考古題

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1. Let

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$$

An eigenvalue of A is 3. Find a basis for the corresponding eigenspace.

1

找 eigenvector

$(\lambda I - A)x = 0$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix} = \begin{bmatrix} \lambda-4 & -2 & -3 \\ 1 & \lambda-1 & 3 \\ -2 & -4 & \lambda-9 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 2 & 3 \\ -2 & -4 & -6 \end{bmatrix} = A'$$

$\lambda = 3$ (題目給的)

$A'x = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{cases} -x_1 - 2x_2 - 3x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0 \\ -2x_1 - 4x_2 - 6x_3 = 0 \end{cases}$$

$-x_1 = 2$

$x_1 = -2t - 3s,\ x_2 = t,\ x_3 = s$

$-x_1 - 2t - 3s = 0$

$$x = \begin{bmatrix} -2t-3s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$(-2, 1, 0),\ (-3, 0, 1)$

2. Let

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

If we know the eigenvalues of A are 5,1,1. Find a matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

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$P = [\, x_1 \mid x_2 \mid x_3 \,]$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For $\lambda_1 = 5$

$(\lambda I - A)x = 0$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = A'$$

$A'x = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{cases} 3x_1 - 2x_2 + x_3 = 0 \\ -x_1 + 2x_2 + x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0 \end{cases}$$

$x_1 = s, \quad x_3 = -s, \quad x_2 = s$

$$x = \begin{bmatrix} s \\ s \\ -s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

For $\lambda = 1$

$(\lambda I - A)x = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 1 \\ -1 & -2 & 1 \\ 1 & 2 & -1 \end{bmatrix} = A'$$

$A'x = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{cases} -x_1 - 2x_2 + x_3 = 0 \\ -x_1 - 2x_2 + x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases}$$

$x_1 = s, \quad x_2 = t, \quad x_3 = s + 2t$

$$x = \begin{bmatrix} s \\ t \\ s+2t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

3. Let

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

If we know the eigenvalues of A are -4,4,7. Please find an orthogonal matrix Q and diagonal matrix D such that $Q^T A Q = D$.

• orthogonal
단위 eigenspace

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① $\lambda = -4$

$(\lambda I - A)x = 0$

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -1 & -5 \\ -1 & -9 & -1 \\ -5 & -1 & -5 \end{bmatrix} = A'$$

$A'x = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{cases} -5x_1 - x_2 - 5x_3 = 0 \\ -x_1 - 9x_2 - x_3 = 0 \\ -5x_1 - x_2 - 5x_3 = 0 \end{cases}$$

$x_1 = s, \quad x_2 = 0, \quad x_3 = -s$

$$x = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

② $\lambda = 4$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -5 \\ -1 & -1 & -1 \\ -5 & -1 & 3 \end{bmatrix} = A'$$

$A'x = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{cases} 3x_1 - x_2 - 5x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \\ -5x_1 - x_2 + 3x_3 = 0 \end{cases}$$

$x_1 = t, \quad x_2 = -2t, \quad x_3 = t$

$$x = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

③ $\lambda = 7$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 6 \end{bmatrix} = A'$$

$A'x = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{cases} 6x_1 - x_2 - 5x_3 = 0 \\ -x_1 + 2x_2 - x_3 = 0 \\ 5x_1 - x_2 + 6x_3 = 0 \end{cases}$$

$x_1 = s, \quad x_2 = s, \quad x_3 = s$

$$x = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

4. Let

$$A = \begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix} \quad \times -\frac{2}{3}$$

Find the LU decomposition of A.

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$$U = \begin{bmatrix} 6 & 9 \\ 0 & -1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix}$$

5. Given a quadratic form $2x_1^2 - 4x_1x_2 - x_2^2$, please make a change of variable $x = Py$ that transforms this quadratic form into one with no cross-product term.

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$$2x_1^2 - 4x_1x_2 - x_2^2$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}}_{= A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} \lambda-2 & 2 \\ 2 & \lambda+1 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$$

① $\lambda_1 = 3$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = A'$$

$$A'x = 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{cases}$$

$x_1 = 2s, \quad x_2 = -s$

$$x = s\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$x = s\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}$$

② $\lambda = -2$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = A'$$

$$A'x = 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{cases} -4x_1 + 2x_2 = 0 \\ 2x_1 - x_2 = 0 \end{cases}$$

$$x_1 = s, \; x_2 = 2s$$

$$x \quad s\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x = PY \qquad Y = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

$$X^T A X = Y^T D Y$$

$$= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= 3(x')^2 + -2(y')^2$$

6. Construct a spectral decomposition of the matrix A that has the orthogonal diagonalization:

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

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$$A = QDQ^T$$

$$\begin{bmatrix} u_1 \mid u_2 \mid \cdots \mid u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$$

$$\lambda_1 = 8 \quad , \quad u_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\lambda_2 = 3 \quad , \quad u_2 = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$u_1 u_1^T = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$u_2 u_2^T = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

$$A = 8 \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

7. Let

$$A = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Use the Cayley-Hamilton theorem to compute $A^3 - 6A^2$.

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$$\det(\lambda I - A) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-5 & -4 & -3 \\ 1 & \lambda & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$= \lambda^2 - 14\lambda + 32 = P(\lambda)$$

$$P(A) = A^2 - 14A + 32I$$

$$A^3 - 6A^2 = A(\underbrace{A^2 - 14A + 32I}_{}) + 8A^2 - 32I \quad \overset{0}{\parallel}$$

$$= 8A^2 - 32I = 8\begin{bmatrix} 24 & 14 & 6 \\ -8 & 2 & -6 \\ 8 & 2 & 10 \end{bmatrix} - \begin{bmatrix} 32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} = \begin{bmatrix} 160 & 112 & 48 \\ -64 & -16 & -48 \\ 64 & 16 & 48 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 14 & 6 \\ -8 & 2 & -6 \\ 8 & 2 & 10 \end{bmatrix}$$