# MATH50001 Analysis II, Complex Analysis

Lecture 11

Last time:

**Theorem.** (Schwarz reflection principle)

Suppose that $f$ is a holomorphic function in $\Omega^+$ that extends continuously to $I$ and such that $f$ is real-valued on $I$. Then there exists a function $F$ holomorphic in $\Omega$ such that $F|_{\Omega^+} = f$.

*Proof.* Let us define $F(z)$ for $z \in \Omega^-$ by

$$F(z) = \overline{f(\bar{z})}.$$

To prove that $F$ is holomorphic in $\Omega^-$ we note that if $z, z_0 \in \Omega^-$ then $\bar{z}, \bar{z}_0 \in \Omega^+$ and since $f$ is holomorphic in $\Omega^+$ we have

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n (\bar{z} - \bar{z}_0)^n.$$

Therefore

$$F(z) = \sum_{n=0}^{\infty} \overline{a_n} (z - z_0)^n$$

and thus $F$ is holomorphic in $\Omega^-$.
Since $f$ is real valued on $I$ we have $\overline{f(x)} = f(x)$ whenever $x \in I$ and hence $F$ extends continuously up to $I$.

## Section: The complex logarithm.

We have seen that to make sense of the logarithm as a single-valued function, we must restrict the set on which we define it. This is the so-called choice of a branch or sheet of the logarithm.

Theorem. Suppose that $\Omega$ is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in $\Omega$ there is a branch of the logarithm $F(z) = \log_{\Omega}(z)$ so that:

(i) $F$ is holomorphic in $\Omega$,

(ii) $e^{F(z)} = z, \quad \forall z \in \Omega$,

(iii) $F(r) = \log r$ whenever $r$ is a real number and near $1$.

In other words, each branch $\log_{\Omega}(z)$ is an extension of the standard logarithm defined for positive numbers.

*Proof.*
We shall construct $\mathsf{F}$ as a primitive of the function $1/z$. Since $0 \notin \Omega$, the function $\mathsf{f}(z) = 1/z$ is holomorphic in $\Omega$. We define

$$\log_{\Omega}(z) = \mathsf{F}(z) = \int_{\gamma} \mathsf{f}(z)\,\mathrm{d}z,$$

where $\gamma$ is any curve in $\Omega$ connecting $1$ to $z$. Since $\Omega$ is simply connected, this definition does not depend on the path chosen. Then $\mathsf{F}$ is holomorphic and $\mathsf{F}'(z) = 1/z$ for all $z \in \Omega$. This proves (i).

To prove (ii), it suffices to show that $ze^{-\mathsf{F}(z)} = 1$. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(ze^{-\mathsf{F}(z)}\right) = e^{-\mathsf{F}(z)} - z\mathsf{F}'(z)e^{-\mathsf{F}(z)} = (1 - z\mathsf{F}'(z))e^{-\mathsf{F}(z)} = 0.$$

Thus $ze^{-\mathsf{F}(z)}$ is a constant. Using $\mathsf{F}(1) = 0$ we find that this constant must be $1$.

## Section: Zeros of holomorphic functions.

Definition. We say that $f$ has a zero of order $m$ at $z_0 \in \mathbb{C}$ if

$$f^{(k)}(z_0) = 0, \qquad k = 0, 1, \ldots m - 1,$$

and $f^{(m)}(z_0) \neq 0$.

Theorem. A holomorphic function $f$ has a zero of order $m$ at $z_0$ if and only if it can be written in the form

$$f(z) = (z - z_0)^m g(z),$$

where $g$ is holomorphic at $z_0$ and $g(z_0) \neq 0$.

*Proof.*

$$\begin{aligned} f(z) &= \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \dots \\ &= (z-z_0)^m\left(\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \dots\right). \end{aligned}$$

Then $f(z) = (z-z_0)^m g(z)$ where $g$ is defined by

$$g(z) = \frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \dots.$$

The above series converges and thus $g$ is holomorphic at $z_0$.

Conversely, if $f(z) = (z-z_0)^m g(z)$, where $g(z_0) \neq 0$, then $f^{(k)}(z_0) = 0$, $k = 0, 1 \dots, m-1$ and $f^{(m)}(z_0) = m!\, g(z_0) \neq 0$.

Corollary. The zeros of a non-constant holomorphic function are isolated; that is every zero has a neighbourhood inside of which it is the only zero.

*Proof.*

If $z_0$ is a zero of $f$ of order $m$, then $f(z) = (z - z_0)^m g(z)$, where $g$ is holomorphic at $z_0$ and $g(z_0) \neq 0$. This means that $g$ is continuous and therefore there is a neighbourhood of $z_0$ in which $g(z) \neq 0$. Thus $f(z) \neq 0$ except for $z = z_0$.

## Section: Laurent Series.

Definition. The series

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n = \cdots + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1}$$

$$+ a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$$

is called Laurent series for $f$ at $z_0$ where the series converges.

Example.

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n, \qquad z \neq 0.$$

**Theorem.** (Laurent Expansion Theorem)
Let $f$ be holomorphic in the annulus $D = \{z : r < |z - z_0| < R\}$.
Then $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z_0)^{n+1}} \, d\eta,$$

and where $\gamma$ is any simple, closed, piecewise-smooth curve in $D$ that contains $z_0$ in its interior.

Pierre Alphonse Laurent
1813 – 1854 (French)

Thank you