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MATH 50006
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Exercise Sheet 4

1. Let $(X, \mathcal{A})$ be a measurable space.

a) Suppose $f, g : X \to \mathbb{R}$ are measurable. Prove that the sets

$$\{x \in X : f(x) > g(x)\}, \quad \{x \in X : f(x) = g(x)\}$$

are measurable.

b) Prove that the set of points at which a sequence of measurable real-valued functions converges is measurable.

2. For each of the following statements, give a proof or supply a counterexample.

a) Any continuous function on $\mathbb{R}$ is integrable with respect to the Lebesgue measure.

b) Any continuous function on $[0, 1]$ is integrable with respect to the Lebesgue measure.

c) If a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ is such that $\int_{\mathbb{R}} f \, d\lambda = 0$, then $f = 0$ almost-everywhere.

d) If f_n, f are measurable real-valued functions on a measure space $(X, \mathcal{A}, \mu)$, and $f_n \uparrow f$ as $n \to \infty$, then $\int f_n d\mu \uparrow \int f d\mu$ as $n \to \infty$.

e) If $(f_n)_{n \geq 1}$ is a sequence of nonnegative measurable functions on a measure space $(X, \mathcal{A}, \mu)$ such that $\sup_{n \geq 1} \int f_n d\mu < \infty$, and if $f_n \underset{n\to\infty}{\longrightarrow} f$ pointwise, then $\int f d\mu < \infty$.

3. If f, g are real valued integrable functions on a measure space (X, μ), show the following statements hold:

a) If $\mu(A) = 0$ then $\int_A f \, d\mu = 0$.

b) If $\int_A f \, d\mu = 0$ for every measurable set A then $f = 0$ μ almost-everywhere.

4. (Markov's inequality). Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be a nonnegative, measurable function on X. For all $M > 0$, show that $\int f \, d\mu \geq \int f \, 1_{\{f \geq M\}} \, d\mu$, and deduce that

$$\mu(\{f \geq M\}) \leq \frac{\int f \, d\mu}{M}.$$

5. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $(f_n)_{n \geq 1}$ be a sequence of nonnegative integrable functions converging μ-a.e. to an integrable function f. We assume that

$$\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu.$$

Show that $f_n \to f$ in $L^1(X, \mathcal{A}, \mu)$ (**Hint**: first show that $\lim_{n \to \infty} \int (f - f_n)^+ \, d\mu = 0$).

6. (Convergence in measure). Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$, and let f_n, $n \geq 1$, and f be measurable functions from $(X, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We say that f_n converges to f in measure if, for all $\epsilon > 0$,

$$\mu(\{|f_n - f| > \epsilon\}) \underset{n \to \infty}{\longrightarrow} 0.$$

a) Using Markov's inequality (Ex.4), show that if $\int |f_n - f| \, d\mu \underset{n \to \infty}{\longrightarrow} 0$, then f_n converges to f in measure. Show, with a counter-example, that the converse is wrong.

b) Show that if f_n converges to f μ-a.e., then f_n converges to f in measure. Show, with a counter-example, that the converse is wrong.