closable if it has a closed extension. The smallest closed extension of T is called its closure, denoted $\overline{T}$, i.e. for any closed extension T' of T, $\Gamma(\overline{T}) \subseteq \Gamma(T')$.*

Remark 5.4. *The case where an operator T may not be closable arises when $\overline{\Gamma(T)}$ is not the graph of an operator.*

Lemma 5.5. *If T is closable, then $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.*

Proof [2]. Let T' be a closed extension of T. Then $\overline{\Gamma(T)} \subseteq \Gamma(T')$, since $\Gamma(T')$ is closed. Note that for any $x \in \mathscr{H}$, there is at most one element in $\Gamma(T')$ of the form (x, y) where x is the first element of a pair in $\Gamma(T')$. So consider an operator R defined on $D(R) = \{x \,|\, (x, y) \in \overline{\Gamma(T)}\}$ with values $Rx = y$ where y is the unique vector such that $(x, y) \in \overline{\Gamma(T)}$. Since T' is closed, $Rx = T'x$ for all $x \in D(R)$. Finally, $D(T)$ is a linear subspace of $\mathscr{H}$ and $\overline{D(T)} = D(R)$. And the closure of a linear subspace must be a linear subspace, hence R is an operator. $\square$

Definition 5.6. *Let T be a densely defined linear operator on a Hilbert space $\mathscr{H}$, i.e. $D(T)$ is dense in $\mathscr{H}$. Define $D(T^*) := \{y \in \mathscr{H} \,|\, \exists z \in \mathscr{H} : \forall x \in D(T) :$*

$\langle Tx, y\rangle = \langle x, z\rangle\}$. Then by the Riesz Representation Theorem we define $T^*y = z$ for all $y \in D(T^*)$. We call T^* the adjoint of T. Note that if $S \subseteq T$ then $T^* \subseteq S^*$ due to the $\forall x \in D(T)$ requirement in the definition of T^*.

Theorem 5.7. *For a densely defined operator T on a Hilbert space $\mathscr{H}$ the following statements hold. T^* is closed, T is closable if and only if $D(T^*)$ is dense in which case $\overline{T} = T^{**}$, and if T is closable, then $(\overline{T})^* = T^*$.*

Proof [2]. Consider the unitary operator

$$V : \mathscr{H} \times \mathscr{H} \to \mathscr{H} \times \mathscr{H} : (\phi, \psi) \mapsto (-\psi, \phi).$$

Because V is unitary, $V(E^\perp) = (V(E))^\perp$ for any subspace $E \subseteq \mathscr{H}$. Let T be a linear operator on $\mathscr{H}$ and $(\phi, \eta) \in \mathscr{H} \times \mathscr{H}$. Then $(\phi, \eta) \in V(\Gamma(T))^\perp$ if and only if $\langle(\phi, \eta), (-T\psi, \psi)\rangle = 0$ for all $\psi \in D(T)$, i.e. if and only if $(\phi, \eta) \in \Gamma(T^*)$. Therefore $\Gamma(T^*) = V(\Gamma(T))^\perp$, which is an orthogonal complement, and thus closed.

To show the second statement, note that $\Gamma(T) \subseteq \mathscr{H} \times \mathscr{H}$ is a linear subset, so

$$\begin{aligned}\overline{\Gamma(T)} &= (\Gamma(T)^\perp)^\perp = (V^2\Gamma(T)^\perp)^\perp \\ &= (V(V\Gamma(T))^\perp)^\perp = (V\Gamma(T^*))^\perp.\end{aligned}$$

So by our previous arguments, if T^* is densely defined, $\overline{\Gamma(T)}$ is the graph of T^{**}.

Conversely, suppose $D(T^*)$ is not dense and let $\psi \in D(T^*)^\perp$. Then for any $(\phi, T^*\phi) \in \Gamma(T^*)$ we have $\langle(\phi, T^*\phi), (\psi, 0)\rangle = \langle\phi, \psi\rangle + \langle T^*\phi, 0\rangle = 0$. So $(\psi, 0) \in (\Gamma(T^*))^\perp$. Therefore $V(\Gamma(T^*))^\perp$ is not the graph of an operator, as if it were then the corresponding operator would not be well-defined, since $V(\psi, 0) = (0, \psi)$. Now note that $\overline{\Gamma(T)} = (V\Gamma(T^*))^\perp$, so T is not closable.

Finally, notice that if T is closable then $T^* = \overline{(T^*)} = T^{***} = (\overline{T})^*$. □

Definition 5.8. *The resolvent set $\rho(T)$ of a closed operator T on a Hilbert space $\mathscr{H}$ is defined as the set of complex numbers $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is a bijection of $D(T)$ onto $\mathscr{H}$. For $\lambda \in \rho(T)$, we call $R_\lambda(T) := (T - \lambda I)^{-1}$ the resolvent of T at λ. The spectrum, point spectrum, and residual spectrum of an unbounded operator T are defined in the same way as they are for bounded operators.*

Remark 5.9. *The statements of Theorem 2.6 and Corollary 2.7 hold for unbounded operators as well, with identical proofs to the bounded case.*

Definition 5.10. *An operator T is symmetric if $T \subset T^*$ and self-adjoint if $T = T^*$. A densely defined symmetric operator T is always closable, since $T \subset T^*$ and T^* is closed as we have seen. If $\overline{T}$ is self-adjoint then we say that T is essentially self-adjoint.*

Lemma 5.11. *If an operator A is symmetric, then its adjoint A^* is symmetric if and only if A is self-adjoint.*

Proof [2]. Clearly A is symmetric if it is self-adjoint, and if A^* is symmetric then $A^* \subset A$, so $A^* = A$. □

Definition 5.12. *For a symmetric operator A the deficiency subspaces of A are defined as $\mathscr{K}_\pm = \ker(A^* \mp i)$, i.e. $\mathscr{K}_+ = \{\phi \in \mathscr{H} \,|\, A^*\phi = i\phi\}$. The quantities $d_\pm = \dim(\mathscr{K}_\pm)$ are called the deficiency indices of A.*

Lemma 5.13. *If A is a closed symmetric operator, then $\mathrm{Ran}(A \pm i)$ are closed, and $\mathscr{K}_\pm^\perp = \mathrm{Ran}(A \pm i)$.*

Proof [9]. Suppose $\phi \in D(A)$. Then since A is symmetric, $\|(A-i)\phi\|^2 = \langle (A^*+i)(A-i)\phi, \phi\rangle = \langle A^*A\phi + \phi, \phi\rangle = \|A\phi\|^2 + \|\phi\|^2$, in particular this gives us a lower bound $\|\phi\|^2 \leq \|(A-i)\phi\|^2$. Now suppose we have a sequence of vectors ϕ_n in $\mathscr{H}$ such that $(A-i)\phi_n \to \psi$ for some $\psi \in \mathscr{H}$. Then the sequence of vectors $S\phi_n$ is Cauchy, and by our lower bound so is the sequence of vectors ϕ_n, thus by completeness of $\mathscr{H}$ we get $\phi_n \to \phi$ as $n \to \infty$ for some $\phi \in \mathscr{H}$. Now $\phi_n \to \phi$ and $(A-i)\phi_n \to \psi$ together imply that $(A-i)\phi = \psi$, hence $\mathrm{Ran}(A-i)$ is closed. By a similar argument $\mathrm{Ran}(A+i)$ is also closed. That $\mathscr{K}_\pm = \ker(A^* \mp i) = \mathrm{Ran}(A \pm i)^\perp$ holds by Lemma 2.12. □

Definition 5.14. *For a closed symmetric operator A, we define on $D(A^*)$ the inner product $\langle \phi, \psi\rangle_{A^*} = \langle \phi, \psi\rangle + \langle A^*\phi, A^*\psi\rangle$. The norm $\|\phi\|_{A^*}$ induced by this inner product is called the graph norm. We say that a subspace of $D(A^*)$ is A-closed or A-orthogonal to some other subspace of $D(A^*)$ if it is closed or respectively orthogonal in this inner product. We define another sesquilinear form on $D(A^*)$ as $[\phi, \psi]_A = \langle A^*\phi, \psi\rangle - \langle \phi, A^*\psi\rangle$. For a subspace of $D(A^*)$ we say that it is A-symmetric if for all ϕ, ψ in the subspace we have $[\phi, \psi]_A = 0$.*

Lemma 5.15. *For a closed symmetric operator A we have $D(A^*) = D(A) \oplus \mathscr{K}_+ \oplus \mathscr{K}_-$, with $\oplus$ being the orthogonal direct sum in $\langle -, -\rangle_{A^*}$.*

Proof [4], [5]. First let $\phi \in \mathscr{K}_+, \psi \in \mathscr{K}_-$, so $\langle \phi, \psi\rangle_{A^*} = \langle \phi, \psi\rangle + \langle i\phi, -i\psi\rangle = 0$, and we conclude $\mathscr{K}_+ \perp_{A^*} \mathscr{K}_-$. Now we show $D(A) \perp_{A^*} \mathscr{K}_+ \oplus \mathscr{K}_-$. Let $\phi \in D(A), \psi \in \mathscr{K}_\pm$, so $\langle \phi, \psi\rangle_{A^*} = \langle \phi, \psi\rangle - \langle \phi, \psi\rangle = 0$ since $\langle A^*\phi, A^*\psi\rangle = \langle A\phi, \pm i\psi\rangle = \langle \phi, \pm iA^*\psi\rangle = \langle \phi, \pm i(\pm i\psi)\rangle = -\langle \phi, \psi\rangle$. We already know the inclusion $D(A) \oplus \mathscr{K}_+ \oplus \mathscr{K}_- \subseteq D(A^*)$, so all that remains to show is the opposite inclusion.

Let $\eta \in D(A^*)$. We have seen that $\mathrm{Ran}(A+i) = \mathscr{K}_+^\perp$, so $\mathrm{Ran}(A+i) \oplus \mathscr{K}_+ = \mathscr{H}$, and thus we can find $\phi \in D(A)$ and $\psi \in \mathscr{K}_+$ such that $(A^*+i)\eta = (A^*+i)(\phi + \psi) = (A+i)\phi + 2i\psi$, since $\mathscr{K}_+ = \{\psi \in \mathscr{H} \,|\, A^*\psi = i\psi\}$. And then $(A^*+i)(\eta - \phi - \psi) = 0$, so $\eta - \phi - \psi \in \mathscr{K}_-$, i.e. $\eta \in D(A) \oplus \mathscr{K}_+ \oplus \mathscr{K}_-$. □

Corollary 5.16. *A closed symmetric operator A is self-adjoint if and only if $d_+ = d_- = 0$. If all we know is A is symmetric, then A is essentially self-adjoint if and only if $d_+ = d_- = 0$.*

Proof [4], [5]. Let A be a symmetric operator. Suppose A is essentially self-adjoint or self-adjoint. In either case $D(A^*) = \overline{D(A)}$, hence the orthogonal complement of $D(A)$ in $D(A^*)$ must be 0-dimensional, i.e. $\dim(\mathscr{K}_+ \oplus \mathscr{K}_-) = 0$. Conversely, suppose $d_+ = d_- = 0$. Since $D(A^*) = \overline{D(A)} \oplus \mathscr{K}_+ \oplus \mathscr{K}_-$, it must be the case that $D(A^*) = \overline{D(A)}$, as otherwise either $d_+ > 0$ or $d_- > 0$. □

Lemma 5.17. *If A is a closed symmetric operator then B is a closed symmetric extension of A if and only if all the following hold: $B \subseteq A^*$, and $D(B)$ is A-closed and A-symmetric.*

Proof [4], [5]. Clearly B is a closed symmetric extension of A if and only if $A \subseteq B \subseteq B^* \subseteq A^*$. Thus every symmetric extension of A is contained in A^*. Additionally an extension B of A is closed if and only if $D(B)$ is A-closed, and symmetric if and only if $D(B)$ is A-symmetric. □

Corollary 5.18. *If A is a closed symmetric operator then the closed symmetric extensions B of A correspond one-to-one to the A-closed, A-symmetric subspaces S of $\mathscr{K}_+ \oplus \mathscr{K}_-$, and this correspondence is given by $D(B) = D(A) \oplus S$.*

Proof. We have already seen that such operators B are A-closed, A-symmetric restrictions of A^* to subspaces $D(B)$ of $D(A^*)$, and $D(B) \supset D(A)$. Hence by the identity $D(A^*) = D(A) \oplus \mathscr{K}_+ \oplus \mathscr{K}_-$ we must have $D(B) = D(A) \oplus S$ for some subspace $S \subset \mathscr{K}_+ \oplus \mathscr{K}_-$. □

Definition 5.19. *The initial subspace of a partial isometry U is the orthogonal complement of its kernel, denoted $I(U) := \ker(U)^\perp$.*

Theorem 5.20. *Suppose A is a closed symmetric operator. The closed symmetric extensions of A are in one-to-one correspondence with the set of partial isometries of $\mathscr{K}_+$ into $\mathscr{K}_-$ in the usual inner product on $\mathscr{H}$. If $U : \mathscr{K}_+ \to \mathscr{K}_-$ is such a partial isometry with initial space $I(U) := \ker(U)^\perp \subseteq \mathscr{K}_+$, then the closed symmetric extension A_U of A corresponding to U has domain*

$$D(A_U) = \{\phi + \phi_+ + U\phi_+ \,|\, \phi \in D(A), \phi_+ \in I(U)\},$$

and

$$A_U(\phi + \phi_+ + U\phi_+) = A\phi + i\phi_+ - iU_{\phi_+}.$$

Additionally, if $\dim(I(U)) < \infty$, the deficiency indices of A_U are

$$d_\pm(A_U) = d_\pm(A) - \dim(I(U)).$$

Proof [4], [5]. Let A_1 be a closed symmetric extension of A. Now by the previous lemma we have $D(A_1) = D(A) \oplus S_1$ for some A-closed, A-symmetric subspace of $\mathscr{K}_+ \oplus \mathscr{K}_-$. Suppose $\phi \in S_1$, so we can write $\phi = \phi_+ + \phi_-$ uniquely with $\phi_+ \in \mathscr{K}_+, \phi_- \in \mathscr{K}_-$. Additionally, we have $\|\phi_+\|^2 = \|\phi_-\|^2$ since

$$2i\langle \phi_-, \phi_- \rangle - 2i\langle \phi_+, \phi_+ \rangle = \langle A^*\phi, \phi \rangle - \langle \phi, A^*\phi \rangle = 0,$$

where the last equality is due to S_1 being A-symmetric.

Now $\|\phi_+\|^2 = \|\phi_-\|^2$, and the following argument shows that $U : \phi_+ \mapsto \phi_-$ is a well-defined isometry from some subspace of $\mathscr{K}_+$ into $\mathscr{K}_-$, i.e. $U : \mathscr{K}_+ \to \mathscr{K}_-$ is a partial isometry. Suppose we have $(\phi_+, \phi_-), (\phi_+, \phi_-') \in S_1 \subseteq \mathscr{K}_+ \oplus \mathscr{K}_-$, then we would have $(\phi_+, \phi_-) - (\phi_+, \phi_-') = (0, \phi_- - \phi_-') \in S_1$ since S_1 is a subspace. Consider the relation $<_U$ defined by $\phi_+ <_U \phi_-$ whenever $\exists \phi = (\phi_+, \phi_-) \in S_1$, clearly $<_U$ respects addition and scalar multiplication. Thus $0 <_U \phi_- - \phi_-'$, which implies that $0 = \|0\|^2 = \|\phi_- - \phi_-'\|^2$, i.e. $\phi_i = \phi_-'$. Thus

$$D(A_1) = D(A) \oplus S_1 = \{\phi + \phi_+ + U\phi_+ \,|\, \phi \in D(A), \phi_+ \in I(U)\}.$$

Additionally we can calculate

$$A_1(\phi + \phi_+ + U\phi_+) = A^*(\phi + \phi_+ + U\phi_+) = A\phi + i\phi_+ - iU\phi_+.$$

Conversely, suppose U is an isometry from a subspace of $\mathscr{K}_+$ into $\mathscr{K}_-$ and let $D(A_1)$ and A_1 as above. This gives that $D(A_1)$ is an A-closed, A-symmetric subspace of $D(A^*)$, so by Corollary 5.18 A_1 is a closed symmetric extension of A.

Now we calculate:

$$\begin{aligned}
\mathscr{K}_+(A_1) &= \ker(A_1^* - i) = \operatorname{Ran}(A_1 + i)^\perp \\
&= \{A\phi + i\phi_+ - iU\phi_+ + i(\phi + \phi_+ + U\phi_+) \mid \phi \in D(A), \phi_+ \in I(U)\}^\perp \\
&= \{(A+i)\phi + 2i\phi_+ \mid \phi \in D(A), \phi_+ \in I(U)\}^\perp \\
&= (\operatorname{Ran}(A+i) + 2iI(U))^\perp, \text{ and similarly}
\end{aligned}$$

$$\mathscr{K}_-(A_1) = \{(A-i)\phi - 2iU\phi_+ \mid \phi \in D(A), \phi_+ \in I(U)\}^\perp = (\operatorname{Ran}(A-i) - 2iUI(U))^\perp.$$

Finally $I(U) \subseteq \mathscr{K}_+(A) \perp \operatorname{Ran}(A+i)$, and $UI(U) \subseteq \mathscr{K}_-(A) \perp \operatorname{Ran}(A-i)$, thus $d_\pm(A_1) = \dim(\mathscr{K}_\pm(A_1)) = \dim(\mathscr{K}_\pm(A)) - \dim(I(U))$. $\square$

Lemma 5.21. *A closed symmetric operator A has self-adjoint extensions if and only if its deficiency indices are equal.*

Proof [4], [5]. Suppose A_U is a self-adjoint extension $A_U \supseteq A$ of A. Then there is a partial isometry $U : \mathscr{K}_+ \to \mathscr{K}_-$ between the deficiency subspaces of A, and $d_+(A_U) = d_-(A_U) = 0$, so by the formula for $d_\pm(A_U)$ from previous theorem U is an isometry. Now if $d_+(A) = d_-(A)$ then there exists an isometry $U : \mathscr{K}_+ \to \mathscr{K}_-$, and the corresponding self-adjoint extension A_U of A is self-adjoint, since the previous theorem gives $d_+(A_U) = d_-(A_U) = 0$. $\square$

Definition 5.22. *A norm-preserving antilinear map $C : \mathscr{H} \to \mathscr{H}$ (so $C(\alpha\phi + \beta\psi) = \overline{\alpha}C\phi + \overline{\beta}C\psi$) is called a conjugation if $C^2 = I$.*

Theorem 5.23. *If $A : D(A) \to \mathscr{H}$ is a symmetric operator and there exists a conjugation $C : \mathscr{H} \to \mathscr{H}$ such that $C(D(A)) \subseteq D(A)$, and $CA = AC$, then the deficiency indices of A are equal, and hence A has self-adjoint extensions.*

Proof [4], [5]. We have that $C^2 = I$ and $CD(A) \subseteq D(A)$, so $CD(A) = D(A)$. Now let $\phi_+ \in \mathscr{K}_+$, and $\psi \in D(A)$. This implies

$$0 = \overline{\langle \phi_+, (A+i)\psi \rangle} = \langle C\phi_+, C(A+i)\psi \rangle = \langle C\phi_+, (A-i)C\psi \rangle,$$

where the second equality is due to the anti-linearity of C combined with $C^2 = I$. Now note that C maps $D(A)$ onto $D(A)$, and $C\phi_+ \in \mathscr{K}_-$ since C commutes with A, and thus C maps $\mathscr{K}_+$ to $\mathscr{K}_-$. By a similar argument C maps $\mathscr{K}_-$ to $\mathscr{K}_+$. Note that C is norm-preserving and is its own inverse, in other words C induces an isometry between $\mathscr{K}_+$ and $\mathscr{K}_-$, so $\dim(\mathscr{K}_+) = \dim(\mathscr{K}_-)$. $\square$

Chapter 6

More on functional calculi

Our goal in this chapter is to eventually construct a functional calculus for bounded normal operators. In order to achieve this, we first construct a functional calculus for a pair of commuting bounded self-adjoint operators.

Definition 6.1. *[10] The joint spectrum of two self-adjoint operators $A, B \in \mathscr{B}(\mathscr{H})$ such that $AB = BA$ is defined as*

$$\sigma(A, B) := \{(\lambda, \mu) \mid \exists (v_n)_{n=0}^{\infty} : \forall n : \|v_n\| = 1, (A - \lambda)v_n \to 0, \, (B - \mu)v_n \to 0\}.$$

Theorem 6.2. *For A, B commuting self-adjoint operators on a Hilbert space $\mathscr{H}$ there is a unique map $\tilde{\phi} : C(\sigma(A, B)) \to \mathscr{B}(\mathscr{H})$ such that:*

- *$\tilde{\phi}$ is a $*$-homomorphism:*
  - $\tilde{\phi}(f(A, B)g(A, B)) = \phi(f(A, B))\phi(g(A, B))$.
  - $\tilde{\phi}(\lambda f) = \lambda \tilde{\phi}(f)$.
  - $\tilde{\phi}(1) = I$.
  - $\tilde{\phi}(\overline{f}) = \tilde{\phi}(f)^*$.
- *$\tilde{\phi}$ matches up the relevant projections; if $f(x, y) = x$ then $\tilde{\phi}(f(A, B)) = A$, if $g(x, y) = y$ then $\tilde{\phi}(g(A, B)) = B$.*
- *$\tilde{\phi}$ is an isometry, and thus continuous.*
- *$\tilde{\phi}$ has the spectral mapping property: $\sigma(\tilde{\phi}(f)) = \{f(\lambda, \mu) \mid (\lambda, \mu) \in \sigma(A, B)\} =: f(\sigma(A, B))$.*
- *If $f \geq 0$ then $\tilde{\phi}(f) \geq 0$, i.e. $\tilde{\phi}(f)$ is positive and thus self-adjoint.*
- *If $A\psi = \lambda\psi$, and $B\psi = \mu\psi$ then $\tilde{\phi}(f)\psi = f(\lambda, \mu)\psi$ for $(\lambda, \mu) \in \sigma(A, B)$, $\psi \in \mathscr{H}$.*

We proceed similarly as in the case of the functional calculus for a single self-adjoint operator, with the notable difference that at first we define our functional calculus on $C(\sigma(A) \times \sigma(B))$ and show that the corresponding mapping ϕ is a bounded linear operator. Once we have defined our spectral measures through this functional calculus we will a posteriori show that the spectral measures are supported only on $\sigma(A,B)$, and hence we will see that $\tilde{\phi}$ is an isometry.

Similar to the case for a single operator, we first define a $*$-homomorphism $\phi'_{A,B} : \mathbb{C}[X,Y] \to \mathscr{B}(\mathscr{H}) : \sum_{i=0}^{d}\sum_{j=0}^{i} a_{i,j}X^{i-j}Y^j \mapsto \sum_{i=0}^{d}\sum_{j=0}^{i} a_{i,j}A^{i-j}B^j$. We will use the notation $P(A,B) := \phi'_{A,B}(P)$.

To begin with, a weaker version of the Spectral Mapping Theorem will be enough to show that $\phi'_{A,B}$ is bounded, allowing us to extend it to $C(\sigma(A)\times\sigma(B))$.

Lemma 6.3. *For A,B as before, we have $P(\sigma(A,B)) \subseteq \sigma(P(A,B))$.*

Proof. Fix $(\lambda,\mu) \in \sigma(A,B)$ and consider $P(A,B) - P(\lambda,\mu)$. Let $[x-\lambda, y-\mu]$ be the ideal of $\mathbb{C}[x,y]$ generated by the elements $x-\lambda, y-\mu$. Then the quotient ring $\mathbb{C}[x,y]/[x-\lambda,y-\lambda]$ can be identified with $\mathbb{C}$ through the isomorphisms $\mathbb{C}[x,y]/[x-\lambda,y-\lambda] \cong \mathbb{C}[x-\lambda,y-\mu]/[x-\lambda,y-\lambda] \cong \mathbb{C}[x,y]/[x,y] \cong \mathbb{C}$. Thus we can do the factoring $P(A,B) - P(\lambda,\mu) = (A-\lambda)Q_1(A,B,\lambda,\mu) + (B-\mu)Q_2(A,B,\lambda,\mu)$ since we have just shown that every element in $\mathbb{C}[x,y]$ is of this form. Now note that A,B are both bounded, and $Q_1(A,B,\lambda,\mu), Q_2(A,B,\lambda,\mu) \in \operatorname{span}(\{A,B\}) \subseteq \mathscr{B}(\mathscr{H})$, hence $Q_1(A,B,\lambda,\mu), Q_2(A,B,\lambda,\mu)$ are bounded. Then immediately $\exists C_1, C_2 > 0$ such that for any sequence $(v_n)_{n=0}^{\infty}$ as in the definition of $\sigma(A,B)$ we have $\|(P(A,B)-P(\lambda,\mu))v_n\| \leq \|C_1(A-\lambda)v_n + C_2(B-\mu)v_n\|$ and hence $(P(A,B)-P(\lambda,\mu))v_n \to 0$ as $n \to \infty$, so $P(\lambda,\mu) \in \sigma(P(A,B))$. □

Lemma 6.4. *For commuting self-adjoint operators A,B on a Hilbert space $\mathscr{H}$ we have the inclusion $\sigma(P(A,B)) \subseteq P(\sigma(A)\times\sigma(B))$.*

Proof. Let $\rho \in \sigma(P(A,B))$. To show $\rho \in P(\sigma(A,B))$ all we have to do is find one pair $(\lambda,\mu) \in \sigma(A,B)$ such that $P(\lambda,\mu) - \rho = 0$. Fix $\lambda \in \sigma(A)$ and consider $P(\lambda,y) - \rho = 0$. Now by the polynomial Spectral Mapping Theorem for a single bounded self-adjoint operator there is some $\mu \in \sigma(B)$ such that $y = \mu$ is a root of $P(\lambda,y) - \rho = 0$. □

Lemma 6.5. *For bounded self-adjoint operators A,B we have the inequalities*

$$\sup_{(\lambda,\mu)\in\sigma(A,B)} |P(\lambda,\mu)| \leq \|P(A,B)\|^2 \leq \sup_{(\lambda,\mu)\in\sigma(A)\times\sigma(B)} |P(\lambda,\mu)|.$$

Proof. We calculate:

$$\begin{aligned} \sup_{(\lambda,\mu)\in\sigma(A,B)} |\overline{P}P(\lambda,\mu)| &\leq \sup_{\lambda'\in\sigma(\overline{P}P(A,B))} |\lambda'| && \text{by Lemma 6.3, and} \\ &\leq \sup_{(\lambda,\mu)\in\sigma(A)\times\sigma(B)} |\overline{P}P(\lambda,\mu)| && \text{by Lemma 6.4.} \end{aligned}$$

But $\sup_{\lambda'\in\sigma(\overline{P}P(A,B))} |\lambda'| = \|\overline{P}P(A,B)\| = \|P(A,B)^*P(A,B)\| = \|P(A,B)\|^2$ where the first equality is by Theorem 2.10, since Theorem 2.16 implies that $P(A,B)^*P(A,B) \geq 0$ and thus $\overline{P}P(A,B) = P(A,B)^*P(A,B)$ is self-adjoint. The second equality is due to $\phi'_A : P(\lambda,\mu) \mapsto P(A,B)$ being a $*$-homomorphism. □

Lemma 6.6. *The polynomials $\mathbb{C}[X,Y]$ on $\sigma(A)\times\sigma(B)$ are dense in $C(\sigma(A)\times\sigma(B))$.*

Proof. Omitted. This lemma is a consequence of the Stone-Weierstrass Theorem. □

Lemma 6.7. *There exists a unique extension $\phi : C(\sigma(A) \times \sigma(B)) \to \mathscr{B}(\mathscr{H})$ of $\phi'_{A,B}$ that is a bounded linear operator and a $*$-homomorphism, and has the following properties:*

- *If $A\psi = \lambda\psi$ and $B\psi = \mu\psi$ for $\psi \in \mathscr{H}$ then $\phi(f)\psi = f(\lambda, \mu)\psi$.*
- *The restriction of ϕ to real-valued functions is positive.*

Proof. The upper bound from Lemma 6.5 on $\|\phi'_{A,B}\|$ implies that $\phi'_{A,B}$ is bounded. Then the BLT Theorem implies that $\phi'_{A,B}$ has a unique extension $\phi : C(\sigma(A) \times \sigma(B)) \to \mathscr{B}(\mathscr{H})$ and that ϕ is also bounded (i.e. continuous). By continuity, ϕ is a $*$-homomorphism.

Since $\phi(P)\psi = P(\lambda, \mu)\psi$, we can again apply continuity to see that if $A\psi = \lambda\psi$ and $B\psi = \mu\psi$ then $\phi(f)\psi = f(\lambda, \mu)\psi$ for $\lambda \in \sigma(A), \mu \in \sigma(B)$, $\psi \in \mathscr{H}$.

That ϕ, when restricted to real-valued functions, is a positive operator follows by an argument largely identical to the one we used in the case of the functional calculus for a single bounded self-adjoint operator. □

Now we can define the spectral measures of a pair of commuting bounded self-adjoint operators A, B by fixing $\psi \in \mathscr{H}$. The map $f \mapsto \langle f(A, B)\psi, \psi\rangle$ is a positive bounded linear functional. Then, since $\sigma(A) \times \sigma(B)$ is a product of two compact sets, and hence is itself compact under the product topology on $\mathbb{R}^2$, the Riesz-Markov-Kakutani Theorem tells us that there exists a unique, finite, positive, Borel-regular measure μ_ψ on $\sigma(A) \times \sigma(B)$ such that $\langle f(A, B)\psi, \psi\rangle = \int_{\sigma(A)\times\sigma(B)} f(x, y)\, d\mu_\psi(x, y)$.

Lemma 6.8. *For $A, B \in \mathscr{B}(\mathscr{H})$, μ_ψ as above, we have* $\operatorname{supp}(\mu_\psi) \subseteq \sigma(A, B)$.

Proof. Suppose $(x_0, y_0) \in \operatorname{supp}(\mu_\psi)$ so there exist functions $g_n \in C(\sigma(A) \times \sigma(B))$ such that $\|g_n\|_{L^2_{\mu_\psi}} = \int_{\sigma(A)\times\sigma(B)} |g_n|^2\, d\mu_\psi = 1$ and $g_n(x, y)$ vanishes outside the closed ball $B\left((x_0, y_0), \frac{1}{n}\right)$ of radius $\frac{1}{n}$ around (x_0, y_0), for all $n \in \mathbb{N}$. Then for $(x, y) \in B\left((x_0, y_0), \frac{1}{n}\right)$ and $f_n = (x - x_0)g_n$ we have $\|f_n\|_{L^2_{\mu_\psi}} \leq \frac{1}{n}$. Now let $T_n = g_n(A, B)$, so $\|(A - x_0)T_n\psi\|^2 = \langle((A - x_0)T_n)^*((A - x_0)T_n)\psi, \psi\rangle = \langle(\overline{f_n} f_n)(A, B)\psi, \psi\rangle = \int_{\sigma(A)\times\sigma(B)} |f_n|^2\, d\mu_\psi$ and similarly $\|T_n\psi\|^2 = 1$. Then we have a sequence of norm 1 vectors $v_n := T_n\psi$ with $(A - x_0)v_n \to 0$ as $n \to \infty$, and through an identical argument $(B - y_0)v_n \to 0$ as $n \to \infty$. Hence $(x_0, y_0) \in \sigma(A, B)$. □

From this lemma we conclude that in fact

$$\langle f(A, B)\psi, \psi\rangle = \int_{\sigma(A)\times\sigma(B)} f(x, y)\, d\mu_\psi(x, y) = \int_{\sigma(A,B)} f(x, y)\, d\mu_\psi(x, y).$$

Now since the Polarization Identity allows us to find $\langle f(A, B)\phi, \psi\rangle$ for any $\phi \in \mathscr{H}$, we can see that $\|\phi(f) - \phi(f')\| = 0$, so $\phi(f) = \phi(f')$ if $f \restriction_{\sigma(A,B)} = f' \restriction_{\sigma(A,B)}$. Hence for such f, f', we have $\sigma(\phi(f)) = \sigma(\phi(f'))$.

Theorem 6.9 (Spectral Mapping). *$\sigma(\phi(f)) = \{f(\lambda,\mu) \,|\, (\lambda,\mu) \in \sigma(A,B)\} =: f(\sigma(A,B))$ for any $f \in C(\sigma(A,B))$.*

Proof. Omitted, as once we have the previous lemmas the argument is almost entirely identical to the case with a single operator. $\square$

Lemma 6.10. *$\tilde{\phi} := \phi \restriction_{C(\sigma(A,B))}$ is an isometry.*

Proof. We calculate $\|\phi(f)\|^2 = \|\phi(f)^*\phi(f)\| = \|\phi(\overline{f}f)\| = \sup_{\lambda' \in (\overline{f}f)(\sigma(A,B))} |\lambda'| = \sup_{(\lambda,\mu)\in\sigma(A,B)} |(\overline{f}f)(\lambda,\mu)|$ by Theorem 2.10 and the Spectral Mapping Theorem. $\square$

Now we consider bounded normal operators, as the functional calculus for these is what will finally unlock the spectral measures we need to construct in the case where the operator A, as described in our introduction to the Hamburger moment problem, is unbounded.

Definition 6.11. *$T \in \mathscr{B}(\mathscr{H})$ is a normal operator if it commutes with its adjoint: $TT^* = T^*T$. As usual $\mathscr{H}$ denotes a Hilbert space.*

We will see that the operator A corresponding to the Moment Problem is closed and symmetric, and has a self-adjoint extension $\tilde{A}$. Note that in this case by Lemma 5.13 we have $\mathrm{Ran}(A \pm i) = \mathscr{K}_{\pm}^{\perp}$, and Corollary 5.16 then implies $\mathrm{Ran}(A \pm i) = \mathscr{H}$. Then by the Closed Graph Theorem $(A \pm i)^{-1} \in \mathscr{B}(\mathscr{H})$, hence $\pm i \in \rho(A)$, and Corollary 2.7 and Remark 5.9 tell us that $(A \pm i)^{-1}$ commute with each other. Finally, Theorem 2.5 gives that $((A+i)^{-1})^* = (A-i)^{-1}$, so $(A+i)^{-1}$ is a normal operator.

Let $T := (A+i)^{-1}$. We can express T as $T = B + iC$ where $B := \frac{T+T^*}{2}$ and $C := \frac{T-T^*}{2i}$. Now note that B, C are self-adjoint by the following elementary calculations:

$$B^* = \left(\frac{T+T^*}{2}\right)^* = \frac{T^*+T}{2},$$
$$C^* = \left(\frac{T-T^*}{2i}\right)^* = \frac{T^*-T}{-2i} = \frac{T-T^*}{2i}.$$

We have seen that T is normal, so T, T^* commute, and B, C are just linear combinations of T, T^*, so B and C commute with each other. This yields a functional calculus for the normal operator T through the functional calculus for the pair of bounded operators B, C, as: $f(T) := (f \circ z)(B, C)$, where $z : \mathbb{R}^2 \to \mathbb{C} : (x,y) \mapsto z(x,y) := x + iy$, and $f \in C(\sigma(T))$. All that is left to be checked is that $\sigma(T) = z(\sigma(B,C))$, which is an immediate consequence of the Spectral Mapping Theorem (6.9) for the commuting functional calculus.

Remark 6.12. *The class of spectral measures and the functional calculi for both a pair of commuting bounded self-adjoint operators as well as a normal operator can be extended to measures $\mu_{\phi,\psi}$ and to Borel-measurable functions in an identical manner to that described in Remark 3.10 and Definition 3.11. We will use these extensions in Chapter 7.*

Chapter 7

Conditions for existence

With all the required machinery in place, we are ready to prove the following:

Theorem 7.1. *There exists a Borel measure ρ on $\mathbb{R}$ such that*

$$\forall n \in \mathbb{N} : \gamma_n = \int_{\mathbb{R}} x^n \, d\rho(x),$$

if and only if $\forall N \in \mathbb{N} : \det(H_N) \geq 0$.

Definition 7.2. *We define the square matrices H_N, $N \in \mathbb{N} \backslash \{0\}$ by their elements $(H_N)_{ij} = \gamma_{i+j}$.Define the bilinear form polynomials $P, Q \in \mathbb{C}[X]$ with degrees* $\deg P, \deg Q \leq N$, *as $\mathcal{H}(P, Q) := QH_N P$.*

Remark 7.3. *Note that this definition immediately implies that $\mathcal{H}$ is positive semi-definite if and only if H_N is a positive semi-definite matrix for all $N \in \mathbb{N} \setminus \{0\}$.*

Lemma 7.4. *For $N \in \mathbb{N} \backslash \{0\}$ we have that H_N is a positive semi-definite matrix if and only if* $\det H_K \geq 0$ *for $K \in \{1, 2, \ldots, N\}$.*

Proof [4]. A simple inductive argument proves this lemma. The statement clearly holds in the case $N = 1$. Suppose it holds for all $N' \in \{1, 2, \ldots, N-1\}$ for some $N \in \mathbb{N} \setminus \{0\}$. Recall that a matrix is positive semi-definite if and only if all of its eigenvalues are non-negative. Let $\lambda_k \geq 0, k \in \{1, 2, \ldots, N-1\}$ be the eigenvalues of H_{N-1}. By definition H_N agrees with H_{N-1} on the subspace of polynomials $P \in \mathbb{C}[X]$ of degree $\deg P \leq N-1$, which is an $N-1$-dimensional subspace. Hence $N-1$ eigenvalues of H_N are determined as the eigenvalues of H_{N-1}, all of which are non-negative. Then H_N is positive semi-definite if and only if both its one remaining eigenvalue is non-negative, and $\forall K' \in \{1, 2, \ldots, N-1\} : \det(H_{K'}) \geq 0$ by our inductive assumption. This is equivalent to saying that H_N is positive semi-definite if and only if $\forall K \in \{1, 2, \ldots, N\} : \det(H_K) \geq 0$. $\square$

Thus $\mathcal{H}$ is positive semi-definite if and only if $\forall N \in \mathbb{N} \setminus \{0\} : \det(H_N) \geq 0$. Now note that if we assume ρ is a Borel measure on $\mathbb{R}$ with nth moments $\gamma_n =$

$\int_{\mathbb{R}} x^n \, d\rho(x)$ and $P = \sum_{n=0}^{N} a_n X^n \in \mathbb{C}[X]$, then

$$\begin{aligned}\mathcal{H}(P, P) &= \sum_{n=0}^{N} \sum_{m=0}^{N} a_n \overline{a_m} \gamma_{n+m} = \sum_{n=0}^{N} \sum_{m=0}^{N} a_n \overline{a_m} \int_{\mathbb{R}} x^{n+m} \, d\rho(x) \\ &= \int_{\mathbb{R}} \sum_{n=0}^{N} \sum_{m=0}^{N} a_n \overline{a_m} x^{n+m} \, dx = \int_{\mathbb{R}} \left| \sum_{n=0}^{N} a_n x^n \right|^2 dx \geq 0.\end{aligned}$$

In other words, in this case $\mathcal{H}$ is a positive semi-definite bilinear form, and recalling what we have seen so far, we have the following result [4]:

Lemma 7.5. *If there exists a Borel measure ρ on $\mathbb{R}$ such that $\forall n \in \mathbb{N} : \gamma_n = \int_{\mathbb{R}} x^n \, d\rho(x)$, then $\forall N \in \mathbb{N} : \det(H_N) \geq 0$.*

In the remainder of this chapter, we will prove the converse statement, which together with Lemma 7.5 proves Theorem 7.1:

Lemma 7.6. *If $\forall N \in \mathbb{N} : \det(H_N) \geq 0$, then there exists a Borel measure ρ on $\mathbb{R}$ with $\forall n \in \mathbb{N} : \gamma_n = \int_{\mathbb{R}} x^n \, d\rho(x)$.*

Lemma 7.7. *$\mathcal{H}$ is a semi inner product on $\mathbb{C}[X]$ if and only if $\forall N \in \mathbb{N} : \det(H_N) \geq 0$.*

Proof. $\mathcal{H}$ is always conjugate symmetric, $\mathcal{H}(p, q) = \sum_{n=0}^{d_p} \sum_{m=0}^{d_q} a_n \overline{b_m} \gamma_{n+m} = \overline{\sum_{n=0}^{d_p} \sum_{m=0}^{d_q} \overline{a_n} b_m \gamma_{n+m}} = \overline{\mathcal{H}(q, p)}$, and it is clearly linear in the second argument. Finally, we have seen that $\mathcal{H}$ is positive semi-definite if and only if $\forall N \in \mathbb{N} : \det(H_N) \geq 0$. □

For the remainder of this chapter we suppose that $\mathcal{H}$ is a semi inner product on $\mathbb{C}[X]$, and use the notation $\langle P, Q \rangle = \mathcal{H}(P, Q)$, and $\|P\|^2 = \mathcal{H}(P, P)$.

Lemma 7.8. *Given $\mathcal{H}$ is positive semi-definite, for the subspace*

$$K := \ker(\mathcal{H}) = \{V \in \mathbb{C}[X] \mid \|P\|^2 = \langle V, V \rangle = 0\} \subseteq \mathbb{C}[X]$$

we have $\forall V \in K : \forall P \in \mathbb{C}[X] : \mathcal{H}(P, V) = 0$ and $\forall W \notin K : \mathcal{H}(W, W) > 0$

Proof. Let $V \in K$, then by the Cauchy-Schwartz inequality we have

$$0 \leq |\langle P, V \rangle| \leq \|P\| \|V\| = 0,$$

so $\langle P, V \rangle = 0$ for all $P \in \mathbb{C}[X]$, and the second statement holds by definition of K. □

Definition 7.9. *Let $A : \mathbb{C}[X] \to \mathbb{C}[X] : P \mapsto XP$.*

Lemma 7.10. *A is a symmetric operator.*

Proof. Holds by the same argument as in the proof of Lemma 4.5. □

Lemma 7.11. *For K as in the Lemma 7.8, we have $A(K) \subseteq K$.*

Proof. A is symmetric, thus $\|AP\| = \langle A^2P, P\rangle = \langle X^2P, P\rangle \leq \|X^2P\|\|P\| = 0$ by the Cauchy-Schwartz inequality. □

We consider the quotient space $X = \mathbb{C}[x]/\ker(\mathcal{H})$. Now $\mathcal{H}$ induces an inner product on X. For a representative ϕ of the equivalence classes that form the elements of X, denote the corresponding equivalence class as $[\phi]$. Let $\mathscr{H}^\gamma$ be the completion of X with regard to the norm associated with the inner product $\mathcal{H}$.

Definition 7.12. *Denote by C the conjugation*

$$C : \mathbb{C}[X] \to \mathbb{C}[X] : \sum_{i=0}^{d} a_i X^i \mapsto \sum_{i=0}^{d} \overline{a_i} X^i.$$

We can see that A commutes with C, $AC = CA$. Additionally, C is an isometry, so it respects our quotienting in the same sense that A does,

$$C(\ker(\mathcal{H})) \subseteq \ker(\mathcal{H}).$$

Therefore we can consider both A and C as operators on the quotient space $X = \mathbb{C}[x]/\ker(\mathcal{H})$. As they are both bounded, they have extensions to $\mathscr{H}^\gamma$ by the BLT Theorem, which we denote as A' and C' respectively. By continuity C' remains a conjugation, and these new operators also commute, $A'C' = C'A'$. Now the theory we have been developing can bear fruit, as we can apply von Neumann's Theorem 5.23 to see that A' has some (not necessarily unique) self-adjoint extension $\tilde{A}$ on $\mathscr{H}^\gamma$. Although we do not consider the question of uniqueness in the unbounded case of the Hamburger Moment Problem, we note that the possibility that A' has more than one self-adjoint extension is what eventually leads to complications when compared to the bounded case. [4]

What still remains to be shown is that there exists some measure μ, such that $\int_{\mathbb{R}} x^n \, d\mu(x) = \gamma_n$ for all $n \in \mathbb{N}$. In case that A is bounded there is no work remaining, we simply note that the spectral measure $\mu_{\mathbb{1}}$ satisfies the Hamburger Moment Problem, as $\int_{\sigma(A)} x^n \, d\mu_{\mathbb{1}}(x) = \langle (A')^n \mathbb{1}, \mathbb{1}\rangle = \gamma_n$.

When A is unbounded we cannot yet make this construction, as we do not have access to the unbounded functional calculus, which would be the standard way of constructing a solution to the Hamburger Moment Problem [5], [4]. We do however have access to a functional calculus for normal operators, as seen in Chapter 6. In the rest of this chapter we show that the functional calculus, as well as the spectral measures, corresponding to the normal operator $T := (A' + i)^{-1}$ will be sufficient for our purposes. We will denote the map inducing the functional calculus for T as ϕ', and the spectral measure $\mu_{\mathbb{1}}$ is the unique measure such that $\langle f(T)\mathbb{1}, \mathbb{1}\rangle = \int_{\sigma(T)} f \, d\mu_{\mathbb{1}}$ for all bounded Borel-measurable functions f defined on $\sigma(T)$.

Lemma 7.13. *The Borel set $\{0\}$ has measure 0 under $\mu_{\mathbb{1}}$: $\mu_{\mathbb{1}}(\{0\}) = 0$.*

Proof. Let $\mathbb{1}_{\{0\}}$ denote the indicator function of $\{0\}$, i.e. $\mathbb{1}_{\{0\}}(0) = 1$, and for all other $0 \neq x \in \mathbb{R}$, $\mathbb{1}_{\{0\}}(x) = 0$. Then $T\phi'(\mathbb{1}_{\{0\}}) = \phi'(x\mathbb{1}_{\{0\}}) = \phi'(0) = 0$, but $\ker(T) = \{0\}$ since T is injective by definition. Therefore $\phi'(\mathbb{1}_{\{0\}}) = 0$, hence $\int_{\sigma(T)} \mathbb{1}_{\{0\}} \, d\mu_{\mathbb{1}} = \langle \phi'(\mathbb{1}_{\{0\}})\mathbb{1}, \mathbb{1}\rangle = \langle 0, \mathbb{1}\rangle = 0$. □

Lemma 7.14. *Let $f(x) = \frac{1}{x} - i$, and consider the pushforward measure $\mu_{\mathbb{1}} \circ f^{-1}$, where f^{-1} is the set-valued function mapping all elements of* $\mathrm{Ran}(f)$ *to their preimages. The support of this measure* $\mathrm{supp}(\mu_{\mathbb{1}} \circ f^{-1}) \subseteq \mathbb{R}$ *is a subset of the reals.*

Proof. First calculate $(\mu_{\mathbb{1}} \circ f^{-1})(f(supp(\mu_{\mathbb{1}}))) = \mu_{\mathbb{1}}(\mathrm{supp}(\mu_{\mathbb{1}})) = 1$, hence all complex Borel sets disjoint from $\overline{f(\mathrm{supp}(\mu_{\mathbb{1}}))}$ have measure 0 under $\mu_{\mathbb{1}} \circ f^{-1}$. Therefore $\mathrm{supp}(\mu_{\mathbb{1}} \circ f^{-1}) \subseteq \overline{f(\mathrm{supp}(\mu_{\mathbb{1}}))} \subseteq \overline{f(\sigma(T))}$. So all that remains to show is that $f(\sigma(T)) \subseteq \mathbb{R}$.

Suppose $\frac{1}{\lambda} - i \in \mathbb{C} \setminus \mathbb{R}$, we will show that in this case $\lambda \in \rho(T)$, i.e. that $T - \lambda$ has a bounded inverse. Since A' is self-adjoint, its spectrum is a subset of $\mathbb{R}$, and thus $A + i - \frac{1}{\lambda}$ has a bounded inverse $S_\lambda := \left(A + i - \frac{1}{\lambda}\right)^{-1}$. Consider the candidate $R_\lambda := -\frac{1}{\lambda}\left(I + \frac{1}{\lambda} S_\lambda\right)$ for the inverse of $T - \lambda$. Clearly R_λ is bounded since S_λ is bounded. Now we calculate:

$$\begin{aligned}
R_\lambda &= -\frac{1}{\lambda}\left(I + \frac{1}{\lambda}\left(A + i - \frac{1}{\lambda}\right)^{-1}\right) = \frac{1}{\lambda}\left(-I + (-\lambda A - \lambda i + I)^{-1}\right) \\
&= \frac{1}{\lambda}\left(\left((\lambda A + \lambda i - I) + I\right)(-\lambda A - \lambda i + I)^{-1}\right) \\
&= (A + i)\,(-\lambda A - \lambda i + I)^{-1} \\
&= (-\lambda + (A+i)^{-1})^{-1} = (T - \lambda)^{-1}.
\end{aligned}$$

□

Definition 7.15. *We define a linear operator S on $\mathscr{H}^\gamma$. Its domain $D(S)$ consists of those $v \in \mathscr{H}$ such that $\forall w \in \mathscr{H} : \frac{1}{x} - i \in L^1(\sigma(T), \mu_{v,w})$ and $|\int_{\sigma(T)} \frac{1}{x} - i \, d\mu_{v,w}| \leq C\|w\|_{\mathscr{H}^\gamma}$. For $v \in D(S)$ we define Sv as the unique element representing the linear functional $w \mapsto \langle Sv, w\rangle = \int_{\sigma(T)} \frac{1}{x} - i \, d\mu_{v,w}$ by the Riesz Representation Theorem.*

Lemma 7.16. *Let $f \in L^1(\sigma(T), \mu_{\mathbb{1}})$. Then*

$$\|f\|_{L^1(\mu_{\mathbb{1}})} = \sup_{g \in C(\sigma(T)), \|g\|_{L^\infty(\mu_{\mathbb{1}})} \leq 1} \left| \int_{\sigma(T)} f g \, d\mu_{\mathbb{1}} \right|.$$

Proof. Follows by elementary calculation from Hölder's Inequality. □

Lemma 7.17. *A' is an extension of S, i.e. $S \subseteq A'$.*

Proof. Suppose $w \in D(A')$, $v = Tw$, let $\mathbb{1}_{\{|x| \geq \epsilon\}}$ be the indicator function for $\{x \in \sigma(T) : |x| \geq \epsilon\}$, and $g \in C(\sigma(T)), \|g\|_{L^\infty(\mu_{\mathbb{1}})} \leq 1$. We calculate for all $\epsilon > 0$:

$$\begin{aligned}
\left| \int_{\sigma(T)} \left(\frac{1}{x} - i\right) \mathbb{1}_{\{|x| \geq \epsilon\}} g(x) \, d\mu_{v,z}(x) \right| &= \left| \int_{\sigma(T)} x \left(\frac{1}{x} - i\right) \mathbb{1}_{\{|x| \geq \epsilon\}} g(x) \, d\mu_{w,z}(x) \right| \\
&= \left| \int_{\sigma(T)} (1 - ix) \mathbb{1}_{\{|x| \geq \epsilon\}} g(x) \, d\mu_{w,z}(x) \right| \\
&\leq \|g\|_{L^\infty(\mu_{w,z})} \left(\sup_{x \in \sigma(T)} (1 - ix) \right) \|w\| \|z\|.
\end{aligned}$$

The last inequality above is a consequence of the Cauchy-Schwartz Inequality. Then, letting $\epsilon \to 0$, the previous Lemma with $f(x) = \left(\frac{1}{x} - i\right) \mathbb{1}_{|x|\geq\epsilon}$ implies that $w \in D(S)$. So $D(A') \subseteq D(S)$.

Now we will show that $Sw = A'w$. Calculate

$$\int_{\sigma(T)} \frac{1}{x} - i \, d\mu_{v,z}(x) = \int_{\sigma(T)} x\left(\frac{1}{x} - i\right) d\mu_{w,z} = \langle (1 - iT)w, z\rangle.$$

Then we need to see that $\langle (1 - iT)w, z\rangle = \langle A'v, z\rangle$, which holds if and only if

$$A'v = (1 - iT)w \iff A'v = w - iv \iff (A' + i)v = w \iff v = (A' + i)^{-1}w,$$

but $T = (A' + i)^{-1}$ so the last equality holds by assumption. □

Note that A' is self-adjoint, and $A' \subseteq S$, therefore $S^* \subseteq A'^*$, so we have

$$A' = S = S^* = A'^*.$$

Now, we know by definition that $\gamma_n = \langle (A')^n \mathbb{1}, \mathbb{1}\rangle$, and the previous statement implies $\langle (A')^n \mathbb{1}, \mathbb{1}\rangle = \langle S^n \mathbb{1}, \mathbb{1}\rangle$. Since $\mathbb{1} \in D(A')$, $\mathbb{1} \in D(S)$, hence $\frac{1}{x} - i \in L^1(\sigma(T), \mu_{\mathbb{1}})$ by definition of $D(S)$. So the measurable functional calculus for T gives $\langle S^n \mathbb{1}, \mathbb{1}\rangle = \int_{\sigma(T)} \left(\frac{1}{x} - i\right)^n d\mu_{\mathbb{1}}$. Finally, let $f(x) = \frac{1}{x} - i$, and we have seen in Lemma 7.14 that $\operatorname{supp}(\mu_{\mathbb{1}} \circ f^{-1}) \subseteq \mathbb{R}$. Putting everything together, we have for all $n \in \mathbb{N}$:

$$\gamma_n = \langle (A')^n \mathbb{1}, \mathbb{1}\rangle = \langle S^n \mathbb{1}, \mathbb{1}\rangle = \int_{\mathbb{R}} x^n \, d(\mu_{\mathbb{1}} \circ f^{-1})(x).$$

So $\mu_{\mathbb{1}} \circ f^{-1}$ is a solution to the Hamburger Moment Problem.

Bibliography

[1] W. Rudin, *Real and complex analysis.* McGraw-Hill international editions Mathematics series, New York, NY: McGraw-Hill, 3. ed., internat. ed., [nachdr.] ed., 2013.

[2] M. Reed and B. Simon, *Methods of modern mathematical physics. volume I: Functional analysis.* Academic Press, 1980.

[3] J. K. Hunter and B. Nachtergaele, *Applied analysis.* World Scientific, 2007.

[4] B. Simon, "The Classical Moment Problem as a Self-Adjoint Finite Difference Operator," *Advances in Mathematics*, vol. 137, pp. 82–203, July 1998.

[5] M. Reed and B. Simon, *Methods of modern mathematical physics: II: Fourier analysis, self-adjointness.* Academic Press, 1975.

[6] E. Hille and R. S. Phillips, *Functional analysis and Semi-Groups.* American Mathematical Society, 1957.

[7] Ludolila, "Answer to "The range of \$T^*\$ is the orthogonal complement of \$\ker(T)\$"," Mar. 2013.

[8] Ruy, "Answer to "Proving the spectral mapping theorem via the continuous functional calculus - significant progress"," Oct. 2020.

[9] K. R. Murthy, "Answer to "Possibly unbounded operator which is closed and has a lower bound has closed range"," Oct. 2020.

[10] A. KällstrÖm and B. Sleeman, "Joint spectra for commuting operators," *Proceedings of the Edinburgh Mathematical Society*, vol. 28, pp. 233–248, June 1985.