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# Exercise Sheet 7

**1.** Let $X = \{1, \dots, N\}$ be a finite state space and consider the measure space $(X, \mathcal{A}, m)$, where $\mathcal{A} = 2^X$ is the discrete $\sigma$-algebra and $m = \sum_{i=1}^N \delta_i$ is the counting measure on $(X, \mathcal{A})$.

a) What can you say about the sigma-algebra $\mathcal{A} \otimes \mathcal{A}$ on $X \times X$? And about $m \otimes m$?

b) Write down Fubini's Theorem for the specific case of a measurable function on the measure space $(X \times X, \mathcal{A} \otimes \mathcal{A}, m \otimes m)$.

**2.** Consider the measure space $([0,1]^2, \mathcal{B}([0,1]^2), \lambda)$, where $\lambda$ is the Lebesgue measure on $[0,1]^2$. Let $\alpha \in \mathbb{R}$. and, for all $(x,y) \in [0,1]^2$, let

$$f(x,y) = \begin{cases} \frac{1}{|x-y|^\alpha} & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

Compute $\int_{[0,1]\times[0,1]} f d\lambda$. For which values of $\alpha$ is $f$ integrable?

**3.** Let $f(x,y) = e^{-xy} - 2e^{-2xy}$, for $(x,y) \in [0,1] \times [1,+\infty)$. Show that the integrals $\int_{[0,1]} \int_{[1,+\infty)} f(x,y)\, d\lambda(y)\, d\lambda(x)$ and $\int_{[1,+\infty)} \int_{[0,1]} f(x,y)\, d\lambda(x)\, d\lambda(y)$ exist but do not coincide. Deduce therefrom that $f$ is not integrable on $[0,1] \times [1,+\infty)$.

**4.** In the following let $(X, \mathcal{A})$ be a fixed measurable space. Are the following statements true or false? Give a proof of provide a counter-example

a) If $\mu$ and $\nu$ are two measures on $(X, \mathcal{A})$ such that $\mu \leq C\nu$ for some $C > 0$, then $\mu \ll \nu$. How about the converse?

b) If $\mu$ and $\nu$ are two measures on $(X, \mathcal{A})$, there always exists a measure $\xi$ such that $\mu \ll \xi$ and $\nu \ll \xi$.

c) If $m$ is the counting measure on $X$, then every measure $\mu$ on $(X, \mathcal{A})$ is absolutely continuous with respect to $m$.

d) On $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the measure $\mathbf{1}_{[0,1]}\lambda$ is absolutely continuous with respect to $\lambda$.

e) On $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the measure $\lambda$ is absolutely continuous with respect to $\mathbf{1}_{[0,1]}\lambda$.

**5.** For all $h \in \mathbb{R}$, let $N(h,1)$ be the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by

$$N(h,1)(A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-h)^2}{2}} dx.$$

Show that the measures $N(h,1)$ and $N(0,1)$ are both absolutely continuous with respect to each other, and compute the corresponding densities.