MATH50004 Differential Equations
Spring Term 2021/22

Repetition Material 1: Higher-order differential equations

In this course, we focus on first-order differential equation, i.e. the right hand side depends only on the value of the solution, but not on its derivatives. In many applications, differential equations with higher-order derivatives appear naturally, and an example is given by the harmonic oscillator $\dot{x} = -x$, which is solved by the sin and cos function, and linear combinations of these two functions (this works, because this is a linear equation). Remember also that you have encountered higher-order differential equations already in Year 1, and you have learned how to solve (autonomous) linear higher-order differential equations.

In this short note, we show that higher-order differential equations can be equivalently rewritten as first-order differential equation. For this reason, all results in this course are formulated for first-order differential equations. Please note that this document is not required to understand any material of the course, and it is not examinable.

The formal definition of a higher-order differential equation is given as follows.

Definition 1 (Higher-order differential equation)**.** Consider $n, d \in \mathbb{N}$, an open set $D \subset \mathbb{R} \times \mathbb{R}^{nd}$, and a function $f : D \to \mathbb{R}^d$. An equation of the form

$$x^{(n)} = f\big(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}\big) \tag{1}$$

is called a d-dimensional *ordinary differential equation* of order n. An n-times differentiable function $\lambda : I \to \mathbb{R}^d$ on an interval $I \subset \mathbb{R}$ is called a *solution* to the differential equation (1) if $\big(t, \lambda(t), \dot{\lambda}(t), \ddot{\lambda}(t), \ldots, \lambda^{(n-1)}(t)\big) \in D$ and

$$\lambda^{(n)}(t) = f\big(t, \lambda(t), \dot{\lambda}(t), \ddot{\lambda}(t), \ldots, \lambda^{(n-1)}(t)\big) \quad \text{for all } t \in I\,, \tag{2}$$

where $\dot{\lambda}(t) := \frac{\mathrm{d}\lambda}{\mathrm{d}t}(t)$, $\ddot{\lambda}(t) := \frac{\mathrm{d}^2\lambda}{\mathrm{d}t^2}(t)$ and $\lambda^{(k)}(t) := \frac{\mathrm{d}^k\lambda}{\mathrm{d}t^k}(t)$ for $k \in \mathbb{N}$ and $t \in I$.

Note that this definition generalises Definition 1.2. We also consider initial value problems for higher-order differential equations, and here we need to fix not only x-value of a solution at a particular time t_0, but also (some of) its derivatives.

Definition 2 (Initial value problem)**.** Consider $n, d \in \mathbb{N}$, an open set $D \subset \mathbb{R} \times \mathbb{R}^{nd}$, and a function $f : D \to \mathbb{R}^d$. The combination of the ordinary differential equation

$$x^{(n)} = f\big(t, x, \dot{x}, \ldots, x^{(n-1)}\big)$$

with an initial condition of the form

$$x(t_0) = x_0, \quad \dot{x}(t_0) = x_1, \quad \ldots, \quad x^{(n-1)}(t_0) = x_{n-1}\,, \tag{3}$$

where $(t_0, x_0, x_1, \ldots, x_{n-1}) \in D$, is called an *initial value problem*, and (3) is called *initial condition*. A solution to the above initial value problem is a solution $\lambda : I \to \mathbb{R}^d$ to the differential equation such that t_0 is in the interior of I and

$$\lambda(t_0) = x_0, \quad \dot{\lambda}(t_0) = x_1, \quad \ldots, \quad \lambda^{(n-1)}(t_0) = x_{n-1}\,.$$

We now demonstrate how differential equations of higher order can be transformed into first-order differential equations.

Proposition 3 (Reduction to first-order systems)**.** *Given a set $D \subset \mathbb{R}^{1+nd}$ and a function $f : D \to \mathbb{R}^d$. Then the d-dimensional ordinary differential equation of order n*

$$x^{(n)} = f\big(t, x, \dot{x}, \dots, x^{(n-1)}\big) \tag{4}$$

is equivalent to the nd-dimensional first-order differential equation

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ &\vdots \\ \dot{y}_{n-1} &= y_n \\ \dot{y}_n &= f(t, y_1, y_2, \dots, y_n) \end{aligned} \tag{5}$$

with $y_1, y_2, \dots, y_n \in \mathbb{R}^d$ and $(t, y_1, y_2, \dots, y_n) \in D$. The equivalence is given by the following two statements:

(i) If λ is a solution to (4), then $t \mapsto \big(\lambda(t), \dot{\lambda}(t), \dots, \lambda^{(n-1)}(t)\big)$ is a solution to (5).

(ii) If $t \mapsto \big(\mu_1(t), \mu_2(t), \dots, \mu_n(t)\big)$ is a solution to (5), then μ_1 is a solution to (4).

In addition, if for a given $(t_0, x_0, x_1, \dots, x_{n-1}) \in D$, a solution to (4) satisfies the initial condition

$$x(t_0) = x_0, \quad \dot{x}(t_0) = x_1, \quad \dots, \quad x^{(n-1)}(t_0) = x_{n-1}\,,$$

then the corresponding solution to (5) satisfies the initial condition

$$y_1(t_0) = x_0, \quad y_2(t_0) = x_1, \quad \dots, \quad y_n(t_0) = x_{n-1}\,.$$

Proof. (i) If λ is a solution to (4), then we have

$$\lambda^{(n)}(t) = f\big(t, \lambda(t), \dots, \lambda^{(n-1)}(t)\big)\,.$$

Define $(\mu_1(t), \dots, \mu_n(t)) := (\lambda(t), \dots, \lambda^{(n-1)}(t))$. Then we get

$$\dot{\mu}_1(t) = \mu_2(t)\,, \quad \dot{\mu}_2(t) = \mu_3(t)\,, \quad \dots, \quad \dot{\mu}_{n-1}(t) = \mu_n(t)$$

and

$$\dot{\mu}_n(t) = f\big(t, \lambda(t), \dots, \lambda^{(n-1)}(t)\big) = f(t, \mu_1(t), \dots, \mu_n(t))\,.$$

This implies that μ solves (5).
(ii) Let $t \mapsto (\mu_1(t), \dots, \mu_n(t))$ be a solution of (5). This means that

$$\dot{\mu}_1(t) = \mu_2(t)\,, \quad \dot{\mu}_2(t) = \mu_3(t)\,, \quad \dots, \quad \dot{\mu}_n(t) = f(t, \mu_1(t), \dots, \mu_n(t))\,.$$

It follows that

$$\dot{\mu}_1(t) = \mu_2(t)\,, \quad \ddot{\mu}_1(t) = \dot{\mu}_2(t) = \mu_3(t)\,, \quad \dots, \quad \mu_1^{(n-1)} = \mu_n(t)\,,$$

and this implies

$$\mu_1^{(n)}(t) = \dot{\mu}_n(t) = f(t, \mu_1(t), \dots, \mu_n(t)) = f\big(t, \mu_1(t), \dot{\mu}_1(t), \dots, \mu_1^{(n-1)}(t)\big)\,.$$

Consequently, $t \mapsto \mu_1(t)$ is a solution of (4).
The statement regarding the initial conditions is clear. $\square$