6. $I = \int_{-\infty}^{\infty} \exp\left\{\frac{-x^2}{2}\right\} dx$. $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{x^2+y^2}{2}\right\} dy\,dx$.

Podstawienie $x = r\cos\theta$, $y = r\sin\theta$. Pokaż, że $I^2 = 2\pi$.

$x = r\cos\theta \quad y = r\sin\theta$

Jakobian przejścia (z wikipedii)

$$\frac{D(x,y)}{D(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = \boxed{r}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dy\,dx = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}} r\,d\theta\,dr = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}} r\,d\theta\,dr =$$

$$= \int_0^{\infty} \left[\theta e^{-\frac{r^2}{2}} r\right]_0^{2\pi} = \int_0^{\infty} 2\pi e^{-\frac{r^2}{2}} r\,dr = \left[-2\pi e^{-\frac{r^2}{2}}\right]_0^{\infty} = \lim_{r\to\infty} -2\pi \frac{1}{e^{\frac{r^2}{2}}} + 2\pi\frac{1}{e^0}$$

$$= -2\pi \cdot \frac{1}{e^{\infty}} + 2\pi \cdot \frac{1}{e^0} = -2\pi\cdot 0 + 2\pi \cdot 1 = 2\pi \quad \blacksquare$$

$$\int_0^{\infty} 2\pi e^{-\frac{r^2}{2}} r\,dr = \begin{vmatrix} u = -\frac{r^2}{2} \\ du = -r\,dr \end{vmatrix} = \int_0^{\infty} -2\pi e^u\,du = \left[-2\pi e^u\right]_0^{\infty} = \left[-2\pi e^{-\frac{r^2}{2}}\right]_0^{\infty}$$