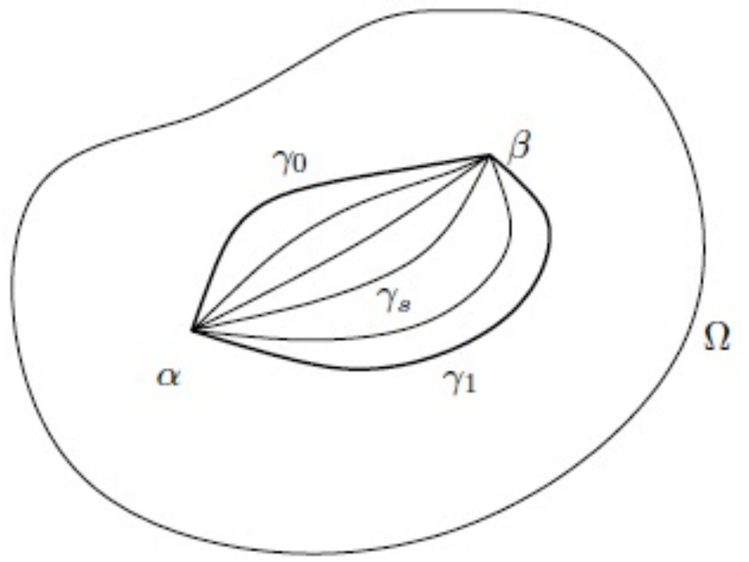
MATH50001 Analysis II, Complex Analysis

Lecture 7

To remind:

In the previous lecture we introduced homotopic curves:



Theorem. If γ_0 and γ_1 are homotopic in Ω and if f is holomorphic in Ω, then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Besides, we had

Definition. An open set $\Omega \subset \mathbb{C}$ is *simply connected* if any two pair of curves in Ω with the same end-points are homotopic.

Theorem. Any holomorphic function in a simply connected domain has a primitive.

Proof. Fix a point z_0 in Ω and define

$$F(z) = \int_\gamma f(w)\,dw,$$

where the integral is taken over any curve in Ω joining z_0 to z. This definition is independent of the curve chosen, since Ω is simply connected. Consider

$$F(z+h) - F(z) = \int_\eta f(w)\,dw,$$

where η is the line segment joining z and $z+h$. Arguing as in the proof of the Theorem where we constructed a primitive to a holomorphic function in a disc, we obtain

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

The proof is complete.

Corollary. (Cauchy-Goursat theorem)
If f is holomorphic in the simply connected open set Ω, then

$$\oint_{\gamma} f(z)\, dz = 0,$$

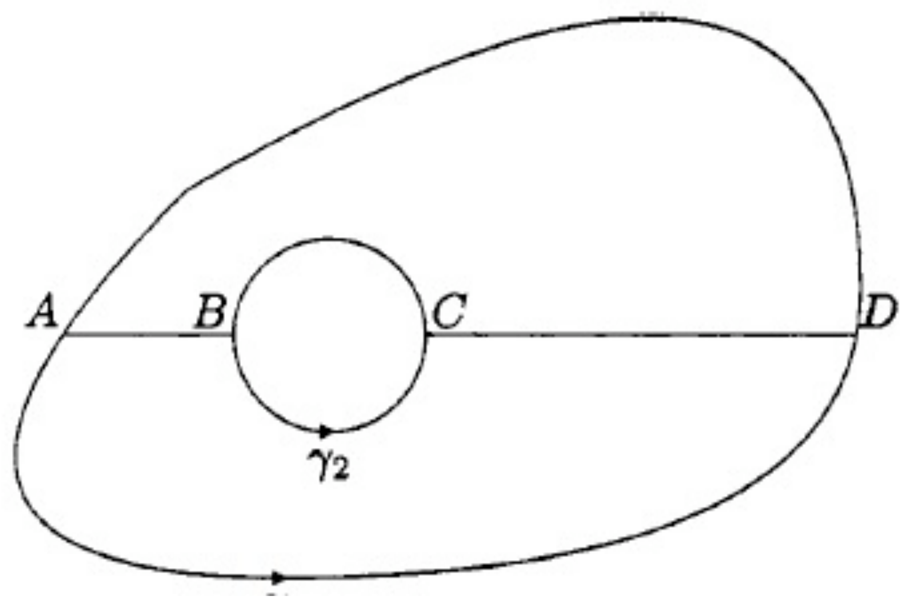
for any closed, piecewise-smooth, curve $\gamma \subset \Omega$.

1858 - 1936, French

Theorem. (Deformation Theorem)
Let γ_1 and γ_2 be two simple, closed, piecewise-smooth curves with γ_2 lying wholly inside γ_1 and suppose f is holomorphic in a domain containing the region between γ_1 and γ_2. Then

$$\oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz.$$

Proof.



Example. Let $\gamma = \{z \in \mathbb{C} : |z-1| = 2\}$. Then

$$\oint_\gamma \frac{1}{z^2-4}\,dz = \oint_\gamma \frac{1}{(z-2)(z+2)}\,dz = \frac{1}{4}\oint_\gamma \left(\frac{1}{z-2} - \frac{1}{z+2}\right) dz.$$

Since $1/(z+2)$ is holomorphic inside and on γ, then

$$\oint_\gamma \frac{1}{z+2}\,dz = 0.$$

On the other hand

$$\oint_\gamma \frac{1}{z-2}\,dz = \oint_{\{z:\,|z-2|=1\}} \frac{1}{z-2}\,dz = 2\pi i.$$

Therefore

$$\oint_\gamma \frac{1}{z^2-4}\,dz = i\,\frac{\pi}{2}.$$

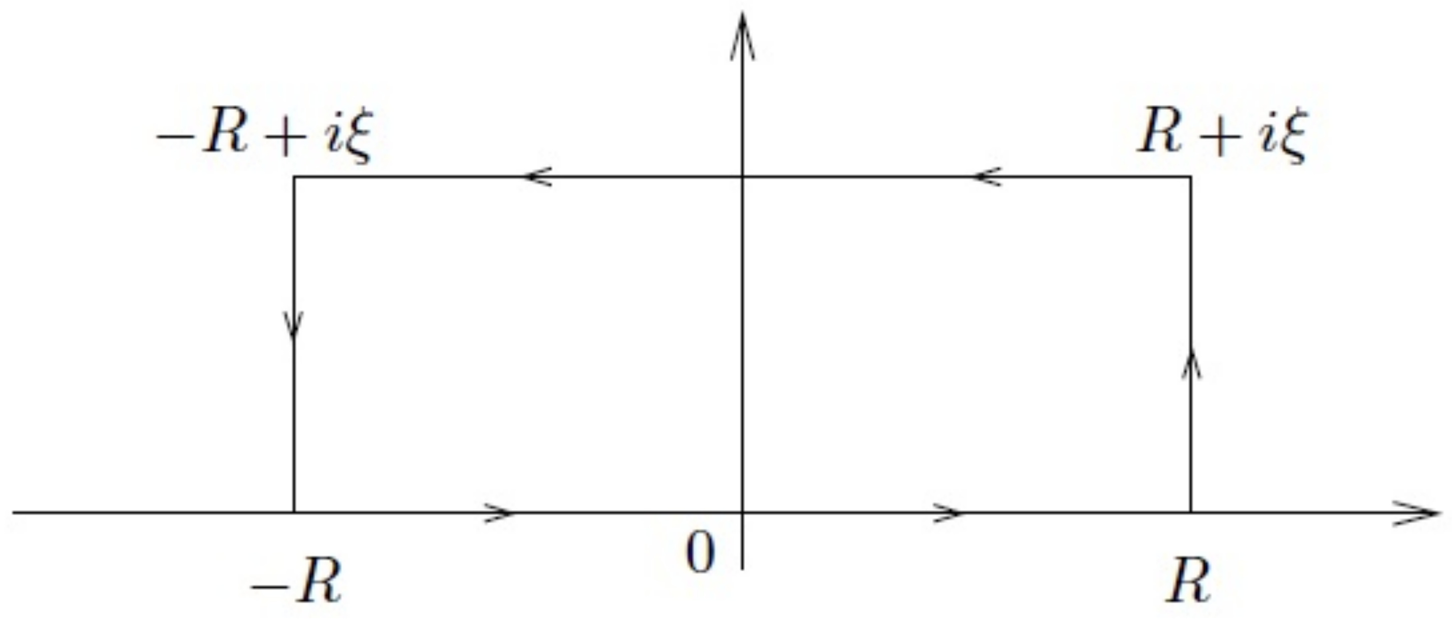
Example. We show that if $\xi \in \mathbb{R}$ then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx.$$

This gives a proof of the fact that $e^{-\pi x^2}$ is its own Fourier transform. If $\xi = 0$, the formula is precisely the known integral

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx.$$

Now suppose that $\xi > 0$, and consider the function $f(z) = e^{-\pi z^2}$, which is entire, and in particular holomorphic in the interior of the contour γ_R



The contour γ_R consists of a rectangle with vertices R, $R + i\xi$, $-R + i\xi$, $-R$ and the positive counterclockwise orientation.
By the Cauchy-Goursat theorem theorem

$$\oint_{\gamma_R} f(z)dz = 0 \qquad (*)$$

The integral over the real segment is simply

$$\int_{-R}^{R} e^{-\pi x^2}\, dx$$

which converges to 1 as $R \to \infty$. The integral on the vertical side on the right is

$$|I(R)| = \left| \int_0^{\xi} f(R+iy)\, i\, dy \right| = \left| \int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)}\, dy \right|$$
$$\leq e^{-\pi R^2} \int_0^{\xi} |e^{-\pi(2iRy-y^2)}|\, dy \leq e^{-\pi R^2}\, \xi\, e^{\pi \xi^2} \to 0,$$

as $R \to \infty$.

Similarly, the integral over the vertical segment on the left also goes to 0 as $R \to \infty$ for the same reasons.

Finally, the integral over the horizontal segment on top is

$$\int_{R}^{-R} e^{-\pi(x+i\xi)^2}\,dx = -\int_{-R}^{R} e^{-\pi(x+i\xi)^2}\,dx$$

$$= -e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx.$$

Therefore, in the limit as $R \to \infty$ we obtain that $(*)$ gives

$$0 = 1 - e^{\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx.$$

Section: Cauchy's integral formulae.

Theorem. Let f be holomorphic inside and on a simple, closed, piecewise-smooth curve γ. Then for any point z_0 interior to γ we have

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Proof. If z_0 is interior to γ then for any $r > 0$ such that $\gamma_r = \{z : |z - z_0| = r\}$ lying wholly inside γ, using the deformation theorem we obtain

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = \oint_{\gamma_r} \frac{f(z)}{z - z_0}\, dz.$$

Then

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_{\gamma_r}\frac{f(z)}{z-z_0}\,dz
&= \frac{1}{2\pi i} f(z_0)\oint_{\gamma_r}\frac{1}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{\gamma_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz \\
&= f(z_0) + \frac{1}{2\pi i}\oint_{\gamma_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz.
\end{aligned}
$$

Since f is holomorphic it is continuous at z_0. Therefore for a given $\varepsilon > 0$ there is $\delta > r > 0$ such that as soon $|z - z_0| < \delta$ we have

$$|f(z) - f(z_0)| < \varepsilon.$$

Then, by using the ML-inequality we have

$$\left|\frac{1}{2\pi i}\oint_{\gamma_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \leq \frac{1}{2\pi}\frac{\varepsilon}{r}2\pi r = \varepsilon.$$

So we have proved that for any $\varepsilon > 0$

$$\left| \oint_\gamma \frac{f(z)}{z - z_0} \, dz - f(z_0) \right| < \varepsilon$$

and hence

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \, dz = f(z_0).$$

The proof is complete.

Example.

$$
\begin{aligned}
\frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{(z-i)(z+i)}\, dz \\
&= \frac{1}{2\pi i} \frac{1}{2i} \oint_{|z|=2} \left(\frac{e^z}{z-i} - \frac{e^z}{z+i} \right) dz \\
&= \frac{1}{2i} \left(e^i - e^{-i} \right) = \sin 1.
\end{aligned}
$$

Theorem. (Generalised Cauchy's integral formula)

Let f be holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, for simple, closed, piecewise-smooth curve $\gamma \subset \Omega$ and any z lying inside γ we have

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}}\, d\eta.$$

Proof. The proof is by induction on n. The case $n = 0$ is simply the Cauchy integral formula. Suppose that f has up to $n-1$ complex derivatives and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n}}\, d\eta.$$

Let $h \in \mathbb{C}$ be small enough, so that $z+h$ is lying inside γ. Then

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \oint_\gamma f(\eta) \frac{1}{h} \left(\frac{1}{(\eta - z - h)^n} - \frac{1}{(\eta - z)^n} \right) d\eta.$$

Recall

$$A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1})$$

and apply it with $A = 1/(\eta - z - h)$ and $B = 1/(\eta - z)$. Then we obtain as $h \to 0$

$$\begin{aligned}
&\frac{1}{h} \left(\frac{1}{(\eta - z - h)^n} - \frac{1}{(\eta - z)^n} \right) \\
&\quad = \frac{1}{h} \frac{h}{(\eta - z - h)(\eta - z)} (A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1}) \\
&\quad \to \frac{1}{(\eta - z)^2} \frac{n}{(\eta - z)^{n-1}}.
\end{aligned}$$

This implies

$$\begin{aligned}\frac{f^{(n-1)}(z+h)-f^{(n-1)}(z)}{h}\\ \to \frac{(n-1)!}{2\pi i}\oint_{\gamma} f(\eta)\,\frac{1}{(\eta-z)^2}\,\frac{n}{(\eta-z)^{n-1}}\,d\eta\\ = \frac{n!}{2\pi i}\oint_{\gamma}\frac{f(\eta)}{(\eta-z)^{n+1}}\,d\eta.\end{aligned}$$

The proof is complete.

Corollary. If f is holomorphic in Ω, then all its defivatives $f', f'', \dots,$ are holomorphic.

Exercise:

Let f be continuous on a piecewise-smooth curve γ At each point $z \notin \gamma$ define the value of a function F by

$$F(z) = \int_\gamma \frac{f(\eta)}{\eta - z}\, d\eta.$$

Show that F is holomorphic at $z \notin \gamma$ and

$$F'(z) = \int_\gamma \frac{f(\eta)}{(\eta - z)^2}\, d\eta.$$

Theorem. (Morera's theorem)

Suppose f is a continuous function in the open disc D such that for any triangle T contained in D

$$\int_T f(z)\, dz = 0,$$

then f is holomorphic.

Giacinto Morera

Italian, 1856 - 1909

Proof. We have proved before that f has a primitive F in D that satisfies $F' = f$. Then F is indefinitely complex differentiable, and therefore f is holomorphic.

Theorem. (Fundamental theorem of Algebra) Every polynomial of degree greater than zero with complex coefficients has at least one zero.

Proof. Assume that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} \cdots + a_0 = 0.$$

has no zeros. Then $1/p(z)$ is entire. Clearly $|1/p(z)| \to 0$ as $|z| \to \infty$. Indeed, given $\varepsilon > 0$ there is R such that

$$\left|\frac{1}{p(z)}\right| < \varepsilon, \qquad \forall z : |z| > R.$$

Since $1/p(z)$ is entire it is also continuous and therefore there is a constant $M > 0$ such that

$$\left|\frac{1}{p(z)}\right| \leq M, \qquad z : |z| \leq R$$

and thus $|1/p(z)|$ is bounded in $\mathbb{C}$. This implies $1/p$ is constant and this contradicts the fact that $p(z)$ is a polynomial of degree greater than zero.

Corollary.

Every polynomial $P(z) = a_n z^n + \dots a_0$ of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If these roots are denoted by $w_1, \dots w_n$, then P can be factored as

$$P(z) = a_n (z - w_1)(z - w_2) \dots (z - w_n).$$

Proof. We now know that P has at least one root, say w_1. Then writing $z = (z - w_1) + w_1$. Substituting this in $P(z) = a_n z^n + \dots a_0$ and using the binomial formula we get

$$P(z) = b_n (z - w_1)^n + \cdots + b_1 (z - w_1) + b_0,$$

where $b_n = a_n$. Since $P(w_1) = 0$ we have $b_0 = 0$ and thus

$$P(z) = (z - w_1) Q(z).$$

Repeating this we find

$$P(z) = a_n (z - w_1)(z - w_2) \dots (z - w_n).$$

Section: Applications of Cauchy's integral formulae.

Joseph Liouville

French 1809 - 1882

Corollary. (Liouville's theorem)

If an entire function is bounded, then it is constant.

Proof. Suppose that f is entire and bounded. Then there is a constant M such that

$$|f(z)| \le M, \qquad \forall z \in \mathbb{C}.$$

Let $z_0 \in \mathbb{C}$ and let $\gamma_r = \{z : |z - z_0| = r\}$. Then

$$|f'(z_0)| = \left| \frac{1!}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{(z-z_0)^2} \, dz \right| \le \frac{M}{r} \to 0 \quad \text{as} \quad r \to \infty.$$

Therefore for any $z_0 \in \mathbb{C}$ we have $f'(z_0) = 0$ and thus f is constant.

Section: The complex logarithm.

We have seen that to make sense of the logarithm as a single-valued function, we must restrict the set on which we define it. This is the so-called choice of a branch or sheet of the logarithm.

Theorem. Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_{\Omega}(z)$ so that:

(i) F is holomorphic in Ω,

(ii) $e^{F}(z) = z, \quad \forall z \in \Omega$,

(iii) $F(r) = \log r$ whenever r is a real number and near 1.

Proof. We shall construct F as a primitive of the function $1/z$. Since $0 \notin \Omega$, the function $f(z) = 1/z$ is holomorphic in Ω. We define

$$\log_{\Omega}(z) = F(z) = \int_{\gamma} f(z)\, dz,$$

where γ is any curve in Ω connecting 1 to z. Since Ω is simply connected, this definition does not depend on the path chosen. Then F is holomorphic and $F'(z) = 1/z$ for all $z \in \Omega$. This proves (i).

To prove (ii), it suffices to show that $ze^{?F(z)} = 1$. Indeed,

$$\frac{d}{dz}\left(ze^{-F(z)}\right) = e^{-F(z)} - zF'(z)e^{-F(z)} = (1 - zF'(z))e^{-F(z)} = 0.$$

Thus $ze^{-F(z)}$ is a constant. Using $F(1) = 0$ we find that this constant must be 1.

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

Proof. Let D be any disc whose closure is contained in Ω and T any triangle in that disc. Then, since each f_n is holomorphic, Goursat's theorem implies

$$\oint_T f_n(z)\, dz = 0, \qquad \text{for all } n.$$

By assumption$f_n \to f$ uniformly in the closure of D, so f is continuous and

$$\oint_T f_n(z)\, dz = \oint_T f(z)\, dz.$$

Therefore

$$\oint_T f(z)\, dz = 0.$$

Using Morera's theorem we find that f is holomorphic in D. Since this conclusion is true for every D whose closure is contained in Ω, we find that f is holomorphic in all of Ω.

Remark. This is not true in the case of real variables: the uniform limit of continuously differentiable functions need not be differentiable. WHY??

Remark. Consider

$$F(z) = \sum_{n=1}^{\infty} f_n(z)$$

where f_n are holomorphic in $\Omega \subset \mathbb{C}$. Assume that the series converges uniformly in compact subsets of Ω, then Theorem guarantees that F is also holomorphic in Ω.

Section: Holomorphic functions defined in terms of integrals.

Theorem. Let $F(z, s)$ be defined for $(z, s) \in \Omega \times [0, 1]$ where $\Omega \subset \mathbb{C}$ is an open set. Suppose F satisfies the following properties:

- $F(z, s)$ is holomorphic in Ω for each s.
- F is continuous on $\Omega \times [0, 1]$.

Then the function f defined on Ω by

$$f(z) = \int_0^1 F(z, s)\, ds$$

is holomorphic.

Proof. To prove this result, it suffices to prove that f is holomorphic in any disc D contained in Ω. By Morera's theorem this could be achieved by showing that for any triangle T contained in D we have

$$\oint_T \int_0^1 F(z,s)\, ds dz = 0.$$

The proof would be trivial if we could change the order of integration that is not clear. In order to go around this problem we consider for each $n \geq 1$ the Riemann sum

$$f_n(z) = \frac{1}{n} \sum_{k=1}^{n} F(z, k/n).$$

Then by the first assumption f_n is holomorphic in Ω.
We can now show that on any disc D such that $\overline{D} \subset \Omega$, the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f .

Indeed, since F is continuous on $\Omega \times [0,1]$ for a given $\varepsilon > 0$ there exists $\delta > 0$ such that as soon $|s_1 - s_2| < \delta$ we have

$$\sup_{z \in D} |F(z, s_1) - F(z, s_2)| < \varepsilon.$$

Then if $n > 1/\delta$ and $z \in D$ we find

$$\begin{aligned}
|f_n(z) - f(z)| &= \left| \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \left(F(z, k/n) - F(z, s)\right) \, ds \right| \\
&\leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \int_{(k-1)/n}^{k/n} |F(z, k/n) - F(z, s)| \, ds < \sum_{k=1}^{n} \frac{\varepsilon}{n} = \varepsilon.
\end{aligned}$$

By Theorem 1.1 we conclude that f is holomorphic in D and thus in Ω.

Section: Schwarz reflection principle.

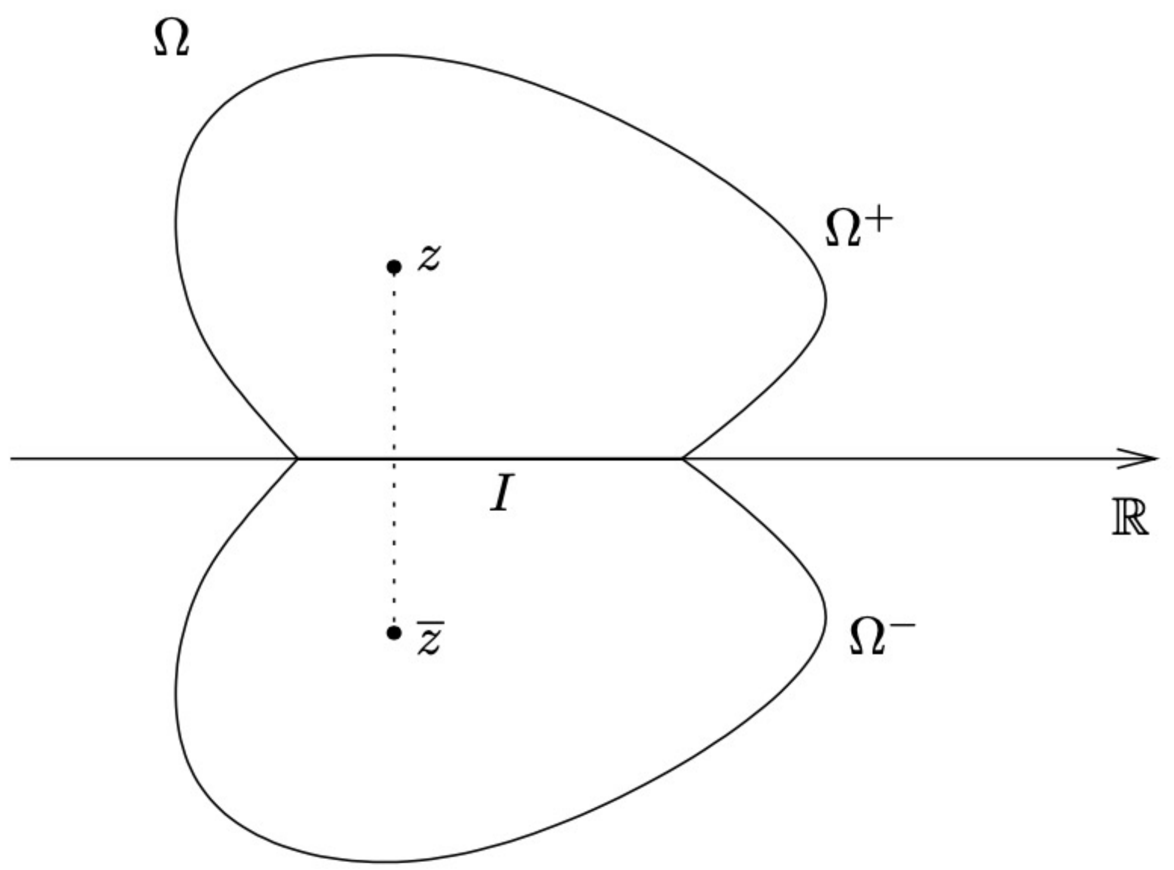
In this section we deal with a simple extension problem for holomorphic functions that is very useful in applications. It is the Schwarz reflection principle that allows one to extend a holomorphic function to a larger domain.

Let $\Omega \subset \mathbb{C}$ be open and symmetric with respect to the real line, that is

$$z \in \Omega \quad \text{iff} \quad \bar{z} \in \Omega.$$

Let

$$\Omega^+ = \{z \in \Omega : \operatorname{Im} z > 0\}, \quad \Omega^- = \{z \in \Omega : \operatorname{Im} z < 0\}$$
$$\text{and} \quad I = \{z \in \Omega : \operatorname{Im} z = 0\}.$$



Thank you