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Exercise Sheet 1

1. Let $(X, \mathcal{A})$ be a measurable space, and let $(A_n)_{n\geq 1}$ be a sequence of measurable sets.

a) Show that the sets $\liminf A_n$ and $\limsup A_n$ are measurable, where

$$\liminf A_n = \bigcup_{N\geq 1} \bigcap_{n\geq N} A_n, \qquad \limsup A_n = \bigcap_{N\geq 1} \bigcup_{n\geq N} A_n.$$

b) Let $B, C \in \mathcal{A}$. Specify $\liminf A_n$ and $\limsup A_n$ in terms of B and C if $A_n = B$ for even n and $A_n = C$ for odd n.

2. Let X, Y be sets and $f : X \to Y$ a map. Let $\mathcal{B}$ be a σ-algebra over Y.

a) Show that the collection of pre-image sets

$$\{f^{-1}(B) : B \in \mathcal{B}\}$$

is a σ-algebra (recall that $f^{-1}(A) \overset{\text{def.}}{=} \{x \in X : f(x) \in B\}$). This σ-algebra is usually denoted by $\sigma(f)$.

b) If $y \in Y$ is fixed, $\{y\} \in \mathcal{B}$ and $f(x) = y$ for all $x \in X$ (i.e. f is the constant function equal to y), what is $\sigma(f)$? If $f : X \to Y$ is arbitrary and $Y = \{0, 1\}$, $\mathcal{B} = 2^Y$, what is $\sigma(f)$?

3. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two σ-algebras over a set X.

a) Give an example showing that $\mathcal{F}_1 \cup \mathcal{F}_2 = \{A : A \in \mathcal{F}_1 \text{ or } A \in \mathcal{F}_2\}$ is in general not a σ-algebra.

b) Let

$$\mathcal{F}_1 \vee \mathcal{F}_2 \stackrel{\text{def.}}{=} \sigma(\mathcal{F}_1 \cup \mathcal{F}_2) \qquad \text{(see (1.5) in the notes for notation)}$$

be the σ-algebra generated by $\mathcal{F}_1 \cup \mathcal{F}_2$. Show that:

$$\mathcal{F}_1 \vee \mathcal{F}_2 = \sigma\{A \cup B \,:\, A \in \mathcal{F}_1 \text{ and } B \in \mathcal{F}_2\},$$
$$\mathcal{F}_1 \vee \mathcal{F}_2 = \sigma\{A \cap B \,:\, A \in \mathcal{F}_1 \text{ and } B \in \mathcal{F}_2\}.$$

4. Let X be a set and let $\mathcal{C}$ be a non-empty collection of subsets of X. Lucy claims: "For any $A \in \sigma(\mathcal{C})$, there must exist a countable sub-collection $\mathcal{D} \subset \mathcal{C}$ such that $A \in \sigma(\mathcal{D})$." Do you agree with Lucy? Prove your claim or give a counter-example.

5. Let X be an uncountable set,

$$\mathcal{A} = \{E \subset X : \; E \text{ is countable or } E^c = X \setminus E \text{ is countable}\}.$$

Show that $\mathcal{A}$ defines a σ-algebra and that $\mu : \mathcal{A} \to [0,1]$ with

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is countable} \\ 1, & \text{else} \end{cases}$$

is a measure on X (cf. Example 1.4,3) in the notes).