10. Dane są zmienne losowe $X_1, \ldots, X_n$. Udowodnić, że:

$$\sum_{k=1}^{n} (X_k - \mu)^2 = \sum_{k=1}^{n} \left(X_k - \bar{\mathbf{X}}\right)^2 + n\,(\bar{\mathbf{X}} - \mu)^2.$$

$$L = \sum_{k=1}^{n} (X_k - \mu)^2 = \sum_{k=1}^{n} X_k^2 - 2X_k\mu - \mu^2$$

$$P = \sum_{k=1}^{n} (X_k - \bar{X})^2 + n(\bar{X} - \mu)^2 = \sum_{k=1}^{n} X_k^2 - 2X_k\bar{X} + \bar{X}^2 + n(\bar{X}^2 - 2\bar{X}\mu + \mu^2)$$

Z zad 7. z listy 1 mamy, że

7. Symbol $\bar{s}$ oznacza srednią ciągu $s_1, \ldots, s_n$. Udowodnić, że:

(a) $\sum_{k=1}^{n} (x_k - \bar{x})^2 = \sum_{k=1}^{n} x_k^2 - n \cdot \bar{x}^2$,

(b) $\sum_{k=1}^{n} (x_k - \bar{x})(y_k - \bar{y}) = \sum_{k=1}^{n} x_k y_k - n\bar{x}\bar{y}$.

Stąd

$$P = \sum_{k=1}^{n} (X_k - \bar{X})^2 + n(\bar{X} - \mu)^2 = \sum_{k=1}^{n} (X_k^2) - n\bar{X}^2 + n(\bar{X} - \mu)^2$$

$$P = \sum_{k=1}^{n} (X_k^2) - \cancel{n\bar{X}^2} + \cancel{n\bar{X}^2} - 2n\bar{X}\mu + n\mu^2 = \sum_{k=1}^{n} (X_k^2) - 2\sum_{k=1}^{n} X_k\mu + \sum_{k=1}^{n} \mu^2$$

$$P = \sum_{k=1}^{n} X_k^2 - 2X_k\mu + \mu^2 = \sum_{k=1}^{n} (X_k - \mu)^2 = L \quad \blacksquare$$