

MATH50001 Analysis II, Complex Analysis

Lecture 1

Section: Syllabus & Historical Remarks

- Holomorphic Functions: Definition using derivative, Cauchy-Riemann equations, Polynomials, Power series.
- Cauchy's Integral Formula: Complex integration along curves, Goursat's theorem, Local existence of primitives and Cauchy's theorem in a disc, Evaluation of some integrals, Homotopies and simply connected domains, Cauchy's integral formulas.
- Applications of Cauchy's integral formula: Morera's theorem, Sequences of holomorphic functions, Holomorphic functions defined in terms of integrals, Schwarz reflection principle.
- Meromorphic Functions: Zeros and poles. Laurent series. The residue formula, Singularities and meromorphic functions, The argument principle and applications, The complex logarithm.
- Harmonic functions: Definition, and basic properties, Maximum modulus principle. Conformal Mappings: Definitions, Preservation of Angles, Statement of the Riemann mapping theorem, Rational functions, Möbius transformations.

Section: Complex numbers

The complex number $i = \sqrt{-1}$ ie associated with solutions of the equation

$$x^2 + 1 = 0$$

that does not have real solutions. However, historically complex numbers came through the cubic equation

$$x^3 - ax - b = 0.$$

In 1515 Scipione del Ferro (1465-1526, Italian) found but not published the solution

$$x = \sqrt{3}\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}} + \sqrt{3}\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}$$

It was interesting that even if $\frac{b^2}{4} - \frac{a^3}{27} < 0$ the equation has real solutions for a, b real. This formula was published by Girolamo Cardano (1501-1576, Italian) in 1545.



Scipione del Ferro



Girolamo Cardano

In 1572, Rafael Bombelli (1526-1572, Italian) published a book which spelled out rules of arithmetic for complex numbers and used them in Cardano's formula for finding real solutions of cubics.

Key later work is by John Wallis (1616 - 1703, English) and Leonhard Euler (1707-1783, Swiss). In particular, Euler clarified complex roots of unity and found the multiple roots. He used complex numbers extensively. He introduced i as the symbol for $\sqrt{-1}$ and linked the exponential and trigonometric functions in the famous formula

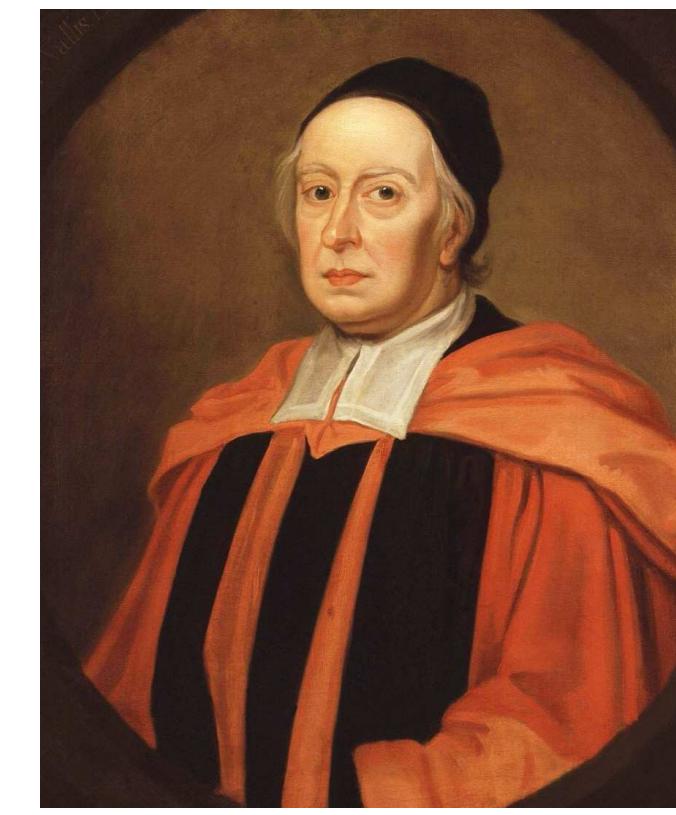
$$e^{it} = \cos t + i \sin t.$$

Carl Friedrich Gauss (1777-1855, German), who gave a proof of the Fundamental Theorem of Algebra in 1799.

It took almost another century before mathematicians as a community fully accepted complex numbers.



Rafael Bombelli



John Wallis



Leonhard Euler



Carl Friedrich Gauss

The founding fathers of complex analysis are:
Augustin-Louis Cauchy, Karl Weierstrass and Bernhard Riemann.

To A.-L. Cauchy - the central aspect is the differential and integral calculus of complex-valued functions of a complex variable. Here the fundamentals are the Cauchy integral theorem and Cauchy integral formula



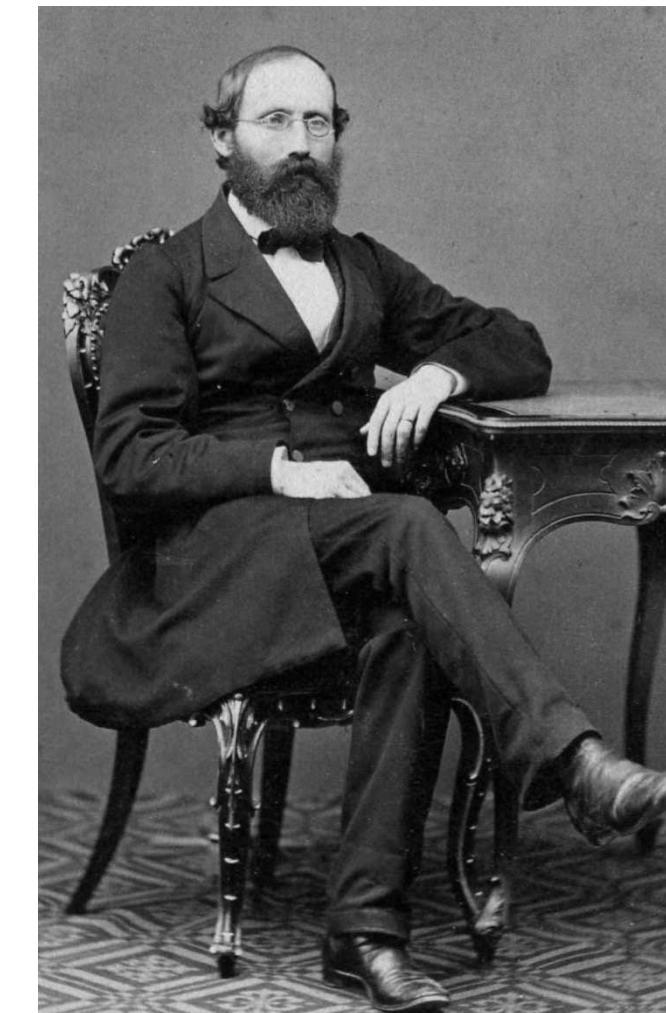
Augustin-Louis Cauchy (1789 -1857) - French

To K. Weierstrass - sums and products and especially power series are the central object.



Karl Weierstrass (1815-1897) - German

To B. Riemann - conformal maps and associated geometry.



Bernhard Riemann (1826-1866) - German

- Riemann Hypothesis is still open (1859)

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-z}}.$$

This series converges if $\operatorname{Re} z > 1$.

If z is a complex number then in the above sum there some cancellation. In particular the Riemann Hypothesis states

$$\zeta(z) = 0 \implies \operatorname{Re} z = 1/2.$$

Modern state of art

- The Mandelbrot set, Complex dynamics:

The Mandelbrot set is the set of complex numbers η for which the function

$$f_\eta(z) = z^2 + \eta$$

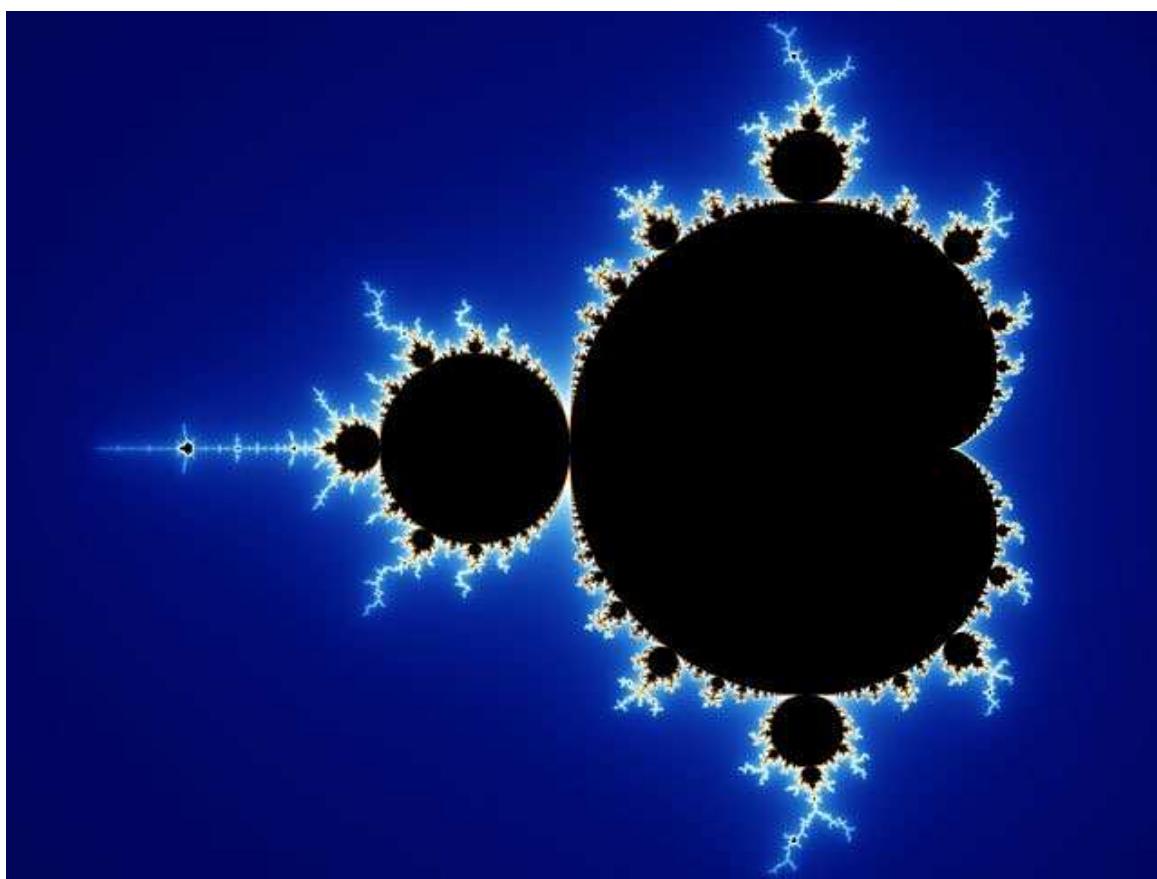
does not diverge when iterating from $z = 0$ so that the sequence

$$f_\eta(0), f_\eta(f_\eta(0)), f_\eta(f_\eta(f_\eta(0))), \dots$$

remains bounded



Benoit Mandelbrot (1924 - 2010)



Section: Basic properties

A complex number takes the form $z = x + iy$, where x and y are real, $x, y \in \mathbb{R}$, and i is an imaginary number that satisfies $i^2 = -1$. We call x and y the real part and the imaginary part of z , respectively, and we write

$$x = \operatorname{Re}(z) \quad \text{and} \quad y = \operatorname{Im}(z).$$

The real numbers are complex numbers with zero imaginary parts. A complex number with zero real part is said to be purely imaginary.

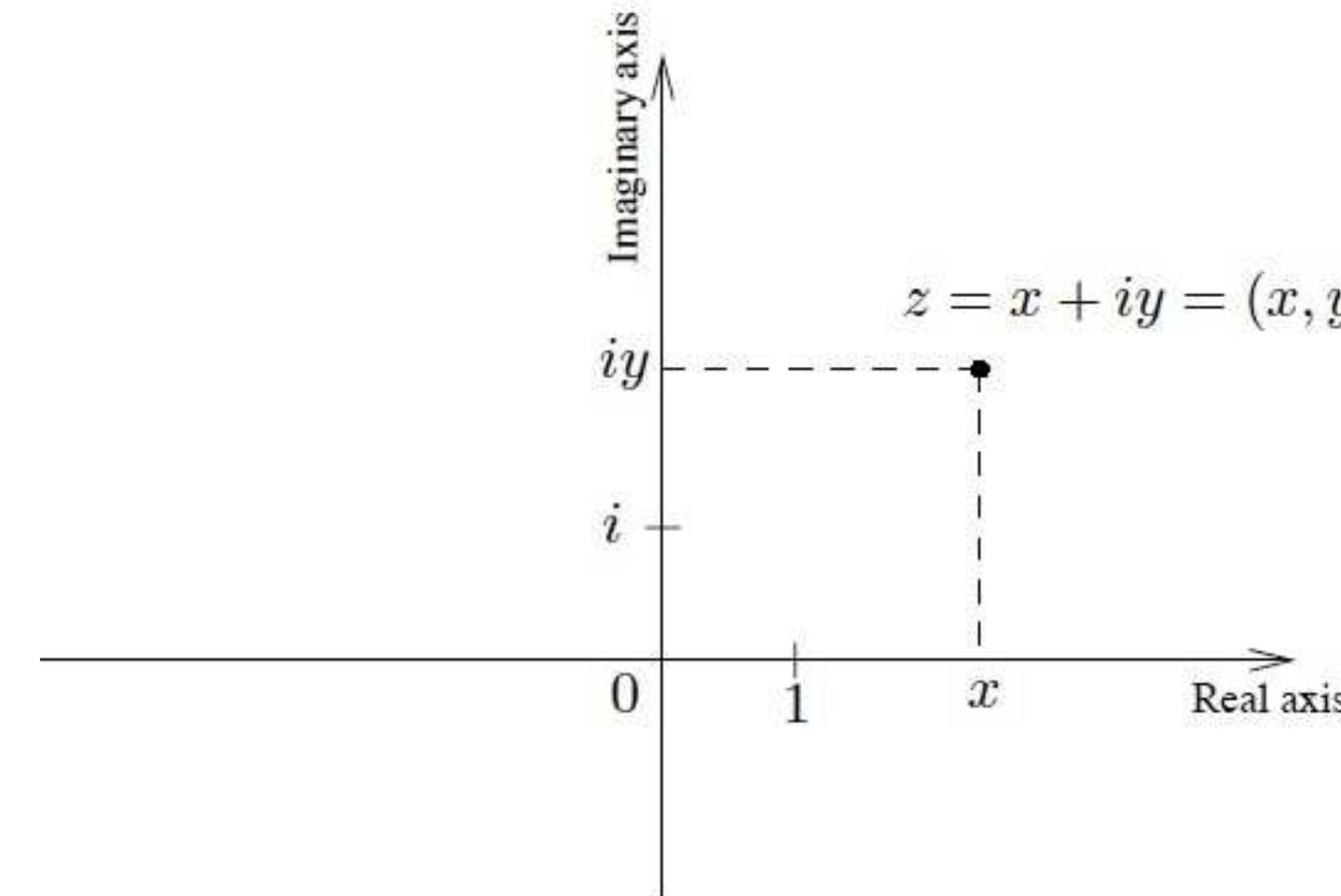
The complex conjugate of $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

The complex numbers can be visualized as the usual Euclidean plane:

$z = x + iy \in \mathbb{C}$ is identified with the point $(x, y) \in \mathbb{R}^2$.

- in this case 0 corresponds to the origin,
- i corresponds to $(0, 1)$.
- the x and y axis of \mathbb{R}^2 are called the real axis and imaginary axis respectively.



- **Polar coordinates.**

$$z = x + iy, \quad r = |z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}},$$

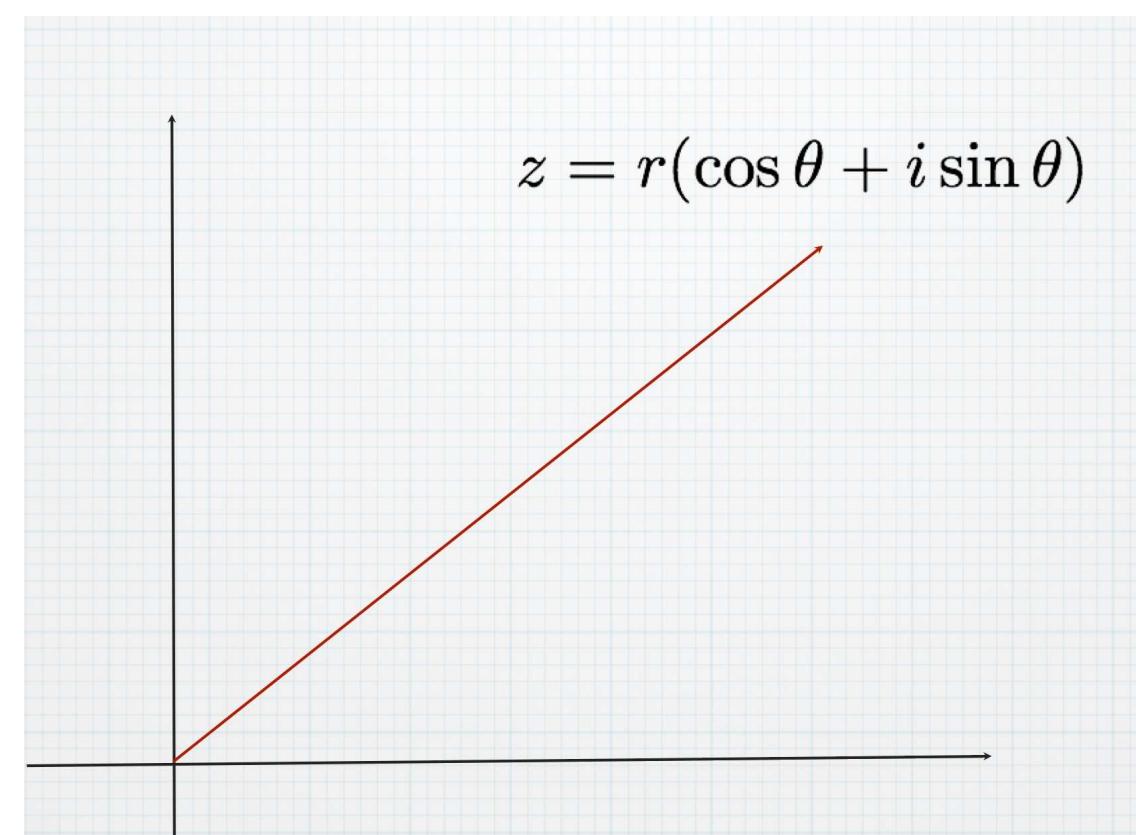
$$x = r \cos \theta, \quad y = r \sin \theta,$$

where

$$\cos \theta = \frac{x}{r} \quad \sin \theta = \frac{y}{r}.$$

and thus

$$z = r(\cos \theta + i \sin \theta).$$



Example. Let $z = 1 - i$. Then $r = \sqrt{2}$ and $\sin \theta = -1/\sqrt{2}$. Then

$$\theta = -\frac{\pi}{4} + 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

So $\arg z = -\pi/4 + 2\pi k$.

Definition. $\text{Arg } z = \theta$ such that $-\pi < \theta \leq \pi$ is called the Principal value of the argument of z .

Example.

$$\text{Arg}(1 - i) = -\frac{\pi}{4}.$$

Theorem. Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$. Then

$$z_1 \cdot z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$

Proof. Use elementary trigonometric formulae.

Corollary. (De Moivres formula)

$$z^n = r^n(\cos n\theta + i \sin n\theta), \quad n = 1, 2, 3, \dots$$



Remark. Theorem implies

Abraham De Moivres (French, 1667-1754)

$$\arg z_1 + \arg z_2 = \arg(z_1 \cdot z_2),$$

however,

$$\text{Arg } z_1 + \text{Arg } z_2 \neq \text{Arg } (z_1 \cdot z_2).$$

WHY ???

Section: Sets in the complex plane

Definition. Let $z_0 \in \mathbb{C}$ and $r > 0$. Define the open disc $D_r(z_0)$

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

The boundary of the open or closed disc is the circle

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

The unit disc is the disc centred at the origin and of radius one

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Given a set $\Omega \subset \mathbb{C}$, a point z_0 is an interior point of Ω if there exists $r > 0$ such that $D_r(z_0) \subset \Omega$. The interior of Ω consists of all its interior points.

Definition. A set Ω is open if every point in that set is an interior point of Ω . This definition coincides precisely with the definition of an open set in \mathbb{R}^2 .

Definition. A set Ω is *closed* if its complement $\Omega^c = \mathbb{C} \setminus \Omega$ is open.

A set is closed if and only if it contains all its limit points. The closure of any set Ω is the union of Ω and its limit points, and is often denoted by $\overline{\Omega}$.

Definition. The *boundary* of a set Ω is equal to its closure minus its interior, and is often denoted by $\partial\Omega$.

Definition. A set Ω is *bounded* if there exists $M > 0$ such that $|z| < M$ whenever $z \in \Omega$.

Definition. If Ω is bounded, we define its *diameter* by

$$\text{diam}(\Omega) = \sup_{z,w \in \Omega} |z - w|.$$

Definition. A set Ω is said to be *compact* if it is closed and bounded. Arguing as in the case of real variables, one can prove the following.

Theorem. The set $\Omega \subset \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω .

An open covering of Ω is a family of open sets $\{U_\alpha\}$ (not necessarily countable) such that

$$\Omega \subset \bigcup_\alpha U_\alpha.$$

In analogy with the situation in \mathbb{R}^2 , we have the following equivalent formulation of compactness.

Theorem. A set Ω is compact if and only if every open covering of Ω has a finite subcovering.

Another property of compactness is that of “nested sets”.

Theorem. If $\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_n \dots$ is a sequence of non-empty compact sets in \mathbb{C} with the property that $\text{diam}(\Omega_n) \rightarrow 0$ as $n \rightarrow \infty$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n .

Proof. Choose a point z_n in each Ω_n . The condition $\text{diam}(\Omega_n) \rightarrow 0$ says that $\{z_n\}$ is a Cauchy sequence, therefore this sequence converges to a limit that we call w . Since each set Ω_n is compact we must have $w \in \Omega_n$ for all n . Finally, w is the unique point satisfying this property, for otherwise, if w' satisfied the same property with $w' \neq w$ we would have $|w' - w| > 0$ and the condition $\text{diam}(\Omega_n) \rightarrow 0$ would be violated.

Definition. An open set Ω is *connected* if and only if any two points in Ω can be joined by a curve γ entirely contained in Ω .

Thank you

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Lecture 2

Section: Complex functions

Definition. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$.

$$f : \Omega_1 \rightarrow \Omega_2$$

is said to be a mapping from Ω_1 to Ω_2 if for any $z = x + iy \in \Omega_1$ there exists only one complex number $w = u + iv \in \Omega_2$ such that

$$w = f(z).$$

We use notations:

$$w = f(z) = u(x, y) + iv(x, y),$$

where u and v are two real functions of two real variables.

Example. Let $w = f(z) = z^2 = x^2 - y^2 + i2xy$, $z \in \mathbb{C}$. Then

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Example. Let $w = f(z) = 1/z = \bar{z}/|z|^2$, $z \in \mathbb{C} \setminus \{0\}$. Then

$$u(x, y) = \frac{x}{x^2 + y^2} \quad \text{and} \quad v(x, y) = -\frac{y}{x^2 + y^2}.$$

Example. Möbius transformation

$$w = f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad cz + d \neq 0.$$

Definition. Let f be a function defined on a set $\Omega \subset \mathbb{C}$. We say that f is continuous at the point $z_0 \in \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$.

Definition. The function f is said to be continuous on Ω if it is continuous at every point of Ω .

Section: Complex derivative

Definition. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open sets and let $f: \Omega_1 \rightarrow \Omega_2$. We say that f is *differentiable (holomorphic)* at $z_0 \in \Omega_1$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges to a limit when $h \rightarrow 0$. Here $h \in \mathbb{C}$, $h \neq 0$ and $z_0 + h \in \Omega_1$. The limit of this quotient, when it exists, is denoted by $f'(z_0)$, and is called the derivative of f at z_0 :

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

This means that for any $\varepsilon > 0$ there is $\delta > 0$ such that as soon $|h| < \delta$ we have

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0) \right| < \varepsilon.$$

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

It should be emphasised that in the above limit that $h = h_1 + ih_2 \in \mathbb{C}$ is a complex number that may approach 0 from any direction.

Remark. The word "holomorphic" was introduced by two of Cauchy's students, Briot (1817-1882) and Bouquet (1819-1895), and derives from the Greek (holos) meaning "entire", and (morphe) meaning "form" or "appearance".

Definition. The function f is said to be holomorphic on open set Ω if f is holomorphic at every point of Ω .

If C is a closed subset of \mathbb{C} , we say that f is holomorphic on C if f is holomorphic in some open set containing C . Finally, if f is holomorphic in all of \mathbb{C} we say that f is entire.

Example. The function $f(z) = z$ is holomorphic on any open set in \mathbb{C} and $f'(z) = 1$.

Example. If $f(z) = z^n$ then $f'(z) = nz^{n-1}$. Indeed we use induction to find:

- If $n = 1$ then $(z)' = 1$.
- Assuming $(z^n)' = nz^{n-1}$ we obtain

$$(z^{n+1})' = (z \cdot z^n)' = z' \cdot z^n + z \cdot (z^n)' = z^n + z \cdot nz^{n-1} = (n+1)z^n.$$

Example. Any polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is holomorphic in the entire complex plane and

$$p'(z) = a_1 + \cdots + n a_n z^{n-1}.$$

Example. The function $1/z$ is holomorphic on any open set in \mathbb{C} that does not contain the origin, and $f'(z) = -1/z^2$.

Proof it.

Example. The function $f(z) = \bar{z}$ is not holomorphic. Indeed, we have

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\bar{h}}{h}.$$

which has no limit as $h \rightarrow 0$, as one can see by first taking h real and then h purely imaginary.

Proposition. A function f is holomorphic at $z_0 \in \Omega$ if and only if there exists a complex number a such that

$$f(z_0 + h) - f(z_0) - ah = h\psi(h),$$

where ψ is a function defined for all small h and

$$\lim_{h \rightarrow 0} \psi(h) = 0.$$

In this case

$$a = f'(z_0).$$

Proof. The proof follows directly from the Definition. Indeed, dividing by h we have

$$\frac{f(z_0 + h) - f(z_0)}{h} - a = \psi(h) \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Corollary. If a function f is holomorphic then it is continuous.

Proposition. If f and g are holomorphic in Ω then:

- (i) $f + g$ is holomorphic in Ω and $(f + g)' = f' + g'$.
- (ii) fg is holomorphic in Ω and $(fg)' = f'g + fg'$.
- (iii) If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

- (iv) Moreover, if $f : \Omega \rightarrow U$ and $g : U \rightarrow \mathbb{C}$ are holomorphic, the chain rule holds

$$(g \circ f)(z) = g'(f(z))f'(z), \quad \forall z \in \Omega.$$

Proof. Arguing as in the case of one real variable, use the expression

$$f(z_0 + h) - f(z_0) - ah = h\psi(h).$$

Section: Cauchy-Riemann equations

Consider first

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}, \quad h = h_1 + ih_2,$$

assuming that $h = h_1$ (namely that $h_2 = 0$). Then if

$$f(z_0) = f(x_0 + iy_0) = u(x_0, y_0) + iv(x_0, y_0),$$

we have

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h_1 \rightarrow 0} \frac{u(x_0 + h_1, y_0) + iv(x_0 + h_1, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h_1} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) = u'_x(x_0, y_0) + iv'_x(x_0, y_0). \end{aligned}$$

Let now $h = ih_2$ (namely that $h_1 = 0$). Then

$$\begin{aligned}
 f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \\
 &= \lim_{h_2 \rightarrow 0} \frac{u(x_0, y_0 + h_2) + iv(x_0, y_0 + h_2) - u(x_0, y_0) - iv(x_0, y_0)}{ih_2} \\
 &= \frac{1}{i} \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) = \frac{1}{i} u'_y(x_0, y_0) + v'_y(x_0, y_0) \\
 &\quad = -iu'_y(x_0, y_0) + v'_y(x_0, y_0).
 \end{aligned}$$

Thus the function u and v satisfy the following

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- *Cauchy-Riemann equations.*

Example. Let $f(z) = z^2$. Then $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Then

$$u'_x = 2x = v'_y \quad \text{and} \quad u'_y = -2y = -v'_x, \quad -O'K.$$

Example. Let $f(z) = \bar{z}$. Then $u(x, y) = x$ and $v(x, y) = -y$.

$$u'_x = 1 \neq -1 = v'_y.$$

This means that $f(z) = \bar{z}$ is not differentiable.

The Cauchy-Riemann equations link real and complex analysis.

Definition.

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right)$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right).$$

Theorem. Let $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$. If f is holomorphic at z_0 , then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2 \frac{\partial u}{\partial z}(z_0).$$

Proof. Using the Cauchy-Riemann equations $u'_x = v'_y$ and $u'_y = -v'_x$ we obtain

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(u'_x - \frac{1}{i} u'_y \right) + \frac{i}{2} \left(v'_x - \frac{1}{i} v'_y \right) = \frac{1}{2} (u'_x + iu'_y + iv'_x - v'_y) = 0.$$

and

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2} \left(u'_x + \frac{1}{i} u'_y \right) + \frac{i}{2} \left(v'_x + \frac{1}{i} v'_y \right) = \frac{1}{2} (u'_x - iu'_y + iv'_x + v'_y) \\ &= \frac{1}{2} (2u'_x - i2u'_y) = u'_x + \frac{1}{i} u'_y = 2 \frac{\partial u}{\partial z}. \end{aligned}$$

The fact that $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$ follows from our computations before. Indeed, we have seen that

$$f'(z_0) = u'_x(x_0, y_0) + iv'_x(x_0, y_0) = u'_x(x_0, y_0) - iu'_y(x_0, y_0) = 2 \frac{\partial u}{\partial z}(x_0, y_0).$$

The proof is complete.

The next theorem contains an important converse.

Theorem. Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω . If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω , then f is holomorphic on Ω and $f'(z) = \partial f(z)/\partial z$.

Thank you

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Lecture 3

Cauchy-Riemann equations.

The next theorem contains an important converse.

Theorem. Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω . If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω , then f is holomorphic on Ω and $f'(z) = \partial f(z)/\partial z$.

Proof. Assuming $h = h_1 + ih_2$ we have

$$u(x+h_1, y+h_2) - u(x, y) = u'_x(x, y) h_1 + u'_y(x, y) h_2 + |h|\psi_1(h),$$

where $\psi_1(h) \rightarrow 0$ as $h \rightarrow 0$. Indeed,

$$\begin{aligned} u(x+h_1, y+h_2) - u(x, y) &= u(x+h_1, y+h_2) - u(x, y+h_2) + u(x, y+h_2) - u(x, y) \\ &= u'_x(x, y+h_2) h_1 + h_1 \varphi_1(h) + u'_y(x, y) h_2 + h_2 \varphi_2(h). \end{aligned}$$

Since $u'_x(x, y+h_2)$ is continuous we have

$$u'_x(x, y+h_2) - u'_x(x, y) = \varphi_3(h) \rightarrow 0 \quad \text{as } h_2 \rightarrow 0$$

and thus

$$\begin{aligned} u(x+h_1, y+h_2) - u(x, y) &= u'_x(x, y) h_1 + u'_y(x, y) h_2 \\ &\quad + h_1(\varphi_3(h) + \varphi_1(h)) + h_2 \varphi_2(h) = |h|\psi_1(h), \end{aligned}$$

where $\psi_1(h) = |h|^{-1}(h_1(\varphi_3(h) + \varphi_1(h)) + h_2 \varphi_2(h)) \rightarrow 0$, $h \rightarrow 0$.

Similarly

$$v(x + h_1, y + h_2) - v(x, y) = v'_x(x, y) h_1 + v'_y(x, y) h_2 + |h|\psi_2(h),$$

where $\psi_2(h) \rightarrow 0$ as $h \rightarrow 0$.

Using the Cauchy-Riemann equations $v'_x = -u'_y$ and $v'_y = u'_x$, we find

$$\begin{aligned} f(z + h) - f(z) &= u(x + h_1, y + h_2) + iv(x + h_1, y + h_2) - u(x, y) - iv(x, y) \\ &= u'_x(x, y) h_1 + u'_y(x, y) h_2 + i(v'_x(x, y) h_1 + v'_y(x, y) h_2) + |h|\psi(h) \\ &= u'_x(x, y) h_1 + u'_y(x, y) h_2 - iu'_y(x, y) h_1 + iu'_x(x, y) h_2 + |h|\psi(h) \\ &= (u'_x - iu'_y)(h_1 + ih_2) + |h|\psi(h), \end{aligned}$$

where $\psi(h) = \psi_1(h) + i\psi_2(h) \rightarrow 0$, as $h \rightarrow 0$. Therefore f is holomorphic and

$$f'(z) = 2 \frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}.$$

Section: Cauchy-Riemann equations in polar coordinates

Usual Cauchy-Riemann equations for a holomorphic function $f = u + iv$ as they were defined before are:

$$u'_x = v'_y \quad u'_y = -v'_x$$

Introduce polar coordinate

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}, \quad \theta = \arctan y/x.$$

Then

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2 + y^2}} = \cos \theta, & \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2 + y^2}} = \sin \theta, \\ \frac{\partial \theta}{\partial x} &= \frac{1}{1 + (y/x)^2} (-1) \frac{y}{x^2} = -\frac{\sin \theta}{r}, & \frac{\partial \theta}{\partial y} &= \frac{1}{1 + (y/x)^2} \frac{1}{x} = \frac{\cos \theta}{r}. \end{aligned}$$

Therefore

$$\begin{aligned} u'_x &= u'_r \cos \theta + u'_\theta \frac{-\sin \theta}{r}, & v'_y &= v'_r \sin \theta + v'_\theta \frac{\cos \theta}{r}, \\ u'_y &= u'_r \sin \theta + u'_\theta \frac{\cos \theta}{r}, & v'_x &= v'_r \cos \theta + v'_\theta \frac{-\sin \theta}{r}. \end{aligned}$$

Multiplying u'_x by $\cos \theta$ and u'_y by $\sin \theta$ and adding the results we find

$$u'_r = u'_r \cos^2 \theta + u'_r \sin^2 \theta = u'_x \cos \theta + u'_y \sin \theta.$$

Using $u'_x = v'_y$ and $u'_y = -v'_x$ we conclude

$$\begin{aligned} u'_x \cos \theta + u'_y \sin \theta &= v'_y \cos \theta - v'_x \sin \theta \\ &= \left(v'_r \sin \theta + v'_\theta \frac{\cos \theta}{r} \right) \cos \theta - \left(v'_r \cos \theta - v'_\theta \frac{\sin \theta}{r} \right) \sin \theta = v'_\theta \frac{1}{r}. \end{aligned}$$

Then

$$u'_r = \frac{1}{r} v'_\theta \quad \text{and similarly} \quad v'_r = -\frac{1}{r} u'_\theta.$$

Example. Let

$$\begin{aligned} f(z) = u(x, y) + iv(x, y) &= \ln(x^2 + y^2) + 2i \arctan \frac{y}{x} \\ &= \ln |z|^2 + 2i \operatorname{Arg}(z) = 2(\ln r + i\theta), \end{aligned}$$

where $z = r(\cos \theta + i \sin \theta)$. Then

$$u'_r = \frac{2}{r} = \frac{1}{r} \cdot 2 = \frac{1}{r} v'_\theta \quad \text{and} \quad 0 = v'_r = -\frac{1}{r} u'_\theta = 0.$$

Section: Power series

Definition. A power series is an expansion of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $a_n \in \mathbb{C}$.

The series is convergent at z if the partial sum $S_N(z) = \sum_{n=0}^N a_n z^n$ has a limit

$$S(z) = \lim_{N \rightarrow \infty} S_N(z).$$

In this case we write $S(z) = \sum_{n=0}^{\infty} a_n z^n$.

For its absolute convergence we consider

$$\sum_{n=0}^{\infty} |a_n| |z|^n.$$

Proposition. If $S(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\lim_{N \rightarrow \infty} (S(z) - S_N(z)) = 0$.

Example. The complex exponential function, which is defined by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely for any $z \in \mathbb{C}$ and $R = \infty$.

Example. The geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

converges absolutely $|z| < 1$ and its radius of convergence $R = 1$.

Theorem. Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \leq R \leq \infty$ such that:

- (i) If $|z| < R$ the series converges absolutely.
- (ii) If $|z| > R$ the series diverges.

Moreover, R is given by the formula

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$$

The number R is called the *radius of convergence* of the power series, and the domain $|z| < R$ the *disc of convergence*.

Proof. Let $L = 1/R$ and suppose that $L \neq 0, \infty$. If $|z| < R$, choose $\varepsilon > 0$ so that

$$(L + \varepsilon)|z| = r < 1.$$

By the definition L , we have $|a_n|^{1/n} \leq L + \varepsilon$ for all large n , therefore

$$|a_n||z|^n \leq ((L + \varepsilon)|z|)^n = r^n$$

Comparison with the geometric series $\sum_{n=0}^{\infty} r^n$ shows that $\sum_{n=0}^{\infty} a_n z^n$ converges.

If $|z| > R$, then a similar argument proves that there exists a sequence of terms in the series whose absolute value goes to infinity, hence the series diverges.

Remark. Prove the above result for $R = 0$ and $R = \infty$ ($L = \infty$ and $L = 0$ respectively).

Remark. On the boundary of the disc of convergence, $|z| = R$, one can have either convergence or divergence.

Power series provide an important class of holomorphic functions.

Theorem. The power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

defines a holomorphic function in its disc of convergence. The derivative of f is also a power series obtained by differentiating term by term the series for f , that is,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

Moreover, f' has the same radius of convergence as f .

Proof. Indeed, note that

$$\lim_{n \rightarrow \infty} n^{1/n} = \lim_{n \rightarrow \infty} e^{\frac{1}{n} \ln n} = e^0 = 1.$$

Therefore

$$\sum_{n=1}^{\infty} a_n z^{n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} n a_n z^n$$

have the same radius of convergence and thus this is also true for $\sum_{n=1}^{\infty} a_n z^{n-1}$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$.

It remains to show that $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ coincides with $f'(z)$.

Let R be the radius of convergence of f , $|z_0| < r < R$ and let

$$S_N(z) = \sum_{n=0}^N a_n z^n, \quad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Then if h is chosen so that $|z_0 + h| < r$ we have

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) &= \left(\frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right) \\ &\quad + (S'_N(z_0) - g(z_0)) + \left(\frac{E_N(z_0 + h) - E_N(z_0)}{h} \right). \end{aligned}$$

We find that

$$\begin{aligned} \left| \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right| &\leq \sum_{n=N+1}^{\infty} |a_n| \left| \frac{(z_0 + h)^n - z_0^n}{h} \right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1} \rightarrow 0, \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Given $\varepsilon > 0$ there is N_1 such that for any $N > N_1$ we have

$$\left| \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right| < \varepsilon.$$

Since $\lim_{N \rightarrow \infty} S'_N(z_0) \rightarrow g(z_0)$ there is N_2 such that for any $N > N_2$ we have

$$|S'_N(z_0) - g(z_0)| < \varepsilon$$

Finally for any fixed $N > \max(N_1, N_2)$ we choose $\delta > 0$ such that if $|h| < \delta$

$$\left| \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right| < \varepsilon.$$

We now conclude

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < 3\varepsilon, \quad |h| < \delta.$$

The proof is complete.

Corollary. A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwize differentiation.

Thank you

MATH50001 Analysis II, Complex Analysis

Lecture 4

Section: Elementary functions.

1. Exponential function.

Definition. We define exponential e^z ($z = x + iy \in \mathbb{C}$) as:

$$e^z = e^x \cos y + ie^x \sin y.$$

Properties:

- a) If $y = 0$ then $e^z = e^x$.
- b) e^z is entire (holomorphic for any $z \in \mathbb{C}$)

Indeed, for that we check the C-R equations. Since $u = \operatorname{Re} f = e^x \cos y$ and $v = \operatorname{Im} f = e^x \sin y$, we have

$$u'_x = e^x \cos y = v'_y \quad \text{and} \quad u'_y = e^x(-\sin y) = -v'_x.$$

c)

$$\frac{\partial}{\partial z} e^z = \frac{\partial}{\partial x} e^x \cos y + i \frac{\partial}{\partial x} e^x \sin y = e^z.$$

d) Let $g(z)$ be holomorphic. Then

$$\frac{\partial}{\partial z} e^{g(z)} = e^{g(z)} g'(z).$$

e) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} e^{z_1+z_2} &= e^{x_1+x_2} (\cos(y_1+y_2) + i \sin(y_1+y_2)) \\ &= e^{x_1+x_2} (\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)) \\ &= e^{x_1+x_2} (\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) = e^{z_1} e^{z_2}. \end{aligned}$$

f) $|e^z| = |e^x| |e^{iy}| = e^x \sqrt{\cos^2 y + \sin^2 y} = e^x.$

The function e^z is 2π -periodic with respect to y .

g) Applying the De Moivres formula

$$(\cos y + i \sin y)^n = \cos ny + i \sin ny$$

we obtain

$$(e^{iy})^n = e^{iny}.$$

h) Since $\arg z = \arctan y/x$

$$\arg e^z = \arctan \frac{e^x \sin y}{e^x \cos y} = \arctan(\tan y) = y + 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

Definition. If f is holomorphic for all $z \in \mathbb{C}$ then it calls *entire*.

Clearly the exponential function e^z is entire.

2. Trigonometric functions.

$$\begin{cases} e^{i\theta} = \cos \theta + i \sin \theta \\ e^{-i\theta} = \cos \theta - i \sin \theta \end{cases} \Rightarrow \begin{cases} \cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta}) \\ \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \end{cases}$$

Definition. For any $z \in \mathbb{C}$ we define

$$\sin z = \frac{1}{2i} (e^{iz} - e^{-iz}), \quad \cos z = \frac{1}{2} (e^{iz} + e^{-iz}).$$

Properties:

- a) $\sin z$ and $\cos z$ are entire functions
- b) $\frac{\partial}{\partial z} \sin z = \cos z$ and $\frac{\partial}{\partial z} \cos z = -\sin z$.
- c) $\sin^2 z + \cos^2 z = 1$.

Indeed:

$$-\frac{1}{4} (e^{iz} - e^{-iz})^2 + \frac{1}{4} (e^{iz} + e^{-iz})^2 = \dots = 1.$$

d)

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2,$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2.$$

3. Logarithmic functions.

Let $z = r(\cos \theta + i \sin \theta) = r e^{i\theta}$.

Definition. $\log z = \ln |z| + i \arg z = \log r + i(\theta + 2\pi k), \quad z \neq 0,$

where $k = 0, \pm 1, \pm 2, \dots$

Clearly:

$$e^{\log z} = e^{\ln r + i(\theta + 2\pi k)} = r e^{i(\theta + 2\pi k)} = r (\cos \theta + i \sin \theta) = x + iy = z.$$

Remark. The function \log is a multi-valued function.

Definition. We define $\text{Log } z$ as the single-valued function:

$$\text{Log } z = \ln |z| + i \operatorname{Arg} z,$$

where $\operatorname{Arg} z$ is the principal value of the argument, namely, $-\pi < \operatorname{Arg} z \leq \pi$.

Remark. The function Log is a single-valued function.

Examples.

$$\text{Log}(-1) = i\pi,$$

$$\text{Log}(2i) = \ln 2 + i\pi/2,$$

$$\text{Log}(1-i) = \ln \sqrt{2} - i\pi/4.$$

Properties:

a) $\log(z_1 \cdot z_2) = \log(z_1) + \log(z_2)$. Indeed

$$\begin{aligned}\log(z_1 \cdot z_2) &= \ln |z_1 z_2| + i \arg(z_1 \cdot z_2) \\ &= \ln |z_1| + \ln |z_2| + i \arg z_1 + i \arg z_2 = \log z_1 + \log z_2.\end{aligned}$$

Remark. $\text{Log}(z_1 \cdot z_2) \neq \text{Log} z_1 + \text{Log} z_2$, because $\text{Arg}(z_1 \cdot z_2) \neq \text{Arg} z_1 + \text{Arg} z_2$.

b) The function $\text{Log } z$ is holomorphic in $\mathbb{C} \setminus \{(-\infty, 0]\}$.

Indeed, we have already checked that the C-R equations are satisfied:

$$\text{Log } z = \ln r + i\theta = u + iv, \quad -\pi < \theta \leq \pi.$$

Therefore we have

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot 1 = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = 0 = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

Exercise. Compute $(\text{Log } z)'$.

4. Powers.

Definition. For any $\alpha \in \mathbb{C}$, we define $z^\alpha = e^{\alpha \log z}$ as a multi-valued function.

Example. $i^i = e^{i \log i} = e^{i(i\pi/2 + i2\pi k)} = e^{-\pi/2} e^{-2\pi k}$, $k = 0, \pm 1, \pm 2, \dots$

Definition. We define the principal value of z^α , $\alpha \in \mathbb{C}$, as

$$z^\alpha = e^{\alpha \operatorname{Log} z}.$$

Property:

a) $z^{\alpha_1} \cdot z^{\alpha_2} = e^{\alpha_1 \operatorname{Log} z} e^{\alpha_2 \operatorname{Log} z} = e^{(\alpha_1 + \alpha_2) \operatorname{Log} z} = z^{\alpha_1 + \alpha_2}.$

Section: Parametrised curve.

Definition. A *parametrised curve* is a function $z(t)$ which maps a closed interval $[a, b] \subset \mathbb{R}$ to the complex plane. We say that the parametrised curve is smooth if $z'(t)$ exists and is continuous on $[a, b]$, and $z'(t) \neq 0$ for $t \in [a, b]$. At the points $t = a$ and $t = b$, the quantities $z'(a)$ and $z'(b)$ are interpreted as the one-sided limits

$$z'(a) = \lim_{h \rightarrow 0, h > 0} \frac{z(a + h) - z(a)}{h}, \quad z'(b) = \lim_{h \rightarrow 0, h < 0} \frac{z(b + h) - z(b)}{h}.$$

Similarly we say that the parametrised curve is piecewise - smooth if z is continuous on $[a, b]$ and if there exist a finite number of points $a = a_0 < a_1 < \dots < a_n = b$, where $z(t)$ is smooth in the intervals $[a_k, a_{k+1}]$. In particular, the righthand derivative at a_k may differ from the left-hand derivative at a_k for $k = 1, 2, \dots, n - 1$.

Two parametrisations,

$$z : [a, b] \rightarrow \mathbb{C} \quad \text{and} \quad \tilde{z} : [c, d] \rightarrow \mathbb{C},$$

are equivalent if there exists a continuously differentiable bijection $s \rightarrow t(s)$ from $[c, d]$ to $[a, b]$ so that $t'(s) > 0$ and

$$\tilde{z}(s) = z(t(s)).$$

The condition $t'(s) > 0$ says precisely that the orientation is preserved: as s travels from c to d , then $t(s)$ travels from a to b .

Given a smooth curve γ in \mathbb{C} parametrised by $z : [a, b] \rightarrow \mathbb{C}$, and f a continuous function on γ we define the integral of f along γ by

$$\int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt.$$

In order for this definition to be meaningful, we must show that the right-hand integral is independent of the parametrisation chosen for γ . Say that \tilde{z} is an equivalent parametrisation as above. Then the change of variables formula and the chain rule imply that

$$\begin{aligned} \int_a^b f(z(t)) z'(t) dt &= \int_c^d f(z(t(s))) z'(t(s)) t'(s) ds \\ &= \int_c^d f(\tilde{z}(s)) \tilde{z}'(s) ds. \end{aligned}$$

This proves that the integral of f over γ is well defined.

If γ is piecewise smooth, then the integral of f over γ is the sum of the integrals of f over the smooth parts of γ , so if $z(t)$ is a piecewise-smooth parametrisation as before, then

$$\int_{\gamma} f(z) dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t)) z'(t) dt.$$

We can define a curve γ^- obtained from the curve γ by reversing the orientation (so that γ and γ^- consist of the same points in the plane). As a particular parametrisation for γ^- we can take $z^- : [a, b] \rightarrow \mathbb{C}$ defined by

$$z^-(t) = z(b + a - t).$$

A smooth or piecewise-smooth curve is closed if $z(a) = z(b)$ for any of its parametrisations. A smooth or piecewise-smooth curve is simple if it is not self-intersecting, that is, $z(t) \neq z(s)$ unless $s = t$, $s, t \in (a, b)$.

A basic example consists of a circle. Consider the circle $C_r(z_0)$ centred at z_0 and of radius r , which by definition is the set

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

The positive orientation (counterclockwise) is the one that is given by the standard parametrisation

$$z(t) = z_0 + r e^{it}, \quad \text{where } t \in [0, 2\pi],$$

while the negative orientation (clockwise) is given by

$$z(t) = z_0 + r e^{-it}, \quad \text{where } t \in [0, 2\pi].$$

Section: Integration along curves.

By definition, the length of the smooth curve γ is

$$\text{length } (\gamma) = \int_a^b |z'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

Thank you

MATH50001 Analysis II, Complex Analysis

Lecture 5

Section: Integration along curves.

By definition, the length of the smooth curve γ is

$$\text{length } (\gamma) = \int_a^b |z'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

Theorem. Integration of continuous functions over curves satisfies the following properties:

- $$\int_{\gamma} (\alpha f(z) + \beta g(z)) dz = \alpha \int_{\gamma} f(z) dz + \beta \int_{\gamma} g(z) dz.$$
- If γ^- is γ with the reverse orientation, then

$$\int_{\gamma} f(z) dz = - \int_{\gamma^-} f(z) dz.$$

- (ML-inequality)

$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma).$$

Proof. The first property follows from the definition and the linearity of the Riemann integral. The second property is left as an exercise. For the third one, we note that

$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{t \in [a, b]} |f(z(t))| \int_a^b |z'(t)| dt = \sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma).$$

Section: Primitive functions.

Definition. A primitive for f on $\Omega \subset \mathbb{C}$ is a function F that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in \Omega$.

Theorem. If a continuous function f has a primitive F in an open set Ω , and γ is a curve in Ω that begins at w_1 and ends at w_2 , then

$$\int_{\gamma} f(z) dz = F(w_2) - F(w_1).$$

Proof. If γ is smooth, the proof is a simple application of the chain rule and the fundamental theorem of calculus. Indeed, if $z(t) : [a, b] \rightarrow \mathbb{C}$ is a parametrization for γ , then $z(a) = w_1$ and $z(b) = w_2$, and we have

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_a^b f(z(t)) z'(t) dt = \int_a^b F'(z(t)) z'(t) dt \\ &= \int_a^b \frac{d}{dt} F(z(t)) dt = F(z(b)) - F(z(a)). \end{aligned}$$

If γ is only piecewise-smooth then arguing the same as we did we have

$$\begin{aligned} \int_{\gamma} f(z) dz &= \sum_{k=0}^{n-1} (F(z(a_{k+1})) - F(z(a_k))) \\ &= F(z(a_n)) - F(z(a_0)) = F(z(b)) - F(z(a)). \end{aligned}$$

Corollary. If γ is a closed curve in an open set Ω , f is continuous and has a primitive in Ω , then

$$\oint_{\gamma} f(z) dz = 0.$$

Proof. This is immediate since the end-points of a closed curve coincide.

For example, the function $f(z) = 1/z$ does not have a primitive in the open set $\mathbb{C} \setminus \{0\}$, since if C is the unit circle parametrized by $z(t) = e^{it}$, $0 \leq t \leq 2\pi$, we have

$$\oint_C f(z) dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i \neq 0.$$

Corollary. If f is holomorphic in an open connected set Ω and $f' = 0$, then f is constant.

Proof. Fix a point $w_0 \in \Omega$. It suffices to show that $f(w) = f(w_0)$ for all $w \in \Omega$. Since Ω is connected, for any $w \in \Omega$, there exists a curve γ which joins w_0 to w . Since f is clearly a primitive for f' , we have

$$\int_{\gamma} f'(z) dz = f(w) - f(w_0),$$

By assumption, $f' = 0$ so the integral on the left is 0, and we conclude that $f(w) = f(w_0)$ as desired.

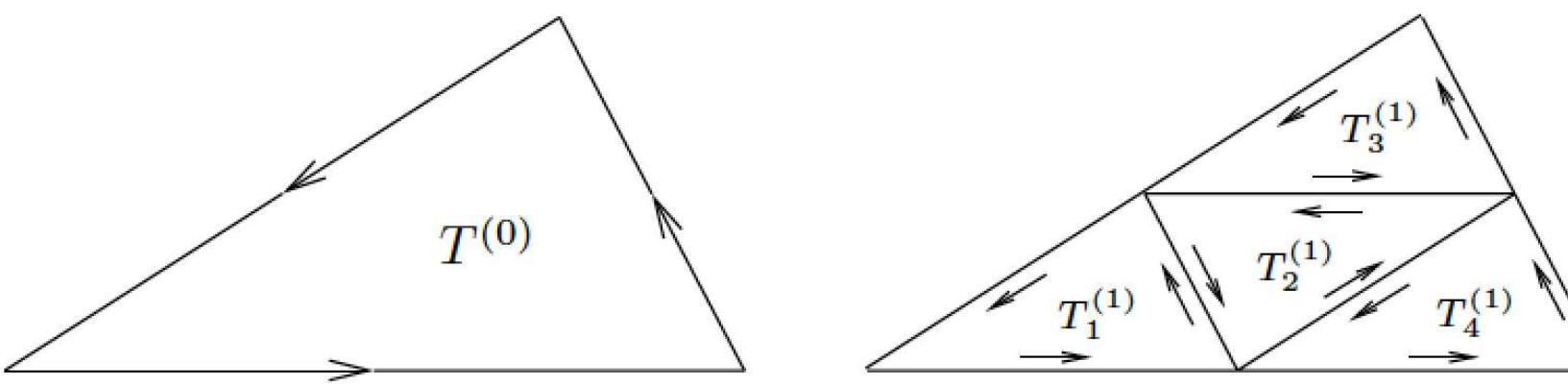
Section: Properties of holomorphic functions.

Theorem. Let $\Omega \subset \mathbb{C}$ be an open set and $T \subset \Omega$ be a triangle whose interior is also contained in Ω , then

$$\oint_T f(z) dz = 0,$$

whenever f is holomorphic in Ω .

Proof. Let $T^{(0)}$ be our original triangle (with a fixed orientation which we choose to be positive), and let $d^{(0)}$ and $p^{(0)}$ denote the diameter and perimeter of $T^{(0)}$, respectively. At the first step we find middle point of each side of $T^{(0)}$ and introduce four triangles $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}, T_4^{(1)}$ that are similar to the original triangle as follows:



Then

$$\oint_{T^{(0)}} f(z) dz = \oint_{T_1^{(1)}} f(z) dz + \oint_{T_2^{(1)}} f(z) dz + \oint_{T_3^{(1)}} f(z) dz + \oint_{T_4^{(1)}} f(z) dz.$$

There is some $j \in \{1, 2, 3, 4\}$ such that (WHY?)

$$\left| \oint_{T^{(0)}} f(z) dz \right| \leq 4 \left| \oint_{T_j^{(1)}} f(z) dz \right|.$$

We choose a triangle that satisfies this inequality, and rename it $T^{(1)}$. Observe that if $d^{(1)}$ and $p^{(1)}$ denote the diameter and perimeter of $T^{(1)}$, respectively. Then

$$d^{(1)} = \frac{1}{2} d^{(0)} \quad \text{and} \quad p^{(1)} = \frac{1}{2} p^{(0)}.$$

We now repeat this process for the triangle $T^{(1)}$. Continuing this process, we obtain a sequence of triangles

$$T^{(1)}, T^{(1)}, T^{(2)}, \dots, T^{(n)}, \dots$$

with the properties that

$$\left| \oint_{T^{(0)}} f(z) dz \right| \leq 4^n \left| \oint_{T_j^{(n)}} f(z) dz \right|$$

and

$$d^{(n)} = 2^{-n} d^{(0)} \quad \text{and} \quad p^{(n)} = 2^{-n} p^{(0)},$$

where $d^{(n)}$ and $p^{(n)}$ denote the diameter and perimeter of $T^{(n)}$.

Let $\Omega^{(n)}$ be the closed triangle such that $\partial\Omega^{(n)} = T^{(n)}$. Clearly we have a sequence of compact nested sets

$$\Omega^{(0)} \supset \Omega^{(1)} \supset \dots \supset \Omega^{(n)} \supset \dots,$$

whose diameter goes to 0. Then there exists a unique point z_0 that belongs to all triangles $\Omega^{(n)}$. Since f is holomorphic then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\psi(z),$$

where $\psi(z) \rightarrow 0$ as $z \rightarrow z_0$.

Since the constant $f(z_0)$ and the linear function $f'(z_0)(z - z_0)$ have primitives, we can integrate the above equality over $T^{(n)}$ and obtain

$$\oint_{T^{(n)}} f(z) dz = \oint_{T^{(n)}} \psi(z)(z - z_0) dz.$$

Since z_0 belongs to all triangles we have $|z - z_0| \leq d^{(n)}$ and using the ML-inequality we arrive at

$$\left| \oint_{T^{(n)}} f(z) dz \right| \leq \varepsilon_n d^{(n)} p^{(n)},$$

where $\varepsilon_n = \sup_{z \in T^{(n)}} |\psi(z)| \rightarrow 0$ as $n \rightarrow \infty$. Therefore

$$\left| \oint_{T^{(n)}} f(z) dz \right| \leq \varepsilon_n 4^{-n} d^{(0)} p^{(0)},$$

and thus finally we obtain

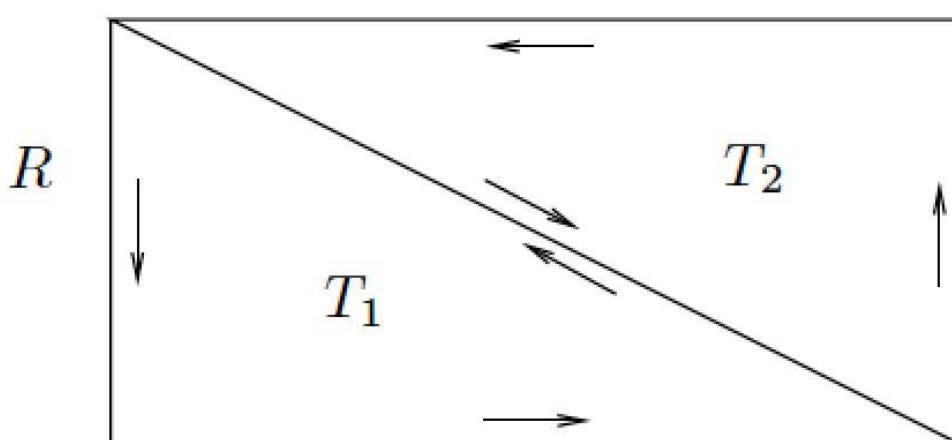
$$\left| \oint_{T^{(0)}} f(z) dz \right| \leq 4^n \left| \oint_{T_j^{(n)}} f(z) dz \right| \leq \varepsilon_n d^{(0)} p^{(0)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Corollary. If f is holomorphic in an open set Ω that contains a rectangle R and its interior, then

$$\oint_R f(z) dz = 0.$$

Proof. This immediately follows from the equality

$$\oint_R f(z) dz = \oint_{T_1} f(z) dz + \oint_{T_2} f(z) dz.$$



Thank you

MATH50001 Analysis II, Complex Analysis

Lecture 6

Section: Properties of holomorphic functions.

Theorem. Let $\Omega \subset \mathbb{C}$ be an open set and $T \subset \Omega$ be a triangle whose interior is also contained in Ω , then

$$\oint_T f(z) dz = 0,$$

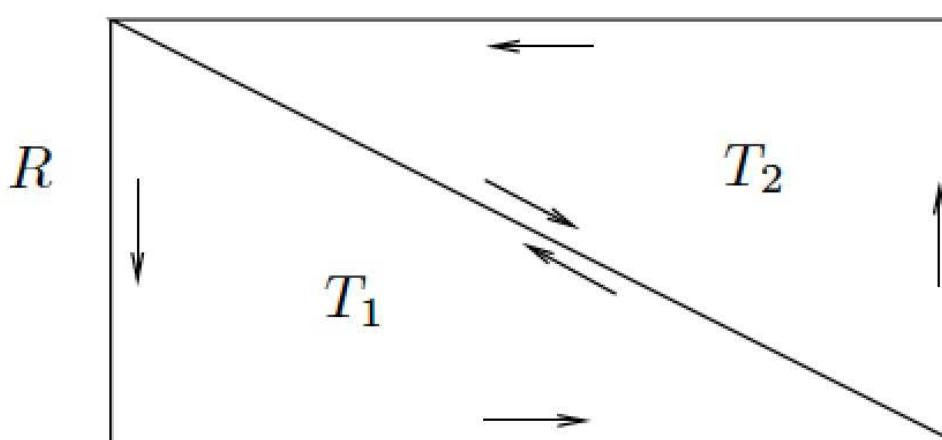
whenever f is holomorphic in Ω .

Corollary. If f is holomorphic in an open set Ω that contains a rectangle R and its interior, then

$$\oint_R f(z) dz = 0.$$

Proof. This immediately follows from the equality

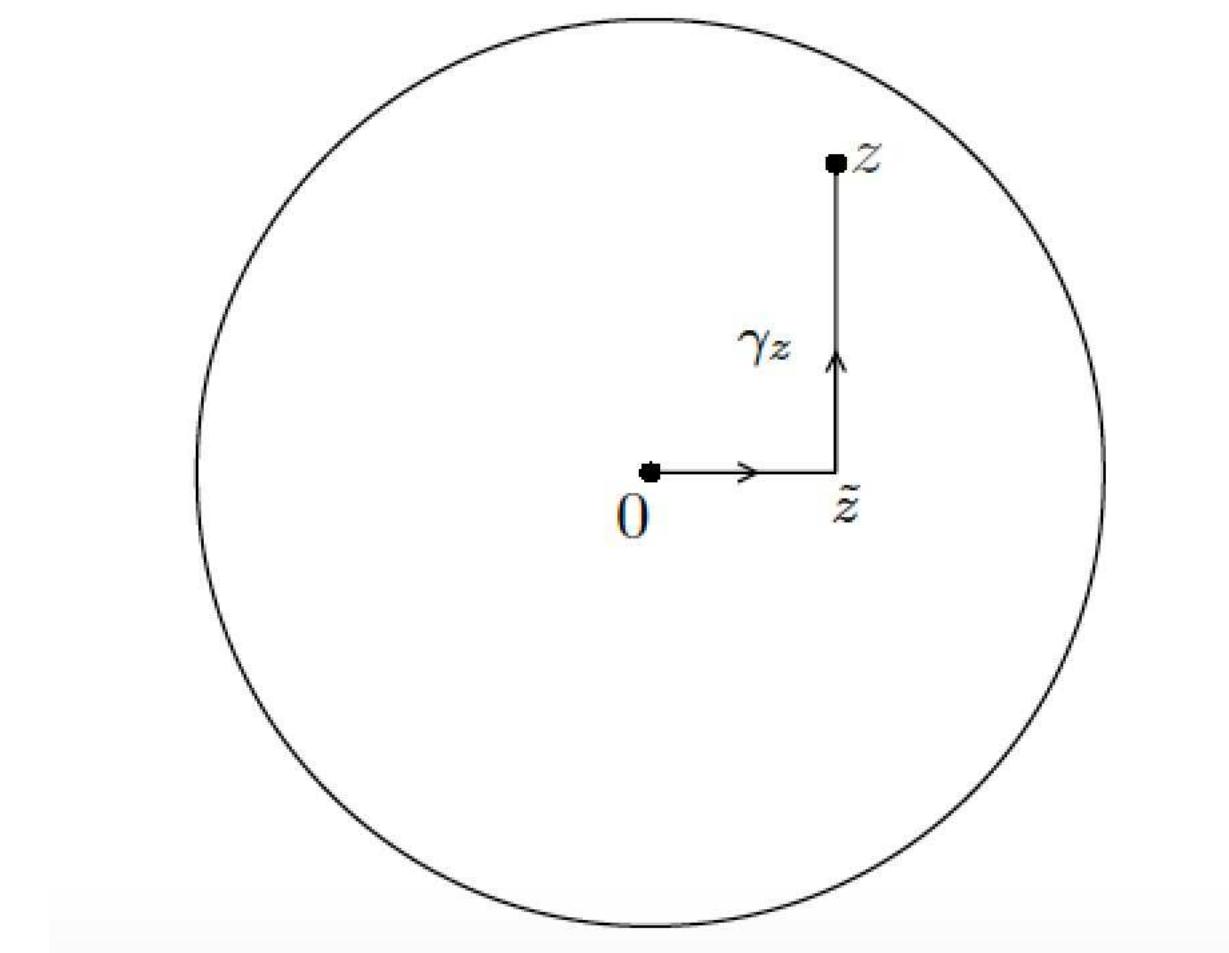
$$\oint_R f(z) dz = \oint_{T_1} f(z) dz + \oint_{T_2} f(z) dz.$$



Section: Local existence of primitives and Cauchy-Goursat theorem in a disc.

Theorem. A holomorphic function in an open disc has a primitive in that disc.

Proof. We may assume that the disc D is centered at the origin. For any $z \in D$ we consider γ_z given by



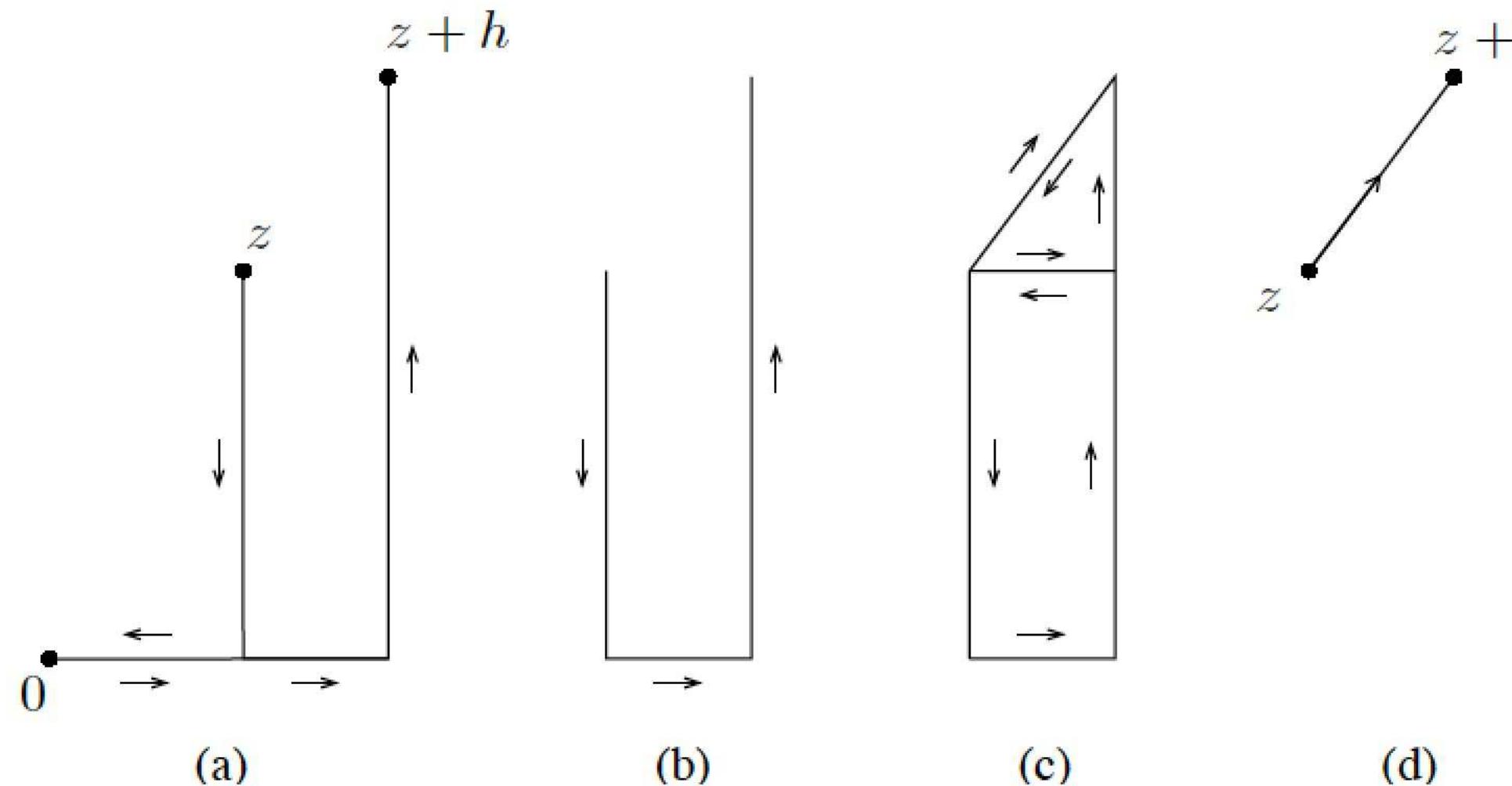
Define

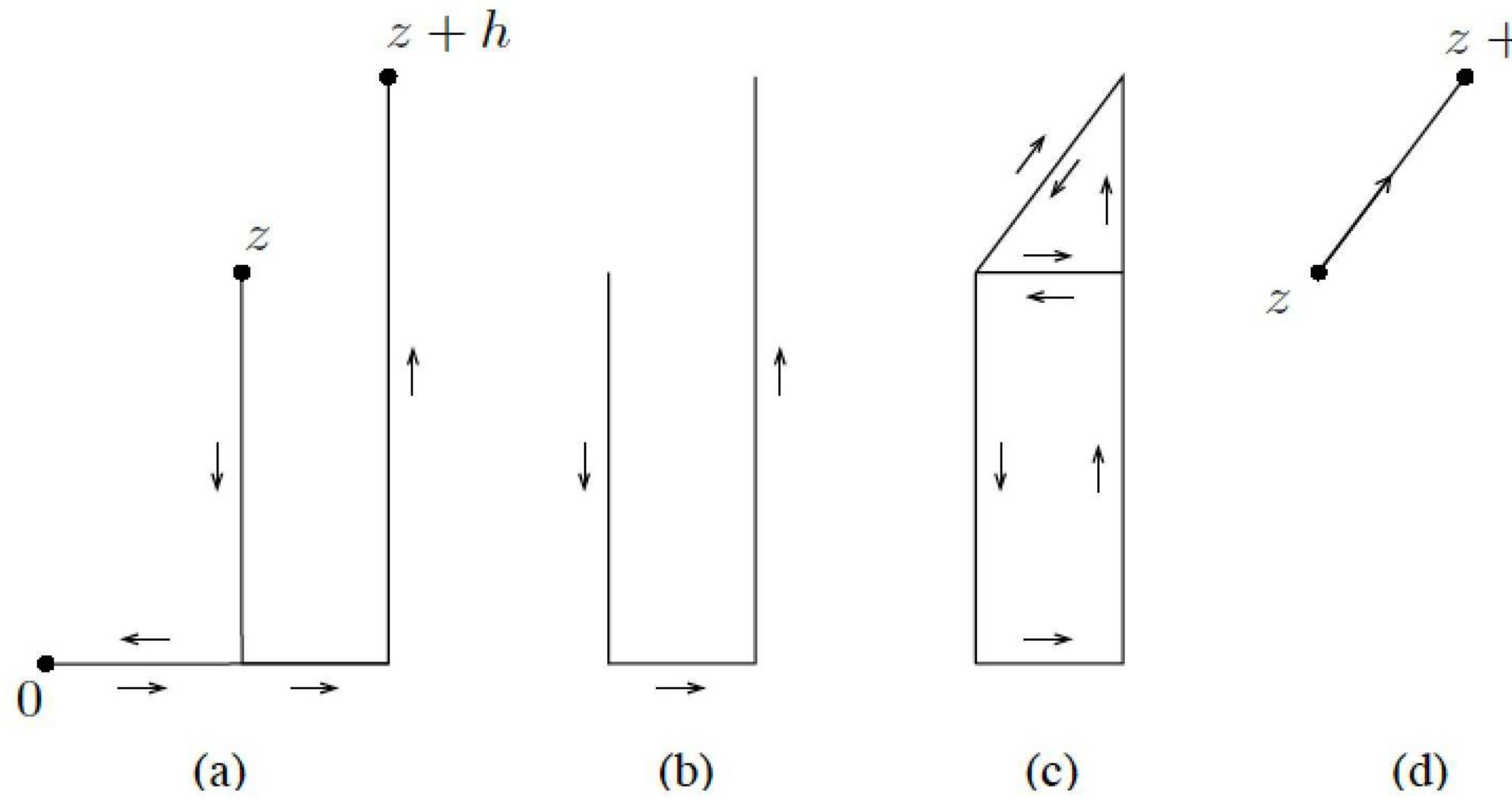
$$F(z) = \int_{\gamma_z} f(w) dw.$$

Consider the difference

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w) dw - \int_{\gamma_z} f(w) dw$$

The function f is first integrated along γ_{z+h} with the original orientation, and then along γ_z with the reverse orientation.





Using the fact that the integration over the triangle and the rectangle equal zero we obtain

$$F(z+h) - F(z) = \int_{\eta} f(w) dw,$$

where η is the straight line segment from z to $z+h$. Since f is continuous at z we can write

$$f(w) = f(z) + \psi(w),$$

where $\psi(w) \rightarrow 0$ as $w \rightarrow z$. Then

$$F(z+h) - F(z) = \int_{\eta} f(z) dw + \int_{\eta} \psi(w) dw = f(z) h + \int_{\eta} \psi(w) dw.$$

Finally we note that using the LM-inequality

$$\left| \int_{\eta} \psi(w) dw \right| \leq |h| \sup_{w \in \eta} |\psi(w)|$$

Since $\psi(w) \rightarrow 0$ as $w \rightarrow z$ we obtain

$$\lim_{h \rightarrow 0} \frac{F(z + h) - F(z)}{h} = f(z).$$

Corollary. (Cauchy-Goursat theorem for a disc)

If f is holomorphic in a disc, then

$$\oint_{\gamma} f(z) dz = 0$$

for any closed curve γ in that disc.

Corollary. Suppose f is holomorphic in an open set containing the circle C and its interior. Then

$$\oint_C f(z) dz = 0.$$

Proof. Let D be the disc with boundary circle C . Then there exists a slightly larger disc $\tilde{D} \supset D$ and so that f is holomorphic on \tilde{D} . We may now apply Cauchy-Goursat theorem in \tilde{D} to conclude that $\oint_C f(z) dz = 0$.

Section: Homotopies and simply connected domains.

Let γ_0 and γ_1 be two curves in an open set Ω with common end-points. That is if γ_0 and γ_1 are two parametrizations defined on $[a, b]$, we have

$$\gamma_0(a) = \gamma_1(a) = \alpha \quad \text{and} \quad \gamma_0(b) = \gamma_1(b) = \beta.$$

Definition. The curves γ_0 and γ_1 are said to be *homotopic* in Ω if for each $0 \leq s \leq 1$ there exists a curve $\gamma_s \subset \Omega$, parametrized by $\gamma_s(t)$ defined on $[a, b]$, such that for every s

$$\gamma_s(a) = \alpha \quad \text{and} \quad \gamma_s(b) = \beta,$$

and for all $t \in [a, b]$

$$\gamma_s(t)|_{s=0} = \gamma_0(t) \quad \text{and} \quad \gamma_s(t)|_{s=1} = \gamma_1(t).$$

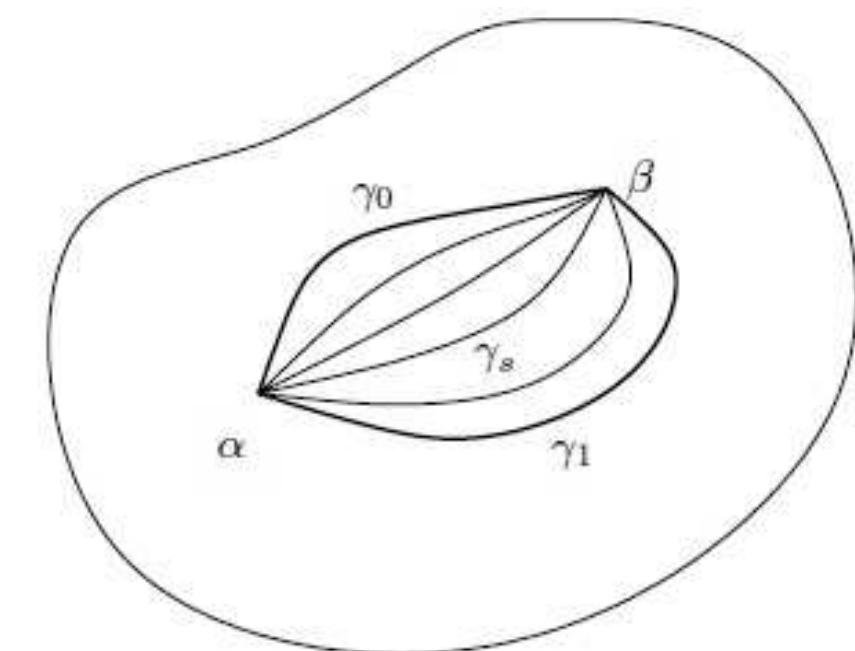
Moreover, $\gamma_s(t)$ should be jointly continuous in $s \in [0, 1]$ and $t \in [a, b]$.

Theorem. If f is holomorphic in Ω , then

$$\int_{\gamma_0} f(z) dz = \int_{\gamma_1} f(z) dz.$$

Proof. We first show that if two curves are close to each other and have the same end-points, then the integrals over them are equal.

Due to definition, the function $F(s, t) = \gamma_s(t)$ is continuous on $[0, 1] \times [a, b]$. Then the image of F denoted by K is compact.



Then there is $\varepsilon > 0$ such that every disc of radius $3\varepsilon > 0$ centred at a point in the image of F is completely contained in Ω .

WHY ??? Show it.

Since F is uniformly continuous we choose δ such that

$$\sup_{t \in [a, b]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \varepsilon \quad \text{whenever} \quad |s_1 - s_2| < \delta.$$

We now choose discs $\{D_0, \dots, D_n\}$ of radius 2ε , and points $\{z_0, \dots, z_{n+1}\}$ on γ_{s_1} and $\{w_0, \dots, w_{n+1}\}$ on γ_{s_2} such that the union of these discs covers both curves, and

$$z_i, z_{i+1}, w_i, w_{i+1} \in D_i.$$

Here $z_0 = w_0 = \gamma_{s_1}(a) = \gamma_{s_2}(a)$ and

$z_{n+1} = w_{n+1} = \gamma_{s_1}(b) = \gamma_{s_2}(b)$.

On each D_i , let F_i be a primitive of f .

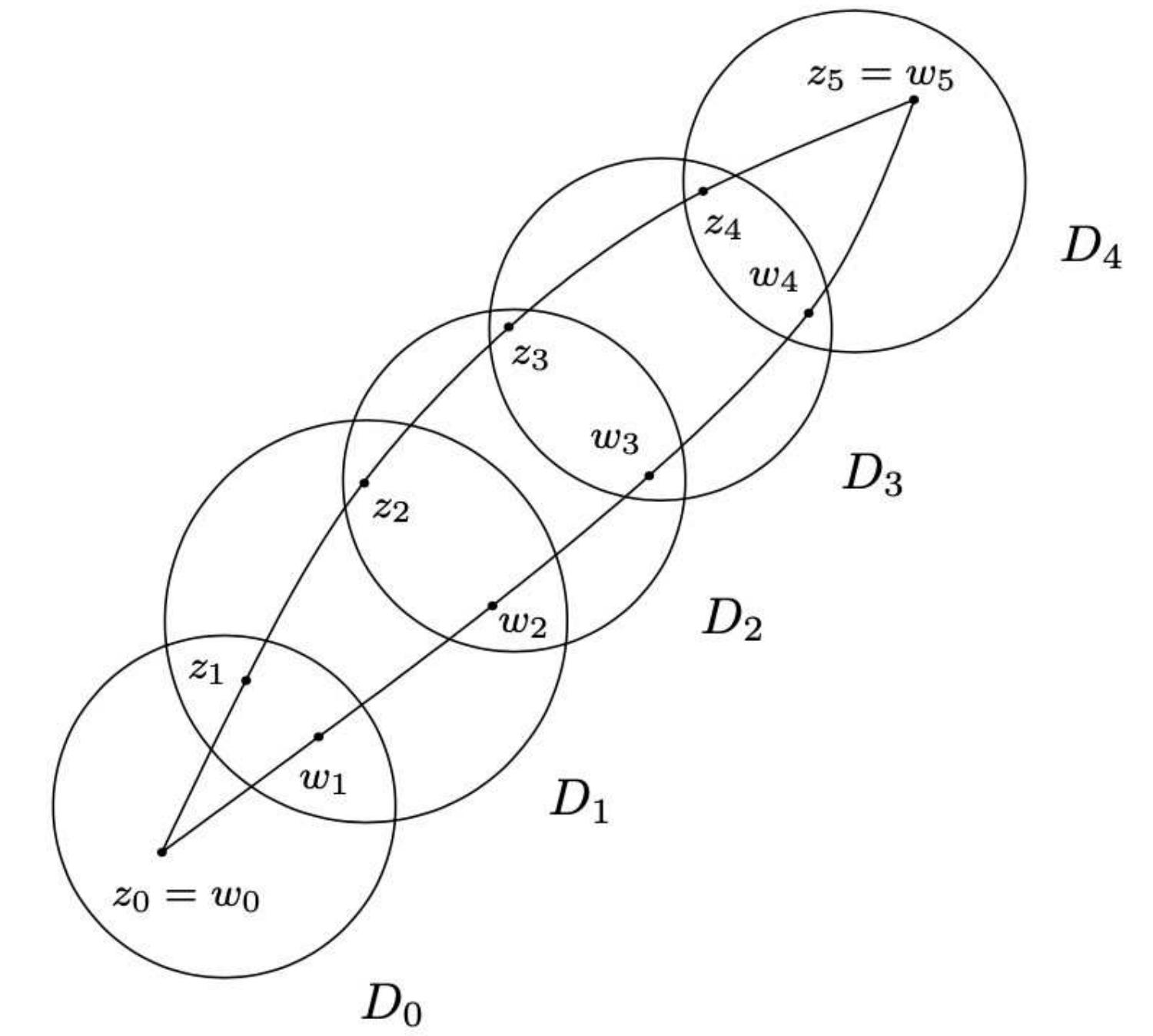
In $D_i \cap D_{i+1}$ the primitives F_i and F_{i+1} are two primitives of the same function, so they must differ by a constant.

Therefore

$$F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1}),$$

or

$$F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1}) = F_i(z_{i+1}) - F_i(w_{i+1}).$$



Finally we have

$$\begin{aligned}
& \int_{\gamma_{s_1}} f(z) dz - \int_{\gamma_{s_2}} f(z) dz \\
&= \sum_{i=0}^{n+1} (F_i(z_{i+1}) - F_i(z_i)) - \sum_{i=0}^{n+1} (F_i(w_{i+1}) - F_i(w_i)) \\
&\quad \sum_{i=0}^{n+1} (F_i(z_{i+1}) - F_i(w_{i+1}) - (F_i(z_i) - F_i(w_i))) \\
&= F_n(z_{n+1}) - F_n(w_{n+1}) - (F_0(z_0) - F_0(w_0)) = 0.
\end{aligned}$$

By subdividing the interval $[0, 1]$ into subintervals $[s_k, s_{k+1}]$, $k = 0, \dots, m$, of length less than δ and using the above arguments for each pair γ_{s_k} and $\gamma_{s_{k+1}}$ with $\gamma_{s_0} = \gamma_0$ and $\gamma_{s_{m+1}} = \gamma_1$ we complete the proof.

Definition. An open set $\Omega \subset \mathbb{C}$ is *simply connected* if any two pair of curves in Ω with the same end-points are homotopic.

Example. A disc D is simply connected. Indeed, let $\gamma_0(t)$ and $\gamma_1(t)$ be two curves lying in D . We can define $\gamma_s(t)$ by $\gamma_s(t) = (1 - s)\gamma_0(t) + s\gamma_1(t)$. Note that if $0 \leq s \leq 1$, then for each t , the point $\gamma_s(t)$ is on the segment joining $\gamma_0(t)$ and $\gamma_1(t)$, and so is in D .

The same argument works if D is replaced any open convex set.

WHY ??? - show it

Example. The set $\mathbb{C} \setminus \{(-\infty, 0]\}$ is simply connected.

WHY ??? - show it

Example. The punctured plane $\mathbb{C} \setminus \{0\}$ is not simply connected.

Theorem. Any holomorphic function in a simply connected domain has a primitive.

Proof. Fix a point z_0 in Ω and define

$$F(z) = \int_{\gamma} f(w) dw,$$

where the integral is taken over any curve in Ω joining z_0 to z . This definition is independent of the curve chosen, since Ω is simply connected. Consider

$$F(z + h) - F(z) = \int_{\eta} f(w) dw,$$

where η is the line segment joining z and $z + h$. Arguing as in the proof of the Theorem where we constructed a primitive to a holomorphic function in a disc, we obtain

$$\lim_{h \rightarrow 0} \frac{F(z + h) - F(z)}{h} = f(z).$$

The proof is complete.

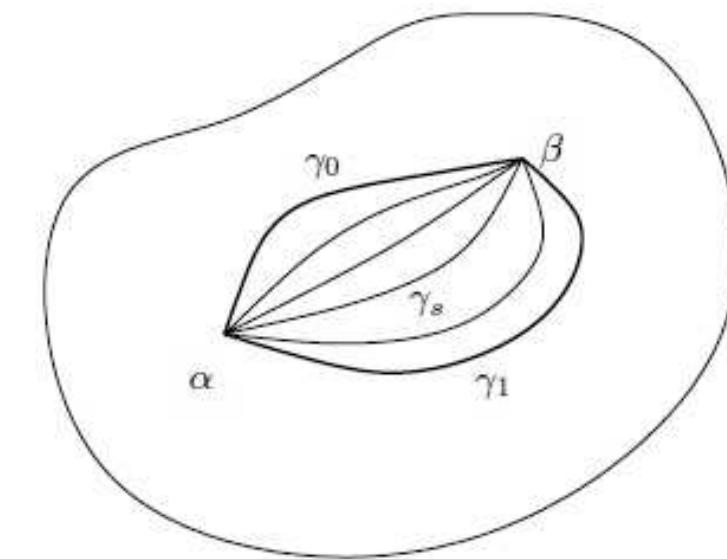
Thank you

MATH50001 Analysis II, Complex Analysis

Lecture 7

To remind:

In the previous lecture we introduced homotopic curves:



Theorem. If γ_0 and γ_1 are homotopic in Ω and if f is holomorphic in Ω , then

$$\int_{\gamma_0} f(z) dz = \int_{\gamma_1} f(z) dz.$$

Besides, we had

Definition. An open set $\Omega \subset \mathbb{C}$ is *simply connected* if any two pair of curves in Ω with the same end-points are homotopic.

Theorem. Any holomorphic function in a simply connected domain has a primitive.

Proof. Fix a point z_0 in Ω and define

$$F(z) = \int_{\gamma} f(w) dw,$$

where the integral is taken over any curve in Ω joining z_0 to z . This definition is independent of the curve chosen, since Ω is simply connected. Consider

$$F(z + h) - F(z) = \int_{\eta} f(w) dw,$$

where η is the line segment joining z and $z + h$. Arguing as in the proof of the Theorem where we constructed a primitive to a holomorphic function in a disc, we obtain

$$\lim_{h \rightarrow 0} \frac{F(z + h) - F(z)}{h} = f(z).$$

The proof is complete.

Corollary. (Cauchy-Goursat theorem)

If f is holomorphic in the simply connected open set Ω , then

$$\oint_{\gamma} f(z) \, dz = 0,$$

for any closed, piecewise-smooth, curve $\gamma \subset \Omega$.



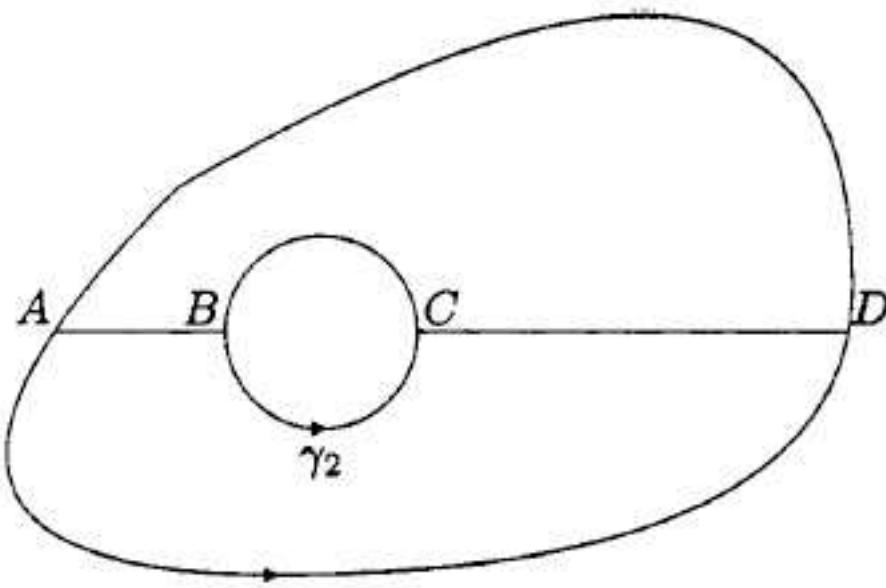
1858 - 1936, French

Theorem. (Deformation Theorem)

Let γ_1 and γ_2 be two simple, closed, piecewise-smooth curves with γ_2 lying wholly inside γ_1 and suppose f is holomorphic in a domain containing the region between γ_1 and γ_2 . Then

$$\oint_{\gamma_1} f(z) dz = \oint_{\gamma_2} f(z) dz.$$

Proof.



Example. Let $\gamma = \{z \in \mathbb{C} : |z - 1| = 2\}$. Then

$$\oint_{\gamma} \frac{1}{z^2 - 4} dz = \oint_{\gamma} \frac{1}{(z-2)(z+2)} dz = \frac{1}{4} \oint_{\gamma} \left(\frac{1}{z-2} - \frac{1}{z+2} \right) dz.$$

Since $1/(z+2)$ is holomorphic inside and on γ , then

$$\oint_{\gamma} \frac{1}{z+2} dz = 0.$$

On the other hand

$$\oint_{\gamma} \frac{1}{z-2} dz = \oint_{\{z : |z-2|=1\}} \frac{1}{z-2} dz = 2\pi i.$$

Therefore

$$\oint_{\gamma} \frac{1}{z^2 - 4} dz = i \frac{\pi}{2}.$$

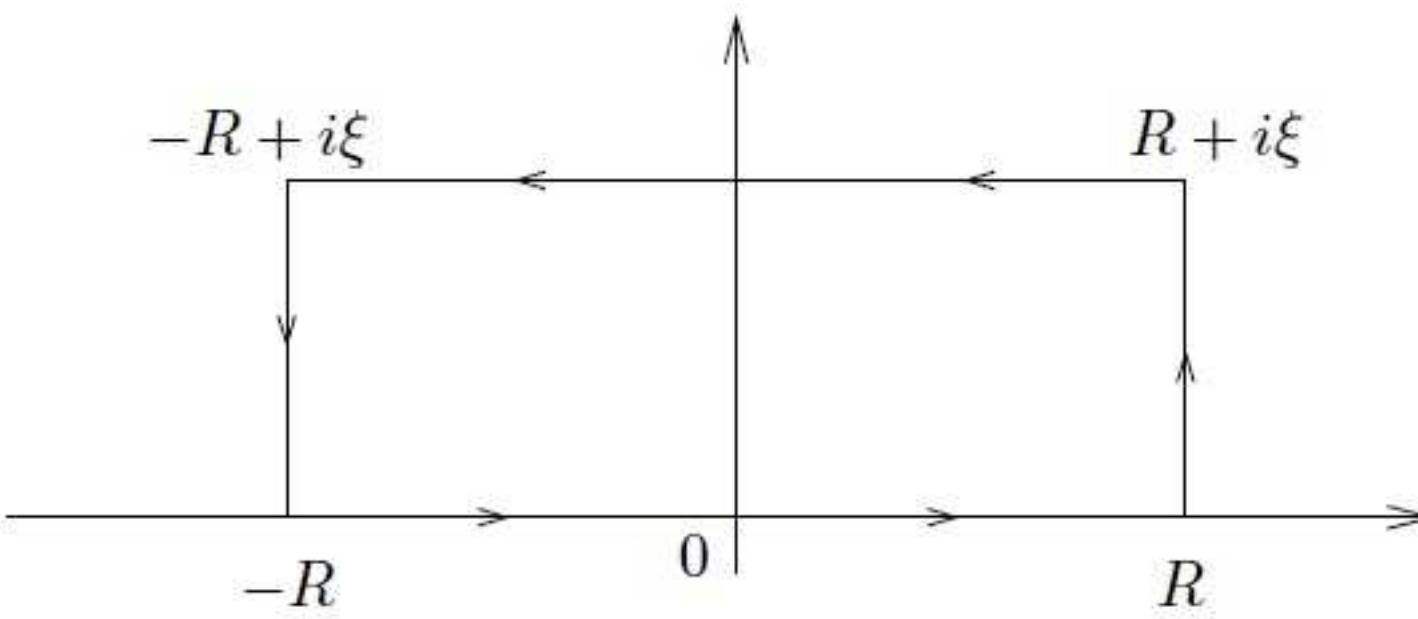
Example. We show that if $\xi \in \mathbb{R}$ then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx.$$

This gives a proof of the fact that $e^{-\pi x^2}$ is its own Fourier transform. If $\xi = 0$, the formula is precisely the known integral

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx.$$

Now suppose that $\xi > 0$, and consider the function $f(z) = e^{-\pi z^2}$, which is entire, and in particular holomorphic in the interior of the contour γ_R



The contour γ_R consists of a rectangle with vertices $R, R + i\xi, -R + i\xi, -R$ and the positive counterclockwise orientation.

By the Cauchy-Goursat theorem

$$\oint_{\gamma_R} f(z) dz = 0 \quad (*)$$

The integral over the real segment is simply

$$\int_{-R}^R e^{-\pi x^2} dx$$

which converges to 1 as $R \rightarrow \infty$. The integral on the vertical side on the right is

$$\begin{aligned} |I(R)| &= \left| \int_0^\xi f(R + iy) i dy \right| = \left| \int_0^\xi e^{-\pi(R^2 + 2iy - y^2)} dy \right| \\ &\leq e^{-\pi R^2} \int_0^\xi |e^{-\pi(2iy - y^2)}| dy \leq e^{-\pi R^2} \xi e^{\pi \xi^2} \rightarrow 0, \end{aligned}$$

as $R \rightarrow \infty$.

Similarly, the integral over the vertical segment on the left also goes to 0 as $R \rightarrow \infty$ for the same reasons.

Finally, the integral over the horizontal segment on top is

$$\begin{aligned} \int_R^{-R} e^{-\pi(x+i\xi)^2} dx &= - \int_{-R}^R e^{-\pi(x+i\xi)^2} dx \\ &= -e^{\pi\xi^2} \int_{-R}^R e^{-\pi x^2} e^{-2\pi i x \xi} dx. \end{aligned}$$

Therefore, in the limit as $R \rightarrow \infty$ we obtain that $(*)$ gives

$$0 = 1 - e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx.$$

Section: Cauchy's integral formulae.

Theorem. Let f be holomorphic inside and on a simple, closed, piecewise-smooth curve γ . Then for any point z_0 interior to γ we have

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz.$$

Proof. If z_0 is interior to γ then for any $r > 0$ such that $\gamma_r = \{z : |z - z_0| = r\}$ lying wholly inside γ , using the deformation theorem we obtain

$$\oint_{\gamma} \frac{f(z)}{z - z_0} dz = \oint_{\gamma_r} \frac{f(z)}{z - z_0} dz.$$

Then

$$\begin{aligned}
& \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{z - z_0} dz \\
&= \frac{1}{2\pi i} f(z_0) \oint_{\gamma_r} \frac{1}{z - z_0} dz + \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} dz \\
&= f(z_0) + \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} dz.
\end{aligned}$$

Since f is holomorphic it is continuous at z_0 . Therefore for a given $\varepsilon > 0$ there is $\delta > r > 0$ such that as soon $|z - z_0| < \delta$ we have

$$|f(z) - f(z_0)| < \varepsilon.$$

Then, by using the ML-inequality we have

$$\left| \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{1}{2\pi} \frac{\varepsilon}{r} 2\pi r = \varepsilon.$$

So we have proved that for any $\varepsilon > 0$

$$\left| \oint_{\gamma} \frac{f(z)}{z - z_0} dz - f(z_0) \right| < \varepsilon$$

and hence

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz = f(z_0).$$

The proof is complete.

Example.

$$\begin{aligned} \frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{(z-i)(z+i)} dz \\ &= \frac{1}{2\pi i} \frac{1}{2i} \oint_{|z|=2} \left(\frac{e^z}{z-i} - \frac{e^z}{z+i} \right) dz \\ &= \frac{1}{2i} (e^i - e^{-i}) = \sin 1. \end{aligned}$$

Theorem. (Generalised Cauchy's integral formula)

Let f be holomorphic in an open set Ω , then f has infinitely many complex derivatives in Ω . Moreover, for simple, closed, piecewise-smooth curve $\gamma \subset \Omega$ and any z lying inside γ we have

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta - z)^{n+1}} d\eta.$$

Proof. The proof is by induction on n . The case $n = 0$ is simply the Cauchy integral formula. Suppose that f has up to $n - 1$ complex derivatives and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta - z)^n} d\eta.$$

Let $h \in \mathbb{C}$ be small enough, so that $z + h$ is lying inside γ . Then

$$\begin{aligned} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} \\ = \frac{(n-1)!}{2\pi i} \oint_{\gamma} f(\eta) \frac{1}{h} \left(\frac{1}{(\eta-z-h)^n} - \frac{1}{(\eta-z)^n} \right) d\eta. \end{aligned}$$

Recall

$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1})$$

and apply it with $A = 1/(\eta - z - h)$ and $B = 1/(\eta - z)$. Then we obtain as $h \rightarrow 0$

$$\begin{aligned} & \frac{1}{h} \left(\frac{1}{(\eta-z-h)^n} - \frac{1}{(\eta-z)^n} \right) \\ &= \frac{1}{h} \frac{h}{(\eta-z-h)(\eta-z)} (A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1}) \\ &\quad \rightarrow \frac{1}{(\eta-z)^2} \frac{n}{(\eta-z)^{n-1}}. \end{aligned}$$

This implies

$$\begin{aligned} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} \\ \rightarrow \frac{(n-1)!}{2\pi i} \oint_{\gamma} f(\eta) \frac{1}{(\eta-z)^2} \frac{n}{(\eta-z)^{n-1}} d\eta \\ = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta-z)^{n+1}} d\eta. \end{aligned}$$

The proof is complete.

Corollary. If f is holomorphic in Ω , then all its derivatives f', f'', \dots , are holomorphic.

Exercise:

Let f be continuous on a piecewise-smooth curve γ . At each point $z \notin \gamma$ define the value of a function F by

$$F(z) = \int_{\gamma} \frac{f(\eta)}{\eta - z} d\eta.$$

Show that F is holomorphic at $z \notin \gamma$ and

$$F'(z) = \int_{\gamma} \frac{f(\eta)}{(\eta - z)^2} d\eta.$$

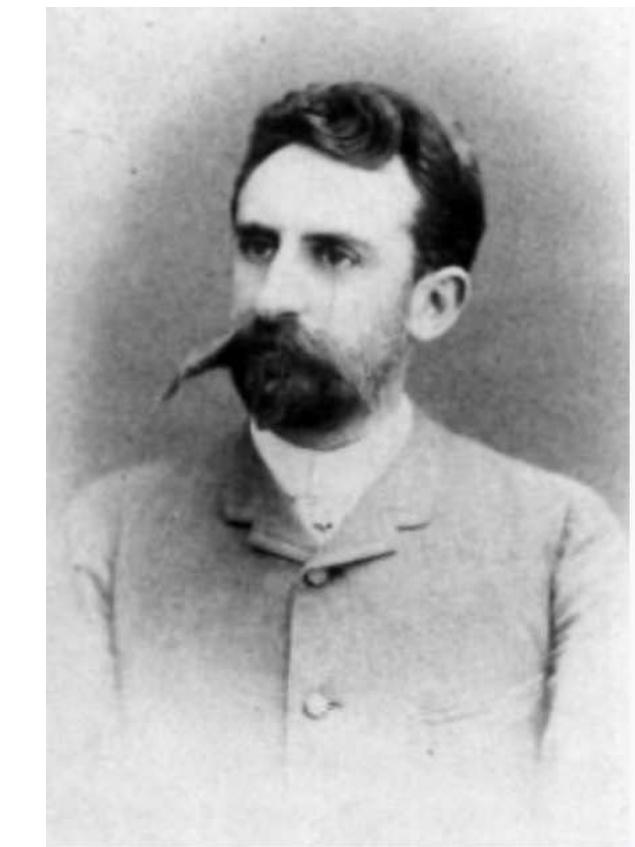
Theorem. (Morera's theorem)

Suppose f is a continuous function in the open disc D such that for any triangle T contained in D

$$\int_T f(z) dz = 0,$$

then f is holomorphic.

Proof. We have proved before that f has a primitive F in D that satisfies $F' = f$. Then F is indefinitely complex differentiable, and therefore f is holomorphic.



Giacinto Morera

Italian, 1856 - 1909

Theorem. (Fundamental theorem of Algebra) Every polynomial of degree greater than zero with complex coefficients has at least one zero.

Proof. Assume that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} \cdots + a_0 = 0.$$

has no zeros. Then $1/p(z)$ is entire. Clearly $|1/p(z)| \rightarrow 0$ as $|z| \rightarrow \infty$. Indeed, given $\varepsilon > 0$ there is R such that

$$\left| \frac{1}{p(z)} \right| < \varepsilon, \quad \forall z : |z| > R.$$

Since $1/p(z)$ is entire it is also continuous and therefore there is a constant $M > 0$ such that

$$\left| \frac{1}{p(z)} \right| \leq M, \quad z : |z| \leq R$$

and thus $|1/p(z)|$ is bounded in \mathbb{C} . This implies $1/p$ is constant and this contradicts the fact that $p(z)$ is a polynomial of degree greater than zero.

Corollary.

Every polynomial $P(z) = a_n z^n + \dots + a_0$ of degree $n \geq 1$ has precisely n roots in \mathbb{C} . If these roots are denoted by w_1, \dots, w_n , then P can be factored as

$$P(z) = a_n(z - w_1)(z - w_2) \dots (z - w_n).$$

Proof. We now know that P has at least one root, say w_1 . Then writing $z = (z - w_1) + w_1$. Substituting this in $P(z) = a_n z^n + \dots + a_0$ and using the binomial formula we get

$$P(z) = b_n(z - w_1)^n + \dots + b_1(z - w_1) + b_0,$$

where $b_n = a_n$. Since $P(w_1) = 0$ we have $b_0 = 0$ and thus

$$P(z) = (z - w_1)Q(z).$$

Repeating this we find

$$P(z) = a_n(z - w_1)(z - w_2) \dots (z - w_n).$$

Section: Applications of Cauchy's integral formulae.

Corollary. (Liouville's theorem)

If an entire function is bounded, then it is constant.



Joseph Liouville
French 1809 - 1882

Proof. Suppose that f is entire and bounded. Then there is a constant M such that

$$|f(z)| \leq M, \quad \forall z \in \mathbb{C}.$$

Let $z_0 \in \mathbb{C}$ and let $\gamma_r = \{z : |z - z_0| = r\}$. Then

$$|f'(z_0)| = \left| \frac{1!}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{(z - z_0)^2} dz \right| \leq \frac{M}{r} \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

Therefore for any $z_0 \in \mathbb{C}$ we have $f'(z_0) = 0$ and thus f is constant.

Section: The complex logarithm.

We have seen that to make sense of the logarithm as a single-valued function, we must restrict the set on which we define it. This is the so-called choice of a branch or sheet of the logarithm.

Theorem. Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_{\Omega}(z)$ so that:

- (i) F is holomorphic in Ω ,
- (ii) $e^F(z) = z$, $\forall z \in \Omega$,
- (iii) $F(r) = \log r$ whenever r is a real number and near 1.

Proof. We shall construct F as a primitive of the function $1/z$. Since $0 \notin \Omega$, the function $f(z) = 1/z$ is holomorphic in Ω . We define

$$\log_{\Omega}(z) = F(z) = \int_{\gamma} f(z) \, dz,$$

where γ is any curve in Ω connecting 1 to z . Since Ω is simply connected, this definition does not depend on the path chosen. Then F is holomorphic and $F'(z) = 1/z$ for all $z \in \Omega$. This proves (i).

To prove (ii), it suffices to show that $ze^{?F(z)} = 1$. Indeed,

$$\frac{d}{dz} \left(ze^{-F(z)} \right) = e^{-F(z)} - zF'(z)e^{-F(z)} = (1 - zF'(z))e^{-F(z)} = 0.$$

Thus $ze^{-F(z)}$ is a constant. Using $F(1) = 0$ we find that this constant must be 1 .

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω , then f is holomorphic in Ω .

Proof. Let D be any disc whose closure is contained in Ω and T any triangle in that disc. Then, since each f_n is holomorphic, Goursat's theorem implies

$$\oint_T f_n(z) dz = 0, \quad \text{for all } n.$$

By assumption $f_n \rightarrow f$ uniformly in the closure of D , so f is continuous and

$$\oint_T f_n(z) dz = \oint_T f(z) dz.$$

Therefore

$$\oint_T f(z) dz = 0.$$

Using Morera's theorem we find that f is holomorphic in D . Since this conclusion is true for every D whose closure is contained in Ω , we find that f is holomorphic in all of Ω .

Remark. This is not true in the case of real variables: the uniform limit of continuously differentiable functions need not be differentiable. WHY??

Remark. Consider

$$F(z) = \sum_{n=1}^{\infty} f_n(z)$$

where f_n are holomorphic in $\Omega \subset \mathbb{C}$. Assume that the series converges uniformly in compact subsets of Ω , then Theorem guarantees that F is also holomorphic in Ω .

Section: Holomorphic functions defined in terms of integrals.

Theorem. Let $F(z, s)$ be defined for $(z, s) \in \Omega \times [0, 1]$ where $\Omega \subset \mathbb{C}$ is an open set. Suppose F satisfies the following properties:

- $F(z, s)$ is holomorphic in Ω for each s .
- F is continuous on $\Omega \times [0, 1]$.

Then the function f defined on Ω by

$$f(z) = \int_0^1 F(z, s) \, ds$$

is holomorphic.

Proof. To prove this result, it suffices to prove that f is holomorphic in any disc D contained in Ω . By Morera's theorem this could be achieved by showing that for any triangle T contained in D we have

$$\oint_T \int_0^1 F(z, s) \, ds \, dz = 0.$$

The proof would be trivial if we could change the order of integration that is not clear. In order to go around this problem we consider for each $n \geq 1$ the Riemann sum

$$f_n(z) = \frac{1}{n} \sum_{k=1}^n F(z, k/n).$$

Then by the first assumption f_n is holomorphic in Ω .

We can now show that on any disc D such that $\overline{D} \subset \Omega$, the sequence $\{f_n\}_{n=1}^\infty$ converges uniformly to f .

Indeed, since F is continuous on $\Omega \times [0, 1]$ for a given $\varepsilon > 0$ there exists $\delta > 0$ such that as soon $|s_1 - s_2| < \delta$ we have

$$\sup_{z \in D} |F(z, s_1) - F(z, s_2)| < \varepsilon.$$

Then if $n > 1/\delta$ and $z \in D$ we find

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{k=1}^n \int_{(k-1)/n}^{k/n} (F(z, k/n) - F(z, s)) \, ds \right| \\ &\leq \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \int_{(k-1)/n}^{k/n} |F(z, k/n) - F(z, s)| \, ds \, dk < \sum_{k=1}^n \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

By Theorem 1.1 we conclude that f is holomorphic in D and thus in Ω .

Section: Schwarz reflection principle.

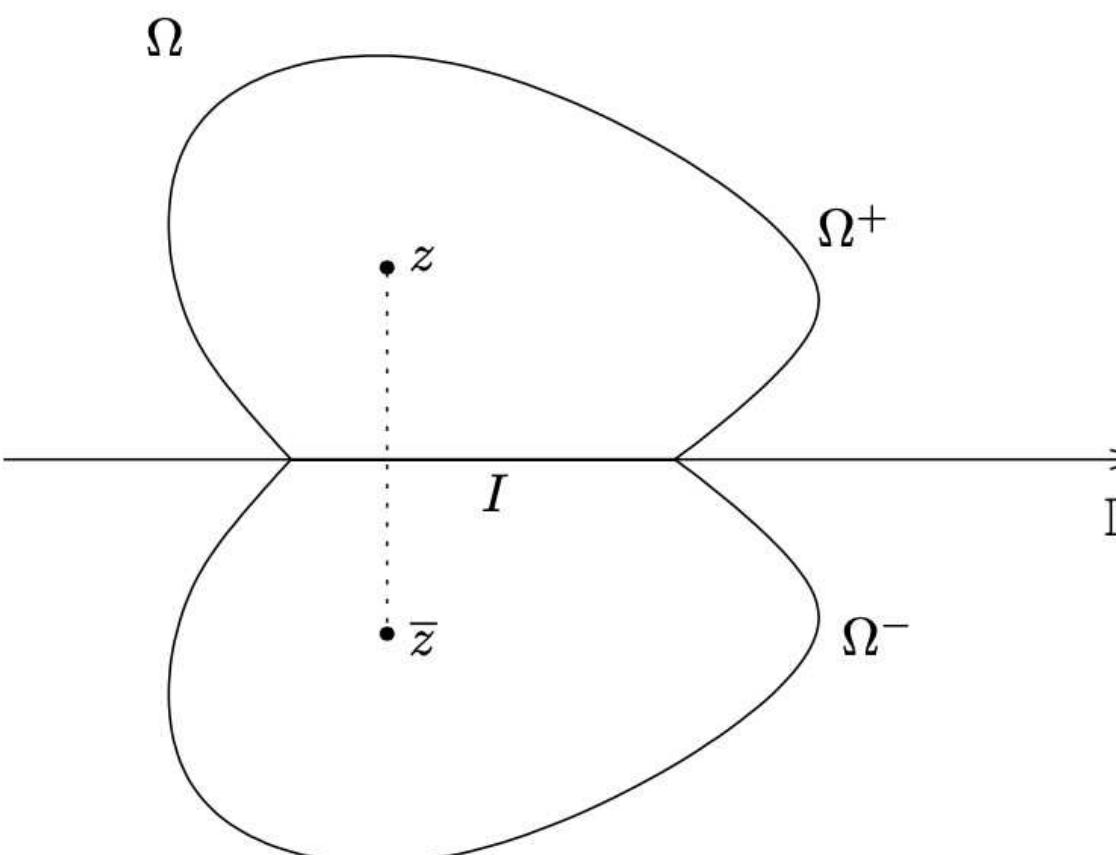
In this section we deal with a simple extension problem for holomorphic functions that is very useful in applications. It is the Schwarz reflection principle that allows one to extend a holomorphic function to a larger domain.

Let $\Omega \subset \mathbb{C}$ be open and symmetric with respect to the real line, that is

$$z \in \Omega \quad \text{iff} \quad \bar{z} \in \Omega.$$

Let

$$\begin{aligned}\Omega^+ &= \{z \in \Omega : \operatorname{Im} z > 0\}, & \Omega^- &= \{z \in \Omega : \operatorname{Im} z < 0\} \\ && \text{and} & I = \{z \in \Omega : \operatorname{Im} z = 0\}.\end{aligned}$$



Thank you

MATH50001 Analysis II, Complex Analysis

Lecture 8

Section: Cauchy's integral formulae.

Theorem. Let f be holomorphic inside and on a simple, closed, piecewise-smooth curve γ . Then for any point z_0 interior to γ we have

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz.$$

Example.

$$\begin{aligned} \frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{(z-i)(z+i)} dz \\ &= \frac{1}{2\pi i} \frac{1}{2i} \oint_{|z|=2} \left(\frac{e^z}{z-i} - \frac{e^z}{z+i} \right) dz \\ &= \frac{1}{2i} (e^i - e^{-i}) = \sin 1. \end{aligned}$$

Theorem. (Generalised Cauchy's integral formula)

Let f be holomorphic in an open set Ω , then f has infinitely many complex derivatives in Ω . Moreover, for simple, closed, piecewise-smooth curve $\gamma \subset \Omega$ and any z lying inside γ we have

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta - z)^{n+1}} d\eta.$$

Proof. The proof is by induction on n . The case $n = 0$ is simply the Cauchy integral formula. Suppose that f has up to $n - 1$ complex derivatives and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta - z)^n} d\eta.$$

Let $h \in \mathbb{C}$ be small enough, so that $z + h$ is lying inside γ . Then

$$\begin{aligned} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} \\ = \frac{(n-1)!}{2\pi i} \oint_{\gamma} f(\eta) \frac{1}{h} \left(\frac{1}{(\eta-z-h)^n} - \frac{1}{(\eta-z)^n} \right) d\eta. \end{aligned}$$

Recall

$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1})$$

and apply it with $A = 1/(\eta - z - h)$ and $B = 1/(\eta - z)$. Then we obtain as $h \rightarrow 0$

$$\begin{aligned} & \frac{1}{h} \left(\frac{1}{(\eta-z-h)^n} - \frac{1}{(\eta-z)^n} \right) \\ &= \frac{1}{h} \frac{h}{(\eta-z-h)(\eta-z)} (A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1}) \\ &\quad \rightarrow \frac{1}{(\eta-z)^2} \frac{n}{(\eta-z)^{n-1}}. \end{aligned}$$

This implies

$$\begin{aligned} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} \\ \rightarrow \frac{(n-1)!}{2\pi i} \oint_{\gamma} f(\eta) \frac{1}{(\eta-z)^2} \frac{n}{(\eta-z)^{n-1}} d\eta \\ = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta-z)^{n+1}} d\eta. \end{aligned}$$

The proof is complete.

Corollary. If f is holomorphic in Ω , then all its derivatives f', f'', \dots , are holomorphic.

Exercise:

Let f be continuous on a piecewise-smooth curve γ . At each point $z \notin \gamma$ define the value of a function F by

$$F(z) = \int_{\gamma} \frac{f(\eta)}{\eta - z} d\eta.$$

Show that F is holomorphic at $z \notin \gamma$ and

$$F'(z) = \int_{\gamma} \frac{f(\eta)}{(\eta - z)^2} d\eta.$$

Section: Applications of Cauchy's integral formulae.

Corollary. (Liouville's theorem)

If an entire function is bounded, then it is constant.



Joseph Liouville
French 1809 - 1882

Proof. Suppose that f is entire and bounded. Then there is a constant M such that

$$|f(z)| \leq M, \quad \forall z \in \mathbb{C}.$$

Let $z_0 \in \mathbb{C}$ and let $\gamma_r = \{z : |z - z_0| = r\}$. Then

$$|f'(z_0)| = \left| \frac{1!}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{(z - z_0)^2} dz \right| \leq \frac{M}{r} \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

Therefore for any $z_0 \in \mathbb{C}$ we have $f'(z_0) = 0$ and thus f is constant.

Theorem. (Fundamental theorem of Algebra) Every polynomial of degree greater than zero with complex coefficients has at least one zero.

Proof. Assume that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} \cdots + a_0 = 0.$$

has no zeros. Then $1/p(z)$ is entire. Clearly $|1/p(z)| \rightarrow 0$ as $|z| \rightarrow \infty$. Indeed, given $\varepsilon > 0$ there is R such that

$$\left| \frac{1}{p(z)} \right| < \varepsilon, \quad \forall z : |z| > R.$$

Since $1/p(z)$ is entire it is also continuous and therefore there is a constant $M > 0$ such that

$$\left| \frac{1}{p(z)} \right| \leq M, \quad z : |z| \leq R$$

and thus $|1/p(z)|$ is bounded in \mathbb{C} . This implies $1/p$ is constant and this contradicts the fact that $p(z)$ is a polynomial of degree greater than zero.

Corollary.

Every polynomial

$$P(z) = a_n z^n + \dots + a_0$$

of degree $n \geq 1$ has precisely n roots in \mathbb{C} . If these roots are denoted by w_1, \dots, w_n , then P can be factored as

$$P(z) = a_n(z - w_1)(z - w_2) \dots (z - w_n).$$

Proof. We now know that P has at least one root, say w_1 . Then writing $z = (z - w_1) + w_1$. Substituting this in $P(z) = a_n z^n + \dots + a_0$ and using the binomial formula we get

$$P(z) = b_n(z - w_1)^n + \dots + b_1(z - w_1) + b_0,$$

where $b_n = a_n$. Since $P(w_1) = 0$ we have $b_0 = 0$ and thus

$$P(z) = (z - w_1)Q(z).$$

Repeating this we find

$$P(z) = a_n(z - w_1)(z - w_2) \dots (z - w_n).$$

Theorem. (Morera's theorem)

Suppose f is a continuous function in the open disc D such that for any triangle T contained in D

$$\int_T f(z) dz = 0,$$

then f is holomorphic.

Proof. We have proved before that f has a primitive F in D that satisfies $F' = f$. Then F is indefinitely complex differentiable, and therefore f is holomorphic.



Giacinto Morera

Italian, 1856 - 1909

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω , then f is holomorphic in Ω .

Thank you

MATH50001 Analysis II, Complex Analysis

Lecture 9

Section: Taylor and Maclaurin series.

Theorem. (Taylor Expansion theorem)

Let f be holomorphic in an open set Ω and let $z_0 \in \Omega$. Then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!} (z - z_0)^2 \dots,$$

valid in all circles $\{z : |z - z_0| < r\} \subset \Omega$.



Brook Taylor
1685 – 1731,
English



Colin Maclaurin
1698 – 1746
Scottish

Proof. Let $\gamma = \{\eta : |\eta - z_0| = r\} \subset \Omega$ and let $z : |z - z_0| < r$.

$$\begin{aligned}
 f(z) &= \frac{1}{2i\pi} \oint_{\gamma} \frac{f(\eta)}{\eta - z} d\eta = \frac{1}{2i\pi} \oint_{\gamma} \frac{f(\eta)}{(\eta - z_0) - (z - z_0)} d\eta \\
 &= \frac{1}{2i\pi} \oint_{\gamma} \frac{f(\eta)}{\eta - z_0} \cdot \frac{1}{1 - \frac{z-z_0}{\eta-z_0}} d\eta \\
 &= \frac{1}{2i\pi} \oint_{\gamma} \frac{f(\eta)}{\eta - z_0} \cdot \left\{ 1 + \frac{z - z_0}{\eta - z_0} + \left(\frac{z - z_0}{\eta - z_0} \right)^2 + \dots \right. \\
 &\quad \left. + \left(\frac{z - z_0}{\eta - z_0} \right)^{n-1} + \frac{\left(\frac{z - z_0}{\eta - z_0} \right)^n}{1 - \frac{z - z_0}{\eta - z_0}} \right\} d\eta
 \end{aligned}$$

Using Cauchy's generalised integral formula applied to the first n terms we obtain

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!} (z - z_0)^{n-1} + R_n,$$

where

$$R_n = \frac{(z - z_0)^n}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta - z)(\eta - z_0)^n} d\eta.$$

Let $M = \max_{\eta \in \gamma} |f(\eta)|$ and let $|z - z_0| = \rho$. Then by using the ML-inequality we obtain

$$|R_n| \leq \frac{\rho^n}{2\pi} \frac{M}{(r - \rho)r^n} (2\pi r) = \frac{rM}{r - \rho} \left(\frac{\rho}{r}\right)^n.$$

Since $\rho < r$ we conclude that $R_n \rightarrow 0$ as $n \rightarrow \infty$.

Definition. The expansion

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!} (z - z_0)^2 \dots,$$

is called the Taylor series of f about z_0 . The special case in which $z_0 = 0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n,$$

is called the Maclaurin series for f .

Example.

$$f(z) = e^z, f^{(n)} \Big|_{z=0} = 1. \text{ Therefore}$$

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad R = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \infty.$$

Example.

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z| < 1 \quad (R = 1).$$

Example.

$\text{Log}(1 - z)$. Note that

$$(\text{Log}(1 - z))' = -\frac{1}{1 - z} = -\sum_{n=0}^{\infty} z^n.$$

Integrating both sides we arrive at

$$\text{Log}(1 - z) = -\sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} + C = -\sum_{n=1}^{\infty} \frac{1}{n} z^n + C,$$

where $C = \text{Log}(1 - 0) = 0$.

Example.

$f(z) = \frac{1}{1+z}$ about $z_0 = i$.

$$\begin{aligned}\frac{1}{1+z} &= \frac{1}{1+i+z-i} = \frac{1}{1+i} \cdot \frac{1}{1 - \left(-\frac{z-i}{1+i}\right)} \\ &= \frac{1}{1+i} \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(1+i)^{n+1}} (z-i)^n.\end{aligned}$$

where R is defined by the inequality

$$\frac{|z-i|}{|1+i|} < 1 \quad \text{or} \quad |z-i| < \sqrt{2}.$$

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω , then f is holomorphic in Ω .

Proof. Let D be any disc whose closure is contained in Ω and T any triangle in that disc. Then, since each f_n is holomorphic, Cauchy-Goursat's theorem implies

$$\oint_T f_n(z) dz = 0, \quad \text{for all } n.$$

By assumption $f_n \rightarrow f$ uniformly in the closure of D , so f is continuous and

$$\oint_T f_n(z) dz = \oint_T f(z) dz.$$

Therefore

$$\oint_T f(z) dz = 0.$$

Using Morera's theorem we find that f is holomorphic in D . Since this conclusion is true for every D whose closure is contained in Ω , we find that f is holomorphic in all of Ω .

Remark. This is not true in the case of real variables: the uniform limit of continuously differentiable functions need not be differentiable. WHY??

Remark. Consider

$$F(z) = \sum_{n=1}^{\infty} f_n(z)$$

where f_n are holomorphic in $\Omega \subset \mathbb{C}$. Assume that the series converges uniformly in compact subsets of Ω , then the theorem guarantees that F is also holomorphic in Ω .

Theorem. Let $\{f_n\}_{n=1}^\infty$ be a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω . Then the sequence of derivatives $\{f'_n\}_{n=1}^\infty$ converges uniformly to f' on every compact subset of Ω .

Proof. For any $\tilde{\Omega} \subset \Omega$ such that $\overline{\tilde{\Omega}} \subset \Omega$ and given $\delta > 0$ we define $\tilde{\Omega}_\delta \subset \tilde{\Omega}$ by

$$\tilde{\Omega}_\delta = \{z \in \tilde{\Omega} : \overline{D_\delta}(z) \subset \tilde{\Omega}\}.$$

By the previous theorem it is enough to show that $\{f'_n\}_{n=1}^\infty$ converges uniformly to f' on $\tilde{\Omega}_\delta$. For any holomorphic function F in Ω_δ we have

$$\begin{aligned} |F'(z)| &= \left| \frac{1}{2\pi i} \oint_{|\eta-z|=\delta} \frac{F(\eta)}{(\eta-z)^2} d\eta \right| \\ &\leq \frac{1}{2\pi} \max_{\eta \in \overline{\tilde{\Omega}}} |F(\eta)| \frac{1}{\delta^2} 2\pi\delta \leq \frac{1}{\delta} \max_{\eta \in \overline{\tilde{\Omega}}} |F(\eta)|. \end{aligned}$$

Applying this inequality to $F(z) = f_n - f$ we conclude the proof.

Corollary.

Let each f_n be holomorphic in a given open set $\Omega \subset \mathbb{C}$ and the series

$$F(z) := \sum_{n=1}^{\infty} f_n(z)$$

converges uniformly in compact subsets of Ω . Then F is holomorphic in Ω .

Thank you

MATH50001 Analysis II, Complex Analysis

Lecture 10

Last time:

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω , then f is holomorphic in Ω .

Corollary.

Let each f_n be holomorphic in a given open set $\Omega \subset \mathbb{C}$ and the series

$$F(z) := \sum_{n=1}^{\infty} f_n(z)$$

converges uniformly in compact subsets of Ω . Then F is holomorphic in Ω .

Section: Holomorphic functions defined in terms of integrals.

Theorem. Let $F(z, s)$ be defined for $(z, s) \in \Omega \times [0, 1]$ where $\Omega \subset \mathbb{C}$ is an open set. Suppose F satisfies the following properties:

- $F(z, s)$ is holomorphic in Ω for each s .
- F is continuous on $\Omega \times [0, 1]$.

Then the function f defined on Ω by

$$f(z) = \int_0^1 F(z, s) \, ds$$

is holomorphic.

Proof. To prove this result, it suffices to prove that f is holomorphic in any disc D contained in Ω . By Morera's theorem this could be achieved by showing that for any triangle T contained in D we have

$$\oint_T \int_0^1 F(z, s) \, ds \, dz = 0.$$

The proof would be trivial if we could change the order of integration that is not clear. In order to go around this problem we consider for each $n \geq 1$ the Riemann sum

$$f_n(z) = \frac{1}{n} \sum_{k=1}^n F(z, k/n).$$

Then by the first assumption f_n is holomorphic in Ω .

We can now show that on any disc D such that $\overline{D} \subset \Omega$, the sequence $\{f_n\}_{n=1}^\infty$ converges uniformly to f .

Indeed, since F is continuous on $\Omega \times [0, 1]$ for a given $\varepsilon > 0$ there exists $\delta > 0$ such that as soon $|s_1 - s_2| < \delta$ we have

$$\sup_{z \in D} |F(z, s_1) - F(z, s_2)| < \varepsilon.$$

Then if $n > 1/\delta$ and $z \in D$ we find

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{k=1}^n \int_{(k-1)/n}^{k/n} (F(z, k/n) - F(z, s)) \, ds \right| \\ &\leq \sum_{k=1}^n \int_{(k-1)/n}^{k/n} |F(z, k/n) - F(z, s)| \, ds < \sum_{k=1}^n \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

By the previous theorem we conclude that f is holomorphic in D and thus in Ω .

Section: Schwarz reflection principle.

In this section we deal with a simple extension problem for holomorphic functions that is very useful in applications. It is the Schwarz reflection principle that allows one to extend a holomorphic function to a larger domain.

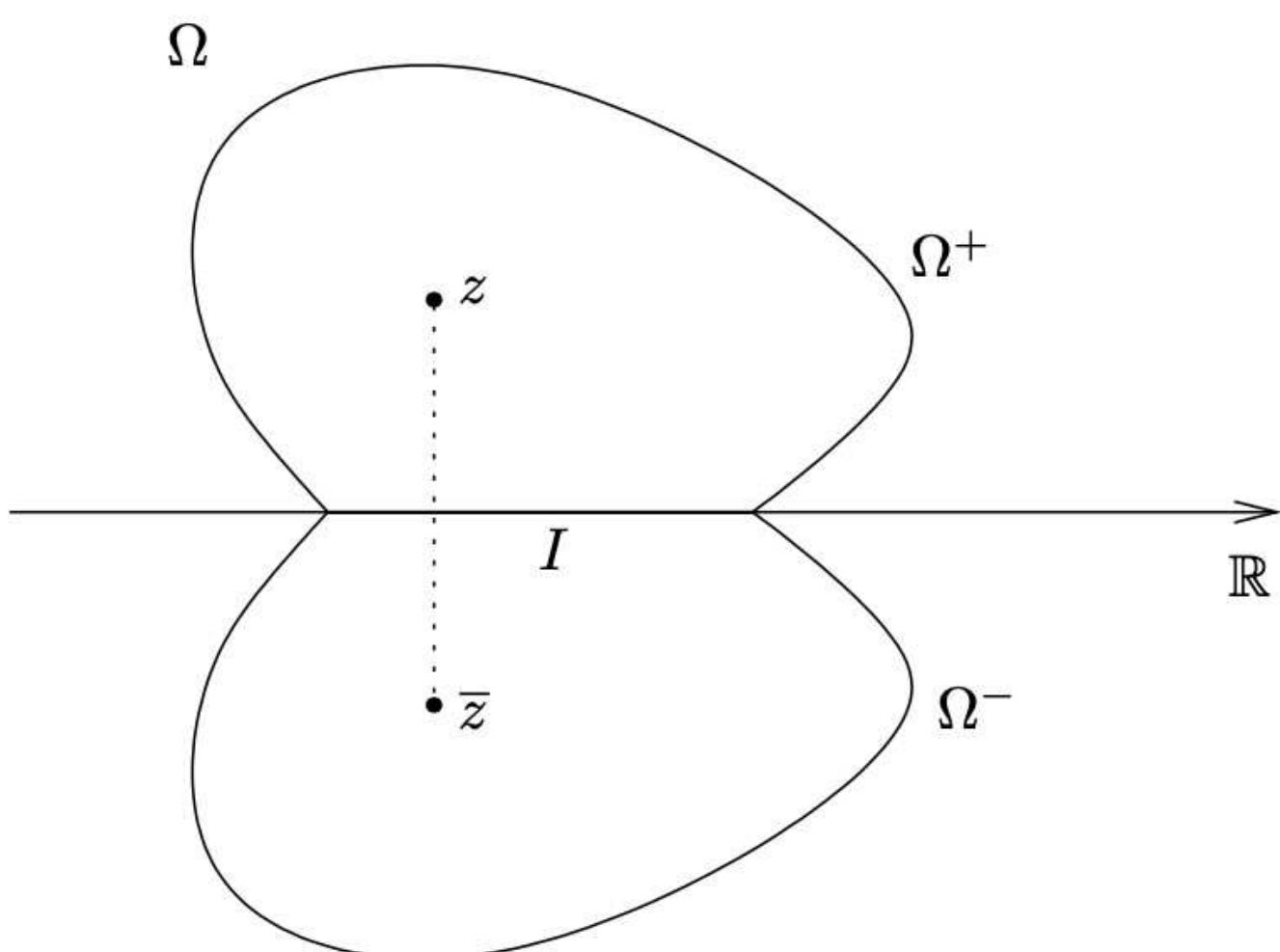
Let $\Omega \subset \mathbb{C}$ be open and symmetric with respect to the real line, that is

$$z \in \Omega \quad \text{iff} \quad \bar{z} \in \Omega.$$

Let

$$\Omega^+ = \{z \in \Omega : \operatorname{Im} z > 0\}, \quad \Omega^- = \{z \in \Omega : \operatorname{Im} z < 0\}$$

$$\text{and} \quad I = \{z \in \Omega : \operatorname{Im} z = 0\}.$$



Theorem. (Symmetry principle)

If f^+ and f^- are holomorphic functions in Ω^+ and Ω^- respectively, that extend continuously to I such that

$$f^+(x) = f^-(x) \quad \text{for all } x \in I,$$

then the function f defined in Ω by

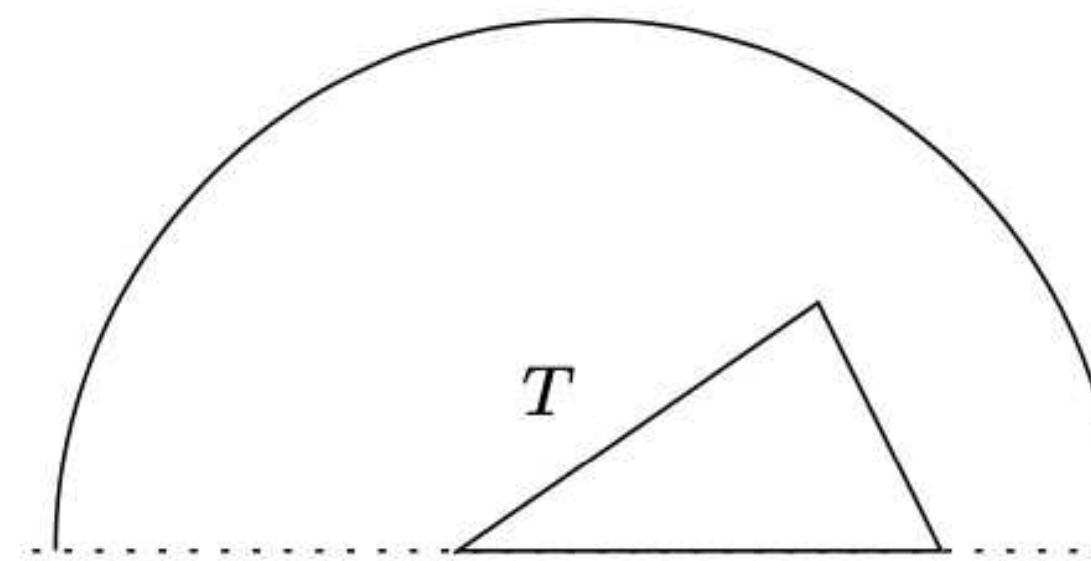
$$f(z) = \begin{cases} f^+(z), & z \in \Omega^+, \\ f^+(z) = f^-(z), & z \in I, \\ f^-(z), & z \in \Omega^-, \end{cases}$$

is holomorphic in Ω .

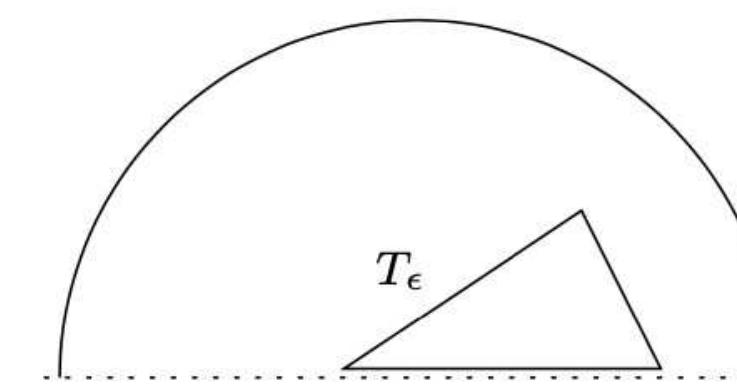
Proof. We only need to prove that f is holomorphic at points of I . Suppose D is a disc centred at a point on I and entirely contained in Ω . We prove that f is holomorphic in D by Morera's theorem. Suppose T is a triangle in D . If T does not intersect I , then

$$\oint_T f(z) dz = 0.$$

Suppose now that one side or vertex of T is contained in I , and the rest of T is in, for ex., the upper half-disc.



If T_ϵ is the triangle obtained from T by slightly raising the edge or vertex which lies on I



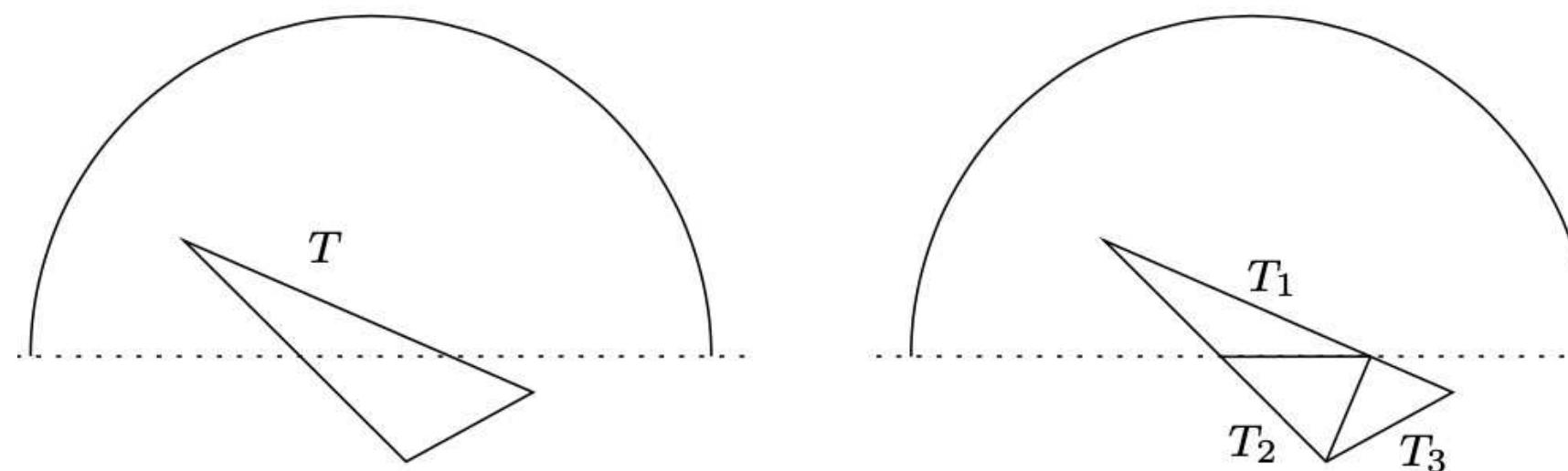
then we have

$$\oint_{T_\epsilon} f(z) dz = 0.$$

since T_ϵ is entirely contained in the upper half-disc. Letting $\epsilon \rightarrow 0$, by continuity we conclude that

$$\oint_T f(z) dz = 0.$$

If the interior of T intersects I , we can reduce the situation to the previous one by splitting T as the union of triangles each of which has an edge or vertex on I



By Morera's theorem we conclude that f is holomorphic in D .

Theorem. (Schwarz reflection principle)

Suppose that f is a holomorphic function in Ω^+ that extends continuously to I and such that f is real-valued on I . Then there exists a function F holomorphic in Ω such that $F|_{\Omega^+} = f$.

Thank you