MATH50001 Analysis II, Complex Analysis

Lecture 19

Conformal mappings.

Section: Preservation of angles.

Let us considers a smooth curve $\gamma \subset \mathbb{C}$ parametrised by $z(t) = x(t) + iy(t)$, $t \in [a, b]$. For each $t_0 \in [a, b]$ there is the direction vector

$$
\begin{aligned}
L_{t_0} &= \{(z(t_0) + tz'(t_0) : t \in \mathbb{R}\} \\
&\qquad\qquad = \{(x(t_0) + tx'((t_0) + i\,(y(t_0) + ty'(t_0)) : t \in \mathbb{R}\}\,.
\end{aligned}
$$

Consider now two curves γ_1 and γ_2 parametrised by the functions $z_1(t)$ and $z_2(t)$, $t \in [0, 1]$, respectively intersecting in the point $t = 0$, namely, $z_1(0) = z_2(0)$.

We then define the angle between the curves γ_1 and γ_2 to be the angle between the tangents, namely

$$\arg z_2'(0) - \arg z_1'(0).$$

We have the following result:

Theorem. (Angle preservation theorem)
Let f be holomorphic in an open subset set $\Omega \subset \mathbb{C}$. Suppose that two curves γ_1 and γ_2 lying inside Ω are parametrised by $z_1(t)$ and $z_2(t)$, $t \in [0, 1]$. Assume that $z_0 = z_1(0) = z_2(0)$ is their intersecting point and $z_1'(0)$, $z_2'(0)$ and also $f'(z_0)$ are all non-zero.

Then the angles between the curves $(z_1(t), z_2(t))$ and $(f(z_1(t)), f(z_2(t))$ at $t = 0$ satisfy

$$\arg z_2'(t) - \arg z_1'(t)\Big|_{t=0} = \arg\left(f(z_2(t))\right)' - \arg\left(f(z_1(t))'\right)\Big|_{t=0} \mod (2\pi).$$

Proof. Indeed,

$$\left.\frac{(f(z_1(t)))'}{(f(z_2(t)))'}\right|_{t=0} = \frac{f'(z_1(0))z_1'(0)}{f'(z_2(0))z_2'(0)} = \frac{f'(z_0)z_1'(0)}{f'(z_0)z_2'(0)} = \frac{z_1'(0)}{z_2'(0)}.$$

This implies

$$\arg\,(f \circ z_2))'(0) - \arg\,(f \circ z_1)'(0)) = \arg z_2'(0) - \arg z_1'(0) \mod (2\pi).$$

Remark.

The condition $f'(z_0) \neq 0$ in the Theorem is essential. For example, consider the holomorphic function $f(z) = z^2$ at $z_0 = 0$. The positive x-axis maps to itself, and the line $\theta = \pi/4$ maps to the positive y-axis. The angle between the lines doubles.

Remark.

The theorem states that it is not only the value of the angle is preserved by f but also its orientation. Consider for example of a (nonholomorphic) f preserving the value of the angle but not the orientation

$$f(z) = \bar{z}$$

One can think of this mapping geometrically as reflection in the x-axis.

Definition. We say that a complex function f is conformal in an open set $\Omega \subset \mathbb{C}$ if it is holomorphic in Ω and if $f'(z) \neq 0$, $\forall z \in \Omega$.

For example, the function $f(z) = z^2$ is conformal in the open set $\mathbb{C} \setminus \{0\}$.

The angle preservation theorem tells us that conformal mappings preserve angles.

Definition. A holomorphic function is a local injection on an open set $\Omega \subset \mathbb{C}$ if for any $z_0 \in \Omega$ there exists $D = \{z : |z - z_0| < r\} \subset \Omega$ such that $f : D \to f(D)$ is injection.

Theorem.

If $f : \Omega \to \mathbb{C}$ is a local injection and holomorphic, then $f'(z) \neq 0$ for all $z \in \Omega$. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.

Proof. We argue by contradiction. Suppose that $f'(z_0) = 0$ for some $z_0 \in \Omega$. Then for a sufficiently small $r > 0$ there is $D = \{z : |z - z_0| < r\}$, $\overline{D} \subset \Omega$, such that

$$f(z) - f(z_0) = a\,(z - z_0)^k + g(z), \qquad z \in D,$$

where $a \neq 0$, $k \geq 2$ and $g(z) = O(|z - z_0|^{k+1})$. For sufficiently small $0 \neq w \in \mathbb{C}$ denote

$$f(z) - f(z_0) - w = F(z) + G(z),$$

where

$$F(z) = a\,(z - z_0)^k - w, \quad G(z) = g(z).$$

If $r > 0$ and $|w|$ are small enough then we have

$$|G(z)| < |F(z)|, \qquad z \in \{z : |z - z_0| = r\},$$

Rouche's theorem implies that $f(z) - f(z_0) - w$ has at least two zeros in D.

Note that since the zeros of holomorphic function are isolated and $f'(z_0) = 0$ then for a sufficiently small r it follows $f'(z) \neq 0$, $z \neq z_0$. Therefore the roots of $\varkappa(z) = f(z) - f(z_0) - w$ are **distinct**. Indeed, $\varkappa(z_0) = w \neq 0$. Hence if $\varkappa(z)$ has a root of degree at least two at some z_1 then $\varkappa'(z_1) = f'(z_1) = 0$ which is impossible.
This finally implies that f is not injective and gives contradiction.

Let $g = f^{-1}$ denote the inverse of f on its range, which we can assume is $V \subset \mathbb{C}$. Suppose $w_0 \in V$ and w is closed to w_0. Assuming $w = f(z)$ and $w_0 = f(z_0)$ with $w \neq w_0$ we find

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\frac{w-w_0}{g(w)-g(w_0)}} = \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}}.$$

Since $f'(z_0) \neq 0$ then letting $z \to z_0$ we conclude that g is holomorphic at w_0 and $g'(w_0) = 1/f'(g(w_0))$.

Section: Möbius Transformations.

Definition.

A Möbius transformation (that is also called a bilinear transformation) is a map

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where} \quad a, b, c, d \in \mathbb{C} \quad \text{and} \quad ad - bc \neq 0.$$

The condition $ad - bc \neq 0$ is necessary for the transformation to be non-trivial. Indeed, $ad - bc = 0$ gives $a/c = b/d =$ const and the transformation reduces to $f(z) =$ const.

It is clear that a Möbius transformation is holomorphic except for a simple pole at $z = -d/c$. Its derivative is the function

$$f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+b)^2} = \frac{ad-bc}{(cz+d)^2}$$

and therefore the mapping is conformal throughout $\mathbb{C} \setminus \{-d/c\}$.

Theorem.

The inverse of a Möbius transformation is a Möbius transformation. The composition of two Möbius transformations is a Möbius transformation.

Proof. It is easily to verify, that the Möbius transformation

$$g(w) = \frac{dw - b}{-cw + a}$$

is the inverse of $f(z) = \frac{az+b}{cz+d}$. Indeed,

$$\begin{aligned} g(f(z)) = \frac{d\,\frac{az+b}{cz+d} - b}{-c\,\frac{az+b}{cz+d} + a} &= \frac{d(az+b) - b(cz+d)}{-c(az+b) + a(cz+d)} \\ &= \frac{adz + db - bcz - db}{-caz - cb + acz + ad} = z. \end{aligned}$$

Composition of two Möbius transformations.

Given two Möbius transformations

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{and} \quad f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

an easy calculation gives

$$f_1 \circ f_2(z) = f_1(f_2(z)) = \frac{Az + B}{Cz + D},$$

where

$$A = a_1 a_2 + b_1 c_2,\ B = a_1 b_2 + b_1 d_2,\ C = c_1 a_2 + d_1 c_2,\ D = c_1 b_2 + d_1 d_2.$$

Thus $f_1 \circ f_2$ is a Möbius transformation. A simple computation gives

$$AD - BC = (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0.$$

Remark.

The composition of Möbius transformations in effect corresponds to matrix multiplication. Indeed,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Besides,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This is essentially the matrix of the inverse mapping $f(z) = \frac{az+b}{cz+d}$, since multiplication of all the coefficients by a non-zero complex constant does not change a Möbius transformation.

Special Möbius transformations.

Let

$$f(z) = \frac{az + b}{cz + d}$$

and consider the following cases:

$$\text{(M1)} \qquad z \mapsto az \quad (b = c = 0,\ d = 1);$$

if $|a| = 1$, $a = e^{i\theta}$, then this is a rotation by θ. If $a > 0$ then f corresponds to a dilation and if $a < 0$ the map consists of a dilation by $|a|$ followed by a rotation of π.

$$\text{(M2)} \qquad z \mapsto z + b \quad (a = d = 1, c = 0 - \text{translation by } b);$$

$$\text{(M3)} \qquad z \mapsto \tfrac{1}{z} \quad (a = d = 0,\ b = c = 1 - \text{inversion}).$$

In (M1), if $a = re^{i\theta}$, the geometrical interpretation is an expansion by the factor r followed by a rotation anticlockwise by the angle θ.

Theorem.

Every Möbius transformation

$$f(z) = \frac{az + b}{cz + d}$$

is a composition of transformations of type (M1), (M2) and (M3).

Thank you