Imperial College London
Department of Mathematics
Lecturer: P.-F. Rodriguez

Tutorials Lead: Y. Shulzhenko
Assistants: R. Carini, J. McCarthy, S. Karwa, W. Turner

MATH 50006
Winter 2022

Exercise Sheet 5

A word on **notation:** the integral of f over the set A with respect to the Lebesgue measure λ is commonly expressed in the literature using any of the following notations (the first one being the one we introduced in class):

$$\int_A f d\lambda, \quad \int_A f(x) d\lambda(x), \quad \int_A f(x) dx, ...$$

These expressions are all equal by definition.

1. (Image measure).

a) Let $(X, \mathcal{A})$, $(Y, \mathcal{A}')$ be measurable spaces and $f : X \to Y$ be a measurable map. Let μ be a measure on $(X, \mathcal{A})$. Show that $\mu \circ f^{-1}$ given by

$$(\mu \circ f^{-1})(A') = \mu(f^{-1}(A')), \quad A' \in \mathcal{A}'$$

defines a measure on $(Y, \mathcal{A}')$ (the image measure of μ under f).

b) Let $n \geq 1$ be an integer and μ denote the uniform measure on $X = \{0, 1\}^n$ (endowed with the σ-algebra $\mathcal{A} = 2^X$), defined by setting $\mu(x) = 2^{-n}$, $x \in X$. For $x = (x_1, \ldots, x_n)$, let $f(x) = \sum_{i=1}^n x_i$. Determine $\mu \circ f^{-1}$. Note: f can be interpreted as counting the number of heads (=1's) when flipping a fair coin n times.

2. (Bounded convergence). Let $f_n : \mathbb{R} \to \mathbb{R}$ be measurable and uniformly bounded, i.e. there exists $M \geq 1$ such that $|f_n(x)| \leq M$ for all $x \in \mathbb{R}$ and $n \geq 0$. Show that if $f_n \to f$ μ-a.e. and f is measurable, then $\int_I |f_n - f| d\lambda \to 0$, for any bounded interval I.

3. (Fundamental theorem of calculus revisited). Let $[a, b] \subset \mathbb{R}$ be an interval, $-\infty < a < b < \infty$, λ the Lebesgue measure on $\mathbb{R}$. Prove the following:

a) If $\varphi : [a, b] \to \mathbb{R}$ is continuous, show that the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_{[a,x]} \varphi \, d\lambda, \qquad x \in [a, b]$$

is differentiable on $[a, b]$, and $F' = \varphi$.

b) If $f : [a, b] \to \mathbb{R}$ is a differentiable function with bounded derivative, show that

$$\int_{[a,b]} f' d\lambda = f(b) - f(a).$$

4. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a non-negative integrable function on X. Let $\epsilon > 0$ be fixed.

a) Show that there exists a $M > 0$ such that

$$\int f \, 1_{\{f \geq M\}} \, d\mu \leq \frac{\epsilon}{2}$$

b) Deduce that there exists a $\delta > 0$ such that, for all $A \in \mathcal{A}$ with $\mu(A) \leq \delta$,

$$\int_A f \, d\mu \leq \epsilon.$$

c) Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. If f is integrable with respect to the Lebesgue measure, show that the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \int_{(-\infty,x]} f \, d\lambda, \qquad x \in \mathbb{R}$$

is uniformly continuous on $\mathbb{R}$.

5. (Dominated Convergence Theorem revisited). Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$ and let f_n, $n \geq 1$, and f be measurable functions from $(X, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The following exercise shows that the conclusions of the DCT continue to hold if one weakens the assumption that f_n converges to f μ-a.e. to convergence in measure.

Assume that f_n converges to f in measure, and that there exists a non-negative integrable function $g : X \to \mathbb{R}$ such that $|f_n| \leq g$ μ-a.e., for all $n \geq 1$.

a) Show that $|f| \leq g$ μ-a.e..

b) Using Exercise 4b), show that

$$\int |f_n - f| \, d\mu \underset{n\to\infty}{\longrightarrow} 0.$$