MATH50001 Analysis II, Complex Analysis

Lecture 14

Section: Residue Theory.

Definition. Let

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n, \qquad 0 < |z - z_0| < R.$$

be the Laurent series for f at z_0. The residue of f at z_0 is

$$\operatorname{Res}[f, z_0] = a_{-1}.$$

Theorem. Let $\gamma \subset \{z : 0 < |z - z_0| < R\}$ be a simple, closed, piecewise-smooth curve that contains z_0. Then

$$\operatorname{Res}[f, z_0] = \frac{1}{2\pi i} \oint_\gamma f(z)\, dz.$$

Let

$$f(z) = a_{-m}(z - z_0)^{-m} + a_{-m+1}(z - z_0)^{-m+1} + \dots$$

and let $g(z) = (z - z_0)^m f(z)$.

$m = 1$. Then $g(z) = a_{-1} + a_0\,(z - z_0) + \dots$ and therefore

$$\operatorname{Res}[f, z_0] = a_{-1} = \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (z - z_0)\, f(z).$$

$m = 2$. Then $g(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + \dots$ and

$$\operatorname{Res}[f, z_0] = a_{-1} = \frac{d}{dz} g(z)\Big|_{z = z_0} = \lim_{z \to z_0} \frac{d}{dz}\left((z - z_0)^2 f(z)\right).$$

m.

$$\operatorname{Res}[f, z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z)\right).$$

Example.

Evaluate

$$\oint_\gamma \frac{1}{z^5 - z^3}\, dz, \qquad \gamma = \{z : |z| = 1/2\}$$

Clearly

$$\frac{1}{z^5 - z^3} = \frac{1}{z^3(z-1)(z+1)}.$$

Since $z = \pm 1$ is outside γ we obtain

$$\oint_\gamma \frac{1}{z^5 - z^3}\, dz = 2\pi i \,\mathrm{Res}\,[f, 0] = 2\pi i \,\frac{1}{2!} \lim_{z\to 0} (z^3 f(z))''$$
$$= \pi i \lim_{z\to 0} \left(\frac{1}{z^2 - 1}\right)'' = \pi i \lim_{z\to 0} \left(\frac{-2z}{(z^2-1)^2}\right)'$$
$$= \pi i \lim_{z\to 0} \left(\frac{-2(z^2-1)^2 - (-2z)\, 2(z^2-1)\, 2z}{(z^2-1)^4}\right) = -2\pi i.$$

Example.

Evaluate

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\,dz, \qquad \gamma = \{z : |z| = 2\}.$$

Because the integrand has singularities at $z = -5$ and $z = \pm 1$ only the last two are interior to γ, we have

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\,dz$$
$$= 2\pi i \left\{ \operatorname{Res}\left[\frac{1}{(z+5)(z^2-1)}, -1\right] + \operatorname{Res}\left[\frac{1}{(z+5)(z^2-1)}, 1\right] \right\}.$$

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\,dz, \qquad \gamma = \{z : |z| = 2\}.$$

Now $z = 1$ is a pole of order 1 and therefore

$$\begin{aligned}
\operatorname{Res}&\left[\frac{1}{(z+5)(z^2-1)}, 1\right] \\
&= \lim_{z\to 1} \frac{z-1}{(z+5)(z^2-1)} = \lim_{z\to 1} \frac{1}{(z+5)(z+1)} = \frac{1}{12}.
\end{aligned}$$

Similarly, $z = -1$ is a simple pole and

$$\begin{aligned}
\operatorname{Res}&\left[\frac{1}{(z+5)(z^2-1)}, -1\right] \\
&= \lim_{z\to -1} \frac{z+1}{(z+5)(z^2-1)} = \lim_{z\to -1} \frac{1}{(z+5)(z-1)} = -\frac{1}{8}.
\end{aligned}$$

Thus,

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\,dz = 2\pi i\left(\frac{1}{12} - \frac{1}{8}\right) = -\frac{\pi i}{12}.$$

Section: The argument principle.

Theorem. (Principle of the Argument)

Let f be holomorphic in an open set Ω except for a finite number of poles and let γ be a simple, closed, piecewise-smooth curve in Ω that does not pass through any poles or zeros of f. Then

$$\oint_{\gamma} \frac{f'(z)}{f(z)}\, dz = 2\pi i(N - P),$$

where N and P are the sums of the orders of the zeros and poles of f inside γ.

Remark. Why Principle of the Argument?

Indeed, let γ be a closed curve. Then

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi i}\oint_\gamma \frac{d}{dz}\log f(z)\,dz = \frac{1}{2\pi i}\log f(z)\Big|_{z_1}^{z_2}\\ &= \frac{1}{2\pi i}\Big(\ln|f(z_2)| - \ln|f(z_1)| + i(\arg f(z_2) - \arg f(z_1))\Big) = \frac{1}{2\pi}\Delta\arg f(z).\end{aligned}$$

Example. Let $f(z) = z^3$ and let $\gamma = \{z : z = e^{i\theta}, \theta \in [0, 2\pi]\}$, then $f(z) = e^{i3\theta}$ and $\frac{1}{2\pi} \Delta_\gamma \arg f = 3$.

Example. Let $f(z) = 1/z$ and let $\gamma = \{z : z = e^{i\theta}, \theta \in [0, 2\pi]\}$. Then $\frac{1}{2\pi} \Delta_\gamma \arg f = -1$.

Example. Let $f(z) = z + 2$ and let $\gamma = \{z : z = e^{i\theta}, \theta \in [0, 2\pi]\}$. Then $\frac{1}{2\pi} \Delta_\gamma \arg f = 0$.

Thank you