Probability and Statistics for JMC
Exercises 3 — Discrete Random Variables

1. An experiment involves tossing two fair coins
   (a) What is the sample space for this experiment?
   (b) What is the probability mass function (pmf) of the random variable X, which takes value 2 if two heads show, 1 if one head shows, and 0 if no heads show?
   (c) What is the probability mass function of the random variable Y, which takes the value 3 if at least one head shows and 1 if no heads show?

2. Suppose that two fair dice are thrown and define a random variable X as the total number of spots showing. Make a table showing the probability mass function, $p(x)$ of X and plot a graph of $p(x)$.

3. In tossing a fair coin four times, what is the probability that one will obtain
   (a) four heads?
   (b) three heads?
   (c) at least two heads?
   (d) not more than one head?

4. An urn holds 5 white and 3 black marbles.
   (a) If two marbles are drawn at random without replacement and X denotes the number of white marbles
      i. find the probability mass function of X, and
      ii. plot the cumulative distribution function of X.
   (b) Repeat 4a if the marbles are drawn with replacement.

5. The probability that a student will pass a particular course is 0.4. Find the probability that, out of 5 students
   (a) none pass
   (b) one passes
   (c) at least one passes

6. (a) If each student in a class of 110 has the same probability, 0.8, of passing an examination, what is
      i. the expected number of passes?
      ii. the standard deviation of the number of passes?
   (b) If each student in a college of 11000 has the same probability, 0.8, of graduating, what is
      i. the expected number of graduates?
      ii. the standard deviation of the number of graduates?

7. Banach match problem. The pipe-smoking mathematician Banach has a matchbox in each of his two pockets. Whenever he needs a match he reaches into one of his pockets at random. Each matchbox starts out with N matches. At the moment Banach discovers one of the boxes is empty what is the probability that the other box contains exactly k matches (for $k = 0, 1, 2, \ldots, N$)? (Discovering a box to be empty means reaching for a box that contains zero matches — it does not mean taking the last match from a box.)

8. Compute the mean, sd, and the skewness for the following binomial distributions, and comment on the trends as n and p change:
   (a) Binomial(100, 0.9)
   (b) Binomial(100, 0.7)
   (c) Binomial(100, 0.5)
   (d) Binomial(1000, 0.9)
   (e) Binomial(1000, 0.7)
   (f) Binomial(1000, 0.5)

9. In a class of 20 students taking an examination,
   2 have probability 0.4 of passing;
   4 have probability 0.6 of passing;
   5 have probability 0.7 of passing;
   7 have probability 0.8 of passing;
   2 have probability 0.9 of passing.
   (a) What is the expected number of passes?
   (b) What is the standard deviation of the number of passes?

10. A computer class has a limited number of terminals available for use. You notice that, on average, there is a 0.4 chance that there will be a free terminal each time you try to use a machine.
   (a) What is the average number of times you will have to try use a machine until you are successful?
   (b) What is your chance of being successful the first time you try?

(c) What is your probability of being successful the first time on each of three different occasions?

11. (a) What is the mean and variance of a sum of n independent Bernoulli random variables, each with parameter p?
    (b) What if they have different parameters, $(p_1, p_2, \ldots, p_n)$?
    (c) What can you say if I now tell you that they are not independent?

12. Let $X \sim \text{Poisson}(\lambda)$, for some $\lambda > 0$.
    (a) Verify that the Poisson distribution has a valid probability mass function (non-negative and normalized).
    (b) Find $E(X^2)$.
    (c) Suppose Z is another discrete random variable with $P(Z = 16) = 0.2$, and $P(Z = z) \propto P(X = z)$ whenever $z = \mathbb{R}\backslash\{16\}$.
    Find the probability mass function of Z.

13. If X is a geometric random variable, show that

$$P(X = n + k \mid X > n) = P(X = k)$$

and give a verbal explanation of why this makes sense, given the situation that a geometric random variable describes.

14. There are n fish in a pond. You catch a fish at random, tag it and then throw it back. You keep doing this until you catch a fish that has already been tagged. What is the pmf for X, the number of fish you catch?

15. We set up a new server that is to run for 1 year. From our experience with similar machines we estimate that crashes occur about once every 130,000 minutes (i.e. we monitored the other machines over a time T, observed k crashes and found that $k/T = 130{,}000$ mins.

    Let X be the number of times our new server will crash.

    (a) What kind of distribution should we use to model the random variable X? Explain your reasoning.
    (b) What is the probability that our server will not crash even once over the course of the year?

Partial answers:

1. (a) $S = \{HH, HT, TH, TT\}$; (b) $p_X(0) = 1/4, p_X(1) = 1/2, p_X(2) = 1/4$; (c) $p_Y(1) = 1/4, p_Y(3) = 3/4$
3. (a) 1/16; (b) 1/4; (c) 11/16; (d) 5/16
5. (a) 0.078; (b) 0.26; (c) 0.922

6. (a.i) 88; (a.ii) 4.195; (b.i) 8800; (b.ii) 41.95
7. $\binom{2N-k}{N-k}\left(\frac{1}{2}\right)^{2N-k}$
8. (a) mean=90, sd=3.0, skew=-0.267; (b) mean=70, sd=4.6, skew=-0.087
9. (a) 14.1; (b) 1.95
10. (a) 2.5; (b) 0.4; (c) 0.064
11. (a) np, $np(1-p)$; (b) $\sum_i p_i$, $\sum_i p_i(1-p_i)$
12. (b) $\lambda(1+\lambda)$
14. $\frac{k-1}{n^k}\frac{n!}{(n-k+1)!}$
15. (a) Poisson; (b) 0.019