Probability and Statistics for JMC
Exercises 2a — Probability

1. A string of 4 letters is generated at random. Let E_j denote the event that the j^{th} letter is L.

   (a) Describe the following events in words?

      i. $E_1 \cap E_2 \cap E_3 \cap E_4$
      ii. $E_1 \cup E_2 \cup E_3 \cup E_4$
      iii. $\overline{E_1} \cap \overline{E_2} \cap \overline{E_3} \cap \overline{E_4}$
      iv. $E_3 \cap E_4$
      v. $E_1 \cap \overline{E_2}$

   (b) Write these events in set notation

      i. L appears at least 3 times
      ii. L appears exactly once
      iii. We do not have two L's in a row

2. Let S be a set and $\mathcal{L}$ be a set of σ-algebras on S. Define

$$\mathcal{E}^* = \{A \subseteq S : A \in \mathcal{F} \text{ for all } \mathcal{F} \in \mathcal{L}\}.$$

   Show that $\mathcal{E}^*$ is a σ-algebra.

3. Let P be a probability measure, S be a sample space and A_1, A_2, A_3 be events (subsets of S). Show that

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2 \cup A_3) - P((A_1 \cap A_2) \cup (A_1 \cap A_3)) \\ &= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) \\ &\quad - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \end{aligned}$$

4. Suppose two events E and F are mutually exclusive. State the precise conditions under which they may also be independent.

5. What is the probability that a single roll of a die will give an odd number if
   (a) no other information is given;
   (b) you are told that the number is less than 4.

6. (a) What's the probability of getting two sixes with two dice?
   (b) What's the probability of getting a total of 3 with two dice?

7. Two students try to solve a problem they've been set. Student A has a probability of $\frac{2}{5}$ of being able to solve the problem, and student B has a probability of $\frac{1}{3}$. If both try it independently, what is the probability that the problem is solved?

8. A straight line AB of unit length is divided internally at a point X, where X is equally likely to be found anywhere along AB. What is the probability that $|AX||XB| < \frac{3}{16}$? ($|AX|$ is the length of the line segment from A to the internal point X and similarly for $|XB|$.)

9. A fair die is thrown repeatedly until a six is obtained. Let A_k be the event that the first six occurs on the k^{th} throw. Assume that

$$P(A_k) = \frac{1}{6}\left(\frac{5}{6}\right)^{k-1} \qquad k = 1, 2, 3, \ldots$$

   Let B be the event that an even number of throws are required to obtain a six. Find $P(B)$.

Partial answers:

1. (a.i) The string is LLLL; (a.ii) the string has at least one L; (b.i) $(E_1 \cap E_2 \cap E_3) \cup (E_1 \cap E_2 \cap E_4) \cup (E_1 \cap E_3 \cap E_4) \cup (E_2 \cap E_3 \cap E_4)$
5. (a) 1/2; (b) 2/3
6. (a) 1/36; (b) 1/18
7. 3/5
8. 1/2
9. 5/11