Probability and Statistics for JMC
Exercises 4 — Continuous Random Variables

1. Suppose X is a continuous random variable with density function f which is symmetric around zero, i.e. $\forall x \in \mathbb{R}, f(-x) = f(x)$.

   Show that the cdf satisfies $F(-x) = 1 - F(x)$.

2. Electrons hit a circular plate with radius r_0. Let X be the random variable representing the distance of a particle strike from the center of the plate. Assuming that a particle is equally likely to strike anywhere on the plate,

   (a) for $0 < r < r_0$ find $\mathrm{P}(X < r)$ and write down the full the cumulative distribution function of X, $F_X(r)$ for all r.

   (b) find $\mathrm{P}(r < X < s)$ for $0 \leq r < s \leq r_0$.

   (c) find the probability density function for X, f_X at all r.

   (d) calculate the mean distance of a particle strike from the center.

3. Prove that the mean and variance of an $\mathrm{Exp}(\lambda)$ random variable are $\dfrac{1}{\lambda}$ and $\dfrac{1}{\lambda^2}$ respectively.

4. Let $X \sim U(0, 1)$. Find the cdf and hence the pdf of the transformed variable $Y = \mathrm{e}^X$.

5. let $X \sim N(\mu, \sigma^2)$, and consider a transformation of variables that defines a new random variable Y, $Y = \dfrac{X - \mu}{\sigma}$. Show that $Y \sim N(0, 1)$.

6. Let X be a continuous random variable, with cdf $F_X(x)$ and pdf $f_X(x)$. Let $Y = aX + b$, where $a \neq 0$ and b are constants.

   (a) Considering in turn the two cases $a > 0$ and $a < 0$, use the definition of a cdf to find expressions for the cdf of Y, $F_Y(y)$, in terms of F_X.

   (b) Using the relationship between a pdf and its cdf, show that the pdf for Y is given by

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y - b}{a}\right).$$

7. If "area" refers to the area under the curve of the standard normal probability density function ϕ, find the value or values of z such that

   (a) the area between 0 and z is 0.3770.

   (b) the area to the left of z is 0.8621.

   (c) the area between -1.5 and z is 0.0217.

8. Find the area under the standard normal pdf

   (a) between $z = 0$ and $z = 1.2$.

   (b) between $z = -0.68$ and $z = 0$.

   (c) between $z = -0.46$ and $z = 2.21$

   (d) between $z = 0.81$ and $z = 1.94$.

   (e) to the right of $z = -1.28$.

9. You arrive at the bus stop at 10am, knowing that the bus will arrive at some time between 10 and 10:30 with uniform probability.

   (a) What is the probability that you will have to wait longer than 10 minutes?

   (b) If, at 10:15, the bus still has not arrived, what is the probability you will be waiting for at least 10 more minutes?

10. The random variable X has pdf

$$f(x) = \begin{cases} ax + bx^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   If $\mathrm{E}(X) = 0.6$, find

   (a) $\mathrm{P}\left(X < \frac{1}{2}\right)$

   (b) $\mathrm{Var}(X)$

Partial answers:

2. (a) $F_X(r) = \begin{cases} 0 & r < 0, \\ (r/r_0)^2 & 0 \leq r < r_0, \\ 1 & r \geq r_0. \end{cases}$ (b) $(s^2 - r^2)/r_0^2$; (c) $f_X(r) = \begin{cases} \frac{2r}{r_0^2} & 0 \leq r \leq r_0, \\ 0 & \text{elsewhere.} \end{cases}$ (d) $(2/3)r_0$
7. (a) $z \approx 1.16$ or -1.16; (b) $z \approx 1.09$; (c) $z \approx -1.35$ or -1.69
8. (a) 0.3849; (b) 0.2517; (c) 0.6637; (d) 0.1828; (e) 0.8997
9. (a) 2/3; (b) 1/3
10. (a) 0.35; (b) 0.06