Analysis 2, Complex Analysis

Solutions, CW1

Q1 (5p)

Let
$$S_1 = \sum_{k=0}^{n} \cos kx \quad \text{and} \quad S_2 = \sum_{k=0}^{n} \sin kx.$$
Then
$$S_1 + iS_2 = \sum_{k=0}^{n} z^k = \frac{z^{n+1}-1}{z-1}, \quad \text{where} \quad z = e^{ix}.$$
Therefore
$$S_1 = \operatorname{Re} \frac{z^{n+1}-1}{z-1} = \operatorname{Re} \frac{\cos(n+1)x + i\sin(n+1)x - 1}{(\cos x - 1) + i \sin x}$$
$$= \frac{\cos nx - \cos(n+1)x - \cos x + 1}{2 - 2\cos x}.$$
Finally we have
$$D_n(x) = \frac{1}{\pi}\left(-\frac{1}{2} + S_1\right) = \frac{1}{2\pi} \frac{\sin\left(\frac{2n+1}{2}x\right)}{\sin\frac{x}{2}}.$$
Note that in the Fourier Analysis the function D_n is called the Dirichlet kernel.

Q2a) (2p)

Let $|a| < 1$. Then introducing the parametrisation $z = e^{i\theta}$, $\theta \in [0, 2\pi]$, and using that $1/\overline{a}$ is outside the disc D we find

$$\oint_{|z|=1} \frac{\overline{f(z)}}{z-a}\, dz = \int_0^{2\pi} \frac{\overline{f(e^{i\theta})}}{e^{i\theta}-a}\, i\, e^{i\theta}\, d\theta = \overline{\int_0^{2\pi} \frac{f(e^{i\theta})}{e^{-i\theta}-\overline{a}}\,(-i)\, e^{-i\theta}\, d\theta}$$
$$= \overline{\frac{1}{\overline{a}} \int_0^{2\pi} \frac{f(e^{i\theta})}{\frac{1}{\overline{a}} - e^{i\theta}}\,(-i)\,\frac{e^{i\theta}}{e^{i\theta}}\, d\theta} = \overline{\frac{1}{\overline{a}} \oint_{|z|=1} \frac{f(z)}{\frac{1}{\overline{a}} - z}\,\frac{-1}{z}\, dz} = 2\pi i\, \overline{f(0)}.$$

Q2b) (3p)

Let $|a| > 1$. Using the same computation and the fact that now $1/\bar{a} \in D$ we have

$$\oint_{|z|=1} \frac{\overline{f(z)}}{z-a}\, dz = \overline{\frac{1}{a} \oint_{|z|=1} \frac{f(z)}{\frac{1}{a} - z} \frac{-1}{z}\, dz}$$
$$= \overline{\frac{-1}{\overline{a}} \oint_{|z|=1} f(z)\, \overline{a} \left(\frac{1}{\frac{1}{\bar{a}} - z} + \frac{1}{z} \right) dz} = 2\pi i \left(\overline{f(0)} - \overline{f\left(\frac{1}{\bar{a}}\right)} \right).$$

Q 3 (5p)

Let T be a triangle in $D = \{z : |z| < 1\}$. Then

$$\oint_T f(z)\, dz = \oint_T \left(\int_0^1 \frac{dt}{1 - zt} \right) dz = \int_0^1 \left(\oint_T \frac{dt}{1 - zt}\, dz \right) dt.$$

The inner integral is zero since for a fixed t, the function $1/(1 - tz)$ is holomorphic in z. Since f is continuous and T is arbitrary $T \subset D$, Moreras theorem applies and f is holomorphic.

Q 4 (5p)

Let $f = u + iv$. Then if $|f(z)| = 0$, we obtain $u^2 + v^2 = 0$ which implies $f(z) \equiv 0$.
Assume that $|f(z)| = C \neq 0$. Hence $f(z)$ is not equal to zero in Ω. Then $|f(z)|^2 = f(z)\overline{f(z)} = C^2$ and thus

$$\overline{f(z)} = \frac{C^2}{f(z)}.$$

Therefore $\overline{f(z)}$ is holomorphic in Ω: Using the C-R equations we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

Hence both functions u and v are constants.