MATH50004
Differential Equations

Lecture Notes

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Contents

Chapter 1

Introduction

An algebraic equation over the real numbers is solved by real numbers. For instance, $x^2 - 1 = 0$ is solved by $x = 1$ and $x = -1$. In contrast, ordinary differential equations have functions as their solutions. Let us look at an example.

Example 1.1 (A first example)**.** We consider the differential equation

$$\dot{x} = ax\,, \tag{1.1}$$

where $a \in \mathbb{R}$ is a constant and $\dot{x}$ means $\frac{\mathrm{d}x}{\mathrm{d}t}$. We say that a function $\lambda : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, solves this differential equation if

$$\dot{\lambda}(t) = \frac{\mathrm{d}\lambda}{\mathrm{d}t}(t) = a\lambda(t) \quad \text{for all } \ t \in I\,.$$

The argument t of the solution λ typically stands for time, and x in (1.1) typically represents the state of a physical, ecological, or other system, so the solution λ describes the *evolution* of the state x in time. It should be noted that there are also lots of applications where t does not stand for time.

The differential equation (1.1) is a very simple but realistic model of applications in nature and society.

For instance, if $a > 0$, this models growth of capital with a interest rate linked to a (note that normally interest rates are given as yearly rates, which correspond to a discrete-time model; a here is a continuous-time interest rate; try as an exercise to convert both rates). For positive capital x, the right hand side of (1.1) is positive, so $\dot{x}$ is increasing, and importantly for the model of capital growth, the increase in capital is *proportional* to the amount of capital available.

In contrast, if $a < 0$, then, if $x > 0$, the right hand side of (1.1) is negative, and $\dot{x}$ is decreasing, proportional to x. This models, for instance, radioactive decay, which describes the decay of certain atoms such as Uranium 238.

It is easy to see that for a given $b \in \mathbb{R}$, the function $\lambda_b : \mathbb{R} \to \mathbb{R}$,

$$\lambda_b(t) = be^{at} \quad \text{for all } \ t \in \mathbb{R}\,,$$

solves (1.1), see Figure 1.1. Are there more solutions to this differential equation? Assume there is another solution $\mu : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mu(t)e^{-at}\big) = \dot{\mu}(t)e^{-at} - \mu(t)ae^{-at} = a\mu(t)e^{-at} - \mu(t)ae^{-at} = 0$$

for all $t \in I$. Hence, $\mu(t)e^{-at} \equiv b$ for some $b \in \mathbb{R}$, so $\mu(t) = be^{at} = \lambda_b(t)$ for all $t \in I$, which is not a new solution, so all solutions to (1.1) are known to us.

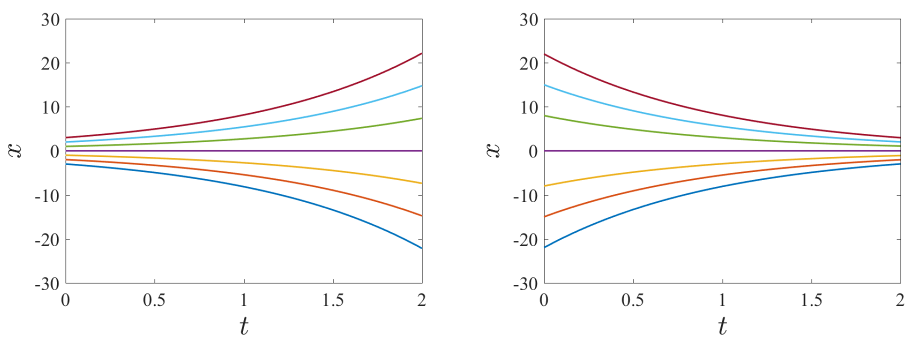

Figure 1.1. Solutions to the differential equation (1.1) for $\alpha = 1 > 0$ (left) and $\alpha = -1 < 0$ (right).

So in contrast to (simple) algebraic equations whose the solutions are in finite-dimensional vector spaces (such as $\mathbb{R}^d$), differential equations are solved by functions, and spaces of functions are typically infinite-dimensional and studied in the mathematical discipline *functional analysis*. Infinite-dimensional spaces are more difficult to grasp in general. However, for a vast majority of the material covered in this course, a finite-dimensional thinking and visualisation is enough to understand the material very well. In places, however, we will need some material from functional analysis to understand differential equations better.

You have encountered ordinary differential equations already last year. In particular, you have learned how to solve certain types of ordinary differential equations. It should be noted that for the most interesting ordinary differential equations, it is not possible to find solutions analytically. In this course, we will learn techniques how to still understand the solutions to these equations without knowing them explicitly. We also address the important question when solutions to ordinary differential equations exist and are unique.

1. Ordinary differential equations and initial value problems

In this section, we look at the definition of an ordinary differential equation and an initial value problem, and we study basic examples.

In Example 1.1, we have studied a differential equation of the form

$$\dot{x} = f(x) \tag{1.2}$$

with $f : \mathbb{R}^d \to \mathbb{R}^d$, where $d = 1$ and $f(x) = \alpha x$. Although such type of equations (i.e. *autonomous* and *first order*) will be mostly studied in this course, we would also like to deal with *nonautonomous differential equations*, i.e. where the right hand side of (1.2) depends on time t, for instance, $\dot{x} = tx^2$ (see Example 1.8 below). We note that *higher-order differential equations* also generalise the situation in (1.2), and you have studied such differential equations already in Year 1. It is demonstrated in *Repetition Material 1* that such differential equations can always be transformed to first-order differential equations, so no separate treatment (with regard to the general theory) is necessary. An example of a higher-order differential equation is given by the harmonic oscillator $\ddot{x} = -x$ (see Example 1.10 below).

The setup for *nonautonomous first-order* differential equations is explained in the next definition.

Definition 1.2 (Ordinary differential equation)**.** *Consider $d \in \mathbb{N}$, an open set $D \subset \mathbb{R} \times \mathbb{R}^d$, and a function $f : D \to \mathbb{R}^d$. An equation of the form*

$$\dot{x} = f(t, x) \tag{1.3}$$

is called a d-dimensional (first-order) ordinary differential equation*. A differentiable function $\lambda : I \to \mathbb{R}^d$ on an interval $I \subset \mathbb{R}$ is called a* solution *to the differential equation* (1.3) *if $\big(t, \lambda(t)\big) \in D$ and*

$$\dot{\lambda}(t) = f\big(t, \lambda(t)\big) \quad \textit{for all } t \in I\,. \tag{1.4}$$

An ordinary differential equation (1.3) is called *autonomous* if the right hand side does not depend on t, i.e. (1.3) is of the form

$$\dot{x} = f(x)\,,$$

where $f : D \to \mathbb{R}^d$ for some open set $D \subset \mathbb{R}^d$. In this case, we also use the symbol D for the domain of the right hand side f, here as a subset of $\mathbb{R}^d$ instead of $\mathbb{R} \times \mathbb{R}^d$, but this should not cause confusion, as it will be clear from the context. We note that any autonomous differential equation can be interpreted as a nonautonomous differential equation (1.3), and the domain

$D \subset \mathbb{R}^d$ then translates to the domain $\mathbb{R} \times D$, which is an open set if and only if D is open.

We will only treat *ordinary* differential equations (ODEs) in this course. Of great importance are also *partial* differential equations (PDEs), which are solved by functions depending on more than one variable, so, in contrast to ordinary differential equations, *partial* differentiation is needed to even define a partial differential equation.

The easiest types of solutions are constant solutions, which are also called equilibrium solutions. If the differential equation is autonomous, they are found algebraically, by zeros of the right hand side.

Proposition 1.3 (Constant solutions to autonomous differential equations)**.** *Consider an open set $D \subset \mathbb{R}^d$ and a function $f : D \to \mathbb{R}^d$, and consider the autonomous differential equation*

$$\dot{x} = f(x)\,.$$

Then there exists a constant solution $\lambda : \mathbb{R} \to \mathbb{R}^d$ of this differential equation with $a \in \mathbb{R}^d$, i.e. $\lambda(t) = a$ for all $t \in \mathbb{R}$, if and only if $f(a) = 0$.

Proof. ($\Rightarrow$) Suppose that $\lambda : I \to \mathbb{R}^d$ is a constant solution, i.e. $\lambda(t) = a$ for all $t \in I$. The solution identity yields

$$\dot{\lambda}(t) = f(\lambda(t)) \quad \text{for all } \; t \in I\,, \tag{1.5}$$

which implies $f(a) = 0$.

($\Leftarrow$) Suppose that $f(a) = 0$ for some $a \in \mathbb{R}^d$. Then for the constant function $\lambda : \mathbb{R} \to \mathbb{R}^d$, $\lambda(t) = a$, the solution identity (1.5) is clearly fulfilled, and thus, the constant function λ is a solution to the given differential equation. □

This proposition says that constant solutions to autonomous ordinary differential equations are easy to find. For many differential equations, constant solutions are the only solutions that can be given explicitly, which means that there are no formulas for all other solutions. In fact, most of the interesting differential equations cannot be solved analytically. There are two approaches to overcome this deficit. Firstly, there are numerous schemes to numerically approximate solutions of differential equations – this will not be covered in this course. Secondly, the so-called *qualitative theory* of ordinary differential equations provides insights into how solutions behave without knowing them explicitly – you will learn some basic elements of this theory in this course.

We are interested now to understand in solutions for a given pair of initial time and initial conditions. For the model discussed in Example 1.1, this would mean that we are interested in the time evolution of capital, given that we have x_0 capital at time t_0.

Definition 1.4 (Initial value problem)**.** *Consider $d \in \mathbb{N}$, an open set $D \subset \mathbb{R} \times \mathbb{R}^d$, and a function $f : D \to \mathbb{R}^d$. The combination of the ordinary differential equation*

$$\dot{x} = f(t, x)$$

with an initial condition of the form

$$x(t_0) = x_0 \,, \tag{1.6}$$

where $(t_0, x_0) \in D$, is called an initial value problem*, and* (1.6) *is called* initial condition*. A solution to the above initial value problem is a solution $\lambda : I \to \mathbb{R}^d$ to the differential equation such that t_0 is in the interior of I and*

$$\lambda(t_0) = x_0 \,.$$

We now solve an initial value problem for the simple differential equation $\dot{x} = ax$.

Example 1.5 (A first example revisited)**.** Consider the ordinary differential equation (1.1) from Example 1.1 with the solutions λ_b for $b \in \mathbb{R}$. For fixed $t_0, x_0 \in \mathbb{R}$, we show that there exists a unique solution $\mu : \mathbb{R} \to \mathbb{R}$ to (1.1) solving the initial condition $x(t_0) = x_0$. This follows from

$$\lambda_b(t_0) = x_0 \Leftrightarrow be^{at_0} = x_0 \Leftrightarrow b = x_0 e^{-at_0} \,.$$

Hence, the solution to this initial value problem is given by $\mu(t) = x_0 e^{a(t-t_0)}$ for all $t \in \mathbb{R}$.

We will later find conditions for ordinary differential equations that guarantee that all initial value problems have a unique solution. These conditions are rather weak and apply to large classes of applications.

2. Examples

We have seen that the differential equation $\dot{x} = x$ behaves as nicely as one can imagine:

(i) a solution *exists* for every initial value problem,

(ii) the solution to each initial value problem is *unique*,

(iii) the solution to each initial value problem *exists globally*, i.e. can be defined on $I = \mathbb{R}$.

In this section, we look at examples for which not all of this properties are satisfied. The first example demonstrates that solutions to initial value problems do not need to exist.

Example 1.6 (No solution to an initial value problem)**.** Consider the one-dimensional initial value problem

$$\dot{x} = f(x) = \begin{cases} 1 & : \quad x < 0 \\ -1 & : \quad x \geq 0 \end{cases}, \qquad x(0) = 0\,,$$

which has a discontinuous right hand side. Show as an exercise that this initial value problem does not have any solutions.

There may exist more than one solution to an initial value problem.

Example 1.7 (Many solutions to an initial value problem)**.** Consider the one-dimensional initial value problem

$$\dot{x} = f(x) := \sqrt{|x|}\,, \qquad x(0) = 0\,. \tag{1.7}$$

Since $f(0) = 0$, Proposition 1.3 implies that there exists a constant solution with value 0. In addition, for any $b \geq 0$, the function $\lambda_b : \mathbb{R} \to \mathbb{R}$,

$$\lambda_b(t) = \begin{cases} 0 & : \quad t \leq b \\ \frac{1}{4}(t-b)^2 & : \quad t > b \end{cases},$$

is a solution to this initial value problem. To check this, we have to verify the solution identity for $t < b$, $t = b$ and $t > b$. Clearly, $\dot{\lambda}(t) = 0 = f(\lambda(t))$ for all $t < b$. Note that $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{4}(t-b)^2 = \frac{1}{2}(t-b)$, which is 0 at $t = b$, so the identity also holds at $t = b$. For $t > b$, we have $\dot{\lambda}(t) = \frac{1}{2}(t-b) = \sqrt{|\lambda(t)|}$, which finishes the proof.

Question: can you find even more solutions to this initial value problem?

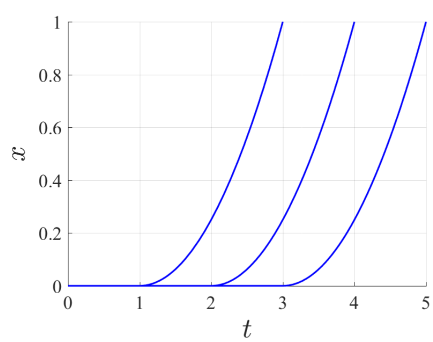

Figure 1.2. Three different solutions to the initial value problem (1.7) ($b = 1, 2, 3$).

We now study a differential equation for which there are solutions that do not exist for all times, i.e. they can only be defined on a proper subset $I \subsetneq \mathbb{R}$ of the real numbers. To solve this differential equation, we need the *separation of variables* technique that you have learned in your first year. This technique is useful to compute initial value problems of the form

$$\dot{x} = g(t)h(x)\,, \qquad x(t_0) = x_0\,, \tag{1.8}$$