The total marks for this test is 10 marks: 3 marks for Problem 1, 3 marks for Problem 2, and 4 marks for Problem 3.

Problem 1. Prove that the set

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < z < 1\}$$

is open in $\mathbb{R}^3$.

Problem 2. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} x^2 \sin(1/x) & \text{if } y = 0, x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Is the map f differentiable at $(0, 0) \in \mathbb{R}^2$? Justify your answer using the definition of the derivative.

Problem 3. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x, y) = \left(x^2 + y^2 \sin x, \sin(xy)\right).$$

Show that f is differentiable at all $(x, y) \in \mathbb{R}^2$. What is the derivative of f at $(x, y) \in \mathbb{R}^2$?

Solution to Problem 1: Let $p = (p^1, p^2, p^3) \in U$ be an arbitrary point. By the definition of U, we have $0 < p^3 < 1$. Define

$$\delta = \min\{p^3, 1 - p^3\} > 0.$$

We claim that $B_\delta(p) \subset U$. To see this, let $q = (q^1, q^2, q^3) \in B_\delta(p)$ be an arbitrary point. We have

$$|q^3 - p^3| \le \|p - q\| < \delta.$$

By the definition of δ, this implies that $0 < q^3 < 1$, and hence $q \in U$.

Total mark for Q1 is 3, with
1 pt for correct understanding of the notion of open sets,
1 pt for correct value of δ,
1 pt for showing that $B_\delta(p) \subset U$.

Solution to Problem 2: Yes, the map f is differentiable at $(0, 0)$. We claim that $Df(0, 0)$ is the linear map $\Lambda \equiv 0$. To see this, let $h = (h^1, h^2) \in \mathbb{R}^2$. We note that

$$\|f((0, 0) + (h^1, h^2)) - f(0, 0) - \Lambda[(h^1, h^2)]\| = \|f(h^1, h^2)\| \le |h^1|^2.$$

Also, by an inequality in the exercises, we have $|h^1| \le \|(h^1, h^2)\|$.

Thus,

$$\frac{\|f((0, 0) + (h^1, h^2)) - f(0, 0) - \Lambda[(h^1, h^2)]\|}{\|(h^1, h^2)\|} \le \frac{|h^1|^2}{|h^1|} = |h^1|.$$

This implies that

$$\lim_{(h^1,h^2)\to 0} \frac{\|f((0, 0) + (h^1, h^2)) - f(0, 0) - \Lambda[(h^1, h^2)]\|}{\|(h^1, h^2)\|} = 0.$$

Total mark for Q2 is 3, with
1 pt for giving the correct answer Yes,
1 pt for showing understanding of the differentiability, that is, setting up the correct ratio and requiring the limit to tend to 0,
1 pt for showing that the limit is 0.

Remark on Question 2: This question is similar to the Examples 1.12 and 1.13 in the typed lecture notes. One can guess from the form of the function f that the derivative at $(0, 0)$ is zero.

If you cannot guess the derivative at $(0, 0)$, it is possible to use the partial derivatives of f to identify the candidate linear map for $Df(0, 0)$. That is,

$$\frac{\partial}{\partial x} f(0, 0) = \lim_{t\to 0} \frac{f((0, 0) + te_1 - f(0, 0)}{t} = \frac{t^2 \sin(1/t) - 0}{t} = \lim_{t\to 0} t \sin(1/t) = 0,$$

where in the last equality we have used that $-1 \le \sin(1/t) \le 1$. Similarly,

$$\frac{\partial}{\partial y} f(0, 0) = \lim_{t\to 0} \frac{f((0, 0) + te_2 - f(0, 0)}{t} = \frac{0 - 0}{t} = \lim_{t\to 0} 0 = 0.$$

Since the derivative of f in both directions e_1 and e_2 is zero, and $\{e_1, e_2\}$ forms a basis for the vector space $\mathbb{R}^2$, we conclude that $Df(0, 0)$ must be zero.

Solution to Problem 3: We compute the partial derivatives of f at $p = (x, y)$ and find

$$D_1 f^1(p) = 2x + y^2 \cos x \qquad D_2 f^1(p) = 2y \sin x,$$
$$D_1 f^2(p) = y \cos(xy), \qquad D_2 f^2(p) = x \cos(xy).$$

All of the above functions are continuous on $\mathbb{R}^2$.

By a theorem in the lectures, if all the first partial derivatives of f exist and are continuous on $\mathbb{R}^2$, then f is differentiable, and its differential is given by the matrix

$$Df(p) = \begin{pmatrix} 2x + y^2 \cos x & 2y \sin x \\ y \cos(xy) & x \cos(xy) \end{pmatrix}.$$

Total mark for Q3 is 4, with
1 pt for calculating the correct partial derivatives,
1 pt for stating the theorem correctly,
2 pt for correct matrix. If there is a mistake in the partial derivatives, but the matrix is correctly formed in terms of those expressions, then 2 pt should be granted.