Mathematics Year I, Linear Algebra Term 1, 2

Most Important theorems, definitions and propositions.

Szymon Kubica

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1 Introduction to matrices and vectors

Definition 1. The standard basis vectors for $\mathbb{R}^n$ are the vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad ... \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{1}$$

Definition 2. Let $v_1, ...v_n$ be vectors in $\mathbb{R}^n$. Any expression of the form

$$\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n$$

is called linear combination of the vectors $v_1, ...v_n$.

Definition 3. The set of all linear combinations of a collection of vectors $v_1, ...v_n$ is called the span of the vectors $v_1, ...v_n$. Notation:

$$span\{v_1, ...v_n\}$$

.

Note 1. R^n is equal to the span of the standard basis vectors.

Definition 4. The norm of v is the non negative real number defined by

$$||v|| = \sqrt{v \cdot v}$$

.

Definition 5. A vector $v \in \mathbb{R}^n$ is called a unit vector if $||v|| = 1$.

Definition 6. Let u and v be vectors in $\mathbb{R}^n$. The distance between u and v is defined by

$$dist(u, v) := ||u - v||$$

.

Definition 7. The (i, j) entry of a matrix is the entry in row i and column j.

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & anm \end{pmatrix}$$

Most often we use the condensed notation $M = (a_{ij})$.

Definition 8. The transpose of an $n \times m$ matrix $A = (a_{ij})$ is the $m \times n$ matrix whose (i, j) entry is a_{ji}. We denote it as A^T.

The leading diagonal of a matrix is the $(1, 1), (2, 2)...$ entries. So the transpose is obtained by doing a reflection in the leading diagonal.

Definition 9. The identity matrix $I_n = (a_{ij})$ is the square matrix such that
$a_{ij} = 0 \; \forall \, i \neq j$, and $a_{ii} = 1$, where $0 < i, j \leq n$.

Definition 10. Let $A = (a_{ij})$ be a $n \times m$ matrix and $\mathbf{b}$ be the column vector of height n and whose ith entry is b_i. Then $(v_1, ..., v_n)$ is a solution to the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1m}x_m & = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2m}x_m & = b_2 \\ \vdots & = \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nm}x_m & = b_n \end{cases}$$

if and only if the vector $\mathbf{v} \in \mathbb{R}^n$ with entries v_i is a solution of the equation

$$Av = b$$

The matrix A is called the coefficient matrix of the system above. The augmented matrix associated to the system is the matrix obtained by adding b as an extra column to A. It is denoted as $(A|b)$.

2 Row operations

Definition 11. A **row operation** is one of the following procedures we can apply to a matrix:

1. $r_i(\lambda)$: Multiply each entry in the ith row by a real number $\lambda \neq 0$.
2. r_{ij} : Swap row i and row j.
3. $r_{ij}(\lambda)$: Add λ times row i to row j.

Proposition 1. Let $Ax = \mathbf{b}$ be a system of linear equations in matrix form. Let r be one of the row operations from Definition 11, and let $(A'|\mathbf{b}')$ be the result of applying r to the augmented matrix $(A|\mathbf{b})$. Then the vector $\mathbf{v}$ is a solution of $Ax = \mathbf{b}$ if and only if it is a solution of $A'x = \mathbf{b}'$.

3 A systematical way of solving linear systems.

Definition 12. The left-most non-zero entry in a non-zero row is called the **leading entry** of that row.

Definition 13. A matrix is in **echelon form** if

1. the leading entry in each non-zero row is 1,
2. the leading 1 in each non-zero row is to the right of the leading 1 in any row above it,
3. the zero rows are below any non-zero rows.

Definition 14. A matrix is in **row reduced echelon form (RRE)** if

1. it is in echelon form,
2. the leading entry in each non zero row is the only non-zero entry in its column.