MATH50001 Analysis II, Complex Analysis

Lecture 15

Section: The argument principle.

Theorem. (Principle of the Argument)

Let f be holomorphic in an open set Ω except for a finite number of poles and let γ be a simple, closed, piecewise-smooth curve in Ω that does not pass through any poles or zeros of f. Then

$$\oint_{\gamma} \frac{f'(z)}{f(z)}\, dz = 2\pi i(N - P),$$

where N and P are the sums of the orders of the zeros and poles of f inside γ.

Remark. Why Principle of the Argument?

Indeed, let γ be a closed curve. Then

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_\gamma \frac{d}{dz}\log f(z)\,dz = \frac{1}{2\pi i}\log f(z)\Big|_{z_1}^{z_2}$$
$$= \frac{1}{2\pi i}\Big(\ln|f(z_2)| - \ln|f(z_1)| + i(\arg f(z_2) - \arg f(z_1))\Big) = \frac{1}{2\pi}\Delta \arg f(z).$$

Proof of Theorem.

Step 1. If z_1 is a zero of order n, then

$$f(z) = (z - z_1)^n g(z),$$

where g is holomorphic at z_1 and $g(z_1) \neq 0$. Consequently

$$f'(z) = n\,(z - z_1)^{n-1}\, g(z) + (z - z_1)^n\, g'(z)$$

and

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_1} + \frac{g'(z)}{g(z)}.$$

Since $g(z_1) \neq 0$ it follows that $g(z) \neq 0$ in some neighborhood of z_1. Therefore there is $r > 0$ such that $g'(z)/g(z)$ is holomorphic for $z : |z - z_1| \leq r$ and we have

$$\oint_{|z-z_1|=r} \frac{f'(z)}{f(z)}\, dz = \oint_{|z-z_1|=r} \frac{n}{z - z_1}\, dz + \oint_{|z-z_1|=r} \frac{g'(z)}{g(z)}\, dz = 2\pi i\, n.$$

Step 2. If z_2 is a pole of order p at z_2, then

$$f(z) = \frac{g(z)}{(z - z_2)^p},$$

where g is holomorphic at z_2 and $g(z_2) \neq 0$. Consequently

$$f'(z) = \frac{-p\, g(z)}{(z - z_2)^{p+1}} + \frac{g'(z)}{(z - z_2)^p}$$

and

$$\frac{f'(z)}{f(z)} = \frac{-p}{z - z_2} + \frac{g'(z)}{g(z)}.$$

Since $g(z_2) \neq 0$ it follows that $g(z) \neq 0$ in some neighborhood of z_2. Therefore there is $r > 0$ such that $g'(z)/g(z)$ is holomorphic for $z : |z - z_2| \leq r$ and we have

$$\oint_{|z-z_2|=r} \frac{f'(z)}{f(z)}\, dz = \oint_{|z-z_2|=r} \frac{-p}{z - z_2}\, dz + \oint_{|z-z_2|=r} \frac{g'(z)}{g(z)}\, dz = -2\pi i p.$$

Finally we complete the proof by locating finite number of zeros and poles and using the Deformation theorem.

Example. Let $f(z) = (1+z)/z = 1 + 1/z$, where $\gamma = \{z : z = 2\, e^{i\theta},\ \theta \in [0, 2\pi]\}$. Then $N - P = 0$. Indeed,

$$w = f(z) = 1 + \frac{1}{2}\, e^{-i\theta} = 1 + \frac{1}{2}\, \cos\theta - \frac{i}{2}\, \sin\theta$$

and finally we have $\frac{1}{2\pi}\, \Delta_\gamma \arg f = 0$.

Example. The same problem with $\gamma = \{z : |z| = 1/2\}$ implies $w = f(z) = 1 + 2\cos\theta - 2i\sin\theta$. Thus $\frac{1}{2\pi}\, \Delta_\gamma \arg f = -1$.

Theorem. (Rouche's Theorem)

Let f and g be holomorphic in an open set Ω and let $\gamma \subset \Omega$ be a simple, closed, piecewise-smooth curve that contains in its interior only points of Ω.
If $|g(z)| < |f(z)|$, $z \in \gamma$, then the sums of the orders of the zeros of $f + g$ and f inside γ are the same.

Eugène Rouché

1832 - 1910 (France)

Published in Journal of the École Polytechnique, 1862.

Proof.

Let us consider the function

$$f_t(z) = f(z) + t\,g(z), \qquad t \in [0,1].$$

Clearly $f_0(z) = f(z)$ and $f_1(z) = f(z) + g(z)$. Let $n(t)$ be the number of zeros of f_t inside γ counted with multiplicities. The inequality $|f(z)| > |g(z)|$, $z \in \gamma$, implies that f_t has no zeros on γ and hence

$$F_t(z) = \frac{f_t'(z)}{f_t(z)}$$

has no poles on γ. Therefore the argument principle implies

$$n(t) = \frac{1}{2\pi i} \oint_\gamma F_t(z)\,dz = \frac{1}{2\pi i} \oint_\gamma \frac{f_t'(z)}{f_t(z)}\,dz.$$

Since $n(t) \in \mathbb{Z}$, in order to prove that $N(f) = N(f+g)$ it is enough to show that $n(t)$ is continuous.

Indeed, from $|f(z)| > |g(z)|$ we obtain that there is $\delta > 0$ such that $|f_t| = |f+tg| > \delta$, $z \in \gamma$, $t \in [0,1]$. Thus for any $t_1, t_2 \in [0,1]$ we have

$$\begin{aligned}
|n(t_2) - n(t_1)| &= \left| \frac{1}{2\pi i} \int_\gamma \left(\frac{f'(z) + t_2\, g'(z)}{f(z) + t_2\, g(z)} - \frac{f'(z) + t_1\, g'(z)}{f(z) + t_1\, g(z)} \right) dz \right| \\
&\leq \frac{1}{2\pi} \max_\gamma \left| \frac{(t_2 - t_1)(f(z)g'(z) - f'(z)g(z))}{(f(z) + t_2\, g(z))f((z) + t_1\, g(z))} \right| \cdot \operatorname{length} \gamma \\
&\leq C\, \frac{1}{\delta^2}\, |t_2 - t_1|.
\end{aligned}$$

Thank you