RPiS Lista 1 Krystian Jasionek

1. (2p)

a) $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$

Zauważmy, że ze wzoru dwumianowego Newtona $(a+b)^n = \sum_{k=0}^{n} a^k b^{n-k}$, zatem jeśli $a=p$, $b=p-1$, wtedy

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+1-p)^n = 1^n = 1$$

b) $\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$

$$\sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} k \cdot \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k} =$$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k} = \sum_{k=0}^{n-1} \frac{n!}{(n-k-1)!\cdot k!} p^{k+1} (1-p)^{n-1-k} = np\sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-k-1)!} p^k (1-p)^{n-1-k} =$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} = np \cdot (p+1-p)^{n-1} = np$$

2. (a) $\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = 1$

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

Rozwiņy e^x w szereg Taylora.

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, stąd $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e$, zatem

$$e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e = 1$$

(b) $\sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \lambda$

$$\sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$$

3. $\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\,dt, \ p>0$. $\quad \Gamma(n) = (n-1)!, \ n \in \mathbb{N}$.

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\,dt = \left| \begin{matrix} \text{możemy całkować} \\ \text{przez części} \end{matrix} \right| = \left| \begin{matrix} f = t^{n-1} & g' = e^{-t} \\ f' = (n-1)t^{n-2} & g = -e^{-t} \end{matrix} \right| = -t^{n-1} \cdot e^{-t} - \int_0^\infty (n-1)t^{n-2} \cdot -e^{-t}\,dt$$

$$\Gamma(n) = -t^{n-1} e^{-t} + (n-1) \int_0^\infty t^{n-2} e^{-t}$$

Możemy to zapisać całką oznaczoną.

$$\Gamma(n) = \left[-t^{n-1} e^{-t}\right]_0^\infty + (n-1) \int_0^\infty t^{n-2} e^{-t}$$

$$\Gamma(n) = -\lim_{t\to\infty} t^{n-1} e^{-t} - (-0 \cdot 1) + (n-1) \int_0^\infty t^{n-2} e^{-t}$$

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^t} = \left| \frac{\infty}{\infty} \text{ - zatem d'l'Hospital} \right| = \lim_{t\to\infty} \frac{t^{n-2}}{e^t} = \left| \begin{matrix} \text{taka sama sytuacja,} \\ \text{więc powtarzamy to n-1 razy} \end{matrix} \right| = \ldots = \lim_{t\to\infty} \frac{1}{e^t} = 0$$

$$\boxed{\Gamma(n) = (n-1) \int_0^\infty t^{n-2} e^{-t}}$$, ponownie całkujemy przez części

$$\Gamma(n) = (n-1)\left[\left[-t^{n-2} e^{-t}\right]_0^\infty + (n-2) \int_0^\infty t^{n-3} e^{-t}\,dt \right]$$, widać, że będziemy to kontynuować, aż dojdziemy do

$$(n-1)(n-2) \cdots (2) \int_0^\infty t^0 e^{-t}\,dt = (n-1)!$$ .. Jak to poprawnie sformułować?

Zauważmy, że $\Gamma(n) = (n-1)\,\Gamma(n-1)$, zatem możemy obliczyć $\Gamma(n)$ rekurencyjnie. Ile wynosi podstawa rekurencji, tj. $\Gamma(1)$?

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\,dt = \int_0^\infty e^{-t}\,dt = \left[-e^{-t}\right]_0^\infty = \left[\frac{-1}{e^t}\right]_0^\infty = 0 + 1 = 1$$, zatem

$\Gamma(n) = (n-1)\,\Gamma(n-1)$; $\Gamma(1) = 1$, stąd $\Gamma(n) = (n-1)!$

4) $f(x) = \lambda \exp(-\lambda x)$, $\lambda > 0$.

(a) $\int_0^\infty \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-\lambda x} dx = \lambda \cdot \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty = \lambda \left[-\lim_{x \to \infty} \frac{1}{\lambda \cdot e^{\lambda x}} + \frac{1}{\lambda} e^{-\lambda \cdot 0} \right] =$

$= \lambda \left[0 + \frac{1}{\lambda} \right] = 1$

(b) $\int_0^\infty x f(x) dx = \int_0^\infty x \lambda e^{-\lambda x} dx = \lambda \int_0^\infty x e^{-\lambda x} dx = \left| \begin{matrix} f = x & g' = e^{-\lambda x} \\ f' = 1 & g = \frac{-1}{\lambda} \cdot e^{-\lambda x} \end{matrix} \right| =$

$= \lambda \left[\left[x \cdot \frac{-1}{\lambda} e^{-\lambda x} \right]_0^\infty - \int_0^\infty \frac{-1}{\lambda} \cdot e^{-\lambda x} dx \right] = -\left[x \cdot e^{-\lambda x} \right]_0^\infty + \int_0^\infty e^{-\lambda x} dx =$

$= -\lim_{x \to \infty} \frac{x}{e^{\lambda x}} + 0 \cdot e^{-\lambda \cdot 0} + \left[\frac{-1}{\lambda} e^{-\lambda x} \right]_0^\infty = 0 + 0 - \underbrace{\lim_{x \to \infty} \frac{1}{\lambda e^{\lambda x}}}_{= 0} + \frac{1}{\lambda} \cdot e^{-\lambda \cdot 0} = \frac{1}{\lambda}$

5. $D_n = n$

$$D_1 = |1| = 1$$

$$D_n = \begin{vmatrix} 1 & -1 & \cdots & -1 & -1 \\ 1 & 1 & & & \\ \vdots & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{vmatrix}$$

$$D_n = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

$$a_{00} A_{00} = \begin{vmatrix} 1 & -1 & \cdots & -1 & -1 \\ 1 & 1 & & & \\ \vdots & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{vmatrix} \cdot 1 = 1 \cdot \begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \ 1 \end{vmatrix} = 1 \cdot 1 = 1$$

$$\begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{bmatrix} \xrightarrow[\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \\ \vdots \\ R_n = R_n - R_1}]{\text{do schodkowej}} \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 2 & 1 & 1 \cdots & 1 \\ 0 & 1 & 2 & 1 \cdots & 1 \\ 0 & 1 & 1 & 2 \ 1 \cdots & 1 \\ \vdots & & & \ddots & \\ 0 & & & & 2 \end{bmatrix} \xrightarrow[\substack{R_3 = R_3 - \frac{1}{2}R_2 \\ R_4 = R_4 - \frac{1}{2}R_2 \\ \vdots \\ R_n = R_n - \frac{1}{2}R_2}]{} \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 2 & 1 & \cdots & 1 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} \cdots & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \cdots & \frac{1}{2} \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & \frac{3}{2} \end{bmatrix}$$

$$\xrightarrow[\substack{R_4 = R_4 - \frac{1}{3}R_3 \\ R_5 = R_5 - \frac{1}{3}R_3 \\ \vdots \\ R_n = R_n - \frac{1}{3}R_3}]{} \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 2 & 1 & \cdots & 1 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} \cdots & \frac{1}{2} \\ 0 & 0 & 0 & \frac{4}{3} & \vdots \\ \vdots & \vdots & 0 & \frac{1}{3} & \frac{1}{3} \\ \vdots & \vdots & \vdots & \vdots \ddots & \frac{4}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{4}{3} \end{bmatrix} \longrightarrow \cdots \longrightarrow \xrightarrow[\substack{R_k = R_k - \frac{1}{k-1} R_{k-1} \\ \vdots \\ R_n = R_n - \frac{1}{k-1} R_{k-1}}]{} \cdots \longrightarrow$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 2 & 1 & \cdots & 1 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} \cdots & \frac{1}{2} \\ 0 & 0 & 0 & \frac{4}{3} \quad \frac{1}{3} \cdots & \frac{1}{3} \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & & \frac{n}{n-1} \end{bmatrix} = A$$

$$\det A = 1 \cdot \cancel{2} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{5}}{\cancel{4}} \cdot \ldots \cdot \frac{\cancel{n-1}}{\cancel{n-2}} \cdot \frac{n}{\cancel{n-1}}$$

$$\det A = 1 \cdot n = n$$

6. $I = \int_{-\infty}^{\infty} \exp\left\{\frac{-x^2}{2}\right\} dx$. $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{x^2+y^2}{2}\right\} dy\,dx$.

Podstawiamy $x = r\cos\theta$, $y = r\sin\theta$. Pokażę, że $I^2 = 2\pi$.

$x = r\cos\theta \qquad y = r\sin\theta$

Jakobian przejścia (z wikipedii)

$$\frac{D(x,y)}{D(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = \boxed{r}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dy\,dx = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}}\, r\,d\theta\,dr = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}}\, r\,d\theta\,dr =$$

$$= \int_0^{\infty}\left[\theta e^{-\frac{r^2}{2}} r\right]_0^{2\pi} = \int_0^{\infty} 2\pi e^{-\frac{r^2}{2}} r\,dr = \left[-2\pi e^{-\frac{r^2}{2}}\right]_0^{\infty} = \lim_{r\to\infty} -2\pi\frac{1}{e^{\frac{r^2}{2}}} + 2\pi\frac{1}{e^0}$$

$$= -2\pi\cdot\frac{1}{e^{\infty}} + 2\pi\cdot\frac{1}{e^0} = -2\pi\cdot 0 + 2\pi\cdot 1 = 2\pi \quad \blacksquare$$

$$\int_0^{\infty} 2\pi e^{-\frac{r^2}{2}} r\,dr = \left| \begin{matrix} u = \frac{-r^2}{2} \\ du = -r\,dr \end{matrix} \right| = \int_0^{\infty} -2\pi e^u\,du = \left[-2\pi e^u\right]_0^{\infty} = \left[-2\pi e^{\frac{-r^2}{2}}\right]_0^{\infty}$$

7. (a) $\sum\limits_{k=1}^{n}(x_k-\overline{x})^2=\sum\limits_{k=1}^{n}x_k^2-n\cdot\overline{x}^2$, $\quad$ $\overline{x}$ - sredina brojev $x_1, x_2, \dots, x_n$

$$\sum_{k=1}^{n}(x_k-\overline{x})^2=\sum_{k=1}^{n}x_k^2-2x_k\overline{x}+\overline{x}^2=\sum_{k=1}^{n}x_k^2-\overline{x}\sum_{k=1}^{n}(2x_k-\overline{x})=\sum_{k=1}^{n}x_k^2-\overline{x}\left(\sum_{k=1}^{n}2x_k-\sum_{j=1}^{n}\frac{x_j}{n}\right)=$$

$$=\sum_{k=1}^{n}x_k^2-\overline{x}\left[\sum_{k=1}^{n}2x_k-\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{x_j}{n}\right]=\sum_{k=1}^{n}x_k^2-\overline{x}\left[2n\overline{x}-n\overline{x}\right]=\sum_{k=1}^{n}x_k^2-\overline{x}\,n\overline{x}=\sum_{k=1}^{n}x_k^2-n\overline{x}\ \blacksquare$$

(b) $\sum\limits_{k=1}^{n}(x_k-\overline{x})(y_k-\overline{y})=\sum\limits_{k=1}^{n}x_ky_k-n\overline{x}\overline{y}$

$$\sum_{k=1}^{n}(x_k-\overline{x})(y_k-\overline{y})=\sum_{k=1}^{n}x_ky_k-x_k\overline{y}-\overline{x}y_k+\overline{x}\overline{y}=\sum_{k=1}^{n}x_ky_k-x_k\overline{y}+\frac{1}{2}\overline{x}\overline{y}+\frac{1}{2}\overline{x}\overline{y}-\overline{x}y_k=$$

$$=\sum_{k=1}^{n}x_ky_k-\overline{y}\sum_{k=1}^{n}\left(x_k-\frac{1}{2}\overline{x}\right)-\overline{x}\sum_{k=1}^{n}\left(y_k-\frac{1}{2}\overline{y}\right)=\sum_{k=1}^{n}x_ky_k-\overline{x}\left[\sum_{k=1}^{n}y_k-\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{y_j}{n}\right]-\overline{y}\left[\sum_{k=1}^{n}x_k-\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{x_j}{n}\right]=$$

$$=\sum_{k=1}^{n}x_ky_k-\overline{x}\left[\overline{y}n-\frac{1}{2}\overline{y}n\right]-\overline{y}\left[\overline{x}n-\frac{1}{2}\overline{x}n\right]=\sum_{k=1}^{n}x_ky_k-\overline{x}\overline{y}\cdot n$$

8. $\vec{\mu}, X \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n \times n}$. Niech $S = (X - \vec{\mu})^T \Sigma^{-1} (X - \vec{\mu})$ oraz $Y = AX$, gdzie A jest odwracalna. Pokażmy, że $S = (Y - A\vec{\mu})^T (A \Sigma A^T)^{-1} (Y - A\vec{\mu})$.

$$(Y - A\vec{\mu})^T (A \Sigma A^T)^{-1} (Y - A\vec{\mu}) = (Ax - A\vec{\mu})^T (A \Sigma A^T)^{-1} (AX - A\vec{\mu}) =$$

$$= \left((Ax)^T - (A\mu)^T\right) \left(A (A \Sigma^T)^T\right)^{-1} A(X - \vec{\mu}) =$$

$$= (x^T A^T - \mu^T A^T) \left((A \Sigma^T)^T\right)^{-1} A^{-1} A(x - \vec{\mu}) = (x^T - \mu^T) A^T \left(\left(A \Sigma^T\right)^{-1}\right)^T I (X - \vec{\mu}) =$$

$$= (x - \mu)^T A^T \left((\Sigma^T)^{-1} A^{-1}\right)^T (X - \vec{\mu}) = (X - \mu)^T A^T (A^{-1})^T \left((\Sigma^{-1})^T\right)^T (X - \vec{\mu}) =$$

$$= (X - \mu)^T (A^{-1} A)^T \Sigma^{-1} (X - \vec{\mu}) = (X - \mu)^T \cdot I^T \Sigma^{-1} (X - \vec{\mu}) = (x - \mu^T) \Sigma^{-1} (X - \vec{\mu}) \quad \blacksquare$$