M2PM3 Complex Analysis Mid-term test

2021

Solutions

1. [5p]

Let $z = x + iy$. Then $|z-1| \geq 2\,|z-i|$ is equivalent to

$$|x-1+iy| \geq 2\,|x+i(y-1)| \iff \sqrt{(x-1)^2+y^2} \geq 2\,\sqrt{x^2+(y-1)^2}.$$

By squaring both sides we have

$$3x^2+2x+3y^2-8y+3 \leq 0,$$

or

$$3\left(x^2+\frac{2}{3}\,x+\frac{1}{9}\right)+3\left(y^2-2\,\frac{4}{3}\,y+\frac{16}{9}\right)-\frac{8}{3} \leq 0.$$

Finally we arrive at

$$\left(x+\frac{1}{3}\right)^2+\left(y-\frac{4}{3}\right)^2 \leq \frac{8}{9}.$$

Therefore

$$\Omega = \left\{z \in \mathbb{C}:\ |z-z_0| \leq \frac{2\sqrt{2}}{3}\right\}, \quad \text{where} \quad z_0 = -\frac{1}{3}+i\,\frac{4}{3},$$

is a closed disc whose centre is $z_0 = -\frac{1}{3}+i\,\frac{4}{3}$ and radius $\frac{2\sqrt{2}}{3}$.

2. [5p]

Let us introduce the parametrisation $z = e^{it}$, $t \in [0, 2\pi]$. Then

$$\overline{\oint_\gamma f(z)\,dz} = \int_0^{2\pi} \overline{f(e^{it})}e^{-it}\,(-i)\,dt = -\int_0^{2\pi} \overline{f(e^{it})}e^{-2it}\,e^{it}\,i\,dt = -\oint_\gamma \frac{\overline{f(z)}}{z^2}\,dz.$$

3. There are two cases:

3a. [3p]
If $|w| < 1$, then

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{dz}{z(z-w)} = \frac{1}{2\pi i}\,\frac{1}{w}\oint_{\gamma}\left(\frac{1}{z-w} - \frac{1}{z}\right)\,dz$$
$$= \frac{1}{2\pi i}\,\frac{1}{w}\,(2\pi i - 2\pi i) = 0.$$

3b. [2p]
If $|w| > 1$, then

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{dz}{z(z-w)} = \frac{1}{2\pi i}\,\frac{1}{w}\oint_{\gamma}\left(\frac{1}{z-w} - \frac{1}{z}\right)\,dz$$
$$= -\frac{1}{2\pi i}\,\frac{1}{w}\oint_{\gamma}\frac{1}{z}\,dz = -\frac{1}{w}.$$

4. [5p]

We argue by contradiction. Assume that for any $\varepsilon_n = 1/n$ there is a polynomial p_n, such that

$$\max_{z\in A} |p_n(z) - z^{-1}| < \frac{1}{n}.$$

This implies that p_n convergens uniformly on A to $1/z$. Let

$$\gamma = \left\{z :\ |z| = \frac{r+R}{2}\right\}.$$

Since p_n is holomorphic

$$\oint_{\gamma} p_n(z)\,dz = 0.$$

Using that $p_n \to 1/z$ uniformly on A, we have

$$0 = \oint_{\gamma} p_n(z)\,dz \to \oint_{\gamma}\frac{1}{z}\,dz = 2\pi i.$$