# MATH50004 Differential Equations
## Spring Term 2021/22

### Repetition Material 5: The trace-determinant rule

The trace-determinant rule helps to quickly obtain information about real parts of eigenvalues of $2 \times 2$ matrices, which is crucial for stability analysis. It does not require you to look at the characteristic polynomial, and you only need to compute the trace and determinant of the matrix.

**Proposition 1** (Trace-determinant rule for two-dimensional linear systems)**.** *Consider a matrix $A \in \mathbb{R}^{2\times 2}$, and let $p := \operatorname{tr} A$ be its trace and $q := \det A$ be its determinant. Then all eigenvalues of $A$*

*(i) have negative real part if and only if $p < 0$ and $q > 0$ (stability),*

*(ii) are non-real if and only if $p^2 - 4q < 0$ (focus or centre),*

*(iii) have positive real part if and only if $p > 0$ and $q > 0$ (instability),*

*(iv) are real and have opposite signs if and only if $q < 0$ (saddle).*

*Proof.* We first show that if $\lambda_1, \lambda_2$ are the eigenvalues of $A$ (i.e. the roots of the characteristic polynomial), then $q = \lambda_1\lambda_2$ and $p = \lambda_1 + \lambda_2$. Here we use that $\det(A - \lambda \operatorname{Id}_2)$ can be written as $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - p\lambda + q$.

Assume that $\lambda_1$ and $\lambda_2$ are the eigenvalues of $A$. Then the characteristic polynomial reads as

$$\chi(\lambda) = \det(A - \lambda \operatorname{Id}) = (\lambda - \lambda_1)(\lambda - \lambda_2) \quad \text{for all } \lambda \in \mathbb{C}\,.$$

For $\lambda = 0$, we get $q = \det(A) = \lambda_1\lambda_2$. To prove $p = \lambda_1 + \lambda_2$, note that the two matrices $A$ and $T^{-1}AT$ have the same trace, and we choose $T$ such that $T^{-1}AT$ is in complex Jordan form, having $\lambda_1$ and $\lambda_2$ on the diagonal. This implies that $p = \lambda_1 + \lambda_2$. It follows that $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - p\lambda + q$, and note that either $\lambda_1$ and $\lambda_2$ are both real, or they are complex conjugate, which implies in both cases that $p$ is real.

(i) If $\lambda_1$ and $\lambda_2$ have negative real part, then $p = \lambda_1 + \lambda_2 < 0$ and $\lambda_1\lambda_2 = q > 0$. Conversely, if $p < 0$, $q > 0$ and $\lambda_1$, $\lambda_2$ are real, then $q > 0$ implies they have the same sign and $p < 0$ that they are both negative. If $p < 0$, $q > 0$ and $\lambda_1$, $\lambda_2$ are complex conjugate, then $p < 0$ implies that the real part of $\lambda_1$ and $\lambda_2$ is negative.

(ii) Since $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - p\lambda + q$, the roots of this equation are $\frac{1}{2}\big(p \pm \sqrt{p^2 - 4q}\big)$ are therefore non-real if and only if $p^2 - 4q < 0$.

(iii) If $\lambda_1$ and $\lambda_2$ have positive real part, then $p = \lambda_1 + \lambda_2 > 0$ and $\lambda_1\lambda_2 = q > 0$. Conversely, if $p > 0$, $q > 0$ and $\lambda_1$, $\lambda_2$ are real, then $q > 0$ implies they have the same sign and $p > 0$ that they are both positive. If $p > 0$, $q > 0$ and $\lambda_1$, $\lambda_2$ are complex conjugate, then $p > 0$ implies that the real part of $\lambda_1$ and $\lambda_2$ is positive.

(iv) If $\lambda_1$ and $\lambda_2$ are real and have opposite signs, then $q = \lambda_1\lambda_2 < 0$. If $q < 0$, then $\lambda_1$ and $\lambda_2$ cannot be complex conjugate, so they must be both real and have opposite signs due to $q = \lambda_1\lambda_2 < 0$. $\square$