Remark 1.6 Let $X = \mathbb{N}$ and μ = counting measure on $(\mathbb{N}, 2^{\mathbb{N}})$, see Example 1.4, 1). Then defining $A_k = \{k, k+1, \dots\}$ for $k \geq 1$, one has $A_k \supset A_{k+1}$ for all k, and $\bigcap_{k=1}^{\infty} A_k = \emptyset$, whence $\infty = \lim_k \mu(A_k) \neq \mu(\bigcap_{k=1}^{\infty} A_k) = 0$. The condition $\mu(A_1) < \infty$ is thus needed in (1.12).

§ I.2 Construction of measures

Let $X \neq \emptyset$ be arbitrary. We now provide a tool to construct measures on X. This will roughly work as follows:

given: $\underbrace{\tilde{\mu}}_{\text{"easy" to define}}$: a pre-measure $\xrightarrow[\text{Step1}]{\text{extend}}$ μ^* : an outer measure $\xrightarrow[\text{Step2}]{\text{restrict}}$ μ: a measure

(see Theorem 1.13 below).

Definition 1.7 i) Let $\mathcal{A} \subset 2^X$ be an algebra. A function $\mu : \mathcal{A} \to [0,\infty]$ satisfying (1.7) and (1.8)* is called a pre-measure (on X).

ii) A function $\mu : 2^X \to [0,\infty]$ satisfying (1.7) and (1.13) with $\mathcal{A} = 2^X$ is called an outer measure (on X).

Step 1 in the above "construction" will be driven by the next proposition

Definition 1.8 A family $\mathcal{K} \subset 2^X$ is called a cover of X if

(1.14) $\emptyset \in \mathcal{K}$, and

(1.15) $\exists (K_n)_{n\geq 1} \subset \mathcal{K}$ s.t. $X = \bigcup_{n=1}^{\infty} K_n$.

Example 1.9 1) The open "intervals" $I = \prod_{k=1}^{n} (a_k, b_k) \overset{\text{def.}}{=} \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : a_k < x_k < b_k \text{ for all } 1 \leq k \leq n\}$, with $a_k \leq b_k \in \mathbb{R}$, form a cover

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* whenever $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

$X = \mathbb{R}^n$; similarly for closed intervals $\prod_{k=1}^{n} [a_k, b_k]$ (require $a_k \le x_k \le b_k$ instead) or the half-open intervals $\prod_{k=1}^{n} (a_k, b_k]$ or $\prod_{k=1}^{n} [a_k, b_k)$.

2) Every algebra $\mathcal{A} \subset 2^X$ is a cover of X, since $\emptyset, X \in \mathcal{A}$ by (1.1), (1.2).

Proposition 1.10 Let $\mathcal{K}$ be a cover of X, $\tilde{\mu} : \mathcal{K} \longrightarrow [0,\infty]$ be a map with $\tilde{\mu}(\emptyset) = 0$. Then

(1.16) $$\mu^*(A) \overset{\text{def.}}{=} \inf \left\{ \sum_{j=1}^{\infty} \tilde{\mu}(K_j) : K_j \in \mathcal{K},\ A \subset \bigcup_{j=1}^{\infty} K_j \right\}, \quad A \in 2^X$$

defines an outer measure on X.

Proof: μ^* is well-defined, as the sequence $(K_n)_{n\in\mathbb{N}}$ supplied by (1.15) is always a valid choice on the right-hand side. Clearly $\mu^*(A) \in [0,\infty]$ for any $A \in 2^X$ and $\mu^*(\emptyset) = 0$ follows by choosing $K_j = \emptyset$, all j, and using $\tilde{\mu}(\emptyset) = 0$. Thus, (1.7) holds. It remains to show (1.13) (for μ^* in place of μ and with $\mathcal{A} = 2^X$).
Let $A_k \in 2^X$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ and $\varepsilon > 0$, by (1.16), we can find a sequence $(K_{k,j})_{j\in\mathbb{N}} \subset \mathcal{K}$ such that $A_k \subset \bigcup_{j=1}^{\infty} K_{k,j}$ and

(1.17) $$\sum_{j=1}^{\infty} \tilde{\mu}(K_{k,j}) < \mu^*(A_k) + 2^{-k}\varepsilon.$$

Now assume $A \in 2^X$, $A \subset \bigcup_{k=1}^{\infty} A_k$ (as in (1.13)). Then $A \subset \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} K_{k,j}$, thus

$$\mu^*(A) \overset{(1.16)}{\le} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \tilde{\mu}(K_{k,j}) \overset{(1.17)}{<} \sum_{k=1}^{\infty} \mu^*(A_k) + \varepsilon. \quad \square$$

We now highlight one key property of outer measures.

Lemma 1.11 If μ^* is an outer measure on X, then

(1.18) $$\Sigma \overset{\text{def.}}{=} \{ A \subset X : \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A) \text{ for all } B \subset X \}$$

is a σ-algebra on X.

Remark 1.12 By subadditivity of μ (cf. (1.13)), Σ is equivalently defined by requiring $\mu^*(B) \ge \mu^*(B \cap A) + \mu^*(B \setminus A)$ in (1.18).

We defer the proof of Lemma 1.8 and proceed to the main result of this section. Its usefulness will be witnessed in the next section.

Theorem 1.13 (Hahn – Carathéodory extension theorem)
Let X be an arbitrary set, $\mathcal{A}$ an algebra over X and $\tilde{\mu}: \mathcal{A} \to [0,\infty]$ a pre-measure on X. Then, defining μ^* by (1.16) with $\mathcal{K} = \mathcal{A}$, Σ by (1.18) and $\mu \overset{\text{def}}{=} \mu^*|_\Sigma$ (i.e. $\mu : \Sigma \to [0,\infty]$, $\mu(A) = \mu^*(A)$ for all $A \in \Sigma$), one has:

(1.19) (X, Σ, μ) is a measure space.

(1.20) $\mathcal{A} \subset \Sigma$.

(1.21) $\mu^*(A) = \tilde{\mu}(A)$, for all $A \in \mathcal{A}$.

Proof: next week.

$\square$

We now supply the:

Proof of Lemma 1.8.
We need to verify (1.1), (1.2), (1.3'). We omit the superscipt * from μ^*.
(1.1): $\mu(B \cap X) + \mu(B \setminus X) = \mu(B) + \mu(\emptyset) \overset{(1.7)}{=} \mu(B)$,
for all $B \subset X$, hence $X \in \Sigma$.

(1.2): Suppose $A \in \Sigma$. Then, for all $B \subset X$, one has

$$\mu(B \cap A^c) + \mu(B \setminus A^c) = \mu(B \setminus A) + \mu(B \cap A) \overset{A \in \Sigma}{=} \mu(B)$$

hence $A^c \in \Sigma$.

(1.3'): We first show (1.3) by induction over $m \geq 1$. The case $m=1$ is trivial. Assume (1.3) holds for $m-1$, for some integer $m \geq 2$. Let $A_1, \dots, A_m \in \Sigma$. Define $A = \bigcup_{k=1}^{m-1} A_k$. By induction assumption, $A \in \Sigma$. Now for arbitrary $B \subset X$, in view of (1.14), we have

$$\text{(1.26)} \qquad \mu(B) = \mu(B \cap A) + \mu(B \setminus A) \qquad (\text{since } A \in \Sigma)$$

$$\text{(1.27)} \qquad \mu(B \setminus A) = \mu((B \setminus A) \cap A_m) + \mu((B \setminus A) \setminus A_m) \qquad (\text{since } A_m \in \Sigma).$$

Hence,

$$\mu(B) \overset{(1.26)(1.27)}{=} \mu(B \cap A) + \mu((B \setminus A) \cap A_m) + \mu((B \setminus A) \setminus A_m)$$
$$= \mu(B \cap (A \cup A_m)) + \mu(B \setminus (A \cup A_m)),$$

i.e. $A \cup A_m = \bigcup_{k=1}^{m} A_k \in \Sigma$.

We now show (1.3'). Let $A_1, A_2, \dots \in \Sigma$. We need to show that $\bigcup_{k=1}^{\infty} A_k \in \Sigma$ and we may assume to that effect that $A_k \cap A_\ell = \emptyset$, $k \neq \ell$. (Indeed, otherwise consider $\tilde{A}_1 = A_1$, $\tilde{A}_k = A_k \setminus \bigcup_{\ell=1}^{k-1} A_\ell$, which satisfy $\tilde{A}_k \in \Sigma$, $k \geq 1$, and observe that $\bigcup_{k=1}^{\infty} \tilde{A}_k = \bigcup_{k=1}^{\infty} A_k$.). Now, for all $m \geq 1$, $B \subset X$, we have

$$\text{(1.28)} \quad \mu\Big(B \cap \big(\bigcup_{k=1}^{m} A_k\big)\Big) \overset{A_m \in \Sigma}{=} \mu\Big(\big(B \cap (\bigcup_{k=1}^{m} A_k)\big) \cap A_m\Big) + \mu\Big(\big(B \cap (\bigcup_{k=1}^{m} A_k)\big) \setminus A_m\Big)$$
$$\overset{A_k\text{'s disj.}}{=} \mu(B \cap A_m) + \mu\Big(B \cap \big(\bigcup_{k=1}^{m-1} A_k\big)\Big)$$
$$\overset{\text{iterat.}}{=} \dots = \sum_{k=1}^{m} \mu(B \cap A_k)$$

Using (1.28) and the fact that $\mu(B) \geq \mu(A)$ for all $A, B \subset X$ with $A \subset B$ (this follows immediately by considering $A_1 = B$, $A_k = \emptyset$, $k \geq 2$ in (1.13) and using (1.7)) yields, for all $B \subset X$, $m \geq 1$,

$$\mu(B) \overset{\bigcup_{k=1}^{m} A_k \in \Sigma}{=} \mu\Big(B \cap \big(\bigcup_{k=1}^{m} A_k\big)\Big) + \mu\Big(B \setminus \big(\bigcup_{k=1}^{m} A_k\big)\Big) \geq \sum_{k=1}^{m} \mu(B \cap A_k) + \mu\Big(B \setminus \big(\bigcup_{k=1}^{\infty} A_k\big)\Big).$$

Thus, letting $n \to \infty$, we find that

$$\mu(B) \geq \sum_{k=1}^{\infty} \mu(B \cap A_k) + \mu\Big(B \setminus \Big(\bigcup_{k=1}^{\infty} A_k\Big)\Big) \overset{(1.13)}{\geq} \mu\Big(B \cap \Big(\bigcup_{k=1}^{\infty} A_k\Big)\Big) + \mu\Big(B \setminus \Big(\bigcup_{k=1}^{\infty} A_k\Big)\Big).$$

On account of Remark 1.9, this implies that $\bigcup_{k=1}^{\infty} A_k \in \Sigma$. $\square$