MATH50001 Analysis II, Complex Analysis

Lecture 13

Section: Poles of holomorphic functions.

Definition. Suppose a holomorphic function f has an isolated singularity at z_0 and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent expansion of f valid in some annulus $0 < |z - z_0| < R$. Then

- If $a_n = 0$ for all $n < 0$, z_0 is called a removable singularity
- If $a_n = 0$ for $n < -m$ where m a fix positive integer, but $a_{-m} \neq 0$, z_0 is called a pole of order m.
- If $a_n \neq 0$ for infinitely many negative n's, z_0 is called an essential singularity.

Example.
$f(z) = \frac{\sin z}{z}$; $f(z) = e^{1/z}$; $f(z) = \frac{1}{z^3(z+2)^2}$.

Theorem. A function f has a pole of order m at z_0 if and only if it can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Proof. If g is holomorphic at z_0 and $g(z_0) \neq 0$ then for some $R > 0$

$$g(z) = a_0 + a_1(z - z_0) + \dots, \qquad |z - z_0| < R,$$

where $a_0 = g(z_0) \neq 0$. Then

$$f(z) = \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \dots, \qquad 0 < |z - z_0| < R.$$

This implies that z_0 is a pole of order m.
Conversely, if f has a pole of order m at z_0, then the Laurent expansion of f about z_0 equals

$$\begin{aligned} f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots \\ + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots \\ = \frac{1}{(z - z_0)^m}\Big(a_{-m} + a_{-m+1}(z - z_0) + \dots\Big). \end{aligned}$$

Section: Residue Theory.

Definition. Let

$$f(z) = \sum_{-\infty}^{\infty} a_n \, (z - z_0)^n, \qquad 0 < |z - z_0| < R.$$

be the Laurent series for f at z_0. The residue of f at z_0 is

$$\mathrm{Res}\,[f, z_0] = a_{-1}.$$

Theorem. Let $\gamma \subset \{z : 0 < |z - z_0| < R\}$ be a simple, closed, piecewise-smooth curve that contains z_0. Then

$$\operatorname{Res}[f, z_0] = \frac{1}{2\pi i} \oint_{\gamma} f(z)\, dz.$$

Proof. Let $0 < r < R$. By using the Deformation theorem we obtain

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{\gamma} f(z)\, dz &= \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)\, dz \\
&= \frac{1}{2\pi i} \oint_{|z-z_0|=r} \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n\, dz \\
&= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} a_n\, r^n\, e^{in\theta}\, i\, r\, e^{i\theta}\, d\theta = a_{-1}.
\end{aligned}$$

Theorem. Let f be holomorphic function inside and on a simple, closed, piecewise-smooth curve γ except at the singularities $z_1, \ldots, z_n$ in its interior. Then

$$\oint_{\gamma} f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}[f, z_j].$$

Proof. Let $\gamma_j = \{z : |z - z_j| = r_j\} \subset \Omega$. Then by using the Deformation theorem we find

$$\oint_{\gamma} f(z)\,dz = \sum_{j=1}^{n} \oint_{\gamma_j} f(z)\,dz.$$

Example. Evaluate $\oint_{|z|=1} e^{1/z}\,dz$.
Clearly
$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \dots$$
Therefore
$$\oint_{|z|=1} e^{1/z}\,dz = 2\pi i.$$

Let

$$f(z) = a_{-m}(z - z_0)^{-m} + a_{-m+1}(z - z_0)^{-m+1} + \dots$$

and let $g(z) = (z - z_0)^m f(z)$.

$m = 1.$ Then $g(z) = a_{-1} + a_0\,(z - z_0) + \dots$ and therefore

$$\mathrm{Res}\,[f, z_0] = a_{-1} = \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (z - z_0)\, f(z).$$

$m = 2.$ Then $g(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + \dots$ and

$$\mathrm{Res}\,[f, z_0] = a_{-1} = \frac{d}{dz} g(z)\Big|_{z = z_0} = \lim_{z \to z_0} \frac{d}{dz}((z - z_0)^2\, f(z)).$$

$m.$

$$\mathrm{Res}\,[f, z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m\, f(z)\right).$$

Thank you