Mathematics Year I, Linear Algebra Term 1, 2

Most Important theorems, definitions and propositions.

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1 Introduction to matrices and vectors

Definition 1. The standard basis vectors for $\mathbb{R}^n$ are the vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{1}$$

Definition 2. Let $v_1, ...v_n$ be vectors in $\mathbb{R}^n$. Any expression of the form

$$\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n$$

is called linear combination of the vectors $v_1, ...v_n$.

Definition 3. The set of all linear combinations of a collection of vectors $v_1, ...v_n$ is called the span of the vectors $v_1, ...v_n$. Notation:

$$span\{v_1, ...v_n\}$$

.

Note 1. R^n is equal to the span of the standard basis vectors.

Definition 4. The norm of v is the non negative real number defined by

$$||v|| = \sqrt{v \cdot v}$$

.

Definition 5. A vector $v \in \mathbb{R}^n$ is called a unit vector if $||v|| = 1$.

Definition 6. Let u and v be vectors in $\mathbb{R}^n$. The distance between u and v is defined by

$$dist(u, v) := ||u - v||$$

.

Definition 7. The (i, j) entry of a matrix is the entry in row i and column j.

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & anm \end{pmatrix}$$

Most often we use the condensed notation $M = (a_{ij})$.

Definition 8. The transpose of an $n \times m$ matrix $A = (a_{ij})$ is the $m \times n$ matrix whose (i, j) entry is a_{ji}. We denote it as A^T.

The leading diagonal of a matrix is the $(1, 1), (2, 2)...$ entries. So the transpose is obtained by doing a reflection in the leading diagonal.

Definition 9. The identity matrix $I_n = (a_{ij})$ is the square matrix such that
$a_{ij} = 0 \; \forall \, i \neq j$, and $a_{ii} = 1$, where $0 < i, j \leq n$.

Definition 10. Let $A = (a_{ij})$ be a $n \times m$ matrix and $\mathbf{b}$ be the column vector of height n and whose ith entry is b_i. Then $(v_1, ..., v_n)$ is a solution to the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1m}x_m & = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2m}x_m & = b_2 \\ \vdots & = \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nm}x_m & = b_n \end{cases}$$

if and only if the vector $\mathbf{v} \in \mathbb{R}^n$ with entries v_i is a solution of the equation

$$Av = b$$

The matrix A is called the coefficient matrix of the system above. The augmented matrix associated to the system is the matrix obtained by adding b as an extra column to A. It is denoted as $(A|b)$.

2 Row operations

Definition 11. A **row operation** is one of the following procedures we can apply to a matrix:

1. $r_i(\lambda)$: Multiply each entry in the ith row by a real number $\lambda \neq 0$.
2. r_{ij} : Swap row i and row j.
3. $r_{ij}(\lambda)$: Add λ times row i to row j.

Proposition 1. Let $Ax = \mathbf{b}$ be a system of linear equations in matrix form. Let r be one of the row operations from Definition 11, and let $(A'|\mathbf{b}')$ be the result of applying r to the augmented matrix $(A|\mathbf{b})$. Then the vector $\mathbf{v}$ is a solution of $Ax = \mathbf{b}$ if and only if it is a solution of $A'x = \mathbf{b}'$.

3 A systematical way of solving linear systems.

Definition 12. The left-most non-zero entry in a non-zero row is called the **leading entry** of that row.

Definition 13. A matrix is in **echelon form** if

1. the leading entry in each non-zero row is 1,
2. the leading 1 in each non-zero row is to the right of the leading 1 in any row above it,
3. the zero rows are below any non-zero rows.

Definition 14. A matrix is in **row reduced echelon form (RRE)** if

1. it is in echelon form,
2. the leading entry in each non zero row is the only non-zero entry in its column.

Solution algorithm

If we have a system of equations

$$Ax = b$$

and A is in RRE form, then we can easily read off the solutions (if any exist). There are four cases to consider.

- **Case 1.** Every column of A contains a leading 1, and there are no zero rows. In this cas the only possibility is that $A = I_n$. Then the equations are simply

$$\begin{aligned} x_1 &= b_1 \\ x_2 &= b_2 \\ &\vdots \\ x_n &= b_n \end{aligned} \tag{2}$$

  and they have a unique solution, i.e. the entries of b.

- **Case 2.** Every column of A contains a leading 1, and there are some zero rows. Then A must have more rows than column, and it must be a matrix of the from

$$A = \begin{pmatrix} I_n \\ \mathbf{0}_{k\times n} \end{pmatrix} \tag{3}$$

  Which looks like an identity matrix with a block of zero rows underneath. In this case the respective equations are

$$\begin{aligned} x_1 &= b_1 \\ &\vdots \\ x_n &= b_n \end{aligned} \tag{4}$$

And the last k of them are

$$\begin{aligned} 0 &= b_{n+1} \\ &\vdots \\ x_n &= b_{n+k}. \end{aligned} \tag{5}$$

Now there are two possibilities:

1. If any of the last k entries of $\mathbf{b}$ are non-zero then this system has no solutions, i.e. it is inconsistent.
2. If the last k entries of $\mathbf{b}$ are all zero then the system has a unique solution, given by setting $x_i = b_i \; \forall \; i \in [1, n]$.

- **Case 3.** Some columns of A do not contain a leading 1, but there are no zero rows. For example

$$A = \begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

  If the ith column of A does not contain a leading 1, then the corresponding x_i is called a **free variable**. The remaining variables are called basic variables. This kind of system has infinitely many solutions, we call such system underdetermined.

See the full version of the notes for a step by step proof-algorithm how to turn any matrix into RRE form.

Now we have a systematic procedure for solving a system of simultaneous linear equations $Ax = \mathbf{b}$.

1. Form the augmented matrix $(A|\mathbf{b})$.
2. Apply the algorithm to put the augmented matrix into RRE form $(A'|\mathbf{b}')$.
3. Read off the solutions to $A'x = \mathbf{b}'$.

Note 2. In fact we only need to put the left block A' to be able to read off the solutions.

Proposition 2. The number of solutions to a system $Ax = \mathbf{b}$ is always either 0, 1 or ∞.

4 Matrix Multiplication

Remark 1. The easiest way to remember how to multiply matrices is as follows
In order to multiply AB where $A \in M_{n\times m}(\mathbb{R})$ and $B \in M_{m\times p}(\mathbb{R})$ we write $r_1, r_2, ..., r_n \in \mathbb{R}^m$ for the rows of A and $c_1, c_2, ..., c_n \in \mathbb{R}^m$ for the columns of B. Then the definitions states that the (i, j) entry of $AB \in M_{n\times p}(\mathbb{R})$ is the dot product

$$r_i^T \cdot c_j$$

We can view this as in order to determine one column of AB we take the respective column of B and scan A with it from top to bottom multiplying each row of A by the current column.

Definition 15. A matrix $A \in M_{n\times m}(R)$ defines a function from $\mathbb{R}^m$ to $\mathbb{R}^n$, which we denote by

$$T_A : \mathbb{R}^m \to \mathbb{R}^n$$
$$\mathbf{v} \mapsto A\mathbf{v}.$$

Provided that two matrices have correct dimensions, we can compose the functions defined by them.

Lemma 1. *$T_A \circ T_B = T_{AB}$, i.e for any $\mathbf{v} \in \mathbb{R}^p$ we have*

$$A(B\mathbf{v}) = (AB)\mathbf{v}.$$

Where $A \in M_{n\times m}(\mathbb{R})$ and $B \in M_{m\times p}(\mathbb{R})$

4.1 Matrix multiplication properties.

Proposition 3. Let $A, A' \in M_{m\times n}(\mathbb{R})$, let $B, B' \in M_{n\times p}(\mathbb{R})$ let $C \in M_{p\times q}(\mathbb{R})$ Then the following holds.

1. $A(BC) = (AB)C$ (associativity).
2. $A(B + B') = AB + AB'$

   and

   $(A + A')B = AB + A'B$

   (left and right distributivity of multiplication over addition.)
3. $\forall\, \lambda \in \mathbb{R}(\lambda A)B = \lambda(AB) = A(\lambda B)$.

The usual rules about multiplication by zero and one translate onto matrix multiplication with a certain degree of caution.

Lemma 2. *Let $A \in M_{n\times m}(\mathbb{R})$.*

1. *$\forall\, k \in \mathbb{N}$ $\mathbf{0}_{k\times n}A = \mathbf{0}_{k\times m}$ and $A\mathbf{0}_{m\times k} = \mathbf{0}_{n\times k}$.*
2. *$I_nA = A = AI_m$.*

Lemma 3. *Let $diag(d_1, ..., d_n)$ and $diag(d'_1, ..., d'_n)$ be two diagonal matrices $\in M_{n\times n}(\mathbb{R})$. Then their product is $diag(d_1d'_1, ..., d_nd'_n)$.*

5 Inverse of a matrix

Definition 16. Let $A \in M_{n\times m}(\mathbb{R})$ An $n \times n$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I_n$$

is called an **inverse** of A. A matrix A is called **invertible** if the inverse exists, **singular** if not.

Lemma 4. *1. If A is invertible, then the inverse is unique.*

2. If A is invertible and either $AB = I_n$ or $BA = I_n$ for some $B \in M_{n\times n}(\mathbb{R})$, then $B = A^{-1}$.

Lemma 5. *Suppose $A, B \in M_{n\times n}(\mathbb{R})$ are invertible. Then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Lemma 6. *Let $A \in M_{n\times n}(\mathbb{R})$.*

1. *If there exists $\mathbf{v} \neq 0 \in \mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$ then A is not invertible.*

2. *If there exists a non zero matrix $B \in M_{n\times n}(\mathbb{R})$ such that $AB = \mathbf{0}_{n\times n}$ or $BA = \mathbf{0}_{n\times n}$ then A is not invertible.*

Corollary 1. If $A \in M_{n\times n}(\mathbb{R})$ has a column of zeros, then A is not invertible.

Corollary 2. If $A \in M_{n\times n}(\mathbb{R})$ has a row of zeros, then A is not invertible.

Lemma 7. *If $A \in M_{n\times n}(\mathbb{R})$ and A' is obtained from A by a row operation, then A' is invertible if and only if A is.*

Lemma 8. *If $A \in M_{n\times n}(\mathbb{R})$ and A' is the RRE form of A , then A' is invertible if and only if it has no zero rows.*

Lemma 9. *A matrix $A \in M_{n\times n}(\mathbb{R})$ is invertible if and only if its RRE form is the identity matrix.*

Proposition 4. A matrix $A \in M_{n\times n}(\mathbb{R})$ is invertible if and only if there is no non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$.

A following algorithm can be used for computing the inverse of a matrix.

1. Write the augmented matrix $(A|I_n)$.

2. Row reduce $(A|I_n)$ and bring it into RRE form.

3. If A is invertible this process will transform the A to its RRE form and I_n to A^{-1}.

Remark 2. The algorithm described above gives us another method for finding solutions of linear systems of equations. Assuming we have such a system in matrix form $Ax = b$, and A is an invertible matrix, we can compute A^{-1} and rewrite our system as $x = A^{-1}b$. Note that it works if and only if A is invertible and therefore the system has a unique solution.

6 Vector spaces

Definition 17. A **vector space** is the following data:

- A set V. We will refer to the elements of V as vectors.
- A binary operation $+ : V \times V \to V$, which we call addition.
- A function $\mathbb{R} \times V \to V$, which we call scalar multiplication. We usually omit the symbol of it.

We require that the following axioms hold:

1. The set V with the binary operation $+$ forms an Abelian group. We denote the identity element of it as $\mathbf{0}_V$
2. $\forall\, \mathbf{v} \in V$ we have $1\mathbf{v} = \mathbf{v}$.
3. $\forall\, \lambda, \mu \in \mathbb{R}\ \forall\, \mathbf{v} \in V$ we have $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$.
4. $\forall\, \lambda, \in \mathbb{R}\ \forall\, \mathbf{u}, \mathbf{v} \in V$ we have $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$.

   (distributivity of scalar multiplication over addition)
5. $\forall\, \lambda, \mu \in \mathbb{R}\ \forall\, \mathbf{v} \in V$ we have $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$.

   (distributivity of scalar multiplication over scalar addition)

Lemma 10. *Let V be a vector space and let $\mathbf{x} \in V$.*

1. *For any positive integer $n \in \mathbb{R}$, we have*

$$n\mathbf{x} = \mathbf{x} + \mathbf{x} + ... + \mathbf{x}$$

   where there are n terms on the right-hand-side.

2. $0\mathbf{x} = (0)_V$.
3. $(-1)\mathbf{x}$ *is the additive inverse of* $\mathbf{x}$.

7 Subspaces

Definition 18. Let V be a vector space. A subset $U \subset V$ is called a subspace of V if:

1. If $\mathbf{x} + \mathbf{y} \in U$ then $\mathbf{x} + \mathbf{y} \in U$ (Closure under vector addition).
2. $\mathbf{0}_V \in U$.
3. If $\mathbf{x} \in U$ then $\forall\, \lambda \in \mathbb{R}$. $\lambda\mathbf{x} \in U$.

8 Spanning sets

Definition 19. Let $\mathcal{S} \subset V$ be any subset. A linear combination of elements in $\mathcal{S}$ is a vector $\mathbf{x} \in V$ which can be written as

$$\mathbf{x} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + ... + \lambda_n \mathbf{v_n}$$

where $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ are vectors in $\mathcal{S}$, and $\lambda_1, \lambda_\mathbf{2}, ..., \lambda_\mathbf{n}$ are real numbers.

Lemma 11. *A non-empty subset U of a vector space V is a subspace if and only if every linear combination of elements of U is again in U.*

Definition 20. Let V be a vector space and let $\mathcal{S} \subset V$ be a non-empty subset.
Then **span** of $\mathcal{S}$ written

$$\text{span } \mathcal{S} \subset V$$

is the set of all linear combinations of elements of $\mathcal{S}$. If $\mathcal{S} = \emptyset$ then we define span $\mathcal{S}$ to be $\{\mathbf{0}_V\}$.

Lemma 12. *Let V be a vector space and let $\mathcal{S} \subset V$ be any subset. Then* span $\mathcal{S}$ *is a subspace of V.*

Definition 21. Let V be a vector space. A subset $\mathcal{S} \subset V$ is called a **spanning set** if span $\mathcal{S} = V$.

Lemma 13. *Let V be a vector space, and let $\mathcal{S} \subset V$ be any subset. Suppose $\mathcal{S} \subset U$ for some subspace $U \subset V$. Then* span $\mathcal{S} \subset U$. *So if $\mathcal{S}$ is a spanning set for V, then $\mathcal{S}$ cannot be contained in any proper subspace $U \subsetneq V$*

Definition 22. A vector space is called **finite-dimensional** if it has a finite spanning set.

Definition 23. Let V be a finite-dimensional vector space. The dimension of V, denoted as dim V, is the smallest $n \in \mathbb{N}$ such that V has a spanning set of size n.

9 Linear independence

Definition 24. A subset $\mathcal{L}$ of a vector space V is called linearly dependent if we can find **distinct** vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n} \in \mathcal{L}$ and non-zero scalars $\lambda_1 \neq 0,\ \lambda_2 \neq 0,\ \dots, \lambda_n \neq 0$, such that

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + ... + \lambda_n \mathbf{v}_n = \mathbf{0}_V.$$

Note 3. Any subset of a linearly independent set is also linearly-independent.

Definition 25. A **basis** of a vector space is a linearly independent spanning set.

Proposition 5. Let $\mathcal{B} = \{\mathbf{v_1}, ..., \mathbf{v_n}\} \subset V$ be a finite basis of a vector space V. Then every vector in V can be written as as linear combination of elements in $\mathcal{B}$, in a unique way. Conversely, any finite subset $\mathcal{B}$ with this property is a basis.

10 Bases an dimension

Lemma 14. *Let $\mathcal{S} \subset V$ be a spanning set, and suppose that $\mathcal{S}$ is not linearly independent. Then there exists a vector $\mathbf{x} \in \mathcal{S}$ such that $\mathcal{S}' = \mathcal{S} \setminus \{\mathbf{x}\}$ is still a spanning set.*

Corollary 3. Any finite spanning set contains a basis.

Corollary 4. Any finite-dimensional vector space V has a basis.

Proposition 6 (Steinitz exchange lemma - easy verion)**.** Let $\mathcal{S} \subset V$ be a spanning set, and let $\mathbf{x} \in V$ be any non-zero vector. Then there exists a vector $\mathbf{y} \in \mathcal{S}$ such that the set

$$\mathcal{S}' = (\mathcal{S} \backslash \{\mathbf{y}\}) \cup \{\mathbf{x}\}$$

is still a spanning set.

Proposition 7 (Steinitz exchange lemma - full verion)**.** Let $\mathcal{S} \subset V$ be a spanning set, and let $\mathcal{L}\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ be a finite linearly independent subset of V. Then there exists a subset $\mathcal{T}\{\mathbf{y}_1, ..., \mathbf{y}_n\} \subset \mathcal{S}$, with the same size as $\mathcal{L}$, such that

$$\mathcal{S}' = (\mathcal{S} \backslash \mathcal{T}) \cup \mathcal{L}$$

is still a spanning set.

Corollary 5. Let V be a finite-dimensional vector space, let $\mathcal{S} \subset V$ be a finite spanning set and let $\mathcal{L} \subset V$ be a linearly independent subset. Then $\mathcal{L}$ is finite and $\#\mathcal{L} \leq \#\mathcal{S}$.

Theorem 1. *Let V be a finite-dimensional vector space with $\dim V = n$. Then any basis of V is finite and has size n.*

11 Dimensions of subspaces

Lemma 15. *Suppose $\mathcal{L} \subset V$ is a linearly independent subset. Let $\mathbf{v} \in V$ be a vector which does not lie in $\operatorname{span} \mathcal{L}$. Then $\mathcal{L} \cup \{\mathbf{v}\}$ is linearly independent.*

Lemma 16. *If V is not finite-dimensional, then for any $n \in \mathbb{N}$ we can find linearly independent subset $\mathcal{L} \subset V$ of size n.*

Lemma 17. *Let V be a finite-dimensional vector space with $\dim V = n$. Then any linearly independent subset $\mathcal{L} \subset V$ of size n must be a basis.*

Lemma 18. *If V is finite-dimensional then any linearly independent subset is contained in a basis.*

Proposition 8. Let V be a finite-dimensional vector space and let $U \subset V$ be a subspace.

1. U is finite-dimensional.
2. $\dim U \leq \dim V$.
3. If $\dim U = \dim V$ then $U = V$.

12 Linear maps

Definition 26. Let U and V be vector spaces. A function $f : U \to V$ is called a **linear map** if

- $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in U$,
- $f(\lambda \mathbf{x}) = \lambda f(\mathbf{x})$ for all $\mathbf{x} \in U$ and all $\lambda \in \mathbb{R}$.

Lemma 19. *If $f : U \to V$ is linear then $f(\mathbf{0}_U) = \mathbf{0}_V$.*

Lemma 20. *A function composition of linear maps is also a linear map.*

Definition 27. Let $f : U \to V$ be a linear map.

- The **image** of f, denoted as $\operatorname{Im} f$ is defined to be the subset

$$\{f(\mathbf{x}) \,|\, \mathbf{x} \in U\} \subset V$$

- The **ker** of f, denoted as $\ker f$ is defined to be the subset

$$\{\mathbf{x} \in U \,|\, f(\mathbf{x}) = \mathbf{0}_V\} \subset U$$

Lemma 21. *Let $f : U \to V$ be a linear map between vector spaces. Then $\ker f$ is a subspace of U and $\operatorname{Im} f$ is a subspace of V.*

Lemma 22. *A linear map $f : U \to V$ is injective if and only if $\ker f = \{\mathbf{0}_U\}$.*

Lemma 23. *Let $f : U \to V$ be a linear map, and fix $\mathbf{y} \in V$. Suppose $\mathbf{x} \in U$ is such that $f(\mathbf{x}) = \mathbf{y}$. Then*

$$f^{-1}(\mathbf{y}) = \{\mathbf{x} + \mathbf{v} \,|\, \mathbf{v} \in \ker f\}$$

13 Linear maps and bases

Proposition 9. Let $f : \mathbb{R}^k \to \mathbb{R}^n$ be linear. Then $f = T_A$ for some matrix $A \in M_{n\times k}(\mathbb{R})$

Proposition 10. Let $f : U \to V$ and $g : U \to V$ be two linear maps, and let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_k\}$ be a basis for U. Suppose $f(\mathbf{b}_i) = g(\mathbf{b}_i)$ for each $i = 1, ..., k$. Then $f = g$.

Proposition 11. Let U and V be vector spaces. Let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_k\}$ be a basis for U, and let $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be any set of k vectors in V. Then there is a unique linear map $f : U \to V$ such that $f(\mathbf{b}_i) = \mathbf{v}_i$ for each i.

14 Isomorphisms