MATH50001 Analysis II, Complex Analysis

Lecture 10

Last time:

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

Corollary.

Let each f_n be holomorphic in a given open set $\Omega \subset \mathbb{C}$ and the series

$$F(z) := \sum_{n=1}^{\infty} f_n(z)$$

converges uniformly in compact subsets of Ω. Then F is holomorphic in Ω.

Section: Holomorphic functions defined in terms of integrals.

Theorem. Let $F(z, s)$ be defined for $(z, s) \in \Omega \times [0, 1]$ where $\Omega \subset \mathbb{C}$ is an open set. Suppose F satisfies the following properties:

- $F(z, s)$ is holomorphic in Ω for each s.
- F is continuous on $\Omega \times [0, 1]$.

Then the function f defined on Ω by

$$f(z) = \int_0^1 F(z, s)\, ds$$

is holomorphic.

Proof. To prove this result, it suffices to prove that f is holomorphic in any disc D contained in Ω. By Morera's theorem this could be achieved by showing that for any triangle T contained in D we have

$$\oint_T \int_0^1 F(z, s)\, ds dz = 0.$$

The proof would be trivial if we could change the order of integration that is not clear. In order to go around this problem we consider for each $n \geq 1$ the Riemann sum

$$f_n(z) = \frac{1}{n} \sum_{k=1}^{n} F(z, k/n).$$

Then by the first assumption f_n is holomorphic in Ω.
We can now show that on any disc D such that $\overline{D} \subset \Omega$, the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f .

Indeed, since F is continuous on $\Omega \times [0,1]$ for a given $\varepsilon > 0$ there exists $\delta > 0$ such that as soon $|s_1 - s_2| < \delta$ we have

$$\sup_{z \in D} |F(z, s_1) - F(z, s_2)| < \varepsilon.$$

Then if $n > 1/\delta$ and $z \in D$ we find

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \left(F(z, k/n) - F(z, s)\right) \, ds \right| \\ &\leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} |F(z, k/n) - F(z, s)| \, ds < \sum_{k=1}^{n} \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

By the previous theorem we conclude that f is holomorphic in D and thus in Ω.

Section: Schwarz reflection principle.

In this section we deal with a simple extension problem for holomorphic functions that is very useful in applications. It is the Schwarz reflection principle that allows one to extend a holomorphic function to a larger domain.

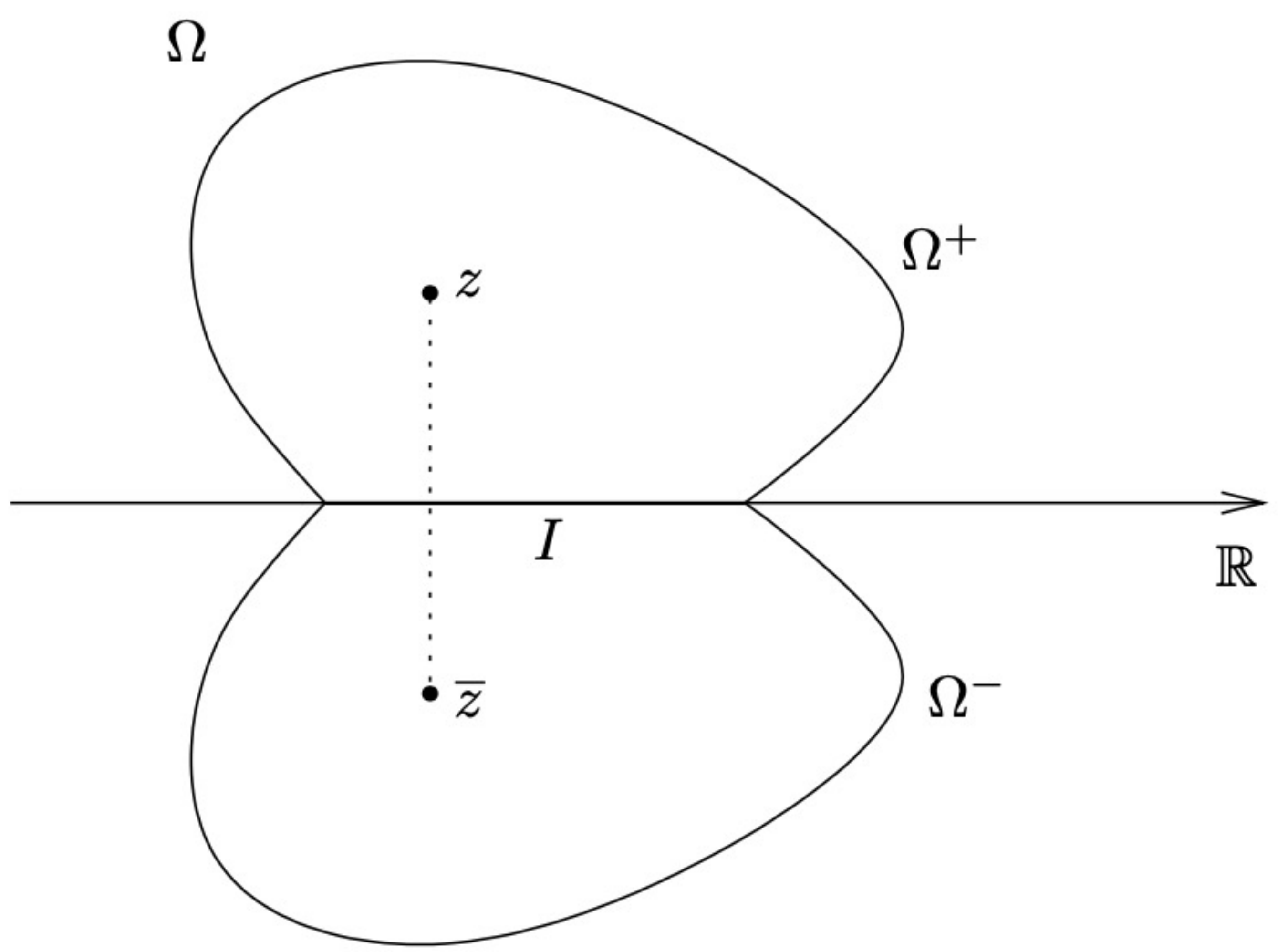
Let $\Omega \subset \mathbb{C}$ be open and symmetric with respect to the real line, that is

$$z \in \Omega \quad \text{iff} \quad \bar{z} \in \Omega.$$

Let

$$\Omega^+ = \{z \in \Omega : \operatorname{Im} z > 0\}, \quad \Omega^- = \{z \in \Omega : \operatorname{Im} z < 0\}$$

and $\quad \mathrm{I} = \{z \in \Omega : \operatorname{Im} z = 0\}$.



Theorem. (Symmetry principle)

If f^+ and f^- are holomorphic functions in Ω^+ and Ω^- respectively, that extend continuously to I such that

$$f^+(x) = f^-(x) \quad \text{for all} \quad x \in I,$$

then the function f defined in Ω by

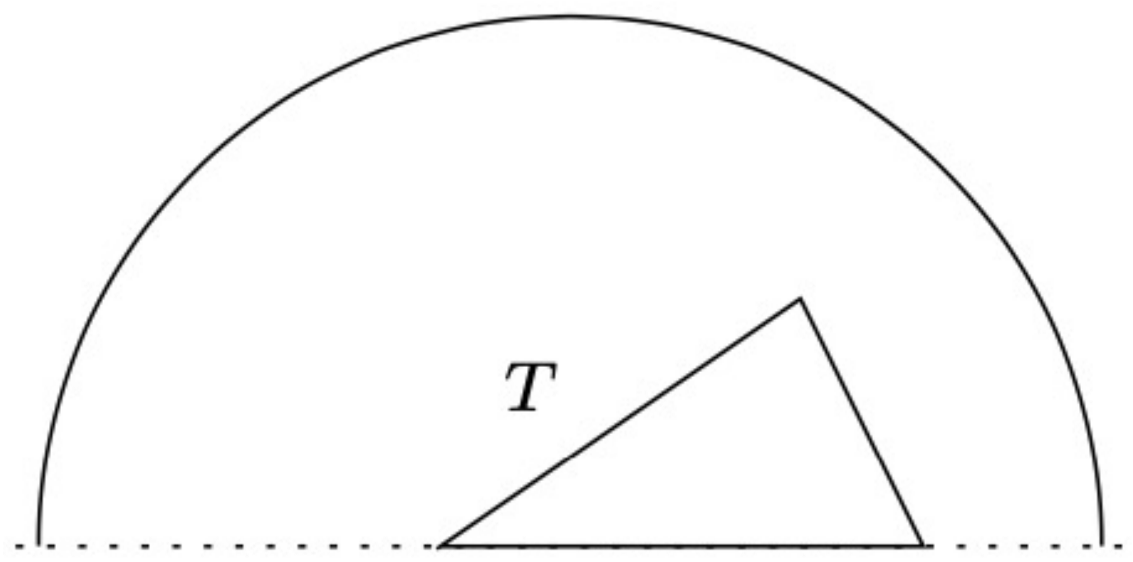
$$f(z) = \begin{cases} f^+(z), & z \in \Omega^+, \\ f^+(z) = f^-(z), & z \in I, \\ f^-(z), & z \in \Omega^-, \end{cases}$$

is holomorphic in Ω.

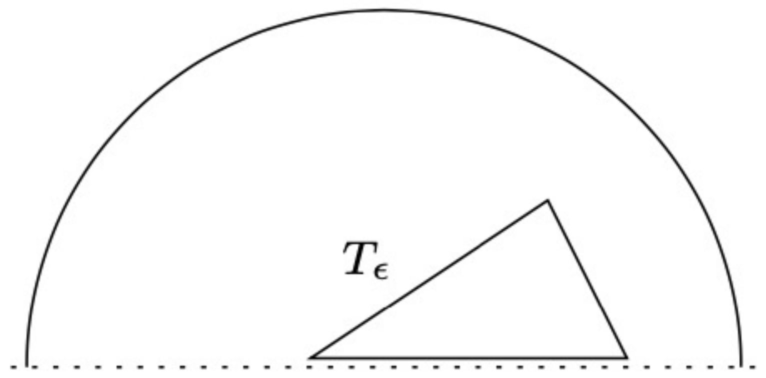
Proof. We only need to prove that f is holomorphic at points of I. Suppose D is a disc centred at a point on I and entirely contained in Ω. We prove that f is holomorphic in D by Morera's theorem. Suppose T is a triangle in D. If T does not intersect I, then

$$\oint_T f(z)\, dz = 0.$$

Suppose now that one side or vertex of T is contained in I, and the rest of T is in, for ex., the upper half-disc.



If T_ε is the triangle obtained from T by slightly raising the edge or vertex which lies on I



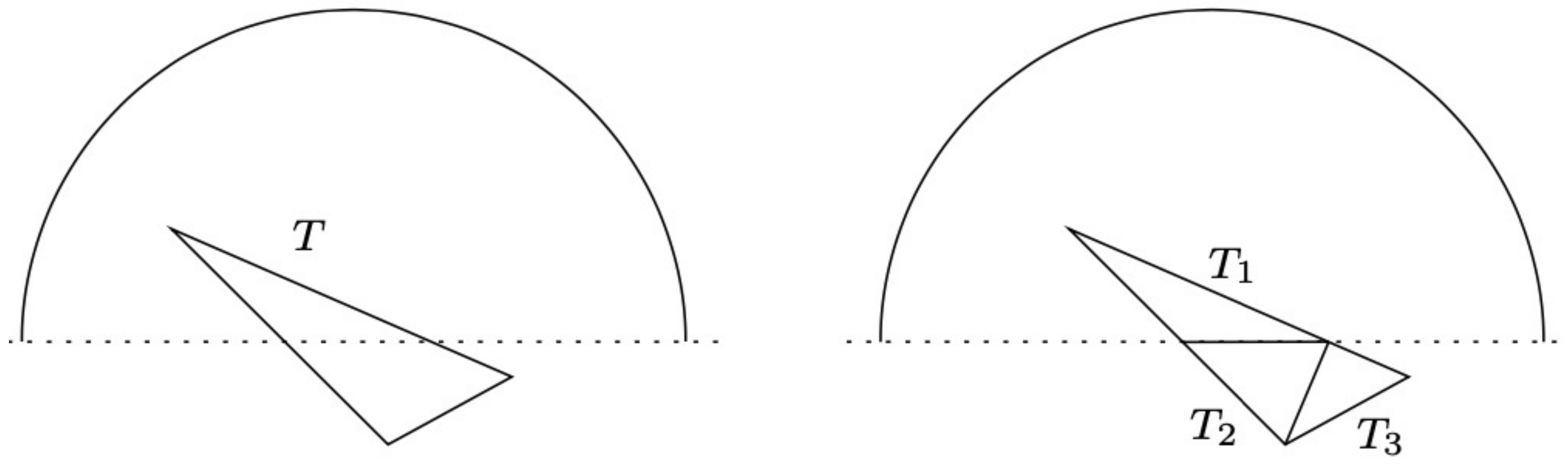
then we have

$$\oint_{\mathsf{T}_\varepsilon} \mathsf{f}(z)\,\mathrm{d}z = 0.$$

since T_ε is entirely contained in the upper half-disc. Letting $\varepsilon \to 0$, by continuity we conclude that

$$\oint_{\mathsf{T}} \mathsf{f}(z)\,\mathrm{d}z = 0.$$

If the interior of T intersects I, we can reduce the situation to the previous one by splitting T as the union of triangles each of which has an edge or vertex on I



By Morera's theorem we conclude that f is holomorphic in D.

Theorem. (Schwarz reflection principle)

Suppose that f is a holomorphic function in Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists a function F holomorphic in Ω such that $F|_{\Omega^+} = f$.

Thank you