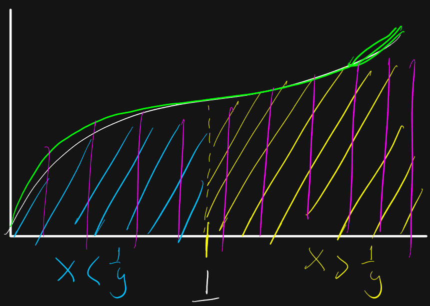
11. Niech X podlega standardowemu rozkładowi Cauchy'ego, $f_X(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$. Udowodnić, że $Y = \frac{1}{X}$ ma również standardowy rozkład Cauchy'ego.

$$F_Y(y) = P(Y<y) = P\left(\frac{1}{x} < y\right) = \left|\begin{matrix} \frac{1}{x} < y \ / \cdot \frac{\cancel{x}}{y} \\ x > \frac{1}{y} \end{matrix}\right| = P\left(X > \frac{1}{y}\right)$$



$$P\left(X>\frac{1}{y}\right) = 1 - P\left(X<\frac{1}{y}\right), \text{ bo } \int_{\mathbb{R}} f_X(x)\,dx = 1.$$

$$F_Y(y) = P\left(X>\frac{1}{y}\right) = 1 - P\left(X<\frac{1}{y}\right) = 1 - \int_{-\infty}^{\frac{1}{y}} \frac{1}{\pi(1+x^2)}\,dx = 1 - \frac{1}{\pi}\int_{-\infty}^{\frac{1}{y}} \frac{1}{1+x^2}\,dx = \left|\begin{matrix} x = \tan(\phi) \\ dx = \frac{1}{\cos^2\phi}\,d\phi \\ \operatorname{atan} x = \phi \end{matrix}\right| =$$

$$= 1 - \frac{1}{\pi}\int_{-\infty}^{\frac{1}{y}} \frac{\frac{1}{\cos^2\phi}}{1+\tan^2\phi}\,d\phi = \left|\begin{matrix} \frac{1}{\cos^2\varphi} = \frac{1}{\cos^2\varphi}\cdot 1 \\ \frac{1}{\cos^2\varphi} = \frac{1}{\cos^2\varphi}\cdot(\sin^2\varphi + \cos^2\varphi) \\ \frac{1}{\cos^2\varphi} = \frac{\cos^2\varphi}{\cos^2\varphi} + \frac{\sin^2\varphi}{\cos^2\varphi} = 1+\tan^2\varphi \end{matrix}\right| = 1 - \frac{1}{\pi}\int_{-\infty}^{\frac{1}{y}} \frac{1+\tan^2\phi}{1+\tan^2\phi}\,d\phi = 1 - \frac{1}{\pi}\int_{-\infty}^{\frac{1}{y}} d\phi =$$

$$= 1 - \frac{1}{\pi}\left[\phi\right]_{-\infty}^{\frac{1}{y}} = 1 - \frac{1}{\pi}\left[\operatorname{atan} x\right]_{-\infty}^{\frac{1}{y}} = 1 - \frac{1}{\pi}\left[\operatorname{atan}\frac{1}{y} - \lim_{x\to-\infty}\operatorname{atan}(\varphi)\right] = 1 - \frac{1}{\pi}\left[\operatorname{atan}\frac{1}{y} + \frac{\pi}{2}\right] =$$

$$= 1 - \frac{1}{\pi}\operatorname{atan}\frac{1}{y} - \frac{1}{2} = \frac{1}{2} - \frac{1}{\pi}\operatorname{atan}\frac{1}{y}$$

$$f_Y(y) = F_Y'(y) = \left(\frac{1}{2} - \frac{1}{\pi}\operatorname{atan}\frac{1}{y}\right)' = 0 - \frac{1}{\pi}\cdot\frac{1}{1+\left(\frac{1}{y}\right)^2}\cdot\left(-\frac{1}{y^2}\right) = \frac{1}{\pi}\,\frac{1}{y^2\left(1+\frac{1}{y^2}\right)} = \frac{1}{\pi}\cdot\frac{1}{1+y^2}$$

Zatem $Y = \frac{1}{X}$ także ma rozkład Cauchy'ego.