MATH50001 Analysis II, Complex Analysis

Lecture 6

Section: Properties of holomorphic functions.

Theorem. Let $\Omega \subset \mathbb{C}$ be an open set and $T \subset \Omega$ be a triangle whose interior is also contained in Ω, then

$$\oint_T f(z)\, dz = 0,$$

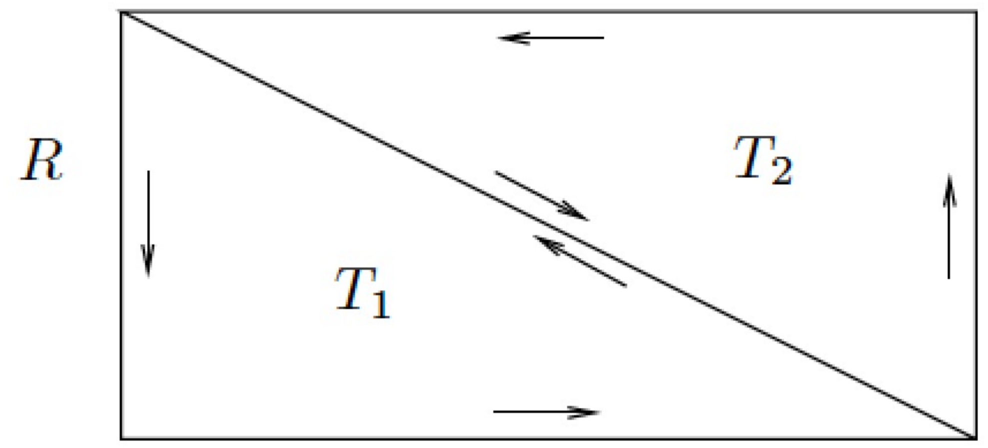
whenever f is holomorphic in Ω.

Corollary. If f is holomorphic in an open set Ω that contains a rectangle R and its interior, then

$$\oint_R f(z)\,dz = 0.$$

Proof. This immediately follows from the equality

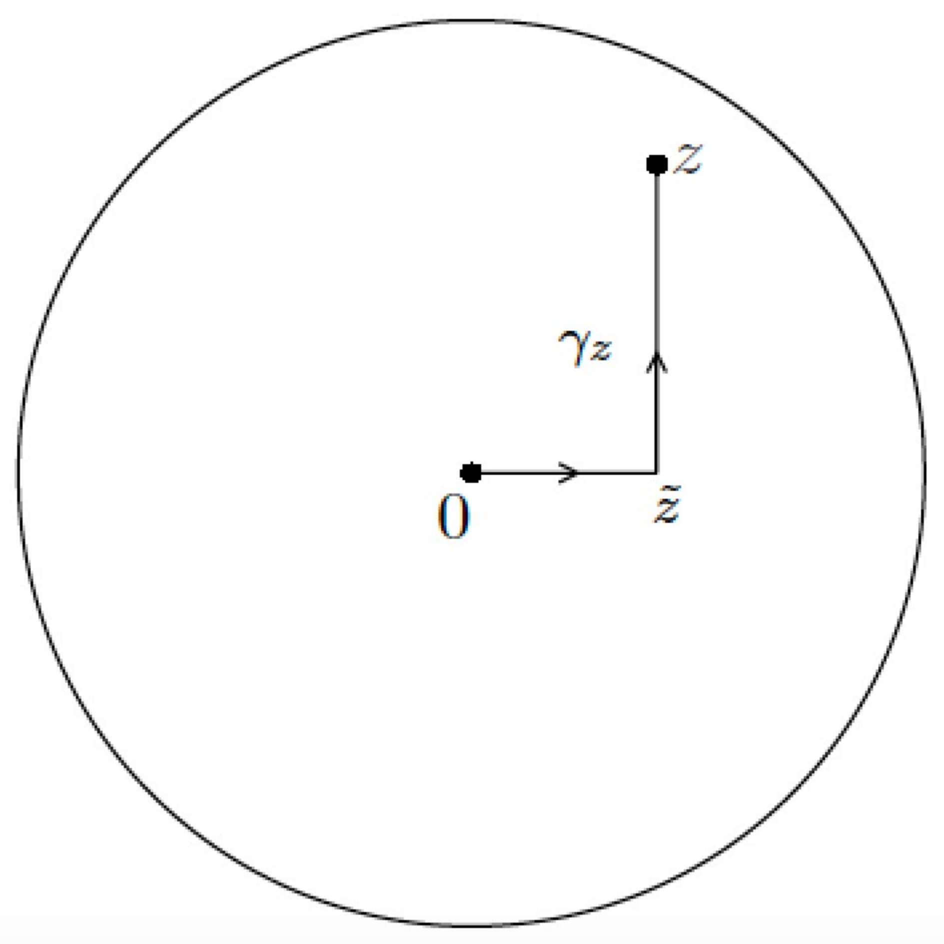
$$\oint_R f(z)\,dz = \oint_{T_1} f(z)\,dz + \oint_{T_2} f(z)\,dz.$$



Section: Local existence of primitives and Cauchy-Goursat theorem in a disc.

Theorem. A holomorphic function in an open disc has a primitive in that disc.

Proof. We may assume that the disc D is centered at the origin. For any $z \in D$ we consider γ_z given by



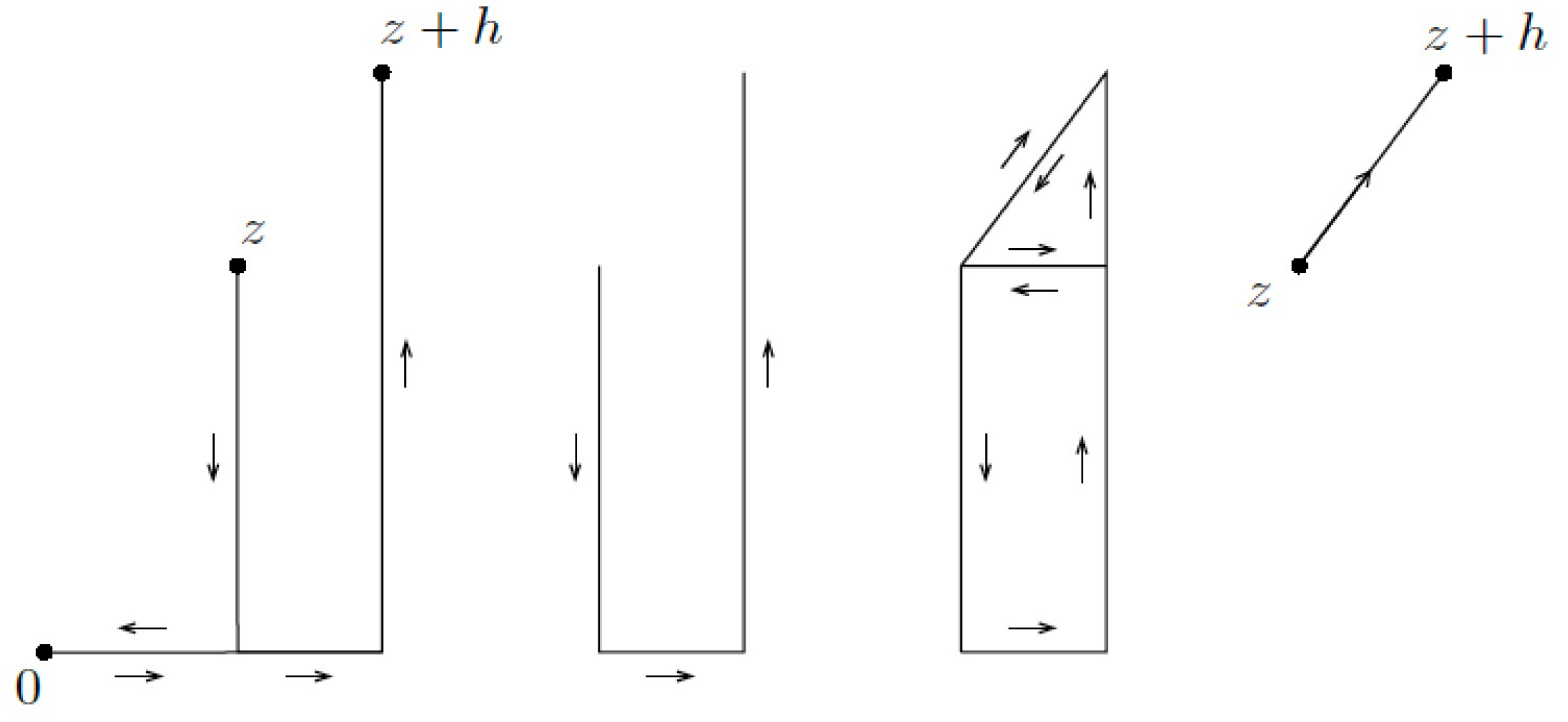
Define

$$F(z) = \int_{\gamma_z} f(w)\,dw.$$

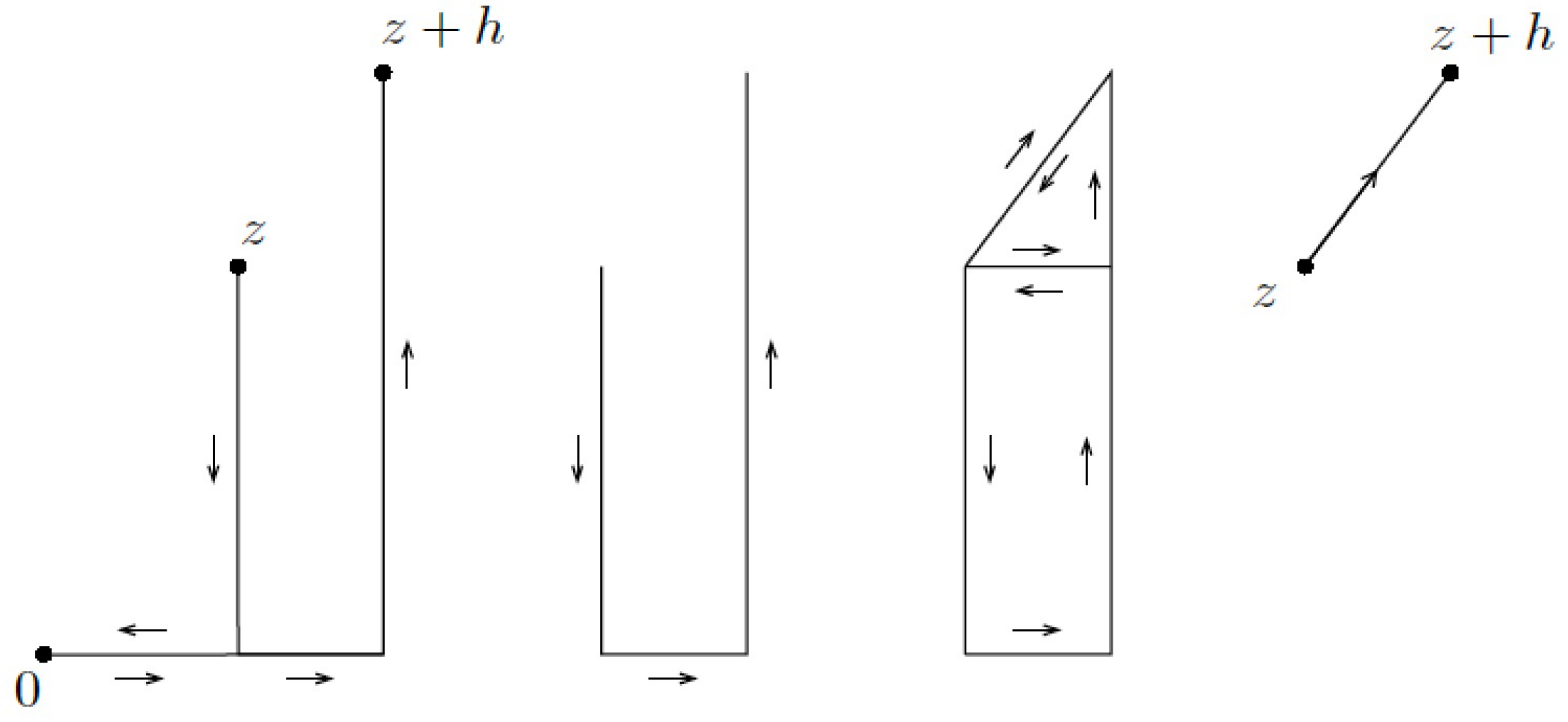
Consider the difference

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw$$

The function f is first integrated along γ_{z+h} with the original orientation, and then along γ_z with the reverse orientation.



(a) (b) (c) (d)



(a) (b) (c) (d)

Using the fact that the integration over the triangle and the rectangle equal zero we obtain

$$F(z+h) - F(z) = \int_\eta f(w)\,dw,$$

where η is the straight line segment from z to $z+h$. Since f is continuous at z we can write

$$f(w) = f(z) + \psi(w),$$

where $\psi(w) \to 0$ as $w \to z$. Then

$$F(z+h) - F(z) = \int_\eta f(z)\,dw + \int_\eta \psi(w)\,dw = f(z)\,h + \int_\eta \psi(w)\,dw.$$

Finally we note that using the LM-inequality

$$\left|\int_\eta \psi(w)\,dw\right| \leq |h| \sup_{w\in\eta} |\psi(w)|$$

Since $\psi(w) \to 0$ as $w \to z$ we obtain

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z).$$

Corollary. (Cauchy-Goursat theorem for a disc)
If f is holomorphic in a disc, then

$$\oint_{\gamma} f(z)\,dz = 0$$

for any closed curve γ in that disc.

Corollary. Suppose f is holomorphic in an open set containing the circle C and its interior. Then

$$\oint_C f(z)\,dz = 0.$$

Proof. Let D be the disc with boundary circle C. Then there exists a slightly larger disc $\tilde{D} \supset D$ and so that f is holomorphic on $\tilde{D}$. We may now apply Cauchy-Goursat theorem in $\tilde{D}$ to conclude that $\oint_C f(z)\,dz = 0$.

Section: Homotopies and simply connected domains.

Let γ_0 and γ_1 be two curves in an open set Ω with common end-points. That is if γ_0 and γ_1 are two parametrizations defined on $[a, b]$, we have

$$\gamma_0(a) = \gamma_1(a) = \alpha \quad \text{and} \quad \gamma_0(b) = \gamma_1(b) = \beta.$$

Definition. The curves γ_0 and γ_1 are said to be *homotopic* in Ω if for each $0 \leq s \leq 1$ there exists a curve $\gamma_s \subset \Omega$, parametrized by $\gamma_s(t)$ defined on $[a, b]$, such that for every s

$$\gamma_s(a) = \alpha \quad \text{and} \quad \gamma_s(b) = \beta,$$

and for all $t \in [a, b]$

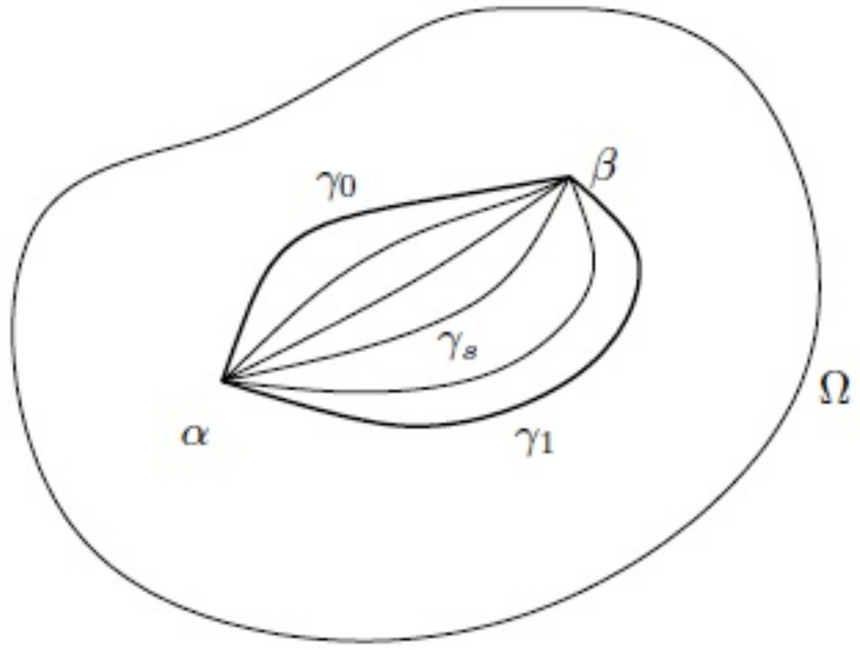
$$\gamma_s(t)|_{s=0} = \gamma_0(t) \quad \text{and} \quad \gamma_s(t)|_{s=1} = \gamma_1(t).$$

Moreover, $\gamma_s(t)$ should be jointly continuous in $s \in [0, 1]$ and $t \in [a, b]$.

Theorem. If f is holomorphic in Ω, then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Proof. We first show that if two curves are close to each other and have the same end-points, then the integrals over them are equal.
Due to definition, the function $F(s,t) = \gamma_s(t)$ is continuous on $[0,1] \times [a,b]$.
Then the image of F denoted by K is compact.



Then there is $\varepsilon > 0$ such that every disc of radius $3\varepsilon > 0$ centred at a point in the image of F is completely contained in Ω.

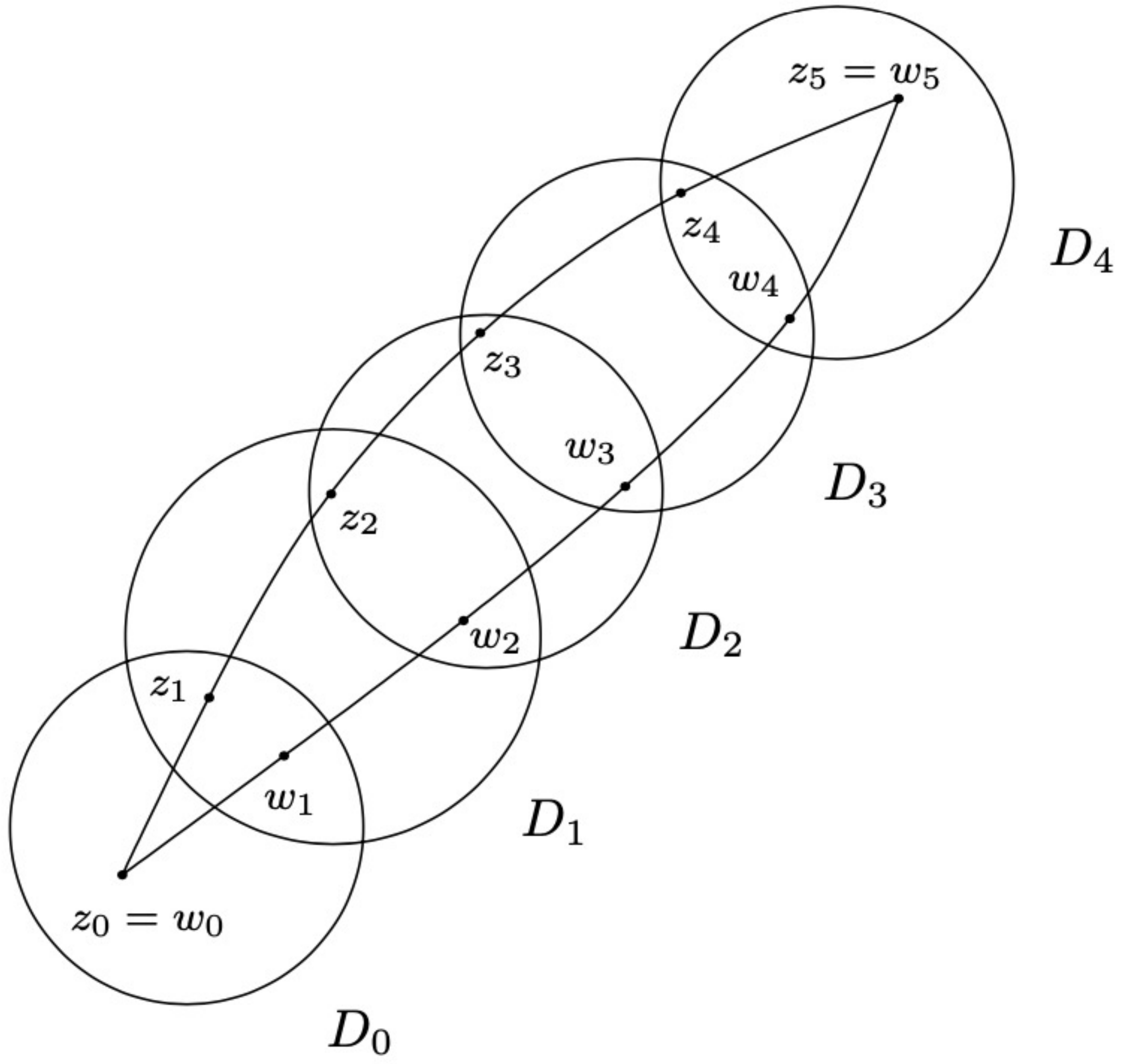
WHY ??? Show it.

Since F is uniformly continuous we choose δ such that

$$\sup_{t\in[a,b]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \varepsilon \quad \text{whenever} \quad |s_1 - s_2| < \delta.$$

We now choose discs $\{D_0, \ldots, D_n\}$ of radius 2ε, and points $\{z_0, \ldots, z_{n+1}\}$ on γ_{s_1} and $\{w_0, \ldots, w_{n+1}\}$ on γ_{s_2} such that the union of these discs covers both curves, and

$$z_i, z_{i+1}, w_i, w_{i+1} \in D_i.$$



Here $z_0 = w_0 = \gamma_{s_1}(a) = \gamma_{s_2}(a)$ and
$z_{n+1} = w_{n+1} = \gamma_{s_1}(b) = \gamma_{s_2}(b)$.
On each D_i, let F_i be a primitive of f.
In $D_i \cap D_{i+1}$ the primitives F_i and F_{i+1}
are two primitives of the same function, so they must
differ by a constant.

Therefore

$$F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1}),$$

or

$$F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1}) = F_i(z_{i+1}) - F_i(w_{i+1}).$$

Finally we have

$$
\begin{aligned}
\int_{\gamma_{s_1}} f(z)\,dz - \int_{\gamma_{s_2}} f(z)\,dz
&= \sum_{i=0}^{n+1} (F_i(z_{i+1}) - F_i(z_i)) - \sum_{i=0}^{n+1} (F_i(w_{i+1}) - F_i(w_i)) \\
&\quad \sum_{i=0}^{n+1} (F_i(z_{i+1}) - F_i(w_{i+1}) - (F_i(z_i) - F_i(w_i))) \\
&= F_n(z_{n+1}) - F_n(w_{n+1}) - (F_0(z_0) - F_0(w_0)) = 0.
\end{aligned}
$$

By subdividing the interval $[0,1]$ into subintervals $[s_k, s_{k+1}]$, $k = 0, \ldots m$, of length less than δ and using the above arguments for each pair γ_{s_k} and $\gamma_{s_{k+1}}$ with $\gamma_{s_0} = \gamma_0$ and $\gamma_{s_{m+1}} = \gamma_1$ we complete the proof.

Definition. An open set $\Omega \subset \mathbb{C}$ is *simply connected* if any two pair of curves in Ω with the same end-points are homotopic.

Example. A disc D is simply connected. Indeed, let $\gamma_0(t)$ and $\gamma_1(t)$ be two curves lying in D. We can define $\gamma_s(t)$ by $\gamma_s(t) = (1-s)\gamma_0(t) + s\gamma_1(t)$. Note that if $0 \leq s \leq 1$, then for each t, the point $\gamma_s(t)$ is on the segment joining $\gamma_0(t)$ and $\gamma_1(t)$, and so is in D.

The same argument works if D is replaced any open convex set.
WHY ??? - show it

Example. The set $\mathbb{C} \setminus \{(-\infty, 0]\}$ is simply connected.
WHY ??? - show it

Example. The punctured plane $\mathbb{C} \setminus \{0\}$ is not simply connected.

Theorem. Any holomorphic function in a simply connected domain has a primitive.

Proof. Fix a point z_0 in Ω and define

$$F(z) = \int_\gamma f(w)\, dw,$$

where the integral is taken over any curve in Ω joining z_0 to z. This definition is independent of the curve chosen, since Ω is simply connected. Consider

$$F(z+h) - F(z) = \int_\eta f(w)\, dw,$$

where η is the line segment joining z and $z+h$. Arguing as in the proof of the Theorem where we constructed a primitive to a holomorphic function in a disc, we obtain

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

The proof is complete.

Thank you