MATH50001 Analysis II, Complex Analysis

Lecture 17

Section: Evaluation of Definite integrals.

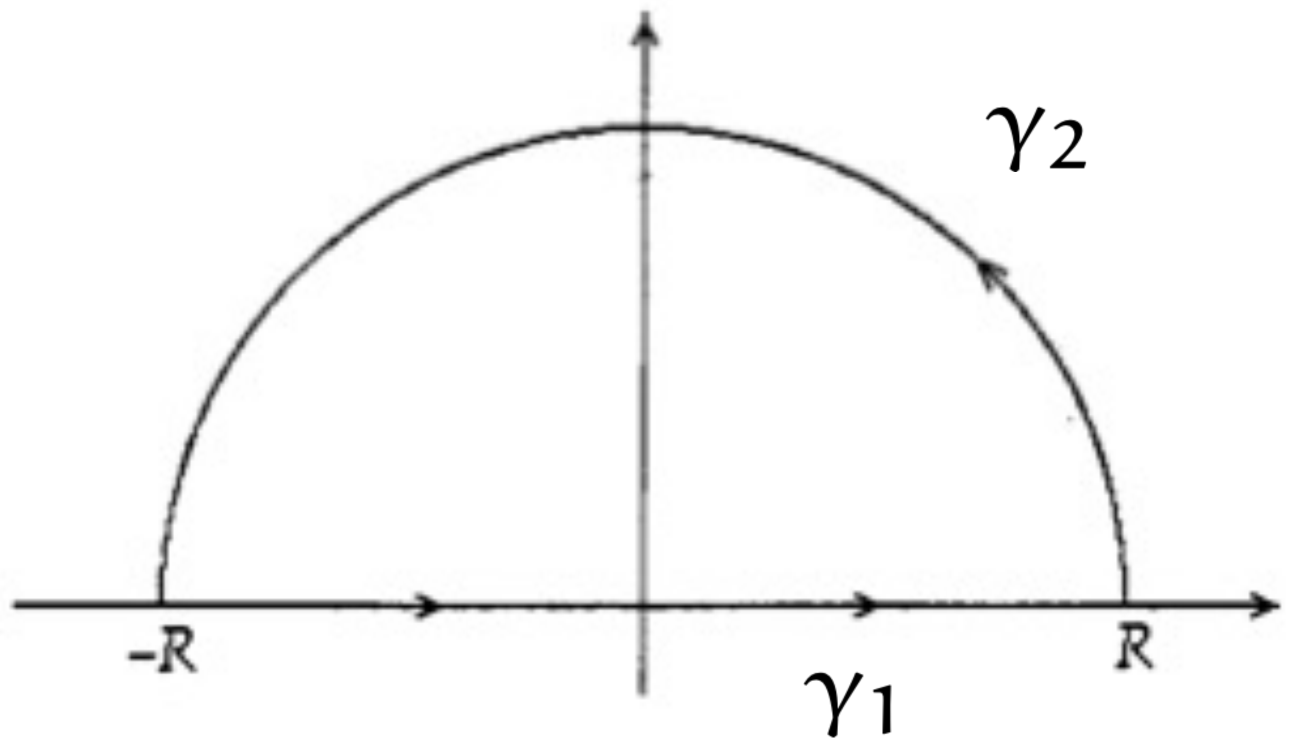
Example. Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

Solution. Consider

$$\oint_{\gamma} \frac{1}{1+z^2}\, dz,$$

where $\gamma = \gamma_1 \cup \gamma_2$.



$$\gamma_1 = \{z : z = x + i0,\ -R < x < R\},$$

$$\text{and} \quad \gamma_2 = \{z : z = R\,e^{i\theta},\ 0 \le \theta \le \pi\}, \quad R > 1.$$

The integrant $(1+z^2)^{-1}$ has simple poles at $\pm i$ and only the pole at i is interior to γ. Therefore

$$\oint_\gamma \frac{1}{1+z^2}\,dz = 2\pi i \operatorname{Res}\left[\frac{1}{1+z^2}, i\right] = 2\pi i \lim_{z\to i} \frac{z-i}{1+z^2} = 2\pi i\,\frac{1}{2i} = \pi.$$

Then

$$\pi = \int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{\gamma_2} \frac{1}{1+z^2}\,dz.$$

Note that by using the ML-inequality we have

$$\left|\int_{\gamma_2} \frac{1}{1+z^2}\,dz\right| \le \frac{1}{R^2-1} R\pi \to 0, \qquad R \to \infty.$$

Finally we have

$$\pi = \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{\gamma_2} \frac{1}{1+z^2}\,dz\right) = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

Example. Evaluate

$$\int_0^\infty \frac{1}{1+x^3}\,dx.$$

Solution. Consider

$$\oint_\gamma \frac{1}{1+z^3}\,dz, \qquad \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3,$$

where

$$\gamma_1 = \{z : z = x + iy, x \in [0, R], y = 0\}, \quad R > 1,$$
$$\gamma_2 = \{z : z = R\,e^{i\theta}, 0 \le \theta \le 2\pi/3\},$$
$$\gamma_3 = \{z : z = r\,e^{i2\pi/3}, r \in [R, 0]\}$$

The function $1+z^3$ has three zeros

$$z_1 = e^{i\pi/3}, \quad z_2 = e^{i\pi} \quad \text{and} \quad z_3 = e^{5i\pi/3},$$

of which only z_1 is internal for γ. Therefore

$$\begin{aligned}\oint_\gamma \frac{1}{1+z^3}\,dz &= 2\pi\,i\,\mathrm{Res}\left[\frac{1}{1+z^3}, e^{i\pi/3}\right] \\ &= 2\pi\,i \lim_{z\to e^{i\pi/3}} \frac{z-e^{i\pi/3}}{1+z^3} \\ &= 2\pi\,i \lim_{z\to e^{i\pi/3}} \frac{1}{3z^2} = 2\pi\,i\,\frac{1}{3}\,e^{-2i\pi/3} = \frac{2}{3}\,\pi\,i\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\ &= \frac{\pi\sqrt{3}}{3} - i\,\frac{\pi}{3}.\end{aligned}$$

Note that

$$\lim_{R\to\infty}\int_{\gamma_1}\frac{1}{1+z^3}\,dz=\lim_{R\to\infty}\int_0^R\frac{1}{1+x^3}\,dx=\int_0^\infty\frac{1}{1+x^3}\,dx.$$

Moreover by using that $|1+R^3e^{i3\theta}|>|R^3-1|$ and the ML-inequality we have

$$\begin{aligned}\Big|\int_{\gamma_2}\frac{1}{1+z^3}\,dz\Big|&=\Big|\int_0^{2\pi/3}\frac{1}{1+R^3\,e^{i3\theta}}\,iR\,e^{i\theta}\,d\theta\Big|\\&\le\frac{R}{R^3-1}\cdot\frac{2\pi}{3}\to 0,\quad\text{as}\quad R\to\infty.\end{aligned}$$

The integral over γ_3 equals

$$\begin{aligned}\int_{\gamma_3}\frac{1}{1+z^3}\,dz&=\int_R^0\frac{1}{1+r^3\,e^{i2\pi3/3}}\,e^{i2\pi/3}\,dr\\&=-\Big(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\Big)\int_0^R\frac{1}{1+r^3}\,dr\to\Big(\frac{1}{2}-i\frac{\sqrt{3}}{2}\Big)\int_0^\infty\frac{1}{1+r^3}\,dr,\\&\qquad\text{as}\quad R\to\infty.\end{aligned}$$

Finally we obtain

$$
\begin{aligned}
\frac{\pi\sqrt{3}}{3} - i\,\frac{\pi}{3} = \frac{\pi}{3}\,(\sqrt{3} - i) \\
&= \int_0^\infty \frac{1}{1+x^3}\,dx + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \int_0^\infty \frac{1}{1+r^3}\,dr \\
&= \left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right) \int_0^\infty \frac{1}{1+x^3}\,dx = \frac{\sqrt{3}}{2}\,(\sqrt{3} - i) \int_0^\infty \frac{1}{1+x^3}\,dx.
\end{aligned}
$$

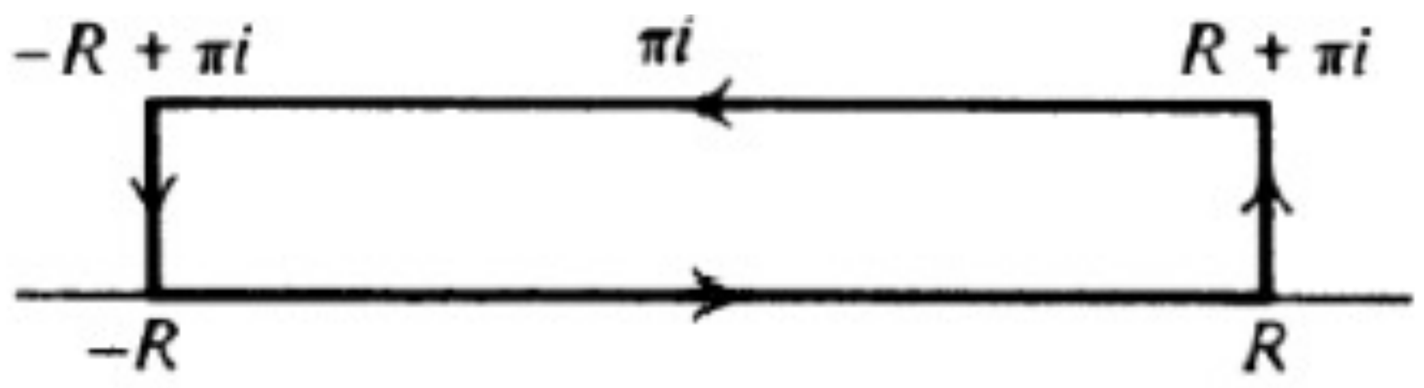
This implies

$$
\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi}{3\sqrt{3}}.
$$

Example. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}}\, dx.$$

Solution. Let introduce the contour



$$\begin{aligned}\gamma &= \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \\ &= [-R, R] \cup [R, R+i\pi] \cup [R+i\pi, -R+i\pi] \cup [-R+i\pi, -R]\end{aligned}$$

Let $f(z) = e^{iz}/(e^z + e^{-z})$. The singularities of f are solutions of the equation $e^z + e^{-z} = 0$, or

$$e^{2x}e^{2iy} = -1.$$

Solutions of this equation are $x = 0$, $y = \pi/2 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. The only singularity of f in the interior of the counter γ is at $z_0 = i\pi/2$ and

$$\mathrm{Res}\left[\frac{e^{iz}}{e^z + e^{-z}}, i\pi/2\right] = \lim_{z \to i\pi/2} \frac{(z - i\pi/2)e^{-\pi/2}}{e^z + e^{-z}} = \frac{e^{i(i\pi/2)}}{2i}.$$

Therefore

$$\oint_\gamma \frac{e^{iz}}{e^z + e^{-z}}\, dz = 2\pi i \cdot \frac{e^{i(i\pi/2)}}{2i} = \pi e^{-\pi/2}.$$

The integral over γ_2 can be estimated as follows

$$\Big|\int_{\gamma_2} \frac{e^{iz}}{e^z+e^{-z}}\,dz\Big| \le \pi \max_{0\le y\le\pi} \left|\frac{e^{iR}\,e^{-y}}{e^R\,e^{iy}+e^{-R}\,e^{-iy}}\right|$$

$$\le \pi \max_{0\le y\le\pi} \frac{e^{-y}}{e^R|e^{iy}+e^{-2R}e^{-iy}|} \le \frac{1}{e^R(1-e^{-2R})} \to 0,$$

as $R\to\infty$.

A similar argument proves the same result for the integral of f over γ_4.

$$\begin{aligned}\int_{\gamma_3} \frac{e^{iz}}{e^z + e^{-z}}\, dz &= \int_R^{-R} \frac{e^{ix-\pi}}{e^{x+i\pi} + e^{-x-i\pi}}\, dx \\ &= e^{-\pi} \int_R^{-R} \frac{e^{ix}}{-e^x - e^{-x}}\, dx = e^{-\pi} \int_{-R}^{R} \frac{e^{ix}}{e^x + e^{-x}}\, dx \\ &= e^{-\pi} \int_{-R}^{R} \frac{\cos x}{e^x + e^{-x}}\, dx.\end{aligned}$$

Therefore

$$(1 + e^{-\pi}) \int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}}\, dx = \pi\, e^{-\pi/2}$$

and finally

$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}}\, dx = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}.$$

Example. Evaluate

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx.$$

Solution. Introduce the following function

$$f(z) = \frac{(\log z - i\pi/2)^2}{1+z^2}$$

and take the branch of the logarithm given by the cut $-\pi/2 < \theta \leq 3\pi/2$.
Consider $\gamma = \gamma_R \cup \gamma_1 \cup \gamma_r \cup \gamma_2$, where

$$\gamma_R = R\,e^{i\theta}, \quad R >> 1, \quad \theta \in [0,\pi],$$

$$\gamma_1 = \{z :\ z = x + i0,\ x \in [-R,-r]\}, \qquad r << 1,$$

$$\gamma_r = r\,e^{i\theta}, \qquad \theta \in [\pi, 0],$$

$$\gamma_2 = \{z :\ z = x + i0,\ x \in [r, R]\}.$$

The only singularity of f which is internal for γ is $z_0 = i$ and

$$\operatorname{Res}\left[\frac{(\log z - i\pi/2)^2}{1+z^2}, i\right] = \frac{2(\log i - i\pi/2)}{2i\, i} = 0.$$

This explains why we have the strange constant $i\pi/2$ in the definition of f. So

$$\oint_\gamma \frac{(\log z - i\pi/2)^2}{1+z^2}\, dz = 0.$$

Note that $\log z - i\pi/2 = \ln|z| + i(\theta - \pi/2)$, where $\theta \in (-\pi/2, 3\pi/2]$.
By using the ML-inequality we obtain

$$\left|\int_{\gamma_R} \frac{(\log z - i\pi/2)^2}{1+z^2}\, dz\right| \leq \frac{(\ln R)^2 + \pi^2}{R^2 - 1} \cdot \pi R \to 0,$$

as $R \to \infty$.

The integral over γ_r equals

$$\Big|\int_{\gamma_r} \frac{(\log z - i\pi/2)^2}{1+z^2}\, dz\Big| \le \frac{(\ln r)^2 + \pi^2}{1-r^2} \cdot \pi\, r \to 0,$$

as $r \to 0$.

$$\int_{\gamma_1} \frac{(\log z - i\pi/2)^2}{1+z^2}\, dz = \int_{-R}^{-r} \frac{(\ln|x| + i\pi/2)^2}{1+x^2}\, dx = \int_{r}^{R} \frac{(\ln|x| + i\pi/2)^2}{1+x^2}\, dx$$

and

$$\int_{\gamma_2} \frac{(\log z - i\pi/2)^2}{1+z^2}\, dz = \int_{r}^{R} \frac{(\ln|x| - i\pi/2)^2}{1+x^2}\, dx.$$

Letting $R \to \infty$ and $r \to 0$ we get

$$2\int_0^\infty \frac{(\ln|x|)^2}{1+x^2}\,dx - 2\,\frac{\pi^2}{4}\int_0^\infty \frac{dx}{x^2+1} = 0.$$

Therefore

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^2}{4}\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi^2}{4}\,\arctan x\Big|_0^\infty = \frac{\pi^3}{8}.$$

Thank you