# MATH50001 Analysis II, Complex Analysis

Lecture 4

# Section: Elementary functions.

## 1. Exponential function.

Definition. We define exponential $e^z$ $(z = x + iy \in \mathbb{C})$ as:

$$e^z = e^x \cos y + ie^x \sin y.$$

Properties:

a) If $y = 0$ then $e^z = e^x$.

b) $e^z$ is entire (holomorphic for any $z \in \mathbb{C}$)

Indeed, for that we check the C-R equations. Since $u = \operatorname{Re} f = e^x \cos y$ and $v = \operatorname{Im} f = e^x \sin y$, we have

$$u'_x = e^x \cos y = v'_y \quad \text{and} \quad u'_y = e^x(-\sin y) = -v'_x.$$

c)

$$\frac{\partial}{\partial z} e^z = \frac{\partial}{\partial x} e^x \cos y + i \frac{\partial}{\partial x} e^x \sin y = e^z.$$

d) Let $g(z)$ be holomorphic. Then

$$\frac{\partial}{\partial z} e^{g(z)} = e^{g(z)} g'(z).$$

e) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} e^{z_1+z_2} &= e^{x_1+x_2}\big(\cos(y_1+y_2) + i\sin(y_1+y_2)\big) \\ &= e^{x_1+x_2}\big(\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)\big) \\ &= e^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) = e^{z_1} e^{z_2}. \end{aligned}$$

f) $|e^z| = |e^x|\,|e^{iy}| = e^x \sqrt{\cos^2 y + \sin^2 y} = e^x.$

The function $e^z$ is $2\pi$-periodic with respect to $y$.

g) Applying the De Moivres formula

$$(\cos y + i \sin y)^n = \cos ny + i \sin ny$$

we obtain

$$\left(e^{iy}\right)^n = e^{iny}.$$

h) Since $\arg z = \arctan y/x$

$$\arg e^z = \arctan \frac{e^x \sin y}{e^x \cos y} = \arctan(\tan y) = y + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

**Definition.** If $f$ is holomorphic for all $z \in \mathbb{C}$ then it calls *entire*.

Clearly the exponential function $e^z$ is entire.

## 2. Trigonometric functions.

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \quad \Rightarrow \quad \begin{cases} \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\ \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \end{cases}$$

Definition. For any $z \in \mathbb{C}$ we define

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \qquad \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right).$$

Properties:

a) $\sin z$ and $\cos z$ are entire functions

b) $\frac{\partial}{\partial z} \sin z = \cos z \quad$ and $\quad \frac{\partial}{\partial z} \cos z = -\sin z$.

c) $\sin^2 z + \cos^2 z = 1$.
Indeed:

$$-\frac{1}{4}\left(e^{iz} - e^{-iz}\right)^2 + \frac{1}{4}\left(e^{iz} + e^{-iz}\right)^2 = \cdots = 1.$$

d)

$$\sin(z_1 \pm z_2) = \sin z_1 \, \cos z_2 \pm \cos z_1 \, \sin z_2,$$

$$\cos(z_1 \pm z_2) = \cos z_1 \, \cos z_2 \mp \sin z_1 \, \sin z_2.$$

**3. Logarithmic functions.**

Let $z = r(\cos\theta + i\sin\theta) = r\,e^{i\,\theta}$.

Definition. $\log z = \ln|z| + i\arg z = \log r + i(\theta + 2\pi k), \quad z \neq 0,$

where $\quad k = 0, \pm 1, \pm 2, \ldots.$

Clearly:

$$e^{\log z} = e^{\ln r + i\,(\theta + 2\pi k)} = r\,e^{i\,(\theta+2\pi k)} = r\,(\cos\theta + i\,\sin\theta) = x + i\,y = z.$$

Remark. The function log is a multi-valued function.

**Definition.** We define $\operatorname{Log} z$ as the singe-valued function:

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z,$$

where $\operatorname{Arg} z$ is the principal value of the argument, namely, $-\pi < \operatorname{Arg} z \leq \pi$.

**Remark.** The function Log is a single-valued function.

**Examples.**

$\operatorname{Log}(-1) = i\pi,$
$\operatorname{Log}(2i) = \ln 2 + i\pi/2,$
$\operatorname{Log}(1 - i) = \ln\sqrt{2} - i\pi/4.$

Properties:

a) $\log(z_1 \cdot z_2) = \log(z_1) + \log(z_2)$. Indeed

$$\begin{aligned}\log(z_1 \cdot z_2) &= \ln|z_1 z_2| + i \arg(z_1 \cdot z_2)\\ &= \ln|z_1| + \ln|z_2| + i \arg z_1 + i \arg z_2 = \log z_1 + \log z_2.\end{aligned}$$

**Remark.** $\operatorname{Log}(z_1 \cdot z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$, because $\operatorname{Arg}(z_1 \cdot z_2) \neq \operatorname{Arg} z_1 + \operatorname{Arg} z_2$.

b) The function $\operatorname{Log} z$ is holomorphic in $\mathbb{C} \setminus \{(-\infty, 0]\}$.

Indeed, we have already checked that the C-R equations are satisfied:

$$\operatorname{Log} z = \ln r + i\,\theta = u + iv, \qquad -\pi < \theta \leq \pi.$$

Therefore we have

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot 1 = \frac{1}{r}\,\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = 0 = -\frac{1}{r}\,\frac{\partial u}{\partial \theta}.$$

**Exercise.** Compute $(\operatorname{Log} z)'$.

## 4. Powers.

Definition. For any $\alpha \in \mathbb{C}$, we define $z^\alpha = e^{\alpha \log z}$ as a multi-valued function.

Example. $i^i = e^{i \log i} = e^{i\,(i\pi/2 + i2\pi k)} = e^{-\pi/2}\, e^{-2\pi k}$, $k = 0, \pm 1, \pm 2, \ldots$.

Definition. We define the principal value of $z^\alpha$, $\alpha \in \mathbb{C}$, as

$$z^\alpha = e^{\alpha \operatorname{Log} z}.$$

Property:

a) $z^{\alpha_1} \cdot z^{\alpha_2} = e^{\alpha_1 \operatorname{Log} z}\, e^{\alpha_2 \operatorname{Log} z} = e^{(\alpha_1 + \alpha_2) \operatorname{Log} z} = z^{\alpha_1 + \alpha_2}$.

# Section: Parametrised curve.

Definition. A *parametrised curve* is a function $z(t)$ which maps a closed interval $[a, b] \subset \mathbb{R}$ to the complex plane. We say that the parametrised curve is smooth if $z'(t)$ exists and is continuous on $[a, b]$, and $z'(t) \neq 0$ for $t \in [a, b]$. At the points $t = a$ and $t = b$, the quantities $z'(a)$ and $z'(b)$ are interpreted as the one-sided limits

$$z'(a) = \lim_{h \to 0,\, h>0} \frac{z(a+h) - z(a)}{h}, \quad z'(b) = \lim_{h \to 0,\, h<0} \frac{z(b+h) - z(b)}{h}.$$

Similarly we say that the parametrised curve is piecewise - smooth if $z$ is continuous on $[a, b]$ and if there exist a finite number of points $a = a_0 < a_1 < \cdots < a_n = b$ , where $z(t)$ is smooth in the intervals $[a_k, a_{k+1}]$. In particular, the righthand derivative at $a_k$ may differ from the left-hand derivative at $a_k$ for $k = 1, 2, \ldots, n-1$.

Two parametrisations,

$$z\colon [a,b] \to \mathbb{C} \quad \text{and} \quad \tilde{z}\colon [c,d] \to \mathbb{C},$$

are equivalent if there exists a continuously differentiable bijection $s \to t(s)$ from $[c,d]$ to $[a,b]$ so that $t'(s) > 0$ and

$$\tilde{z}(s) = z(t(s)).$$

The condition $t'(s) > 0$ says precisely that the orientation is preserved: as $s$ travels from $c$ to $d$, then $t(s)$ travels from $a$ to $b$.

Given a smooth curve $\gamma$ in $\mathbb{C}$ parametrised by $z : [a, b] \to \mathbb{C}$ , and $f$ a continuous function on $\gamma$ we define the integral of $f$ along $\gamma$ by

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

In order for this definition to be meaningful, we must show that the right-hand integral is independent of the parametrisation chosen for $\gamma$. Say that $\tilde{z}$ is an equivalent parametrisation as above. Then the change of variables formula and the chain rule imply that

$$\begin{aligned}\int_a^b f(z(t))\,z'(t)\,dt = \int_c^d f(z(t(s)))\,z'(t(s))\,t'(s)\,ds \\ = \int_c^d f(\tilde{z}(s))\,\tilde{z}'(s)\,ds.\end{aligned}$$

This proves that the integral of $f$ over $\gamma$ is well defined.

If $\gamma$ is piecewise smooth, then the integral of $f$ over $\gamma$ is the sum of the integrals of $f$ over the smooth parts of $\gamma$, so if $z(t)$ is a piecewise-smooth parametrisation as before, then

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))\,z'(t)\,dt.$$

We can define a curve $\gamma^-$ obtained from the curve $\gamma$ by reversing the orientation (so that $\gamma$ and $\gamma^-$ consist of the same points in the plane). As a particular parametrisation for $\gamma^-$ we can take $z^- : [a, b] \to \mathbb{C}$ defined by

$$z^-(t) = z(b + a - t).$$

A smooth or piecewise-smooth curve is closed if $z(a) = z(b)$ for any of its parametrisations. A smooth or piecewise-smooth curve is simple if it is not self-intersecting, that is, $z(t) \neq z(s)$ unless $s = t$, $s, t \in (a, b)$.

A basic example consists of a circle. Consider the circle $C_r(z_0)$ centred at $z_0$ and of radius $r$, which by definition is the set

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

The positive orientation (counterclockwise) is the one that is given by the standard parametrisation

$$z(t) = z_0 + r\,e^{it}, \quad \text{where} \quad t \in [0, 2\pi],$$

while the negative orientation (clockwise) is given by

$$z(t) = z_0 + r\,e^{-it}, \quad \text{where} \quad t \in [0, 2\pi].$$

## Section: Integration along curves.

By definition, the length of the smooth curve $\gamma$ is

$$\operatorname{length}(\gamma) = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Thank you