Probability and Statistics for JMC
Exercises 5 — Joint Random Variables

1. Suppose the joint pdf of a pair of continuous RVs is given by

$$f(x,y) = \begin{cases} k(x+y), & 0 < x < 2,\ 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the constant k.
   (b) Find the marginal pdfs of X and Y.
   (c) Are X and Y independent?

2. A manufacturer has been using two different manufacturing processes to make computer memory chips. Let X and Y be two continuous random variables, where X denotes the time to failure for chips made by process A and Y denotes the time to failure for chips made by process B. Assuming that the joint pdf of (X,Y) is

$$f(x,y) = \begin{cases} (ab)e^{-(ax+by)} & x,y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

   where $a = 10^{-4}$ and $b = 1.2 \times 10^{-4}$, determine $\mathrm{P}(X > Y)$.

3. The joint probability mass function of two discrete random variables X and Y is given by $p(x,y) = cxy$ for $x = 1,2,3$ and $y = 1,2,3$, and zero otherwise. Find

   (a) The constant c;
   (b) $\mathrm{P}(X = 2, Y = 3)$;
   (c) $\mathrm{P}(X \leq 2, Y \leq 2)$;
   (d) $\mathrm{P}(X \geq 2)$;
   (e) $\mathrm{P}(Y < 2)$;
   (f) $\mathrm{P}(X = 1)$;
   (g) $\mathrm{P}(Y = 3)$.

4. Let X and Y be continuous random variables having joint density function $f(x,y) = c\,(x^2 + y^2)$ when $0 \leq x \leq 1$ and $0 \leq y \leq 1$, and $f(x,y) = 0$ otherwise. Determine

   (a) the constant c;
   (b) $\mathrm{P}(X < 1/2, Y > 1/2)$;

(c) $\mathrm{P}(1/4 < X < 3/4)$;

(d) $\mathrm{P}(Y < 1/2)$;

(e) whether X and Y are independent.

5. If $X_1 \sim \text{Gamma}(\alpha_1, \beta)$ and $X_2 \sim \text{Gamma}(\alpha_2, \beta)$, with X_1 and X_2 independent, prove that $Y = X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$.

6. Let X and Y be independent exponential random variables with parameter 1. Find the joint density function of $U = X+Y$ and $V = X/(X+Y)$, and deduce that V is uniformly distributed on $[0, 1]$.

7. X and Y have the joint density function $f(x, y) = 1/(x^2y^2)$ when $x \geq 1$ and $y \geq 1$, and $f(x, y) = 0$ elsewhere.

   (a) Compute the joint density function of U, V, where $U = XY$ and $V = X/Y$.

   (b) What are the marginal densities of U and V?

8. Prove the Law of Total Expectation: if X and Y are two random variables then

$$\mathrm{E}(X) = \mathrm{E}\left(\mathrm{E}(X \mid Y)\right),$$

   where $\mathrm{E}(X \mid Y)$ is the conditional expectation of X given Y (and should be thought of as a function of the random variable Y), and the outer expectation is with respect to the marginal distribution of Y.

9. Prove the Law of Total Variance: if X and Y are two random variables then

$$\mathrm{Var}(X) = \mathrm{E}\left(\mathrm{Var}(X \mid Y)\right) + \mathrm{Var}\left(\mathrm{E}(X \mid Y)\right),$$

   where $\mathrm{Var}(X \mid Y)$ is the conditional variance of X given Y (i.e. it is the variance of X conditioned on $Y = y$, and is considered to be a function of the random variable Y).

10. Consider the 2-class mixture model:

$$Z \sim \text{Bernoulli}(p),$$
$$X|Z \sim \begin{cases} \mathrm{N}(\mu_0, \sigma_0^2) & \text{if } Z = 0, \\ \mathrm{N}(\mu_1, \sigma_1^2) & \text{if } Z = 1, \end{cases}$$

   where the second line with $X|Z$ is specifying the conditional distribution of X given Z, i.e. $f_{X|Z}(x \mid z)$. A concrete example might be that there are two populations whose X values are distributed according to the two normal distributions. We flip a biased coin to determine Z and then, depending on whether we got heads or tails, we measure X from one or the other of the populations.

(a) Sketch the marginal pdf of X assuming the parameters $p = 0.1, \mu_0 = 0, \sigma_0^2 = 1, \mu_1 = 2, \sigma_1^2 = 2$.

(b) What are the mean and variance of X? [Hint: you can either find the marginal distribution of X first or use the laws of total expectation and total variance.]

Partial answers:

1. (a) $\frac{1}{8}$; (b) $\frac{1}{4}(x+1)$, $\frac{1}{4}(y+1)$; (c) no.
2. 0.54545
3. (a) $\frac{1}{36}$; (b) $\frac{1}{6}$; (c) $\frac{1}{4}$; (d) $\frac{5}{6}$; (e) $\frac{1}{6}$; (f) $\frac{1}{6}$; (g) $\frac{1}{2}$.
4. (a) $\frac{3}{2}$; (b) $\frac{1}{4}$; (c) $\frac{29}{64}$; (d) $\frac{5}{16}$; not indep.
6. $f(u, v) = ue^{-u}$ for $u \in (0, \infty), v \in (0, 1)$.
7. (a) $1/(2u^2v)$ for $u \geq 1$ and $1/u \leq v \leq u$; (b) $f_U(u) = \log(u)/u^2$ for $u \geq 1$, $f_V(v) = \begin{cases} 1/2 & 0 < v \leq 1 \\ 1/(2v^2) & v \geq 1 \end{cases}$.

10 (b) $\mathrm{E}(X) = p\mu_1 + (1-p)\mu_0$ and $\mathrm{Var}(X) = p\sigma_1^2 + (1-p)\sigma_0^2 + p(1-p)(\mu_1 - \mu_0)^2$.