Analysis 2, Complex Analysis

Solutions, CW2

Q1 (5) Let $f(z) = \frac{z^2-2z+3}{z-2}$. First we we deal with the term $1/(z-2)$

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{1}{z-1}\,\frac{1}{1-\frac{1}{z-1}} = \frac{1}{z-1}\sum_{n=0}^{\infty}\frac{1}{(z-1)^n}.$$

Then using that $z^2-2z+3 = (z-1)^2+2$ we obtain

$$f(z) = \frac{z^2-2z+3}{z-2} = \frac{(z-1)^2+2}{z-1}\sum_{n=0}^{\infty}\frac{1}{(z-1)^n}$$

$$= \left((z-1)+1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\dots\right)+\left(\frac{2}{z-1}+\frac{2}{(z-1)^2}+\dots\right)$$

$$= (z-1)+1+\sum_{n=1}^{\infty}\frac{3}{(z-1)^n}.$$

Q2 (5)

a) (2) Note that the function u is not differentiable at the origin. Therefore $\Omega = \mathbb{R}^2 \setminus \{0\}$. Clearly

$$u(x,y) = \frac{x^2+y^2+x}{x^2+y^2} = 1+\frac{x}{x^2+y^2}.$$

Hence

$$u'_x = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and

$$u'_y(x,y) = \frac{-2xy}{(x^2+y^2)^2}.$$

Moreover

$$u''_{xx}(x,y) = \frac{-2x}{(x^2+y^2)^2} - 2\frac{(y^2-x^2)2x}{(x^2+y^2)^3} = \frac{2x^3-6y^2x}{(x^2+y^2)^3},$$

$$u''_{yy}(x,y) = \frac{-2x}{(x^2+y^2)^2} - 2\,\frac{(-2xy)2y}{(x^2+y^2)^3} = \frac{-2x^3+6y^2x}{(x^2+y^2)^3}.$$

This implies $u''_{xx}+u''_{yy} = 0$.

b) (2) Introduce polar coordinate $x = r\cos\theta$, $y = r\sin\theta$. Then

$$u = 1 + \frac{\cos\theta}{r} \quad \text{and} \quad u'_r = -\frac{\cos\theta}{r^2}, \quad u'_\theta = -\frac{\sin\theta}{r}.$$

Using the Cauchy-Riemann equation in polar coordinate $u'_r = v'_\theta / r$ we find $v'_\theta = r\,u'_r$ and thus

$$v = -\frac{1}{r}\int \cos\theta\, d\theta = -\frac{\sin\theta}{r} + C(r).$$

Using $v'_r = -u'_\theta / r$ we find

$$v'_r = \frac{\sin\theta}{r^2} + C'(r) = -\frac{u'_\theta}{r} = \frac{\sin\theta}{r^2}.$$

Therfore $C(r) = const$. Finally we obtain

$$v(x,y) = -\frac{\sin\theta}{r} + \text{const} = \frac{-y}{x^2+y^2} + const.$$

c) (1)

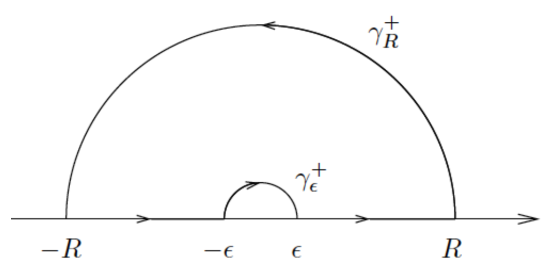
$$\begin{aligned} f(z) = 1 + \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2} + i\,\text{const} &= \frac{x - iy}{x^2+y^2} + 1 + i\,\text{const} \\ &= \frac{\bar{z}}{z\,\bar{z}} + 1 + i\,\text{const} = \frac{1}{z} + 1 + i\,\text{const}. \end{aligned}$$

Q3 (5) Consider

$$f(z) = \frac{1 - e^{iz}}{z^2}$$

and integrate this function over the curve

$$\gamma = [-R, -\varepsilon] \cup \gamma_\varepsilon^+ \cup [\varepsilon, R] \cup \gamma_R^+$$



Function f is holomorphic inside and on the curve Γ. Therefore

$$0 = \oint_\gamma f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz + \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz.$$

(2p)

First observe that by using the ML-inequality we have

$$\begin{aligned} \Big|\int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz\Big| &\le \pi R \max_{z\in\gamma_R^+}\Big|\frac{1-e^{iz}}{z^2}\Big| \\ &\le \frac{\pi}{R}\max_{\theta\in[0,\pi]}\Big|1-e^{iR(\cos\theta+i\sin\theta)}\Big| \\ &\le \frac{\pi}{R}\left(1+\max_{\theta\in[0,\pi]}\Big|e^{-R\sin\theta}\Big|\right) \le \frac{2\pi}{R}\to 0, \quad \text{as } R\to\infty. \end{aligned}$$

Therefore

$$\int_{|x|\ge\varepsilon} \frac{1-e^{ix}}{x^2}\,dx = -\int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz.$$

(1p)

Next note that

$$f(z) = \frac{1-e^{iz}}{z^2} = -\frac{iz}{z^2} + g(z) = -\frac{i}{z} + g(z),$$

where g is bounded. Parametrizing γ_ε^+ by $z = \varepsilon\, e^{i\theta}$, $\theta\in[\pi,0]$ we obtain

$$\begin{aligned} -\int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz \\ &= \int_\pi^0 \frac{i}{\varepsilon\, e^{i\theta}}\, i\varepsilon\, e^{i\theta}\,d\theta - \int_\pi^0 g(\varepsilon e^{i\theta})\, i\,\varepsilon\, e^{i\theta}\,d\theta \\ &= \pi - \int_\pi^0 g(\varepsilon e^{i\theta})\, i\,\varepsilon\, e^{i\theta}\,d\theta \to \pi, \quad \text{as } \varepsilon\to 0. \end{aligned}$$

(1p)

Taking the real part we arrive at

$$\lim_{\varepsilon\to 0} \operatorname{Re}\int_{|x|>\varepsilon} \frac{1-e^{ix}}{x^2}\,dx = \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \pi. \qquad (*)$$

Finally
$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}.$$
(1p)

Remark. Concerning $(*)$ note that
$$\operatorname{Im}\int_{\varepsilon\le|x|\le R} \frac{1-e^{ix}}{x^2}\,dx = -\int_{\varepsilon\le|x|\le R} \frac{\sin x}{x^2}\,dx = 0.$$

Q4 (5)

a) (3) Let $M = \max_{z:\,|z|=1} |\psi(z)|$. Consider the equation
$$z - w\,\psi(z) = 0$$
and denote $f(z) = z$ and $g(z) = -w\,\psi(z)$.
Clearly
$$|f(z)| = |z| = 1 \quad \text{and} \quad |g(z)| = |w\,\psi(z)| \le |w|\,M, \qquad z \in \partial D.$$
If now $|w| < 1/M$ then $|g(z)| < 1$. Then denoting by $\rho = 1/M$ and applying Rouche's theorem we obtain the proof.

b) (2) We use the Cross-Ratio Möbius theorem
$$\left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right) = \left(\frac{w-w_1}{w-w_3}\right)\left(\frac{w_2-w_3}{w_2-w_1}\right)$$
with $z_1 = 1$, $z_2 = i$ and $z_3 = -i$ and $w_1 = -i$, $w_2 = i$ and $w_3 = \infty$. Then
$$\begin{aligned}\left(\frac{z-1}{z+i}\right)\left(\frac{i+i}{i-1}\right) &= \lim_{t\to\infty}\left(\frac{w+i}{w-t}\right)\left(\frac{i-t}{i+i}\right)\\ &= \lim_{t\to\infty}\left(\frac{w+i}{w/t-1}\right)\left(\frac{i/t-1}{2i}\right) = \frac{w+i}{2i}.\end{aligned}$$
This implies
$$w = \frac{(2+i)z-(1+2i)}{z+i}.$$