MATH50001 Analysis II, Complex Analysis

Lecture 18

Section: Harmonic functions.

Definition. Let $\varphi = \varphi(x, y)$, $x, y \in \mathbb{R}^2$ be a real function of two variables. It said to be *harmonic* in an open set $\Omega \subset \mathbb{R}^2$ if

$$\Delta\varphi(x,y) := \frac{\partial^2\varphi}{\partial x^2}(x,y) + \frac{\partial^2\varphi}{\partial y^2}(x,y) = \varphi''_{xx}(x,y) + \varphi''_{yy}(x,y) = 0.$$

Usually Δ is called the Laplace operator.

Theorem. Let $f(z) = u(x, y) + iv(x, y)$ be holomorphic in an open set $\Omega \subset \mathbb{C}$. Then u and v are harmonic.

Proof.
Since $f = u + iv$ is holomorphic it is infinitely differentiable. In particular, the functions u and v have continuous second derivatives that allows us to change the order of the second derivatives and using the Cauchy-Riemann equations to obtain

$$u''_{xx} = (u'_x)'_x = (v'_y)'_x = (v'_x)'_y = (-u'_y)'_y = -u''_{yy}.$$

Therefore

$$u''_{xx} + u''_{yy} = 0.$$

Similarly we find that $\Delta v = 0$.

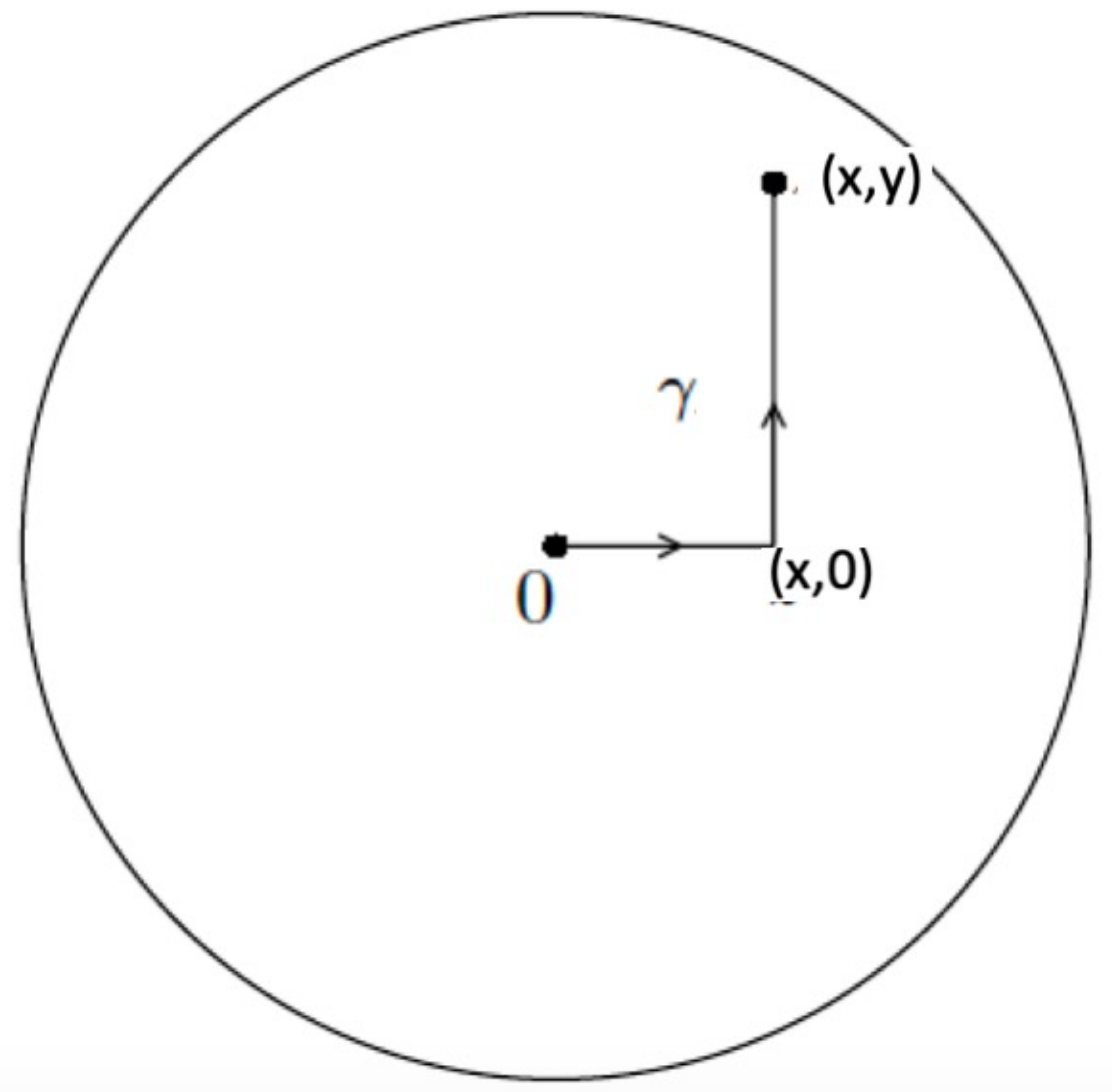
Theorem. (Harmonic conjugate)

Let u be harmonic in an open disc $D \subset \mathbb{C}$. Then there exists a harmonic function v such that $f = u + iv$ is holomorphic in D. In this case v is called harmonic conjugate to u.

Proof.

We can assume that $D = D_R = \{(x, y) \in \mathbb{R}^2 : |z| < R\}$, $R > 0$. Let $(x, y) \in D_R$ and let $\gamma = \gamma_1 \cup \gamma_2$, where

$$\gamma_1 = \{(t, s) \in \mathbb{R}^2 : t \in (0, x),\ s = 0\},$$
$$\gamma_2 = \{(t, s) : t = x,\ s \in (0, y)\},$$



We now define

$$v(x,y) = \int_\gamma \left(-\frac{\partial u}{\partial y}\,dt + \frac{\partial u}{\partial x}\,ds\right) = -\int_0^x \frac{\partial u(t,0)}{\partial y}\,dt + \int_0^y \frac{\partial u(x,s)}{\partial x}\,ds.$$

Using $u''_{xx} = -u''_{yy}$ we obtain

$$\begin{aligned} v'_x(x,y) &= -u'_y(x,0) + \int_0^y \frac{\partial^2 u(x,s)}{\partial x^2}\,ds = -u'_y(x,0) - \int_0^y \frac{\partial^2 u(x,s)}{\partial s^2}\,ds \\ &= -u'_y(x,0) + u'_y(x,0) - u'_y(x,y) = -u'_y(x,y). \end{aligned}$$

Differentiating v with respect to y we have

$$v'_y(x,y) = \frac{\partial}{\partial y}\left(-\int_0^x \frac{\partial u(t,0)}{\partial y}\,dt + \int_0^y \frac{\partial u(x,s)}{\partial x}\,ds\right) = 0 + u'_x(x,y).$$

Thus the C-R equations are satisfied and we conclude that $f(z) = u(x,y) + iv(x,y)$ is holomorphic inside D.

Remark.

In a simply connected domain $\Omega \subset \mathbb{R}^2$ every harmonic function u has a harmonic conjugate v defined by the line integral

$$v(x, y) = \int_\gamma \left(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy \right),$$

where the path of integration γ is a curve starting at a fixed base-point $(x_0, y_0) \in \Omega$ with the end point at $(x, y) \in \Omega$. The integral in independent of path by Green's theorem because u is harmonic and Ω is simply connected.

We leave this statement without the proof because it requires Green's theorem that we did not have in our course.

Example. Let $u(x, y) = \ln(x^2 + y^2)$ defined in $\mathbb{R}^2 \setminus \{0\}$ and let

$$\Omega = \mathbb{C} \setminus \{z = x + iy : x \in (-\infty, 0], \, y = 0\}.$$

Find in Ω a harmonic conjugate v to u and thus a holomorphic function $f = u + iv$.

Step 1. We first check that $\ln(x^2 + y^2)$ is harmonic in $\mathbb{R} \setminus \{0\}$. Indeed,

$$u'_x = \frac{2x}{x^2 + y^2}, \qquad u''_{xx} = \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2}$$

and

$$u'_y = \frac{2y}{x^2 + y^2}, \qquad u''_{yy} = \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2}.$$

Thus $\Delta u = 0$.

Step 2. In order to find u's harmonic conjugate we use the Cauchy-Riemann equations.

a) $v'_y = u'_x = 2x/(x^2+y^2)$ implies

$$v(x,y) = \int \frac{2x}{x^2+y^2}\,dy = 2\arctan\frac{y}{x} + C(x).$$

b) $u'_y = -v'_x$ implies

$$\frac{2y}{x^2+y^2} = -\frac{2}{1+y^2/x^2}\cdot\frac{-y}{x^2} + C'(x) \implies C'(x) = 0$$

and thus $C(x) = C \in \mathbb{R}$.

Solution: $v = 2\arctan\frac{y}{x} + C$ and hence

$$f(z) = \ln(x^2+y^2) + 2i\arctan\frac{y}{x} + iC = 2(\ln|z| + i\operatorname{Arg} z) + iC.$$

Example. Let $u(x,y) = x^3 - 3xy^2 + y$.

i. Verify that the function u is harmonic.

ii. Find all harmonic conjugates v of u.

iii. Find the holomorphic function f, $\operatorname{Re} f = u$, as a function of z, s.t. $f(1) = 1 + i$.

Step 1. For $u = x^3 - 3xy^2 + y$ we have $u'_x = 3x^2 - 3y^2$, $u''_{xx} = 6x$ and $u'_y = -6xy + 1$, $u''_{yy} = -6x$. Thus we have

$$\Delta u(x,y) = u''_{xx} + u''_{yy} = 6x - 6x = 0.$$

Step 2. Cauchy-Riemann equations imply

$$v'_y = u'_x = 3x^2 - 3y^2.$$

Integrating the latter w.r.t. y we find

$$v = 3x^2y - y^3 + F(x),$$

and differentiating it w.r.t. x we have

$$v'_x = 6xy + F'(x) = -u'_y = 6xy - 1.$$

So $F'(x) = -1$ and $F(x) = -x + c$, $c \in \mathbb{R}$. This implies

$$\begin{multline*} v = 3x^2y - y^3 - x + c, \\ f = u + iv = x^3 - 3xy^2 + y + 3ix^2y - iy^3 - ix + ic \\ = (x+iy)^3 - i(x+iy) + ic. \end{multline*}$$

Step 3.

We find $f(z) = z^3 - iz + ic$. Solving the equation

$$f(1) = 1 + i = \left(z^3 - iz + ic\right)_{z=1} = 1 - i + ic$$

we find $c = 2$.

Section: Properties of real and imaginary parts of holomorphic functions.

Theorem.

Assume that $f = u + iv$ is a holomorphic function defined on an open connected set $\Omega \subset \mathbb{C}$. Consider two equations

$$a) \quad u(x, y) = C \qquad \text{and} \qquad b) \quad v(x, y) = K,$$

where C, K are two real constants.

Assume that the equations $a)$ and $b)$ have the same solution (x_0, y_0) and that $f'(z_0) \neq 0$ at $z_0 = x_0 + iy_0$. Then the curve defined by the equation $a)$ is orthogonal to the curve defined by the equation $b)$ at (x_0, y_0).

Proof. It is enough to show that the gradient ∇u and ∇v are orthogonal at z_0. We use C-R equations and obtain

$$\nabla u \cdot \nabla v = u'_x v'_x + u'_y v'_y = v'_y v'_x - v'_x v'_y = 0.$$

Example. Let $f(z) = \ln(x^2 + y^2) + 2i \arctan \frac{y}{x}$. Consider

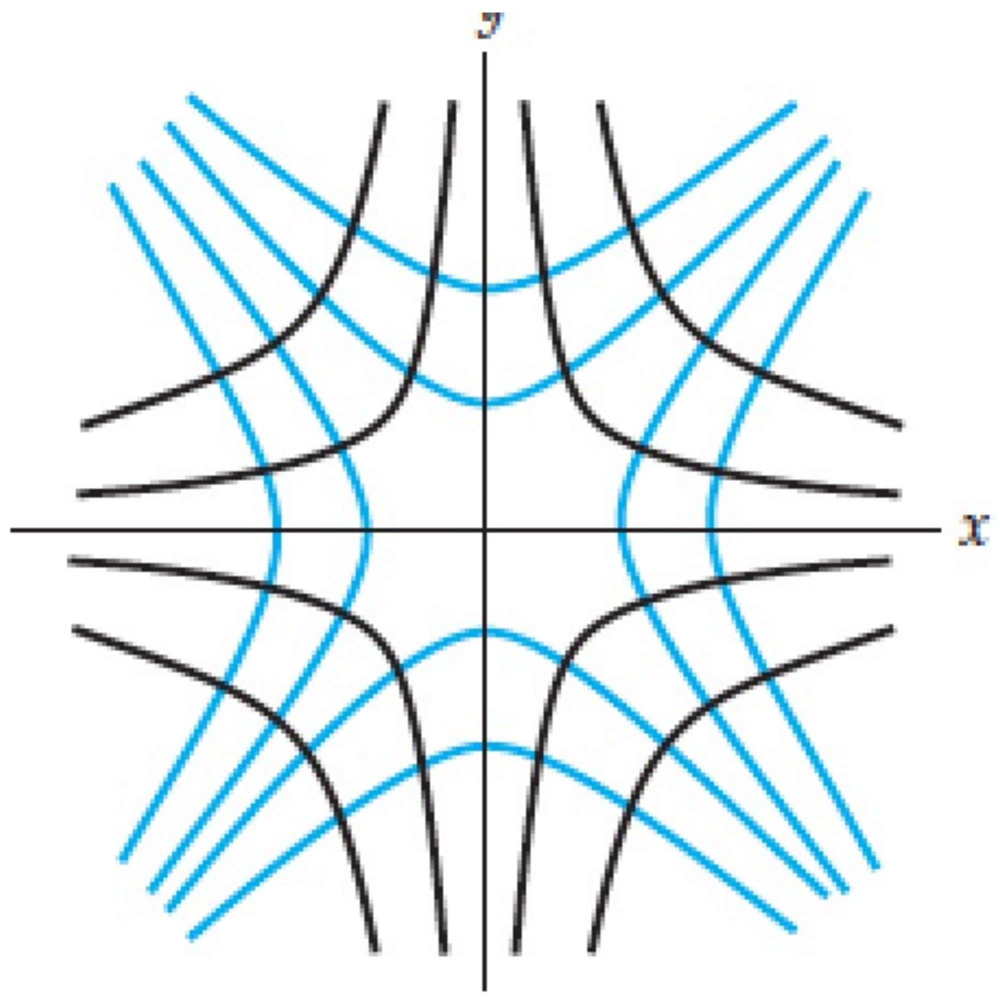
$$\ln(x^2 + y^2) = C \quad \Longrightarrow \quad x^2 + y^2 = e^C.$$

This is a circle whose radius is $e^{C/2}$.
The second equation

$$2 \arctan \frac{y}{x} \; = \; K \quad \Longrightarrow \quad \frac{y}{x} \; = \; \tan(K/2) \quad \Longrightarrow \quad y \; = \; \tan(K/2) \cdot x$$

and this equation describes a straight line going through the origin.

Example. Let $f(z) = z^2 = x^2 - y^2 + 2ixy$. Then we have



Thank you