Imperial College London
Department of Mathematics
Lecturer: P.-F. Rodriguez
p.rodriguez@imperial.ac.uk

Tutorials Lead (Senior GTA): Yuriy Shulzhenko
yuriy.shulzhenko16@imperial.ac.uk

Graduate Teaching Assistants (GTAs):

John McCarthy j.mccarthy18@imperial.ac.uk
Riccardo Carini r.carini19@imperial.ac.uk
Soham Karwa s.karwa19@imperial.ac.uk
William Turner william.turner17@imperial.ac.uk

MATH 50006
Winter 2022

Exercise Sheet 2

1. Show that the countable additive property of a measure μ is equivalent to μ being additive and continuous from below. That is (using the numbering of the lecture notes),

$$(1.8) \quad \Longleftrightarrow \quad (1.10) \text{ and } (1.11).$$

2. Let $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ denote the set of integers. A measure μ on $(\mathbb{Z}, 2^{\mathbb{Z}})$ is siad to be invariant under translation if, for any $A \subset \mathbb{Z}$ and $x \in \mathbb{Z}$, $\mu(x + A) = \mu(A)$. Find all finite measures on $(\mathbb{Z}, 2^{\mathbb{Z}})$ (i.e. such that $\mu(\mathbb{Z}) < \infty$) which are invariant under translation.

3. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, with λ the Lebesgue measure on $\mathbb{R}$.

a) Show that, for any $x \in \mathbb{R}$, we have $\lambda(\{x\}) = 0$.

b) John claims : "One can write $\mathbb{R}$ as the disjoint union, over all $x \in \mathbb{R}$, of the singleton $\{x\}$, and therefore:

$$\lambda(\mathbb{R}) = \sum_{x \in \mathbb{R}} \lambda(\{x\}) = \sum_{x \in \mathbb{R}} 0 = 0."$$

What is wrong with John's argument?

c) Show that any countable subset of $\mathbb{R}$ is Borel measurable and has Lebesgue measure zero.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space and $A_n \in \mathcal{A}$, $n \geq 1$, satisfy $\mu(\bigcup_{n=1}^{\infty}) < \infty$. Recall the event $\limsup A_n$ from Exercise sheet 1.

a) Show that
$$\limsup \mu(A_n) \leq \mu(\limsup A_n).$$

b) Prove the following (first Borel-Cantelli lemma):
$$\text{if } \sum_{n=1}^{\infty} \mu(A_n) < \infty, \text{ then } \mu(\limsup A_n) = 0.$$

In words, what does this conclusion mean?

5. Let μ be a Borel measure on $\mathbb{R}^n$, i.e. a measure $\mu : \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$. We set
$$S := \{x \in \mathbb{R}^n,\ \mu(B(x,r)) > 0 \text{ for all } r > 0\},$$
where $B(x,r) = \{y \in \mathbb{R}^n : |y = x| < r\}$ is the open ball of radius r around $x \in \mathbb{R}^n$. Show that:

a) S is a closed subset of $\mathbb{R}^n$,

b) $\mu(S^c) = 0$,

c) any strict closed subset F of S satisfies $\mu(S \setminus F) > 0$.

S is called the *support* of the measure μ.