Mathematics Year I, Analysis I Term 1

Theorems, propositions.

Szymon Kubica

April 13, 2021

1 Order Axioms

1.$\forall x \in \mathbb{Q}$ exactly one of the following holds: $x > 0$ or $x = 0$ or $-x > 0$ (Trichotomy axiom)

2.$\forall x \in \mathbb{Q} \exists n \in \mathbb{N}$ such that $n > x$ (Archimedean axiom)

2 Decimals

We define an eventually periodic decimal $a_0.a_1...a_i\overline{a_{i+1}a_{i+2}...a_j}$ for $a_0 \in \mathbb{N}$, $a_{i>0} \in \{0, 1, ..., 9\}$ as the rational number

$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + ... + \frac{a_i}{10^i} + \left(\frac{a_{i+1}a_{i+2}...a_j}{10^j}\right)\left(\frac{1}{1 - 10^{i-j}}\right)$$

Theorem 1. *Any $x \in \mathbb{Q}$ is equal to an eventually periodic decimal expansion:*
$x = a_0.a_1...a_i\overline{a_{i+1}a_{i+2}...a_j}$ *for* $a_0 \in \mathbb{N}$, $a_{i>0} \in \{0, 1, ..., 9\}$.

3 The Completeness Axiom

Definition 1. Suppose $\emptyset \neq S \subset \mathbb{R}$ is bounded above. We define $x \in \mathbb{R}$ to be the **supremum** of S iff:

- X is an upper bound for S (i.e. $x \geq s \forall s \in S$),
- $x \leq y$ for any y which is an upper bound for S ($y \geq s\, \forall s \in S \implies x \leq y$).

4 Dedekind cuts

Definition 2. A nonempty subset $S \in \mathbb{Q}$ is a Dedekind cut if it satisfies the following properties:

- If $s \in S$ and $s > t \in \mathbb{Q}$ then $t \in S$ (S is a semi-infinite interval to the left).
- S is bounded above but has no maximum.

5 Sequences

Definition 3. $a_n \to a$ as $n \to \infty$ iff $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \epsilon$.

Note 1. It is important to remember that N can depend on ϵ.

Definition 4. A sequence a_n converges iff $\exists a \in \mathbb{R}$ such that $\forall \epsilon\ \exists N \in \mathbb{N}$ such that $\forall n \geq N\ |a_n - a| < \epsilon$.

Definition 5. A sequence a_n diverges iff $\forall a \in \mathbb{R}\ \exists\, \epsilon > 0$ s.t. $\forall\, N \in \mathbb{N}\ \exists n \geq N$ such that $|a_n - a| \geq \epsilon$.

Theorem 2. *Limits are unique.* $a_n \to a \wedge a_n \to b \implies a = b$.

Theorem 3. *If (a_n) is bounded above and monotonically increasing then a_n converges to $a := sup\{a_i | i \in \mathbb{N}\}$. We write $a_n \uparrow a$.*

Definition 6. $(a_n)_{n\geq 1}$ is called a Cauchy sequence iff:

$$\forall\, \epsilon > 0\ \exists\ N \in \mathbb{N} \text{ such that } \forall\, n, m \geq N\ |a_n - a_m| < \epsilon.$$

Theorem 4. *If (a_n) is a Cauchy sequence of real numbers then a_n converges.*

6 Subsequences

Theorem 5 (Bolzano-Weierstrass)**.** *If (a_n) is a bounded sequence of real numbers then it has a convergent subsequence.*

Definition 7. We say $a_n \to +\infty$ if and only if

$$\forall R > 0\ \exists N \in \mathbb{N} \text{ such that } \forall\ n \geq N a_n > R$$

7 Series

Definition 8. The sequence of partial sums (s_n) of a series is given by:

$$s_n = \sum_{i=1}^{n} a_i.$$

Definition 9. We say that the series $\sum a_n$ converges to $A \in \mathbb{R}$ if and only if the sequence of partial sums converges to A:

$$\sum_{n=1}^{\infty} a_n = A \iff s_n \to A$$

Theorem 6. $\sum_{n=0}^{\infty} a_n$ *is convergent* $\implies a_n \to 0$.

Proposition 1. Suppose $a_n \geq 0\ \forall n$ (i.e. the sequence of partial sums is monotonically increasing), then the following facts are true:

1. $\sum_{n=0}^{\infty} a_n$ converges iff. (s_n) is bounded above.

2. Similarly $\sum_{n=0}^{\infty} a_n \to +\infty$ iff. (s_n) is unbounded.

Theorem 7 (Comparison Test). *If $0 \leq a_n \leq b_n \, \forall n$ and $\sum b_n$ converges then $\sum a_n$ converges. What is more $0 \leq \sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} b_n$.*

8 Absolute Convergence

Theorem 8. *Let $(a_n)_{n\geq 0}$ be a real or complex sequence. If $\sum a_n$ is absolutely convergent then it is also convergent.*

Theorem 9 (Comparison II (Sandwich Test)). *Suppose $c_n \leq a_n \leq b_n \forall n$ and $\sum c_n$, $\sum b_n$ are both convergent. Then $\sum a_n$ is convergent.*

Theorem 10 (Comparison III). *If $\frac{a_n}{b_n} \to L \in \mathbb{R}$ and $\sum b_n$ is absolutely convergent, then $\sum a_n$ is absolutely convergent.*

Theorem 11 (Alternating Series Test). *Suppose a_n is alternating with $|a_n| \downarrow 0$. Then $\sum a_n$ converges.*

Theorem 12 (Ratio Test). *If a_n is a sequence such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.*

9 Rearrangement of Series

Theorem 13. *$\sum a_n$ is absolutely convergent $\iff (1) \wedge (2) \implies (3) \wedge (4)$, where*

(1) $\sum_{a_n \geq 0} a_n$ is convergent (to A say),

(2) $\sum_{a_n < 0} a_n$ is convergent (to B say),

(3) $\sum a_n = A + B$,

(4) $\sum b_m = A + B$, where (b_m) is any rearrangement of (a_n).

10 Power Series

Let $[0, \infty]$ denote the set $[0, \infty) \cup \{+\infty\}$.

Theorem 14 (Radius of Convergence). *Fix a real or complex series (a_n) and consider the series $\sum a_n z^n$ for $\in \mathbb{C}$. Then $\exists R \in [0, \infty]$ such that*

- *$|z| < R \implies \sum a_n z^n$ is absolutely convergent,*
- *$|z| > R \implies \sum a_n z^n$ is divergent.*

11 Products of Series

Definition 10. The Cauchy Product of two series $\sum a_n$ and $\sum b_n$ is defined as the series $\sum c_n$ where $c_n := \sum_{n=0}^{\infty} a_i b_{n-i}$

Theorem 15 (Cauchy Product)**.** *If $\sum a_n$ and $\sum b_n$ are absolutely convergent, then their Cauchy product $\sum c_n$ is absolutely convergent to $(\sum a_n) \cdot (\sum b_n)$*

12 Exponential Power Series

Definition 11 (Exponential Series)**.** For any $z \in \mathbb{C}$ we set

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Proposition 2. $E(x)$ has the following properties for $x \in \mathbb{R}$:

1. $\forall x \in \mathbb{R} E(x) > 0$,
2. $x \geq 0 \implies E(x) \geq 1$ and $x < 0 \implies E(x) < 1$,
3. $E(x)$ is strictly increasing for $x \in \mathbb{R}$,
4. $|E(x) - 1| \leq \frac{|x|}{1-|x|}$ for $|x| < 1$

13 Continuity

Definition 12. Fix a function $f : \mathbb{R} \to \mathbb{R}$ and points $a, b \in \mathbb{R}$.
We say that $f(x) \to b$ as $x \to a$ (or $\lim_{x \to a} f(x) = b$) if and only if

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - b| < \epsilon$$

Theorem 16. *$f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if and only if*

$$\forall \epsilon > 0 \, \exists \delta > 0 \textit{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Theorem 17 (Sequential Continuity)**.** *$f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if and only if $f(x_n) \to f(a) \, \forall$ sequences $x_n \to a$.*