Functional Analysis Autumn 2022 Coursework 1

CID: 01871147

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Problem Set I

Exercise I.1.4

Claim: the following set is not a linear space, because it is not closed under $\oplus$.

$$V := \left\{ f(z) \text{ - analytic} \,\middle|\, \frac{d^2}{dz^2} f - \frac{d}{dz} f - 2z = 0 \right\}$$

Where $\oplus : V \times V \to V$ is the usual addition of functions:

$$\forall x \in \mathbb{C} \text{ we have } (f \oplus g)(x) = f(x) + g(x)$$

Proof:

Let $f, g \in V$ be arbitrary, suppose for contradiciton that $f \oplus g \in V$, then by definition of V we have that:

$$\frac{d^2}{dz^2}(f \oplus g) - \frac{d}{dz}(f \oplus g) - 2z = 0 \tag{1}$$

But now also $f, g \in V$ so the following holds

$$\frac{d^2}{dz^2} f - \frac{d}{dz} f - 2z = 0$$

$$\frac{d^2}{dz^2} g - \frac{d}{dz} g - 2z = 0$$

If we add the two equations above and exploit linearity of differentiation we get:

$$\frac{d^2}{dz^2}(f \oplus g) - \frac{d}{dz}(f \oplus g) - 4z = 0 \tag{2}$$

Now if we subtract the equation (2) from (1) we obtain:

$$\forall z \in \mathbb{C} \ 2z = 0,$$

which is a contradiction and hence we deduce that $f \oplus g \notin V$ and so V is not closed under vector addition and hence is not a linear space. $\blacksquare$

Problem Set II

Exercise I.7

Let (X, ρ) be a metric space, let $\tilde{\rho_z}$ for some $z \in X$ denote the function described in the question:

$$\begin{aligned}\tilde{\rho_z} : X &\to \mathbb{R}^+ \\ x &\mapsto \rho(z, x)\end{aligned}$$

We'll show continuity of $\tilde{\rho_z}$ using seqential continuity. Let $(x_n)_{n\in N}$ be an arbitrary sequence in X convergent to some x with respect to ρ. We need to show that as $n \to \infty$ we have $\tilde{\rho_z}(x_n) \to \tilde{\rho_z}(x)$ with respect to the Euclidean metric on $\mathbb{R}$. Note that since ρ is a metric it satisfies the triangle inequality and hence $\forall n \in \mathbb{N}$

$$\rho(z, x_n) \leq \rho(z, x) + \rho(x, x_n) \implies \rho(z, x_n) - \rho(z, x) \leq \rho(x, x_n). \tag{3}$$

Similarly using the symmetry of a metric:

$$\rho(z, x) \leq \rho(z, x_n) + \rho(x_n, x) \implies \rho(z, x) - \rho(z, x_n) \leq \rho(x, x_n). \tag{4}$$

Hence, by combining the inequalities (3) and (4) above, we have:

$$|\rho(z, x) - \rho(z, x_n)| \leq \rho(x, x_n). \tag{5}$$

Now let $\epsilon > 0$ be arbitrary, since $(x_n)_{n\in\mathbb{N}}$ converges to x w.r.t ρ we can pick $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $\rho(x, x_n) < \epsilon$.
Now also by definition of $\tilde{\rho_z}$ and inequality (5), we have:

$$|\tilde{\rho_z}(x) - \tilde{\rho_z}(x_n)| = |\rho(z, x) - \rho(z, x_n)| \leq \rho(x, x_n) < \epsilon.$$

And so $\tilde{\rho_z}(x_n) \to \tilde{\rho_z}(x)$ as $n \to \infty$ in $\mathbb{R}$ which implies that $\tilde{\rho_z}$ is continuous. ■

Problem Set III

Exercise I.5 (iii)

Fix $j \in \mathbb{N}$ and denote $\ell_{p,j} := \{x \in \ell_p \mid x_j = 0\}$. We'll show that $\ell_{p,j}$ is closed in ℓ_p and that it doesn't contain any open ball from ℓ_p.
Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\ell_{p,j}$ which converges to $x \in \ell_p$ in ℓ_p w.r.t. $\|.\|_p$. We need to show that x belongs to $\ell_{p,j}$.

Let > 0 be arbitrary, pick $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $(\|x_n - x\|_p)^p < \epsilon$. We can do that by convergence of $(x_n)_{n\in\mathbb{N}}$. Now by the definiton of $\|.\|_p$. We have:

$$\sum_{i\in N}^{\infty} |x_{n_i} - x_i|^p < \epsilon,$$

where x_{n_i} denotes the i-th term of the sequence x_n which is the n-th entry of $(x_n)_{n\in\mathbb{N}}$

Now since each $|x_{n_i} - x_i|$ is non-negative, it implies that

$$\forall n \geq N \; \forall i \in N \; |x_{n_i} - x_i| < \epsilon.$$

Now fix $i := j$ (the j that we fixed at the beginning). Note that since each x_n belongs to $\ell_{p,j}$ we have that $\forall n \in \mathbb{N} \; x_{n_j} = 0$. Hence, we may conclude that $\forall n \in \mathbb{N} \; |x_j| < \epsilon$. But that inequality doesn't depend on n, so we have: $|x_j| < \epsilon$. Since ϵ was arbitrary, we can deduce that:

$$\forall \epsilon > 0 \; |x_j| < \epsilon.$$

Which implies that $x_j = 0$, and hence $x \in \ell_{p,j}$. Since $(x_n)_{n\in\mathbb{N}}$ was arbitrary, we deduce that $\ell_{p,j}$ is closed.

Now suppose for contradiction $\ell_{p,j}$ contained an arbitrary open ball $B(x, \delta)$ for $x \in \ell_p$, $\delta > 0$.

Note that in this case $x \in \ell_{p,j}$ and by definition of an open ball:

$$\forall y \in \ell_p \; \|x - y\|_p < \delta \implies y \in B(x, \delta) \implies y \in \ell_{p,j}.$$

If we now define:

$$y := \begin{cases} x_k & k \neq j \\ \frac{\delta}{2} & k = j \end{cases}.$$

And check that indeed $y \in \ell_p$:

$$\|y\|_p \leq \|y - x\|_p + \|x\|_p = \frac{\delta}{2} + \|x\|_p < \infty,$$

because $x \in \ell_p$. We can observe that:

$$\|x - y\|_p = \left(\sum_{i\in N}^{\infty} |x_i - y_i|^p \right)^{\frac{1}{p}} = \frac{\delta}{2} < \delta.$$

Which implies that $y \in B(x, \delta) \subset \ell_{p,j}$. But clearly $y \notin \ell_{p,j}$, as, by definition, $y_j = \frac{\delta}{2} > 0$ Which is a contradiction hence $\ell_{p,j}$ doesn't contain any open ball in ℓ_p. ∎

Exercise I.5 (iv)

Let $S := \left\{ x \in \ell_p \mid \forall j \in \mathbb{N} \; |x_j| \leq Cj^{-\frac{2}{p}} \right\}$ for some $C \in (0, \infty)$, we'll show that S is closed in ℓ_p.
Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in S which converges to some $x \in \ell_p$ in ℓ_p w.r.t. $\|.\|_p$. We need to show that x belongs to S. By definition of S, take $j \in \mathbb{N}$ arbitrary. We need to show that

$$|x_j| \leq Cj^{-\frac{2}{p}}.$$

Let $\epsilon > 0$ be arbitrary, pick $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sum_{i\in N}^{\infty} |x_{n_i} - x_i|^p < \epsilon^p.$$

Now in particular, since each term of the sum above is non-negative, we must have that:

$$|x_{n_j} - x_j|^p < \epsilon^p \text{ and so } |x_{n_j} - x_j| < \epsilon. \tag{6}$$

Now using the triangle inequality, symmetry, the fact that $x_n \in S$, and inequality (6) we have $\forall n \geq N$:

$$|x_j| \leq |x_j - x_{n_j}| + |x_{n_j}| = |x_{n_j} - x_j| + |x_{n_j}| \leq |x_{n_j} - x_j| + Cj^{-\frac{2}{p}} < \epsilon + Cj^{-\frac{2}{p}}.$$

Hence we obtain that:

$$\forall \epsilon > 0 \; |x_j| < \epsilon + Cj^{-\frac{2}{p}}.$$

Therefore $|x_j| \leq Cj^{-\frac{2}{p}}$ and so $x \in S$. Since $(x_n)_{n\in\mathbb{N}}$ was arbitrary, we deduce that S is closed in ℓ_p