Mathematics Year I, Calculus and Applications I Term 1

Most Important theorems, definitions and propositions.

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1 Limits

Note 1. The following basic trigonometric limits are useful:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \qquad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

2 Derivative of a function

Theorem 1. *If $f(x)$ is differentiable at $x = x_0$, then it is also continuous there.*

3 MVT and IVT

Theorem 2. *Let f be a function which is defined and differentiable on the open interval (a, b). Let c be a number in the interval which is a maximum for the function. Then $f'(c) = 0$.*

Theorem 3 (Extreme Value Theorem)**.** *Let $f(x)$ be continuous on the closed interval $[a, b]$. Then $f(x)$ has a maximum and a minimum on this interval. i.e. There exists c_1 and c_2 such that $f(c_1) \geq f(x)$ and $f(c_2) \leq f(x) \; \forall \, x \in [a, b]$.*

Theorem 4. *Let $f(x)$ be continuous over the closed interval $a \leq x \leq b$ and differentiable on the open interval $a < x < b$. Assume also that $f(a) = f(b) = 0$. Then there exists a point c, $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem 5 (Mean Value Theorem)**.** *Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 6. *Let $f(x)$ be continuous on $[a, b]$. Given any number y between $f(a)$ and $f(b)$, there exists a point $x_y \in (a, b)$ such that $f(x_y) = y$.*

4 Inverse functions

Theorem 7. *Let $f(x)$ be a stricty increasing or decreasing function. Then the inverse fuction exists.*

Theorem 8. *If $f(x)$ is continuous on $[a, b]$ and strictly increasing (or decreasing), and $f(a) = y_a$ and $f(b) = y_b$, then $x = g(y)$ is defined on $[y_a, y_b]$.*

Theorem 9. *Let $f(x)$ be differentiable on (a, b) and $f'(x) > 0$ or $f'(x) < 0$ for all $x \in (a, b)$. Then the inverse function exists and we have*

$$g'(y) \left(= f^{-1}(y)\right) = \frac{1}{f'(x)}.$$

5 Exponentials and Logarithms

Definition 1 (Natural Logarithm)**.** $\log(x)$ is the area under the curve $\frac{1}{x}$ between 1 and x if $x \geq 1$ and negative of the area under the curve $\frac{1}{x}$ between 1 and x if $x \in (0, 1)$ In particular, $\log(0) = 1$.

6 Function Estimates

Theorem 10. *Let a be any positive number. Then $\frac{(1+a)^n}{n} \to \infty$ as $n \to \infty$.*

Theorem 11. *The function $f(x) = \frac{e^x}{x}$ is strictly increasing for $x > 1$ and $\lim_{x\to\infty} f(x) = \infty$ (exp beats x).*

Corollary 1. The functions $x = \log(x)$ and $\frac{x}{\log(x)}$ become arbitrarily large as x becomes arbitrarily large (x beats log).

Corollary 2. As x becomes large, $x^{1/x}$ approaches 1.

Theorem 12 ($\exp(x)$ beats any power of x)**.** *Let m be a positive integer. Then the function $f(x) = \frac{e^x}{x^m}$ is strictly increasing for $x > m$ and becomes arbitrarily large as x becomes arbitrarily large.*

7 L'Hopital's Rule

Theorem 13. *Let $f(x)$ and $g(x)$ be differentiable on an open interval containing x_0 (except possibly at x_0). Assume that $g(x) \neq 0$ and $g'(x) \neq 0$ for x in an interval about x_0 but with $x \neq x_0$. Assume also that f, g are continuous at x_0 with $f(x_0) = g(x_0) = 0$, and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = l$. Then also:*

$$\lim_{x\to x_0} \frac{f(x)}{g(x) = l}.$$

Theorem 14 (Cauchy Mean Value Theorem)**.** *Let f, g be continuous on $[a, b]$ and differentiable on (a, b) with $g(a) \neq g(b)$. Then there exists $c \in (a, b)$ such that*

$$g'(c)\frac{f(b) - f(a)}{g(b) - g(a)} = f'(c).$$

8 Integration

Definition 2 (Riemann Sum)**.** Let $(x_0, x_1, ..., x_n)$ be a partition of the interval (a, b). Let $f(x)$ be a function defined on (a, b). Take any sub-interval $[x_{i-1}, x_i]$ and let $x_i^* \in [x_{i-1}, x_i]$. Then the Riemann sum is defined as

$$\sum_{i=1}^{n} f(x_i^*)h,$$

where $h = x_i - x_{i-1}$.

9 MVT for Integrals

Theorem 15. *Let f be continuous on $[a, b]$. Then there exists a point $x_0 \in (a, b)$ such that*

$$f(x_0) = \frac{1}{b-a} \int_a^b f(x)\ dx.$$

10 Applications of Integration

10.1 Length of curves

The length of curve defined by a function $f(x)$ on the interval $[a, b]$ is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ \mathrm{d}x.$$

We can also derive a formula in parametric form

$$L = \int_{t_0}^{t_1} \left[\left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 \right]^{\frac{1}{2}} \mathrm{d}t.$$

10.2 Volumes of revolution

Given a area bounded by $x = a$, $x = b$, $y = f(x)$, $y = 0$ we want to find the corresponding solids of revolution.

The volume of the solid produced by revolving $y = f(x)$ about the x-axis is given by

$$V = \int_a^b \pi (f(x))^2 \mathrm{d}x$$

The volume of the solid produced by revolving $y = f(x)$ about the y-axis is given by

$$V = \int_a^b 2\pi x f(x) \mathrm{d}x$$

10.3 Surface areas of revolution

Given a curve defined by a function $f(x)$ on the interval $[a, b]$ we want to find the respective surface area of revolution obtained by revolving the curve about x-axis. It is given by

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x)^2}\ \mathrm{d}x$$

10.4 Centres of mass

First let us consider a simple one dimensional case with discrete masses in order to gain intuition for developing more sophisticated formulas. In this case if the centre of mass is at $x = \bar{x}$, then we must have a zero total moment at this point. That is

$$\sum m_k(\bar{x} - x_k) \text{ i.e. } \bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}$$

Now let us consider a two dimensional case with a continuous mass distribution. Given a area bounded by $x = a$, $x = b$, $y = f(x)$, $y = 0$ we want to find its center of mass. Suppose we have a center of mass at $(\bar{x}, \bar{y})$, we can consider the moments about x and y-axis separately. Then the x-coordinate of the center of mass is given by

$$\bar{x} = \frac{\int_a^b \frac{1}{2}(f(x))^2 \, \mathrm{d}x}{\int_a^b f(x) \, \mathrm{d}x}.$$

The y-coordinate of the center of mass is given by

$$\bar{y} = \frac{\int_a^b x f(x) \, \mathrm{d}x}{\int_a^b f(x) \, \mathrm{d}x}.$$

10.5 Length of curves and areas using polar coordinates

We can use the parametric form of the formula for the length of a curve in order to derive a formula in polar coordinates.

$$L = \int_{t_0}^{t_1} \left[\left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 \right]^{\frac{1}{2}} \mathrm{d}t.$$

Now in polar coordinates we have curves $r = f(\theta)$ so we use θ as a parameter. Since we know that

$$x = r\cos(\theta) = f(\theta)\cos(\theta) \text{ and } y = r\sin(\theta) = f(\theta)\sin(\theta),$$

We can substitute these equations into the formula above and obtain

$$L = \int_\alpha^\beta \left[(f'(\theta)\cos(\theta) - f(\theta)\sin(\theta))^2) + (f'(\theta)\sin(\theta) + f(\theta)\cos(\theta))^2 \right] \mathrm{d}\theta.$$

As we simplify the expression above, we obtain:

$$L = \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2} \, \mathrm{d}\theta.$$

Which can be written as

$$L = \int_\alpha^\beta \left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta} \right)^2 + r^2 \right]^{\frac{1}{2}} \mathrm{d}\theta.$$

11 Series, Power Series and Taylor's Theorem

11.1 Series and Convergence

Theorem 16. *If $\alpha > 1$ is a rational number, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \text{ converges.}$$

Proposition 1. If $\sum_{n=1}^{\infty} a_n$ converges, then for every N, the series $\sum_{n=N}^{\infty} \to 0$ as $N \to \infty$. Intuition behind this is that the tail of the series needs to go to zero if the series converges.

Theorem 17. *Suppose $(a_n)_{n\geq 1}$ is a decreasing sequence of positive numbers with $a_n \to 0$ as $n \to \infty$. Then the series $\sum_{n=1}^{\infty}(-1)^{n-1}\, a_n = a_1 - a_2 + a_3 - a_4 + ...$ converges.*

Theorem 18 (Comparison Test)**.** *Let $\sum_{n=1}^{\infty} b_n$ be convergent with b_n non-negative. If $|a_n| \leq b_n\ \forall n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ converges.*

Theorem 19 (Integral Test)**.** *Let $f(x)$ be a function which is defined for all $x \geq 1$, and is positive and decreasing. Then the series*

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ converges.

Theorem 20 (Ratio Test)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a series satisfying*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then:

1. *If $L < 1$ the series converges absolutely.*
2. *If $L > 1$ the series diverges.*
3. *If $L = 1$ the test is inconclusive.*

Theorem 21 (Root Test)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a series satisfying*

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L.$$

1. *If $L < 1$ the series converges absolutely.*
2. *If $L > 1$ the series diverges.*
3. *If $L = 1$ the test is inconclusive.*

11.2 Power Series

Theorem 22. *Assume that there is a number $R > 0$ such that $\sum_{n=0}^{\infty} |a_n| R^n$ converges. Then for all $|x| < R$, the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.*

Definition 3. The greatest value of R for which we get convergence is called the radius of convergence and $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x| < R$. It is very important to check $x = \pm R$ separately.

Theorem 23 (Ratio Test for power series)**.** *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and assume that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ exists. Let $R = \frac{1}{L}$. (If $L = 0$ let $R = \infty$, if $L = \infty$ let $R = 0$.)*
Then

1. *If $|x| < R$ the series converges absolutely.*
2. *If $|x| > R$ the power series diverges.*
3. *If $x = \pm R$ the power series could converge or diverge.*

Theorem 24 (Root Test for power series)**.** *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and assume that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$ exists. Then the radius of convergence of the power series is $R = \frac{1}{L}$.*

Theorem 25. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series which converges absolutely for $|x| < R$. Then $f(x)$ is differentiable for $|x| < R$, and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Theorem 26. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series which converges absolutely for $|x| < R$. Then in the interval $|x| < R$, we have*

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}.$$

We can summarize the two theorems above by stating that whenever we are within the radius of convergence, we can both differentiate and integrate the series term by term.

Theorem 27 (Algebraic operations)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R_1, and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ is another power series with radius of convergence R_2. Let*

$$R = min(R_1, R_2).$$

Then

1. *$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$ for $|x| < R$.*
2. *$cf(x) = \sum_{n=0}^{\infty} c a_n x^n$ for $|x| < R_1 (c \neq 0)$.*
3. *$f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_m b_{n-m}\right) x^n$ for $|x| < R$.*

11.3 Taylor Series

Theorem 28 (Taylor's theorem)**.** *Let f be a function defined on a closed interval between two numbers x_0 and x. Assume that the function has $n+1$ derivatives on the interval and that they are all continuous. Then*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0) + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n$$

Where the remainder R_n is given by

$$R_n = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

Using the integral MVT we can also derive an alternative form of the remainder term:

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Where c is a number between x_0 and x.

Note 2. The following Taylor's expansions are common and important to remember.

The Maclaurin series of e^x

$$e^x = \sum_{i=0}^{n} \frac{x^n}{n!} + R_n$$

Where the remainder term is given by (for c between 0 and x)

$$R_n = \frac{e^c}{(n+1)!} x^{n+1}.$$

The expansion of $\log(1+x)$

$$\log(1+x) = \sum_{i=1}^{n} (-1)^{i-1} \frac{x^i}{i} + R_{n+1}$$

Where R_n is given by

$$R_n = (-1)^n \int_0^x \frac{t^n}{1+t} \mathrm{d}t$$

11.3.1 Binomial Expansion

If $|x| < 1$ we have (for any real α)

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + ... = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!} x^n.$$

11.4 Fourier Series

11.4.1 Orthogonal and orthonormal function spaces

Definition 4. If f, g are real valued functions that are Riemann integrable on $[a, b]$, then we define the inner product of f and g, denoted by (f, g), by

$$(f, g) := \int_a^b f(x)g(x)\,\mathrm{d}x$$

$$\text{Note } (f,f)^{\frac{1}{2}} = \left(\int_a^b f^2 \mathrm{d}x \right)^{\frac{1}{2}} := ||f|| \geq 0.$$

Definition 5. Let $\mathbb{S} = \{\phi_0, \phi_1, ...\}$ be a collection fo functions that are Riemann integrable on $[a, b]$. If

$$(\phi_n, \phi_m) = 0 \text{ whenever } m \neq n$$

then $\mathbb{S}$ is an orthogonal system on $[a, b]$. If in addition, $||\phi_n|| = 1 \, \forall \, n$, then $\mathbb{S}$ is said to be orthonormal on $[a, b]$.

11.5 Periodic functions and periodic extensions

Definition 6. Given a function extended periodically over a given interval, we define the value at the points of discontinuity to be

$$f(\xi) = \frac{1}{2}\left[f(\xi_+) + f(\xi_-)\right]$$

11.5.1 Trigonometric Polynomials

The general form of a trigonometric polynomial is given by

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} \left[a_k \cos(k\omega x) + b_k \sin(k\omega x)\right]$$

11.5.2 Euler's relation

The following Euler's relation is very useful

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}.$$

From it we can derive the definitions for sin and cos in terms of e

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

And similarly for sin

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

11.5.3 Orthogonality of complex exponentials

$$\int_{-\pi}^{\pi} e^{inx} \mathrm{d}x = \begin{cases} 0 & n \neq 0, \\ 2\pi & n = 0. \end{cases}$$

Similarly for any integers m, n we have

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx} \mathrm{d}x = \begin{cases} 0 & n \neq m, \\ 2\pi & n = m. \end{cases}$$

11.5.4 Complex notation for trigonometric polynomials

Using the general form of a trigonometric polynomial and expressions for sin and cos in terms of e, we can derive the following complex form

$$S_n(x) = \sum_{k=-n}^{n} \gamma_k e^{ikx}$$

Where

$$\gamma_k = \begin{cases} \frac{1}{2}a_0 & k = 0, \\ \frac{1}{2}(a_k - ib_k) & k > 0, \\ \frac{1}{2}(a_{|k|} + ib_{|k|}) & k < 0. \end{cases}$$

11.6 Fourier series

Given a function $f(x)$ we want to represent it as a trigonometric polynomial

$$f(x) = S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)]$$

We can determine the coefficients using

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \mathrm{d}x$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \mathrm{d}x$$

Theorem 29. *The Fourier series* $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$ *formed by the Fourier coefficients* a_n *and* b_n *given above converges to the value* $f(x)$ *for any piecewise continuous function* $f(x)$ *of period* 2π *which has piecewise continuous derivatives of first and second order*. At any discontinuities, the value of the function bust be defined by* $f(\xi) = \frac{1}{2}[f(\xi_+) + f(\xi_-)]$.

Note 3 (*)**.** Actually we can relax the assumption of the second derivative. It is enough to have $f'(x)$ be piecewise continuous, i.e. the function is piecewise smooth. If $f(x)$ is continuous, the convergence is absolute and uniform. If it is discontinuous, we get absolute and uniform convergence everywhere except at the points of discontinuity.

Theorem 30 (Riemann-Lebesgue Lemma)**.** *If the function* $g(x)$ *is integrable on* $[a, b]$, *(e.g. it is piecewise continuous), then*

$$I_\lambda = \int_a^b g(x) sin(\lambda x)\, dx$$

tends to zero as $\lambda \to \infty$.

11.6.1 Complex form of Fourier series

We have already shown that we can express a trigonometric polynomial in terms of e. We can determine the formula for the γ_n coefficients using the formulas obtained before.

$$\begin{aligned}\gamma_n = \frac{1}{2}(a_n - ib_n) &= \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x)\cos(nx) - if(x)i\sin(nx))\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{d}x.\end{aligned}$$

Similarly

$$\begin{aligned}\gamma_{-n} = \frac{1}{2}(a_n + ib_n) &= \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x)\cos(nx) + if(x)i\sin(nx))\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{+inx}\mathrm{d}x.\end{aligned}$$

11.6.2 Fourier series on $2L$-periodic domains

Given a function $f(x)$ extended over a $2L$-periodic domain, the Fourier series is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

where

$$\begin{aligned}a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\end{aligned}$$

The complex form is

$$f(x) = \sum_{n=-\infty}^{\infty}\gamma_n e^{in\pi x/L} \quad |x| \leq L$$

where

$$\gamma_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\mathrm{d}x \quad n = 0, \pm 1, \pm 2, ...$$

11.7 Parseval's theorem

If $f(x)$ is represented by its Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right], \quad -\pi \leq x \leq \pi$$

then we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2\mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$