# MATH50001 Analysis II, Complex Analysis

Lecture 12

## Section: Laurent Series.

**Definition.** The series

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n = \cdots + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

is called Laurent series for $f$ at $z_0$ where the series converges.

**Theorem.** (Laurent Expansion Theorem)
Let $f$ be holomorphic in the annulus $D = \{z : r < |z - z_0| < R\}$.
Then $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta - z_0)^{n+1}}\, d\eta,$$

and where $\gamma$ is any simple, closed, piecewise-smooth curve in $D$ that contains $z_0$ in its interior.

Pierre Alphonse Laurent
1813 – 1854 (French)

*Proof.* Let us for simplicity assume that $z_0 = 0$ and consider

$$\gamma_1 = \{z : |z| = R' < R\} \quad \text{and} \quad \gamma_2 = \{z : |z| = r' > r\}$$

and such that $z \in D' = \{z : r' < |z| < R'\}$. Then

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta - z} \, d\eta - \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\eta)}{\eta - z} \, d\eta := I_1 - I_2.$$

If $\eta \in \gamma_1$ then $|\eta| > |z|$ and we have

$$I_1 = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta - z} \, d\eta = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta(1 - z/\eta)} \, d\eta$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_{\gamma_1} \frac{f(\eta)}{\eta^{n+1}} \, d\eta \, z^n.$$

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(\eta)}{\eta - z}\, d\eta - \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\eta)}{\eta - z}\, d\eta := I_1 - I_2.$$

If $\eta \in \gamma_2$ then $|\eta| < |z|$ and thus

$$\begin{aligned} -I_2 = -\frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\eta)}{\eta - z}\, d\eta = \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\eta)}{z(1-\eta/z)}\, d\eta \\ = \frac{1}{2\pi i}\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \oint_{\gamma_2} f(\eta)\, \eta^n \, d\eta = [n+1 = -k] \\ = \frac{1}{2\pi i}\sum_{k=-\infty}^{-1} \oint_{\gamma_2} \frac{f(\eta)}{\eta^{k+1}}\, d\eta \; z^k. \end{aligned}$$

Finally we obtain

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\eta)}{\eta^{n+1}}\, d\eta, \quad n = -1, -2, \dots,$$

and

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta^{n+1}}\, d\eta, \quad n = 0, 1, 2, \dots.$$

It remains to show that

$$a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{\eta^{n+1}}\, d\eta, \quad n = 0, \pm 1, \pm 2, \dots.$$

Indeed,

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{\eta^{n+1}}\, d\eta = \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} a_k \oint_{\gamma} \frac{\eta^k}{\eta^{n+1}}\, d\eta = a_n.$$

Example.

Find Laurent series at $z_0 = 0$ for $f(z) = 1/(z-1)$ for $z : |z| > 1$.

$$\frac{1}{z-1} = \frac{1}{z(1-1/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{k=1}^{\infty}\frac{1}{z^k}.$$

This series converges for $|z| > 1$.

Example.

Find Laurent series at $z_0 = 0$ for $f(z) = \frac{1}{z(z+2)}$ for $0 < |z| < 2$.

$$\frac{1}{z(z+2)} = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2}\right) = \frac{1}{2} \cdot \frac{1}{z} - \frac{1}{4(1+z/2)}$$
$$= \frac{1}{2} \cdot \frac{1}{z} - \frac{1}{4} \sum_{n=0}^{\infty} \left(-\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^n}{2^{n+2}} + \frac{1}{2} \cdot \frac{1}{z}.$$

## Section: Poles of holomorphic functions.

**Definition.** A point $z_0$ is called a singularity of a complex function $f$ if $f$ is not holomorphic at $z_0$, but every neighbourhood of $z_0$ contains at least one point at which $f$ is holomorphic.

**Definition.** A singularity $z_0$ of a complex function is said to be isolated if there exists a neighbourhood of $z_0$ in which $z_0$ is the only singularity of $f$.

**Examples.** $f(z) = \frac{1}{1-z}$, $z_0 = 1$; $f(z) = e^{1/z^2}$, $z_0 = 0$; $f(z) = \frac{1}{(z+2)^2}$, $z_0 = -2$.

**Definition.** Suppose a holomorphic function $f$ has an isolated singularity at $z_0$ and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent expansion of $f$ valid in some annulus $0 < |z - z_0| < R$. Then

- If $a_n = 0$ for all $n < 0$, $z_0$ is called a removable singularity
- If $a_n = 0$ for $n < -m$ where $m$ a fix positive integer, but $a_{-m} \neq 0$, $z_0$ is called a pole of order $m$.
- If $a_n \neq 0$ for infinitely many negative $n$'s, $z_0$ is called an essential singularity.

**Examples.**
$f(z) = \frac{\sin z}{z}$; $f(z) = e^{1/z}$; $f(z) = \frac{1}{z^3(z+2)^2}$.

**Theorem.** A function $f$ has a pole of order $m$ at $z_0$ if and only if it can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $g$ is holomorphic at $z_0$ and $g(z_0) \neq 0$.

Thank you