Chapter 2. Review of Elementary Set Theory

This chapter gives a review of set theory, on top of which will be built the formal mathematical model of probability. We introduce several important notions such as elements, empty sets, intersection, union, and disjoint sets.

Before continuing we list some basic mathematical shorthand:

$\implies$	-	"implies" or "if – then –" (e.g. s1 $\implies$ s2 means "If s1, then s2.")
$\iff$	-	"if and only if" (equivalence)
$\exists$	-	"there exists"
$\forall$	-	"for all"
s.t. *or* $\mid$	-	"such that" (conditional)
wrt	-	"with respect to"

2.1 Sets, subsets and complements

2.1.1 Sets

A **set** is any collection of distinct objects and is a fundamental object of mathematics. We use curly braces when we write a set, e.g. $\{1, 2, 3\}$ is the set containing the **elements** 1, 2, and 3.

The set with no elements is the **empty set**, denoted by $\{\}$ or $\emptyset$.

The objects in a set can be anything, for example integers, real numbers, or even other sets. Elements can be more abstract objects like different possible outcomes of an experiment.

2.1.2 Membership, subsets, equality, complements, and singletons

The symbol $\in$ means "in" and denotes set membership. For example, $x \in A$ means that x is an element of the set A. $x \notin A$ means that x is not in A.

If every element of a set A is also in set B we say that A is a **subset** of B and write $A \subseteq B$. We can write out this definition explicitly using shorthand notation: $x \in A \implies x \in B$.

Equality: If $A \subseteq B$ and $B \subseteq A$ then the sets A and B contain exactly the same elements and we write $A = B$. If A is a subset of B but there are elements of B that are not in A ($B \nsubseteq A$) we write $A \subset B$. Compare with the familiar $\leq$ and $<$.

The **complement** of a set A wrt a universal set Ω (say, of "all possible values") is the set of all elements of Ω that are *not* in A. The complement is denoted by $\overline{A}$ or sometimes A^c. Using mathematical notation: $\overline{A} = \{\omega \in \Omega \mid \omega \notin A\}$.

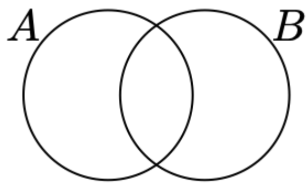
A **singleton** is a set with exactly one element, i.e. $\{\omega\}$.

2.2 Set operations

2.2.1 Venn diagrams, Unions and Intersections

Consider two sets A and B.

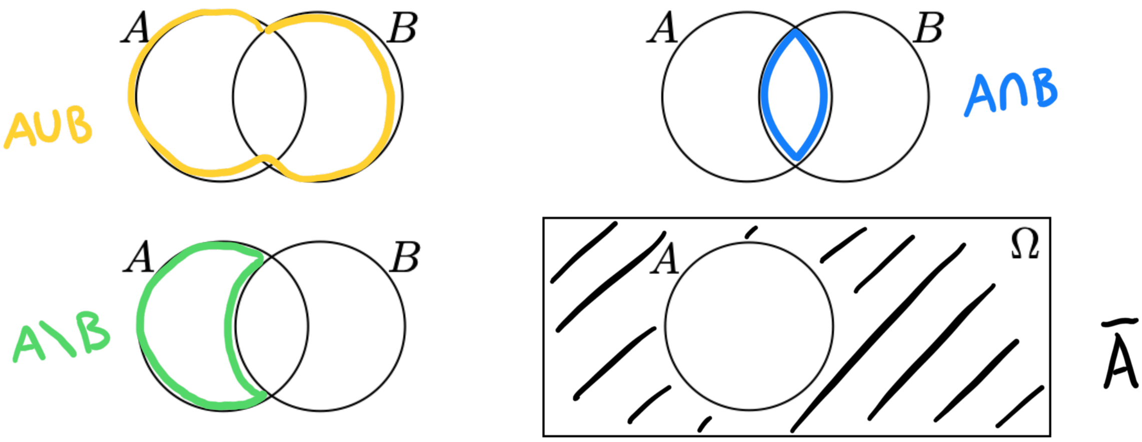
Venn diagram



The **union** of A and B, $A \cup B = \{\omega \,|\, \omega \in A \text{ or } \omega \in B\}$, i.e. all the objects that are in *either* A or B (or both).

The **intersection** of A and B, $A \cap B = \{\omega \,|\, \omega \in A \text{ and } \omega \in B\}$.

The **set difference** of A and B is $A \backslash B = A \cap \overline{B} = \{\omega \,|\, \omega \in A \text{ and } \omega \notin B\}$.



More generally, for sets $A_1, A_2, \dots$ we define

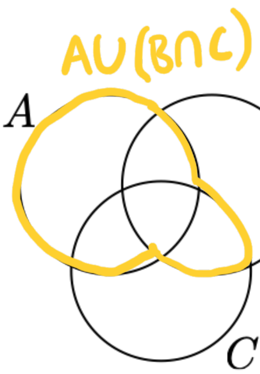
$$\bigcup_i A_i = \{\omega \,|\, \exists i \text{ s.t. } \omega \in A_i\},$$
$$\bigcap_i A_i = \{\omega \,|\, \forall i, \omega \in A_i\}.$$

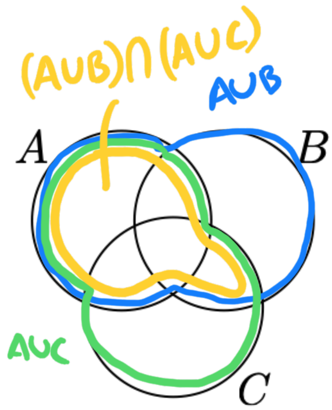
If $A \cap B = \emptyset$, then we say the sets are **disjoint**, i.e. the sets have no element in common.

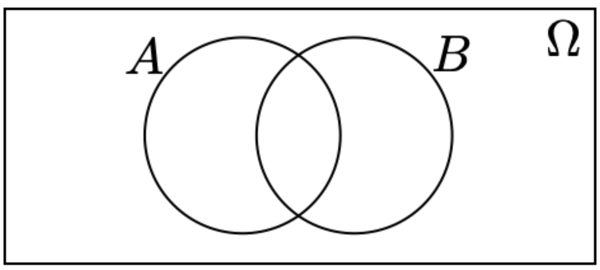
Properties of Union and Intersection Operators

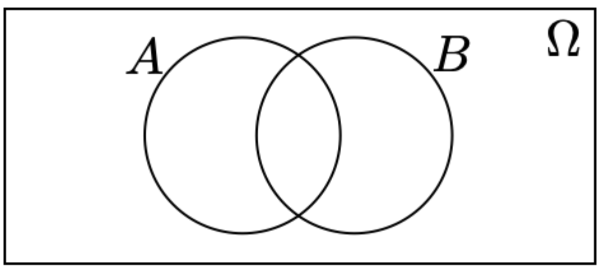
Consider the sets $A, B, C \subseteq \Omega$

COMMUTATIVITY	$A \cup B = B \cup A$ $A \cap B = B \cap A$
ASSOCIATIVITY	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
DISTRIBUTIVITY	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
DE MORGAN'S LAWS	$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$







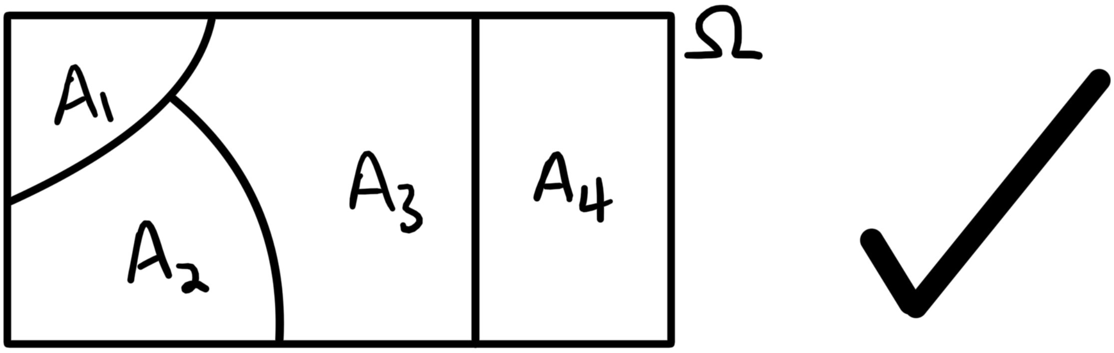


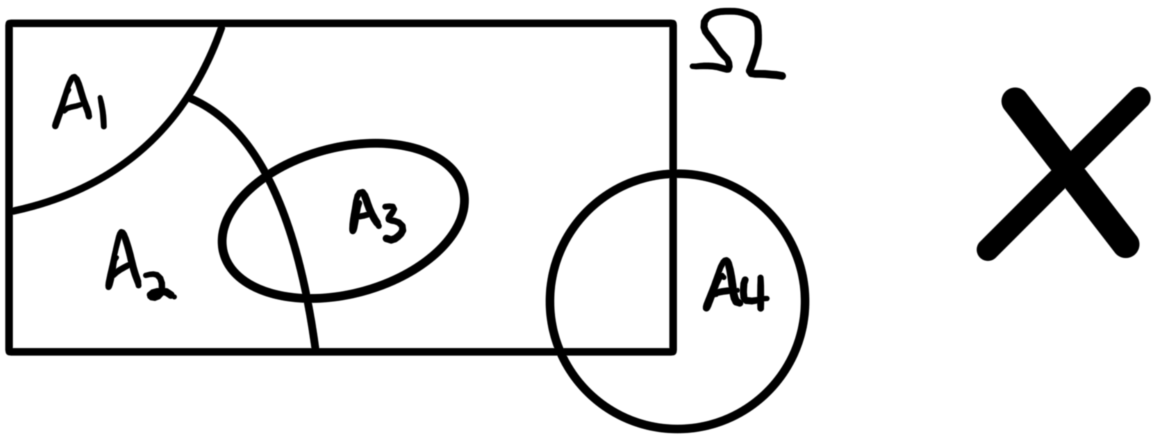
Partitions

The sets $A_1, \dots, A_k$ form a **partition** of the set Ω if

(a) $A_i \cap A_j = \emptyset$ for $i \neq j$, $i, j = 1, \dots, k$ (pairwise disjoint)

(b) $\bigcup_{i=1}^{k} A_i = \Omega$ (the A_i's cover Ω)





2.2.2 Cartesian Products

For two sets Ω_1, Ω_2, their **Cartesian product** is the set of all ordered pairs of their elements. That is,

$$\Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) \mid \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$$

More generally, the Cartesian product for sets $\Omega_1, \Omega_2, \dots$ is written $\prod_i \Omega_i$.

2.3 Cardinality

A useful *measure* of a set is the size, or **cardinality**. The cardinality of a finite set is simply the number of elements it contains. For infinite sets, there are an infinite number of different cardinalities they can take. However, amongst these there is a most important distinction: Between sets which are **countable** and those which are not.

A set Ω is countable if $\exists$ a function $f : \mathbb{N} \to \Omega$ s.t. $\Omega \subseteq f(\mathbb{N})$. That is, the elements of Ω can be *enumerated*, i.e. written out as a possibly unending list $\{\omega_1, \omega_2, \omega_3, \ldots\}$. Note that all finite sets are countable.

A set is **countably infinite** if it is countable but not finite. Clearly $\mathbb{N}$ is countably infinite. So is the set of integers, $\mathbb{N} \times \mathbb{N}$, and even the set of all rational numbers.

A set which is not countable is **uncountable**. The classic example of an uncountable set is the real numbers, $\mathbb{R}$.

The empty set $\emptyset$ has zero cardinality,

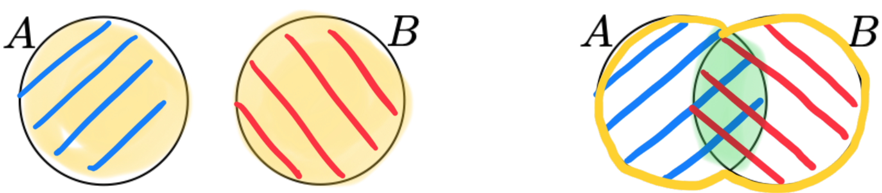
$$|\emptyset| = 0.$$

For finite sets A and B, if A and B are disjoint (that is $A \cap B = \emptyset$), then

$$|A \cup B| = |A| + |B|$$

otherwise,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$



Summary of Notation

Ω	universal set
$\emptyset$	empty set
$A \subseteq \Omega$	Subset of Ω
$\overline{A}$	Complement of A
$\lvert A \rvert$	Cardinality (or size) of A
$A \cup B$	union (A or B)
$A \cap B$	intersection (A and B)
$A = B$	both sets have exactly the same elements
$A \backslash B$	set difference (elements in A that are not in B)

Example Let Ω, the universal set, be all the integers between 1 and 10 (inclusive). A will be the set of odd integers in the universal set, $A = \{x \in \Omega \,|\, x \text{ is odd}\}$, and B will be the set of integers in the universal set greater than 5, $B = \{x \in \Omega \,|\, x > 5\}$.

- $A = \{1, 3, 5, 7, 9\}, \qquad |A| = 5$
- $B = \{6, 7, 8, 9, 10\}, \qquad |B| = 5$
- $A \cup B = \{1, 3, 5, 6, 7, 8, 9, 10\}, \qquad |A \cup B| = 8$
- $A \cap B = \{7, 9\}, \qquad |A \cap B| = 2$
- $A \backslash B = \{1, 3, 5\}, \qquad |A \backslash B| = 3$
- Show $|A \cup B| = |A| + |B| - |A \cap B|$
  $8 = 5 + 5 - 2$
- Show $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
  $\{2, 4\} = \{2, 4, 6, 8, 10\} \cap \{1, 2, 3, 4, 5\}$

Example Let Ω_1 be the set of possible outcomes of one coin flip.

- What is the set of possible outcomes of two flips?
  $\Omega_2 = \Omega_1 \times \Omega_1 = \{(x, y) \mid x \in \Omega_1, y \in \Omega_1\} = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$
- What is the set of possible outcomes of n flips? What is the size of this set?
  $\Omega_n = \underbrace{\Omega_1 \times \Omega_1 \times \cdots \times \Omega_1}_{n \text{ times}} =$ all binary sequences of length n, $\qquad |\Omega_n| = 2^n$
- Of these, how many possible outcomes have at least one heads?
  For any subset A of a universal set Ω, $\{A, \overline{A}\}$ is a partition of Ω (i.e. $A \cap \overline{A} = \emptyset$ and $A \cup \overline{A} = \Omega$). Then $|\Omega| = |A \cup \overline{A}| = |A| + |\overline{A}| - |A \cap \overline{A}| = |A| + |\overline{A}| - |\emptyset| = |A| + |\overline{A}| - 0 = |A| + |\overline{A}|$.

  Let A be the outcomes with at least one heads. A and $\overline{A}$ are a partition of Ω_n. The above relation gives $|A| = |\Omega_n| - |\overline{A}| = 2^n - 1$.