Complex Analysis Spring 2022 Coursework

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Q1.

Since we know that the function Log z is holomorphic everywhere in $\mathbb{C} \setminus (-\infty, 0)$, and also that e^z and sin z are entire, we may deduce that $\sqrt{z}$ defined as $e^{\frac{1}{2}\text{Log}z}$ is holomorphic everywhere in $\mathbb{C} \setminus (-\infty, 0)$ and so by composition the function $f(z) = 2\text{sin}(\sqrt{z})$ is holomorphic inside of that set. Now since $i\frac{\pi^2}{2} \in \mathbb{C} \setminus (-\infty, 0)$ we can apply the theorem in the notes:

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0)$$

And so we may calculate:

$$f'\left(i\frac{\pi^2}{2}\right) = \frac{\cos(\sqrt{i\frac{\pi^2}{2}})}{\sqrt{i\frac{\pi^2}{2}}}$$

Now let us compute

$$\sqrt{i\frac{\pi^2}{2}} = e^{\frac{1}{2}\text{Log}(i\frac{\pi^2}{2})} = e^{\frac{1}{2}\left(\log(\frac{\pi^2}{2})+i\frac{\pi}{2}\right)} = \frac{\pi}{\sqrt{2}}e^{i\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}(1+i).$$

Our expression becomes:

$$f'\left(i\frac{\pi^2}{2}\right) = \frac{\cos(\frac{\pi}{2}(1+i))}{\frac{\pi}{2}(1+i)}.$$

We can simplify it using the definition of complex cosine in the numerator and the multiplication by the conjugate in the denominator.

$$= \frac{1}{2}\frac{\left(e^{(i\frac{\pi}{2}(1+i))} + e^{(-i\frac{\pi}{2}(1+i))}\right)(1-i)}{\frac{\pi}{2}(1+i)(1-i)} = \frac{1}{\pi}\frac{\left(e^{(\frac{\pi}{2}(i-1))} + e^{(-\frac{\pi}{2}(i-1))}\right)(1-i)}{2} = \frac{1}{2\pi}\left(e^{(\frac{\pi}{2}(i-1))} + e^{(-\frac{\pi}{2}(i-1))}\right)(1-i)$$

If we now simplify it even further, we get:

$$= \frac{1}{2\pi}\left(e^{-\frac{\pi}{2}}\left(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\right) + e^{\frac{\pi}{2}}\left(\cos(\frac{\pi}{2}) - i\sin(\frac{\pi}{2})\right)\right)(1-i).$$

Now by evaluating sin and cos in the expression above, we get:

$$= \frac{1}{2\pi}\left(e^{-\frac{\pi}{2}}i - e^{\frac{\pi}{2}}i\right)(1-i) = \frac{e^{-\frac{\pi}{2}} - e^{\frac{\pi}{2}}}{2\pi} + i\frac{e^{-\frac{\pi}{2}} - e^{\frac{\pi}{2}}}{2\pi}.$$

If we note that $\sinh x = \frac{e^x - e^{-x}}{2}$ we can rewrite the expression above as:

$$-\frac{\sinh(\frac{\pi}{2})}{\pi} - i\frac{\sinh(\frac{\pi}{2})}{\pi}$$

Q2.

(a)

Consider the following parameterisation of γ:

$$\gamma := \{z = \rho e^{i\theta} | \theta \in [0, 2\pi]\}$$

Now since $z = z(r, \theta)$ using the total differentiation, we obtain:

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{\partial z}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}\theta}.$$

Since γ is a circle, the radius is constant and so $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ is 0. Therefore the expression above becomes:

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\rho e^{i\theta}\right) = i\rho e^{i\theta}.$$

Hence we can deduce that the following holds:

$$\mathrm{d}z = i\rho e^{i\theta}\mathrm{d}\theta. \tag{1}$$

Now let us consider $|\mathrm{d}z|$

$$|\mathrm{d}z| = |i\rho e^{i\theta}\mathrm{d}\theta|.$$

Since we have parameterised θ to range from 0 to 2π we know that $\mathrm{d}\theta$ is real and positive and hence we can simplify:

$$|\mathrm{d}z| = |i\rho e^{i\theta}|\mathrm{d}\theta = |i||\rho e^{i\theta}|\mathrm{d}\theta = \rho\mathrm{d}\theta.$$

Now from (1) we can also deduce that $\mathrm{d}\theta = \dfrac{\mathrm{d}z}{i\rho e^{i\theta}} = \dfrac{\mathrm{d}z}{iz}$ Combining the above two together, we deduce:

$$|\mathrm{d}z| = \rho\frac{\mathrm{d}z}{iz} = -i\rho\frac{\mathrm{d}z}{z}.$$

(b)

In order to compute the integral in question we first substitute the result from part (a).

$$\oint_\gamma \frac{|\mathrm{d}z|}{|z-a|^2} = -\oint_\gamma \frac{\rho i}{z|z-a|^2}\mathrm{d}z.$$

We can apply the definition of the complex norm to the expression in the denominator to obtain:

$$= -\oint_\gamma \frac{\rho i}{z(z-a)(\overline{z-a})}\mathrm{d}z.$$

By the properties of the conjugate, it becomes:

$$= -\oint_\gamma \frac{\rho i}{z(z-a)(\bar{z}-\bar{a})}\mathrm{d}z.$$

Now if we note that $\bar{z} = \frac{|z|^2}{z}$ we can rewrite the integral as:

$$= -\frac{\rho i}{\bar{a}}\oint_\gamma \frac{\mathrm{d}z}{(z-a)(\frac{\rho^2}{\bar{a}}-z)}.$$

In order to compute the integral above, we need to consider two cases, the first one being when a is interior to γ and the second one when it is outside of the region enclosed by γ. Observe that in the first case, we clearly have $|a| < \rho$ and $|a| = |\bar{a}|$ hence for all z on and inside γ we have:

$$\left|\frac{\rho^2}{\bar{a}} - z\right| \geq \left|\frac{\rho^2}{\bar{a}}\right| - |z| = \frac{\rho^2}{|a|} - |z| > \rho - |z| \geq \rho - \rho = 0.$$

The last transition in the expression above is because if z is on or inside γ then necessarily $|z| \leq \rho$ Therefore we have managed to show that the norm of $(\frac{\rho^2}{\bar{a}} - z)$ is greater than zero for all z on and inside γ and therefore it is never zero, and so we may deduce that $\dfrac{1}{\frac{\rho^2}{\bar{a}} - z}$ is holomorphic on and inside γ and so we may apply the Cauchy's integral formula to evaluate:

$$-\frac{\rho i}{\bar{a}}\oint_\gamma \frac{\mathrm{d}z}{(z-a)(\frac{\rho^2}{\bar{a}}-z)} = -\frac{\rho i}{\bar{a}}\left(2\pi i \frac{1}{\frac{\rho^2}{\bar{a}} - a}\right) = \frac{2\pi\rho}{\rho^2 - |a|^2}.$$

Now in the case when a is outside the region enclosed by γ, then clearly $(a-z)$ is not zero for all points z inside and on γ. Therefore $\frac{1}{z-a}$ is holomorphic on and inside γ. Now we just need to show that $\frac{\rho^2}{\bar{a}}$ is interior to γ in order to be able to apply the Cauchy's integral formula. Observe that since a is outside of the region enclosed by γ, we have $|\bar{a}| > \rho$ therefore we may deduce that $\frac{\rho^2}{|\bar{a}|} < \rho$ and so necessarily $\frac{\rho^2}{\bar{a}}$ is interior to γ and so we may apply the Cauchy's integral formula by letting $f(z) = \frac{1}{z-a}$ around the point $\frac{\rho^2}{\bar{a}}$. Hence we obtain:

$$-\frac{\rho i}{\bar{a}}\oint_\gamma \frac{\mathrm{d}z}{(z-a)(\frac{\rho^2}{\bar{a}}-z)} = \frac{\rho i}{\bar{a}}\oint_\gamma \frac{\mathrm{d}z}{(z-a)(z-\frac{\rho^2}{\bar{a}})} = \frac{\rho i}{\bar{a}}\left(2\pi i \frac{1}{\frac{\rho^2}{\bar{a}} - a}\right) = -\frac{2\pi\rho}{\rho^2 - |a|^2}.$$

Q3.

(a)

In order to show the identities required in the question let us consider the following series for $0 < \theta < 2\pi$

$$\sum_{k=0}^{n} e^{ik\theta} = \sum_{k=0}^{n} \cos(k\theta) + i \sum_{k=0}^{n} \sin(k\theta). \tag{2}$$

If we consider the left-hand side of the equation above as a geometric series, we obtain:

$$\sum_{k=0}^{n} e^{ik\theta} = \frac{1 - (e^{i\theta})^{n+1}}{1 - (e^{i\theta})}.$$

We can now rewrite it and remove the imaginary numbers from the denominator, by multiplying by the conjugate:

$$= \frac{1 - \cos((n+1)\theta) - i\sin((n+1)\theta)}{(1 - \cos\theta) - i\sin\theta} \frac{(1 - \cos\theta) + i\sin\theta}{(1 - \cos\theta) + i\sin\theta}.$$

After multiplying out and simplifying terms using the following trigonometric identities:

$$\cos\theta\cos((n+1)\theta) + \sin\theta\sin((n+1)\theta) = \cos((n+1)\theta - \theta) = \cos(n\theta),$$

$$\sin((n+1)\theta)\cos\theta - \cos((n+1)\theta)\sin\theta = \sin((n+1)\theta - \theta) = \sin(n\theta),$$

we get:

$$= \frac{1 - \cos\theta + \cos(n\theta) - \cos((n+1)\theta) + i(\sin\theta + \sin(n\theta) - \sin((n+1)\theta)}{2(1 - \cos\theta)}. \tag{3}$$

Let us now consider the real part of the expression above. Observe that we can factor out $\frac{1}{2}$:

$$\frac{1 - \cos\theta + \cos(n\theta) - \cos((n+1)\theta)}{2(1 - \cos\theta)} = \frac{1}{2} + \frac{\cos(n\theta) - \cos((n+1)\theta)}{1 - \cos\theta}. \tag{4}$$

Now let us use the following trigonometric identities:

$$\cos(2\theta) = 1 - 2\sin^2\theta \implies 1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right).$$

$$\cos((n+1)\theta) = \cos\left(\left(n + \frac{1}{2}\right)\theta + \frac{\theta}{2}\right) = \cos\left(\left(n + \frac{1}{2}\right)\theta\right)\cos\left(\frac{\theta}{2}\right) - \sin\left(\left(n + \frac{1}{2}\right)\theta\right)\sin\left(\frac{\theta}{2}\right).$$

$$\cos(n\theta) = \cos\left(\left(n + \frac{1}{2}\right)\theta - \frac{\theta}{2}\right) = \cos\left(\left(n + \frac{1}{2}\right)\theta\right)\cos\left(\frac{\theta}{2}\right) + \sin\left(\left(n + \frac{1}{2}\right)\theta\right)\sin\left(\frac{\theta}{2}\right).$$

If we apply the above to (4), we obtain:

$$= \frac{2\sin\left(\left(n + \frac{1}{2}\right)\theta\right)\sin\left(\frac{\theta}{2}\right)}{4\sin^2\left(\frac{\theta}{2}\right)} = \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}.$$

And now by (2) we know that the expression above being the real part of the geometric sum has to be equal to $\sum_{n=0}^{n} \cos(n\theta)$

For the second identity we need to consider the imaginary part of the expression (3).

$$\frac{(\sin\theta + \sin(n\theta) - \sin((n+1)\theta)}{2(1-\cos\theta)}.$$

For the denominator we use the same identity as before and for the numerator we use the following two identities

$$\sin((n+1)\theta) = \sin\left(\left(n+\frac{1}{2}\right)\theta + \frac{\theta}{2}\right) = \sin\left(\left(n+\frac{1}{2}\right)\theta\right)\cos\left(\frac{\theta}{2}\right) + \cos\left(\left(n+\frac{1}{2}\right)\theta\right)\sin\left(\frac{\theta}{2}\right).$$

$$\sin(n\theta) = \sin\left(\left(n+\frac{1}{2}\right)\theta - \frac{\theta}{2}\right) = \sin\left(\left(n+\frac{1}{2}\right)\theta\right)\cos\left(\frac{\theta}{2}\right) - \cos\left(\left(n+\frac{1}{2}\right)\theta\right)\sin\left(\frac{\theta}{2}\right).$$

Applying those yields:

$$\frac{(\sin\theta - 2\cos\big((n+\frac{1}{2})\theta\big)\sin\big(\frac{\theta}{2}\big)}{4\sin^2(\frac{\theta}{2})} = \frac{\sin\theta}{4\sin^2(\frac{\theta}{2})} - \frac{\cos\big((n+\frac{1}{2})\theta\big)}{2\sin\big(\frac{\theta}{2}\big)}.$$

Now since $\sin\theta = 2\sin\big(\frac{\theta}{2}\big)\cos\big(\frac{\theta}{2}\big)$, we obtain

$$\frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \frac{\cos\big((n+\frac{1}{2})\theta\big)}{2\sin\big(\frac{\theta}{2}\big)}.$$

And we deduce that it must be equal to the imaginary part of (2) which we had to show.

(b)

In order to find the Taylor series in question, let us first split $f(z)$ using partial fractions

$$\frac{1}{(z+1)(z+2)} = \frac{1}{(z+1)} - \frac{1}{(z+2)}.$$

Note that now we can compute the n-th derivative of $f(z)$

$$f^{(n)} = (-1)^n n! \left[\frac{1}{(z+1)^{n+1}} - \frac{1}{(z+2)^{n+1}}\right].$$

Hence the Taylor series of $f(z)$ is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n n! \left[\frac{1}{(i+1)^{n+1}} - \frac{1}{(i+2)^{n+1}}\right]}{n!}(z-i)^n.$$

That in turn simplifies into

$$f(z) = \sum_{n=0}^{\infty}(-1)^n \left[\frac{1}{(i+1)^{n+1}} - \frac{1}{(i+2)^{n+1}}\right](z-i)^n.$$

The radius of convergence of that expansion is given by the distance from i to the closest point where $f(z)$ is not holomorphic. Clearly, $f(z)$ is not holomorphic at -1 and -2 and so the closest to i of the two is -1. Hence, as $|i-(-1)| = \sqrt{2}$ we deduce that the disc of convergence is given by

$$|z-i| < \sqrt{2}.$$

Q4.

First observe that for all $n \in \mathbb{N}$ we have:

$$\sqrt{\frac{n}{2+n}} < 1.$$

That is because, clearly $\frac{n}{2+n} < \frac{n}{n}$ and the square root is monotone increasing. Now for all $n \in \mathbb{N}$ define

$$\gamma_n = \{z = re^{i\theta} | r = \sqrt{\frac{n}{2+n}} \text{ and } \theta \in [0, 2\pi]\}.$$

Clearly, each one of those paths is contained in $\mathbb{D}$ and so we deduce that for all $n \in \mathbb{N}$, $f(z)$ is holomorphic on and inside γ_n, hence we may apply the Cauchy's integral formula around 0:

$$|f^{(n)}(0)| = \left| \frac{n!}{2\pi i} \oint_{\gamma_n} \frac{f(\eta)}{(\eta)^{n+1}} \mathrm{d}\eta \right|.$$

By using the properties of the complex norm, the M-L inequality, and the fact that on γ_n we have $|\eta| = r = \sqrt{\frac{n}{2+n}}$ we may bound the norm of the integral above.

$$\left| \frac{n!}{2\pi i} \oint_{\gamma_n} \frac{f(\eta)}{(\eta)^{n+1}} \mathrm{d}\eta \right| \leq \frac{n!}{2\pi} \sup_{\eta \in \gamma_n} \left| \frac{f(\eta)}{(\eta)^{n+1}} \right| 2\pi r = n! \sup_{\eta \in \gamma_n} \frac{|f(\eta)|}{r^n}.$$

Now by the given assumption for all η in γ_n we have $|f(\eta)| \leq \frac{1}{1-|\eta|^2} = \frac{1}{1-r^2}$ And so we get

$$n! \sup_{\eta \in \gamma_n} \frac{|f(\eta)|}{r^n} \leq n! \frac{1}{1-r^2} \frac{1}{r^n}.$$

After substituting our defined value of $r = \sqrt{\frac{n}{2+n}}$ we get

$$n! \frac{1}{1-r^2} \frac{1}{r^n} = n! \frac{1}{1-\frac{n}{2+n}} \left(\frac{2+n}{n} \right)^{\frac{n}{2}} = n! \frac{2+n}{2} \left(\frac{2+n}{n} \right)^{\frac{n}{2}} = \frac{n!(2+n)^{\frac{(2+n)}{2}}}{2n^{\frac{n}{2}}}.$$

Which we had to show.