Probability and Statistics for JMC
Solutions 2b — More probability

1. (a) In one spin of a European roulette wheel (which has pockets numbered 0, 1, 2, up to and including 36) what is the probability that the outcome is odd?

   $$\text{Classical interpretation: } \mathrm{P}(\text{odd}) = \frac{|\{1,3,5,7,\ldots,35\}|}{|\{0,1,2,3,\ldots,36\}|} = \frac{18}{37}.$$

   (b) An urn contains x red balls and y green ones (both x and y are larger than 2). You remove them one at a time until the urn is empty.

   i. What is the chance that the first is red? $= \frac{x}{x+y}$

   ii. What is the chance that the second is red?

   $\mathrm{P}(\text{2nd red}) = \mathrm{P}(\text{2nd red} \mid \text{1st red})\mathrm{P}(\text{1st red}) + \mathrm{P}(\text{2nd red} \mid \text{1st green})\mathrm{P}(\text{1st green})$
   $= \frac{x-1}{x+y-1}\frac{x}{x+y} + \frac{x}{x+y-1}\frac{y}{x+y} = \frac{x}{x+y}.$

   iii. What is the chance that the first two are red?

   $\mathrm{P}(\text{1st red and 2nd red}) = \mathrm{P}(\text{2nd red} \mid \text{1st red})\mathrm{P}(\text{1st red}) = \frac{x-1}{x+y-1}\frac{x}{x+y}.$

   iv. What is the chance that the second to last one is red?

   Parts i and ii hint that the probability that the k^{th} ball is red is the same for any k. One way to see it is to realize every sequence of draws with x red and y green has the same probability. There is complete symmetry, i.e. the number of possible sequences where the j^{th} ball is red is the same as the number of possible sequences where the k^{th} ball is red.

2. (a) An experiment consists of tossing a fair coin and rolling a fair die. What is the probability of the joint event "heads with an odd number of dots"?

   The events concerning the coin are independent of those concerning the die.
   $\mathrm{P}(\text{heads and odd number}) = \mathrm{P}(\text{heads})\mathrm{P}(\text{odd}) = \frac{1}{2}\frac{3}{6} = \frac{1}{4}.$

   (b) In a particular class, 30% of students were female, and 90% of the males and 80% of the females passed the examination. What percentage of the class passed the examination altogether?

   $(0.3)(0.8) + (1\text{-}0.3)(0.9) = 0.87$. 87% of class passed.

3. On any day the chance of rain is 25%. The chance of rain on two consecutive days is 10%.

   (a) Does this mean that the events of rain on two consecutive days are independent or dependent events?

   Dependent. If independent then
   $\mathrm{P}(\text{rain on day 1 and rain on day 2}) = \mathrm{P}(\text{rain on day 1})\mathrm{P}(\text{rain on day 2}) = (0.25)(0.25)$
   $=0.0625$, which is not equal to 0.10.

   (b) Given that it is raining today, what is the chance of rain tomorrow?

   We want $\mathrm{P}(\text{rain tomorrow} \mid \text{rain today})$.
   $\mathrm{P}(\text{rain tomorrow} \mid \text{rain today}) = \mathrm{P}(\text{rain today and tomorrow})/\mathrm{P}(\text{rain today}) = 0.10/0.25$
   $= 0.4.$

(c) Given that it will rain tomorrow, what is the chance of rain today? Same logic as part (b), 0.4.

4. You forget your umbrella with probability $\frac{1}{4}$ every time you visit a shop (and, once you leave it behind, you do not collect it again).

(a) You set out with your umbrella to visit four different shops. What is the probability that you will leave it in the fourth shop?

This is the probability that you do not leave it in shops 1 through 3 and then leave it in shop 4. $(1-\frac{1}{4})(1-\frac{1}{4})(1-\frac{1}{4})\frac{1}{4} = \frac{27}{256}$.

(b) If you arrive home without your umbrella, what is the probability that you left it in the fourth shop?

P(left in 4 | left somewhere) = P(left in 4 and left somewhere)/P(left somewhere). The numerator is just P(left in 4) which was found in part (a). The denominator can be found as P(left somewhere)=1-P(not left somewhere) $= 1 - \frac{3}{4}\frac{3}{4}\frac{3}{4}\frac{3}{4} = 1 - \frac{81}{256}$. So answer is $\frac{27/256}{175/256} = 0.15$.

(c) If you arrive home without it, but were seen carrying it after leaving the first shop, what is the probability that you left it in the fourth shop?

This is the same as part (b) but if there were only 3 shops. $\frac{\frac{3}{4}\frac{3}{4}\frac{1}{4}}{1-\frac{3}{4}\frac{3}{4}\frac{3}{4}} = \frac{9}{37} = 0.24$. Or you can write everything out conditional on not leaving it in the first shop P(left in 4 | left somewhere and not left in 1).

5. A warehouse contains packs of electronic components. Forty percent of the packs contain components of low quality for which the probability that any given component will prove satisfactory is 0.8; forty percent contain components of medium quality for which this probability is 0.9; and the remaining twenty percent contain high quality components which are certain to be satisfactory.

(a) If a pack is chosen at random and one component from it is tested, what is the probability that this component is satisfactory?

P(satisfactory) = P(s | low qual)P(low qual)+ P(s | med qual)P(med qual) + P(s | high qual)P(high qual)
$= 0.8 \times 0.4 + 0.9 \times 0.4 + 1 \times 0.2 = 0.88$.

(b) If a pack is chosen at random and two components from it are tested, what is the probability that exactly one of the components tested is satisfactory?

If the pack has components with probability p of testing satisfactory then the probability that both components test satisfactory is p^2, the probability that both test unsatisfactory is $(1-p)^2$, and the probability that exactly one tests satisfactory is $2p(1-p)$ (see this either as $p(1-p) + (1-p)p$ or as $1 - p^2 - (1-p)^2$). We use the same partition as in part (a) into low, medium and high quality to get
P(exactly 1 is satis)$= 0.4\times2(0.8)(1-0.8)+0.4\times2(0.9)(1-0.9)+0.2\times2(1)(1-1) = 0.2$.

(c) If it was found that just one of the components tested was satisfactory, what is the probability that the selected pack contained medium quality components?

P(med qual | exactly 1 was satis) = P(med qual and exactly 1 satis)/P(exactly 1 satis). The denominator is from part (b), the numerator is $0.4 \times 2(0.9)(1-0.9) = 0.072$ so answer is 0.36.

(d) If both components were found to be satisfactory, what is the probability that the selected pack contained high quality components?

P(high qual | both satis) = P(high qual and both satis)/P(both satis). The numerator is $0.2 \times 1^2 = 0.2$. Denominator is same logic as part (b):
P(both satis) $= 0.4\times(0.8)^2+0.4\times(0.9)^2+0.2\times1^2 = 0.78$. So answer is $0.2/0.78 = 0.26$.

6. Prove that if $P(A) > P(B)$ then $P(A \mid B) > P(B \mid A)$.

   Bayes's theorem tells us that $\mathrm{P}(A \mid B) = \mathrm{P}(B \mid A)\frac{\mathrm{P}(A)}{\mathrm{P}(B)}$ and the ratio $\mathrm{P}(A)/\mathrm{P}(B) > 1$, so $\mathrm{P}(A \mid B) > \mathrm{P}(B \mid A)$.

7. Show that if three events A, B, and C are independent, then A and $B \cup C$ are independent.

   We want to show $\mathrm{P}\left(A \cap (B \cup C)\right) = \mathrm{P}(A)\mathrm{P}(B \cup C)$. The distributive property of intersections and unions lets us write $A \cap (B \cup C)$ as $(A \cap B) \cup (A \cap C)$. Now we use the identity $\mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F)$ to get $\mathrm{P}\left((A \cap B) \cup (A \cap C)\right) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) - \mathrm{P}(A \cap B \cap C)$. Then independence lets us break up all of these probabilities of intersections: $\mathrm{P}\left((A \cap B) \cup (A \cap C)\right) = \mathrm{P}(A)\mathrm{P}(B) + \mathrm{P}(A)\mathrm{P}(C) - \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C) = \mathrm{P}(A)\left(\mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(B)\mathrm{P}(C)\right)$. The second factor, using that same identity, is seen to be $\mathrm{P}(B \cup C)$, just what we needed.

8. Two factories produce similar parts. Factory 1 produces 1000 parts, 100 of which are defective. Factory 2 produces 2000 parts, 150 of which are defective. A part is selected at random and found to be defective. What is the probability that it came from factory 1?

   We want the conditional probability P(factory 1 | defective). The simplest way is to restrict the sample space to the event that the part is defective. There are 250 defective parts, 100 of which came from factory 1. Each of the defective parts is equally likely to be selected. Therefore, the probability that the defective part came from factory 1 is 100/250=0.4.

   We could have also used Bayes's theorem to invert the conditional P(defective | factory 1) by multiplying by P(factory 1)/P(defective). We have P(defective | factory 1) = 100/1000, P(factory 1)=1000/(1000+2000)=1/3, and P(defective)=(100+150)/(1000+2000) = 1/12.

9. In an experiment in which two fair dice are thrown, let A be the event that the first die is odd, let B be the event that the second die is odd, and let C be the event that the sum is odd. Show that events A, B, and C are pairwise independent, but A, B, and C are not jointly independent.

   A and B are a priori independent.

   For the sum to be odd, one of the dice must be even and the other must be odd. Therefore we can write $C = (A \cap \overline{B}) \cup (\overline{A} \cap B)$. This is a disjoint union so $\mathrm{P}(C) = \mathrm{P}(A)\mathrm{P}(\overline{B}) + \mathrm{P}(\overline{A})\mathrm{P}(B) = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}$ (in words: P(sum is odd)=P(first odd, second even) + P(first even, second odd)). The event $A \cap C$ is equivalent to the event $A \cap \overline{B}$ (first die is odd and second die is even), which has a probability of $\frac{1}{4}$. Therefore, $\mathrm{P}(A \cap C) = \frac{1}{4} = \mathrm{P}(A)\mathrm{P}(C)$ so A and C are independent.

   For B and C the logic is identical. $\mathrm{P}(B \cap C) =$ P(first even and second odd) $= \frac{1}{4} = \mathrm{P}(B)\mathrm{P}(C)$.

   Now consider $A \cap B \cap C$. This event is the empty set: there is no way the first die can be odd, the second die is odd, and the sum is odd. So $\mathrm{P}(A \cap B \cap C) = 0 \neq \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C)$.

10. In a gambling game called craps, a pair of dice is rolled and the outcome is the sum of the dice. The player wins on the first roll if the sum is 7 or 11 and loses if the sum is 2, 3, or 12. If the sum is 4, 5, 6, 8, 9, or 10, that number is called the player's "point". Once the point is established, the rule is: If the player rolls a 7 before the point, the player loses; but if the point is rolled before a 7, the player wins. Compute the probability of winning in the game of craps.

    On the first roll you can either win, lose, or establish the point. These are mutually exclusive events that exhaust all possibilities (i.e. they are a partition of the sample space). Therefore we can write:

P(win) = P(win and win on first roll) + P(win and lose on first roll) + P(win and point on first roll). The middle term is 0 since P(win and lose on first roll)=0. The probability of winning on the first roll is P(7 or 11) = 8/36 (i.e. rolling (1,6),(2,5),(3,4),(4,3),(5,2),(6,1),(5,6),(6,5)).

We have P(win) = 8/36 + P(win and point on first roll).

Now we need to find the probability of winning the sub-game, P(win | point on first roll). We can describe one turn of the sub-game as having outcomes win (roll point), lose (roll 7), or neither. Let the probabilities be p, q, and $1 - p - q$ respectively. We can further partition the event of winning the sub-game as {win} = {win on the first roll} + {win on the second roll} + {win on the third roll}+ ... . These probabilities are $p+(1-p-q)p+(1-p-q)^2p+(1-p-q)^3p+\ldots$. This is the sum of a geometric series with first term p and ratio between successive terms $1-p-q$. Therefore the sum, P(win | point on first roll), is $\frac{p}{1-(1-p-q)} = \frac{p}{p+q}$.

We know $q = \mathrm{P}(7) = 6/36$ but p depends on the very first roll of the game. Therefore, let's break up the event {point on first roll} by the different possible "points" (4,5,6,8,9,10).

$$\begin{aligned}
\mathrm{P}(\text{win and point on first roll}) &= \sum_{k\in\{4,5,6,8,9,10\}} \mathrm{P}(\text{win} \mid k \text{ on first roll})\, \mathrm{P}(k \text{ on first roll}) \\
&= \sum_{k\in\{4,5,6,8,9,10\}} \frac{p(k)}{p(k)+q} p(k),
\end{aligned}$$

where $p(k)$ is the probability of rolling two dice and getting a sum of k: $p(4) = 3/36$, $p(5) = 4/36$, $p(6) = 5/36$, $p(8) = 5/36$, $p(9) = 4/36$, and $p(10) = 3/36$. Plugging everything in we get

$$\begin{aligned}
\mathrm{P}(\text{win and point on first roll}) &= \frac{\left(\frac{3}{36}\right)^2}{\frac{3}{36}+\frac{6}{36}} + \frac{\left(\frac{4}{36}\right)^2}{\frac{4}{36}+\frac{6}{36}} + \frac{\left(\frac{5}{36}\right)^2}{\frac{5}{36}+\frac{6}{36}} + \frac{\left(\frac{5}{36}\right)^2}{\frac{5}{36}+\frac{6}{36}} + \frac{\left(\frac{4}{36}\right)^2}{\frac{4}{36}+\frac{6}{36}} + \frac{\left(\frac{3}{36}\right)^2}{\frac{3}{36}+\frac{6}{36}} \\
&= 0.270707\ldots
\end{aligned}$$

Therefore, P(win) = 8/36 + 0.270707 = 0.492929..., extremely close to even chance but the house has a slight advantage.