Imperial College London
Department of Mathematics
Lecturer: P.-F. Rodriguez

Tutorials Lead: Y. Shulzhenko
Assistants: R. Carini, J. McCarthy, S. Karwa, W. Turner

MATH 50006
Winter 2022

Exercise Sheet 6

1. Let $1 \leq p \leq \infty$. For which $\alpha \in \mathbb{R}$ is $f(x) = x^{\alpha}$ in $L^p(X)$ when $X = (0,1)$, $(1,\infty)$ and $(0,\infty)$?

2. Show Young's inequality (Proposition 2.41 in the notes) by investigating the function $f(a) = ab - \frac{a^p}{p}$ for $a, b > 0$.

3. Show Minkowski's inequality (Corollary 2.42(ii) in the notes). *Hint:* to argue $f+g \in L^p$ when $f, g \in L^p$, first show the inequality

$$|a+b|^p \leq 2^p(|a|^p + |b|^p)$$

valid for all $a, b \in \mathbb{R}$. To show the actual inequality, write $|f+g|^p = |f+g|^{p-1}|f+g|$ and apply Hölder's inequality with suitable exponents. Be sure to verify the required integrability assumptions needed for it to apply.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space, $f, f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, measurable and $\mu(X) < \infty$. Show that if f_n converges to f in measure, there exists a subsequence $1 \leq n_1 < n_2 < \ldots$ such that $f_{n_k} \to f$ μ-a.e. as $k \to \infty$. *Hint:* consider the event

$$\{|f_l - f| > 2^{-k}\}$$

for positive integers k, l and choose l suitably (in a manner depending on k) as to make its measure suitably small.

5. Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is Riemann-integrable but not (Lebesgue)-integrable, i.e. $\int |f| d\lambda = \infty$.