# MATH50001 Analysis II, Complex Analysis

Lecture 20

## Section: Möbius Transformations.

### Definition.

A Möbius transformation (that is also called a bilinear transformation) is a map

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where} \quad a, b, c, d \in \mathbb{C} \quad \text{and} \quad ad - bc \neq 0.$$

*Special Möbius transformations.*

Let

$$f(z) = \frac{az+b}{cz+d}$$

and consider the following cases:

$$\text{(M1)} \qquad z \mapsto az \quad (b = c = 0,\ d = 1);$$

if $|a| = 1$, $a = e^{i\theta}$, then this is a rotation by $\theta$. If $a > 0$ then f corresponds to a dilation and if $a < 0$ the map consists of a dilation by $|a|$ followed by a rotation of $\pi$.

$$\text{(M2)} \qquad z \mapsto z + b \quad (a = d = 1, c = 0 - \text{translation by } b);$$

$$\text{(M3)} \qquad z \mapsto \tfrac{1}{z} \quad (a = d = 0,\ b = c = 1 - \text{inversion}).$$

In (M1), if $a = re^{i\theta}$, the geometrical interpretation is an expansion by the factor $r$ followed by a rotation anticlockwise by the angle $\theta$.

**Theorem.**

Every Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

is a composition of transformations of type (M1), (M2) and (M3).

*Proof.*

1. If $c = 0$ and $d \neq 0$, then

$$f(z) = \frac{az+b}{d} = g_2 \circ g_1(z),$$

where

$$g_1(z) = \frac{a}{d}\,z, \qquad g_2(z) = z + \frac{b}{d}.$$

2. If $c \neq 0$, then $f(z) = g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1(z)$, where

$$g_1(z) = cz, \qquad g_2(z) = z + d, \qquad g_3 = \frac{1}{z},$$

$$g_4(z) = \frac{1}{c}(bc - ad)z \qquad g_5(z) = z + \frac{a}{c}.$$

Indeed,

$$g_1(z) = c\,z, \qquad g_2 \circ g_1(z) = c\,z + d, \qquad g_3 \circ g_2 \circ g_1(z) = \frac{1}{cz + d},$$

$$g_4 \circ g_3 \circ g_2 \circ g_1(z) = \frac{bc - ad}{c(cz + d)},$$

$$g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1(z) = \frac{a}{c} + \frac{bc - ad}{c(cz + d)} = \frac{az + b}{cz + d} = f(z).$$

**Corollary.**

A Möbius transformation transforms circles into circles, and interior points into interior points. (Here we mean that straight lines are also circles whose radius equal infinity).

*Proof.* Each of the transformations (M1), (M2) and (M3) transform circles into circles.

## Section: Cross-Ratios Möbius Transformation.

### Theorem.

If $w = f(z)$ is a Möbius transformation that maps the distinct points $(z_1, z_2, z_3)$ into the distinct points $(w_1, w_2, w_3)$ respectively, then

$$\left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right) = \left(\frac{w - w_1}{w - w_3}\right)\left(\frac{w_2 - w_3}{w_2 - w_1}\right), \text{ for all } z.$$

*Proof.*

The Möbius transformation

$$g(z) = \left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right)$$

maps $z_1, z_2, z_3$ to $0, 1, \infty$ respectively. Similarly the Möbius transformation

$$h(w) = \left(\frac{w - w_1}{w - w_3}\right)\left(\frac{w_2 - w_3}{w_2 - w_1}\right)$$

maps $w_1, w_2, w_3$ to $0, 1, \infty$ respectively. Therefore $h^{-1} \circ g$ maps $(z_1, z_2, z_3)$ into $(w_1, w_2, w_3)$.

Example. Find a Möbius transformation $w = f(z)$ that maps the points $1$, $i$, and $-1$ on the unit circle $|z| = 1$ onto the points $-1$, $0$, $1$ on the real axis. Determine the image of the interior $|z| < 1$ under this transformation.

*Proof.* Let $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and $w_1 = -1$, $w_2 = 0$, $w_3 = 1$. The the mapping $w = f(z)$ must satisfy the Cross-Ratios Möbius Transformation

$$
\begin{aligned}
\frac{z-1}{z-(-1)} \cdot \frac{i-(-1)}{i-1} &= \frac{w-(-1)}{w-1} \cdot \frac{0-1}{0-(-1)} \\
\implies \quad \frac{z-1}{z+1} \cdot \frac{i+1}{i-1} = -\frac{w+1}{w-1} &\implies \frac{z-1}{z+1}(-i) = -\frac{w+1}{w-1} \\
\implies \quad (w-1)(z-1)\,i &= (w+1)(z+1) \\
\implies \quad w\,((z-1)i-(z+1)) &= (z-1)i+(z+1) \\
\implies w = \frac{iz-i+z+1}{zi-i-z-i-1} &= \frac{z(1+i)+(1-i)}{iz(1+i)-(1+i)} = \frac{z-i}{iz-1}.
\end{aligned}
$$

Note that if $z = 0$ then $f(0) = i$.

Example. Find a linear fractional transformation $w = f(z)$ that maps the points $z_1 = -i$, $z_2 = 1$, and $z_3 = \infty$ on the line $y = x - 1$ onto the points $w_1 = 1$, $w_2 = i$, and $w_3 = -1$ on the unit circle $|w| = 1$.

*Proof.* Note that

$$\lim_{z_3 \to \infty} \frac{z+i}{z-z_3} \cdot \frac{1-z_3}{1+i} = \lim_{t \to 0} \frac{z+i}{z-1/t} \cdot \frac{1-1/t}{1+i}$$
$$= \lim_{t \to 0} \frac{z+i}{tz-1} \cdot \frac{t1-1}{1+i} = \frac{z+i}{1+i}.$$

Therefore in this case the cross-ratio could be written

$$\frac{z+i}{1+i} = \frac{w-1}{w+1} \cdot \frac{i+1}{i-1} \quad \Longrightarrow \quad \frac{z+i}{1+i} = -i\,\frac{w-1}{w+1}$$
$$\Longrightarrow \quad w = \frac{-z-1}{z+2i-1}.$$

## Section: Conformal mapping of a half-plane to the unit disc.

The upper half-plane can be mapped by a holomorphic bijection to the disc, and this is given by a Möbius transformation.
Let

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : \operatorname{Im} z = y > 0\}.$$

A remarkable surprising fact is that the unbounded set $\mathbb{H}$ is conformally equivalent to the unit disc. Moreover, an explicit formula giving this equivalence exists. Indeed, let

$$w = f(z) = \frac{i - z}{i + z}, \qquad g(w) = i\,\frac{1 - w}{1 + w}.$$

Theorem. Let $\mathbb{D} = \{z : |z| < 1\}$. Then the map $f : \mathbb{H} \mapsto \mathbb{D}$ is a conformal map with inverse $g : \mathbb{D} \mapsto \mathbb{H}$.

*Proof.* Clearly both functions are holomorphic in their respective domains. If $z = x + iy$, $y > 0$, then

$$\left|\frac{i-z}{i+z}\right|^2 = \left|\frac{x^2+(y-1)^2}{x^2+(y+1)^2}\right| < 1.$$

Let $w = u + iv$, $|w| < 1$. Then

$$\begin{aligned}\operatorname{Im} g(w) = \operatorname{Re}\left(\frac{1-u-iv}{1+u+iv}\right) &= \operatorname{Re}\left(\frac{(1-u-iv)(1+u-iv)}{(1+u)^2+v^2)}\right)\\ &= \frac{1-u^2-v^2}{(1+u)^2+v^2} > 0.\end{aligned}$$

Finally

$$f \circ g(w) = \frac{i - i\,\frac{1-w}{1+w}}{i + i\,\frac{1-w}{1+w}} = \frac{1+w-1+w}{1+w+1-w} = w.$$

Similarly we also have $g \circ f(z) = z$.

Note that $f$ is holomorphic in $\mathbb{C} \setminus \{-i\}$ and, in particular, it is continuous on the boundary of $\partial(\mathbb{H}) = \{z = x + i0 \in \mathbb{C}\}$. Clearly

$$|f(z)|_{z=x+i0} = \left|\frac{i-x}{i+x}\right| = 1.$$

Thus $f$ maps $\mathbb{R}$ onto the boundary of the unit disc $\partial\mathbb{D}$. Moreover,

$$f(z) = \frac{i-x}{i+x} = \frac{1-x^2}{1+x^2} + i\,\frac{2x}{1+x^2}.$$

$$f(z) = \frac{i - x}{i + x} = \frac{1 - x^2}{1 + x^2} + i\,\frac{2x}{1 + x^2}.$$

Let $x = \tan\theta$ with $\theta \in (-\pi/2, \pi/2)$. Since

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} \quad \text{and} \quad \sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$$

we obtain

$$f(z) = \cos 2\theta + i\,\sin 2\theta = e^{2i\theta}.$$

$$f(z) = \cos 2\theta + i\,\sin 2\theta = e^{2i\theta}, \qquad \theta \in (-\pi/2, \pi/2).$$

Therefore the image of the real line is the arc consisting of the circle omitting the point $-1$. Moreover, if the value of $x$ changes from $-\infty$ to $\infty$, $f(x)$ changes along that arc starting from $-1$ and first going through that part of the circle that lies in the lower half-plane. The point $-1$ on the circle corresponds to "infinity" of the upper half-plane.

## Section: Riemann mapping theorem.

Definition. We say that $\Omega \subset \mathbb{C}$ is *proper* if it is non-empty and not the whole of $\mathbb{C}$.

Theorem. Suppose $\Omega$ is proper and simply connected. If $z_0 \in \Omega$, then there exists a unique conformal map $f : \Omega \to \mathbb{D}$ such that

$$f(z_0) = 0 \quad \text{and} \quad f'(z_0) > 0.$$

Corollary Any two proper simply connected open subsets in $\mathbb{C}$ are conformally equivalent.

Thank you