Probability and Statistics for JMC
Exercises 8 — Convergence Concepts

1. Suppose that X is a continuous random variable with pdf
$$f_X(x) = \exp\Big[-(x+2)\Big], \quad \text{for} \quad -2 < x < \infty.$$
Find the mgf of X and then use it to find the expectation and variance of X.

2. Using the Central Limit Theorem, construct normal approximations to each of the following random variables,
   (a) a Binomial distribution $X \sim \text{Binomial}(n, \theta)$;
   (b) a Poisson distribution $X \sim \text{Poisson}(\lambda)$.

3. Show that for any random variable, X with mean μ and variance σ^2,
$$\text{P}\Big(|X-\mu| \geq t\Big) \leq \frac{\sigma^2}{t^2}.$$

4. Suppose $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$. Let
$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$
   (a) Show that $\overline{X}$ is a consistent estimator of λ.
   (b) Suppose $T_n = \exp(-\overline{X})$, show that $T_n \xrightarrow{\mathcal{P}} \exp(-\lambda)$.

5. Markov's inequality states that, for any $r > 0$, and for an arbitrary random variable Y,
$$\text{P}(|Y| > \delta) \leq \frac{\text{E}(|Y|^r)}{\delta^r},$$
for all $\delta > 0$. Use this result to show that for independent and identically distributed random variables $X_1, \ldots X_n$ with mean μ and variance $\sigma^2 < \infty$, the sample mean $\overline{X}$ converges in probability to μ as $n \to \infty$.

6. Find the moment generating function of the random variable X which has density function

$$f_X(x) = \begin{cases} \frac{1}{2}x & 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Write down the moment generating function (MGF) of X, and by expanding the MGF as a power series in t, find the mean and variance of X.

Partial answers:

1. $M(t) = -\exp(-2t)/(t-1)$; $\mathrm{E}(X) = -1$; $\mathrm{Var}(X) = 1$.
2. (a) $\mathrm{N}(n\theta,\, n\theta(1-\theta))$; (b) $\mathrm{N}(\lambda, \lambda)$.
6. $\mathrm{E}(X) = 4/3$, $\mathrm{Var}(X) = 2/9$.