MATH50004 Differential Equations
Spring Term 2021/22

Repetition Material 4: Jordan normal form

The Jordan normal form is crucial for the computation of matrix exponential functions, as demonstrated in Proposition 3.8. We first look at Jordan normal forms of two-dimensional matrices. As explained in the lectures, for a given matrix $A \in \mathbb{R}^{2\times 2}$, we require an invertible transformation matrix $T \in \mathbb{R}^{2\times 2}$ that brings A in Jordan normal form: $J = T^{-1}AT$. There are four different cases for (real) Jordan normal forms of two-dimensional matrices:

(C1) A has two different real eigenvalues $a, b \in \mathbb{R}$: $J = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$.

(C2) A has a double real eigenvalue $a \in \mathbb{R}$ with two linearly independent eigenvectors: $J = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$.

(C3) A has a double real eigenvalue $a \in \mathbb{R}$ with only one eigenvector: $J = \left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$.

(C4) A has complex pair $a \pm ib$ of eigenvalues with $b \neq 0$: $J = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$.

In order to see in which of the above four cases a matrix A is and to compute the Jordan normal form, one needs to first find the eigenvalues and eigenvectors. This gives you all the information you need to classify the situation into the four classes, and it is enough to find the Jordan normal form (with a corresponding matrix T) in the cases (C1), (C2) and (C4); the case (C3) requires a bit more work. The case (C4) will be treated in Lecture 15 (where it is explained how to process the complex eigenvectors to get the real transformation matrix). We now briefly explore the cases (C1) and (C2), before we deal with (C3).

Before doing so, remind yourself that the two eigenvalues are obtained by computing the roots of the characteristic polynomial

$$\chi(\gamma) = \det(A - \gamma \operatorname{Id}_2) \quad \text{for all } \gamma \in \mathbb{R},$$

given by γ_1 and γ_2. Then compute the eigenspaces corresponding to the eigenvalues γ_1 and γ_2, which are given by

$$\ker(A - \gamma_1 \operatorname{Id}_2) \quad \text{and} \quad \ker(A - \gamma_2 \operatorname{Id}_2).$$

The eigenspaces are linear subspaces of $\mathbb{R}^2$ containing all eigenvectors. In this case, they are either one-dimensional or two-dimensional (i.e. equal to $\mathbb{R}^2$).

Consider the case that A has two different real eigenvalues $a = \gamma_1$ and $b = \gamma_2$. It follows that the eigenspaces are one-dimensional and spanned by two eigenvectors $v = (v_1, v_2)^\top$ and $w = (w_1, w_2)^\top$. The transformation matrix T is then given by

$$T := \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix},$$

and it follows using $Av = av$ and $Aw = bw$ that

$$J = T^{-1}AT = T^{-1}\begin{pmatrix} av_1 & bw_1 \\ av_2 & bw_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

since $T(a,0)^\top = (av_1, av_2)^\top$ and $T(0,b)^\top = (bw_1, bw_2)^\top$.

Now consider the case that A has one real eigenvalues a with a two-dimensional eigenspace. In this case, the matrix is already in Jordan normal form, and we can choose T to be the identity matrix.

The slightly more complicated case is given by (C3). Suppose that we have computed a one-dimensional eigenspace spanned by the eigenvector v to the eigenvalue a. We aim to get the Jordan normal form

$$J = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

The same logic as in (C1) means that this eigenvector c needs to be in the first column of the transformation matrix T. The second column needs to be so-called generalised eigenvector, which is a vector w satisfying

$$Aw = aw + v \quad \Longleftrightarrow \quad (A - a\,\mathrm{Id}_2)w = v\,.$$

This follows intuitively from understanding the second column of the above Jordan normal form (make sure you understand the meaning of the entries 1 and a in this column). Such a vector w can always be found in this setting, and the transformation matrix is given by

$$T := \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}.$$

The Jordan normal form is then computed as follows:

$$J = T^{-1}AT = T^{-1}\begin{pmatrix} av_1 & aw_1 + v_1 \\ av_2 & aw_2 + v_2 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix},$$

since $T(a,0)^\top = (av_1, av_2)^\top$ and $T(1,b)^\top = (aw_1 + v_1, aw_2 + v_2)^\top$.

The general complex Jordan normal form theorem is given as follows. Note that the word "complex" here means that we "replace" (C4) above with case (C1), where a and b are the complex eigenvalues, and both the matrix T and J can have complex entries, even though the entries of A are all real numbers. You can take this theorem as a fact, and no knowledge of the proof and how to compute T in more complicated situations than above is required in this course. Note that the more complicated Jordan blocks can be obtained using hierarchies of generalised eigenvectors as above.

Theorem 1 (Complex Jordan normal form)**.** *Consider a matrix $A \in \mathbb{R}^{d\times d}$. Then there exists a matrix $T \in \mathbb{C}^{d\times d}$ so that under a basis transformation with the matrix T, we obtain the* complex Jordan normal form

$$J := T^{-1}AT = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_p \end{pmatrix},$$

with the so-called Jordan blocks

$$J_j = \begin{pmatrix} \rho_j & 1 & & 0 & 0 \\ 0 & \rho_j & 1 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & \rho_j & 1 \\ 0 & 0 & & 0 & \rho_j \end{pmatrix} \quad \textit{for all } j \in \{1,\dots,p\}\,,$$

where the ρ_j, $j \in \{1,\dots,p\}$, are complex eigenvalues of the matrix A (some of which may be the same).

Note that if J_j is a 1×1 matrix, then $J_j = (\rho_j)$, and if J_j is a 2×2 matrix, then

$$J_j = \begin{pmatrix} \rho_j & 1 \\ 0 & \rho_j \end{pmatrix}.$$