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MATH 50006
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Exercise Sheet 3

1. Show the following.

a) Any continuous function from $\mathbb{R}$ to $\mathbb{R}$ is Borel-measurable (i.e. $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable).

b) Any non-decreasing function from $\mathbb{R}$ to $\mathbb{R}$ is Borel-measurable.

c) Let f be a differentiable function from $(0, 1)$ to $\mathbb{R}$. Then f' is Borel measurable.

2. (σ-algebra generated by functions).

a) Let X be a space, $(Y, \mathcal{Y})$ a measurable space and $f : X \to Y$. Define

$$\sigma(f) = \{f^{-1}(A) : A \in \mathcal{Y}\}$$

Show that $\sigma(f)$ is a σ-algebra (the smallest σ-algebra on X such that f is measurable).

b) Let Y be a set and $(Y_i, \mathcal{B}_i)_{i\in I}$ be a family of measurable spaces. In the sequel, for an arbitrary family of functions $f_i : Y \to Y_i$, $i \in I$, we define

$$\sigma(f_i, i \in I) \overset{\text{def.}}{=} \sigma\Big(\bigcup_{i\in I} \sigma(f_i)\Big).$$

Let $(X, \mathcal{A})$ be a measurable space and $g : X \to Y$. Show that g is measurable from $(X, \mathcal{A})$ to $(Y, \sigma(f_i, i \in I))$ if and only if, for all $i \in I$, $f_i \circ g$ is measurable from X to Y_i.

3. Let $(X, \mathcal{A})$ be a measurable space. We consider a sequence of Borel measurable maps $f_n : X \to \mathbb{R}$, $n \geq 1$.

a) Show that the set

$$\{x \in X : (f_n(x))_{n\geq 1} \text{ converges in } \mathbb{R}\}$$

is measurable.

b) Show that if $(f_n)_{n\geq 1}$ converges pointwise, that is, for all $x \in \mathbb{R}$, $(f_n(x))_{n\geq 1}$ converges in $\mathbb{R}$, then the map $\lim_{n\to\infty} f_n$ is Borel measurable from $(X, \mathcal{A})$ to $\mathbb{R}$.

c) Let $a \in \mathbb{R}$. Prove the Borel measurability of the map $g : X \to \mathbb{R}$ defined by

$$g(x) := \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if } (f_n(x))_{n\geq 1} \text{ converges in } \mathbb{R} \\ a & \text{otherwise.} \end{cases}$$

4. (Approximating the measure of a set, cf. Proposition 1.22).

Let $\mathcal{A}$ be an algebra on a space X and μ a measure on $(X, \sigma(\mathcal{A}))$, which is σ-finite on $\mathcal{A}$, i.e. there exists a sequence of sets $S_1, S_2, \ldots$ such that $X = \bigcup_{n=1}^{\infty} S_n$, $S_n \in \mathcal{A}$ and $\mu(S_n) < \infty$ for all $n \geq 1$. Show that for all $\varepsilon > 0$ and $A \in \sigma(\mathcal{A})$, there exist mutually disjoint sets $A_1, A_2, \ldots$ with $A_n \in \mathcal{A}$ for all $n \geq 1$ such that

$$A \subset \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \mu\Big(\bigcup_{n=1}^{\infty} (A_n \setminus A)\Big) < \varepsilon.$$