CW2324

February 19, 2024

1 Fixed Point Grover's algorithm

We have seen that the probability of measuring the marked state a after t steps of Grover's algorithm is $\sin^2((2t+1)\theta)$. This means that if we perform too many steps we decrease the probability of measuring a. It turns out that a modification of Grover's algorithm allows us to avoid this, and produces an algorithm which has a as a fixed point, so that the more we run the algorithm, the more accurate result we get. In this coursework, you will implement such an algorithm and verify its properties.

1.1 Setup

Make sure you have installed Qiskit following https://docs.quantum.ibm.com/start/install and install also the AerSimulator with `pip install qiskit-aer`

Run these cells to import the needed modules

[ ]:
```
# Built-in modules
import math
import warnings

warnings.filterwarnings("ignore")

import qiskit

# Imports from Qiskit
from qiskit import QuantumCircuit
from qiskit.visualization import plot_histogram

from qiskit_aer import AerSimulator
```

1.2 Generalised V

We define an operator $S(\beta)$ that acts on $n+1$ qubits, such that

$$S(\beta)|x\rangle|0\rangle = e^{-i\beta/2}V(\beta)|x\rangle|0\rangle\,, \quad V(\beta) = \exp(i\beta|a\rangle\langle a|) = I - (1-e^{i\beta})|a\rangle\langle a|\,.$$

Here β is a given angle and note that $V(\pm\pi)$ coincides with V of the lecture.

1.2.1 Exercise 1: Implement $S(\beta)$ (40 points)

Create a function which takes as inputs β, a and returns a Qiskit circuit for $S(\beta)$ by implementing the following circuit:

where $U = U_f$ is as usual the oracle such that

$$U_f|a\rangle|y\rangle = |a\rangle|y \oplus 1\rangle\,, \quad U_f|a_\perp\rangle|y\rangle = |a_\perp\rangle|y\rangle$$

for any $\langle a|a_\perp\rangle = 0$ and

$$Z_\beta = \exp\left(-i\frac{\beta}{2}Z\right)$$

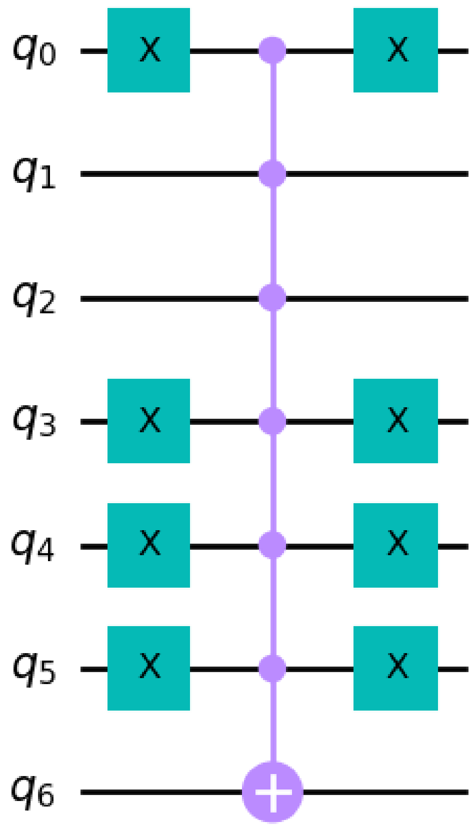
with Z the Pauli matrix.

[ ]:
```python
from qiskit.circuit.library import MCMT, XGate


def grover_oracle(marked_state):
    """Implements the Grover's oracle for a marked state a

    Parameters:
        marked_state (str): Bit string of the marked state (a) of the oracle

    Returns:
        QuantumCircuit: Quantum circuit representing Grover oracle
    """
    num_qubits = len(marked_state) + 1

    qc = QuantumCircuit(num_qubits)

    # We need to reverse the marked state because of the qubit ordering convention
    # used by qiskit. It is needed so that when we want e.g. set a = 10110, then
    # we want the histogram label corresponding to it to be 10110, we opposed to 01101,
    # which would have happened if we didn't reverse the state bit string.
    state_reversed = marked_state[::-1]

    # Find the indices of all the '0' elements in bit-string
    zero_inds = [
        ind for ind in range(num_qubits) if state_reversed.startswith("0", ind)
    ]

    # Add a multi-controlled X-gate with pre- and post-applied X-gates (open-controls)
    # where the target bit-string has a '0' entry
    qc.x(zero_inds)
    qc.compose(MCMT(XGate(), num_qubits - 1, 1), inplace=True)
    qc.x(zero_inds)
```

```
    return qc


def create_s(marked_state, beta):
    """Implement S(\beta)

    Parameters:
        marked_state (str): Marked state of oracle (a)
        beta (float): angle

    Returns:
        QuantumCircuit: Quantum circuit representing S(\beta)
    """
    num_qubits = len(marked_state) + 1
    qc = QuantumCircuit(num_qubits)

    qc.compose(grover_oracle(marked_state), inplace=True)
    qc.barrier()
    # RZ gate implements the exp(-i * beta * Z) transformation.
    qc.rz(beta, num_qubits - 1)
    qc.barrier()
    qc.compose(grover_oracle(marked_state), inplace=True)
    qc.barrier()

    return qc
```

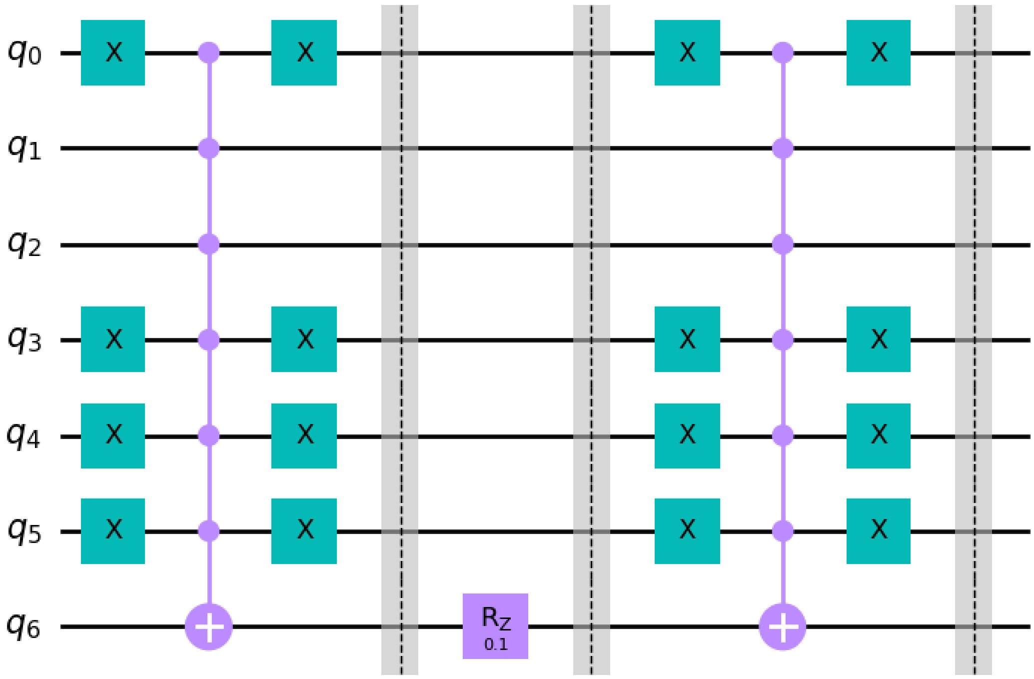
[ ]:
```
# Here we test if the circuits look right
marked_state = "000110"
qc1 = grover_oracle(marked_state)
qc1.draw(output="mpl")
```

[ ]:



[ ]:
```
qc2 = create_s(marked_state, 0.1)
qc2.draw(output="mpl")
```

[ ]:



1.3 Generalised inversion by the mean

We define the generalised inversion by the mean as

$$W(\alpha) = e^{i\alpha/4}(I - (1 - e^{-i\alpha})|\phi\rangle\langle\phi|)$$

with

$$|\phi\rangle = A|0\rangle^{\otimes n}, \quad A = H^{\otimes n}.$$

For $\alpha = \pm$ it coincides with the inversion by the mean seen in the lecture up to a phase.

1.3.1 Exercise 2: Implement $W(\alpha)$ (60 points)

Create a function which takes as inputs α and returns a Qiskit circuit for $W(\alpha)$ by implementing the following circuit:

where $A = H^{\otimes n}$, and

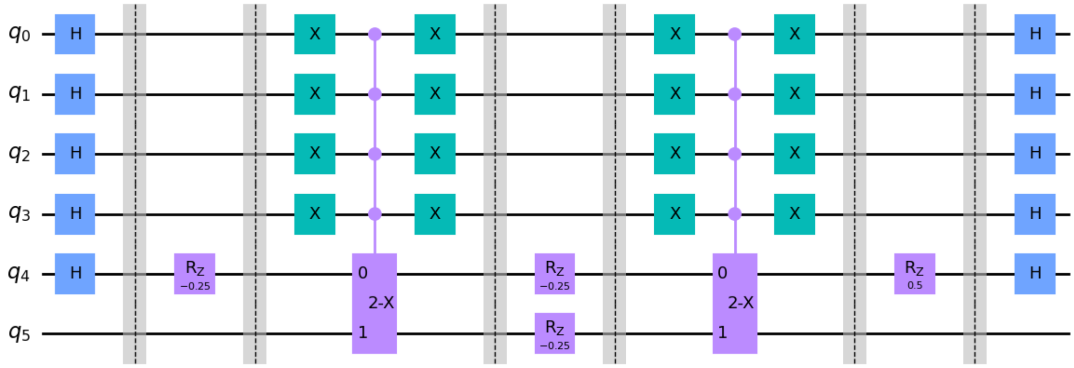
Hint: to implement the multi-controlled CNOT gate you can use the Qiskit method `mcx`

[ ]:
```
from qiskit.circuit.library import MCMT


def create_w(alpha, num_qubits):
    """Implement W(\alpha)

    Parameters:
```

```python
        alpha (float): angle
        num_qubits (int): number of qubits

    Returns:
        QuantumCircuit: Quantum circuit representing W(\alpha)
    """

    qc = QuantumCircuit(num_qubits)

    input_qubits = list(range(num_qubits - 1))
    two_last_qubits = [num_qubits - 2, num_qubits - 1]

    qc.h(input_qubits)
    qc.barrier()
    qc.rz(-alpha / 2, num_qubits - 2)
    # Added barriers for visual grouping of the parts of the circuit
    qc.barrier()
    qc.x(input_qubits[:-1])
    qc.compose(
        MCMT("x", num_ctrl_qubits=num_qubits - 2, num_target_qubits=2),␣
↪inplace=True
    )
    qc.x(input_qubits[:-1])
    qc.barrier()
    qc.rz(-alpha / 2, two_last_qubits)
    qc.barrier()
    qc.x(input_qubits[:-1])
    qc.compose(
        MCMT("x", num_ctrl_qubits=num_qubits - 2, num_target_qubits=2),␣
↪inplace=True
    )
    qc.x(input_qubits[:-1])
    qc.barrier()
    qc.rz(alpha, num_qubits - 2)
    qc.barrier()
    qc.h(input_qubits)

    return qc
```

[ ]:
```python
# Test if the shape of the circuit is correct
qc = create_w(0.5, 6)
qc.draw(output="mpl")
```

[ ]:



1.4 The full algorithm

The full circuit is given by

$$G = \prod_{i=1}^{l} W(\alpha_j) S(\beta_j)$$

The angles α_j, β_j can be chosen in such a way that the success probability in finding a increases with l. We can guarantee a probability greater than $1-\delta^2$ for $\delta \in (0, 1]$ if we choose $L \geq \log(2/\delta)\sqrt{N}$ and the angles as defined below, with $L = 2l + 1$. Note the square root scaling with N as in Grover's, to which the algorithm reduces when $\delta = 1$.

[ ]:
```
def acot(x):
    return math.atan2(1, x)


def cheb(i, x):
    return math.cosh(i * math.acosh(x))


def gamma(L, delta):
    return 1.0 / cheb(1.0 / L, 1.0 / delta)


def alpha_fixed_point(j, l, delta):
    L = 2 * l + 1
    return 2 * acot(math.tan(2 * math.pi * j / L) * math.sqrt(1 - gamma(L,␣
 ↪delta) ** 2))


def beta_fixed_point(j, l, delta):
    return -alpha_fixed_point(l - j + 1, l, delta)
```

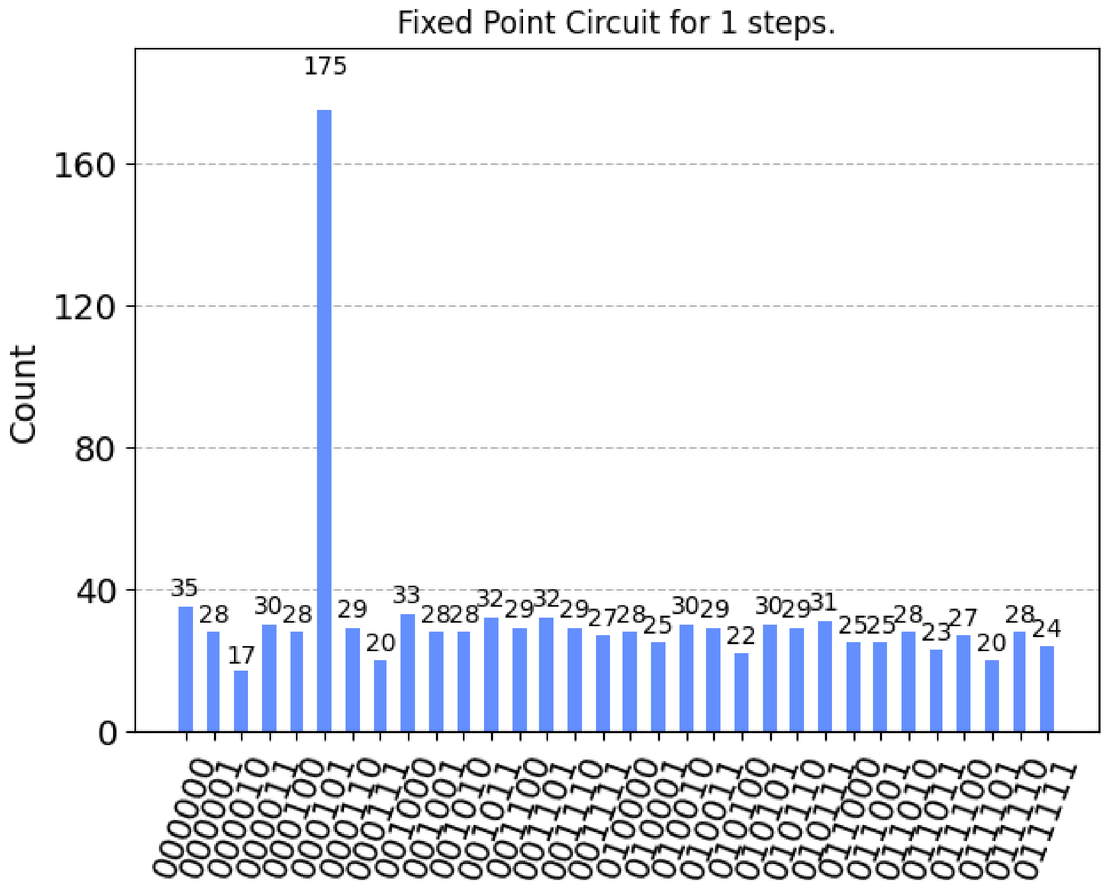
[ ]:
```python
def create_fixed_point_circuit(marked_state, num_steps, delta):
    """Implement G

    Parameters:
        marked_state (str): Marked state of oracle (a)
        num_steps (int): Number of steps of the algorithm (l)
        delta (float): Precision parameter in (0,1]

    Returns:
        QuantumCircuit: Quantum circuit G with input state preparation
    """
    # Compute the number of qubits in circuit
    num_qubits = len(marked_state)

    qc = QuantumCircuit(num_qubits + 1)

    # Prepare input state
    qc.h(list(range(num_qubits)))

    for j in range(num_steps):
        alpha = alpha_fixed_point(j + 1, num_steps, delta)
        beta = beta_fixed_point(j + 1, num_steps, delta)
        qc.compose(create_s(marked_state, beta), inplace=True)
        qc.compose(create_w(alpha, num_qubits + 1), inplace=True)
    return qc
```

Now run this algorithm for $a = 00101$, $\delta = 0.5$ and varying number of steps.

[ ]:
```python
from qiskit import Aer, execute

# As instructed above, running for a varying number of steps
results = {}
for steps in range(10):
    fp_qc = create_fixed_point_circuit("00101", steps + 1, 0.5)
    fp_qc.measure_all()
    # fp_qc.draw(output="mpl", style="iqp")
    simulator = Aer.get_backend("qasm_simulator")
    results[steps] = execute(fp_qc, simulator, shots=1024).result()

DRAW_FULL_CIRCUIT = False

if DRAW_FULL_CIRCUIT:
    fp_qc = create_fixed_point_circuit("00101", 4, 0.5)
    fp_qc.measure_all()
    fp_qc.draw(output="mpl", style="iqp")
```

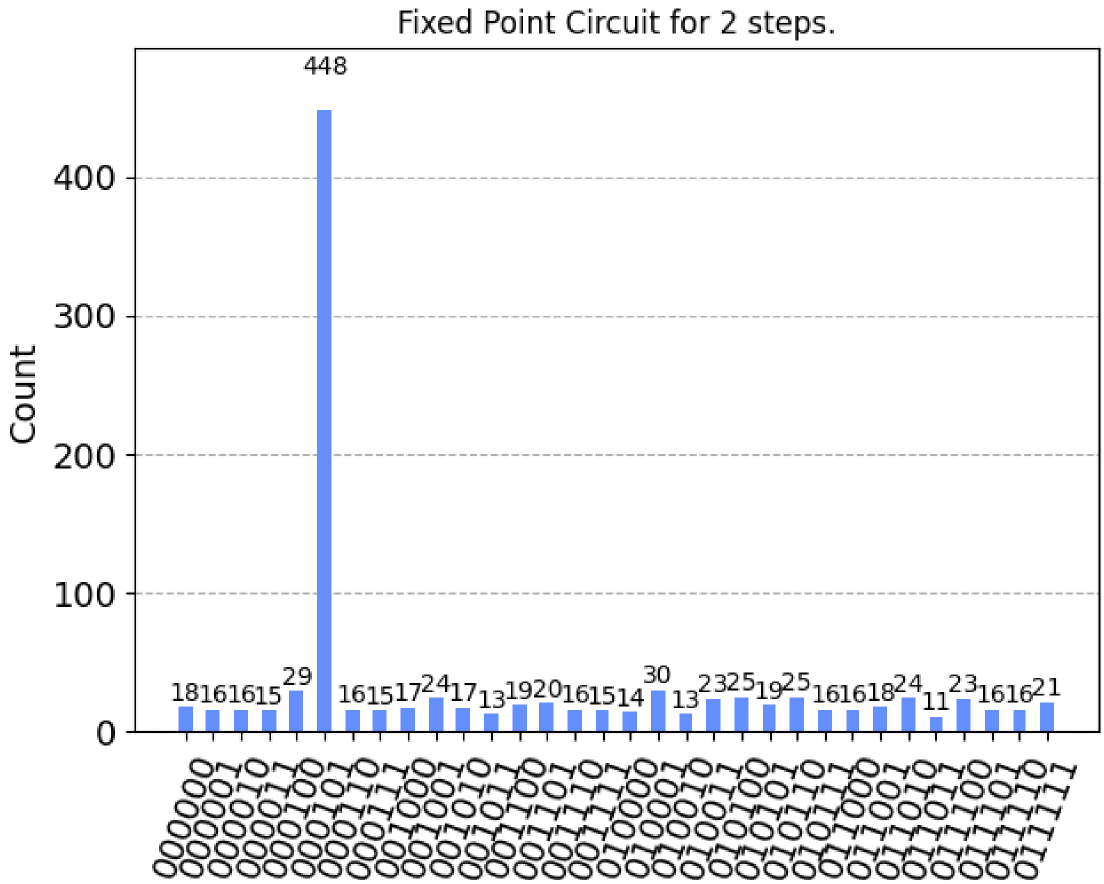
[ ]:
```
# For some reason it doesn't want to plot many histograms in one cell.

counts = results[0].get_counts()
plot_histogram(counts, title=f"Fixed Point Circuit for {1} steps.")
```

[ ]:

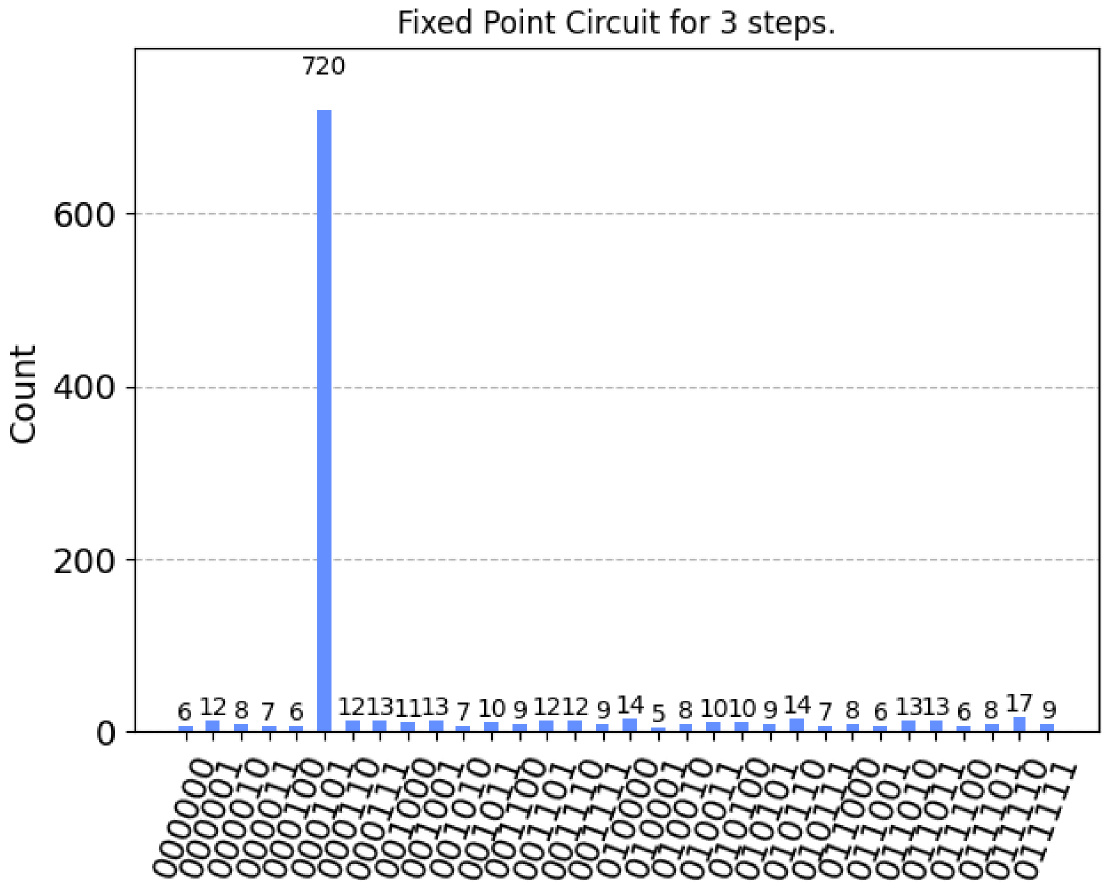


[ ]:
```
# For some reason it doesn't want to plot many histograms in one cell.

counts = results[1].get_counts()
plot_histogram(counts, title=f"Fixed Point Circuit for {2} steps.")
```

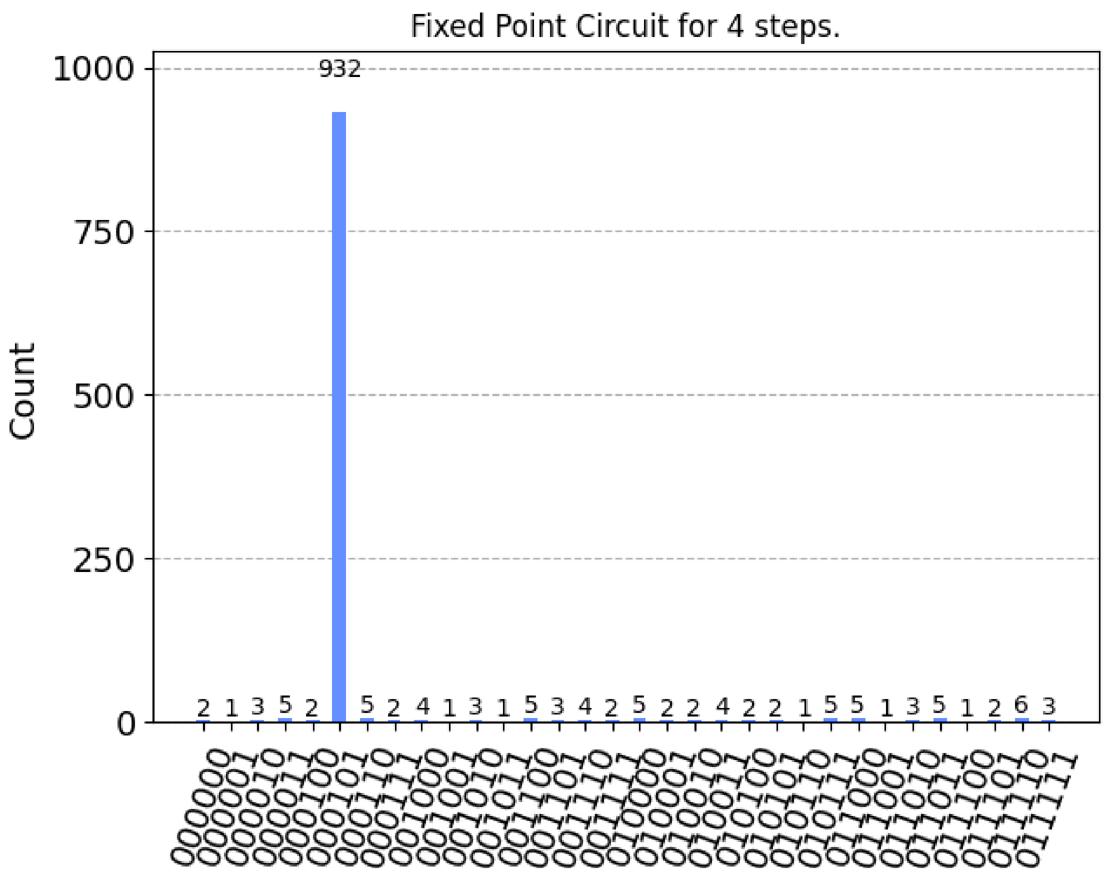
[ ]:



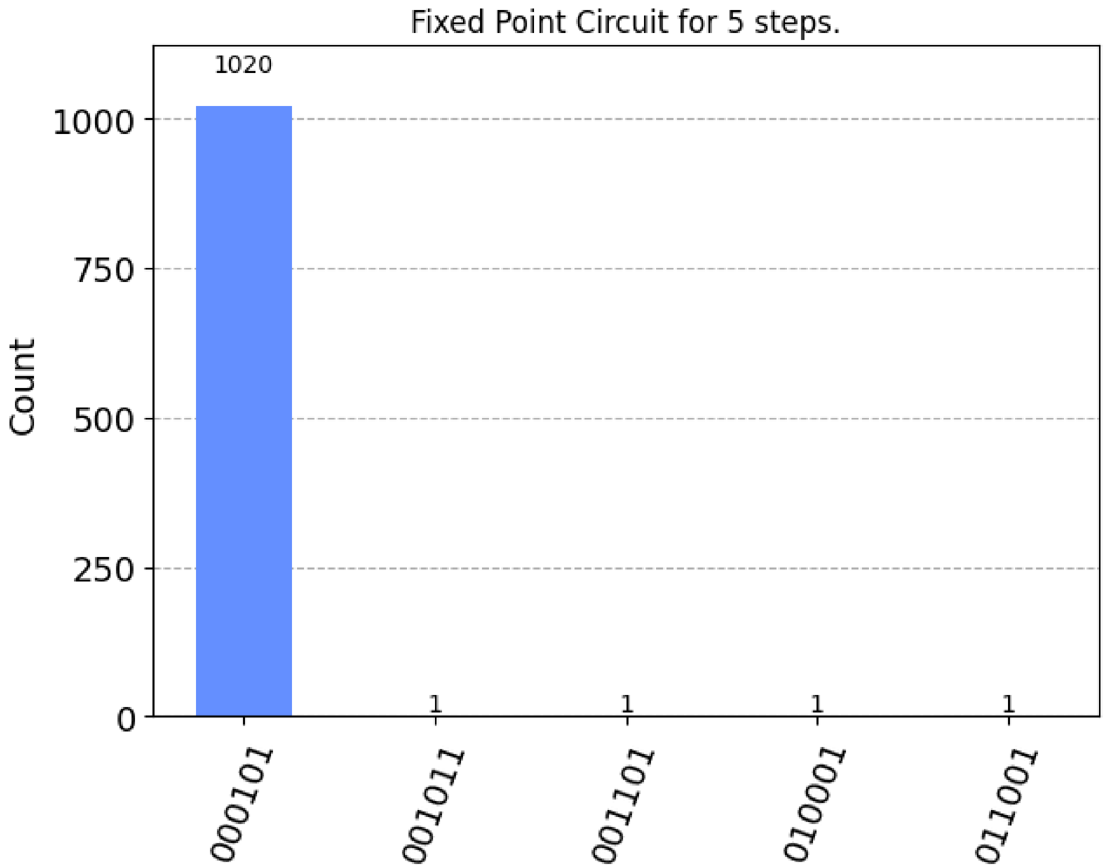

```
[ ]: # For some reason it doesn't want to plot many histograms in one cell.

     counts = results[2].get_counts()
     plot_histogram(counts, title=f"Fixed Point Circuit for {3} steps.")
[ ]:
```




```
[ ]: # For some reason it doesn't want to plot many histograms in one cell.

     counts = results[3].get_counts()
     plot_histogram(counts, title=f"Fixed Point Circuit for {4} steps.")
[ ]:
```

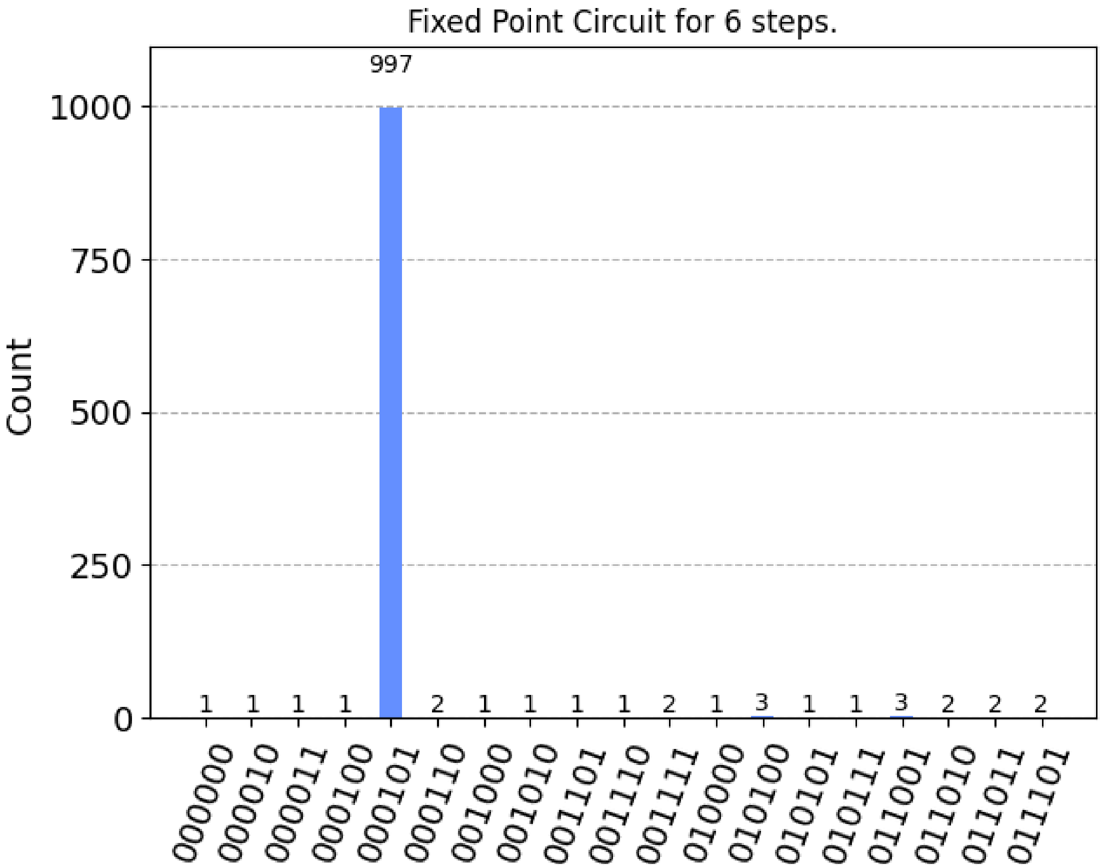



```
[ ]: # For some reason it doesn't want to plot many histograms in one cell.

     counts = results[4].get_counts()
     plot_histogram(counts, title=f"Fixed Point Circuit for {5} steps.")
[ ]:
```



[ ]:
```
# For some reason it doesn't want to plot many histograms in one cell.

counts = results[5].get_counts()
plot_histogram(counts, title=f"Fixed Point Circuit for {6} steps.")
```

[ ]:



Compare with the case $\delta = 1$ which corresponds to Grover's, to check the fixed point property.

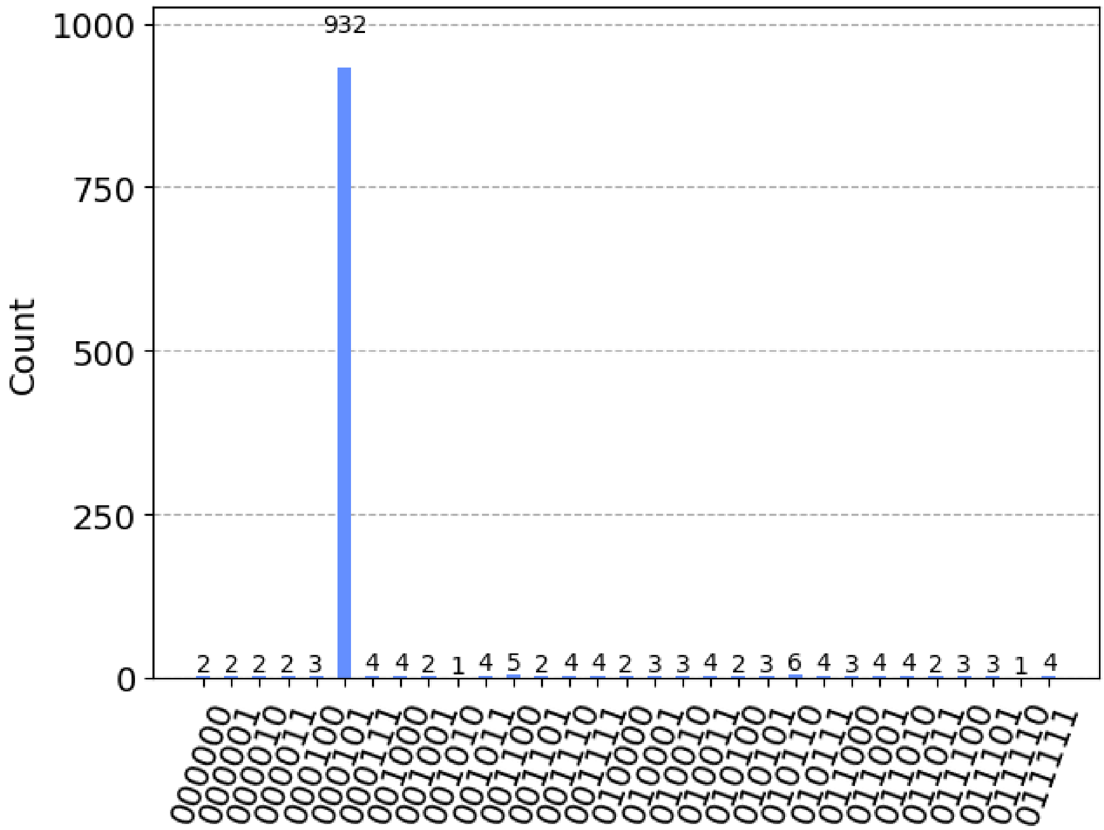
[ ]:
```
# Probability of measuring a in Grover's
[math.sin((2 * t + 1) * math.asin(1 / math.sqrt(32))) ** 2 for t in range(10)]
```

[ ]:
```
[0.031249999999999993,
 0.25830078125,
 0.6024246215820311,
 0.896936535835266,
 0.9991823155432941,
 0.8596366611600389,
 0.5458919990273899,
 0.20991839986584915,
 0.014453075769287534,
 0.054174543096977154]
```

[ ]:
```
# Given the probabilities above, Grover's algorithm achieves the best␣
 ↪performance for 4 steps.
# We can also see in the graph below that it is very good.
grover_qc = create_fixed_point_circuit("00101", 3, 1)
```

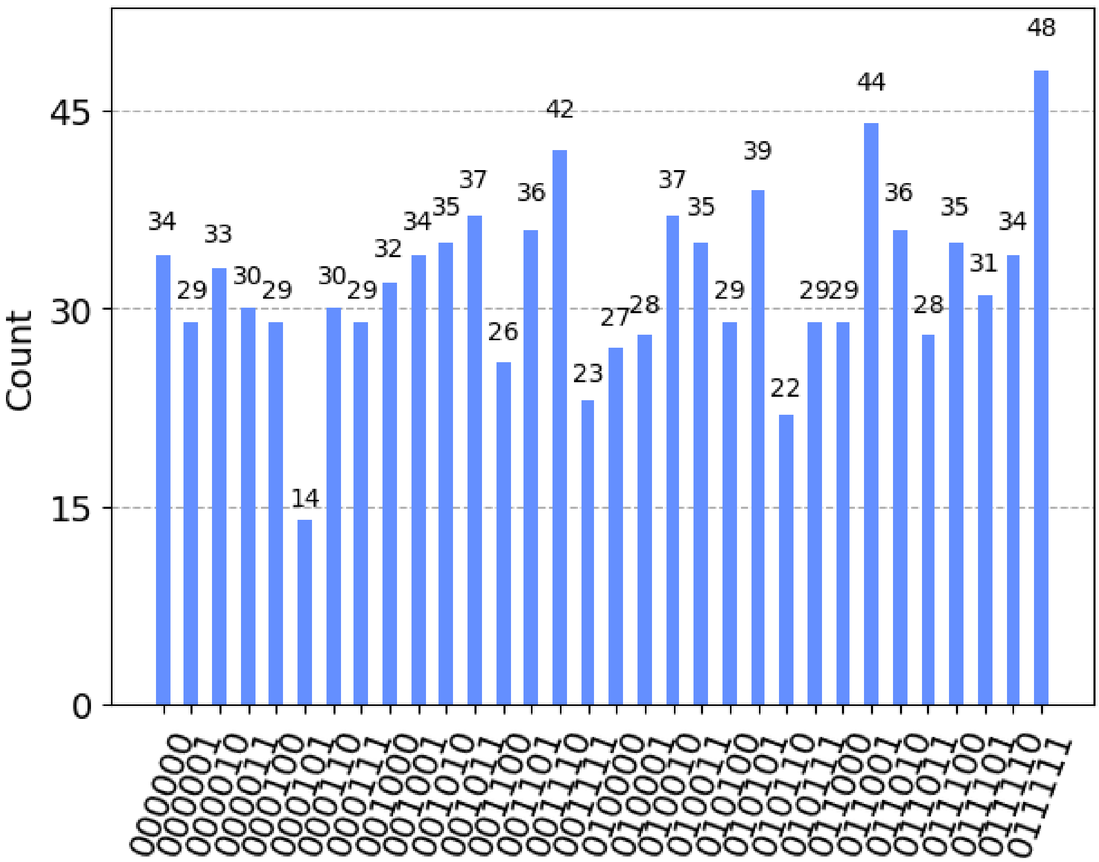
```python
grover_qc.measure_all()
simulator = Aer.get_backend("qasm_simulator")
result = execute(grover_qc, simulator, shots=1024).result()
counts = result.get_counts()
plot_histogram(counts)
```

[ ]:



[ ]:
```python
# If however, we increase the number of steps so that it is larger than
# the optimal 4, we can see that the performance starts to deteriorate.
# 8 here means that overall 9 steps of the algorhthm are performed, which
# corresponds to the lowest success probability above.
grover_qc = create_fixed_point_circuit("00101", 8, 1)
grover_qc.measure_all()
simulator = Aer.get_backend("qasm_simulator")
result = execute(grover_qc, simulator, shots=1024).result()
counts = result.get_counts()
plot_histogram(counts)
```

[ ]:



As expected 8 steps of Grover's algorithm give a much worse estimate than say 5 steps, while the fixed point algorithm does not suffer from this problem.

1.5 References

- https://learning.quantum.ibm.com/tutorial/grovers-algorithm
- `https://arxiv.org/abs/1409.3305`.
- See also `https://arxiv.org/abs/2105.02859` for a more general perspective