RPiS Lista 1 Krystian Jasionek

1. (2p)

a) $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$

Zauważmy, że ze wzoru dwumianowego Newtona $(a+b)^n = \sum_{k=0}^{n} a^k b^{n-k}$, zatem jeśli $a=p$, $b=p-1$, wtedy

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+1-p)^n = 1^n = 1$$

b) $\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$

$$\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} k \cdot \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k} =$$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k} = \sum_{k=0}^{n-1} \frac{n!}{(n-k-1)!\cdot k!} p^{k+1} (1-p)^{n-1-k} = np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-k-1)!} p^k (1-p)^{n-1-k} =$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} = np \cdot (p+1-p)^{n-1} = np$$

2. (a) $\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = 1$

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

Rozwińmy e^x w szereg Taylora.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$, stąd $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e$, zatem

$$e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e = 1$$

(b) $\sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \lambda$

$$\sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$$

3. $\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t}\, dt$, $p>0$. $\Gamma(n) = (n-1)!$, $n \in \mathbb{N}$.

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t}\, dt = \left| \begin{matrix} \text{metodą całkowania} \\ \text{przez części} \end{matrix} \right| = \left| \begin{matrix} f = t^{n-1} & g' = e^{-t} \\ f' = (n-1)t^{n-2} & g = -e^{-t} \end{matrix} \right| = -t^{n-1} \cdot e^{-t} - \int_0^{\infty} (n-1) t^{n-2} \cdot -e^{-t}\, dt$$

$$\Gamma(n) = -t^{n-1} e^{-t} + (n-1) \int_0^{\infty} t^{n-2} e^{-t}$$

Mamy to zależność całką oznaczoną.

$$\Gamma(n) = \left[-t^{n-1} e^{-t}\right]_0^{\infty} + (n-1)\int_0^{\infty} t^{n-2} e^{-t}$$

$$\Gamma(n) = -\lim_{t\to\infty} t^{n-1} e^{-t} - (-0\cdot 1) + (n-1)\int_0^{\infty} t^{n-2} e^{-t}$$

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^t} = \left|\frac{\infty}{\infty} \text{ - zatem d'L'Hospital}\right| = \lim_{t\to\infty}\frac{t^{n-2}}{e^t} = \left|\text{taką sytuację uzyc powtórzymy to } n-1 \text{ razy}\right| = \ldots = \lim_{t\to\infty}\frac{1}{e^t} = 0$$

$\boxed{\Gamma(n) = (n-1)\int_0^{\infty} t^{n-2} e^{-t}}$, ponownie całkowanie przez części

$\Gamma(n) = (n-1)\left[\left[-t^{n-2}e^{-t}\right]_0^{\infty} + (n-2)\int_0^{\infty} t^{n-3} e^{-t}\,dt\right]$, widać, że będziemy tak iterować, aż dojdziemy do

$(n-1)(n-2)\cdot\ldots\cdot(2)\int_0^{\infty} t^0 e^{-t}\,dt = (n-1)!$... Jak to pokazać formalnie?

Zauważmy, że $\Gamma(n) = (n-1)\,\Gamma(n-1)$, zatem możemy obliczyć $\Gamma(n)$ rekurencyjnie. Ile wynosi podstawa rekurencji, tj. $\Gamma(1)$?

$$\Gamma(1) = \int_0^{\infty} t^{1-1} e^{-t}\,dt = \int_0^{\infty} e^{-t}\,dt = \left[-e^{-t}\right]_0^{\infty} = \left[\frac{-1}{e^t}\right]_0^{\infty} = 0 + 1 = 1,$$ zatem

$\Gamma(n) = (n-1)\,\Gamma(n-1)$; $\Gamma(1) = 1$, stąd $\Gamma(n) = (n-1)!$

4) $f(x) = \lambda \exp(-\lambda x)$, $\lambda > 0$.

(a) $$\int_0^{\infty} \lambda e^{-\lambda x}\,dx = \lambda\int_0^{\infty} e^{-\lambda x}\,dx = \lambda\cdot\left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{\infty} = \lambda\left[-\lim_{x\to\infty}\frac{1}{\lambda\cdot e^{\lambda x}} + \frac{1}{\lambda}e^{-\lambda\cdot 0}\right] =$$

$$= \lambda\left[0 + \frac{1}{\lambda}\right] = 1$$

(b) $$\int_0^{\infty} x f(x)\,dx = \int_0^{\infty} x\,\lambda e^{-\lambda x}\,dx = \lambda\int_0^{\infty} x e^{-\lambda x}\,dx = \begin{vmatrix} f = x & g' = e^{-\lambda x} \\ f' = 1 & g = \frac{-1}{\lambda}\cdot e^{-\lambda x}\end{vmatrix} =$$

$$= \lambda\left[\left[x\cdot\frac{-1}{\lambda}e^{-\lambda x}\right]_0^{\infty} - \int_0^{\infty}\frac{-1}{\lambda}\cdot e^{-\lambda x}\,dx\right] = -\left[x\cdot e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx =$$

$$= -\lim_{x\to\infty}\frac{x}{e^{\lambda x}} + 0\cdot e^{-\lambda\cdot 0} + \left[\frac{-1}{\lambda}e^{-\lambda x}\right]_0^{\infty} = 0 + 0 - \lim_{x\to\infty}\underbrace{\frac{1}{\lambda e^{\lambda x}}}_{0} + \frac{1}{\lambda}\cdot e^{-\lambda\cdot 0} = \frac{1}{\lambda}$$

5. $D_n = n$ $\qquad D_1 = |1| = 1$

$$D_n = \begin{vmatrix} 1 & -1 & \cdots & -1 & -1 \\ 1 & 1 & & & \\ \vdots & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{vmatrix} \qquad D_n = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

$$a_{00}A_{00} = \begin{vmatrix} 1 & -1 & \cdots & -1 & -1 \\ 1 & 1 & & & \\ \vdots & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{vmatrix}\cdot 1 = 1\cdot\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{vmatrix} = 1\cdot 1 = 1$$

$$\begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{bmatrix} \xrightarrow[\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \\ \vdots \\ R_n = R_n - R_1}]{\text{do schodkowej}} \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 2 & 1 & 1 \cdots & 1 \\ 0 & 1 & 2 & 1 \cdots & 1 \\ 0 & 1 & 1 & 2 \; 1 \cdots & 1 \\ \vdots & & & \ddots & \\ 0 & & & & 2 \end{bmatrix} \xrightarrow[\substack{R_3 = R_3 - \frac{1}{2}R_2 \\ R_4 = R_4 - \frac{1}{2}R_2 \\ \vdots \\ R_n = R_n - \frac{1}{2}R_2}]{} \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 2 & 1 & \cdots & 1 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} \cdots & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \cdots & \frac{1}{2} \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & \frac{3}{2} \end{bmatrix}$$

$$\xrightarrow[\substack{R_4 = R_4 - \frac{1}{3}R_3 \\ R_5 = R_5 - \frac{1}{3}R_3 \\ \vdots \\ R_n = R_n - \frac{1}{3}R_3}]{} \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 2 & 1 & \cdots & 1 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} \cdots & \frac{1}{2} \\ 0 & 0 & 0 & \frac{4}{3} & \frac{1}{3} \\ \vdots & \vdots & 0 & \frac{1}{3} & \frac{1}{3} \\ \vdots & \vdots & \vdots & \vdots & \frac{4}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \end{bmatrix} \longrightarrow \cdots \longrightarrow \underset{\substack{R_k = R_k - \frac{1}{k-1} R_{k-1} \\ \vdots \\ R_n = R_n - \frac{1}{k-1} R_{n-1}}}{} \longrightarrow \cdots \longrightarrow$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & -1 & \cdots & & -1 \\ 0 & 2 & 1 & \cdots & & 1 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 0 & 0 & 0 & \frac{4}{3} & \frac{1}{3} \cdots & \frac{1}{3} \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & \cdots & & & \frac{n}{n-1} \end{bmatrix} = A$$

$$\det A = 1 \cdot \cancel{2} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{5}}{\cancel{4}} \cdots \frac{\cancel{n-1}}{\cancel{n-2}} \cdot \frac{n}{\cancel{n-1}}$$

$$\det A = 1 \cdot n = n$$

6. $I = \int_{-\infty}^{\infty} \exp\left\{\frac{-x^2}{2}\right\} dx$. $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{x^2+y^2}{2}\right\} dy\,dx$.

Podstawiamy $x = r\cos\theta$, $y = r\sin\theta$. Pokaż, że $I^2 = 2\pi$.

$x = r\cos\theta \quad y = r\sin\theta$

Jakobian przejścia (z wikipedii)

$$\frac{D(x,y)}{D(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = \boxed{r}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dy\,dx = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}} r\,d\theta\,dr = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}} r\,d\theta\,dr =$$

$$= \int_0^{\infty} \left[\theta e^{-\frac{r^2}{2}} r\right]_0^{2\pi} = \int_0^{\infty} 2\pi e^{-\frac{r^2}{2}} r\,dr = \left[-2\pi e^{-\frac{r^2}{2}}\right]_0^{\infty} = \lim_{r\to\infty} -2\pi \frac{1}{e^{\frac{r^2}{2}}} + 2\pi \frac{1}{e^0}$$

$$= -2\pi \cdot \frac{1}{e^{\infty}} + 2\pi \cdot \frac{1}{e^{0}} = -2\pi \cdot 0 + 2\pi \cdot 1 = 2\pi \quad \blacksquare$$

$$\int_0^\infty 2\pi e^{-\frac{r^2}{2}} r \, dr = \left| \begin{matrix} u = -\frac{r^2}{2} \\ du = -r\,dr \end{matrix} \right| = \int_0 -2\pi e^{u} \, du = \left[-2\pi e^{u} \right]_0^\infty = \left[-2\pi e^{-\frac{r^2}{2}} \right]_0^\infty$$

7. (a) $\sum_{k=1}^n (x_k - \bar{x})^2 = \sum_{k=1}^n x_k^2 - n \cdot \bar{x}^2$, $\quad \bar{x}$ - średnia ciągu $x_1, x_2, \dots, x_n$

$$\sum_{k=1}^n (x_k-\bar{x})^2 = \sum_{k=1}^n x_k^2 - 2x_k\bar{x} + \bar{x}^2 = \sum_{k=1}^n x_k^2 - \bar{x}\sum_{k=1}^n (2x_k - \bar{x}) = \sum_{k=1}^n x_k^2 - \bar{x}\left(\sum_{k=1}^n 2x_k - \sum_{j=1}^n \frac{x_j}{n}\right) =$$

$$= \sum_{k=1}^n x_k^2 - \bar{x}\left[\sum_{k=1}^n 2x_k - \sum_{k=1}^n\sum_{j=1}^n \frac{x_j}{n}\right] = \sum_{k=1}^n x_k^2 - \bar{x}\left[2n\bar{x} - n\bar{x}\right] = \sum_{k=1}^n x_k^2 - \bar{x}n\bar{x} = \sum_{k=1}^n x_k^2 - n\bar{x}^2 \quad \blacksquare$$

(b) $\sum_{k=1}^n (x_k - \bar{x})(y_k - \bar{y}) = \sum_{k=1}^n x_k y_k - n\bar{x}\bar{y}$

$$\sum_{k=1}^n (x_k-\bar{x})(y_k-\bar{y}) = \sum_{k=1}^n x_k y_k - x_k\bar{y} - \bar{x}y_k + \bar{x}\bar{y} = \sum_{k=1}^n x_k y_k - x_k\bar{y} + \frac{1}{2}\bar{x}\bar{y} + \frac{1}{2}\bar{x}\bar{y} - \bar{x}y_k =$$

$$= \sum_{k=1}^n x_k y_k - \bar{y}\sum_{k=1}^n \left(x_k - \frac{1}{2}\bar{x}\right) - \bar{x}\sum_{k=1}^n \left(y_k - \frac{1}{2}\bar{y}\right) = \sum_{k=1}^n x_k y_k - \bar{x}\left[\sum_{k=1}^n y_k - \frac{1}{2}\sum_{k=1}^n\sum_{j=1}^n \frac{y_j}{n}\right] - \bar{y}\left[\sum_{k=1}^n x_k - \frac{1}{2}\sum_{k=1}^n\sum_{j=1}^n \frac{x_j}{n}\right] =$$

$$= \sum_{k=1}^n x_k y_k - \bar{x}\left[\bar{y}n - \frac{1}{2}\bar{y}n\right] - \bar{y}\left[\bar{x}n - \frac{1}{2}\bar{x}n\right] = \sum_{k=1}^n x_k y_k - \bar{x}\bar{y}\cdot n$$

8. $\vec{\mu}, X \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$. Niech $S = (X-\vec{\mu})^T \Sigma^{-1} (X - \vec{\mu})$ oraz $Y = AX$, gdzie A jest odwracalna. Pokażmy, że $S = (Y - A\vec{\mu})^T (A\Sigma A^T)^{-1} (Y - A\vec{\mu})$.

$$(Y - A\vec{\mu})^T (A\Sigma A^T)^{-1} (Y - A\vec{\mu}) = (Ax - A\vec{\mu})^T (A\Sigma A^T)^{-1} (AX - A\vec{\mu}) =$$

$$= \left((Ax)^T - (A\mu)^T\right) \left(A(A\Sigma^T)^T\right)^{-1} A(X - \vec{\mu}) =$$

$$= \left(x^T A^T - \mu^T A^T\right)\left((A\Sigma^T)^T\right)^{-1} A^{-1} A(x - \vec{\mu}) = (x^T - \mu^T)A^T\left((A\Sigma^T)^{-1}\right)^T I (X - \vec{\mu}) =$$

$$= (x-\mu)^T A^T \left((\Sigma^T)^{-1} A^{-1}\right)^T (X - \vec{\mu}) = (X-\mu)^T A^T (A^{-1})^T \left((\Sigma^{-1})^T\right)^T (X - \vec{\mu}) =$$

$$= (X-\mu)^T (A^{-1}A)^T \Sigma^{-1} (X - \vec{\mu}) = (X-\mu)^T \cdot I^T \Sigma^{-1} (X - \vec{\mu}) = (x - \mu^T)\Sigma^{-1}(X - \vec{\mu}) \quad \blacksquare$$